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A Dissertation Submitted in the Partial Fulfillment of the Requirements for the Degree of

MASTER OF PHILOSOPHY

IN

MATHEMATICS

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## CERTIFICATE

A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTER OF PHILOSOPHY

**We accept this dissertation as conforming to the required standard**

1. ____________________
Prof. Dr. Muhammad Ayub
(Chairman)

2. ____________________
Dr. Malik Muhammad Yousaf
(Supervisor) 10/7/08

3. ____________________
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(External Examiner)

**Department of Mathematics**
**Quaid-i-Azam University**
**Islamabad, Pakistan**
**2008**

# Dedicated

## To

## The Holy Prophet
## Hazrat
## Muhammad(PBUH)

**Source of hope, inspiration,**
**contentment, knowledge**
**and guidance to all the Humanity.**
**Well wisher of all human beings.**

# Acknowledgement

All praises and reverences to Almighty Allah, the merciful, the most bounteous and the creator of all the creatures in the universe. Thanks to Him who adorned us with the crown of humanity. Many many thanks to Him as He blessed us with the Holy Prophet, Hazrat Muhammad (PBUH) for Whom the whole universe is created. Whatever the respect I have, it is because of my Prophet Muhammad ( PBUH) Grace and kindness.

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**Azad Hussain**

# Preface

The subject of non-Newtonian fluid mechanics is very popular and an area of active research especially in industrial and technological problems. Examples of non-Newtonian fluids include tomato sauce, mustard, mayonnaise, toothpaste, asphalt, lava and ice, mud slides, snow avalanches etc. The flow characteristics of non-Newtonian fluids are quite different from those of Newtonian fluids. The order of the governing equations for the non-Newtonian flow problem is in general higher than the corresponding Newtonian problem. Hence one needs the additional boundary/ initial condition(s) for a unique solution. This issue of extra conditions has been discussed in detail by Rajagopal [1], Rajagopal and Gupta [2], Rajagopal et al. [3] and Rajagopal and Kaloni [4]. Furthermore the equations of non-Newtonian fluids are more non-linear than the Newtonian fluids and to obtain an analytic solution is not an easy task. Despite all these challenges, various workers [5-15] are recently engaged in finding the analytical solutions for flows involving non-Newtonian fluids. Literature survey further indicates that very less attention has been given to the flows of non-Newtonian fluids with variable viscosity. The works of Massoudi and Christie [16], Pakdermirli and Yilbas [17] and Pantikrators [18] may be mentioned in this direction. The objective of this dissertation is to model and analyze the flow between two coaxial cylinders. So the dissertation is arranged in the following way:

In chapter one, we have discussed the basic definitions and the governing equations for third grade fluid in cylindrical coordinate system have been derived.

Chapter two deals with the approximate analytical solutions obtained by means of perturbation method for the flow of a third grade fluid in a pipe.

Chapter three is devoted to the study of flow of a third grade fluid between coaxial cylinders with variable viscosity (temperature dependent). The analytical expressions of velocity and temperature are developed in each case by using a powerful technique namely Homotopy analysis method (HAM).

# Contents

# Chapter 1

# Basics of Fluids

## 1.1 Introduction

This chapter deals with some basic definitions of fluid mechanics. The equation of continuity is illustrated briefly, the energy and momentum equations for third grade fluid are derived in cylindrical coordinate system.

## 1.2 Fluid mechanics

It is that branch of applied mathematics in which we study the behavior of fluids in the states of rest as well as in motion.

## 1.3 Fluid

Fluid is the material that flows or a fluid is a substance that deforms continuously under the application of a shear stress no matter how small the shear stress may be .

### 1.3.1 Rest

A body is said to be at rest with respect to an observer if it's position does not change according to the observer.

### 1.3.2 Motion

A body is said to be in motion with respect to an observer if it's position changes according to the observer.

### 1.3.3 Types of fluids

#### Newtonian fluids

A fluid in which the shear stress is directly proportional to the rate of deformation. For such fluids Newton's law of viscosity holds i.e,

$$\tau_{yx} = \mu \frac{du}{dy}, \tag{1.1}$$

where the constant of proportionality $\mu$ is known as the viscosity of the fluid.

**Examples:** Water, sugar solutions, glycerin, silicone oils, light-hydrocarbons oils and air.

#### Non-Newtonian fluids

A fluid in which the shear stress is not directly proportional to the rate of deformation is called non-Newtonian fluid.

**Examples:** Toothpaste, blood, ketchup, leucite paint, drilling muds and biological fluids etc.

For such fluids power law model hold i.e,

$$\tau_{yx} = k \left( \frac{du}{dy} \right)^n, n \neq 1, \tag{1.2}$$

where "$n$" is flow behavior index and $k$ is the consistency index. Note that $Eq.$ (1.2) reduces to Newton's law of viscosity for $n = 1$ with $k = \mu$.

$$\tau_{yx} = \eta \frac{du}{dy}, \tag{1.3}$$

where

$$\eta = k \left( \frac{du}{dy} \right)^{n-1}, \tag{1.4}$$

is apparent viscosity.

## 1.4 Flow

A material goes under deformation when different forces act upon it. If the deformation continuously increases without limit, then the phenomenon is known as flow.

The treatment of fluid mechanics can be divided into two broad categories:

1. Internal flow systems.

2. External flow systems.

### 1.4.1 Internal flow systems

Internal flow systems are those where fluid flows through confined spaces, such as pipes and open channels.

### 1.4.2 External flow systems

External flow systems are those where confining boundaries are at relatively larger or infinite distances such as the atmosphere through which aeroplanes, missile and space vehicles travel.

### 1.4.3 Types of flows

#### Steady flow

The type of flow in which the time rate of change of velocity of the fluid at a point is zero is called steady flow.

#### Unsteady flow

The type of flow in which the velocity of the fluid at a point does not remain constant with respect to time is called unsteady flow.

### Uniform flow

The type of flow in which the velocity of the fluid is of the same magnitude and direction at every point in the fluid is called uniform flow.

### Non-uniform flow

The type of flow in which the velocity of the fluid is not of the same magnitude and direction at every point in the fluid is called non-uniform flow.

### Laminar flow

The flow is said to be streamline or laminar, if every particle that passes a particular point, moves along exactly the same path, as followed by particles which passed that points earlier. In this type of flow the fluid travels smoothly or in regular paths. The velocity, pressure, and other flow properties at each point in the fluid remain constant. Laminar flow over a horizontal surface may be thought of as consisting of thin layers, all parallel to each other, that slide over each other. In the laminar flow each liquid particle has a definite path. In this case each particle of the fluid moves along a smooth path called a streamline. The different streamlines can not cross each other.

### Turbulent flow

Above a certain velocity of the fluid flow, the motion of the fluid becomes unsteady and irregular. Under this condition the velocity of the fluid changes abruptly. In this case the exact path of the particles of the fluid can not be predicted. The irregular or unsteady flow of the fluid is called turbulent. In the turbulent flow each liquid particle does not have a definite path. In this type of flow the fluid undergoes irregular fluctuations, or mixing. The speed of the fluid at a point is continuously undergoing changes in magnitude and direction, which results in swirling and eddying as the bulk of the fluid moves in a specific direction. Common examples of turbulent flow include atmospheric and ocean currents, blood flow in arteries, oil transport in pipelines, lava flow, the flow through pumps and turbines, the flow in boat wakes and around aircraft wing tips.

#### Compressible flows

These are flows in which density of the fluid can not be supposed to be constant.

#### Incompressible flows

These are flows in which density of the fluid can be supposed to be constant.

## 1.5 Thermodynamic properties

### 1.5.1 Temperature

Degree of hotness and coldness of a body is termed as temperature.

### 1.5.2 Heat

Heat may be defined as energy in transit from a high temperature object to a lower temperature object.

### 1.5.3 Heat flux

Heat flux is defined as rate of heat transfer per unit cross-sectional area .

### 1.5.4 Thermal conductivity

Thermal conductivity $k$, is the intensive property of a material that indicates its ability to conduct heat. It is defined as the quantity of heat $Q$, transmitted in time $t$ through a thickness $L$, in a direction normal to a surface of area $A$, due to a temperature difference $\Delta T$, under steady state conditions and when the heat transfer is dependent only on the temperature gradient.

Mathematically it can be written as

$$k = \frac{Q \times L}{t \times A \times \Delta T}, \tag{1.4a}$$

where $Q$ is the heat flux, $t$ is the time, $A$ is the area, $\Delta T$ is the characteristic temperature and $L$ is the thickness.

### 1.5.5 Fourier law of heat conduction

The law states that the rate of heat flow through a substance is proportional to the area normal to the direction of flow and to the negative of the rate of change of temperature with distance along the direction of flow. The heat flux $Q$, at a point in a medium is directly proportional to the temperature gradient at the point.

$$Q = -k\Delta T. \tag{1.4b}$$

The negative sign indicates that heat flows in the direction of decreasing temperature.

### 1.5.6 Brinkman number

It is the ratio of viscous dissipation and conduction due to change in temperature.

$$\Gamma = \frac{\eta U^2}{k\left(T_w - T_0\right)}, \tag{1.4c}$$

where $\Gamma$ is the the Brinkman number, $\eta$ is the fluid viscosity, $U$ is the fluid velocity, $T_0$ is the bulk fluid temperature and $T_w$ is the wall temperature.

### 1.5.7 Internal Energy

Internal energy of a system, is the energy content of the system due to its thermodynamic properties such as temperature and pressure. The change of internal energy of a system depends only on the initial and final states of the system and not in any way by the path or manner of the change. This concept is used to define the first law of thermodynamics.

## 1.6 Surface forces

Surface forces include all forces acting on the boundaries of a medium through direct contact. Pressure is an example of surface force. These are called short range forces.

## 1.7 Body force

Forces developed without physical contact and distributed over the volume of the fluid are termed as body forces. Gravitational and electromagnetic forces are examples of body forces arising in a fluid. These are long range forces.

## 1.8 Stress

The concept of stress provides a convenient way to describe the manner in which forces acting on the boundaries of the medium are transmitted through the medium. Force per unit area is known as stress. Stresses are surface forces.

## 1.9 Viscosity

Viscosity is a physical property of fluids associated with shearing deformation of fluid particles subjected to the action of applied forces or the viscosity of a fluid is a measure of its resistance to shear or angular deformation or viscosity is the resistance of a fluid to its motion.

The frictional effect between different layers of a flowing fluid is described in terms of viscosity of the fluid. Viscosity measures, how much force is required to slide one layer of the liquid over another layer.

### 1.9.1 Kinematic viscosity ($\nu$)

The ratio of absolute viscosity to density is called kinematic viscosity i.e,

$$\nu = \frac{\mu}{\rho}. \tag{1.5}$$

where $\rho$ is density.

## 1.10 Cylindrical coordinates

In cylindrical coordinates system a point in three-dimensional space is represented by the order triple$(r, \theta, z)$, where $r$ and $\theta$ are polar coordinates of the projection of point on the $xy-$plane.

To convert from rectangular coordinates to cylindrical coordinates and cylindrical coordinates to rectangular coordinates, we use the following equation

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z, \tag{1.5a}$$

$$r^2 = x^2 + y^2,\ tan\theta = \frac{y}{x},\ z = z. \tag{1.5b}$$

## 1.11 Equation of continuity

The mathematical expression of law of conservation of mass is known as equation of continuity. It is defined as

$$\frac{\partial\rho}{\partial t} + \nabla.(\rho\mathbf{V}) = 0. \tag{1.6}$$

For an incompressible fluid it can be written as

$$\nabla.\mathbf{V} = 0. \tag{1.7}$$

## 1.12 Equation of motion

We construct the governing equation for third grade fluid in cylindrical coordinates. The equation of motion in the presence of body forces is

$$\rho\frac{d\mathbf{V}}{dt} = div(\tau) + \rho\mathbf{B}, \tag{1.8}$$

where $\rho$ is the density of the fluid, $d/dt$ is the total derivative, $\tau$ is the Cauchy stress tensor and $\mathbf{B}$ is the body force. For third grade fluid the Cauchy stress tensor which satisfy the thermodynamic property, is defined as

$$\tau = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_3\left(tr(\mathbf{A}_1^2)\mathbf{A}_1\right), \tag{1.9}$$

where $p$ is the pressure, $\mu$ is the dynamic viscosity, $\alpha_1, \alpha_2$ and $\beta_3$ are material constants such that $\alpha_1 > 0,\ \beta_3 > 0,\ |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3}$. The *Rivlin Erickson* tensors $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$...are defined as

$$\mathbf{A}_1 = \text{grad}\mathbf{V} + [\text{grad}\mathbf{V}]^{t_1}, \tag{1.10}$$

$$\mathbf{A}_n = \frac{d\mathbf{A}_{n-1}}{dt} + \mathbf{A}_{n-1}(\text{grad}\mathbf{V}) + (\text{grad}\mathbf{V})^{t_1}\,\mathbf{A}_{n-1},\ n > 1, \tag{1.11}$$

in which $\mathbf{V}$ is the velocity, $t$ is the time and $t_1$ denotes the transpose.

We seek the velocity field of the following form

$$\mathbf{V} = [0, 0, w(r)], \tag{1.12}$$

where $w(r)$ is the $z-$component of velocity.

With the help of $Eq.\,(1.12)$, we can calculate

$$L = grad\mathbf{V} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial w}{\partial r} & 0 & 0 \end{bmatrix}, \tag{1.13}$$

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & \frac{\partial w}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial w}{\partial r} & 0 & 0 \end{bmatrix}, \tag{1.14}$$

$$\mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1(grad\mathbf{V}) + (grad\mathbf{V})^{t_1}\,\mathbf{A}_1, \tag{1.15}$$

$$\mathbf{A}_1(grad\mathbf{V}) = \begin{bmatrix} 0 & 0 & \frac{\partial w}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial w}{\partial r} & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial w}{\partial r} & 0 & 0 \end{bmatrix} = \begin{bmatrix} (\frac{\partial w}{\partial r})^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.16}$$

$$(\text{grad}\mathbf{V})^{t_1}\,\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & \frac{\partial w}{\partial r} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & \frac{\partial w}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial w}{\partial r} & 0 & 0 \end{bmatrix} = \begin{bmatrix} (\frac{\partial w}{\partial r})^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.16a}$$

$$\frac{d\mathbf{A}_1}{dt} = \left[\frac{\partial}{\partial t} + (\mathbf{V}.\nabla)\right]\mathbf{A}_1, \tag{1.16b}$$

$$\frac{d\mathbf{A}_1}{dt} = \left[\frac{\partial}{\partial t} + (\mathbf{V}.\nabla)\right]\mathbf{A}_1, \tag{1.16b}$$

$$\frac{d\mathbf{A}_1}{dt} = \left[\frac{\partial}{\partial t} + \left([0,0,w\,(r)]\,.\left(\frac{\partial}{\partial r},\frac{1}{r}\frac{\partial}{\partial \theta},\frac{\partial}{\partial z}\right)\right)\right]\mathbf{A}_1, \tag{1.17}$$

$$\frac{d\mathbf{A}_1}{dt} = 0, \tag{1.18}$$

$$\mathbf{A}_2 = \begin{bmatrix} 2(\frac{\partial w}{\partial r})^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.19}$$

$$\mathbf{A}_1^2 = \begin{bmatrix} 0 & 0 & \frac{\partial w}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial w}{\partial r} & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & \frac{\partial w}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial w}{\partial r} & 0 & 0 \end{bmatrix}, \tag{1.20}$$

$$\mathbf{A}_1^2 = \begin{bmatrix} (\frac{\partial w}{\partial r})^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (\frac{\partial w}{\partial r})^2 \end{bmatrix}, \tag{1.21}$$

$$tr(\mathbf{A}_1^2) = 2(\frac{\partial w}{\partial r})^2, \tag{1.22}$$

$$div\,(\mu\mathbf{A}_1) = div\left(\begin{bmatrix} 0 & 0 & \mu\frac{\partial w}{\partial r} \\ 0 & 0 & 0 \\ \mu\frac{\partial w}{\partial r} & 0 & 0 \end{bmatrix}\right), \tag{1.23}$$

$$div\,(\mu\mathbf{A}_1)_r = \frac{1}{1.r.1}\left[\frac{\partial}{\partial r}(r.1.0) + \frac{\partial}{\partial \theta'}(1.1.0) + \frac{\partial}{\partial z}\left(1.r.\mu\frac{\partial w}{\partial r}\right)\right], \tag{1.24}$$

$$div\,(\mu\mathbf{A}_1)_r = \frac{1}{r}(0+0+0) = 0, \tag{1.25}$$

$$div\,(\mu\mathbf{A}_1)_{\theta^*} = \frac{1}{1.r.1}\left[\frac{\partial}{\partial r}(r.1.0) + \frac{\partial}{\partial \theta^*}(1.1.0) + \frac{\partial}{\partial z}(0)\right], \tag{1.26}$$

$$div\,(\mu\mathbf{A}_1)_{\theta^*} = 0, \tag{1.27}$$

$$div\,(\mu\mathbf{A}_1)_z = \frac{1}{1.r.1}\left[\frac{\partial}{\partial r}\left(r.1.\mu\frac{\partial w}{\partial r}\right) + 0 + 0\right], \tag{1.28}$$

$$div\,(\mu \mathbf{A}_1)_z = \frac{1}{r}\frac{d}{dr}\left(r.\mu\frac{dw}{dr}\right), \tag{1.29}$$

