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A Dissertation Submitted in the Partial Fulfillment of the Requirement for the Degree of

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in

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CERTIFICATE

A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTER OF PHILOSOPHY

We accept this dissertation as conforming to the required standard

1. Prof. Dr. Muhammad Ayub (Chairman)

2. Dr. Tasawar Hayat (Supervisor)

3. Dr. Khalid Hanif (External Examiner) 04/06/2009

Department of Mathematics
Quaid-i-Azam University
Islamabad, Pakistan
2009

Sincerely Dedicated To-----

All those who love and care me

especially

The one WHO never die

Acknowledgement

All praise to ALLAH Almighty who guides us in the darkness and helps us in difficulties and all respect and affections to His Holy Prophet Hazrat Muhammad (S.A.W) who enable us to recognize our creator.

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Preface

In general, the Navier-Stokes equations are the non-linear partial differential equations describing the flows of viscous fluid. However, there are several materials, to name as few, such as soaps, drilling mud, ketchup and milk which do not obey the constitutive relationship of viscous fluids. These are categorized as the non-Newtonian fluids. Typical examples of rheological behavior occur in reactions of polymer and food processing. It is also an established fact now that the non-Newtonian fluids cannot be described by a single constitutive relation as for viscous fluids and they thus offer great challenges to modelers of such fluids. Many non-Newtonian fluid models have been proposed up to date in the fluid literature. Non-Newtonian fluid models and their resulting equations are more complicated and of higher order than the Navier-Stokes equations counterpart. Generally, uniqueness of solutions to these equations require additional boundary conditions (see [1,2]). Non-Newtonian flows have been understood from various viewpoints from experimental to the theoretical. Few related attempts on the topic have been presented in the refs. [3-20].

Ever since the seminal work of Sakiadis [21], boundary layer flow engendered by a moving surface has engaged many investigators. Such flows are vital in both viscous and non-Newtonian fluids, e.g. in crystal growth. A non-Newtonian fluid bounded by a porous stretching surface has promising application in polymer processing [22]. These kinds of flows have been looked at through various aspects (see refs. [23-29]).

Literature survey shows that there is little known on the flow of non-Newtonian fluids bounded by a porous shrinking surface. We wish to fill this void. In view of these facts, this dissertation has been arranged as follows

Chapter one presents the relevant basic laws and equations. Chapter two discusses the magnetohydrodynamic (MHD) and mass transfer effects on the flow of an upper convected Maxwell (UCM) fluid. Analytic treatment to the nonlinear mathematical problem is given by using the homotopy analysis method (HAM). The series solutions are discussed in detail. The contents of this chapter provides the review of a paper by Hayat et al. [30]

The purpose of chapter three is to extend the analysis of ref. [30] for a Jeffrey fluid. The relevant mathematical modeling is performed and the problem formulation is completed. Series solution by homotopy analysis method [31-44] is included. The present solution is compared with the previous results. The influence of rheological parameters in a Jeffrey fluid is displayed and discussed.

Contents

Chapter 1

Preliminaries

The purpose of this chapter is to present some dimensionless numbers, fundamental equations and laws required for the subsequent chapters.

1.1 Newtonian and non-Newtonian fluids

The real fluids for which the shear stress is directly and linearly proportional to the deformation rate are called Newtonian fluids. In mathematical notation one can represent as

$$\tau_{yx} = \mu \frac{du}{dy}, \tag{1.1}$$

where τ_{yx} is the shear stress, u is the x-component of velocity and μ is the dynamic viscosity.

The real fluids that do not obey the Newtons law of viscosity are known as Non-Newtonian fluids. For such fluids shear stress is not linearly proportional to the deformation rate.

Mathematically

$$\tau_{yx} = \eta \cdot \frac{du}{dy}, \tag{1.2}$$

where $\eta = k \left|\frac{du}{dy}\right|^{n-1}$ is the apparent viscosity, n is flow behavior and k is consistency index.

1.2 Fundamental equations

The equations that will be useful for the flow descriptions in the subsequent analysis are given below.

1.2.1 Equation of continuity

In fluid mechanics, continuity equation is an equation of conservation of mass i.e. the rate at which the mass enters into the system is equal to the rate at which mass leaves the system.

Mathematically

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{V}) = 0, \tag{1.3}$$

which for incompressible flows reduces to the following expression

$$\boldsymbol{\nabla}.\mathbf{V} = 0. \tag{1.4}$$

1.2.2 Equation of Momentum

The equation of motion is

$$\rho \frac{d\mathbf{V}}{dt} = \boldsymbol{\nabla}.\mathbf{T}+\rho \mathbf{b}, \tag{1.5}$$

For Navier-Stokes equations

$$\mathbf{T} = -p\mathbf{I} + \mu \mathbf{A}_1, \tag{1.6}$$

$$\mathbf{A}_1 = \operatorname{grad} \mathbf{V} + (\operatorname{grad} \mathbf{V})^t, \tag{1.7}$$

where ρ is the fluid density, $\mathbf{V}$ is the velocity field, $\mathbf{T}$ is the Cauchy's stress tensor, $\mathbf{B}$ is the body force, p is the pressure and μ is the dynamic viscosity.

The Cauchy's stress tensor can be expressed in matrix form as

$$\mathbf{T} = \begin{bmatrix} \sigma_{xx} & T_{xy} & T_{xz} \\ T_{yx} & \sigma_{yy} & T_{yz} \\ T_{zx} & T_{zy} & \sigma_{zz} \end{bmatrix}, \tag{1.8}$$

where σ_{xx}, σ_{yy} and σ_{zz} are normal stresses while all others are shear stresses.

Eq. (1.5) can be expressed in scalar form as

$$\rho\frac{du}{dt} = \frac{\partial\left(\sigma_{xx}\right)}{\partial x} + \frac{\partial\left(T_{xy}\right)}{\partial y} + \frac{\partial\left(T_{xz}\right)}{\partial z} + \rho b_x, \tag{1.9}$$

$$\rho\frac{dv}{dt} = \frac{\partial\left(T_{yx}\right)}{\partial x} + \frac{\partial\left(\sigma_{yy}\right)}{\partial y} + \frac{\partial\left(T_{yz}\right)}{\partial z} + \rho b_y, \tag{1.10}$$

$$\rho\frac{dw}{dt} = \frac{\partial\left(T_{zx}\right)}{\partial x} + \frac{\partial\left(T_{zy}\right)}{\partial y} + \frac{\partial\left(\sigma_{zz}\right)}{\partial z} + \rho b_z, \tag{1.11}$$

where b_x, b_y and b_z are the body forces in x, y and z direction respectively.

1.2.3 Equation of mass transfer

If a fluid contains species A which are slightly soluble in it then there will be relative transport of species. The species A may be transported by advection (with the mean velocity of mixture) and by diffusion (relative to the mean motion) in each of the coordinate directions. The concentration C_A may also be affected by chemical reaction. Let $\dot{N}_A$ be the rate at which the mass of species A is generated per unit volume due to such reaction and D is the coefficient of diffusing species.

Through boundary layer approximation, the governing equation for the concentration field is

$$u\frac{\partial C_A}{\partial x} + v\frac{\partial C_A}{\partial y} = D\frac{\partial^2 C_A}{\partial y^2} + \dot{N}_A, \tag{1.12}$$

1.2.4 Maxwell's equations

The Maxwell's equations can be expressed as

$$\operatorname{div}\mathbf{E}_{app} = \frac{\rho_e}{\epsilon_0}, \tag{1.13}$$

$$\operatorname{div}\mathbf{B} = 0, \tag{1.14}$$

$$\operatorname{curl}\mathbf{E}_{ind} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.15}$$

$$\operatorname{curl}\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial \mathbf{E}_{app}}{\partial t}. \tag{1.16}$$

In above equations the total magnetic field is $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$ ($\mathbf{B}_0$ and $\mathbf{b}$ are the applied and induced magnetic fields respectively), $\mathbf{E}_{app}$, $\mathbf{E}_{ind}$ is the applied and induced electric field respectively, ρ_e is the charge density, $\mathbf{J}$ is the current density, μ_0 is the magnetic permeability, ϵ_0 is the permittivity of the free space.

Ohm's and the Lorentz force laws are respectively given by

$$\mathbf{J} = \sigma(\mathbf{E}_{app} + \mathbf{V}\times\mathbf{B}), \tag{1.17}$$

$$\mathbf{F} = \mathbf{J}\times\mathbf{B}, \tag{1.18}$$

where σ is the electrical conductivity of the fluid.

In absence of applied electric field $\mathbf{E}_{app}$, Eq. (1.17) yields

$$\mathbf{J} = \sigma(\mathbf{V}\times\mathbf{B}). \tag{1.19}$$

For small magnetic Reynolds number, Eq. (1.18) simplifies to the following expression

$$\mathbf{J} \times \mathbf{B} = -\sigma B_0^2 \mathbf{V}. \tag{1.20}$$

1.3 Some useful dimensionless numbers

1.3.1 Reynolds number (Re)

It gives a measure of the ratio of inertial forces to viscous forces. It is further used to characterize different flow regimes, such as laminar or turbulent flow. Note that laminar flow occurs at low Reynolds numbers, where viscous forces are dominant, and is characterized by smooth, constant fluid motion, while turbulent flow occurs at high Reynolds numbers and is dominated by inertial forces.

