بسم الله الرحمن الرحيم

IN THE NAME OF

ALLAH

THE MOST BENIFICIENT

THE MOST MERCIFUL

On Topological Rings

By

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Supervised

By

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Quaid-i-Azam University, Islamabad
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On Topological Rings

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CERTIFICATE

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WE ACCEPT THIS DISSERTATION AS CONFORMING TO THE REQUIRED STANDARD

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Dedication

To my
Parents,
whose prayers, love and guidance are
unforgettable and priceless for me,
and specially to my
beloved ***sister*** and cute little niece ***Maha***

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The most important acknowledgement is to my **Lord** the Most Merciful, the Most Compassionate, the Creator of all things, by Whose mercy I the unworthy slave is given the ability to begin and complete this work. **Allah** states in the Quran **"Then remember Me; I will remember you. Be grateful to Me, and reject not faith"** (al-Baqarah 2: 152). I am grateful towards all He has given me.

May Allah's peace and blessing be upon our last beloved **Prophet Muhammad**, the most noble, the most generous, having the highest level of exemplary character and who was a mercy upon us from Allah. May Allah shower his rain of blessings upon the family and companions of his most beloved Prophet (Peace be upon him).

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*May **Almighty Allah** reward all of them with great honor in this world and the world hereafter.*

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Preface

A systematic investigation of topological rings had started since 1940, by using the frame of topological algebra. Many parts of the theory of topological rings have been exposed in mathematical text (for example, p-adic numbers etc.). It was L. S. Pontrya who obtained the first fundamental result in the classification of locally compact skew fields by using topological rings. After that algebraic geometrists used topological rings and topological modules as a tool for solving many problems of algebraic geometry. Amongst the founders of modern algebraic geometry Oskar Zariski was the first to realized that the subject needed a proper foundation. It was he who laid a proper foundation of the subject in 1937. Mainly, his work concentrated on fundamental groups. He used the notions of integral independence, valuation rings, Zariski rings and regular local rings in algebraic geometry. In 1949, he published a short paper "A simple analytic proof of fundamental property of birational transformation" in which he proved his main theorem using the completion and valuation of a local ring. Later on a more simple proof of this theorem was given using some standard facts of commutative algebra.

In algebraic geometry Zariski topology is important for studying the polynomial equations.

Zariski ring was first introduced by Zariski itself in 1960 for the sake of generalization of Zariski topology in which he discussed the pair (R, I), where R is a Noetherian ring with unity such that every submodule F of every finitely generated R-module E is closed and I is an ideal of R.

This dissertation consists of three chapters. In chapter one we give some introductory concepts of ring theory and topology. In chapter two we give basic definitions and results of topological rings and topological modules, especially, the I-adic completion of a ring and Zariski topology.

In ring theory, ascend and descend of various properties for ring (domain) extensions has been discussed frequently, that is, conditions are found under which these properties ascend or descend. In the papers [14] and [16], ascend and descend of factorization properties for atomic domains, domains satisfying $ACCP$, bounded factorization domains, Half factorization domains, Pre-Schreier and semi-rigid domains has been discussed. We studied the ascend and descend conditions for Zariski rings and proved some valuable results in this respect. In view of the papers [10] and [11], we have discussed domain $R + XT[[X]]$, where $R \subset T$ be the unitary ring (domain) extension. This has provides us with some good examples of Zariski rings. We also have discussed for a ring the conditions under which its ring of fractions, polynomial ring and the ring of formal power series becomes a Zariski ring.

In chapter three we define Zariski ring and give conditions for the ascend and descend of Zariski ring property for unitary ring extensions. We also give some examples of Zariski ring and find the conditions for some extensions of a ring with identity to be Zariski.

Contents

Chapter 1

Preliminaries

1.1 Introduction

This chapter consists of basic concepts of commutative ring theory, module theory and theory of topological spaces. We have partitioned it into three sections. In first section we discuss the basic definitions, examples and structures relating to the commutative ring theory. Though we can not review whole of the subject but we have tried our best to include those topics which are necessary and are used in the forthcoming chapters. Thus we introduced specific terminology and its explanation. While selecting the material for this chapter, we have followed mostly ([3], [11], [13], [14]) and have mentioned otherwise. In section 2, we have discussed basic concepts, definitions, examples and structures related to the module theory. The last Section consists of basic definitions and examples of the structures related to the theory of topological spaces.

1.2 Commutative Rings

This section includes the basic concepts and discussion over the commutative ring theory. For explanation we give examples, as elementary as possible. Thus in the following all the basic definitions and concepts that we use in the later chapters are discussed.

1.2.1 Basic Concepts

Ring

A non-empty set R together with two binary operations, addition and multiplication is said to be ring if;

(i) $(R,+)$ is an abelian group.

(ii) $(R,.)$ is a semigroup.

(iii) The multiplication is distributive over addition, that is $(x+y)z = xz + yz$ and $z(x+y) = zx + zy$ for all $x, y, z \in R$. Rings as defined above are also called associative rings, a non associative ring only possess the properties (i) and (iii).

Identity element

An element say 1 is called identity element if $1.x = x = x.1$ for all $x \in R$. The identity element is also called unity and a ring with 1 is known as ring with unity or ring with identity.

Invertible element

Let R be a ring with unity then $a \in R$ is

(1) Left invertible, if there exist some $a' \in R$ such that $a'a = 1_R$.

(2) Right invertible, if there exist some $a' \in R$ such that $aa' = 1_R$.

(3) Invertible if it is both left and right invertible.

An invertible element is also called a unit.

Commutative ring

A ring R is said to be commutative ring if multiplication is commutative, that is $ab = ba$ for all $a, b \in R$.

Examples

(a) If $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ represent the set of integers, rationals and real numbers

respectively, then the structures $(\mathbb{Z},+,.)$, $(\mathbb{Q},+,.)$ and $(\mathbb{R},+,.)$ are the examples of commutative rings with identity.

(b) Consider the set $M_n(\mathbb{R})$ of all $n \times n$ matrices with real entries. Under the usual addition and multiplication of matrices, $(M_n(\mathbb{R}),+,.)$ forms a non-commutative ring since multiplication is non-commutative in matrices.

Zero divisor

Let R be a ring and $0 \neq a \in R$, then a is said to be a left (right) zero divisor in R if there exists some element $0 \neq b \in R$ such that $ab = 0$ $(ba = 0)$. A zero divisor is any element of R that is, either a left or right zero divisor.

According to this definition unit elements of a ring cannot be the zero divisors. An obvious example of a ring with zero divisor is $\mathbb{Z}_n$, where the integer $n > 1$ is composite; if $n_1n_2 = n$ in $\mathbb{Z}\,(0 < n_1, n_2 < n)$, then the product $n_1n_2 = 0$ in $\mathbb{Z}_n$.

1.2.2 Integral Domain

A commutative ring with identity $1 \neq 0$ is said to be an integral domain if it has no zero divisors.

Note that some authors defines integral domain without identity 1, but throughout we take an integral domain a commutative ring with identity $1 \neq 0$.

Examples

The ring of integers $\mathbb{Z}$ and the Gaussian integers ring $\mathbb{Z}[i]$ are the examples of integral domains.

Homomorphism

Let R and S be rings. A ring homomorphism is a map $\phi : R \rightarrow S$ which satisfies the following for all x, $y \in R$

(i). $\phi(x + y) = \phi(x) + \phi(y)$.

(ii). $\phi(xy) = \phi(x)\phi(y)$.

If R and S contain identity element, then the homomorphism of R into S is usually called a homomorphism of rings with identity, which also preserve the identity element; that is $\phi(1_R) = 1_S$.

A one-one and onto ring homomorphism $\phi : R \rightarrow S$ is called ring isomorphism. In this case the rings R and S are said to be isomorphic and we write it as, $R \cong S$.

Example

(a) The identity mapping 1_R on a ring R is a ring homomorphism.

(b) The composition of two ring homomorphism $\phi : R \to S$, $\upsilon : S \to T$ is again a ring homomorphism $\upsilon \circ \phi : R \to T$.

Subring

A non-empty subset S of a ring R is called a subring of R. If S is itself a ring (using the induced operations).

A subring is called unitary if it contains the identity element of the ring.

Examples

The set of integers $\mathbb{Z}$ and $\mathbb{Q}$ are both subring of $\mathbb{R}$.

Remark 1 A subset S of a ring R is a subring of R if and only if S is a subgroup of $(R, +)$, and closed under multiplication.

Ideal

Let R be a ring and I be a subgroup of $(R, +)$ then I is called a left ideal(resp. right ideal) of R if $RI \subseteq I$ (resp. $IR \subseteq I$).

Two sided ideal (ideal) is both left and right ideal of the ring.

Prime element

A non-zero element p in a commutative ring R is said to be prime if and only if p is not invertible and p divides ab implies either p divides a or p divides b.

Prime ideal

Let P be an ideal in a commutative ring R with 1 such that $P \neq R$ and for all $a, b \in R$

$$ab \in P \implies a \in P \text{ or } b \in P$$

then P is a prime ideal.

Spectrum

The set of all prime ideals in a ring R is called the spectrum of R and it is regarded as $\text{Spec}(R)$.

$$\text{Spec}(R) = \{P : P \text{ is prime ideal of } R\}.$$

Example

The prime ideals of the ring of integers $(\mathbb{Z}, +, .)$ are precisely the ideals (p), where p is a prime number, together with the two trivial ideals $\{0\}$ and Z.

Nil Radical of a Ring

Nil radical of a ring R is the intersection of all prime ideals of the ring. It is denoted by $N(R)$, i.e.

$$N(R) = \cap_{P \in \text{Spec}(R)} P.$$

Principal ideal

An ideal I of a commutative ring R with 1 is said to be principal if it is generated by a single element that is, $I = < a > = \{ar : r \in R\}$.

Maximal ideal

An ideal M of the commutative ring with 1 is said to be maximal if $M \neq R$ and for every ideal N such that $M \subseteq N \subseteq R$, either $N = R$ or $N = M$.

Theorem 2 *[11] Let I be a proper ideal of a commutative ring R. Then I is maximal ideal if and only if $(I, a) = R$ for any element $a \in R$, where (I, a) denotes the ideal generated by $I \cup \{a\}$.*

Example

Let $(\mathbb{Z}, +, .)$ be the ring of integers. Then the maximal ideals of Z correspond to the prime numbers. More precisely, the principal ideal (p), $p > 1$, is maximal if and only if p is prime.

The prime ideals can be characterized in the following manner.

Theorem 3 *Let I be a proper ideal of the ring R. Then I is a prime ideal (resp. maximal) if and only if the quotient ring R/I is an integral domain (resp. Field).*

MAX of a Ring (Max(R))

The set of all maximal ideals of a ring R is called the Max of R. It is regarded as $\text{Max}(R)$, i.e.

$$\text{Max}(R) = \{M : M \text{ is maximal ideal of } R\}.$$

Jacobson Radical of a Ring

Jacobson radical of a ring R is the intersection of all maximal ideals of the ring. It is denoted and defined by

$$J(R) = \cap_{M \in \text{Max}(R)} M.$$

To know the relationship between the maximal and prime ideals the following result gives very important information.

Theorem 4 *In a commutative ring R with identity every maximal ideal is a prime ideal.*

Note that the converse of above theorem does not hold, as in Z although $\{0\}$ is a prime ideal but it is not a maximal ideal of Z.