$$div\,(\alpha \mathbf{A}_2) = div\left(\begin{bmatrix} 2\alpha_1(\frac{\partial w}{\partial r})^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right), \tag{1.30}$$

$$div\,(\alpha_1 \mathbf{A}_2)_r = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r.1.2\alpha_1(\frac{\partial w}{\partial r})^2\right) + 0 + 0\right], \tag{1.31}$$

$$div\,(\alpha_1 \mathbf{A}_2)_r = \frac{1}{r}\frac{d}{dr}\left(.2r\alpha_1(\frac{\partial w}{\partial r})^2\right), \tag{1.32}$$

$$div\,(\alpha_1 \mathbf{A}_2)_{\theta^*} = 0 = div\,(\alpha_1 \mathbf{A}_2)_z\,, \tag{1.33}$$

$$div\,(\alpha_2 \mathbf{A}_1^2) = \begin{bmatrix} \alpha_2(\frac{\partial w}{\partial r})^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha_2(\frac{\partial w}{\partial r})^2 \end{bmatrix}, \tag{1.34}$$

$$div\,(\alpha_2 \mathbf{A}_1^2)_r = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r.\alpha_2(\frac{\partial w}{\partial r})^2\right) + 0 + 0\right], \tag{1.35}$$

$$div\,(\alpha_2 \mathbf{A}_1^2)_r = \frac{1}{r}\frac{d}{dr}\left(.r\alpha_2(\frac{\partial w}{\partial r})^2\right), \tag{1.36}$$

$$div\,(\alpha_2 \mathbf{A}_1^2)_{\theta^*} = 0 = div\,(\alpha_2 \mathbf{A}_1^2)_z\,, \tag{1.37}$$

$$div\,(\beta_3\,(tr(\mathbf{A}_1^2)\mathbf{A}_1)) = \begin{bmatrix} 0 & 0 & 2\beta_3(\frac{\partial w}{\partial r})^3 \\ 0 & 0 & 0 \\ 2\beta_3(\frac{\partial w}{\partial r})^3 & 0 & 0 \end{bmatrix}, \tag{1.38}$$

$$div\,(\beta_3\,(tr(\mathbf{A}_1^2)\mathbf{A}_1))_r = 0 = div\,(\beta_3\,(tr(\mathbf{A}_1^2)\mathbf{A}_1))_{\theta^*}\,, \tag{1.39}$$

$$div\,(\beta_3\,(tr(\mathbf{A}_1^2)\mathbf{A}_1))_z = \frac{1}{r}\frac{d}{dr}\left(.2r\beta_3(\frac{\partial w}{\partial r})^3\right), \tag{1.40}$$

$$\frac{d\mathbf{V}}{dt} = \left[\frac{\partial}{\partial t} + (\mathbf{V}.\nabla)\right]\mathbf{V}, \tag{1.41}$$

$$\frac{d\mathbf{V}}{dt} = \left[[0, 0, w\,(r)]\,.\left[\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta^*}, \frac{\partial}{\partial z}\right]\right]\mathbf{V}, \tag{1.42}$$

$$\frac{d\mathbf{V}}{dt} = 0. \tag{1.43}$$

Using *Eqs.* (1.13) to (1.43), the component form of *Eq.* (1.8) in the absence of body forces, take the following form

$$\frac{\partial p}{\partial r}=\frac{1}{r}\frac{d}{dr}\left[(2\alpha_1+.\alpha_2)\,r(\frac{\partial w}{\partial r})^2\right], \tag{1.44}$$

$$\frac{\partial p}{\partial \theta^*}=0, \tag{1.45}$$

$$\frac{\partial p}{\partial z}=\frac{1}{r}\frac{d}{dr}\left(\mu r\frac{dw}{dr}\right)+\frac{1}{r}\frac{d}{dr}\left(2r\beta_3(\frac{dw}{dr})^3\right). \tag{1.46}$$

*Eq.* (1.44) shows $p\neq p(\theta^*)$, the response of the velocity field occur only in the $z-$component of velocity, therefore, we can write *Eq.* (1.46) again in (bars) the following form

$$\frac{d\bar{\mu}}{d\bar{r}}\frac{d\bar{v}}{d\bar{r}}+\frac{\bar{\mu}}{\bar{r}}\left(\frac{d\bar{v}}{d\bar{r}}+\bar{r}\frac{d^2\bar{v}}{d\bar{r}^2}\right)+\frac{2}{\bar{r}}\beta_3(\frac{d\bar{v}}{d\bar{r}})^3+6\beta_3(\frac{d\bar{v}}{d\bar{r}})^2\frac{d^2\bar{v}}{d\bar{r}^2}=\frac{\partial p}{\partial z}. \tag{1.47}$$

Introducing the following non-dimensional variables

$$r=\frac{\bar{r}}{R},v=\frac{\bar{v}}{v_0},\mu=\frac{\bar{\mu}}{\mu_*}. \tag{1.48}$$

With the help of *Eq.* (1.48), *Eq.* (1.47) takes the form

$$\frac{d\mu}{dr}\frac{dv}{dr}+\frac{\mu}{r}(\frac{dv}{dr}+r\frac{d^2v}{dr^2})+\frac{\Lambda}{r}(\frac{dv}{dr})^2\left(\frac{dv}{dr}+3r\frac{d^2v}{dr^2}\right)=C, \tag{1.49}$$

where

$$\Lambda=\frac{2v_0^2\beta_3}{\mu_*R^2},\ C_1=\frac{\partial p}{\partial z},\ C=\frac{C_1R^2}{\mu_*v_0}. \tag{1.50}$$

## 1.13 Energy equation

The law of energy states

$$\rho\frac{de}{dt}=\tau.\mathrm{L}-\mathrm{div}\,(q)+\rho r^*, \tag{1.51}$$

where $r^*$ is radiant heating. In absence of radiant heating ,

$$\rho\frac{de}{dt}=\tau.\mathrm{L}-\mathrm{div}\,(\mathrm{q}). \tag{1.52}$$

According to Fourier's law

$$q = -k \operatorname{grad} \theta, \tag{1.53}$$

where $q$ is heat flux vector, e $= C_p \theta$, $\theta$ is temperature and $C_p$ is specific heat. Assuming $\theta = \theta(r)$

$$\operatorname{div}(\mathbf{q}) = -k \nabla . (\nabla \theta), \tag{1.54}$$

$$\operatorname{div}(\mathbf{q}) = -\frac{k}{r}\left[\frac{dr}{dr}\frac{d\theta}{dr} + r\frac{d^2\theta}{dr^2}\right], \tag{1.55}$$

$$\operatorname{div}(\mathbf{q}) = -k\left[\frac{1}{r}\frac{d\theta}{dr} + \frac{d^2\theta}{dr^2}\right], \tag{1.56}$$

$$\boldsymbol{\tau}.\mathbf{L} = tr(\boldsymbol{\tau}\mathbf{L}). \tag{1.57}$$

With the help of these equations, *Eq.* (1.52) can be written as

$$\boldsymbol{\tau}.\mathbf{L} = (\frac{\partial v}{\partial r})^2[\mu + 2\beta_3(\frac{\partial v}{\partial r})^2], \tag{1.58}$$

$$\mu(\frac{\partial v}{\partial r})^2 + 2\beta_3(\frac{\partial v}{\partial r})^4 + k\left[\frac{1}{r}\frac{d}{dr}(r\frac{d\theta}{dr})\right] = 0. \tag{1.59}$$

Introducing bars

$$\bar{\mu}(\frac{\partial \bar{v}}{\partial \bar{r}})^2 + 2\beta_3(\frac{\partial \bar{v}}{\partial \bar{r}})^4 + k\left[\frac{1}{\bar{r}}\frac{d}{d\bar{r}}(r\frac{d\bar{\theta}}{d\bar{r}})\right] = 0. \tag{1.60}$$

The above equation in nondimensional form can be written as

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} + \Gamma(\frac{dv}{dr})^2\left[\mu + \Lambda(\frac{dv}{dr})^2\right] = 0, \tag{1.61}$$

where

$$\Lambda = \frac{2v_0^2\beta_3}{\mu_* R^2}, \; \Gamma = \frac{\mu_* v_0^2}{k(\theta_m - \theta_w)}, \; \theta = \frac{\bar{\theta} - \theta_w}{\theta_m - \theta_w}. \tag{1.62}$$

## 1.14 Method of solution

### 1.14.1 Perturbation technique

The purpose of this section is to describe the application of perturbation expansion technique to the solution of differential equations and the approximation of integrals. Approximate expressions are generated in the form of asymptotic series. They may or may not converge but, in truncated form two or three terms provide the good approximation to the original problem.

The techniques, being analytical rather than numerical, provide an alternative to a direct computer solution. An awareness of the perturbation approach is some times essential even when a direct numerical approach is adopted. An example of this occurs in boundary layer problems where there are regions of rapid change of quantities such as fluid velocity, temperature or concentration.

### 1.14.2 Homotopy

A homotopy between two continuous functions $f$ and $g$ from a topological space $X$ to a topological space $Y$ is defined to be a continuous function

$$H : X \times [0,1] \to Y. \tag{1.63}$$

From the product of the space $X$ with the unit interval $[0,1]$ to $Y$ such that, for all points in $X$,

$$H(x,0) = f(x), H(x,1) = g(x). \tag{1.64}$$

If $f$ is homotopic to $g$ we write

$$f \simeq g. \tag{1.65}$$

In topology, two continuous functions from one topological space to another are called homotopic if one can be "continuously deformed" into the other, such a deformation being called a homotopy between the two functions.

# Chapter 2

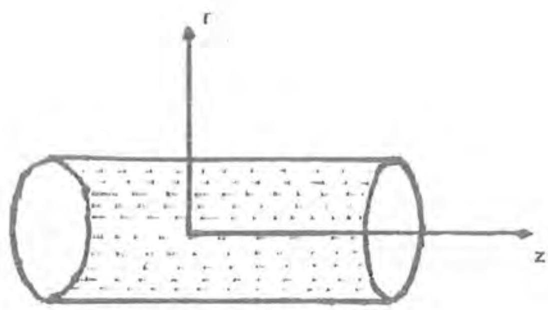
# Approximate analytical solutions for the flow of a third grade fluid in a pipe

## 2.1 Introduction

In this chapter, we have discussed the analytical solutions for a third grade fluid in a pipe with variable viscosity. The temperature of the pipe is assumed to be higher than the temperature of the fluid. Introducing a scale parameter in the governing equations. The governing equations are then solved using regular perturbation keeping $\in$ (appeared in scale parameter) to be small. Three models of viscosity namely constant, Reynolds and Vogels model are considered for the variable viscosity. The perturbation results are compared with the numerical results obtained by Massoudi and Christie [16] . This work is due to Yurusoy and Pakdemirli [31].

## 2.2 Mathematical formulation

Consider an incompressible flow of a third grade fluid in a pipe. The temperature dependent viscosity is taken into account. Three models of variable viscosity are considered (constant model, Reynolds' model and Vogels' model) . The pipe is at rest and the disturbance in the flow occurs due to constant pressure gradient. It is also assumed that the temperature




of the pipe is greater than the temperature of the fluid.

The governing equations of momentum and energy in nondimensional form can be written as

$$\frac{d\mu}{dr}\frac{dv}{dr}+\frac{\mu}{r}\left(\frac{dv}{dr}+r\frac{d^2v}{dr^2}\right)+\frac{\Lambda}{r}\left(\frac{dv}{dr}\right)^2\left(\frac{dv}{dr}+3r\frac{d^2v}{dr^2}\right)=C, \tag{2.1}$$

$$\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}+\Gamma\left(\frac{dv}{dr}\right)^2\left[\mu+\Lambda\left(\frac{dv}{dr}\right)^2\right]=0. \tag{2.2}$$

The corresponding boundary conditions are

$$v(1)=\theta(1)=0,\frac{dv}{dr}(0)=\frac{d\theta}{dr}(0)=0. \tag{2.3}$$

## 2.3 Solution of the problem

### 2.3.1 Constant viscosity case

Introducing

$$\mu=1,\Lambda=\epsilon\lambda. \tag{2.4}$$

Defining

$$v=v_o+\in v_1, \tag{2.5}$$

$$\theta=\theta_o+\in\theta_1. \tag{2.6}$$

Substituting *Eqs.* (2.5) and (2.6) into *Eqs.* (2.1) to (2.3) and comparing the like powers of $\in$, we obtain the following systems.

**Zeroth order system**

$$rv_0'' + v_0' = Cr, \tag{2.7}$$

$$\theta_0'' + \frac{1}{r}\theta_0' + \Gamma v_0'^2 = 0, \tag{2.8}$$

$$v_0(1) = 0,\ \theta_0(1) = 0,\ v_0'(0) = 0,\ \theta_0'(0) = 0. \tag{2.9}$$

**First order system**

$$rv_1'' + v_1' = -\lambda(v_0'^3 + 3rv_0'^2 v_0''), \tag{2.10}$$

$$\theta_1'' + \frac{1}{r}\theta_1' + 2\Gamma v_0' v_1' + \lambda\Gamma v_0'^4 = 0, \tag{2.11}$$

$$v_1(1) = 0, \theta_1(1) = 0, v_1'(0) = 0, \theta_1'(0) = 0. \tag{2.12}$$

**Solution of zeroth order system**

*Eq.* (2.7) can be written as

$$r^2 v_0'' + rv_0' = Cr^2. \tag{2.13}$$

*Eq.* (2.13) is a Cauchy Euler equation and its solution is written as

$$v_0 = c_1 + c_2 \ln r + \frac{Cr^2}{4}, \tag{2.14}$$

where $c_1$ and $c_2$ are constants. With the help of boundary conditions, we obtain $c_1 = \frac{-C}{4}$ and $c_2 = 0$. Thus solution (2.14) takes the form

$$v_0 = \frac{-C}{4}\left(1 - r^2\right). \tag{2.14a}$$

Using $Eq.\,(2.14a)$, the solution of $Eq.$ (2.8) satisfying the boundary conditions (2.9) can be written as

$$\theta_0 = \frac{\Gamma C^2}{64}\left(1 - r^4\right). \tag{2.15}$$

**First order solution**

Using $(2.14a)$, the solution of $Eqs.$ (2.10) and (2.11) satisfying the boundary condition (2.12) can be written as

$$v_1 = \frac{\lambda C3}{32}\left(1 - r^4\right) \tag{2.16}$$

$$\theta_1 = -\frac{\lambda \Gamma c^4}{576}\left(1 - r^6\right). \tag{2.17}$$

Substituting (2.14) to (2.17) into Eqs. (2.5) and (2.6), returning back to the original parameters, since $\in$ is artificially introduced, one has

$$v = \frac{-C}{4}\left(1 - r^2\right) + \frac{\Lambda C^3}{32}\left(1 - r^4\right), \tag{2.17a}$$

$$\theta = \frac{\Gamma C^2}{64}\left(1 - r^4\right) - \frac{\Lambda \Gamma C^4}{576}\left(1 - r^6\right). \tag{2.17b}$$

The second terms are the correction terms in the perturbation expansion and hence should be much smaller than the preceding terms to assure the validity of the expansion. Hence in absolute values

$$\frac{\Lambda C^3}{32} \ll \frac{C}{4}, \tag{2.17c}$$

$$\frac{\Lambda \Gamma C^4}{576} \ll \frac{\Gamma C^2}{64}, \tag{2.17d}$$

or more precisely selecting the larger one

$$Cr_1 = \frac{\Lambda C^2}{8} \ll 1. \tag{2.17e}$$

Eq. $(2.17e)$ is the final criterion for the approximate solution to be valid.