One may expressed it as

$$\begin{aligned}
\text{Reynolds number} &= \frac{\text{inertial forces}}{\text{viscous forces}}, \\
&= \frac{\text{mass} * \text{acceleration}}{\text{shear stress} * \text{cross sectional area}}, \\
&= \frac{\text{volume} * \text{density} * \text{velocity}}{\text{shear stress} * \text{cross sectional area} * \text{time}}, \\
&= \frac{\text{cross sectional area} * \text{linear dimension} * \rho * V}{\text{shear stress} * \text{cross sectional area} * \text{time}}, \\
\text{or} \qquad \text{Re} &= \frac{V^2 * \rho}{\mu \frac{du}{dy}}, \\
&= \frac{V^2 * \rho}{\mu \frac{V}{L}}, \\
&= \frac{V * L}{\nu}.
\end{aligned} \tag{1.21}$$

where L and V denote length and velocity respectively.

1.3.2 Magnetic Reynolds number ($\mathbf{R}_m$)

It is a number that occurs in magnetohydrodynamics. It gives an estimate of the effects of magnetic advection to magnetic diffusion

$$R_m = \frac{UL}{\eta}, \tag{1.22}$$

Here η is magnetic diffusivity.

1.3.3 Schmidt number (Sc)

Schmidt number Sc is a dimensionless number which is defined as the ratio of momentum diffusivity (viscosity) and mass diffusivity. It physically relates the relative thickness of the hydrodynamic layer and mass-transfer boundary layer. It is expressed as

$$Sc = \frac{\nu}{D}, \tag{1.23}$$

where ν is kinematic viscosity and D is the mass diffusivity.

1.3.4 Deborah number (De)

The ratio of a relaxation time t_r and the characteristic time t_c is known as the Deborah number. In mathematical notation we have

$$De = \frac{t_r}{t_c} \tag{1.24}$$

1.3.5 Hartman number

It is the ratio of magnetic body forces to the viscous forces.

Chapter 2

Magnetohydrodynamic flow of an upper-convected Maxwell (UCM) fluid past a porous shrinking sheet with mass transfer and chemical reaction species

2.1 Introduction

This chapter looks at the mass transfer of the steady two-dimensional magnetohydrodynamic (MHD) boundary layer flow of an upper-convected Maxwell (UCM) fluid past a porous shrinking sheet in the presence of chemical reaction. The nonlinear partial differential equations are transformed into the system of nonlinear ordinary differential equations by invoking similarity transformations. Velocity and concentration fields are derived by a homotopy analysis method (HAM). Convergence of the derived series is shown. The gradient of mass transfer and the surface mass transfer are also tabulated. Discussion is made by sketching graphs. The content of this chapter are the review of a paper by Hayat et al.[30]

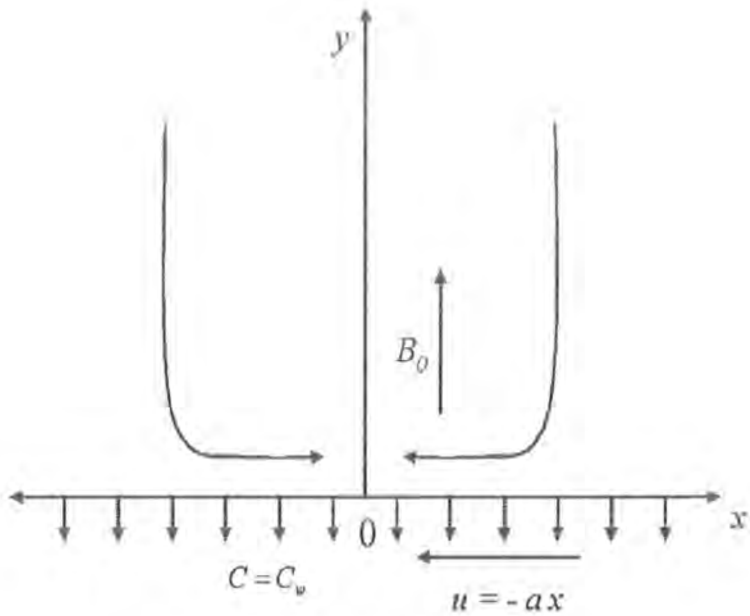

Fig. 1: Geometry of the problem

2.2 Mathematical formulation

Consider the steady, incompressible, MHD flow of two-dimensional upper-convected Maxwell (UCM) fluid over a porous shrinking sheet with suction. The sheet coincides with the plane $(y = 0)$ and the flow occupies the region $(y > 0)$. The x and y axes are taken along and perpendicular to the sheet, respectively (Fig. 1). A constant magnetic field of strength B_0 acts along the y-axis. The induced magnetic field is negligible. The external electric field is zero. The continuity, momentum and constitutive equations for an upper convected Maxwell (UCM) fluid are given by

$$\nabla.\mathbf{V} = 0, \tag{2.1}$$

$$\rho a_i = \nabla.\mathbf{T}, \tag{2.2}$$

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \tag{2.3}$$

$$\left(1+\lambda\frac{D}{Dt}\right)\mathbf{S}=\mu\mathbf{A}_1, \tag{2.4}$$

$$\mathbf{A}_1=\mathbf{L}+\mathbf{L}^T, \tag{2.5}$$

where the velocity field is

$$\mathbf{V}=[u(x,y),\ v(x,y),\ 0]. \tag{2.6}$$

Here $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{S}$ is the extra stress tensor, $\mathbf{I}$ is the identity tensor, $\mathbf{A}_1$ is the Rivilin-Ericksen tensor, $\mathbf{L}$ is the velocity gradient, μ is the dynamic viscosity, λ is the relaxation time and D/Dt is the covariant derivative.

The first Rivilin-Ericksen tensor is

$$\mathbf{A}_1=\begin{bmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} & 2\frac{\partial v}{\partial y} \end{bmatrix}. \tag{2.7}$$

Using Eqs.(2.3) – (2.7) in Eq. (2.2) we get

$$\rho\left(1+\lambda\frac{D}{Dt}\right)a_i=-\left(1+\lambda\frac{D}{Dt}\right)\nabla p+\mu\nabla\cdot\mathbf{A}_1. \tag{2.8}$$

We have assumed that the flow is caused only due to shrinking of the sheet therefore the pressure gradient is neglected

$$\rho\left(1+\lambda\frac{D}{Dt}\right)a_i=\mu\nabla\cdot\mathbf{A}_1, \tag{2.9}$$

where the general form for covariant derivative D/Dt is

$$\frac{Da_i}{Dt} = \frac{\partial a_i}{\partial t} + u_r a_{i,r} - u_{i,r} a_r. \tag{2.10}$$

For $i = 1$,

$$\left(1 + \lambda \frac{D}{Dt}\right) a_1 = \nu \nabla \cdot \mathbf{A}_1, \tag{2.11}$$

$$\frac{Da_1}{Dt} = \frac{\partial a_1}{\partial t} + u_1 a_{1,1} + u_2 a_{1,2} - u_{1,1} a_1 - u_{1,2} a_2, \tag{2.12}$$

$$\begin{aligned} a_1 &= \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}, & (2.13) \\ a_2 &= \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}, & (2.14) \end{aligned}$$

using Eqs. (2.12) – (2.14) in Eq. (2.11) we get

$$\begin{aligned} & u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \lambda \left[\begin{array}{c} u\frac{\partial}{\partial x}\left\{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right\} + v\frac{\partial}{\partial y}\left\{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right\} \\ -\frac{\partial u}{\partial x}\left\{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right\} - \frac{\partial u}{\partial y}\left\{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right\} \end{array} \right] \\ & = \nu \left[2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right], \end{aligned} \tag{2.15}$$

which by continuity equation yields

$$\begin{aligned} & u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \lambda \left[u^2\frac{\partial^2 u}{\partial x^2} + v^2\frac{\partial^2 u}{\partial y^2} + 2uv\frac{\partial^2 u}{\partial x \partial y}\right] \\ & = \nu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]. \end{aligned} \tag{2.16}$$

Similarly for $i = 2$, we get

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \lambda\left[u^2\frac{\partial^2 v}{\partial x^2} + v^2\frac{\partial^2 v}{\partial y^2} + 2uv\frac{\partial^2 v}{\partial x \partial y}\right]$$
$$= \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]. \tag{2.17}$$

Under order analysis in the boundary layer

$$u = O(1),\ x = O(1),\ v = O(\delta),\ y = O(\delta), \tag{2.18}$$

equation (2.17) satisfies identically and equation (2.16) yields

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \lambda\left[u^2\frac{\partial^2 u}{\partial x^2} + v^2\frac{\partial^2 u}{\partial y^2} + 2uv\frac{\partial^2 u}{\partial x \partial y}\right] = \nu\frac{\partial^2 u}{\partial y^2}. \tag{2.19}$$

The magnetic force under the stated assumptions gives

$$\mathbf{J} \times \mathbf{B} = -\sigma B_0^2 \mathbf{V}, \tag{2.20}$$

where σ is electrical conductivity of fluid, Eq. (2.19) in magnetohydrodynamics becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \lambda\left[u^2\frac{\partial^2 u}{\partial x^2} + v^2\frac{\partial^2 u}{\partial y^2} + 2uv\frac{\partial^2 u}{\partial x \partial y}\right] = \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2}{\rho}u. \tag{2.21}$$

Furthermore mass transfer is the flow along a sheet that contains a species A slightly soluble in the fluid B. Let C_w be the concentration at the sheet surface and the solubility of A in B and concentration of A far away from the sheet is C_∞. Also the reaction of a species A with B be the first order homogeneous chemical reaction of rate constant K_1. The concentration of dissolved A is considered small enough. Through boundary layer approximations, the governing equation for concentration fields is

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\frac{\partial^2 C}{\partial y^2} - K_1 C, \tag{2.22}$$

where C, D and δ are respectively the concentration of the species, the diffusion coefficient of the diffusing species in the fluid and the boundary layer thickness.