Factor Ring

If I be an ideal of the ring R, then the equivalence classes of $y \in R$ for the relation $\sim$ is the set

$$\begin{aligned} [y] &= \{x \in R : \ x - y \in I\} \\ &= \{x \in R : \ x - y = i, \ i \in I\} \\ &= \{x \in R : \ x = y + i, \ i \in I\} \\ &= y + I = \{y + i, \ i \in I\}. \end{aligned}$$

Let I is an ideal of the commutative ring R with identity 1, then the factor ring of R is denoted by R/I and is the collection of all distinct equivalence classes of I in R; that is,

$$R/I = \{a + I : a \in R\}.$$

It is easy to verify that R/I is again a ring and R/I is commutative if R is commutative.

Field

A commutative ring F with 1 (having at least two elements) whose every non-zero element is invertible is called a field.

Examples

There are some standard examples of fields, that is rational field $\mathbb{Q}$, the real field $\mathbb{R}$ and the finite field $\mathbb{Z}_p$, where p is a prime integer.

Cancellative law

Let R be a commutative ring and $a \in R$ where $a \neq 0$, then a is said to be cancellative if $ab = ac \Longrightarrow b = c$ and $ba = ca \Longrightarrow b = c$.

If R is a ring with unity and $o \neq a \in R$ is invertible , then a is cancelable.

Remark 5 Cancellation law holds in a ring R if and only if R has no zero divisor.

Division ring (Skew field)

If every non-zero element of a ring R ,with identity, is a unit then R is called a division ring

.

In a ring R with identity a unit can not be a zero divisor.

Nilpotent element

If R is a ring with identity then an element $a \in R$ is said to be nilpotent if there is a positive integer n such that $a^n = 0$, where a^n stands for $a \cdot a \cdots a$ (n factors). 0_R is the trivial nilpotent element.

If R is a ring with identity then unit elements of R can not be nilpotent.

1.2.3 Formal Power Series Ring

Let R be a commutative ring and $\mathbb{Z}_0$ be the additive monoid of non-negative integers. Set $R^{\mathbb{Z}_0} = \{f : \mathbb{Z}_0 \to R\}$ to represent the collection of all infinite sequences from $\mathbb{Z}_0$ to R, then we have

$$f(0) = f_0,\ f(1) = f_1, ...,\ f(n) = f_n, ...$$

This can be written as

$$f = (f_0, f_1, f_2, ..., f_k, ...),\ \text{where } f_{i's} \in R,$$

and is called formal power series.

Now we will introduce operations in the set $R^{\mathbb{Z}_0}$ such that $R^{\mathbb{Z}_0}$ is a ring containing R as a subring. Let us consider $f, g \in R^{\mathbb{Z}_0}$ such that

$$f = (f_0, f_1, ...) \text{ and } g = (g_0, g_1, ...)$$

and $f = g$ if and only if $f_n = g_n$ for all $n \geqslant 0$.

The addition and multiplication of formal power series is defined as follows:

$$f + g = (f_0 + g_0, f_1 + g_1, ...).$$

$$fg = (h_0, h_1, ...),$$

where for each $n \geqslant 0$,

$$h_n = \Sigma_{n=i+j} f_i g_j.$$

$(0, 0, 0, ...)$ is the zero element of $R^{\mathbb{Z}_0}$ and the additive inverse of $(f_0, f_1, ...)$ is $(-f_0, -f_1, ...)$. Hence $(R^{\mathbb{Z}_0}, +)$ becomes an abelian group. Moreover, $(R^{\mathbb{Z}_0}, .)$ is semigroup and multiplication is distributive over addition, therefore $(R^{\mathbb{Z}_0}, +, .)$ forms a ring structure known as the ring of formal power series in one indeterminate over R.

Isomorphism

There is an imbedding $\theta : R \to R^{\mathbb{Z}_0}$ defined by

$$\theta(a) = (a, 0, 0, 0, ...).$$

So, an element $a \in R$ has a representation $(a, 0, 0, 0, ...)$ in $R^{\mathbb{Z}_0}$.

Formation of power series

Now we define a power series in a formal way, we have

$$X = (0, 1, 0, ...)$$

and

$$f_0 X = (0, f_0, 0, ...), \text{ where } f_0 \in R.$$

$f_0 X \in R^{\mathbb{Z}_0}$ which has the element f_0 for its second term and 0 for all other terms. In general $f_n X^n$, $n \geqslant 1$ denotes the sequence

$$(0, 0, ..., 0, f_n, 0, ...),$$

where f_n is the element at $(n+1)th$ term in this sequence. Now we have

$$f_2 X^2 = (0, 0, f_2, 0, ...),$$

$$f_3 X^3 = (0, 0, 0, f_3, 0, ...) \text{ and so on.}$$

Thus

$$f(X) = (f_0, f_1, ..., f_n, ...)$$

can be uniquely expressed in the form

$$f = f_0 + f_1 X + f_2 X^2 + ... + f_n X^n + ... = \sum f_k X^k.$$

To indicate the indeterminate X, usually we denote $R^{\mathbb{Z}_0}$ by $R[[X]]$.

Remark 6 *If the ring R has a multiplicative identity* 1, *then* $X \in R[[X]]$.

Proposition 7 *[11] Let R be a ring and denote by $R[[X]]$ the set of all sequences of elements $(f_0, f_1, ...)$ of R.*

(1) *$R[[X]]$ is a ring with addition and multiplication defined by:* $(f_0, f_1, ...) + (g_0, g_1, ...) = (f_0 + g_0,\ f_1 + g_1, ...)$ *and* $(f_0, f_1, ..)(g_0, g_1, ...) = (h_0, h_1, ..)$, *where* $h_n = \sum_{i=0}^{n} f_i g_{n-i} = \sum_{k+j=n}^{n} f_k g_j$.

(2) *If R is commutative ring, then so is $R[[X]]$.*

Remark 8 If R is an integral domain, then so is its power series ring $R[[X]]$.

Lemma 9 *Let R be a commutative ring with identity. A formal power series $f(X) = \sum f_k X^k$ is invertible in $R[[X]]$ if and only if the constant term f_0 has an inverse in R.*

Corollary 10 *A power series $f(X) = \sum f_k X^k \in K[[X]]$, where K is a field, has an inverse in $K[[X]]$ if and only if its constant term $f_0 \neq 0$.*

Theorem 11 *Let R be a commutative ring with* 1. *Then there is a one to one correspondence between the maximal ideals M of the ring R and the maximal ideals $M[[X]]$ of its power series ring $R[[X]]$ in such a way that $M[[X]]$ corresponds to M if and only if $M[[X]]$ is generated by M and X; that is $M[[X]] = \langle M, X \rangle$.*

1.2.4 Polynomial Rings

Let $R[X]$ denote the set of all power series in $R[[X]]$, whose finite number of coefficients are nonzero. So,

$$R[X] = \{f_0 + f_1 X + ... + f_n X^n : f_n \in R,\ n \geqslant 0\}.$$

An element of $R[X]$ is called polynomial in indeterminate X over the ring R.

Proposition 12 *The polynomial ring $R[X]$ is a subring of $R[[X]]$.*

Remark 13 If $f \in R[[X]]$ is actually a polynomial with irreducible [resp. unit] constant term then f need not be irreducible [resp. a unit] in the polynomial ring R.

1.2.5 Localization

Localization generalizes the construction of the field of fraction of a domain but applies to any commutative ring.

Multiplicative System

Let R be a commutative ring with 1. A subset T of R with 1, is called a multiplicative system in R if $s, t \in T$ implies $st \in T$.

Examples

Let R be a commutative ring with identity. Then followings are few examples of multiplicative systems in R.

(a) $\{1\}$.

(b) $U(R)$, unit elements of ring R.

(c) $R \backslash P$ is multiplicative system if P is prime ideal in R.

(d) $\{1, a, ..., a^n, ...\}$, where $a \in R\backslash\{0\}$ is a nonzero divisor.

(e) Intersection of multiplicative systems is again a multiplicative system.

Saturated multiplicative system

A multiplicative system T is said to be saturated if any factor of an element of T again lies in T. For example $\{1, a, ..., a^n, ...\}$, where $a \in R\backslash\{0\}$ is a nonzero divisor.

Ring of fractions

Let R be a commutative ring with identity and T be a multiplicative system in R.

Define an equivalence relation $\sim$ on $R \times T$ by;

$(a, s) \sim (b, t) \Leftrightarrow atu = bsu$ for some $u \in T$.

Where the equivalence class of $(a, s) \in R \times T$ is denoted by the fraction a/s.

The ring of fractions of R with denominators in T is the set $T^{-1}R = (R \times T)/\sim$ of all fractions with operations given by

$$(a/s) + (b/t) = (at + bs)/st \text{ and}$$

$$(a/s)(b/t) = ab/st, \text{ where } a/s, b/t \in T^{-1}R.$$

It is straight forward to show that, the operations on $T^{-1}R$ are well defined and that $T^{-1}R$ is a ring with zero element $0/1$ and identity element $1/1$. For all $s, t \in S$, s/t is a unit in $T^{-1}R$, with $(s/t)^{-1} = t/s$.

1.2.6 Factorization Domains

Atomic domain

An integral domain D is called atomic domain if every non zero non unit element of D can be written as a product of irreducibles(atoms).

Half factorial domain(HFD)

We define D to be a *half factorial domain* (HFD) if D is atomic and whenever $x_1....x_m = y_1....y_m$ with each $x_i, y_i \in R$ irreducible, then $m = n$.

Bounded factorization domain(BFD)

An integral domain D is a BFD if D is atomic and for each nonzero non-unit of D there is a bound on the length of factorization into products of irreducible elements.

Finite factorization domain(FFD)

An integral domain R is an FFD if every nonzero element of R has only a finite number of nonassociate divisors.

In general,

$$HFD \Longrightarrow BFD \Longrightarrow Atomic.$$

But none of the above implication is reversible.

Noetherian domain

We say that a ring R is Noetherian if it satisfies one of the following three equivalent conditions:

(1) Every non-empty set of ideals in R has maximal element.

(2) Every ascending chain of ideals in R is stationary.

(3) Every ideal in R is finitely generated.

Theorem 14 *[11] (Hilbert's Basis Theorem)If R is Noetherian then the polynomial ring $R[X]$ is also Noetherian.*

Theorem 15 *[11] If R is Noetherian then the factor ring R/I is also Noetherian for any ideal I of R.*

1.3 Module

Definition

Let R be a ring with 1, a (left) R-module is an additive abelian group M together with a function $R \times M \to M$(the image of (r, a) being denoted by ra) such that for all $r, s \in R$ and $m, n \in M$:

(i). $r(m + n) = rm + rn$.

(ii). $(r + s)m = rm + sm$.

(iii). $r(sa) = (rs)a$.

(iv). $1_R m = m$ for all $m \in M$,

then M is said to be unitary R-module.

A (unitary) right module is defined similarly. An R-module is both left and right R-module.

Examples

Every abelian group is a module over the ring of integers $\mathbb{Z}$ in a unique way. For $n > 0$, let $nx = x + x + ... + x$ (n summands), $0x = 0$, and $(-n)x = -(nx)$.

If R is any ring and n a natural number, then the Cartesian product R^n is both a left and a right module over R if we use the component-wise operations. Hence when $n = 1$, R is an R-module, where the scalar multiplication is just ring multiplication.

Submodule

Suppose M is a left R-module and N is a subgroup of M. Then N is a submodule (or R-submodule) if, for any n in N and any r in R, the product rn is in N (or nr for a right module).