### 2.3.2 Reynolds' model of viscosity

For Reynold model we consider

$$\mu = e^{-M\theta}. \tag{2.18}$$

*Eq.* (2.18) can be written as (Maclaurin's series)

$$\mu = 1 - M\theta + O\left(M^2\right). \tag{2.19}$$

Considering $M = \in m$, *Eq.* (2.19) can be written as

$$\mu \tilde{=} 1 - \in m\theta. \tag{2.19a}$$

Invoking *Eqs.* (2.5), (2.6) and (2.19*a*) into *Eqs.* (2.1) to (2.3) and comparing the like powers of $\in$, we obtain the following systems

**Zeroth order system**

$$r^2 v_0'' + r v_0' = C r^2, \tag{2.20}$$

$$\theta_0'' + \frac{1}{r}\theta_0' + \Gamma v_0'^2 = 0. \tag{2.21}$$

**First order system**

$$r v_1'' + v_1' - rM\theta_0' v_0' - M\theta_0 v_0'' = -\lambda(v_0'^3 + 3r v_0'^2 v_0''), \tag{2.21a}$$

$$\theta_1'' + \frac{1}{r}\theta_1' + 2\Gamma v_0' v_1' + \Gamma v_0'^2\left(-M\theta_0 + \lambda v_0'^2\right) = 0. \tag{2.22}$$

**Zeroth order solution**

The solution of the zeroth order system can be written as

$$v_0 = \frac{-C}{4}\left(1 - r^2\right), \tag{2.23}$$

$$\theta_0 = \frac{\Gamma C^2}{64}\left(1 - r^4\right). \tag{2.24}$$

#### First order solution

Using the procedure discussed in previous section the solution of first order system can be written as

$$v_1 = \lambda \frac{C^3}{32}\left(1-r^4\right) - \frac{\Gamma m C^3}{768}\left(2-3r^2+r^6\right), \tag{2.25}$$

$$\theta_1 = \frac{m\Gamma^2 C^4}{16384}\left(3-4r^4+r^8\right) - \frac{\lambda\Gamma C^4}{576}(1-r^6). \tag{2.26}$$

Substituting solutions at each order to the expansion and returning back to the original parameters since $\in$ is artificially introduced, one finally has the following approximate solutions

$$v = \frac{-C}{4}\left(1-r^2\right) + \Lambda\frac{C^3}{32}\left(1-r^4\right) - \frac{\Gamma M C^3}{768}\left(2-3r^2+r^6\right), \tag{2.26a}$$

$$\theta = \frac{\Gamma C^2}{64}\left(1-r^4\right) + \frac{M\Gamma^2 C^4}{16384}\left(3-4r^4+r^8\right) - \frac{\Lambda\Gamma C^4}{576}(1-r^6). \tag{2.26b}$$

Restrictions for the expansions to be valid therefore comes out to be

$$Cr_1 = \frac{\Lambda C^2}{8} \ll 1, Cr_2 = \frac{3M\Gamma C^2}{256} \ll 1. \tag{2.26c}$$

### 2.3.3 Vogels' model of viscosity

In this case viscosity is expressed as

$$\mu = \mu_* e^{\left(\frac{A}{(B+\theta)} - \theta_w\right)}. \tag{2.27}$$

Maclaurin's series of $\mu$ can be simplified as [17]

$$\mu = \frac{C}{C_*}\left(1-\frac{A\theta}{B^2}\right), \text{where } C^* = \frac{C}{\mu_* e^{\left(\frac{A}{B} - \theta_w\right)}}. \tag{2.28}$$

Since $\mu$ and $\theta$ are related in a complex manner, expansions presented in the previous sections for velocity and temperature would not yield simple equations at the first order. To achieve this, in addition to $\Lambda$, $\Gamma$ should be selected small,

$$\Gamma = \in \gamma, \Lambda = \in \lambda. \tag{2.28a}$$

However, this choice dictates expansion of the below form for velocity and temperature

$$v = v_0 + \in v_1, \tag{2.28b}$$

$$\theta = \in \theta_0 + \in^2 \theta_1 \tag{2.28c}$$

Making use of *Eqs.* (2.28) to (2.28*c*), adopting the similar procedure as discussed in previous section, the solution of *Eqs.* (2.1) and (2.2) can be written as

$$v = -\frac{C^*}{4}\left(1 - r^2\right) + -\frac{A\Gamma C C^{*2}}{768 B^2}\left(2 - 3r^2 + r^6\right) + \frac{\Lambda C^{*4}}{32 C}\left(1 - r^4\right), \tag{2.29}$$

$$\theta = \frac{\Gamma C C^*}{64}\left(1 - r^4\right) + \frac{C^{*4}\Gamma\Lambda}{576}\left(1 - r^6\right) + \frac{A\Gamma^2 C^2 C^{*2}}{16384 B^2}\left(r^8 - 4r^4 + 3\right)]. \tag{2.30}$$

The conditions for valid expansions require

$$Cr_1 = \frac{\Lambda C^{*3}}{8C} \ll 1, Cr_2 = \frac{3A\Gamma C C^*}{256 B^2} \ll 1. \tag{2.30a}$$

## 2.4 Numerical results and comparisons.

Approximate analytical solutions obtained in sections 2.4 and 2.5 are compared with finite difference results presented in Ref. [16]. Tables 1-4 are referred to Reynolds' model. Table 1 shows pressure gradient variation. It can be observed that results are not in agreement with the finite difference results for $|C| \geq 2$ where as results are in good agreement with the finite -difference results for $|C| = 1$. In Table 2 $\Gamma-$ variation is considered. Analytic results are in very good agreement with the finite -difference results since both criteria are much smaller than 1. $\Lambda$-variation is shown in Table 3. Results are not in accordance with the finite -difference results for $|\Lambda| \geq 5$ since criteria 1 is not satisfied. However, results are same up to three digits for $\Lambda = 0$. Variation of $M$ is considered in Table 4. Results are in good agreement with the finite -difference results for $0 \leq M \leq 20$, $C = -1$, $\Gamma = \Lambda = 1$.

Tables 5-7 are for Vogels' model. In Table 5, A-variation is shown. It reveals that results agree because the criteria is extremely small for these range of $A$ values. Table 6 depicts variation in $B$. It is obvious that results do not match due to the fact that first criteria is violated for $B \geq 2$. Table 7 is prepared to see the effects of variation of $\theta_w$. For $\theta_w \leq 1$, both criteria meet and the results agree. For $\theta_w \geq 1$ deviations become substantial.

**Table 1**

Effect of decreasing $C$ in Reynolds' model $\Gamma = M = \Lambda = 1$

| $C$ | $v_{\max(Pert)}$ | $v_{\max(FDM)}$ | $\theta_{\max(Pert)}$ | $\theta_{\max(FDM)}$ | $Cr_1$ | $Cr_2$ |
|---|---|---|---|---|---|---|
| -1 | 0.221 | 0.230 | 0.014 | 0.014 | 0.125 | 0.012 |
| -2 | 0.271 | 0.404 | 0.038 | 0.051 | 0.500 | 0.047 |
| -3 | -0.023 | 0.541 | 0.015 | 0.102 | 1.125 | 0.106 |
| -4 | -0.833 | 0.656 | -0.0148 | 0.165 | 2.000 | 0.188 |
| -5 | -2.331 | 0.755 | -0.580 | 0.236 | 3.125 | 0.293 |

**Table 2**

Effect of increasing $\Gamma$ in Reynolds' model $C = -1$, $M = \Lambda = 1$

| $\Gamma$ | $v_{\max(Pert)}$ | $v_{\max(FDM)}$ | $\theta_{\max(Pert)}$ | $\theta_{\max(FDM)}$ | $Cr_1$ | $Cr_2$ |
|---|---|---|---|---|---|---|
| 0 | 0.219 | 0.228 | 0.000 | 0.000 | 0.125 | 0.000 |
| 5 | 0.232 | 0.238 | 0.074 | 0.075 | 0.125 | 0.259 |
| 10 | 0.245 | 0.249 | 0.157 | 0.158 | 0.125 | 0.117 |
| 15 | 0.258 | 0.261 | 0.249 | 0.249 | 0.125 | 0.176 |
| 20 | 0.271 | 0.275 | 0.351 | 0.351 | 0.125 | 0.234 |

**Table 3**

Effect of increasing $\Lambda$ in Reynolds' model $C = -1$, $M = \Gamma = 1$

| $\Lambda$ | $v_{\max(Pert)}$ | $v_{\max(FDM)}$ | $\theta_{\max(Pert)}$ | $\theta_{\max(FDM)}$ | $Cr_1$ | $Cr_2$ |
|---|---|---|---|---|---|---|
| 0 | 0.253 | 0.253 | 0.016 | 0.016 | 0.000 | 0.012 |
| 5 | 0.096 | 0.192 | 0.007 | 0.012 | 0.625 | 0.012 |
| 10 | -0.060 | 0.172 | -0.002 | 0.011 | 1.250 | 0.012 |
| 15 | -0.216 | 0.159 | -0.010 | 0.010 | 1.875 | 0.012 |
| 20 | -0.372 | 0.150 | -0.009 | 0.009 | 2.500 | 0.012 |

Table 4

Effect of increasing $\Lambda$ in Reynolds' model $C = -1$, $\Lambda = \Gamma = 1$

| $M$ | $v_{\max(Pert)}$ | $v_{\max(FDM)}$ | $\theta_{\max(Pert)}$ | $\theta_{\max(FDM)}$ | $Cr_1$ | $Cr_2$ |
|---|---|---|---|---|---|---|
| 0 | 0.219 | 0.228 | 0.014 | 0.014 | 0.125 | 0.000 |
| 5 | 0.232 | 0.238 | 0.015 | 0.015 | 0.125 | 0.059 |
| 10 | 0.245 | 0.249 | 0.016 | 0.016 | 0.125 | 0.117 |
| 15 | 0.258 | 0.261 | 0.017 | 0.017 | 0.125 | 0.176 |
| 20 | 0.271 | 0.275 | 0.018 | 0.018 | 0.125 | 0.234 |

Table 5

Effect of increasing $A$ in vogel's model $C = -1, \Lambda = \Gamma = \theta_w = B = 1$

| $A$ | $v_{\max(Pert)}$ | $v_{\max(FDM)}$ | $\theta_{\max(Pert)}$ | $\theta_{\max(FDM)}$ | $Cr_1$ | $Cr_2$ |
|---|---|---|---|---|---|---|
| 1 | 0.221 | 0.230 | 0.014 | 0.014 | 0.125 | 0.112 |
| 2 | 0.092 | 0.092 | 0.006 | 0.006 | $6.22\times10^{-3}$ | $8.62\times10^{-3}$ |
| 3 | 0.034 | 0.034 | 0.002 | 0.002 | $3.10\times10^{-4}$ | $4.67\times10^{-3}$ |
| 4 | 0.012 | 0.012 | 0.001 | 0.001 | $1.54\times10^{-5}$ | $2.33\times10^{-3}$ |

Table 6

Effect of increasing $B$ in vogels' model $C = -1, \Lambda = \Gamma = \theta_w = A = 1$

| $B$ | $v_{\max(Pert)}$ | $v_{\max(FDM)}$ | $\theta_{\max(Pert)}$ | $\theta_{\max(FDM)}$ | $Cr_1$ | $Cr_2$ |
|---|---|---|---|---|---|---|
| 1 | 0.221 | 0.230 | 0.014 | 0.014 | 0.125 | 0.012 |
| 2 | 0.183 | 0.321 | 0.013 | 0.020 | 0.560 | $4.83\times10^{-3}$ |
| 3 | 0.038 | 0.350 | 0.006 | 0.022 | 0.924 | $2.54\times10^{-3}$ |
| 4 | -0.098 | 0.365 | -0.002 | 0.023 | 1.186 | $1.55\times10^{-3}$ |

Table 7

Effect of increasing $\theta_w$in vogels' model $C = -1, \Lambda = \Gamma = B = A = 1$

| $\theta_w$ | $v_{\max(Pert)}$ | $v_{\max(FDM)}$ | $\theta_{\max(Pert)}$ | $\theta_{\max(FDM)}$ | $Cr_1$ | $Cr_2$ |
|---|---|---|---|---|---|---|
| 0.5 | 0.148 | 0.149 | 0.009 | 0.009 | 0.028 | $7.11\times10^{-3}$ |
| 1 | 0.221 | 0.230 | 0.014 | 0.014 | 0.125 | 0.012 |
| 1.5 | 0.188 | 0.324 | 0.013 | 0.020 | 0.560 | 0.019 |
| 2 | -1.007 | 0.408 | -0,051 | 0.026 | 2.511 | 0.032 |

Figs 2.1 to 2.14 are drawn in order to depict velocity and temperature profiles. Figs 2.1 to 2.8 correspond to Reynolds' model and Figs 9 to 14 to Vogels' model. Fig. 1 and Fig. 2 are prepared for variation in velocity and temperature profiles with the pressure gradient. It is observed that when pressure gradient decreases absolutely, velocity and temperature profiles also decrease. In Fig. 3 and Fig. 4 variation of $\Gamma$ parameter within the admissible range is considered. As $\Gamma$ increases both quantities also increase. When non-Newtonian effect increases (as $\Lambda$ increases) , in Fig. 5 and Fig. 6, velocity and temperature decreases. In order to remain in the admissible range, $\Lambda$ is taken smaller than 2. In Fig. 7and Fig. 8 variation of $M$ is taken into account. It is observed that as $M$ increases, velocity and temperature profiles also increase. In Figs 9 to 14 Vogels' model is considered. Figs. 9 and 10 are plotted in order to depict the variation of velocity and temperature profiles for different values of $A$. It is noted that as $A$ increases both profiles decrease. Figs. 11 and 12 are for parameter $B$. As $B$ increases both profiles increase. Finally in Figs. 13 and 14 effect of $\theta_w$ variation is considered with in the admissible range $\theta_w \leq 1$. As $\theta_w$ increases, both profiles increase.

All profiles are qualitatively in agreement with those presented in [16] . Within the admissible parameter range, for common parameters considered in [16] quantitative match is also achieved.

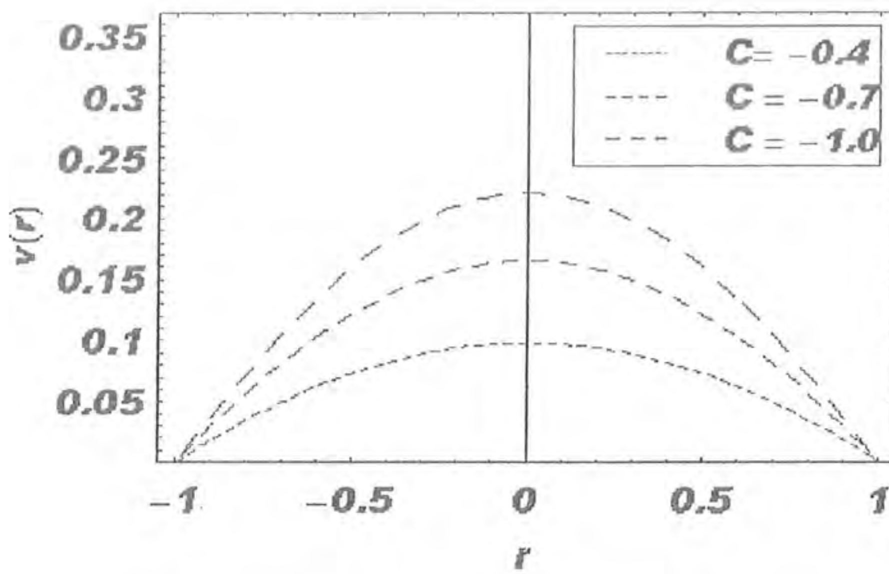



Figure 2-1: Effect of pressure drop (C) on velocity profile

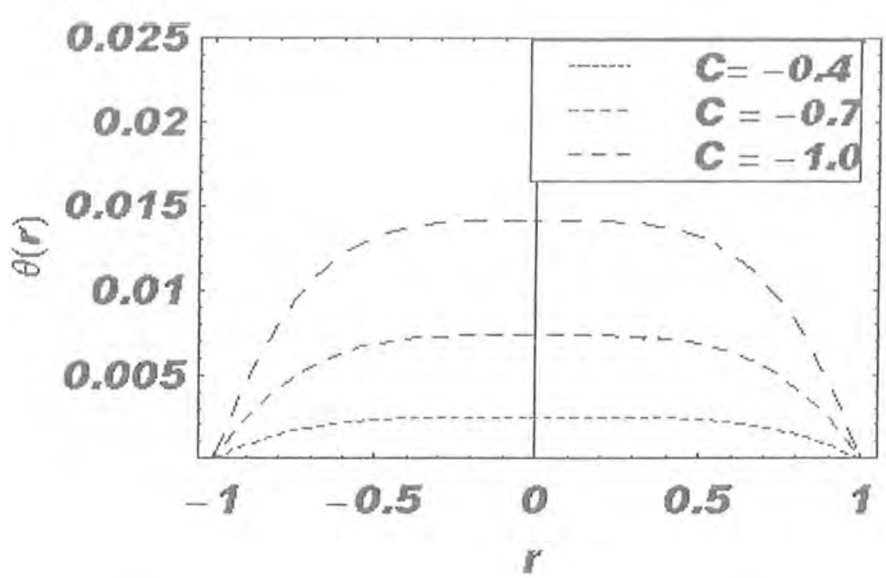



Figure 2-2: Effect of pressure drop (C) on temperature profile.

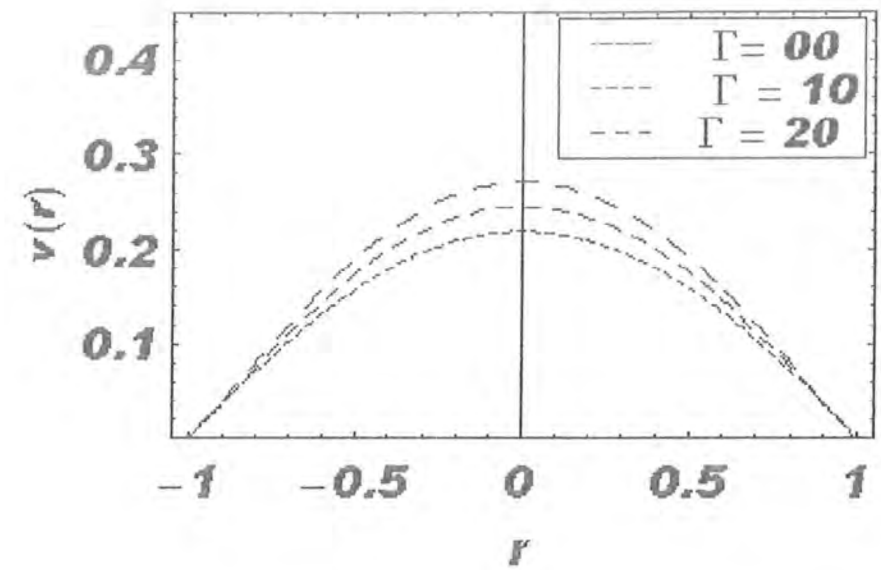



Figure 2-3: Effect of *Brinkman number* Γon velocity profile.