The boundary conditions are

$$u(x,y) = -cx,\ v(x,y) = -V_0,\ C(x,y) = C_w \quad at\ \ y = 0, \tag{2.23}$$

$$u(x,y) \to 0,\ C(x,y) \to C_\infty \quad as\ \ y \to \infty. \tag{2.24}$$

Here $c > 0$ is the rate of shrinking and $V_0 > 0$ is the suction velocity at the surface.

Let us define the dimensionless quantities

$$\eta = \sqrt{\frac{c}{\nu}}y,\ u = cxf'(\eta),\ v = -\sqrt{c\nu}f(\eta),\ \phi = \frac{C - C_\infty}{C_w - C_\infty}. \tag{2.25}$$

By using the above quantities, Eq. (2.1) is identically satisfied and the Eqs. (2.21) – (2.24) becomes

$$f''' - M^2 f' - f'^2 + f\ f'' + \beta\left(2ff'f'' - f^2 f'''\right) = 0, \tag{2.26}$$

$$\phi'' + Scf\phi' - Sc\gamma\phi = 0, \tag{2.27}$$

with the subjected boundary conditions

$$f = S,\ f' = -1,\ \phi = 1 \quad \text{at}\ \ \eta = 0, \tag{2.28}$$

$$f' = 0,\ \phi = 0 \quad \text{at}\ \ \eta = \infty, \tag{2.29}$$

in which the primes denote the derivative with respect to η and

$$S=\frac{V_0}{\sqrt{\nu c}},\ M^2=\frac{\sigma B_0^2}{\rho c},\ \beta=\lambda c,\ Sc=\frac{\nu}{D},\ \gamma=\frac{K_1}{c}. \tag{2.30}$$

Here S, M, Sc and γ are the suction, Hartman, Schmidt and chemical reaction parameters respectively. Moreover β is the Deborah number in terms of relaxation time. It is noted that for destructive/generative chemical reaction $\gamma > 0/\gamma < 0$ respectively and $\gamma = 0$ corresponds to non-reactive species. The surface mass transfer is

$$\phi'(0)=\left(\frac{\partial\phi}{\partial\eta}\right)_{\eta=0}. \tag{2.31}$$

2.2.1 Solution by homotopy analysis method (HAM)

In this section we will construct the HAM solution. For that we select

$$f_0(\eta)=S-1+\exp(-\eta),\ g_0(\eta)=\exp(-\eta), \tag{2.32}$$

with the following operator

$$\mathcal{L}_f=f'''-f',\ \mathcal{L}_\phi=\phi''-\phi, \tag{2.33}$$

which satisfy

$$\mathcal{L}_f\left(C_1+C_2e^{\eta}+C_3e^{-\eta}\right)=0,\ \ \mathcal{L}_\phi\left(C_4e^{\eta}+C_5e^{-\eta}\right)=0, \tag{2.34}$$

where C_i $(i=1-5)$ are the arbitrary constants.

2.2.2 Zeroth order deformation equation

The corresponding problems at the zeroth order are

$$(1-p)\,\mathcal{L}_f\left[\widehat{f}\,(\eta;p)-f_0(\eta)\right]=ph_fN_f\left[\widehat{f}(\eta;p),\ \widehat{\phi}(\eta;p)\right],\tag{2.35}$$

$$(1-p)\mathcal{L}_\phi\left[\widehat{\phi}(\eta;p)-\phi_0(\eta)\right]=ph_\phi N_\phi\left[\widehat{f}(\eta;p),\ \widehat{\phi}(\eta;p)\right],\tag{2.36}$$

$$\widehat{f}(0;p)=S,\ \widehat{f}'(0;p)=-1,\ \widehat{f}'(\infty,p)=0,\quad \widehat{\phi}(0;p)=1,\ \widehat{\phi}(\infty;p)=0,\tag{2.37}$$

where p is an embedding parameter and h_f, h_ϕ are the non-zero auxiliary parameters. Furthermore the non-linear operators are

$$\begin{aligned}N_f[f(\eta;p)]=&\frac{\partial^3 f(\eta;p)}{\partial\eta^3}-M^2\frac{\partial f(\eta;p)}{\partial\eta}+f(\eta;p)\frac{\partial^2 f(\eta;p)}{\partial\eta^2}-\left(\frac{\partial f(\eta;p)}{\partial\eta}\right)^2\\&+\beta\left(2f(\eta;p)\frac{\partial f(\eta;p)}{\partial\eta}\frac{\partial^2 f(\eta;p)}{\partial\eta^2}-f^{\,2}(\eta;p)\frac{\partial^3 f(\eta;p)}{\partial\eta^3}\right),\end{aligned}\tag{2.38}$$

$$N_\phi[\phi(\eta;p),f(\eta;p)]=\frac{\partial^2\phi(\eta;p)}{\partial\eta^2}+Scf(\eta;p)\frac{\partial\phi(\eta;p)}{\partial\eta}-Sc\gamma\phi(\eta;p).\tag{2.39}$$

When $p=0$ and $p=1$ then

$$\widehat{f}\,(\eta;0)=f_0\,(\eta)\,,\widehat{\phi}\,(\eta;0)=\phi_0\,(\eta)\ \text{ and }\widehat{f}\,(\eta;1)=f\,(\eta)\,,\widehat{\phi}\,(\eta;1)=\phi\,(\eta)\,.\tag{2.40}$$

It is noticed that when p increases from 0 to 1 then $f(\eta;p)$and $\phi\,(\eta;p)$ vary from the initial guesses $f_0\,(\eta)$, $\phi_0\,(\eta)$ to the final solutions $f\,(\eta)$ and $\phi\,(\eta)$. Using Taylor series we may write

$$f(\eta;p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta)p^m, \quad (2.41)$$

$$\phi(\eta;p) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta)p^m, \quad (2.42)$$

$$f_m(\eta) = \frac{1}{m!}\frac{\partial^m f(\eta;p)}{\partial \eta^m}\bigg|_{p=0}, \qquad \phi_m(\eta) = \frac{1}{m!}\frac{\partial^m \phi(\eta;p)}{\partial \eta^m}\bigg|_{p=0}. \quad (2.43)$$

Obviously Eqs. (2.35) and (2.36) have two non-zero auxiliary parameters h_f and h_ϕ. The convergence of the series (2.41) and (2.42) is dependent upon h_f and h_ϕ. The values of $h_i(i = f, \phi)$ are chosen properly so that Eqs. (2.41) and (2.42) are convergent at $p = 1$. In view of Eq. (2.40) we have

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \quad (2.44)$$

$$\phi(\eta) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta). \quad (2.45)$$

2.2.3 mth order deformation equations

The m^{th} order deformation problems are obtained by differentiating Eqs. (2.35) and (2.36) m times with respect to p and then setting $p = 0$. These are given by

$$\mathcal{L}_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = h_f R_m^f(\eta), \quad (2.46)$$

$$\mathcal{L}_f[\phi_m(\eta) - \chi_m \phi_{m-1}(\eta)] = h_\phi R_m^\phi(\eta), \quad (2.47)$$

$$f_m(0) = f_m'(0) = f_m'(\infty) = 0 \quad \text{and} \quad \phi_m(0) = \phi_m(\infty) = 0, \tag{2.48}$$

$$\begin{aligned} R_m^f(\eta) &= f_{m-1}''' - M^2 f_{m-1}' + \sum_{k=0}^{m-1} [f_{m-1-k} f_k'' - f_{m-1-k}' f_k'] \\ &\quad + \beta \sum_{k=0}^{m-1} f_{m-1-k} \sum_{l=0}^{k} [2 f_{k-l}' f_l'' - f_{k-l} f_l'''], \end{aligned} \tag{2.49}$$

$$R_m^\phi(\eta) = \phi_{m-1}''(\eta) - Sc\gamma\phi_{m-1} + Sc \sum_{k=0}^{m-1} \phi_{m-1-k}' f_k, \tag{2.50}$$