Examples

An ideal I of the ring R is an R-submodule of the R-module R.

Homomorphism

If M and N are left R-modules, then a map $f : M \to N$ is a homomorphism of R-modules if, for any m, n in M and r, s in R, $f(rm + sn) = rf(m) + sf(n)$.

A bijective module homomorphism is an isomorphism of modules, and the two modules are called isomorphic.

Finitely generated

A module M is finitely generated if there exist finitely many elements $x_1, ..., x_n$ in M such that every element of M is a linear combination of those elements with coefficients from the scalar ring R.

Any ring R with 1 is a finitely generated R-module.

1.4 Topology

Definition

Let X be a non-empty set, a collection τ of subsets of X is called a topology if

(1). ϕ and X are in τ.

(2). Finite intersection of the members of τ is again in τ.

(3). Union of any number of members of τ is again in τ.

then the set X with topology τ on it is called a topological space and is denoted by (X, τ).

Product topology

Let I be a (possibly infinite) index set and suppose X_i is a topological space for every i in I. Set $X = \Pi Xi$, the Cartesian product of the sets X_i. For every i in I, we have a canonical projection $p_i : X \to X_i$. The product topology on X is defined to be the coarsest topology (i.e. the topology with the fewest open sets) for which all the projections p_i are continuous.

The open sets in the product topology are unions (finite or infinite) of sets of the form ΠU_i, where $U_i \neq X_i$ only finitely many times.

The product topology on X is the topology generated by sets of the form $p_i^{-1}(U)$, where i in I and U is an open subset of X_i. In other words, the sets $\{p_i^{-1}(U)\}$ form a subbase for the topology on X. A subset of X is open if and only if it is a (possibly infinite) union of intersections of finitely many sets of the form $p_i^{-1}(U)$. The $p_i^{-1}(U)$ are sometimes called open cylinders, and their intersections are cylinder sets.

Examples

If one starts with the standard topology on the real line R and defines a topology on the product of n copies of R in this fashion, one obtains the ordinary Euclidean topology on R^n.

The Cantor set is homeomorphic to the product of countably many copies of the discrete space $\{0, 1\}$ and the space of irrational numbers is homeomorphic to the product of countably many copies of the natural numbers, where again each copy carries the discrete topology.

Kolmogorov space

A T_0 space is a topological space X in which for every pair of distinct points x and y, there is an open set U which contains precisely one of the points.

Hausdorff Space

Suppose that X is a topological space. Let x and y be points in X. We say that x and y can be separated by neighborhoods if there exists a neighborhood U of x and a neighborhood V of y such that U and V are disjoint ($U \cap V = \phi$). X is a Hausdorff space if any two distinct points of X can be separated by neighborhoods.

Examples

Almost all spaces encountered in analysis are Hausdorff; most importantly, the real numbers (under the standard metric topology on real numbers) are a Hausdorff space. More generally, all metric spaces are Hausdorff. In fact, many spaces of use in analysis, such as topological groups and topological manifolds, have the Hausdorff condition explicitly stated in their definitions.

Compactness of topological spaces

A topological space X is defined as compact if all its open covers have a finite subcover. Formally, this means that for every arbitrary collection $\{V_i\}_{i \in I}$ of open subsets of X such that $U_{i \in I} V_i \supseteq X$, there is a finite subset $J \subset I$ such that $U_{j \in J} V_j \supseteq X$.

An often used equivalent definition is given in terms of the finite intersection property: if any collection of closed sets satisfying the finite intersection property has non-empty intersection, then the space is compact.

Examples

Any finite topological space, including the empty set, is compact. Slightly more generally, any space with a finite topology (only finitely many open sets) is compact; this includes in particular the trivial topology.

Homeomorphism

A function f between two topological spaces X and Y is called a homeomorphism if it has the following properties:

(i). f is a bijection ($1-1$ and onto),

(ii). f is continuous,

(iii). the inverse function f^{-1} is continuous (f is an open mapping).

A function with these three properties is sometimes called bicontinuous. If such a function exists, we say X and Y are homeomorphic. A self-homeomorphism is a homeomorphism of a topological space onto itself. The homeomorphisms form an equivalence relation on the class of all topological spaces. The resulting equivalence classes are called homeomorphism classes.

Examples

(1). The open interval $(-1, 1)$ is homeomorphic to the real numbers R.

(2). Let A be a commutative ring with unity and let S be a multiplicative subset of A. Then Spec (A_S) is homeomorphic to $\{p \in SpecA : p \cap S = \Phi\}$

(3). $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic for $n \neq m$.

Continuous Mapping

Several equivalent definitions for a topological structure exist and thus there are several equivalent ways to define a continuous function.

Open and closed set definition

A function for which the preimages of open sets are open is continuous. Similar to the open set formulation is the closed set formulation, which says that preimages of closed sets are closed.

Neighborhood definition

Definitions based on preimages are often difficult to use directly. Instead, suppose we have a function f from X to Y, where X, Y are topological spaces. We say f is continuous at x for some x in X if for any neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U)$ is a subset of V. Although this definition appears complicated, the intuition is that no matter how "small" V becomes, we can always find a U containing x that will map inside it. If f is continuous at every x in X, then we simply say f is continuous.

Open and closed maps

In topology, an open map is a function between two topological spaces which maps open sets to open sets. That is, a function $f : X \to Y$ is open if for any open set U in X, the image $f(U)$ is open in Y. Likewise, a closed map is a function which maps closed sets to closed sets.

Neither open nor closed maps are required to be continuous. Although their definitions seem natural, open and closed maps are much less important than continuous maps. Recall that a function $f : X \to Y$ is continuous if the preimage of every open set of Y is open in X. (Equivalently, if the preimage of every closed set of Y is closed in X).

Examples

Every homeomorphism is open, closed, and continuous. In fact, a bijective continuous map is a homeomorphism if and only if it is open, or equivalently, if and only if it is closed.

If Y has the discrete topology (i.e. all subsets are open and closed) then every function $f : X \to Y$ is both open and closed (but not necessarily continuous). For example, the floor function from R to Z is open and closed, but not continuous. This example shows that the image of a connected space under an open or closed map need not be connected.

Whenever we have a product of topological spaces $X = \Pi X_i$, the natural projections $p_i : X \to X_i$ are open (as well as continuous). Projections need not be closed however. Consider for instance the projection $p_1 : R^2 \to R$ on the first component; $A = \{(x, 1/x) : x \neq 0\}$ is closed in R^2, but $p_1(A) = R \setminus \{0\}$ is not closed. However, for compact Y, the projection $X \times Y \to X$ is closed. This is essentially the tube lemma.

The function $f : R \to R$ with $f(x) = x^2$ is continuous and closed, but not open.

Properties

A function $f : X \to Y$ is open if and only if for every x in X and every neighborhood U of x (however small), there exists a neighborhood V of $f(x)$ such that $V \subset f(U)$.

It suffices to check openness on an basis for X. That is, a function $f : X \to Y$ is open if and only if it maps basic open sets to open sets.

Open and closed maps can also be characterized by the interior and closure operators. Let $f : X \to Y$ be a function. Then

(a). f is open if and only if $f(A^\circ) \subset f(A)^\circ$ for all $A \subset X$

(b). f is closed if and only if $f(A)^- \subset f(A^-)$ for all $A \subset X$

The composition of two open maps is again open; the composition of two closed maps is again closed.

The product of two open maps is open, however the product of two closed maps need not be closed.

A bijective map is open if and only if it is closed. The inverse of a bijective continuous map is a bijective open/closed map (and vice-versa).

Let $f : X \to Y$ be a continuous map which is either open or closed. Then

(a). if f is a surjection, then it is a quotient map,

(b). if f is an injection, then it is a topological embedding, and

(c). if f is a bijection, then it is a homeomorphism.

In the first two cases, being open or closed is merely a sufficient condition for the result to follow. In the third case it is necessary as well.

Dense set

A subset A of a topological space X is called dense (in X) if, intuitively, any point in X can be "well-approximated" by points in A. Formally, A is dense in X if for any point x in X, any neighborhood of x contains at least one point from A.

Equivalently, A is dense in X if the only closed subset of X containing A is X itself. This can also be expressed by saying that the closure of A is X, or that the interior of the complement of A is empty.

Examples

(1). Every topological space is dense in itself.

(2). The real numbers with the usual topology have the rational numbers and the irrational numbers as dense subsets.

(3). A metric space M is dense in its completion $\widehat{M}$.

Chapter 2

Topological Rings And Modules

2.1 Introduction

This chapter consists of four sections. In section 1 we discuss basic definitions examples and results of topological group. Section 2 is devoted to the basic definitions, examples and results of topological rings and topological fields. Section 3 consists of definitions, examples and results of topological modules. It also includes discussion on the I-adic completion. In the last section we discuss Zariski topology and its related topics. Most of the material included in this chapter is taken from [4].

2.2 Topological Group

Definition 16 A topological group is an abelian group G together with a topology on G such that the group's binary operation and the group's inverse function are continuous. i.e. the maps $G \times G \to G : (x, y) \to x+y$ (addition continuity condition (AC)) and $G \to G : x \to -x$ (additive inverse continuity condition (AIC)) are continuous or the map $G \times G \to G : (x, y) \to x - y$ (subtraction continuity condition (SC)) is continuous. Here, $G \times G$ is viewed as a topological space by using the product topology. *(see [4, Def. 1.1.1])*

Remark 17 *[4, Remark 1.1.2]* In the neighborhoods sense the above definition is defined as follows:

For any two elements $x, y \in G$ and arbitrary neighborhood U of the element $x - y$ there exist neighborhoods V and W of elements x and y respectively such that $V - W \subset U$.

The following are some examples of topological groups.

Example 18 *[4]* Every group can be trivially made into a topological group by considering it with the discrete(resp. anti-discrete) topology; such groups are called discrete(resp. anti-discrete) groups. In this sense, the theory of topological groups subsumes that of ordinary groups.

Example 19 *[4]* The real numbers $\mathbb{R}$, together with addition as operation and its ordinary topology, form a topological group. More generally, Euclidean n-space $\mathbb{R}^n$ with addition and standard topology is a topological group. More generally yet, the additive groups of all topological vector spaces, such as Banach spaces or Hilbert spaces, are topological groups.

Remark 20 *[4]* The above examples are all abelian. Examples of non-abelian topological groups are given by Lie groups (topological groups that are also manifolds). For instance, the general linear group $GL(n, \mathbb{R})$ of all invertible $n \times n$ matrices with real entries can be viewed as a topological group with the topology defined by viewing $GL(n, \mathbb{R})$ as a subset of Euclidean space $\mathbb{R}^{n \times n}$.

Example 21 *[4]* An example of a topological group which is not a Lie group is given by the rational numbers $\mathbb{Q}$ with the topology inherited from $\mathbb{R}$. This is a countable space and it does not have the discrete topology. For a non-abelian example, consider the subgroup of rotations of $\mathbb{R}^3$ generated by two rotations by irrational multiples of 2π about different axes.

Example 22 *[4]* In every Banach algebra with multiplicative identity, the set of invertible elements forms a topological group under multiplication.

Although we do not do so here, many authors require that the topology on G be Hausdorff. This is not a serious restriction, any topological group can be made Hausdorff in a canonical fashion.