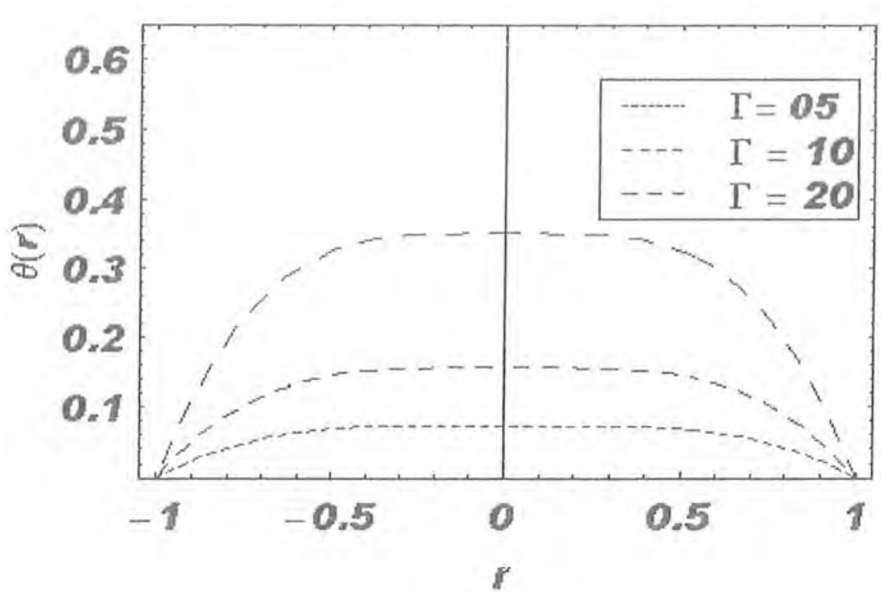



Figure 2-4: Effect of *Brinkman number* Γon temperature profile.

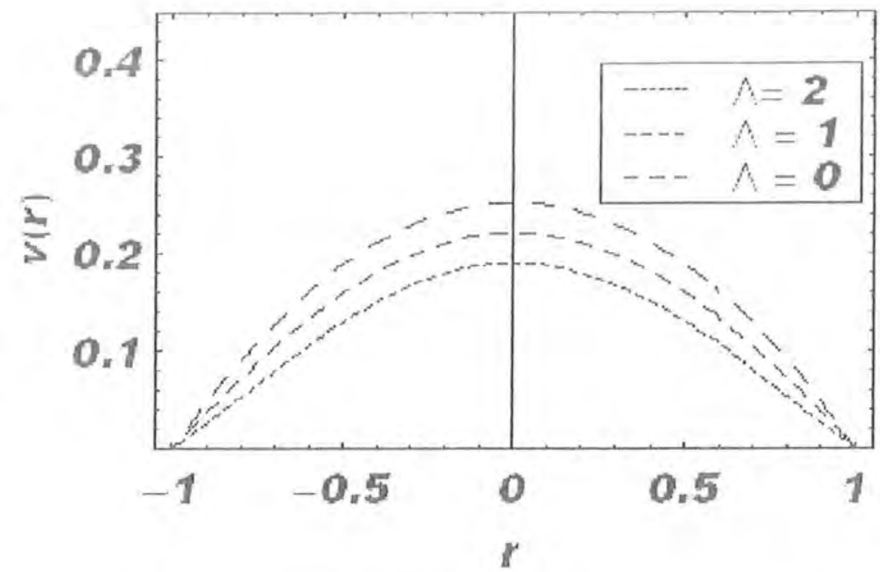



Figure 2-5: Effect of non-Newtonian parameter Λ on velocity profile.

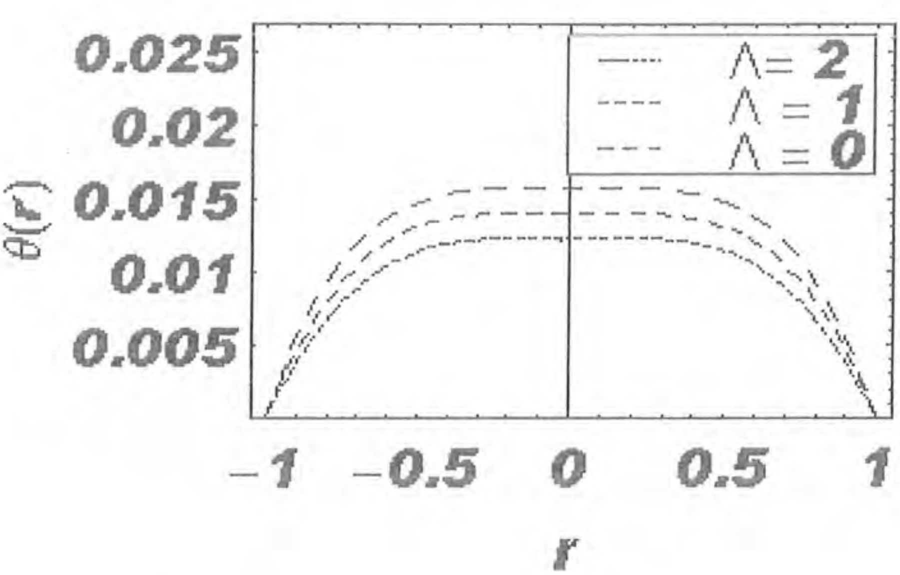



Figure 2-6: Effect of non-Newtonian parameter Λ on temperature profile.

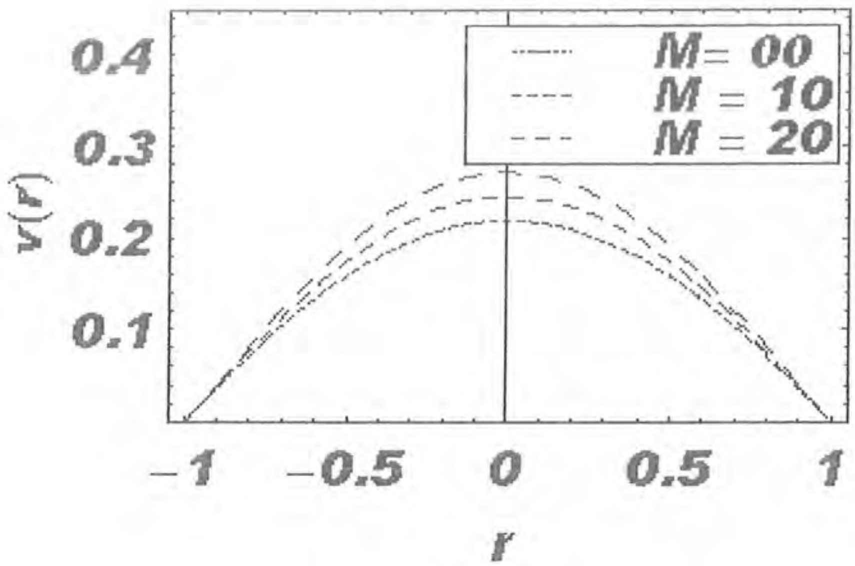



Figure 2-7: Effect of viscosity parameter $M$ on velocity profile.

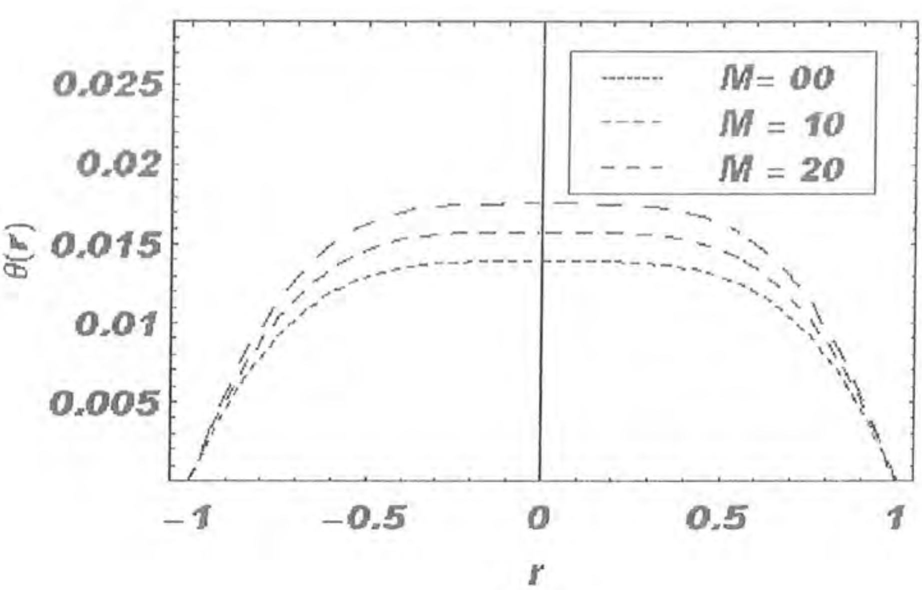



Figure 2-8: Effect of viscosity parameter $M$ on temperature profile.

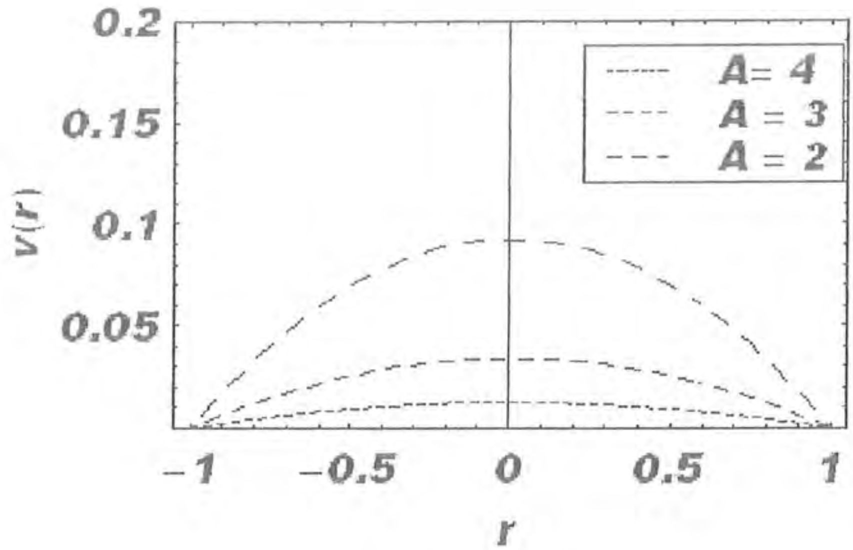



Figure 2-9: Effect of viscosity parameter $A$ on velocity profile.

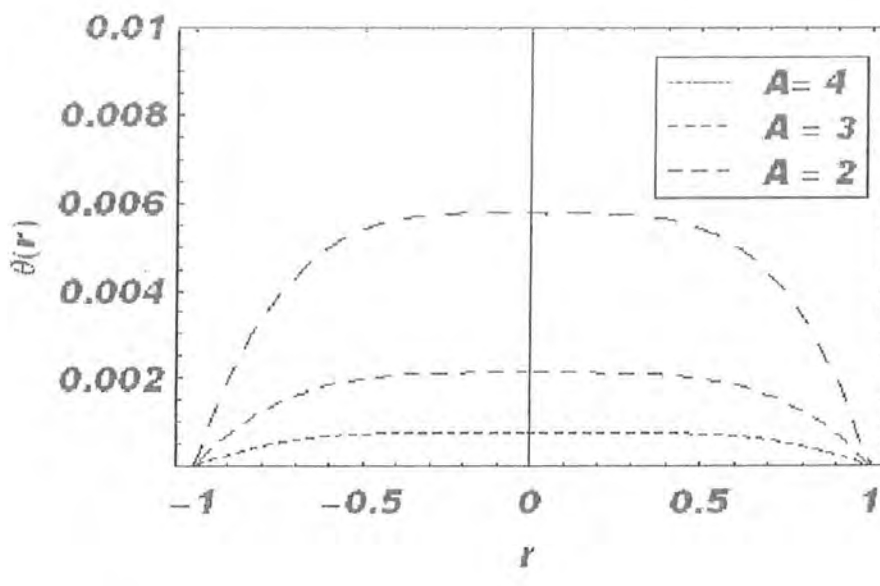



Figure 2-10: Effect of viscosity parameter $A$ on temperature profile.

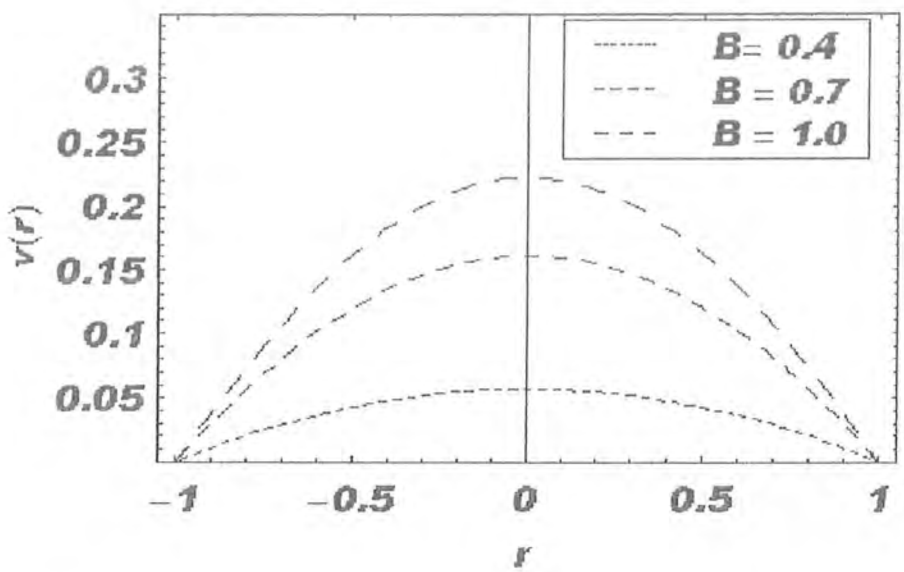



Figure 2-11: Effect of viscosity parameter $B$ on velocity profile.

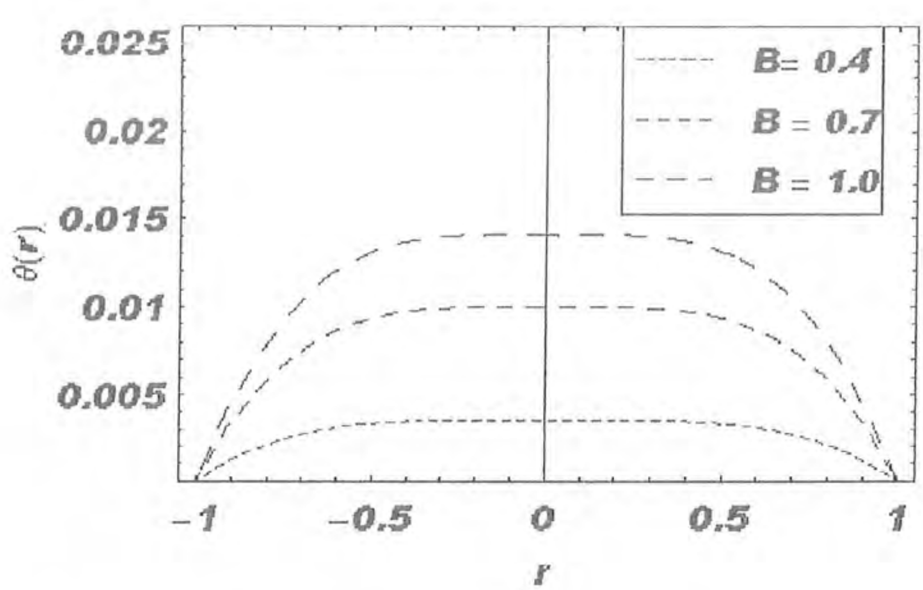



Figure 2-12: Effect of viscosity parameter $B$ on temperature profile.

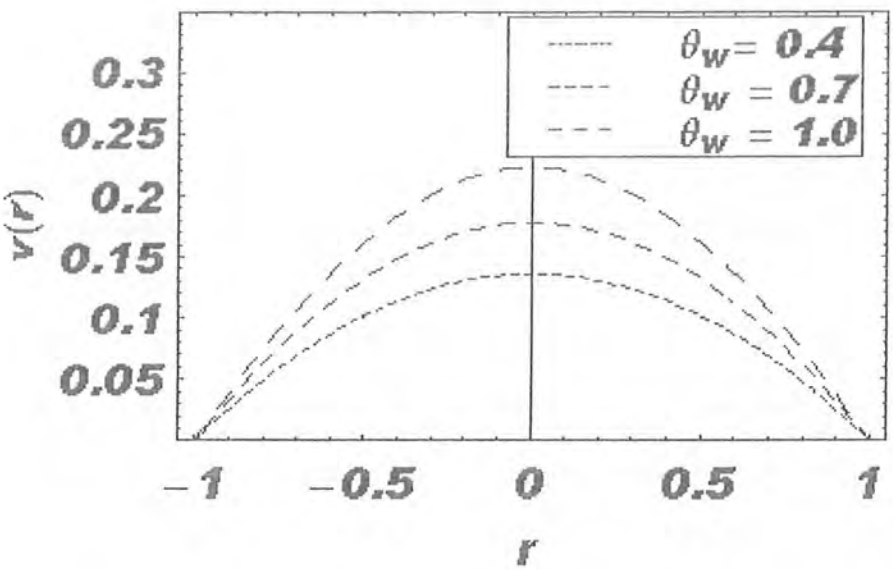



Figure 2-13: Effect of $\theta_w$ on velocity profile.