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{2.51}$$

Denoting $f_m^*(\eta)$ and $\phi_m^*(\eta)$ as the special solutions, we obtain the following general solutions:

$$f_m(\eta) = f_m^*(\eta) + C_1 + C_2 \exp(\eta) + C_3 \exp(-\eta), \tag{2.52}$$

$$\phi_m(\eta) = \phi_m^*(\eta) + C_4 \exp(\eta) + C_5 \exp(-\eta), \tag{2.53}$$

where

$$\begin{aligned} &C_2 = C_4 = 0, \; C_3 = \left. \frac{\partial f_m^*(\eta)}{\partial \eta} \right|_{\eta=0}, \\ &C_1 = -C_3 - f_m^*(0), \; C_5 = -\phi_m^*(0). \end{aligned} \tag{2.54}$$

The symbolic computation software MATHEMATICA is employed to obtain series solutions upto first few order of approximations. The relevant series solutions can be written in the form

$$f_m(\eta) = \sum_{n=0}^{2m+1} \sum_{q=0}^{2m+1-n} a_{m,n}^q \eta^q \exp(-n\eta), \quad m \geq 0, \tag{2.55}$$

$$\phi_m(\eta) = \sum_{n=0}^{2m+1} \sum_{q=0}^{2m+1-n} b_{m,n}^q \eta^q \exp(-n\eta), \tag{2.56}$$

where the coefficients $a_{m,n}^q$ and $b_{m,n}^q$ can be determined in the next section.

2.2.4 Derivation of the coefficients

First of all we calculate the derivatives involving in the Eqs. (2.49) and (2.50). From Eqs. (2.55) and (2.56) we have

$$\begin{aligned} f'_m(\eta) &= \sum_{n=0}^{2m+1} \left[\sum_{q=1}^{2m+1-n} q a_{m,n}^q \eta^{q-1} e^{-n\eta} - \sum_{q=0}^{2m+1-n} n a_{m,n}^q \eta^q e^{-n\eta} \right], \\ &= \sum_{n=0}^{2m+1} \left[\sum_{q=0}^{2m+1-n} (q+1) a_{m,n}^{q+1} \eta^q e^{-n\eta} - \sum_{q=0}^{2m+1-n} n a_{m,n}^q \eta^q e^{-n\eta} \right], \\ &= \sum_{n=0}^{2m+1} \sum_{q=0}^{2m+1-n} \left[(q+1) a_{m,n}^{q+1} - n a_{m,n}^q \right] \eta^q e^{-n\eta}, \\ &= \sum_{n=0}^{2m+1} \sum_{q=0}^{2m+1-n} a1_{m,n}^q \eta^q e^{-n\eta}, \end{aligned} \tag{2.57}$$

$$\begin{aligned} g'_m(\eta) &= \sum_{n=0}^{2m+1} \left[\sum_{q=1}^{2m+1-n} q b_{m,n}^q \eta^{q-1} e^{-n\eta} - \sum_{q=0}^{2m+1-n} n b_{m,n}^q \eta^q e^{-n\eta} \right], \\ &= \sum_{n=0}^{2m+1} \left[\sum_{q=0}^{2m+1-n} (q+1) b_{m,n}^{q+1} \eta^q e^{-n\eta} - \sum_{q=0}^{2m+1-n} n b_{m,n}^q \eta^q e^{-n\eta} \right], \\ &= \sum_{n=0}^{2m+1} \sum_{q=0}^{2m+1-n} \left[(q+1) b_{m,n}^{q+1} - n b_{m,n}^q \right] \eta^q e^{-n\eta}, \\ &= \sum_{n=0}^{2m+1} \sum_{q=0}^{2m+1-n} b1_{m,n}^q \eta^q e^{-n\eta}, \end{aligned} \tag{2.58}$$

where

$$a1^q_{m,n} = \left[(q+1)a^{q+1}_{m,n} - na^q_{m,n}\right], \tag{2.59}$$

$$b1^q_{m,n} = \left[(q+1)b^{q+1}_{m,n} - nb^q_{m,n}\right] \tag{2.60}$$

Similarly

$$f''_m(\eta) = \sum_{n=0}^{2m+1}\sum_{q=0}^{2m+1-n} a2^q_{m,n}\eta^q e^{-n\eta}, \tag{2.61}$$

$$f'''_m(\eta) = \sum_{n=0}^{2m+1}\sum_{q=0}^{2m+1-n} a3^q_{m,n}\eta^q e^{-n\eta}, \tag{2.62}$$

$$g''_m(\eta) = \sum_{n=0}^{2m+1}\sum_{q=0}^{2m+1-n} b2^q_{m,n}\eta^q e^{-n\eta}, \tag{2.63}$$

in which

$$a2^q_{m,n} = \left[(q+1)a1^{q+1}_{m,n} - na1^q_{m,n}\right], \tag{2.64}$$

$$a3^q_{m,n} = \left[(q+1)a2^{q+1}_{m,n} - na2^q_{m,n}\right], \tag{2.65}$$

$$b2^q_{m,n} = \left[(q+1)b1^{q+1}_{m,n} - nb1^q_{m,n}\right]. \tag{2.66}$$

For the product terms, let us consider

$$\begin{aligned}
f_{m-1-k}f_k'' &= \sum_{r=0}^{2m-2k-1}\sum_{s=0}^{2m-2k-1-r} a_{m-1-k,r}^{s}\eta^{s}e^{-r\eta}\times\sum_{i=0}^{2k+1}\sum_{j=0}^{2k+1-i} a2_{k,i}^{j}\eta^{j}e^{-i\eta},\\
&= \sum_{r=0}^{2m-2k-1}\sum_{i=0}^{2k+1} e^{-(r+i)\eta}\sum_{s=0}^{2m-2k-1-r}\sum_{j=0}^{2k+1-i} a_{m-1-k,r}^{s}a2_{k,i}^{j}\eta^{j+s},\\
&= \sum_{n=0}^{2m} e^{-n\eta}\sum_{i=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+1\}}\left[\sum_{s=0}^{2m-2k-1-r}\sum_{j=0}^{2k+1-i} a_{m-1-k,r}^{s}a2_{k,i}^{j}\right]\eta^{j+s},\\
&= \sum_{n=0}^{2m} e^{-n\eta}\sum_{i=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+1\}}\left[\sum_{q=0}^{2m-n}\eta^{q}\sum_{j=\max\{0,q-2m+2k+1+n-i\}}^{\min\{q,2k+1-i\}} a_{m-1-k,n-i}^{q-j}a2_{k,i}^{j}\right],\\
&= \sum_{n=0}^{2m}\sum_{q=0}^{2m-n}\left[\sum_{i=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+1\}}\sum_{j=\max\{0,q-2m+2k+1+n-i\}}^{\min\{q,2k+1-i\}} a_{m-1-k,n-i}^{q-j}a2_{k,i}^{j}\right]\eta^{q}e^{-n\eta},
\end{aligned} \tag{2.67}$$

$$\sum_{k=0}^{m-1} f_{m-1-k}f_k'' = \sum_{n=0}^{2m}\sum_{q=0}^{2m-n}\left[\sum_{k=0}^{m-1}\sum_{i=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+1\}}\sum_{j=\max\{0,q-2m+2k+1+n-i\}}^{\min\{q,2k+1-i\}} a_{m-1-k,n-i}^{q-j}a2_{k,i}^{j}\right]\eta^{q}e^{-n\eta}, \tag{2.68}$$

$$\sum_{k=0}^{m-1} f_{m-1-k}f_k'' = \sum_{n=0}^{2m}\sum_{q=0}^{2m-n}\alpha_{m,n}^{q}\eta^{q}e^{-n\eta}, \tag{2.69}$$

where

$$\alpha_{m,n}^{q} = \sum_{k=0}^{m-1}\sum_{i=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+1\}}\sum_{j=\max\{0,q-2m+2k+1+n-i\}}^{\min\{q,2k+1-i\}} a_{m-1-k,n-i}^{q-j}a2_{k,i}^{j}. \tag{2.70}$$

Similarly we can calculate the other terms involving in Eqs. (2.49) and (2.50) i.e.

$$\sum_{k=0}^{m-1} f'_{m-1-k} f'_k = \sum_{n=0}^{2m} \sum_{q=0}^{2m-n} \alpha 1^q_{m,n} \eta^q e^{-n\eta}, \tag{2.71}$$

$$\sum_{k=0}^{m-1} \phi'_{m-1-k} f_k = \sum_{n=0}^{2m} \sum_{q=0}^{2m-n} \alpha 2^q_{m,n} \eta^q e^{-n\eta}, \tag{2.72}$$

where

$$\alpha 1^q_{m,n} = \sum_{k=0}^{m-1} \sum_{i=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+1\}} \sum_{j=\max\{0,q-2m+2k+1+n-i\}}^{\min\{q,2k+1-i\}} a1^{q-j}_{m-1-k,n-i} a1^j_{k,i}, \tag{2.73}$$

$$\alpha 2^q_{m,n} = \sum_{k=0}^{m-1} \sum_{i=\max\{0,n-2m+2k+2\}}^{\min\{n,2k+1\}} \sum_{j=\max\{0,q-2m+2k+2+n-i\}}^{\min\{q,2k+1-i\}} b1^{q-j}_{m-1-k,n-i} a^j_{k,i}. \tag{2.74}$$