Definition 23 Let (G, τ) be a topological abelian group. A subset H of G is called topological subgroup of a topological group G if H is a subgroup of G and H is endowed with the topology $\tau |_G$ induced by the topology τ.*(see [4, Def. 1.4.1])*

Remark 24 *[4, Remark 1.4.2]* A subgroup of a topological abelian group is a topological abelian group itself.

Proposition 25 *[4, Prop. 1.4.5] Let H be a subgroup of a topological abelian group G. Then $[H]_G$(closure of H in G) is a subgroup of the topological abelian group G.*

Homomorphisms

Definition 26 A homomorphism between two topological groups G and H is just a continuous group homomorphism $G \to H$. An isomorphism of topological groups is a group isomorphism which is also a homeomorphism of the underlying topological spaces. This is stronger than simply requiring a continuous group isomorphism, the inverse must also be continuous. There are examples of topological groups which are isomorphic as ordinary groups but not as topological groups. Indeed, any indiscrete topological group is also a topological group when considered with the discrete topology. The underlying groups are the same, but as topological groups there is not an isomorphism.*(see [4, Def. 1.5.1])*

Remark 27 *[4, Remark.1.5.2]* Let G and G' be topological abelian groups and $\theta : G \to G'$ be an isomorphisms of these groups. Then the following conditions are equivalent:

(i). θ is open mapping.

(ii). $\theta^{-1} : G' \to G$ is continuous mapping.

Therefore, a topological isomorphism of topological groups is an isomorphism of these groups, being homeomorphism of the corresponding spaces.

Properties

As a uniform space, every topological group is completely regular. It follows that if a topological group is T_0 (Kolmogorov) then it is already T_2 (Hausdorff).

If H is a subgroup of G the set of left or right cosets G/H is a topological space when given the quotient topology (the finest topology on G/H which makes the natural projection $q : G \to G/H$ continuous). One can show that the quotient map $q : G \to G/H$ is always open.

If H is a normal subgroup of G, then the factor group, G/H becomes a topological group when given the quotient topology. However, if H is not closed in the topology of G, then G/H will not be T_0 even if G is. It is therefore natural to restrict oneself to the category of T_0 topological groups, and restrict the definition of normal to normal and closed.

The isomorphism theorems known from ordinary group theory are not always true in the topological setting. This is because a bijective homomorphism need not be an isomorphism of topological groups. The theorems are valid if one places certain restrictions on the maps involved. For example, the first isomorphism theorem states that if $f : G \to H$ is a homomorphism then $G/ker(f)$ is isomorphic to $im(f)$ if and only if the map f is open onto its image.

A topological group G is Hausdorff if and only if the identity subgroup is closed in G. If G is not Hausdorff then one can obtain a Hausdorff group by passing to the quotient space G/K where K is the closure of the identity.

The fundamental group of a topological group is always abelian. This is a special case of the fact that the fundamental group of an H-space is abelian, since topological groups are H-spaces.(H-space is a topological space X together with a continuous map $\mu : X \times X \to X$ with an identity element e so that $\mu(e, x) = \mu(x, e) = x$ for all $x \in X$. Every topological group is an H-space; however, in the general case, as compared to a topological group, H-spaces may lack associativity and inverses.)

Proposition 28 *[4, Prop. 1.1.34] Let G be a topological Abelian group $g \in G$, and suppose H and K are subsets of G, then the following statements are equivalent.*

(1). $\phi_g : G \to G$ and $\phi : G \to G$ with $\phi_g(x) = x + g$ and $\phi(x) = -x$, are homeomorphic mappings of G.

(2). The following are equivalent:

(i). Subset H is open (closed);

(ii). Subset $-H$ is open (closed);

(iii). Subset $H + g$ is open (closed). (Among other things a subset $H \subseteq G$ is a neighborhood of the element a if and only if $H - a$ is a neighborhood of 0).

(3). If subset H is open, then $H + K$ is also an open subset.

Corollary 29 *[4, Corollary 1.1.35] Any topological Abelian group is a homogeneous space (As for any $a, b \in G$, $\exists$ a homeomorphism $\phi_{(b-a)}$ such that $\phi_{(b-a)}(a) = b$.*

Proposition 30 *[4, Prop. 1.1.38] Let G be a topological Abelian group and H and K be subsets of G, then the following statements are equivalent.*

(1). If H and K are compact subsets then $H + K$ is a compact subset of G

(2). If H is closed subset and K is compact subset then $H + K$ is closed subset of G.

Corollary 31 *[4, Corollary 1.1.39] The sum $H + K$ of a closed subset H and a finite subset K of a topological Abelian group G is a closed subset of G.*

Proposition 32 *[4, Prop. 1.1.41] Let H and K are subsets of topological Abelian group G, then*

(1). $[H] + [K] \subseteq [H + K]$

(2). $[-H] = -[H]$

(3). $[H] - [K] \subseteq [H - K]$

(4). If K is the compact subset then $[H] + [K] = [H + K] = [H] + K$ *and* $[H] - [K] = [H - K] = [H] - K$

2.3 Topological ring

Definition 33 A topological ring is a ring R with a topology such that the additive group of the ring R is topological group in this topology and the map $R \times R \to R : (x, y) \to xy$ is continuous. (multiplication continuity condition (MC)). (see *[4, Def. 1.1.6]*)

Remark 34 *[4, Remark 1.1.7]* In the neighborhood sense we say that R is a topological ring if for any two elements $x, y \in R$ and arbitrary neighborhood U of the element xy there exist neighborhoods V and W of elements x and y respectively such that $VW \subseteq U$.

The group of units of R may not be a topological group using the subspace topology, as inversion on the unit group need not be continuous with the subspace topology. Embedding the unit group of R into the product $R \times R$ as $(x, x-1)$ does make the unit group a topological group. (If inversion on the unit group is continuous in the subspace topology of R then the topology on the unit group viewed in R or in $R \times R$ as above are the same.)

If one does not require a ring to have a unit, then one has to add the requirement of continuity of the additive inverse, or equivalently, to define the topological ring as a ring which is a topological group (for $+$) in which multiplication is continuous, too.

In the following we are giving some examples of topological rings.

Example 35 *[4]* Let R be a ring, then its additive group could be transformed into a topological abelian group by endowing R with the discrete or anti-discrete topology. It is easy to verify that the ring R satisfies condition (MC) in both topologies. In this manner any ring could be considered as a topological ring.

Example 36 *[4]* In algebra, the following construction is common: one starts with a commutative ring R containing an ideal I, and then considers the I-adic topology on R: a subset U of R is open if and only if for every x in U there exists a natural number n such that $x + I^n \subseteq U$. This turns R into a topological ring.

Definition 37 Let R be a topological ring, a subset I of the topological ring R is called a topological subring, if it is a subring of the ring R and I is endowed with the topology induced by the topology of the ring R. *(see [4, Def. 1.4.3])*

Remark 38 *[4, Remark 1.4.4]* A topological subring of a topological ring is a topological ring itself.

Proposition 39 *[4, Prop. 1.1.44] Let R be a topological ring with unity,and M be a topological R-module,let $r \in R$ be an invertible element, then mappings $\phi_r : M \to M$, $\Psi_r : R \to R$ and $\Psi'_r : R \to R$ are homeomorphic mappings of the topological spaces M and R correspondingly onto themselves.*

Corollary 40 *[4, Corollary 1.1.45] Let R be a topological ring with the unitary element, $r \in R$ be an invertible element and $x \in R$, then the following statements are equivalent:*

(1). U is the neighborhood of the element x in R.

(2). Ur is the neighborhood of the element xr in R.

(3). rU is the neighborhood of the element rx in R.

Corollary 41 *[4, Corollary 1.1.47] Let R be a topological ring with the unitary element, $r \in R$ be an invertible element and let $H \subseteq R$, then the following statements are equivalent:*

(1). H is open (closed).

(2). rH is open (closed).

(3). Hr is open (closed).

Topological fields

Definition 42 A skew field (field) K is called a topological skew field (field) if it is a topological ring and the mapping $x \to x^{-1}$ of the subspace $K \setminus \{0\}$ onto itself is continuous,(multiplicative inversion continuity condition (MIC)) i.e. for any non-zero element $x \in K$ and any neighborhood U of x^{-1} there exist a neighborhood V of the element x such that $(V \setminus \{0\})^{-1} \subset U$.*(see [4])*

Remark 43 *[4, Remark 1.1.20]* The multiplication group of the topological field is a topological abelian group.

The following are some examples of topological fields.

Example 44 *[4]* Some of the most important examples are also fields F. To have a topological field we should also specify that inversion is continuous, when restricted to $F\setminus\{0\}$.

Example 45 *[4]* Let K be a skew field (field). Consider discrete or anti-discrete topology on K. In both cases condition (MIC) is satisfied, and, hence, any skew field (field) is a topological skew field (field) in the discrete or anti-discrete topology.

Proposition 46 *[4, Prop. 1.1.48] Let K be a topological skew field and let $0 \neq a \in K$. If element a is an accumulation point (a limit) of the sequence of non-zero elements $a_1, a_2, ... \in K$, then the element a^{-1} is an accumulation point (a limit) of the sequence $a_1^{-1}, a_2^{-1}, ...$ in the skew field K.*

Proposition 47 *[4, Prop. 1.1.49] Let K be a topological skew field. Then the mapping $\theta : K\setminus\{0\} \to K\setminus\{0\}$, where $\theta(x) = x^{-1}$, for $x \neq 0$, is a topological homomorphism of the topological subspace $K\setminus\{0\}$ onto itself.*

Topological Vector Space

Definition 48 Let K be a topological skew field. A unitary topological K-module is called a topological vector space over K. *(see [4, Def. 1.1.30])*

The following are some examples of topological vector spaces.

Example 49 *[4]* In the natural way, the additive group of a topological skew field K is a topological vector space over K.

Example 50 *[4]* The field of complex numbers $\mathbb{C}$ with the topology specified by the norm $|\cdot|$ are vector spaces over the field $\mathbb{R}$, of real numbers, endowed with the internal topology.

2.4 Topological module

Definition 51 Let R be a topological ring. A left R-mdule M is called a topological left module if on M is specified a topology such that M is a topological abelian group and the mapping $(r, m) \rightarrow rm$ of the topological space $R \times M$ to the topological space M is continuous.(see *[4, Def. 1.1.24]*)

Remark 52 *[4]* In similar way we can define topological right module.

Remark 53 *[4, 1.1.25]* An R-module M. is a topological module if for any $r \in R$ and $m \in M$ and arbitrary neighborhood U of the element rm in M there exist a neighborhood V of the element r in R and a neighborhood W of the element m in M such that $VW \subset U$.

Remark 54 *[4]* A topological left and right module is called topological module.

In the following we are indicating some examples of topological R-modules.

Example 55 *[4]* A topological vector space is a topological module over a topological field.

Example 56 *[4, Example 1.1.27]* An abelian topological group can be considered as a topological module over $\mathbb{Z}$, where $\mathbb{Z}$ is the ring of integers with the discrete topology.

Indeed, let $b = k \cdot a$, where $k \in \mathbb{Z}$ and $a \in A$. Taking into consideration the definition of topological abelian group, we can suppose that $k > 0$, i.e.

$$b = a + a + ... + a. \quad (k \text{ summands})$$

Let U be a neighborhood of the element b in A. In compliance with condition (AC), there exists a neighborhood W of the element a in A such that

$$W + W + ... + W \subseteq U \quad (k \text{ summands}).$$

Since the discrete topology is introduced onto $\mathbb{Z}$, the subset $V = \{k\}$ is a neighborhood of the element k in $\mathbb{Z}$. Hence

$$V \cdot W = \{k\} \cdot W \subseteq W + W + ... + W \subseteq U \quad (k \text{ summands})$$

i.e. condition (RMC) is satisfied.