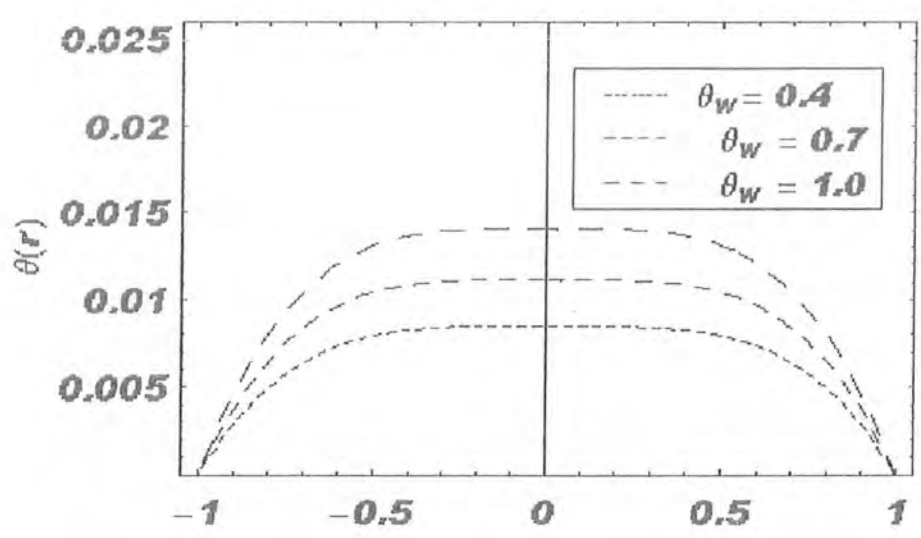



Figure 2-14: Effect of $\theta_w$ on temperature profile.

# Chapter 3

# Flow of a third grade fluid between coaxial cylinders with variable viscosity

## 3.1 Introduction

This chapter deals with the steady flow of a third grade fluid between two coaxial cylinders with variable viscosity. Two models of temperature dependent viscosity namely the Reynolds and Vogels are considered. The flow is driven by the motion of an inner cylinder and a constant pressure gradient. The heat transfer analysis is also carried out. The analytical expressions of velocity and temperature are developed in each case by using a newly powerful technique namely the homotopy analysis method [19-30]. Convergence of the series solution is carefully checked. Finally, the solution are discussed with the help of graphs.

## 3.2 Problem statement

Consider an incompressible and thermodynamic third grade fluid between two infinite coaxial cylinders. The flow is induced by a constant pressure gradient and motion of an inner cylinder. The outer cylinder is kept fixed. The heat transfer analysis is also taken into account.

The dimensionless problems which can describe the flow and heat transfer are

$$\frac{d\mu}{dr}\frac{dv}{dr}+\frac{\mu}{r}[\frac{dv}{dr}+r\frac{d^2v}{dr^2}]+\frac{\Lambda}{r}(\frac{dv}{dr})^2[\frac{dv}{dr}+3r\frac{d^2v}{dr^2}]=C, \tag{3.1}$$

$$\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}+\Gamma(\frac{dv}{dr})^2[\mu+\Lambda(\frac{dv}{dr})^2]=0, \tag{3.2}$$

$$v(r)=1,\theta(r)=1 \ ; \qquad r=1, \tag{3.3}$$

$$v(r)=0,\theta(r)=0 \ ; \qquad r=b, \tag{3.4}$$

whence

$$\begin{aligned} r &= \frac{\bar{r}}{R},\ \Gamma=\frac{\mu_* V_0^2}{k(\theta_m-\theta_w)},\ \theta=\frac{\left(\bar{\theta}-\theta_w\right)}{(\theta_m-\theta_w)},\ \Lambda=\frac{2\beta_3 V_0^2}{R^2\mu_*},\ C_1=\frac{\partial p}{\partial z}, \\ C &= \frac{C_1R^2}{\mu_* V_0}, v=\frac{\bar{v}}{v_0}, \mu=\frac{\bar{\mu}}{\mu_*}, \end{aligned} \tag{3.5}$$

where $\bar{r}$, $\mu_*$, $\theta_m$, $\theta_w$, $V_0$, $\Lambda$, $\beta_3$, and $R$ are defined in chapter 1.

## 3.3 Series solutions for Reynolds' model

Here the viscosity is expressed in the form

$$\mu=e^{-M\theta}, \tag{3.6}$$

which by Maclaurin's series can be written as

$$\mu=1-M\theta+O(\theta^2). \tag{3.7}$$

Note that $M=0$, corresponds to the case of constant viscosity. Invoking above equation into *Eqs.* (3.1) and (3.2), one has

$$-M\frac{d\theta}{dr}\frac{dv}{dr}+\frac{1}{r}\frac{dv}{dr}-\frac{M}{r}\theta\frac{dv}{dr}+\frac{d^2v}{dr^2}-M\theta\frac{d^2v}{dr^2}+\frac{\Lambda}{r}(\frac{dv}{dr})^3+3\Lambda\frac{d^2v}{dr^2}(\frac{dv}{dr})^2=C, \tag{3.8}$$

$$\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}+\Gamma\left(\frac{dv}{dr}\right)^2-\Gamma M\theta(\frac{dv}{dr})^2+\Lambda\Gamma\left(\frac{dv}{dr}\right)^4=0. \tag{3.9}$$

For HAM solution, we choose the following initial guesses

$$v_0(r)=\frac{(r-b)}{(1-b)}, \tag{3.10}$$

$$\theta_0(r)=\frac{(r-b)}{(1-b)}. \tag{3.11}$$

The auxiliary linear operators are in the form

$$\mathcal{L}_{vr}(v)=v'', \tag{3.12}$$

$$\mathcal{L}_{\theta r}(\theta)=\theta'', \tag{3.13}$$

which satisfy

$$\mathcal{L}_{vr}(A_1+B_1r)=0, \tag{3.14}$$

$$\mathcal{L}_{\theta r}(A_2+B_2r)=0, \tag{3.15}$$

where $A_1$, $A_2$, $B_1$, $B_2$ are the constants.

If $p\in[0,1]$ is an embedding parameter and $h_v$ and $h_\theta$ are auxiliary parameters then the problems at the zero and $mth$ order are respectively given by

$$(1-p)\mathcal{L}_v[\bar{v}(r,p)-v_0(r)]=p\hbar_vN_v[\bar{v}(r,p),\bar{\theta}(r,p)], \tag{3.16}$$

$$(1-p)\mathcal{L}_\theta[\bar{\theta}(r,p)-\theta_0(r)]=p\hbar_\theta N_\theta[\bar{v}(r,p),\bar{\theta}(r,p)], \tag{3.17}$$

$$\mathcal{L}_v[v_m(r)-\chi_mv_{m-1}(r)]=\hbar_vR_{vr}(r), \tag{3.18}$$

$$\mathcal{L}_\theta[\theta_m(r)-\chi_m\theta_{m-1}(r)]=\hbar_\theta R_{\theta r}(r), \tag{3.19}$$

$$\bar{v}(r,p)=\bar{\theta}(r,p)=1, \qquad r=1, \tag{3.20}$$

$$\bar{v}(r,p) = \bar{\theta}(r,p) = 0, \quad r = b, \tag{3.21}$$

boundary conditions for the *mth* order are

$$\bar{v}_m(r,p) = \bar{\theta}_m(r,p) = 0, \; r = 1, \tag{3.22}$$

$$\bar{v}_m(r,p) = \bar{\theta}_m(r,p) = 0, \; r = b, \tag{3.23}$$

$$\begin{aligned} N_v[\;\bar{v}\;(r,p),\bar{\theta}(r,p)] \;&=\; -M\frac{d\theta}{dr}\frac{dv}{dr} + \frac{1}{r}\frac{dv}{dr} - \frac{M}{r}\theta\frac{dv}{dr} + \frac{d^2v}{dr^2} \\ &\quad - M\theta\frac{d^2v}{dr^2} + \frac{\Lambda}{r}(\frac{dv}{dr})^3 + 3\Lambda\frac{d^2v}{dr^2}(\frac{dv}{dr})^2 - C, \end{aligned} \tag{3.24}$$

$$\begin{aligned} N_\theta[\;\bar{v}\;(r,p),\bar{\theta}(r,p)] \;&=\; \frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} + \Gamma\left(\frac{dv}{dr}\right)^2 \\ &\quad -\Gamma M\theta(\frac{dv}{dr})^2 + \Lambda\Gamma\left(\frac{dv}{dr}\right)^4, \end{aligned} \tag{3.25}$$

$$\begin{aligned} R_{vr} = -M\sum_{k=0}^{m-1} v'_{m-1-k}\theta'_k + \frac{1}{r}v'_{m-1} - \frac{M}{r}\sum_{k=0}^{m-1} v'_{m-1-k}\theta_k + v''_{m-1} \\ -M\sum_{k=0}^{m-1} v''_{m-1}\theta_k + \frac{\Lambda}{r}\sum_{k=0}^{m-1} v'_{m-1-k}\sum_{l=0}^{k} v'_{k-l}v'_l \\ +3\Lambda\sum_{k=0}^{m-1} v'_{m-1-k}\sum_{l=0}^{k} v'_{k-l}v''_l - C(1-\chi_m\;), \end{aligned} \tag{3.26}$$

$$\begin{aligned} R_{\theta r} \;&=\; \frac{1}{r}\theta'_{m-1} + \theta''_{m-1} + \Gamma\sum_{k=0}^{m-1} v'_{m-1-k}v'_k \\ &\quad +\Gamma M\sum_{k=0}^{m-1} v'_{m-1-k}\sum_{l=0}^{k} v'_{k-l}\theta_l \\ &\quad -\Lambda\Gamma\sum_{k=0}^{m-1} v'_{m-1-k}\sum_{k=0}^{m-1} v'_{k-l}\sum_{s=0}^{k} v'_{l-s}v'_s. \end{aligned} \tag{3.27}$$

By Mathematica the solution of Eqs. (3.26) and (3.27) can be written as

$$v_m(r) = \sum_{n=0}^{3m} a_{m,n} r^n, m \geq 0, \tag{3.28}$$

$$\theta_m(r) = \sum_{n=0}^{3m+1} d_{m,n} r^n, \; m \geq 0, \tag{3.29}$$

where $a_{m,n}$ and $d_{m,n}$ are constants and are computed as

$$v'_m(r) = \sum_{n=0}^{3m} b_{m,n} r^n, m \geq 0, \; \theta'_m(r) = \sum_{n=0}^{3m+1} e_{m,n} r^n, \; m \geq 0. \tag{3.30}$$

$$b_{m,n} = (n+1)\, a_{m,n+1} \text{ and } \; e_{m,n} = (n+1)\, d_{m,n+1}, \tag{3.31}$$

$$\begin{aligned} v''_m(r) &= \sum_{n=0}^{3m} c_{m,n} r^n, m \geq 0, \\ v'_{k-l}(r) &= \sum_{n=0}^{3(k-l)} b_{k-l,n} r^n, \\ v'_{m-k-1}(r) &= \sum_{n=0}^{3m-3k-3} b_{m-k-1,n} r^n, \end{aligned} \tag{3.32}$$

$$c_{m,n} = (n+1)\, d_{m,n+1}, \tag{3.32a}$$

$$\begin{aligned} v'_{m-1}(r) &= \sum_{n=0}^{3m-3} b_{m-1,n} r^n, \\ v''_{m-1}(r) &= \sum_{n=0}^{3m-3} c_{m-1,n} r^n, \\ v''_l(r) &= \sum_{n=0}^{3l} c_{l,n} r^n, \end{aligned} \tag{3.33}$$

$$v'_{m-k-1}(r)\,\theta'_k(r) = \sum_{n=0}^{3m-3k-3} b_{m-k-1,n}\, r^n \sum_{n=0}^{3k+1} e_{k,n} r^n \tag{3.34}$$
$$= \sum_{i=0}^{3m-3k-3} b_{m-k-1,i} r^i \sum_{j=0}^{3k+1} e_{k,j} r^j,$$

$$v'_{m-k-1}(r)\,\theta'_k(r) = \sum_{i=0}^{3m-3k-3} \sum_{j=0}^{3k+1} b_{m-k-1,i} e_{k,j} r^{i+j}, \tag{3.35}$$

put

$$\begin{aligned} i+j &= p, \\ 0 &\leq p \leq 3m-2, \\ i &= p-j, \\ 0 &\leq p-j \leq 3m-3k-3, \\ p-3m+3k+3 &\leq j \leq p, \end{aligned} \tag{3.36}$$

this shows

$$j = max\{0, p+3k-3m+3\} \ \ to \quad \min\{p, 3k+1\}. \tag{3.37}$$

Then

$$v'_{m-k-1}(r)\,\theta'_k(r) = \sum_{p=0}^{3m-2} \sum_{j=max\{0,p+3k-3m+3\}}^{\min\{p,3k+1\}} b_{m-k-1,p-j} e_{k,j} r^p, \tag{3.38}$$

$$\sum_{k=0}^{m-1} v'_{m-k-1}(r)\,\theta'_k(r) = \sum_{k=0}^{m-1} \sum_{n=0}^{3m-2} \times \sum_{j=max\{0,n+3k-3m+3\}}^{\min\{n,3k+1\}} b_{m-k-1,n-j} e_{k,j} r^n, \tag{3.39}$$

$$\sum_{k=0}^{m-1} v'_{m-k-1}(r)\,\theta'_k(r) = \sum_{n=0}^{3m-2} \Delta_m^n r^n, \tag{3.40}$$

$$\text{where} \qquad \Delta_m^n = \sum_{k=0}^{m-1} \sum_{j=max\{0,n+3k-3m+3\}}^{\min\{n,3k+1\}} b_{m-k-1,n-j} e_{k,j}\,,$$

$$\begin{aligned} v'_{k-l}(r)\,v'_l(r) &= \sum_{n=0}^{3(k-l)} b_{k-l,n} r^n \sum_{n=0}^{3l} b_{l,n} r^n \\ &= \sum_{i=0}^{3(k-l)} b_{k-l,i} r^i \sum_{j=0}^{3l} b_{l,j} r^j \\ &= \sum_{i=0}^{3(k-l)} \sum_{j=0}^{3l} b_{k-l,i} b_{l,j} r^{i+j}, \end{aligned} \tag{3.41}$$

now

$$\begin{aligned} i+j &= p, \\ 0 &\leq p \leq 3k, \\ i &= p-j, \\ 0 &\leq p-j \leq 3k-3l, \\ p-3k+3l &\leq j \leq p. \end{aligned} \tag{3.42}$$

This gives

$$j = max\{0, p-3k+3l\} \;\; to \quad \min\{p, 3l\}\,. \tag{3.43}$$

Therefore,

$$v'_{k-l}(r)\,v'_l(r) = \sum_{p=0}^{3k} \sum_{j=max\{0,p-3k+3l\}}^{\min\{p,3l\}} b_{k-l,p-j} b_{l,j} r^p. \tag{3.44}$$

$$v'_{m-k-1}v'_{k-l}(r)\,v'_l(r) = \sum_{s=0}^{3m-3k-3} b_{m-k-1,s}r^s \sum_{p=0}^{3k} \times \sum_{j=max\{0,p-3k+3l\}}^{\min\{p,3l\}} b_{k-l,p-j}b_{l,j}r^p, \tag{3.45}$$

$$v'_{m-k-1}v'_{k-l}(r)\,v'_l(r) = \sum_{s=0}^{3m-3k-3}\sum_{p=0}^{3k} \times \sum_{j=max\{0,p-3k+3l\}}^{\min\{p,3l\}} b_{m-k-1,s}b_{k-l,p-j}b_{l,j}r^{p+s}, \tag{3.46}$$

if

$$\begin{aligned} p+s &= a, \\ 0 &\leq a \leq 3m-3, \\ s &= a-p, \\ 0 &\leq a-p \leq 3m-3k-3, \\ a-3m+3k+3 &\leq p \leq a. \end{aligned} \tag{3.47}$$