For the terms which involve the product of three functions we have

$$\begin{aligned}
f'_{k-l} f''_l &= \sum_{r=0}^{2k-2l+1} \sum_{s=0}^{2k-2l+1-r} a1^s_{k-l,r} \eta^s e^{-r\eta} \times \sum_{i=0}^{2l+1} \sum_{j=0}^{2l+1-i} a2^l_{l,i} \eta^j e^{-i\eta}, \\
&= \sum_{r=0}^{2k-2l+1} \sum_{i=0}^{2l+1} e^{-(r+i)\eta} \sum_{s=0}^{2k-2l+1-r} \sum_{j=0}^{2l+1-i} a1^s_{k-l,r} a2^j_{l,i} \eta^{j+s}, \\
&= \sum_{p=0}^{2k+2} e^{-p\eta} \left[\sum_{i=\max\{0,p-2k+2l-1\}}^{\min\{p,2l+1\}} \sum_{s=0}^{2k-2l+1-r} \sum_{j=0}^{2l+1-i} a1^s_{k-l,r} a2^j_{l,i} \right] \eta^{j+s}, \\
&= \sum_{p=0}^{2k+2} e^{-p\eta} \sum_{i=\max\{0,p-2k+2l-1\}}^{\min\{p,2l+1\}} \left[\sum_{t=0}^{2k+2-p} \eta^t \sum_{j=\max\{0,t-2k+2l-1+p-i\}}^{\min\{t,2l+1-i\}} a1^{t-j}_{k-l,p-i} a2^j_{l,i} \right], \\
&= \sum_{p=0}^{2k+2} \sum_{t=0}^{2k+2-p} \left[\sum_{i=\max\{0,p-2k+2l-1\}}^{\min\{p,2l+1\}} \sum_{j=\max\{0,t-2k+2l-1+p-i\}}^{\min\{t,2l+1-i\}} a1^{t-j}_{k-l,p-i} a2^j_{l,i} \right] \eta^t e^{-p\eta},
\end{aligned} \tag{2.75}$$

$$\sum_{k=0}^{m-1} f'_{k-l} f''_l = \sum_{p=0}^{2k+2} \sum_{l=0}^{2k+2-p} \left[\sum_{k=0}^{m-1} \sum_{i=\max\{0,p-2k+2l-1\}}^{\min\{p,2l+1\}} \sum_{j=\max\{0,t-2k+2l-1+p-i\}}^{\min\{t,2l+1-i\}} a1_{k-l,p-i}^{t-j} a2_{l,i}^{j} \right] \eta^t e^{-p\eta} \tag{2.76}$$

$$\sum_{k=0}^{m-1} f'_{k-l} f''_l = \sum_{p=0}^{2k+2} \sum_{t=0}^{2k+2-p} \alpha 3_{k,p}^{t} \eta^t e^{-p\eta}, \tag{2.77}$$

$$\sum_{k=0}^{m-1} f_{k-l} f'''_l = \sum_{p=0}^{2k+2} \sum_{t=0}^{2k+2-p} \alpha 4_{k,p}^{t} \eta^t e^{-p\eta}, \tag{2.78}$$

$$\alpha 3_{k,p}^{t} = \sum_{k=0}^{m-1} \sum_{i=\max\{0,p-2k+2l-1\}}^{\min\{p,2l+1\}} \sum_{j=\max\{0,t-2k+2l-1+p-i\}}^{\min\{t,2l+1-i\}} a1_{k-l,p-i}^{t-j} a2_{l,i}^{j} \tag{2.79}$$

$$\alpha 4_{k,p}^{t} = \sum_{k=0}^{m-1} \sum_{i=\max\{0,p-2k+2l-1\}}^{\min\{p,2l+1\}} \sum_{j=\max\{0,t-2k+2l-1+p-i\}}^{\min\{t,2l+1-i\}} a_{k-l,p-i}^{t-j} a3_{l,i}^{j} \tag{2.80}$$

$$\begin{aligned} f_{m-1-k} f'_{k-l} f''_l &= \sum_{x=0}^{2m-2k-1} \sum_{y=0}^{2m-2k-1-x} a_{m-1-k,x}^{y} \eta^y e^{-x\eta} \times \sum_{p=0}^{2k+1} \sum_{t=0}^{2k+2-p} \alpha 3_{k,p}^{t} \eta^t e^{-p\eta} \\ &= \sum_{x=0}^{2m-2k-1} \sum_{p=0}^{2k+1} e^{-(x+p)\eta} \sum_{y=0}^{2m-2k-1-x} \sum_{t=0}^{2k+2-p} a_{m-1-k,x}^{y} \alpha 3_{k,p}^{t} \eta^{y+t}, \\ &= \sum_{n=0}^{2m+1} e^{-n\eta} \sum_{p=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+2\}} \left[\sum_{y=0}^{2m-2k-1-x} \sum_{t=0}^{2k+2-p} a_{m-1-k,x}^{y} \alpha 3_{k,p}^{t} \right] \eta^{y+t}, \\ &= \sum_{n=0}^{2m+1} e^{-n\eta} \sum_{p=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+2\}} \left[\sum_{q=0}^{2m+1-n} \eta^q \sum_{t=\max\{0,q-2m+2k+1+n-p\}}^{\min\{q,2k+2-p\}} a_{m-1-k,n-p}^{q-t} \alpha 3_{k,p}^{t} \right], \\ &= \sum_{n=0}^{2m+1} \sum_{q=0}^{2m+1-n} \left[\sum_{p=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+2\}} \sum_{t=\max\{0,q-2m+2k+1+n-p\}}^{\min\{q,2k+2-p\}} a_{m-1-k,n-p}^{q-t} \alpha 3_{k,p}^{t} \right] \eta^q e^{-n\eta}, \end{aligned} \tag{2.81}$$

$$\sum_{k=0}^{m-1} f_{m-1-k}f'_{k-l}f''_l = \sum_{n=0}^{2m+1}\sum_{q=0}^{2m+1-n}\left[\sum_{k=0}^{m-1}\sum_{p=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+2\}}\sum_{t=\max\{0,q-2m+2k+1+n-p\}}^{\min\{q,2k+2-p\}} a_{m-1-k,n-p}^{q-t}\alpha3_{k,p}^{t}\right]\eta^q e^{-n\eta} \tag{2.82}$$

$$\sum_{k=0}^{m-1} f_{m-1-k}f'_{k-l}f''_l = \sum_{n=0}^{2m+1}\sum_{q=0}^{2m+1-n}\alpha5_{m,n}^{q}\eta^q e^{-n\eta} \tag{2.83}$$

where

$$\alpha5_{k,p}^{t} = \sum_{k=0}^{m-1}\sum_{p=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+2\}}\sum_{t=\max\{0,q-2m+2k+1+n-p\}}^{\min\{q,2k+2-p\}} a_{m-1-k,n-p}^{q-t}\alpha3_{l,i}^{j} \tag{2.84}$$

similarly

$$\sum_{k=0}^{m-1} f_{m-1-k}f_{k-l}f'''_l = \sum_{n=0}^{2m+1}\sum_{q=0}^{2m+1-n}\alpha6_{m,n}^{q}\eta^q e^{-n\eta} \tag{2.85}$$

where

$$\alpha6_{k,p}^{t} = \sum_{k=0}^{m-1}\sum_{p=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+2\}}\sum_{t=\max\{0,q-2m+2k+1+n-p\}}^{\min\{q,2k+2-p\}} a_{m-1-k,n-p}^{q-t}\alpha4_{l,i}^{j}. \tag{2.86}$$