By this means any topological abelian group in the natural way is the topological $\mathbb{Z}$-module over the ring of integers $\mathbb{Z}$ with the discrete topology.

Topological Submodule

Definition 57 Let R be a topological ring and M be a topological R-module, a subset N of the topological R-module is called a topological submodule, if it is an R-submodule of the R-module M and N is endowed with the topology of the topological R-module M. *(see [4, Def. 1.4.3])*

Remark 58 *[4, Remark 1.4.4]* A submodule of a topological module is a topological module itself.

Proposition 59 *[4, Prop. 1.4.7] Let R be a topological ring and M be a topological R-module. Let S be a subring of the ring R and N be a S-submodule of R-module M, then*

(1). $[S]_R$ is subring of R.

(2). $[N]_M$ is an $[S]_R$-module.

Corollary 60 *[4, Corollary 1.4.8] Let S be a dense subring of a topological ring R and N be S-submodule of a topological R-module M. Then $[N]_M$ is a submodule of a topological R-module M. In particular, the closure of any submodule of a topological R-module is also a topological R-module.*

Corollary 61 *[4, Corollary 1.4.9] Let S be a dense subring of a topological ring R and I be left(right, two-sided) ideal of the ring R. Then $[I]_R$ is a left(right, two-sided) ideal of the ring R. In particular, the closure of any left(right, two-sided) ideal of the ring R is also a left(right, two-sided) ideal of the ring R.*

Corollary 62 *[4, Corollary 1.4.11] Let B_0 be a basis of neighborhoods of zero of topological R-module M then $M_0 = \cap_{V \in B_0} V$ is the smallest closed submodule of M.*

Remark 63 *[4, Remark 1.4.12]* Let B_0 be a basis of neighborhoods of zero of topological ring R then $R_0 = \cap_{V \in B_0} V$ is the smallest closed two sided ideal of R.

Remark 64 *[4, Remark 1.4.14]* Any topological ring without closed proper ideal is hausdorff or anti-discrete.

Proposition 65 *[4, Prop. 1.1.42] Let R be a topological ring, M a topological R-module, $r \in R$, $m \in M$, and Q a subset in R, B a subset in M, then the following statements are true:*

(1). The mapping $\phi_r : M \to M$, where $\phi_r(m) = rm$, $m \in M$, is continuous mapping of the topological space M into itself.

(2). The mapping $\phi_a : R \to M$, where $\phi_a(x) = xa$, $a \in R$, is continuous mapping.

(3). $[Q]_R[B]_M \subseteq [QB]_M$.

(4). If subset Q and R are compact, then QR is a compact subset.

Corollary 66 *[4, Corollary 1.1.43] Let R be a topological ring, $r \in R$, and let H and K are subset in R, then the following statements are true:*

(1). The mapping $\Psi_r : R \to R$ and $\Psi'_r : R \to R$ where $\Psi_r(x) = xr$ and $\Psi'_r(x) = rx$ for $x \in R$, are continuous mappings of the topological space R into itself.

(2). The mapping $\phi_a : R \to M$, where $\phi_a(x) = xa$, $a \in R$ is continuous mapping.

(3). $[H]_R[K]_R \subseteq [HK]_R$.

(4). If subset Q and R are compact, then QR is a compact subset in R.

2.4.1 Completion

The set R of real numbers is a complete metric space in which the set Q of rationals is dense. In fact any metric space can be embedded as a dense subset of a complete metric space. The construction is a familiar one involving equivalence classes of Cauchy sequences. We will see that under appropriate conditions, this procedure can be generalized to modules.

Definitions and Comments

A graded ring is a ring R that is expressible as $\oplus_{n\geq 0}R_n$ where the R_n are additive subgroups such that $R_mR_n \subseteq R_{m+n}$. A graded module over a graded ring R is a module M expressible as $\oplus_{n\geq 0}M_n$, where $R_mM_n \subseteq M_{m+n}$.

Now suppose that $\{R_n\}$ is a filtration of the ring R, in other words, the R_n are additive subgroups such that $R = R_0 \supseteq R_1 \supseteq \cdots \supseteq R_n \supseteq \cdots$ with $R_mR_n \subseteq R_{m+n}$. We call R a filtered ring. A filtered module $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq \cdots$ over the filtered ring R may be defined similarly. In this case, each M_n is a submodule and we require that $R_mM_n \subseteq M_{m+n}$.

If I is an ideal of the ring R and M is an R-module, we will be interested in the I-adic filtrations of R and of M, given respectively by $R_n = I^n$ and $M^n = I^nM$. (Take $I^0 = R$, so that $M_0 = M$.)

Inverse Limits

Suppose we have countably many R-modules $M_0, M_1, \ldots,$ with R-module homomorphisms $\theta_n : M_n \to M_{n-1}$, $n \geq 1$. (We are restricting to the countable case to simplify the notation, but the ideas carry over to an arbitrary family of modules, indexed by a directed set. If $i \leq j$, we have a homomorphism f_{ij} from M_j to M_i. We assume that if $i \leq j \leq k$, then $f_{ij} \circ f_{jk} = f_{ik}$.) The collection of modules and maps is called an inverse system. A sequence (x_i) in the direct product M_i is said to be coherent if it respects the maps θ_n in the sense that for every i we have $\theta_{i+1}(x_{i+1}) = x_i$. The collection M of all coherent sequences is called the inverse limit of the inverse system and is denoted by $\varprojlim M_n$.

Note that M becomes an R-module with componentwise addition and scalar multiplication of coherent sequences, in other words, $(x_i) + (y_i) = (x_i + y_i)$ and $r(x_i) = (rx_i)$.

Now suppose that we have homomorphisms g_i from an R-module M to M_i, $i = 0, 1, \ldots$. Call the g_i coherent if $\theta_{i+1} \circ g_{i+1} = g_i$ for all i. Then the g_i can be lifted to a homomorphism g from M to M. Explicitly, $g(x) = (g_i(x))$, and the coherence of the g_i forces the sequence $(g_i(x))$ to be coherent.

An inverse limit of an inverse system of rings can be constructed in a similar fashion, as coherent sequences can be multiplied componentwise, that is, $(x_i)(y_i) = (x_i y_i)$.

Examples

1. Take $R = \mathbb{Z}$, and let I be the ideal (p) where p is a fixed prime. Take $M_n = \mathbb{Z}/I^n$ and $\theta_{n+1}(a + I^{n+1}) = a + I^n$. The inverse limit of the M_n is the ring $\mathbb{Z}_p$ of p-adic integers.

COMPLETION OF A MODULE

2. Let $R = A[x_1, \ldots, x_n]$ be a polynomial ring in n variables, and I the maximal ideal $(x_1, \ldots, x_n)$. Let $M_n = R/I^n$ and $\theta_n(f + I^n) = f + I^{n-1}$, $n = 1, 2, \ldots$. An element of M_n is represented by a polynomial f of degree at most $n-1$. (We take the degree of f to be the maximum degree of a monomial in f.) The image of f in I^{n-1} is represented by the same polynomial with the terms of degree $n-1$ deleted. Thus the inverse limit can be identified with the ring $A[[x_1, \ldots, x_n]]$ of formal power series.

Now let M be a filtered R-module with filtration $\{Mn\}$. The filtration determines a topology on M with the M_n forming a base for the neighborhoods of 0.

Definition of the Completion

Let $\{Mn\}$ be a filtration of the R-module M. If we go far out in a Cauchy sequence, the difference between terms becomes small. Thus we can define a Cauchy sequence $\{x_n\}$ in M by the requirement that for every positive integer r there is a positive integer N such that $x_n - x_m \in M_r$ for $n, m \geq N$. We identify the Cauchy sequences $\{x_n\}$ and $\{y_n\}$ if they get close to each other for large n. More precisely, given a positive integer r there exists a positive integer N such that $x_n - y_n \in M_r$ for all $n \geq N$. Notice that the condition $x_n - x_m \in M_r$ is equivalent to $x_n + M_r = x_m + M_r$. This suggests that the essential feature of the Cauchy condition is that the sequence is coherent with respect to the maps $\theta_n : M/M_n \to M/M_{n-1}$. Motivated by this observation, we define the completion of M as

$$\widehat{M} = \varprojlim(M/M_n).$$

Examples

(1). The ring of p-adic integers is the inverse limit of the rings $\mathbb{Z}/p^n\mathbb{Z}$ with the index set being the natural numbers with the usual order, and the morphisms being "take remainder". The natural topology on the p-adic integers is the same as the one described here.

(2). The ring $R[[t]]$ of formal power series over a commutative ring R can be thought of as the inverse limit of the rings $R[t]]/t^nR[t]$, indexed by the natural numbers as usually ordered, with the morphisms from $R[t]/t^{n+j}R[t]$ to $R[t]/t^nR[t]$ given by the natural projection.

(3). Let the index set I of an inverse system (X_i, f_{ij}) have a greatest element m. Then the natural projection $\pi_m : X \to X_m$ is an isomorphism.

2.5 Zariski Topology

modern definition

Modern algebraic geometry takes the spectrum of a ring as its starting point. In this formulation, the Zariski-closed sets are taken to be the sets $V(I) = \{P \in \text{Spec}(A) : I \subseteq P\}$ where A is a fixed commutative ring and I is an ideal. To see the connection with the classical picture, note that for any set S of polynomials (over an algebraically closed field), it follows from Hilbert's Nullstellensatz that the points of $V(S)$ are exactly the tuples $(a_1, ..., a_n)$ such that $(x_1 - a_1, ..., x_n - a_n)$ contains S.

Examples

(i). Spec(K), the spectrum of a field K is the topological space with one element.

(ii). Spec($\mathbb{Z}$), the spectrum of the integers has a closed point for every prime number p corresponding to the maximal ideal $(p) \subset \mathbb{Z}$, and one non-closed generic point (i.e., whose closure is the whole space) corresponding to the zero ideal (0). So the closed subsets of Spec($\mathbb{Z}$) are precisely finite unions of closed points and the whole space.

Chapter 3

On Zariski Rings

3.1 Introduction

This chapter consists of four sections. The first section consists of the basic definitions, examples and results of Zariski ring. In section two It is shown that the fraction ring of a Noetherian ring is a Zariski ring and a compact Noetherian domain is itselg a Zariski ring. Also it is shown that the Noetherian intersection of finite Zariski rings is again a Zariski ring. The Zariski ring as a polynomial ring and power series ring is discussed in third section. In the last section we discuss ascent and descent of some properties of Zariski rings for (unitary) commutative ring extension $R \subseteq T$ and related examples.

3.2 Zariski Ring

Let R be a topological ring with identity 1 and M be a topological R-module and $\sum(M)$ a system of open sets in M which contains the zero of M and satisfy the following condition

(a) Any open set in M containing zero contains an element from $\sum(M)$ (i.e. $\sum(M)$ is a local open base at zero). Then we have

(c_1). The set $\{x + U : x \in M, U \in \sum(M)\}$ is a open base for M.

such a set $\sum(M)$ is called basis of neighborhoods of zero for the topological module M.

Let R be a topological ring and $\sum(R)$ be the basis of neighborhoods of the zero, in the sense of the above definition. Then $\sum(R)$ satisfies the following properties

(c_2). The intersection of any two sets of the system $\sum(R)$ contains a third set of that system.