We have

$$p = max\{0, a-3m+3k+3\} \ to \ \min\{a, 3k\}. \tag{3.48}$$

Therefore,

$$v'_{m-k-1}v'_{k-l}(r)\,v'_l(r) = \sum_{a=0}^{3m-3}\sum_{p=max\{0,a-3m+3k+3\}}^{\min\{a,3k\}} \times \sum_{j=max\{0,p-3k+3l\}}^{\min\{p,3l\}} b_{m-k-1,a-p}b_{k-l,p-j}b_{l,j}r^a, \tag{3.49}$$

$$\sum_{k=0}^{m-1} v'_{m-k-1} v'_{k-l}(r)\, v'_l(r) = \sum_{k=0}^{m-1} \sum_{n=0}^{3m-3} \sum_{p=max\{0,n-3m+3k+3\}}^{\min\{n,3k\}} \times \sum_{j=max\{0,p-3k+3l\}}^{\min\{p,3l\}} b_{m-k-1,a-p} b_{k-l,p-j} b_{l,j} r^a, \tag{3.50}$$

$$\sum_{k=0}^{m-1} v'_{m-k-1} v'_{k-l}(r)\, v'_l(r) = \sum_{n=0}^{3m-3} \beta_m^n r^n, \tag{3.51}$$

$$\beta_m^n = \sum_{k=0}^{m-1} \sum_{p=max\{0,a-3m+3k+3\}}^{\min\{a,3k\}} \sum_{j=max\{0,p-3k+3l\}}^{\min\{p,3l\}} b_{m-k-1,a-p} b_{k-l,p-j} b_{l,j}, \tag{3.51a}$$

$$\begin{aligned} v'_{m-k-1}(r)\,\theta_k(r) &= \sum_{n=0}^{3m-3k-3} b_{m-k-1,n} r^n \sum_{n=0}^{3k+1} d_{k,n} r^n \\ &= \sum_{i=0}^{3m-3k-3} b_{m-k-1,i} r^i \sum_{j=0}^{3k+1} d_{k,j} r^j, \end{aligned} \tag{3.52}$$

$$v'_{m-k-1}(r)\,\theta_k(r) = \sum_{i=0}^{3m-3k-3} \sum_{j=0}^{3k+1} b_{m-k-1,i} d_{k,j} r^{i+j}, \tag{3.53}$$

put

$$\begin{aligned} i+j &= p, \\ 0 &\leq p \leq 3m-2, \\ i &= p-j, \\ 0 &\leq p-j \leq 3m-3k-3, \\ p-3m+3k+3 &\leq j \leq p. \end{aligned} \tag{3.54}$$

We obtain

$$j = max\{0, p+3k-3m+3\} \ to \ \min\{p, 3k+1\}. \tag{3.55}$$

Hence

$$v'_{m-k-1}(r)\,\theta_k(r) = \sum_{p=0}^{3m-2} \sum_{j=max\{0,p+3k-3m+3\}}^{\min\{p,3k+1\}} b_{m-k-1,p-j} d_{k,j} r^p, \tag{3.56}$$

$$\begin{aligned} \sum_{k=0}^{m-1} v'_{m-k-1}(r)\,\theta_k(r) &= \sum_{n=0}^{3m-2} \delta_m^n r^n, \\ \delta_m^n &= \sum_{j=max\{0,p+3k-3m+3\}}^{\min\{p,3k+1\}} b_{m-k-1,p-j} d_{k,j}, \end{aligned} \tag{3.57}$$

$$\begin{aligned} v'_{k-l}(r)\,v''_l(r) &= \sum_{n=0}^{3(k-l)} b_{k-l,n} r^n \sum_{n=0}^{3l} c_{l,n} r^n \\ &= \sum_{i=0}^{3(k-l)} b_{k-l,i} r^i \sum_{j=0}^{3l} c_{l,j} r^j \\ &= \sum_{i=0}^{3(k-l)} \sum_{j=0}^{3l} b_{k-l,i} c_{l,j} r^{i+j}, \end{aligned} \tag{3.58}$$

put

$$\begin{aligned} i+j &= p, \\ 0 &\leq p \leq 3k, \\ i &= p-j, \\ 0 &\leq p-j \leq 3k-3l, \\ p-3k+3l &\leq j \leq p. \end{aligned} \tag{3.59}$$

This shows that

$$j = max\{0, p-3k-+3l\} \;\; to \;\; \min\{p, 3l\}, \tag{3.60}$$

$$v'_{k-l}(r)\,v''_l(r) = \sum_{p=0}^{3k} \sum_{j=max\{0,p-3k-+3l\}}^{\min\{p,3l\}} b_{k-l,i} c_{l,j} r^p, \tag{3.61}$$

$$\sum_{k=0}^{m-1} v'_{m-k-1}(r)\, v'_{k-l}(r)\, v''_l(r) = \sum_{n=0}^{3m-3} \zeta_m^n r^n, \tag{3.62}$$

where

$$\begin{aligned} \zeta_m^n &= \sum_{k=0}^{m-1}\sum_{l=0}^{k} \sum_{p=max\{0,n-3m-+3k+3\}}^{\min\{n,3k\}} \\ &\times \sum_{j=max\{0,p-3k-+3l\}}^{\min\{p,3l\}} b_{k-l,p-j} c_{l,j} b_{m-k-1,n-p}. \end{aligned} \tag{3.63}$$

Similarly

$$\begin{aligned} \sum_{k=0}^{m-1} v''_{m-k-1}(r)\, \theta_k(r) &= \sum_{n=0}^{3m-2} \Gamma_m^n r^n, \\ where\ \Gamma_m^n &= \sum_{k=0}^{m-1} \sum_{j=max\{0,n-3m+3\}}^{\min\{n,3k+1\}} d_{k,j} c_{m-k-1,n-j}. \end{aligned} \tag{3.64}$$

Thus,

$$\begin{aligned} R_{vr} &= -M\sum_{n=0}^{3m-2} \Delta_m^n r^n + \frac{1}{r}\sum_{n=0}^{3m-3} b_{m-1,n} r^n - \frac{M}{r}\sum_{n=0}^{3m-2} \delta_m^n r^n \\ &+ \sum_{n=0}^{3m-3} c_{m-1,n} r^n - M\sum_{n=0}^{3m-2} \Gamma_m^n r^n \\ &+ \frac{\Lambda}{r}\sum_{n=0}^{3m-3} \beta_m^n r^n + 3\Lambda \sum_{n=0}^{3m-3} \zeta_m^n r^n - C\left(1-\chi_m\right), \end{aligned} \tag{3.65}$$

$$
\begin{aligned}
R_{vr} \;=\;& -M\sum_{n=0}^{3m-2}\Delta_m^n r^n+\sum_{n=1}^{3m-3}b_{m-1,n}r^{n-1}-M\sum_{n=1}^{3m-2}\delta_m^n r^{n-1} \\
&+\sum_{n=0}^{3m-3}c_{m-1,n}r^n-M\sum_{n=0}^{3m-2}\Gamma_m^n r^n \\
&+\Lambda\sum_{n=1}^{3m-3}\beta_m^n r^{n-1}+3\Lambda\sum_{n=0}^{3m-3}\zeta_m^n r^n-C\left(1-\chi_m\right),
\end{aligned} \tag{3.66}
$$

$$
\begin{aligned}
R_{vr} \;=\;& -M\sum_{n=0}^{3m}\chi_{3m-n}\Delta_m^n r^n+\sum_{n=0}^{3m}\chi_{3m-n}b_{m-1,n+1}r^n \\
&-M\sum_{n=0}^{3m}\chi_{3m-n}\delta_m^{n+1}r^n+\sum_{n=0}^{3m}\chi_{3m-n-1}C_{m-1,n}r^n \\
&-M\sum_{n=0}^{3m}\chi_{3m-n}\Gamma_m^n r^n+\Lambda\sum_{n=0}^{3m}\chi_{3m-n-1}\beta_m^{n+1}r^n \\
&+3\Lambda\sum_{n=0}^{3m}\chi_{3m-n-1}\zeta_m^n r^n-C\left(1-\chi_m\right),
\end{aligned} \tag{3.67}
$$

$$
R_{vr}=\sum_{n=0}^{3m}\pi_m^n r^n-C\left(1-\chi_m\right), \tag{3.68}
$$

where,

$$
\begin{aligned}
\pi_m^n \;=\;& -M\chi_{3m-n}\Delta_m^n+\chi_{3m-n}b_{m-1,n+1} \\
&-M\chi_{3m-n}\delta_m^{n+1}+\chi_{3m-n-1}c_{m-1,n} \\
&-M\chi_{3m-n}\Gamma_m^n+\Lambda\chi_{3m-n-1}\beta_m^{n+1} \\
&+3\Lambda\chi_{3m-n-1}\zeta_m^n,
\end{aligned} \tag{3.69}
$$

Thus $mth$ order deformation takes the form,

$$
\pounds_{vr}\left[v_m(r)-\chi_m v_{m-1}(r)\right]=\hbar_v\left\{\sum_{n=0}^{3m}\pi_m^n r^n-C\left(1-\chi_m\right)\right\}. \tag{3.70}
$$

Let

$$y = v_m(r) - \chi_m v_{m-1}(r). \tag{3.71}$$

Where $\chi_m = \begin{cases} 0 & m \leq 1, \\ 1 & \text{otherwise.} \end{cases}$

We get

$$\frac{d^2y}{dr^2} = \hbar_v \left\{ \sum_{n=0}^{3m} \pi_m^n r^n - C(1-\chi_m) \right\}. \tag{3.72}$$

Integrating twice we get,

$$\begin{aligned} y &= \hbar_v \left\{ \sum_{n=0}^{3m} \pi_m^n r^{n+1} \frac{1}{(n+1)(n+2)} - \frac{r^2}{2} C(1-\chi_m) \right\} \\ &\quad + C_1 r + C_2. \end{aligned} \tag{3.73}$$

Using boundary conditions above expression becomes,

$$\begin{aligned} y &= \hbar_v \left\{ \sum_{n=0}^{3m} \pi_m^n r^{n+1} \frac{1}{(n+1)(n+2)} - \frac{r^2}{2} C(1-\chi_m) \right\} \\ &\quad + r \left\{ \begin{array}{c} \sum\limits_{n=0}^{3m} \frac{\pi_m^n}{(n+1)(n+2)} \left( \frac{b^{n+2}-1}{1-b} \right) \\ + \frac{(b+1)}{2} C(1-\chi_m) \end{array} \right\} \\ &\quad + \frac{C(1-\chi_m)b}{2} - \sum_{n=0}^{3m} \frac{\pi_m^n}{(n+1)(n+2)} \left( \frac{b^{n+2}-1}{1-b} \right) \\ &\quad - \sum_{n=0}^{3m} \frac{\pi_m^n}{(n+1)(n+2)}. \end{aligned} \tag{3.74}$$

As

$$v_{m-1}(r) = \sum_{n=0}^{3m} \chi_{3m-n-1} a_{m-1,n} r^n, \tag{3.75}$$

$$\begin{aligned}
\sum_{n=0}^{3m} a_{m,n} r^n &= \sum_{n=0}^{3m} \chi_m \chi_{3m-n-1} a_{m-1,n} r^n \\
&+ \hbar_v \left\{ \begin{array}{c} \sum\limits_{n=0}^{3m} \pi_m^n r^{n+2} \frac{1}{(n+1)(n+2)} \\ -\frac{r^2}{2} C (1-\chi_m) \end{array} \right\} \\
&+ r \left\{ \begin{array}{c} \sum\limits_{n=0}^{3m} \frac{\pi_m^n}{(n+1)(n+2)} \left( \frac{b^{n+2}-1}{1-b} \right) \\ + \frac{(b+1)}{2} C (1-\chi_m) \end{array} \right\} + \frac{C(1-\chi_m) b}{2} \\
&- \sum_{n=0}^{3m} \frac{\pi_m^n}{(n+1)(n+2)} \left( \frac{b^{n+2}-1}{1-b} \right) - \sum_{n=0}^{3m} \frac{\pi_m^n}{(n+1)(n+2)}.
\end{aligned} \tag{3.76}$$

Comparing like powers of $r$,

$$\begin{aligned}
a_{m,0} &= \chi_m \chi_{3m-1} a_{m-1,0} + \frac{C(1-\chi_m) b}{2} \\
&- \sum_{n=0}^{3m} \frac{\pi_m^n}{(n+1)(n+2)} \left( \frac{b^{n+2}-1}{1-b} \right),
\end{aligned} \tag{3.77}$$

$$\begin{aligned}
a_{m,1} &= \chi_m \chi_{3m-2} a_{m-1,1} + \sum_{n=0}^{3m} \frac{\pi_m^n}{(n+1)(n+2)} \left( \frac{b^{n+2}-1}{1-b} \right) \\
&+ \frac{(b+1)}{2} c (1-\chi_m),
\end{aligned} \tag{3.78}$$

$$a_{m,n} = \sum_{n=0}^{3m} \chi_m \chi_{3m-n-1} a_{m-1,n} + \sum_{n=0}^{3m} \frac{\pi_{m,n-2}}{n(n-1)}. \tag{3.79}$$

$$\begin{aligned}
\sum_{k=0}^{m-1} v'_{m-k-1}(r) v'_k &= \sum_{n=0}^{3m-3} \gamma_m^n r^n, \\
\textit{where } \gamma_m^n &= \sum_{k=0}^{m-1} \sum_{j=max\{0,p+3k-3m+3\}}^{\min\{p,3k\}} b_{k,j} b_{m-k-1,n-j},
\end{aligned} \tag{3.80}$$

$$\sum_{k=0}^{m-1} v'_{m-k-1} \sum_{l=0}^{k} v'_{k-l}\theta_l(r) = \sum_{n=0}^{3m-2} \alpha_m^n r^n, \tag{3.81}$$

$$\begin{aligned} \alpha_m^n &= \sum_{k=0}^{m-1}\sum_{l=0}^{k} \sum_{p=max\{0,n-3m-+3k+3\}}^{\min\{n,3k+1\}} \\ &\times \sum_{j=max\{0,p-3k+3l\}}^{\min\{p,3l+1\}} a'_{l,j} b_{k-l,p-j} b_{m-k-1,n-p}. \end{aligned} \tag{3.82}$$

Also

$$\sum_{k=0}^{m-1} v'_{m-k-1} \sum_{l=0}^{m-1} v'_{k-l} \sum_{s=0}^{k} v'_{l-s} v'_s = \sum_{n=0}^{3m-3} \Omega'^n_m r^n, \tag{3.83}$$

where

$$\begin{aligned} \Omega'^n_m &= \sum_{k=0}^{m-1}\sum_{l=0}^{m-1}\sum_{s=0}^{k} \sum_{p=max\{0,a-3k+3l\}}^{\min\{a,3l\}} \\ \times & \sum_{a=max\{0,n-3m+3k+3\}}^{\min\{n,3k\}} b_{m-k-1,n-a} b_{s,j} b_{k-l,a-p} b_{l-s,p-j}. \end{aligned} \tag{3.84}$$

Using the similar procedure for temperature, we obtain

$$d_{m,0} = \chi_m \chi_{3m} d_{m-1,0} + \hbar_\theta \sum_{n=0}^{3m+1} \frac{\Omega_m^n}{(n+1)(n+2)} \frac{(b^{n+2}-b)}{(b-1)}, \tag{3.85}$$

$$d_{m,1} = \chi_m \chi_{3m-1} d_{m-1,1} + \hbar_\theta \sum_{n=0}^{3m+1} \frac{\Omega_m^n}{(n+1)(n+2)} \frac{(b^{n+2}-1)}{(1-b)}, \tag{3.86}$$

$$d_{m,n} = \chi_m \chi_{3m-n} d_{m-1,n} + \hbar_\theta \sum_{n=2}^{3m+1} \frac{\Omega_m^{n-2}}{n(n-1)}. \tag{3.87}$$

The corresponding $zth$-order approximation of *Eqs.* (1.2) and (1.3) can be written as

$$\sum_{m=0}^{z} v_m(r) = \sum_{m=0}^{z} \left( \sum_{n=0}^{3m} a_{m,n} r^n \right), \tag{3.88}$$

$$\sum_{m=0}^{z}\theta_m(r)=\sum_{m=0}^{z}\left(\sum_{n=0}^{3m+1}d_{m,n}r^n\right). \tag{3.89}$$

Finally, the complete analytic solutions are given as

$$v(r)=\lim_{z\to\infty}\left[\sum_{m=0}^{z}\left(\sum_{n=0}^{3m}a_{m,n}r^n\right)\right], \tag{3.90}$$

$$\theta(r)=\lim_{z\to\infty}\left[\sum_{m=0}^{z}\left(\sum_{n=0}^{3m+1}d_{m,n}r^n\right)\right]. \tag{3.91}$$

## 3.4 Series solutions for Vogels' model.

Here

$$\mu=\mu_*\exp\left[\frac{A}{(B+\theta)}-\theta_w\right], \tag{3.92}$$

which by Maclaurin's series reduces to [17]

$$\mu=\frac{C}{C^*}\left(1-\frac{\theta A}{B^2}\right). \tag{3.93}$$

Invoking above expression, *Eqs.* (3.1) and (3.2) become

$$\frac{-Ac}{C^*B^2}\frac{d\theta}{dr}\frac{dv}{dr}+\frac{C}{rC^*}\frac{dv}{dr}-\frac{AC}{rC^*B^2}\theta\frac{dv}{dr}+\frac{AC}{C^*B^2}\theta\frac{d^2v}{dr^2}+\frac{\Lambda}{r}\left(\frac{dv}{dr}\right)^3+3\Lambda\left(\frac{dv}{dr}\right)^2=C, \tag{3.94}$$

$$\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}+\frac{\Gamma C}{C^*}\left(\frac{dv}{dr}\right)^2-\frac{AC}{C^*B^2}\theta\left(\frac{dv}{dr}\right)^2+\Lambda\Gamma\left(\frac{dv}{dr}\right)^4=0. \tag{3.94a}$$