Using the above relations in Eqs. (2.49) and (2.50) we have

$$
\begin{aligned}
h_f R_m^f(\eta) &= h_f \sum_{n=0}^{2m-1} \sum_{q=0}^{2m-1-n} \left[a3_{m-1,n}^q - M^2 a1_{m-1,n}^q\right] \eta^q e^{-n\eta} \\
&+ h_f \sum_{n=0}^{2m} \sum_{q=0}^{2m-n} \left[\alpha_{m,n}^q - \alpha 1_{m,n}^q\right] \eta^q e^{-n\eta} \\
&+ h_f \sum_{n=0}^{2m+1} \sum_{q=0}^{2m+1-n} \left[2\,(\beta)\,\alpha 5_{m,n}^q - (\beta)\alpha 6_{m,n}^q\right] \eta^q e^{-n\eta},
\end{aligned} \tag{2.87}
$$

$$
\begin{aligned}
h_\phi R_m^\phi(\eta) &= h_\phi \sum_{n=0}^{2m-1} \sum_{q=0}^{2m-1-n} \left[b2_{m-1,n}^q - Sc\gamma b_{m-1,n}^q\right] \eta^q e^{-n\eta} \\
&+ h_\phi \sum_{n=0}^{2m} \sum_{q=0}^{2m-n} \left[(Sc)\,\alpha 2_{m,n}^q\right] \eta^q e^{-n\eta},
\end{aligned} \tag{2.88}
$$

$$
h_f R_m^f(\eta) = \sum_{n=0}^{2m+1} \sum_{q=0}^{2m+1-n} h_f \left[\begin{array}{c} \chi_{2m+1-n-q}\{a3_{m-1,n}^q - M^2 a1_{m-1,n}^q\} \\ +\chi_{2m+1-n-q}\{\alpha_{m,n}^q - \alpha 1_{m,n}^q\} \\ +2(\beta)\alpha 5_{m,n}^q - (\beta)\alpha 6_{m,n}^q \end{array}\right] \eta^q e^{-n\eta}, \tag{2.89}
$$

$$
h_\phi R_m^\phi(\eta) = \sum_{n=0}^{2m} \sum_{q=0}^{2m-n} h_\phi \left[\begin{array}{c} \chi_{2m+1-n-q}\{b2_{m-1,n}^q - Sc\gamma b_{m-1,n}^q\} \\ +(Sc)\,\alpha 2_{m,n}^q \end{array}\right] \eta^q e^{-n\eta}. \tag{2.90}
$$

or we can write

$$
h_f R_m^f(\eta) = \sum_{n=0}^{2m+1} \sum_{q=0}^{2m+1-n} \Delta_{m,n}^q \eta^q e^{-n\eta}, \tag{2.91}
$$

$$
h_\phi R_m^\phi(\eta) = \sum_{n=0}^{2m} \sum_{q=0}^{2m-n} \Gamma_{m,n}^q \eta^q e^{-n\eta}, \tag{2.92}
$$

where $\Delta_{m,n}^q$ and $\Gamma_{m,n}^q$ are

$$\Delta_{m,n}^{q}=h_{f}\left[\begin{array}{c}\chi_{2m+1-n-q}\{a3_{m-1,n}^{q}-M^{2}a1_{m-1,n}^{q}\}\\+\chi_{2m+1-n-q}\{\alpha_{m,n}^{q}-\alpha1_{m,n}^{q}\}\\+2(\beta)\alpha5_{m,n}^{q}-(\beta)\alpha6_{m,n}^{q}\end{array}\right], \tag{2.93}$$

$$\Gamma_{m,n}^{q}=h_{\phi}\left[\begin{array}{c}\chi_{2m+1-n-q}\{b2_{m-1,n}^{q}-Sc\gamma b_{m-1,n}^{q}\}\\+(Sc)\,\alpha2_{m,n}^{q}\end{array}\right]. \tag{2.94}$$

Using Eqs. (2.91) and (2.92), the Eqs. (2.46) and (2.47) become

$$\mathcal{L}_{f}\left[f_{m}(\eta)-\chi_{m}f_{m-1}(\eta)\right]=\sum_{n=0}^{2m+1}\sum_{q=0}^{2m+1-n}\Delta_{m,n}^{q}\eta^{q}e^{-n\eta}, \tag{2.95}$$

$$\mathcal{L}_{\phi}\left[\phi_{m}(\eta)-\chi_{m}\phi_{m-1}(\eta)\right]=\sum_{n=0}^{2m}\sum_{q=0}^{2m-n}\Gamma_{m,n}^{q}\eta^{q}e^{-n\eta}. \tag{2.96}$$

Applying the inverse of the linear operators on both sides, one can write

$$f_{m}(\eta)-\chi_{m}f_{m-1}(\eta)=\sum_{n=0}^{2m+1}\sum_{q=0}^{2m+1-n}\sum_{k=1}^{q+1}\Delta_{m,n}^{q}\mu1_{n,k}^{q}\eta^{q}e^{-n\eta}+C_{1}^{m}+C_{2}^{m}e^{\eta}+C_{3}^{m}e^{-\eta}, \tag{2.97}$$

$$\phi_{m}(\eta)-\chi_{m}\phi_{m-1}(\eta)=\sum_{n=0}^{2m}\sum_{q=0}^{2m-n}\sum_{k=1}^{q+1}\Gamma_{m,n}^{q}\mu2_{n,k}^{q}\eta^{q}e^{-n\eta}+C_{4}^{m}e^{\eta}+C_{5}^{m}e^{-\eta}, \tag{2.98}$$

where C_i $(i=1-5)$ are the constants of integration. Using boundary conditions, one has

$$C_{2}^{m}=0,\ C_{4}^{m}=0 \tag{2.99}$$

$$C_3^m = \sum_{q=0}^{2m+1} \Delta_{m,0}^q \mu 1_{0,1}^q + \sum_{q=0}^{2m} \Delta_{m,1}^q (\mu 1_{1,1}^q - \mu 1_{1,0}^q) + \sum_{n=2}^{2m+1} \sum_{q=0}^{2m+1-n} \Delta_{m,n}^q (\mu 1_{n,1}^q - \mu 1_{n,0}^q), \tag{2.100}$$

$$C_1^m = -\sum_{q=0}^{2m+1} \Delta_{m,0}^q \mu 1_{0,1}^q + \sum_{q=0}^{2m+1} \Delta_{m,1}^q \mu 1_{1,1}^q + \sum_{n=2}^{2m+1} \sum_{q=0}^{2m+1-n} \Delta_{m,n}^q \{(\mu 1_{n,1}^q - (n-1)\, \mu 1_{n,0}^q)\}, \tag{2.101}$$

$$C_5^m = -\left[\sum_{q=0}^{2m+1} \Gamma_{m,0}^q \mu 2_{0,0}^q + \sum_{q=0}^{2m} \Gamma_{m,1}^q \mu 2_{1,0}^q + \sum_{n=2}^{2m+1} \sum_{q=0}^{2m+1-n} \Gamma_{m,n}^q \mu 2_{n,0}^q \right], \tag{2.102}$$

with

$$\mu 1_{n,k}^q = \sum_{r=0}^{q-k} \sum_{p=0}^{q-k-r} \frac{-q!}{k!\,(n-1)^{q+1-k-r-p}\, n^{r+1}\,(n+1)^{p+1}}, \tag{2.103}$$

$$\mu 2_{n,k}^q = \sum_{p=0}^{q-k} \frac{q!}{k!\,(n-1)^{q+1-k-p}\,(n+1)^{p+1}}. \tag{2.104}$$

Substituting the values of constants into Eqs. (2.97) and (2.98) we have

$$\begin{aligned}
&\sum_{n=0}^{2m+1} \sum_{q=0}^{2m+1-n} \left[a_{m,n}^q - \chi_{m+2-n} \chi_{m+2-n-q} a_{m-1,n}^q \right] \eta^q e^{-n\eta} = \sum_{n=0}^{2m+1} \sum_{q=0}^{2m+1-n} \sum_{k=1}^{q+1} \Delta_{m,n}^q \mu 1_{n,k}^q \eta^q e^{-n\eta} \\
&+ \left[\sum_{q=0}^{2m+1} \Delta_{m,0}^q \mu 1_{0,1}^q + \sum_{q=0}^{2m} \Delta_{m,1}^q (\mu 1_{1,1}^q - \mu 1_{1,0}^q) + \sum_{n=2}^{2m+1} \sum_{q=0}^{2m+1-n} \Delta_{m,n}^q (\mu 1_{n,1}^q - \mu 1_{n,0}^q) \right] e^{-\eta} \\
&- \left[\sum_{q=0}^{2m+1} \Delta_{m,0}^q \mu 1_{0,1}^q - \sum_{q=0}^{2m} \Delta_{m,1}^q \mu 1_{1,1}^q - \sum_{n=2}^{2m+1} \sum_{q=0}^{2m+1-n} \Delta_{m,n}^q \{(\mu 1_{n,1}^q - (n-1)\, \mu 1_{n,0}^q)\} \right],
\end{aligned} \tag{2.105}$$

$$\begin{aligned}
&\sum_{n=0}^{2m+1} \sum_{q=0}^{2m+1-n} \left[b_{m,n}^q - \chi_{m+2-n} \chi_{m+2-n-q} b_{m-1,n}^q \right] \eta^q e^{-n\eta} = \sum_{n=0}^{2m+1} \sum_{q=0}^{2m+1-n} \sum_{k=1}^{q+1} \Gamma_{m,n}^q \mu 2_{n,k}^q \eta^q e^{-n\eta} \\
&- \left[\sum_{q=0}^{2m+1} \Gamma_{m,0}^q \mu 2_{0,0}^q + \sum_{q=0}^{2m} \Gamma_{m,1}^q \mu 2_{1,0}^q + \sum_{n=2}^{2m+1} \sum_{q=0}^{2m+1-n} \Gamma_{m,n}^q \mu 2_{n,0}^q \right] e^{-\eta}.
\end{aligned} \tag{2.106}$$