(c_3). If $U \in \sum(R)$, then there exist a set $W \in \sum(R)$ such that $W - W \subseteq U$ and $W^2 \subset U$.

(c_4). For any $U \in \sum(R)$, any $a \in U$ and $b \in R$ there exist $W \in \sum(R)$ such that $W + a \subseteq U$ and $Wb \subseteq U$.

Now for the topological ring R and the topological R-module M with system of neighborhoods $\sum(R)$ and $\sum(M)$ of zeros of R and M respectively, $\sum(M)$ satisfies the following:

(c'_2). The intersection of any two sets of the system $\sum(M)$ contains a third set of that system.

(c'_3). If $U' \in \sum(M)$, then there exist a set $W' \in \sum(M)$ and a set W in $\sum(R)$ such that $W' - W' \subseteq U'$ and $WW' \subseteq U'$.

(c'_4). For any $U' \in \sum(M)$, any $a \in U'$, $y \in M$ and $b \in R$, then there exist a set $W' \in \sum(R)$ and a set W in $\sum(R)$ such that $W' + a \subseteq U'$ and $bW' \subseteq U'$ and $Wy \subseteq U'$.

For the converse process i.e. from a ring with identity how we can get the special system of neighborhoods of zero, we have the following lemma.

Lemma 67 *[4, Theorem 1.2.4 (i)] Let R be a ring with identity and $\sum(R)$ be the set of subsets of R satisfying the conditions c_2, c_3, and c_4 then there exist a topology corresponding to $\sum(R)$, and $\sum(R)$ will be the basis of neighborhoods of zero for that topology on R.*

Proof. Let $\Im = \{B \subseteq R \mid$ for any $b \in B$ there exists $U_b \in \sum(R)$ such that $b + U_b \subseteq B\}$.

It is obvious that ϕ and R belongs to $\Im$. Let $B_1, B_2 \in \Im$ and let $b \in B_1 \cap B_2$. Then there exists $U_1, U_2 \in \sum(R)$ such that $b + U_1 \subseteq B_1$ and $b + U_2 \subseteq B_2$. In view of condition c_2, there exists $U_3 \in \sum(R)$ such that $U_3 \subseteq U_1 \cap U_2$. Hence, $b + U_3 \subseteq B_1 \cap B_2$. Thus $B_1 \cap B_2 \in \Im$.

Let $\phi \neq A' \subseteq \Im, B' = \cup_{B \in A'} B$ and $b \in B'$. Then $b \in \sum(R)$ is true for some $B_0 \in A'$, and, hence, $B + U \subseteq B_0$ for some $U \in \sum(R)$. Consequently, $b + U \subseteq B'$ and, hence, $B' \subseteq \Im$. Thus, on A is defined a topology and $\Im$ is the family of all open subsets in this topology.

Let's show that for any element $a \in A$ the family $B_a = \{a + U \mid U \in \sum(R)\}$ is the basis of neighborhoods of the element a in this topology. Verify first that for any $U \in \sum(R)$ the subset $a + U$ is the neighborhood of the element a.

Let $U \in \sum(R)$ and $V_a = \{x \in R \mid$there exist $U_x \in \sum(R)$ such that $x + U_x \subseteq a + U\}$. Let is obvious that $a \in V_a$ and $V_a \subset a + U$. If $x \in V_a$, then $x + U_x \subseteq a + U$ and $U_x \in \sum(R)$. In view of condition c_3, there exists $U'_x \in \sum(R)$ such that $(U'_x + U'_x) \subseteq U_x$.By virtue of this fact, $(x + U'_x) + U'_x \subseteq x + U_x \subseteq a + U$.and, hence, $x + U'_x \subseteq V_a$. Thus $V_a \in \Im$, i.e. V_a is an open set, and because of this, $a + U$ is a neighborhood of the element a.

Now let's verify that B_a is the bases of neighborhood of the element a. Let W be a neighborhood of the element a in the constructed topology, then there exists $B \in \Im$ such that $a \in B \subseteq W$. On the strength of the definition of $\Im$, there exists $U \in \sum(R)$ such that $a + U \subset B \subset W$. Since $a + U \in B_a$, then, B_a is a basis of neighborhoods of the element a. In particular, $\sum(R)$ is the basis of neighborhoods of zero. ■

Lemma 68 *[4, Theorem 1.2.4 (ii)] Let R be a ring with identity and $\sum(R)$ be the set of subsets of R satisfying the conditions c_2, c_3, and c_4 then there exist one and only one topology corresponding to $\sum(R)$, and $\sum(R)$ will be the basis of neighborhoods of zero for that topology on R, and $(R, +)$ is the topological Abelian group with respect to this topology.*

Proof. Now, let us show that R is topological Abelian group in the constructed topology. For this let's verify that condition (SC) is fulfilled. Let $a, b \in R$, and let W be a neighborhood of the element $a - b$. Then there exists $U \in \sum(R)$ such that $(a - b) + U \subseteq W$. On the strength of condition c_3, there exists $V \in \sum(R)$ such that $V - V \subseteq U$. As it was shown above, the subsets $a + V$ and $b + V$ are neighborhoods of the elements a and b respectively and $(a + V) - (b + V) = (a - b) + (V - V) \subseteq (a - b) + U \subseteq W$. Thus, condition (SC) is fulfilled, and hence, R is a topological group with basis $\sum(R)$ of neighborhoods of zero. Denote this topological group over $(R, \Im)$.

It remains to verify that if some system $\Im'$ of the subsets of the group R defines a topology on R, and this system is the family of all open subsets in the topology, and besides, $(R, \Im')$ is a topological group with the basis $\sum(R)$ of neighborhoods of zero, then $\Im = \Im'$.

Let $B \in \Im$ and $b \in B$. Then $b + U \subseteq B$ for some $U \in \sum(R)$. Since $\sum(R)$ is a basis of neighborhoods of zero in the topological group $(R, \Im')$, so $b + U$ is a neighborhood of the element b in $(R, \Im')$. Consequently, every element $b \in B$ enters in B together with some of its neighborhood relative to the topology defined by system $\Im'$. That means that B is an open subset of the topological group $(R, \Im')$ and, hence, $B \in \Im'$. Thus $\Im \subseteq \Im'$. Now, let $B' \subseteq \Im'$, hence, B' is a neighborhood of any of its elements in the topology defined by system $\Im'$. Then for every $b \in \Im'$ the subset $B' - b$ is a neighborhood of zero in $(R, \Im')$. Because of this, there exists $U_b \in \sum(R)$ such that $U_b \subseteq B' - b$, that is $b + U_b \subseteq B'$. From the definition of $\Im'$ it follows that $\Im' \subseteq \Im$. ■

Theorem 69 *[4, Theorem 1.2.5] Let R be a ring with identity and $\sum(R)$ be the set of subsets of R satisfying the conditions c_2, c_3, and c_4 then there exist a topology corresponding to $\sum(R)$, and $\sum(R)$ will be the basis of neighborhoods of zero for that topology on R, and R is the topological ring with respect to this topology.*

Proof. By Lemma 67 and 68 there exists a unique topology on R and $\sum(R)$ is the system of neighborhoods of zero, corresponding to which $(R, +)$ is the topological Abelian group. So it just remains to verify condition (MC) is satisfied. Let $a, b \in A$ and U be the neighborhood of the element ab. So B_a, B_b and B_{ab}, where $B_x = \{x + V \mid V \in \sum(R)\}$, are bases of neighborhoods respectively of elements a, b and ab in the topological group $A(+)$. Hence, there exists a neighborhood $V \in \sum(R)$ such that $ab + V \subseteq U$. Using conditions c_2, c_3, and c_4, it is possible to choose neighborhoods $V_1, V_2 \in \sum(R)$ such that $aV_2 + V_1 b + V_1 V_2 \subseteq V$. Then $a + V_1$ and $b + V_2$ are neighborhoods of the elements a and b respectively. Besides, $(a + V_1)(b + V_2) \subseteq ab + aV_2 + V_1 b + V_1 V_2 \subseteq ab + V \subseteq U$, i.e. condition (MC) is satisfied. ■

Theorem 70 *[4, Theorem 1.2.6] Let R be a topological ring and $\sum(M)$ be the set of subsets of R-module M satisfying the conditions c'_2, c'_3, and c'_4, then there exist a unique topology on M in which M is a topological R-module with $\sum(M)$ as a basis of neighborhoods of zero of M.*

Lemma 71 *[15, Page 252] For the above system $\sum(M)$, if the zero of a topological module M is a closed set then M is a hausdorff space*

Proof. Let $a \neq b$ are any two elements of M and suppose $V = M - \{0\}$, then $b - a \neq 0$ and $b - a \in V \subseteq V$. This implies V is a neighborhood of $b - a$ that does not contains $\{0\}$.

Now consider the set $U = b - a + V$. As U is open and $b - a \notin U$ also $0 \in U$, so U is a neighborhood of 0 such that $b - a \notin U$

Now let W be a neighborhood of 0 such that $W - W \subseteq U$, then $a + W$ and $b + W$ are disjoint neighborhoods of a and b respectively. Hence M is a hausdorff space. ■

Corollary 72 *[15, Page 253] If $\sum(M)$ is a basis of neighborhood of zero then M is a hausdorff space iff the intersection of the sets of the system $\sum(M)$ is zero, (i.e.$\cap_{V\in\sum(M)}V = \{0\}$).*

We shall be concerned primarily with topologies in R which can be defined by using power of any ideal I of R. i.e.I-adic topologies

Definition 73 A topology of a ring R is said to be the I-adic topology of R for the ideal I(two sided) of R in which fundamental system of neighborhoods of zero consists of all the powers of I *(see [15])*.

Example 74 *[4]* The p-adic topology on the integers is an example of an I-adic topology (with $I = (p)$).

Lemma 75 *[15, Page 253] Let I be an ideal of R, then $\{I^n : n \in Z^0$, where $I^0 = R\}$ forms a system $\sum(R)$ that will satisfy the conditions c_2, c_3, and c_4.*

Proof. Let I^n and I^m be any two sets in $\sum(R)$ then

$$I^n \cap I^m \text{ contains } I^{n+1} \text{ if } n > m,$$
$$I^n \cap I^m \text{ contains } I^{m+1} \text{ if } m > n.$$

Now for any I^n in $\sum(R)$ there exists I^{n+1} in $\sum(R)$ such that, $I^{n+1} - I^{n+1} \subseteq I^n$, since $I^{n+1} \subseteq I^n$ and I^{n+1} is itself an ideal

$$\text{also } (I^{n+1})^2 = I^{2n+2} \subseteq I^n$$

and lastly, for any I^n in $\sum(R)$, $x \in I^n$ and a any element of R, then there exists I^{n+1} in $\sum(R)$ such that $I^{n+1} + x \subseteq I^n$ and $I^{n+1}a \subseteq I^n$. Hence the result follows. ■

Proposition 76 *The I-adic topology is discrete if and only if I is nilpotent ideal of the ring R.*

Proof. Suppose the topology is discrete. This implies that $\{0\}$ is open, so $I^n = \{0\}$ for some positive integer n, which implies that I is nilpotent.

Conversely suppose that I is nilpotent, so $I^n = \{0\}$ for some positive integer n. Let $A \subseteq R$ and let $a \in A$ be any element of A, then $a + \{0\} = \{a\} \subseteq A$.