With the following initial guesses and auxiliary linear operators

$$v_{0v}(r)=\frac{(r-b)}{(1-b)}, \tag{3.95}$$

$$\theta_{0v}(r)=\frac{(r-b)}{(1-b)}, \tag{3.96}$$

$$\pounds_{vv}(v)=v'', \tag{3.97}$$

$$\pounds_{\theta v}(\theta) = \theta'', \tag{3.98}$$

the $mth$ order deformation problems are

$$\pounds_{vr}\left[v_m(r) - \chi_m v_{m-1}(r)\right] = \hbar_v R_{vv}(r), \tag{3.99}$$

$$\pounds_{\theta r}\left[\theta_m(r) - \chi_m \theta_{m-1}(r)\right] = \hbar_\theta \mathrm{R}_{\theta v}(r), \tag{3.100}$$

$$\begin{aligned} R_{vv} = & -\frac{AC}{C^* B^2}\sum_{k=0}^{m-1} v'_{m-k-1}\theta'_k + \frac{C}{rC^*}v'_{m-1} \\ & -\frac{AC}{rC^* B^2}\sum_{k=0}^{m-1} v'_{m-k-1}\theta_k + \frac{C}{C^*}v''_{m-1} \\ & -\frac{AC}{C^* B^2}\sum_{k=0}^{m-1} v''_{m-k-1}\theta_k + \frac{\Lambda}{r}\sum_{k=0}^{m-1} v'_{m-k-1}\sum_{l=0}^{k} v'_{k-l}v'_l \\ & +3\Lambda\sum_{k=0}^{m-1} v'_{m-k-1}\sum_{l=0}^{k} v'_{k-l}v''_l \\ & -C(1-\chi_m), \end{aligned} \tag{3.101}$$

$$\begin{aligned} R_{\theta v} = & \frac{1}{r}\theta'_{m-1} + \theta''_{m-1} + \frac{C}{C^*}\Gamma\sum_{k=0}^{m-1} v'_{m-k-1}v'_k \\ & -\Gamma\frac{AC}{C^* B^2}\sum_{k=0}^{m-1} v'_{m-k-1}\sum_{l=0}^{k} v'_{k-l}\theta_l \\ & +\Lambda\Gamma\sum_{k=0}^{m-1} v'_{m-k-1}\sum_{l=0}^{k} v'_{k-l}\sum_{s=0}^{k} v'_{l-s}v'_s. \end{aligned} \tag{3.102}$$

We have found that $\mathrm{v}_m(r)$ can be written as

$$v_m(r) = \sum_{n=0}^{3m} a'_{m,n}r^n, m \geq 0, \tag{3.103}$$

$$\theta_m(r) = \sum_{n=0}^{3m+1} d'_{m,n}r^n, m \geq 0. \tag{3.104}$$

Where $a'_{m,n}$ and $d'_{m,n}$ can be easily calculated by using the procedure discussed in previous section and are given as

$$a'_{m,n} = \sum_{n=0}^{3m} \chi_m \chi_{3m-n-1} a'_{m-1,n} + \sum_{n=0}^{3m} \frac{\pi'_{m,n-2}}{n(n-1)}. \tag{3.105}$$

$$d'_{m,n} = \chi_m \chi_{3m-n} d'_{m-1,n} + \hbar_\theta \sum_{n=2}^{3m+1} \frac{\Omega''^{n-2}_m}{n(n-1)}. \tag{3.106}$$

Finally, the complete analytic solutions are given as

$$v(r) = \lim_{z\to\infty} \left[ \sum_{m=0}^{z} \left( \sum_{n=0}^{3m} a'_{m,n} r^n \right) \right], \tag{3.107}$$

$$\theta(r) = \lim_{z\to\infty} \left[ \sum_{m=0}^{z} \left( \sum_{n=0}^{3m+1} d'_{m,n} r^n \right) \right]. \tag{3.108}$$

## 3.5 Graphical results and discussion

In order to report the effects of sundry parameters in the present investigation we plotted Figs. 3.1 to 3.19. The Figs. 3.1 to 3.8 have been sketched for the Reynolds' model while Figs. 3.10 to 3.19 correspond to the case of Vogels' model. Figs. (3.2) and (3.3) illustrate the temperature and velocity profiles along the radial distance for different values of $\Lambda$ . It is found that velocity and temperature fields increase by increasing $\Lambda$. Fig. (3.1) is prepared to see the convergence region for different values of $\Lambda$. The effects of constant pressure gradient on the temperature and velocity can be seen in Figs. (3.5) and (3.6). It is observed from the figures that with the increase in $C$, both the velocity and temperature decrease. Fig. (3.7) is prepared for the temperature distribution for different values of $M$. It shows that with the increase in $M$ the temperature decreases. This happens because when we increase $M$ the viscosity decreases and decrease of viscosity effects the viscous dissipation which causes the decrease in temperature. Fig. (3.8) is plotted for the velocity distribution against $r$ for different values of $M$. It is seen from the figure that with the increase in $M$, the velocity decreases. Figs. (3.9) and (3.10) illustrate the temperature and velocity profiles along the radial distance for different values of $\Lambda$. It is found that with the increase in $\Lambda$, both velocity and temperature increase. Figs.

(3.11) and (3.12) illustrate the temperature and velocity profiles along the radial distance for different values of $B$. It is found that with the increase in B, both velocity and temperature increase. The effects of constant pressure gradient on the temperature and velocity can be seen in Figs. (3.14) and (3.15). It is observed from the figures that with the increase in $C$, both the velocity and temperature increases .The effects of Brinkmann number $\Gamma$ are shown in Figs. (3.16) and (3.17). It is found that with the increase in $\Gamma$, velocity decreases and temperature increases. Fig. (3.18) illustrates the temperature profile along the radial distance for different values of $C^*$. It is found that with the increase in $C^*$ temperature increases.

### Convergence of the solution

Fig. (3.1) is prepared to see the convergence region for different values of $\Lambda$. It can be seen that convergence region decreases as we increase the values of $\Lambda$ .The convergence region for different values of pressure drop (C) in case of Reynolds' model can be seen in Fig. (3.4). The convergence region increases as we decrease the values of pressure drop (C) . The convergence region for different values of pressure drop (C) in case of Vogels' model can be seen in Fig. (3.13). The convergence region for $-0.4 \leq C \leq -0.1$ increases as we decrease values of C .

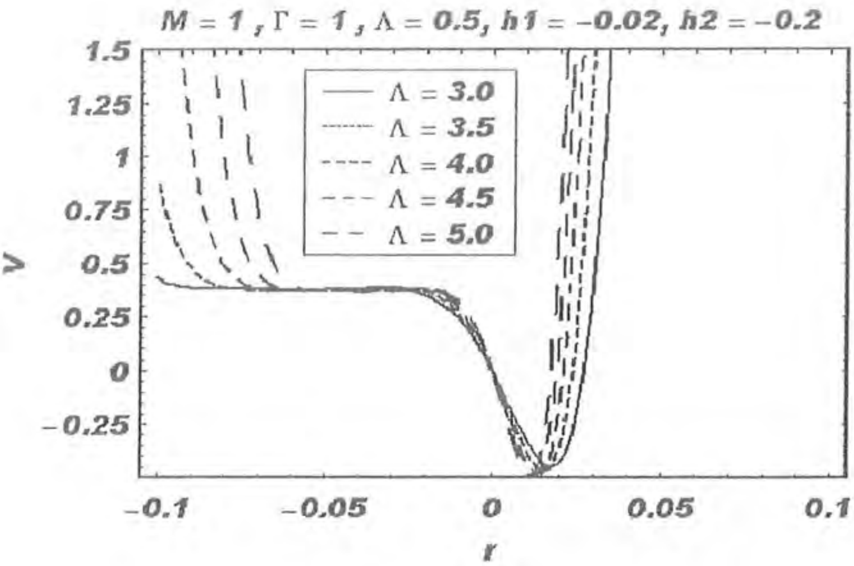



Figure 3-1: h-curve for different values of non-Newton parameter Λ for the Reynolds' model.

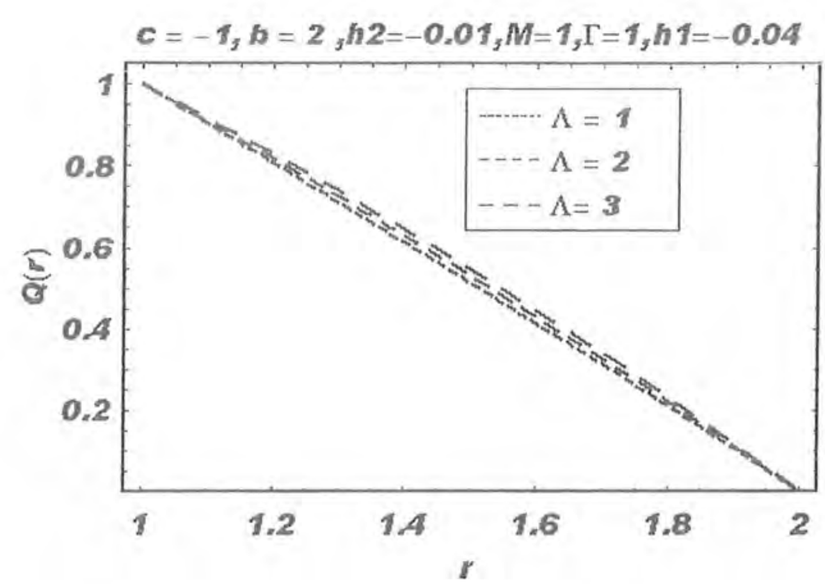



Figure 3-2: Temperature profile along the radial distance for different values of non-Newton parameter Λ for the Reynolds' model.

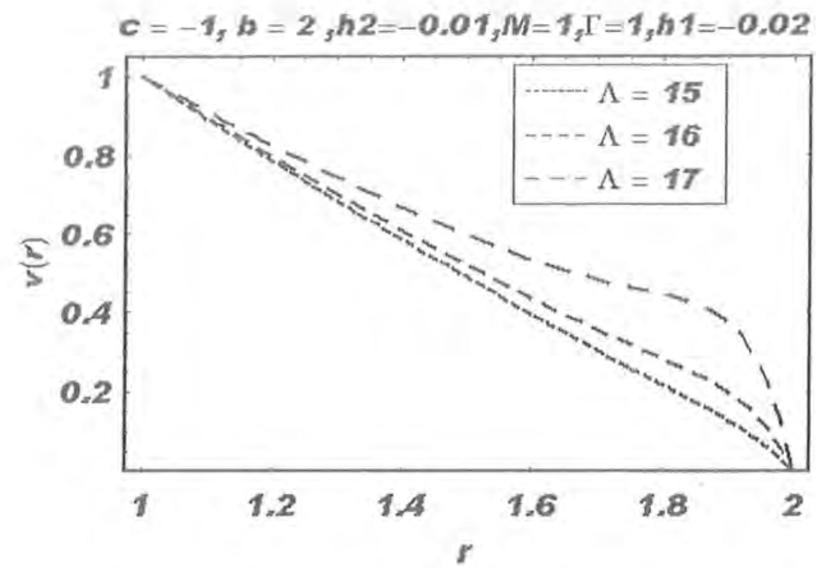



Figure 3-3: Velocity profile along the radial distance for different values of non-Newton parameter Λ for the Reynolds' model.

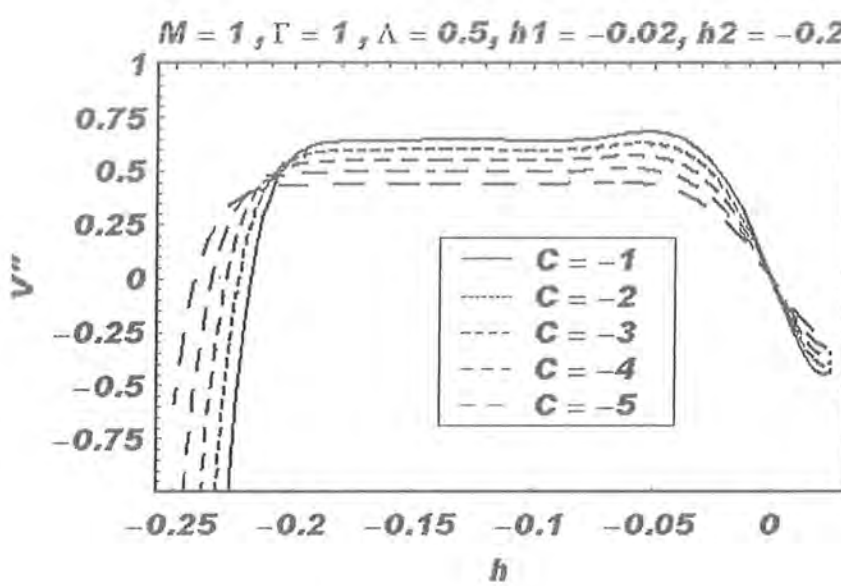



Figure 3-4: The h-curve for different values of pressure drop (C) for the Reynolds' model.

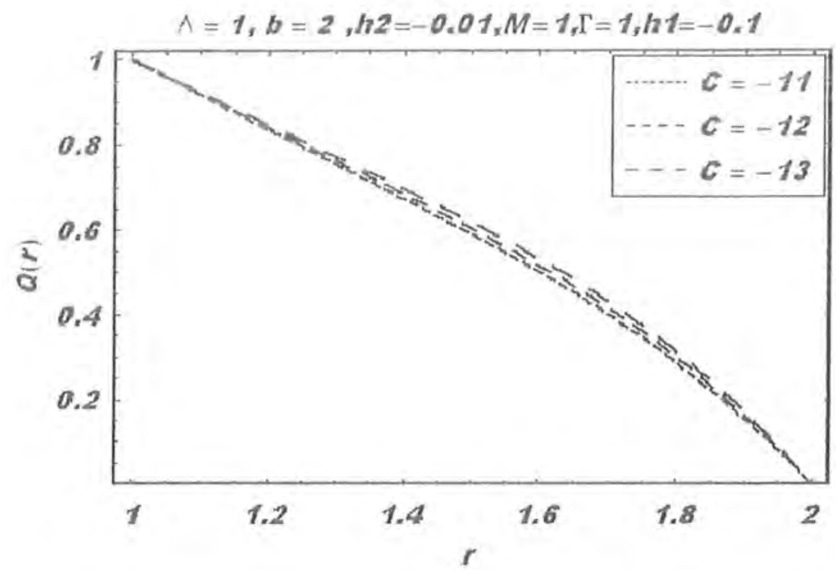



Figure 3-5: Temperature profile along the radial distance for different values of pressure drop( C) for the Reynolds' model.

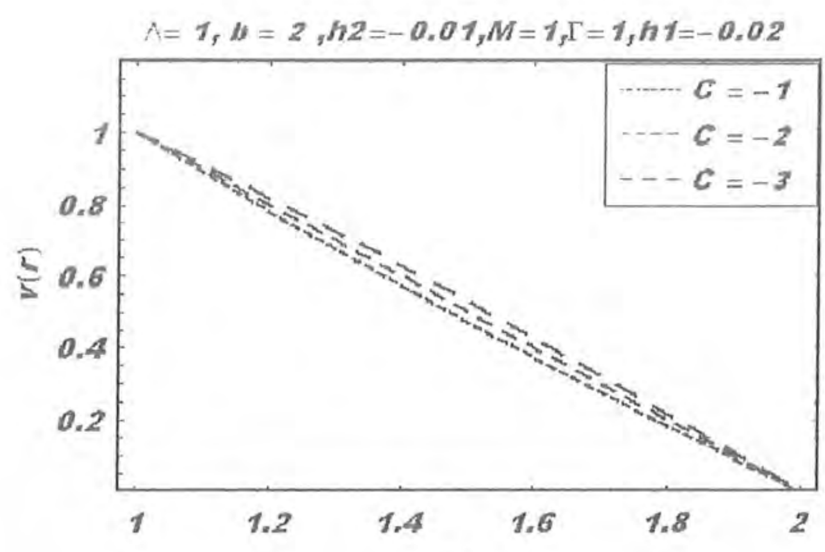



Figure 3-6: Velocity profile along the radial distance for different values of pressure drop( C) for the Reynolds' model.

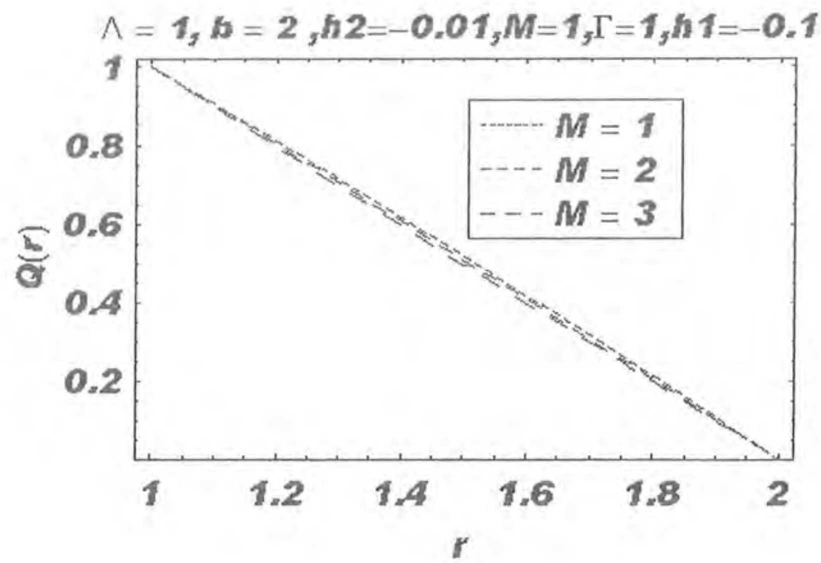



Figure 3-7: Temperature profile along the radial distance for different values of viscosity parameter( M) for the Reynolds' model.