Comparing like powers of η in the above equations we arrive at

$$a_{m,0}^{0} = \chi_m \chi_{m+2} \chi_{m+2-q} a_{m-1,0}^{0} - \sum_{q=0}^{2m} \Delta_{m,0}^{q} \mu 1_{0,1}^{q} - \sum_{n=2}^{2m+1} \sum_{q=0}^{2m+1-n} \Delta_{m,n}^{q} (\mu 1_{n,1}^{q} - (n-1)\, \mu 1_{n,0}^{q}), \tag{2.107}$$

$$a_{m,1}^{0} = \chi_m \chi_{m+1} \chi_{m+1-q} a_{m-1,0}^{0} - \sum_{q=0}^{2m} \Delta_{m,1}^{q} \mu 1_{1,1}^{q} - \sum_{n=2}^{2m+1} \sum_{q=0}^{2m+1-n} \Delta_{m,n}^{q} (\mu 1_{n,1}^{q} - n \mu 1_{n,0}^{q}), \tag{2.108}$$

$$a_{m,n}^{k} = \chi_m \chi_{m+2-n} \chi_{m+2-n-q} a_{m-1,n}^{k} - \sum_{q=0}^{m+1-n} \Delta_{m,n}^{q} \mu 1_{n,k}^{q} - \sum_{n=2}^{2m+1} \sum_{q=k}^{2m+1-n} \Delta_{m,n}^{q} (\mu 1_{n,1}^{q} - (n-1)\, \mu 1_{n,0}^{q}),\ n \geq 2. \tag{2.109}$$

Similarly

$$b_{m,1}^{0} = \chi_m \chi_{m+1} \chi_{m+1-q} b_{m-1,1}^{0} - \sum_{n=2}^{2m+1} \sum_{q=0}^{2m+1-n} \Gamma_{m,n}^{q} \mu 2_{n,0}^{q}, \tag{2.110}$$

$$b_{m,n}^{k} = \chi_m \chi_{m+1} \chi_{m+1-q} b_{m-1,n}^{k} - \sum_{q=k}^{2m+1-n} \Gamma_{m,n}^{q} \mu 2_{n,k}^{q},\quad n \geq 2. \tag{2.111}$$

Using the above recurrence formulas we can calculate all the coefficients $a_{m,n}^{q}$ and $b_{m,n}^{q}$ using only the first few

$$a_{0,0}^{0} = S-1,\ a_{0,1}^{0} = 1,\ a_{0,0}^{1} = 1, \tag{2.112}$$

$$b_{0,0}^{0} = 0,\ b_{0,0}^{1} = 0,\ b_{0,1}^{0} = 1 \tag{2.113}$$

given by the initial guess approximation in Eq. (2.32).

Thus, the explicit analytical solutions are

$$
\begin{aligned}
f(\eta) &= \sum_{m=0}^{\infty} f_m(\eta) \\
&= \lim_{M\to\infty}\left[\sum_{m=0}^{M} a_{m,0}^{0} + \sum_{n=1}^{M+1} e^{-n\eta}\left(\sum_{m=n-1}^{M}\sum_{q=0}^{2m+1-n} a_{m,n}^{q}\eta^{q}\right)\right], \qquad (2.114)
\end{aligned}
$$

$$
\begin{aligned}
\phi(\eta) &= \sum_{m=0}^{\infty} \phi_m(\eta) \\
&= \lim_{M\to\infty}\left[\sum_{m=0}^{M} b_{m,0}^{0} + \sum_{n=1}^{M+1} e^{-n\eta}\left(\sum_{m=n-1}^{M}\sum_{q=0}^{2m+1-n} b_{m,n}^{q}\eta^{q}\right)\right]. \qquad (2.115)
\end{aligned}
$$

2.3 Convergence of HAM solution

The series solutions of the considered problems are presented in Eqs. (2.114) and (2.115). Obviously the convergence of these solutions depend upon the parameters h_f and h_ϕ. To see the admissible values of h_f and h_ϕ, the h curves are plotted for 15th order of approximation in Fig. (2). It is apparent from Fig. (2) that admissible values of h_f is $-1.7 \leq h_f \leq -0.7$ and h_ϕ is $-1.8 \leq h_\phi \leq -0.7$. The series (2.114) and (2.115) converge in the whole region of η when $h_f = h_\phi = -1.0$. Table 1 is made just to decide that how much order of approximation are necessary for a convergent solution. It is noticed that 20th order of approximations are enough.

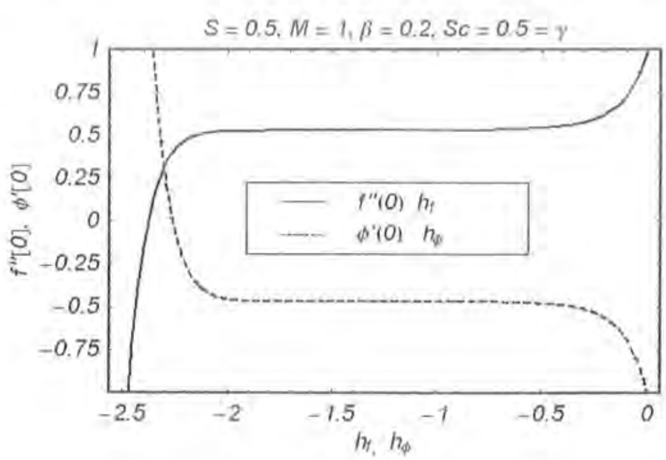

Fig. 2: h curves of $f''(0)$ and $\phi'(0)$ at 15th order of approximation.

Table 1: Convergence of the HAM solutions for different order of approximations when $S = 1$, $M = 1.5$, $Sc = \gamma = 1$ and $\beta = 0.2$.

Order of approximation	$f''(0)$	$-\phi'(0)$
1	1.402500	1.233333
5	1.719884	1.431262
10	1.732064	1.436541
15	1.732260	1.436517
20	1.732268	1.436517
25	1.732268	1.436517
30	1.732268	1.436517
35	1.732268	1.436517
40	1.732268	1.436517
50	1.732268	1.436517
60	1.732268	1.436517

2.4 Results and discussion

In this section Figs. $(3) - (13)$ are displayed for the effects of parameters S, M, β, Sc and γ on the velocity f' and the concentration field ϕ. The surface mass transfer $\phi'(0)$ and the gradient of mass transfer $-\phi'(\eta)$ are also given in Tables 2 and 3, respectively. Figs. $(3) - (6)$ depict the effects of S, M and β on the velocity component f'. From Figs. (3 and 4) it is noticed that the magnitude of velocity and the boundary layer thickness decreases by increasing S and M. The effects of the Deborah number β is similar to S and M on f' (Fig.5). Figs. $(6) - (13)$ have been sketched for concentration field ϕ. Figs (6 and 7) show of that ϕ is a decreasing function of S and M but the change in case of S is larger than M. The boundary layer decreases when S and M are increased. The variations of β on the concentration field ϕ for a non-reactive species $\gamma = 0$ is plotted in Fig. 8. We note that without reactive species the concentration field ϕ decreases when β increases. Figs. (9 and 10) show the concentration field ϕ for various values of Deborah number β for destructive $(\gamma > 0)$ and generative $(\gamma < 0)$ chemical reactions, respectively. These Figs. show that ϕ is a decreasing function of β but the change is larger in case of generative chemical reaction $(\gamma < 0)$ in comparison to the case of destructive chemical reaction ($\gamma > 0$). The variation of the destructive chemical reaction parameter ($\gamma > 0$) on the concentration field ϕ is displayed in Fig.11. The concentration boundary layer decreases in case of destructive chemical reaction. Fig. 12 describes the variation of the generative chemical reaction parameter ($\gamma < 0$) on ϕ. The fluid concentration increases with an increase in the generative chemical reaction parameter. . The fluid concentration ϕ has the opposite behavior for ($\gamma > 0$) when compared with generative chemical reaction parameter ($\gamma < 0$). The change in concentration field is larger for the generative chemical reaction. The variations of the Schmidt number Sc on ϕ is plotted in Fig. 13. From Fig. 13 it is obvious that concentration field ϕ is decreases by increasing Sc. The concentration boundary layer also decreases for large values of Sc.

Table 2: Values of the surface mass transfer $-\phi'(0)$ for some values of S , M and β when $Sc = \gamma = 1$

S	M	β	$-\phi'(0)$
0.0	1.5	0.5	0.860477
0.2			0.949625
0.5			1.106254
0.7			1.227578
1.0			1.438716
0.5	1.0		1.060457
	1.2		1.083708
	1.5		1.106254
	2.0		1.131740
	3.0		1.163358
	1.5	0.0	1.108198
		0.2	1.107432
		0.5	1.106254
		0.7	1.105449
		1.0	1.104212
		2.0	1.099823

Table 3: Values of the surface mass transfer $-\phi'(0)$ and the gradient of mass transfer $-\phi'(\eta)$ for some values of Sc and γ when $S = 1$, $M = 1.5$ and $\beta = 0.2$.