This implies A is open subset. Hence every subset of R is open. Hence the I-adic topology is discrete. ■

The topology defined in above lemma will be called the I-topology hereinafter, also we can generalize this to the R-module M by taking $\sum(M) = \{I^n M : I$ is ideal of R and $n \in Z^+\}$. In fact $I^n M$ are submodules of M, and the topology in this case defined by this $\sum(M)$ will be called the IM-topology for R-module M.(see [15])

Lemma 77 *[15, Page 253] The R-module M with IM-topology is hausdorff iff* $\cap_{n=0}^{\infty} I^n M = \{0\}$.

Lemma 78 *[15, §2 Lemma.1] The closure* $\overline{S}$ *of a subset S of M is equal to* $\cap_{n=0}^{\infty}(S + I^n M)$.

Corollary 79 *A submodule F of M is closed in IM-topology iff* $F = \cap_{n=0}^{\infty}(F + I^n M)$, *particularly an ideal J of R is closed in I-topology iff* $J = \cap_{n=0}^{\infty}(J + I^n)$.

The following is famous Nakayama Lemma.

Lemma 80 *[11, Ch. 8,Lemma 4.5] Let M be a finitely generated R-module, and I be an ideal of R such that* $I \subseteq J(R)$, *then* $IM = M \Rightarrow M = \{0\}$.

We are going to study the pair (R, I) formed by a Noetherian ring R and an ideal I in R with I-topology

Definition 81 Zariski ring is a Noetherian ring R having an ideal I such that, every maximal ideal in R is closed in its I-topology.*(cf [7, §3.3(3)])*

Theorem 82 *[15, §4 Theorem 9] For a Noetherian ring R with I-topology, the following conditions are equivalent*

1. For every finite R-module M and every submodule F of M, F is closed for the IM-topology of M.

2. Every finite R-module M (in particular R itself)is a hausdorff space in its I-topology.

3. Every ideal in R is closed in I-topology of R.

4. The ideal I is contained in Jacobson radical of R.

5. Every element of $1 + I$ is invertible in R.

6. Nakayama lemma holds.

Example 83 *[15, §4 Example (1)]* Let R be a Noetherian local ring, then R will be a Zariski ring with respect to the maximal ideal.

Example 84 *[15, §4 Example (2)]* Suppose a semi-local Noetherian ring R then R will be a Zariski ring with respect to the ideal $M = \cap M_i$, where M_i are the maximal ideals of R.

Example 85 *[15, §4 Example (3)]* A factor ring of a Zariski ring (R, I) with J is Zariski with respect to the ideal $(I + J)/J$.

Theorem 86 *[15, §4 Theorem 10] Let (R, I) be a Zariski ring then R is semi local (resp: local) ring iff R/I satisfy d.c.c (resp:satisfy d.c.c, with only one prime ideal).*

Lemma 87 *[15,] Let R be a topological ring with I-topology, then R is a hausdorff space iff $\cap_{n \in Z^{\circ}} I^n = \{0\}$.*

The following is famous Krull Intersection Theorem.

Theorem 88 *[11, Theorem 4.4] Let R be a Noetherian ring, and I an ideal of R such that $I \subseteq J(R)$ then $\cap I^n = \{0\}$.*

Remark 89 By Theorem *88* and Lemma *87*, we can say that if R is a Noetherian ring and I an ideal of R such that $I \subseteq J(R)$, then R is hausdorff space with respect to the I-adic topology.

The following is famous Domainized Krull Intersection Theorem.

Theorem 90 *For any non-unit ideal J in any Noeherian domain R we have $\cap J^n = \{0\}$, moreover $J^m \neq J^n$, for $m \neq n$ in Z^+.*

Remark 91 By Lemma *87*, a Noetherian domain R is a hausdorff with respect to the I-adic topology, where I is any proper ideal.

Proposition 92 *Let R be a topological Noetherian domain with any I-topology then R is a hausdorff space.*

Proof. Let R be a topological Noetherian domain and let I be any ideal in R, then by domainized krull intersection theorem $\cap I^n = (0)$, which is necessary and sufficient condition for I-topology to be hausdorff. Hence R is a hausdorff space. ■

Lemma 93 *In a Noetherian ring R and for any ideal I, $I \subseteq J(R)$ if and only if $1 + a \in U(R)$, for all $a \in I$.*

Theorem 94 *[2, Theorem 2] Let $R = \cap_\alpha R_\alpha$ be a locally finite intersection of FFD's $\{R_\alpha\}$. Then R is an FFD.*

Proposition 95 *Every Noetherian hausdorff space is a Zariski ring.*

Proof. Let R be a Noetherian topological ring and I be any ideal of R also the I-topology on R is hausdorff, then R can be considered a topological R-module and as R is a hausdorff space which implies that R is a hausdorff space as an R-module.

Hence R is a Zariski ring. ■

3.3 Fraction ring as a Zariski ring

In [8], J. E. Cude have categorized the elements of the compact integral domains into invertible elements and nilpotent elements (in topological sense) in the following lemma.

Lemma 96 *[8, Lemma 1] Let R be a compact integral domain and $J(R)$ be its Jacobson radical, then for $x \in R$, x has an inverse if and only if $x \notin J(R)$, and x is nilpotent if and only if $x \in J(R)$.*

The following theorem is a consequence of lemma 96.

Theorem 97 *A compact Noetherian domain R is a Zariski ring with respect to any proper ideal I of R.*

Proof. Let us consider a proper ideal I of the compact Noetherian domain R, then by Lemma 96, for any element a in I, a must belongs to the Jacobson radical $J(R)$. This implies $I \subseteq J(R)$. Hence by [15, VIII, Theorem 9(d)], R is a Zariski ring with respect to the ideal I. ∎

In general fraction ring of a Noetherian ring is not necessarily be a Zariski ring, as if we consider the multiplicative system $S = \{1, X, X^2, ...\}$ in Noetherian ring $R = \mathbb{Z}[X]$, then the fraction ring R_S is not a Zariski ring with respect to the ideal IR_S, where $I = X\mathbb{Z}[X]$. But localization of a Noetherian ring is a Zariski ring, for instance, the localization $\mathbb{Z}_{(p)}$, where p is prime, of $\mathbb{Z}$ is a Zariski ring with defining ideal $(p\mathbb{Z})\mathbb{Z}_{(p)}$. Now it is natural to ask: Is there exist non local Zariski fraction ring of a Noetherian ring? The following proposition provides an affirmative response.

Proposition 98 *Let R be a Noetherian ring and I be a proper ideal of R. If $S = 1 + I$, then R_S is a Zariski ring with respect to the ideal IR_S.*

Proof. As fraction ring of a Noetherian ring is Noetherian, so R_S is Noetherian. Also S is a multiplicative system, indeed, let $x, y \in S$, so $x = 1 + a$, $y = 1 + b$ for $a, b \in I$ and $xy = 1 + a + b + ab \in S$. Consider

$$\begin{aligned} IR_S &= \{x'(a/s) : a \in R, x' \in I \text{ and } s \in S\} \\ &= \{x'(a/s) : a \in R,\ s = 1 + x \text{ and } x, x' \in I\}. \end{aligned}$$

Now let k be any element of $e + IR_S$, so

$$\begin{aligned} k &= e + x'(a/s),\ e = 1/1, s = 1 + x \text{ and } x', x \in I. \\ \text{So } k &= 1/1 + x'a/(1 + x), \text{ where } x', x \in I \\ &= (1/1) + x'a/(1 + x), \text{ where } x', x \in I \\ &= (1 + x + x'a)/(1 + x), \text{ where } x', x \in I. \\ \text{As } x'a &\in I, \text{ so } x + x'a \in I \text{ and } 1 + x + x'a \in S. \end{aligned}$$

This implies k is invertible. Thus every element of $e + IR_S$ is invertible in R_S. Hence by [15, VIII, Theorem 9(e)], R_S is a Zariski ring. ∎

What should be an appropriate answer when one enquire about the intersection of finite family of Noetherian rings. We extend it and ask the following:

Question. Is the intersection of finite family of Zariski rings Noetherian?

In the following we assume an affirmative response of the question and establish that this finite intersection of Zariski rings is again a Zariski ring. For this we first need the following lemma.

Lemma 99 *Let $\{R_i\}_{i=1}^{n}$ be a finite family of Noetherian rings, then*

$$J(\cap_{i=1}^{n} R_i) = \cap_{i=1}^{n} J(R_i)$$

Proof. Let

$$\begin{aligned} x &\in J(\cap_{i=1}^{n} R_i) \Longleftrightarrow 1 + x \in U(\cap_{i=1}^{n} R_i) \\ &\Longleftrightarrow 1 + x \in U(R_i) \text{ for each } i \\ &\Longleftrightarrow x \in J(R_i) \text{ for each } i \\ &\Longleftrightarrow x \in \cap_{i=1}^{n} J(R_i). \\ \text{Hence } J(\cap_{i=1}^{n} R_i) &= \cap_{i=1}^{n} J(R_i). \end{aligned}$$

■

Proposition 100 *Let $R = \cap_{i=1}^{n} R_j$, where each R_i is a Zariski ring with defining ideal IR_i. If R is Noetherian ring, then R is a Zariski ring with I as a defining ideal.*

Proof. Let I be an ideal in R.This implies $I \subseteq R_i$ for each i and as R_i is Zariski ring with defining ideal IR_i, so by [15, VIII, Theorem 9(d)], $IR_j \subseteq J(R_i)$, this implies $I \subseteq IR_i \subseteq J(R_i)$ for each i. So we have $I \subseteq J(R_i)$ for each i, which implies $I \subseteq \cap_{i=1}^{n} J(R_i)$.

By Lemma 99, $I \subseteq J(\cap_{i=1}^{n} R_i)$, which gives $I \subseteq J(R)$

Hence by [15, VIII, Theorem 9(d)] R is a Zariski ring. ■

Remark 101 An infinite intersection of Zariski rings need not to be a Zariski ring but Noetherian. For example $\mathbb{Z}$, as $\{\mathbb{Z}_{(p)} : p \text{ is prime intger}\}$ is an infinite family of Zariski rings, but $\mathbb{Z} = \cap_p \mathbb{Z}_{(p)}$ is not a Zariski ring.

3.4 Polynomial Ring and Power Series Ring

3.4.1 Polynomial ring as a Zariski ring

The polynomial rings are not behave as Zariski ring, indeed for any ring R (whether Zariski ring or not), the polynomial extension $R[X]$ generally is not a Zariski ring. Since for any ideal I of $R[X]$, the elements of $1+I$ are not all units in $R[X]$.

The following remark provides that under which circumstances a polynomial ring become a Zariski ring.

Remark 102 For any finite field K, the polynomial ring $K[X]$ is Zariski ring. Indeed; since K is field, so $K[X]$ is Noetherian. Also there is one-to-one correspondence between the set of maximal ideals of $K[X]$ and the set $K\setminus\{0\}$(cf. *[12, Page 5]*). This means $K[X]$ is a semi-local Noetherian domain, so by *[15, §4 Example (2)]* $K[X]$ is a Zariski ring with defining ideal $J(K[X])$.

Remark 103 By remark *102*, $GF(p^n)[X]$ is Zariski ring, where p is prime and n is a positive integer.

3.4.2 Formal Power Series Ring and its subrings

Unlike the polynomial ring, the formal power series ring becomes a Zariski ring whenever we impose certain conditions.

We initiate by the restatement of the well known lemma.