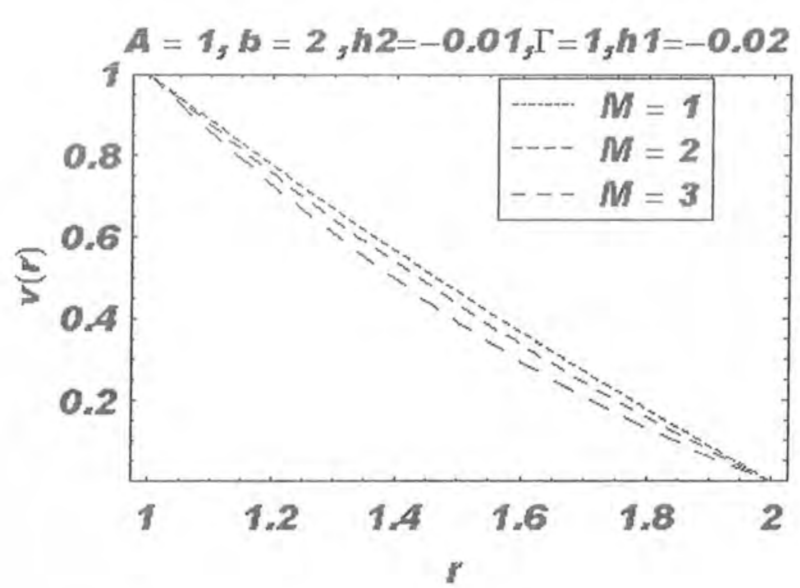



Figure 3-8: Velocity profile along the radial distance for different values of viscosity parameter( M) for the Reynolds' model.

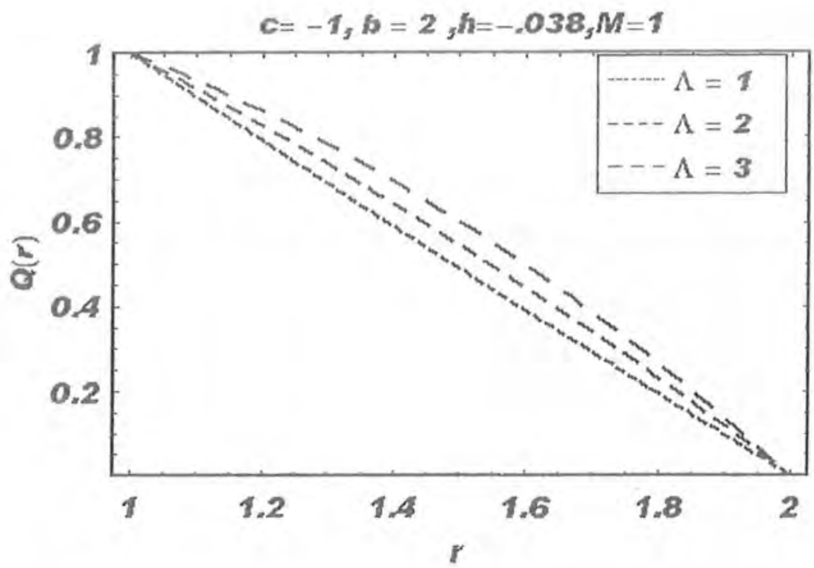



Figure 3-9: Temperature profile along the radial distance for different values of non-Newton parameter Λ for Vogels' model. .

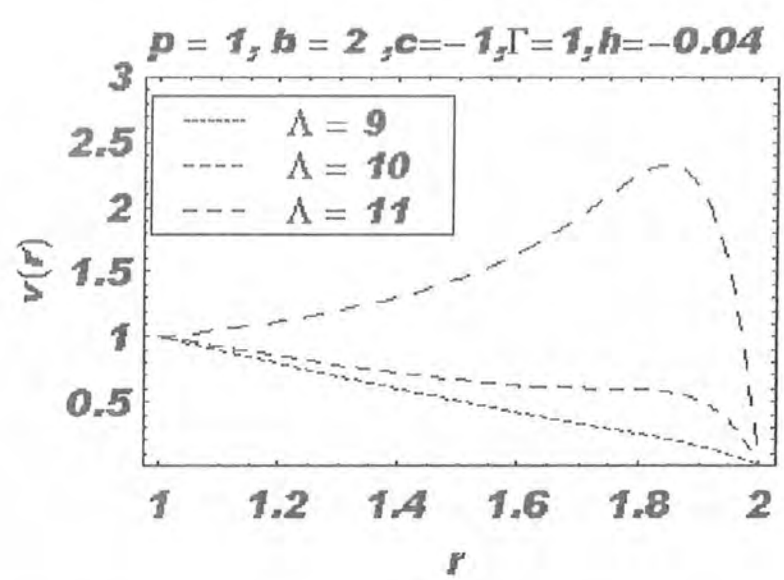



Figure 3-10: Velocity profile along the radial distance for different values of non-Newton parameter Λ for Vogels' model.

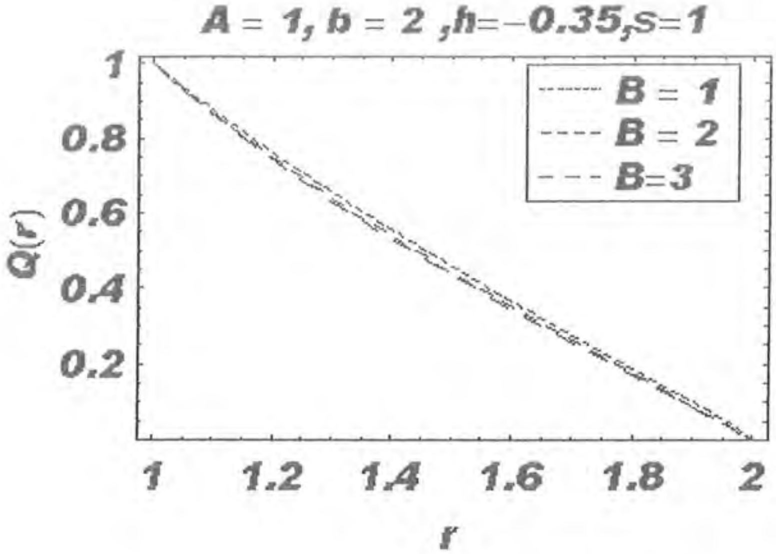



Figure 3-11: Temperature profile along the radial distance for different values of viscosity parameter B for Vogels' model.

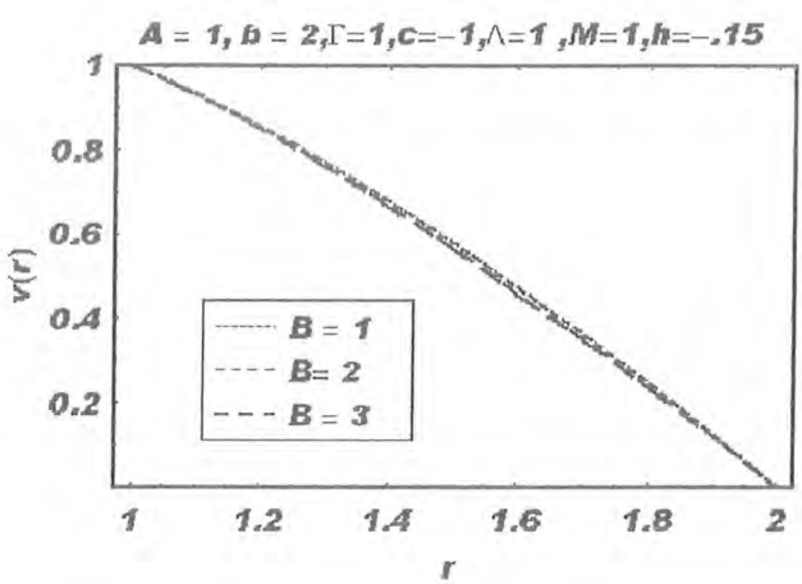



Figure 3-12: Velocity profile along the radial distance for different values of viscosity parameter B for Vogels' model.

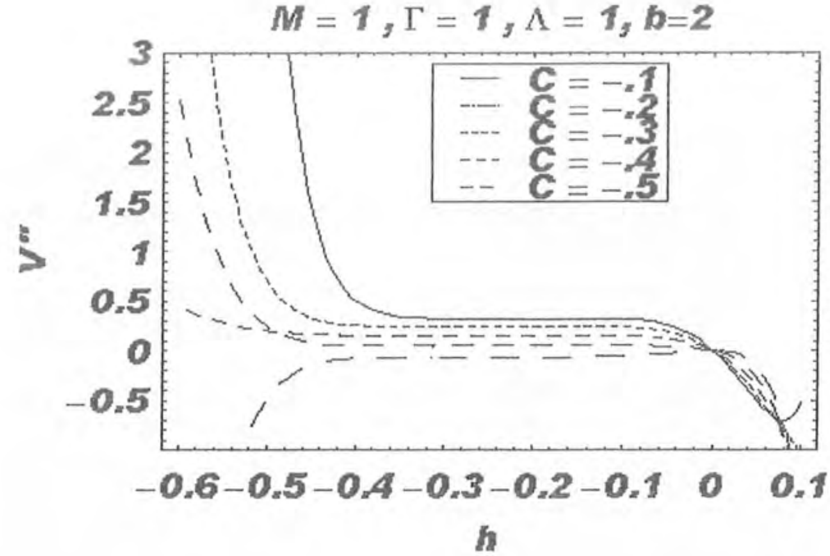



Figure 3-13: h-curve for different values of pressure drop( C) for Vogels' model.

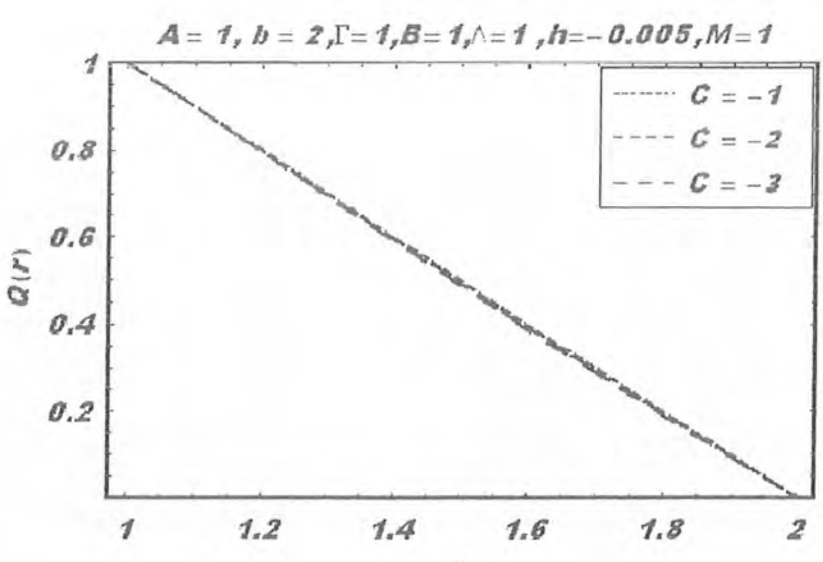



Figure 3-14: Temperature profile along the radial distance for different values of pressure drop( C) for Vogels' model.

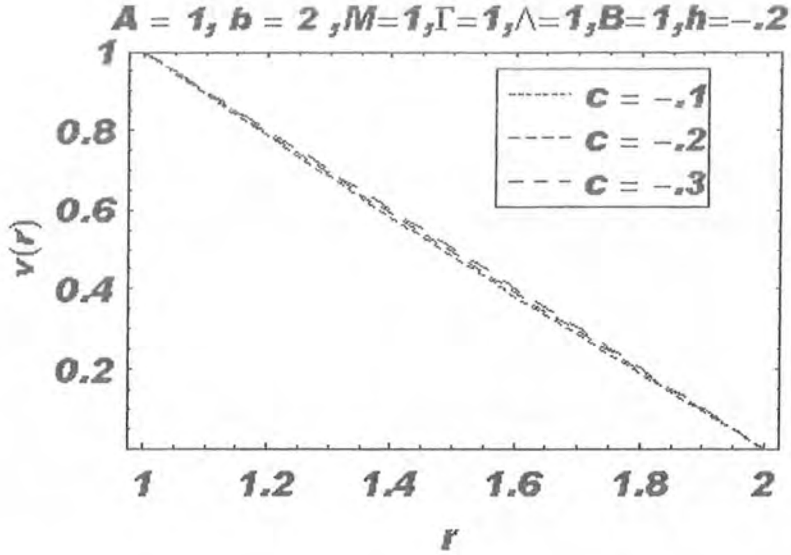



Figure 3-15: Velocity profile along the radial distance for the different values of pressure drop( C) for Vogels' model.

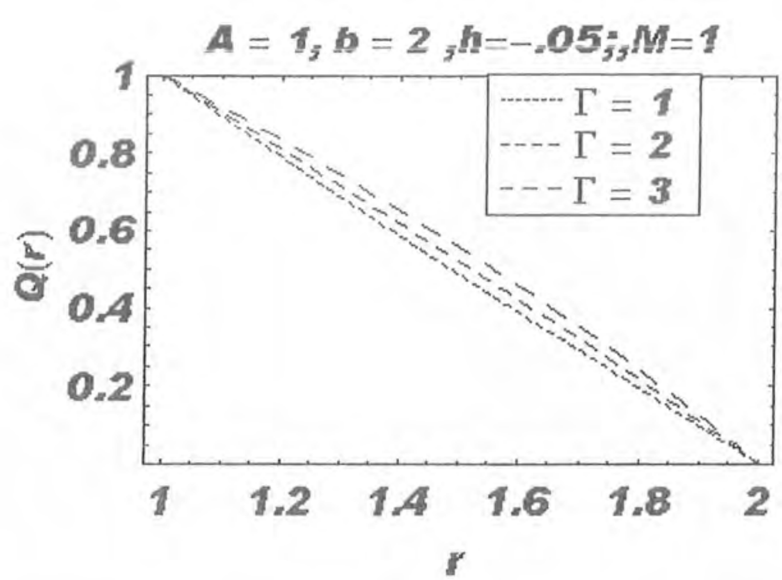



Figure 3-16: Temperature profile along the radial distance for different values of Brinkman number Γ for Vogels' model.

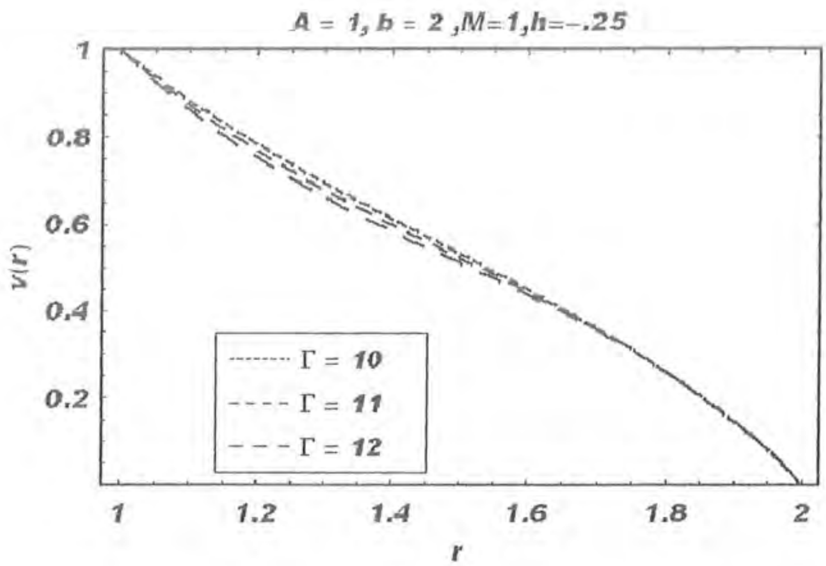



Figure 3-17: Velocity profile along the radial distance for different values of Brinkman number Γ for Vogels' model.

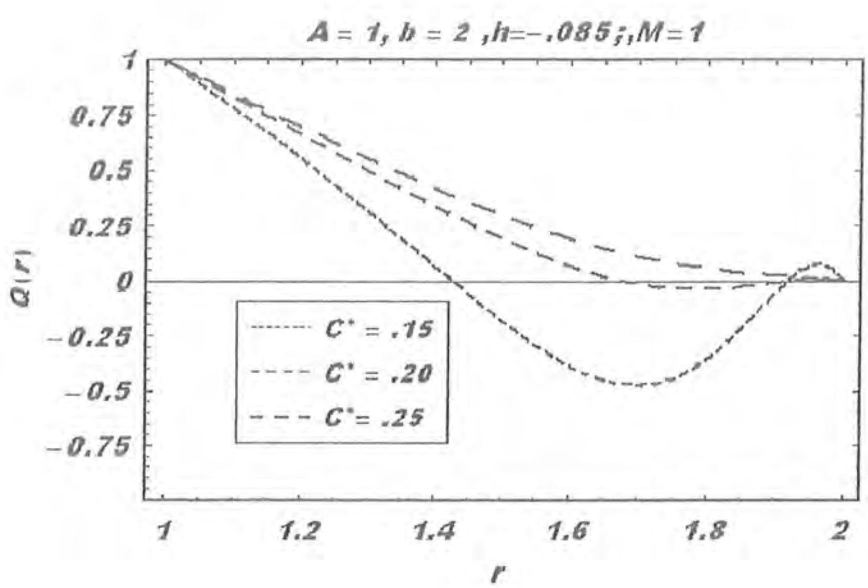



Figure 3-18: Temperature profile along the radial distance for different values of $C^*$ for Vogels' model.

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