Lemma 104 *Let R be a commutative ring with identity, then $x+u \in U(R)$ for $x \in N(R)$ and $u \in U(R)$.*

Proposition 105 *Let R be a commutative Noetherian ring with identity, then $R[[X]]$ is a Zariski ring with respect to the ideal I, where $I = \{\sum_{i=0}^{\infty} a_i X^i : a_i \in N(R)\}$.*

Proof. Let f be any element of $1+I$, then

$$
\begin{aligned}
f &= 1+\sum_{i=0}^{\infty} a_i X^i, \text{ where } a_i \in N(R) \\
f &= 1+a_0+\sum_{i=1}^{\infty} a_i X^i, \text{ where } a_i \in N(R).
\end{aligned}
$$

Since $a_0 \in N(R)$ and as 1 is unit so by Lemma 104, $1 + a_0$ is unit in R. Then by lemma 9 in chapter 1, f is a unit element in $R[[X]]$ and hence every element of $1 + I$ is unit in $R[[X]]$. Thus by [15, VIII, Theorem 9(e)], $R[[X]]$ is a Zariski ring. ■

Corollary 106 *For any field K, the ring $K[[X]]$ is a Zariski ring with defining ideal $XK[[X]]$.*

Example 107 For the field of complex numbers $\mathbb{C}$, the ring $\mathbb{C}[[X]]$ is a Zariski ring with defining ideal $X\mathbb{C}[[X]]$. It can easily be observe that every element of $1 + X\mathbb{C}[[X]]$ is invertible in $\mathbb{C}[[X]]$. Also the subring $R = \mathbb{C}[[X^2, X^5]]$ of $\mathbb{C}[[X]]$, which is quasilocal Noetherian ring *[6, Example 3.4]* is a Zariski ring with defining maximal ideal $M = (X^2, X^5)R$.

3.5 Ascent and Descent of Zariski Ring

In [13], T. Shah has discussed the ascent and descent of factorization properties under certain conditions. In this section we study the stability of Zariski ring in unitary ring extension (resp. the domain extension) $R \subseteq T$. A study has also been made for the composite ring extension (resp. the domain extension) $R + X[[T]] \subseteq T[[X]]$.

3.5.1 Unitary Ring (domain) Extension

Lemma 108 *Let $R \subseteq T$ be a unitary commutative ring extension such that $U(R) = R \cap U(T)$ then, If T is a Zariski ring with respect to the I-topology then for the ideal $J = (R \cap I)$ we have $1 + J \subseteq U(R)$.*

Proof. As

$$1 + J \subseteq R, \text{ since } J \text{ is an ideal of } R \tag{1}$$

and $J \subseteq I$

$$\Rightarrow 1 + J \subseteq 1 + I$$

and as T is a Zariski ring with respect to the ideal I, so every element of $1 + I$ is unit in T, i.e. $1 + I \subseteq U(T)$, this implies $1 + J \subseteq 1 + I \subseteq U(T)$

$$\Rightarrow 1 + J \subseteq U(T) \tag{2}$$

from (1) and (2) we have

$$\begin{aligned} 1+J &\subseteq U(T)\cap R = U(R) \\ &\Rightarrow 1+J\subseteq U(R) \end{aligned}$$

Hence the result follows. ■

Theorem 109 *Let $R\subseteq T$ be a unitary commutative ring extension such that $U(R) = R\cap U(T)$ then, If T is a Zariski ring with respect to the ideal I, then if R is a Noetherian ring then R is a Zariski ring with respect to the ideal $J = R\cap I$.*

Proof. *By Lemma 108 $1+J\subseteq U(R)$, and R is considered to be a Noetherian domain. Hence by Theorem 82 R is a Zariski ring.* ■

The following proposition is about the condition for ascent of Zariski ring.

Proposition 110 *Let $R\subseteq T$ be a unitary commutative Noetherian ring extension and R be a Zariski ring with defining ideal I, and every maximal ideal of T contains the ideal I, then T is a Zariski ring with respect to the ideal IT of T.*

Proof. Since I is contained in every maximal ideal M of T, so $I\subseteq J(T)$ implies $IT\subseteq J(T)$, as $J(T)$ is an ideal of T. Thus by [15, VIII, Theorem 9(d)], T is also a Zariski ring. ■

Example 111 Let R be a Zariski ring with respect to the ideal I and $\widehat{R}$ be the I-adic completion of R, as every maximal ideal of $\widehat{R}$ will contains the ideal I, and so by proposition *110*, $\widehat{R}$ will also be Zariski ring with respect to the ideal $I\widehat{R}$.

Proposition 112 *Let $R\subseteq T$ be a unitary commutative ring extension such that $U(R) = R\cap U(T)$. If T is a Zariski ring with respect to the ideal I and $J = R : T$, then R is a Zariski ring with respect to the ideal generated by the subset IJ of R.*

Proof. As $R : T = \{x\in R : xT\subseteq R\}$, so for every element t in J, $tT\subseteq R$. This implies $JT\subseteq R$ and $IR\subseteq T$ so $t(IR)\subseteq R$, which implies

$$1+tIR\subseteq R. \tag{1}$$

Now as $U(R) = R \cap U(T)$ and for every $t \in J$, we have

Also $tIR \subseteq I \subseteq J(T)$ gives

$$1 + tIR \subseteq U(T) \tag{2}$$

From (1) and (2) we have $1 + tIR \subseteq U(R)$, for all $t \in J$.

This implies $1 + JIR \subseteq U(R)$. Hence by [15, VIII, Theorem 9(e)], R is a Zariski ring. ■

Example 113 Let R be an integral domain with quotient field K and I be a proper ideal of R. The fraction ring R_S, and the power series ring $K[[X]]$ are both Zariski rings, where $S = 1 + I$. Also we have $R_S \subseteq K[[X]]$.

3.5.2 Composite Ring (Domain) Extension

Proposition 114 *Let $R \subseteq T$ be a domain extension satisfies $U(R) = R \cap U(T)$, then the domain extension $A = R + XT[[X]] \subseteq T[[X]]$ satisfies $U(A) = A \cap U(T[[X]])$.*

Proof. As $A = R + XT[[X]] \subseteq T[[X]]$, so $U(A) \subseteq T[[X]]$. Also $U(A) \subseteq A$ implies

$$U(A) \subseteq U(T[[X]]) \cap A \tag{1}$$

Conversely we may see that $U(A) = U(R) + XT[[X]]$.

Now let $f \in U(T[[X]]) \cap A$, this implies $f \in U(T[[X]])$ and $f \in A$. As $U(T[[X]]) = u + XT[[X]]$, where $u \in U(T)$. So

$$f = u + \sum_{i=0}^{\infty} a_i X^i \text{ and } f = a_\circ + \sum_{i=0}^{\infty} a_i X^i,\ a_\circ \in R.$$

$$\text{That is } u + \sum_{i=0}^{\infty} a_i X^i = a_\circ + \sum_{i=0}^{\infty} a_i X^i$$

So comparing the coefficients we have $u = a_\circ \in R$. Also $u = a_\circ \in U(T)$, so we have $u = a_\circ \in U(R)$, and $f = u + \sum_{i=0}^{\infty} a_i X^i \in U(A)$.This means

$$A \cap U(T[[X]]) \subseteq U(A). \tag{2}$$

From (1) and (2), we have $U(A) = A \cap U(T[[X]])$. ■

Remark 115 We may discuss ascent and descent of Zariski ring in composite ring extension $R + XT[[X]] \subseteq T[[X]]$ by using the Proposition *114*, Theorem *109* and Lemma *108*, also we may construct various examples.

Theorem 116 *[15, VIII, Theorem] Let $R \subseteq T$ be a unitary commutative ring extension such that T is a finite R-module, then, If R is a Zariski ring with respect to the I-topology then T is also a Zariski ring with respect to the IT-topology.*

Proof. Since every finite T-module M is also a finite R-module, so M is a hausdorff space in its I-topology because R is a Zariski ring in its I-topology and since I-topology of M coincides with the IT-topology, it follows that every finite T-module M is a hausdorff space for its IT-topology.

Hence by Theorem 82 T is a Zariski ring. ■

We may record [10, Theorem 4] as remark, which ensure that when a composite ring $R + XT[[X]]$ is Noetherian.

Remark 117 *[10, Theorem 4]* Let $R \subseteq T$ be the commutative rings with unity, then the ring $A = R + XT[[X]]$ is Noetherian if and only if R is Noetherian and T is finite R-module.

The following proposition gives us the comparison between the maximal ideals of the ring R and of the ring $R + XT[[X]]$, where $R \subseteq T$ be a domain extension.

Proposition 118 *[9, Proposition 6] Let $R \subseteq T$ be a domain extension then, $Max(A) = \{m + XT[[X]] : m \in Max(R)\}$, where $A = R + XT[[X]]$.*

Example 119 *For the ring extension $\mathbb{Z} \subset \mathbb{Q}$, by Proposition 118 for the ring $A = \mathbb{Z} + X\mathbb{Q}[[X]]$ we have*

$$Max(A) = \{p\mathbb{Z} + X\mathbb{Q}[[X]] : p \text{ is prime integer}\}.$$

Proposition 120 *Let $R \subseteq T$ be the domain extension such that R is semi-local and T is a finite R-module. If R is a Zariski ring then the ring $A = R + XT[[X]]$ is a Zariski ring with respect to the ideal $J(A)$.*

Proof. By Proposition 118 we have that

$$\text{Max}(A) = \{m + XT[[X]] : m \in \text{Max}(R)\}$$

this implies that

$$|\text{Max}(A)| = |\text{Max}(R)|.$$

This shows that A is also semi local and as R is Noetherian and T is a finite R-module so by Theorem 117 A is Noetherian.

Hence by Example 84 A is Zariski ring with respect to the ideal $J(A) = \cap_{M \in \text{Max}(A)} M$. ■

Proposition 121 *Let $R \subset T$ be the domain extension such that R is a Zariski ring with respect to the ideal I and T is a finite R-module, then the ring $A = R + XT[[X]]$ is a Zariski ring with respect to the ideal $J = I + XT[[X]]$.*

Proof. Since R is Zariski ring with respect to the ideal I so

$$1 + I \subseteq U(R). \tag{1}$$

By Theorem 117 T is a Noetherian ring so we just have to show that every element of $1 + J$ is unit in A, where $J = I + XT[[X]]$ is ideal of A.

$$\begin{aligned} 1 + (I + XT[[X]]) &= (1 + I) + XT[[X]] \\ &\subseteq U(R) + XT[[X]] \\ &\subseteq U(A). \end{aligned}$$

Hence by Theorem 82 A is a Zariski ring. ■

Corollary 122 *Let $R \subseteq T$ be the domain extension such that R is a Zariski ring with respect to the ideal I and T is a Noetherian. Then in domain extension $A = R + XT[[X]] \subseteq T[[X]] = B$, A and B both are Zariski rings.*

Remark 123 In proposition *121*, $T[[X]]$ will never be a Zariski ring unless $T[[X]]$ is Noetherian.

Example 124 By example *107* the ring $\mathbb{C}[[X]]$ and $R = \mathbb{C}[[X^2, X^5]]$ are Zariski rings with defining ideals $X\mathbb{C}[[X]]$ and $M = (X^2, X^5)R$ respectively. So the Noetherian ring extension $\mathbb{C}[[X^2, X^5]] \subseteq \mathbb{C}[[X]]$ is a Zariski ring extension.

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