Sc	γ	$-\phi'(0)$	η	Sc	γ	$-\phi'(\eta)$
0.2	1.0	0.505403	0.2	0.2	1	0.449991
0.7		1.103298		0.7		0.853461
1.0		1.412999		1.0		1.018820
1.5		1.905536	0.6	0.2		0.361776
2.0		2.386075		0.7		0.535222
5.0		5.254511		1.0		0.566249
1.0	0.1	0.667048	0.2	1.0	0.2	0.645063
	0.7	1.239376			0.7	0.919963
	1.2	1.513555			1.0	1.018821
	1.7	1.731334	0.6		0.2	0.436890
	2.0	1.845883			0.7	0.543046
	3.0	2.173501			1.0	0.566249

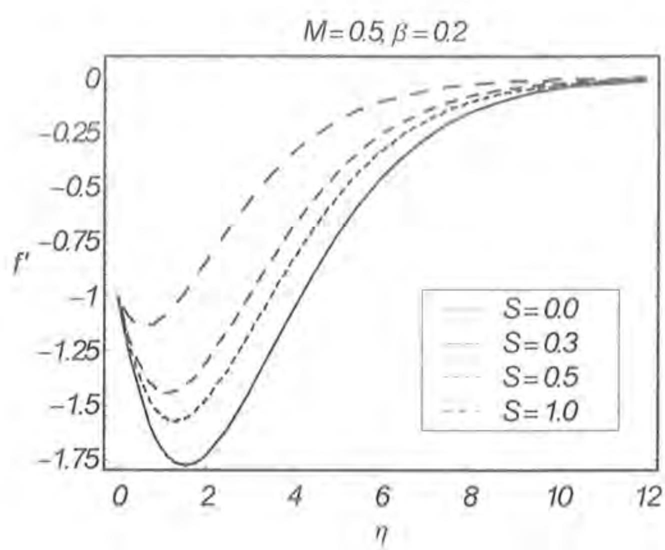

Fig. 3: The variation of suction parameter S on the velocity field f'

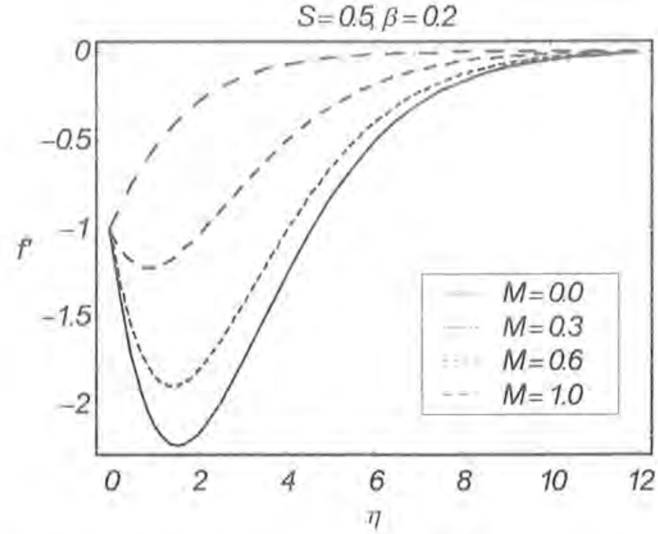

Fig. 4: The variation of Hartman number M on the velocity field f'

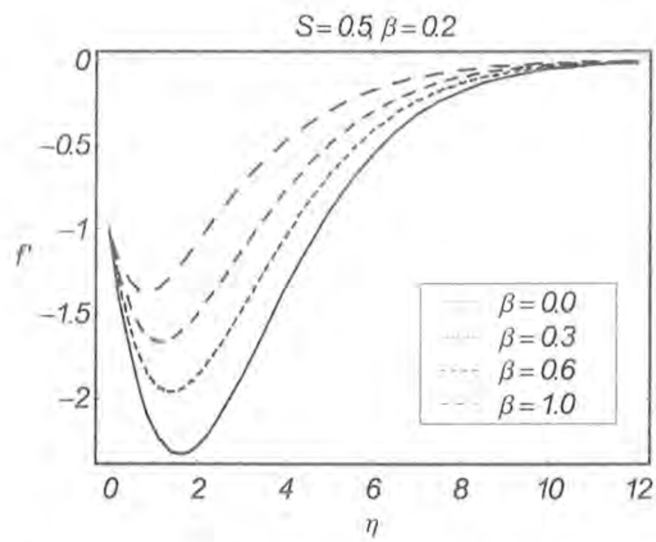

Fig. 5: The variation of Deborah number β on the velocity field f'

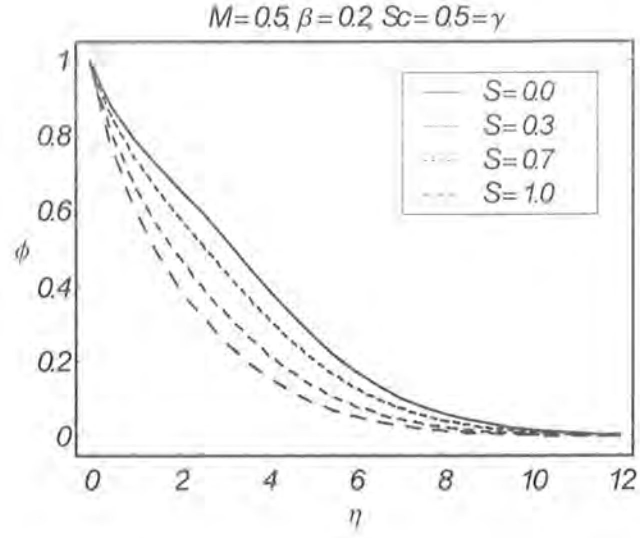

Fig. 6: The variation of suction parameter S on the concentration field ϕ

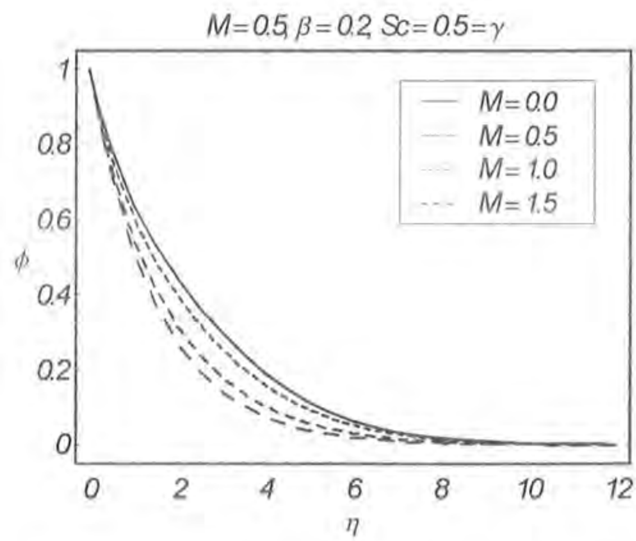

Fig. 7: The variation of Hartman number M on the concentration field ϕ

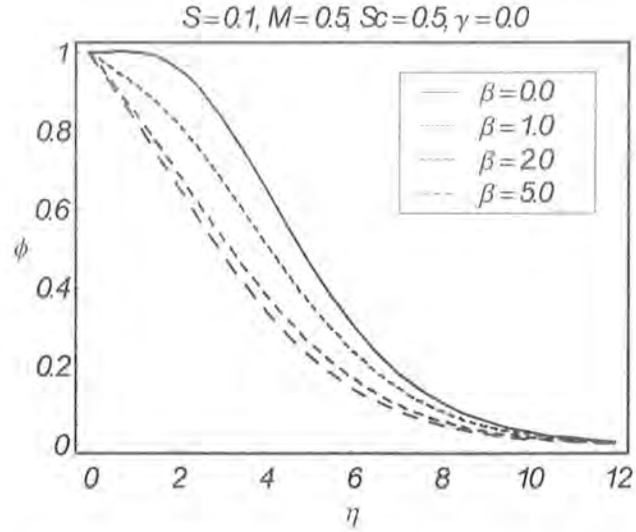

Fig. 8: The variation of Deborah number β on the concentration field ϕ

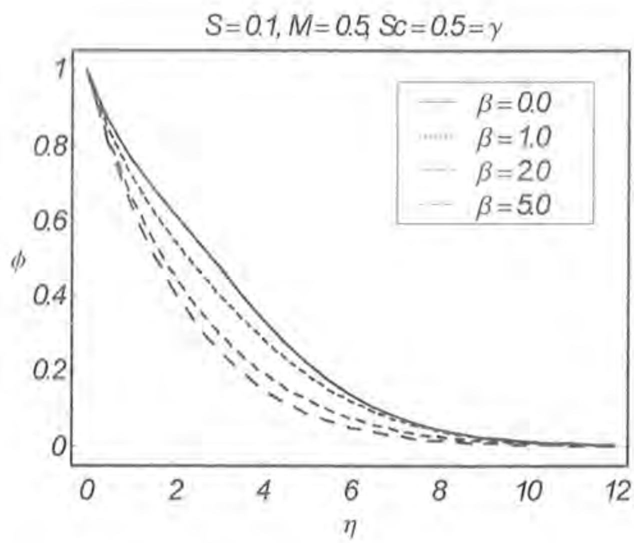

Fig. 9: The variation of Deborah number β on ϕ for destructive chemical reaction

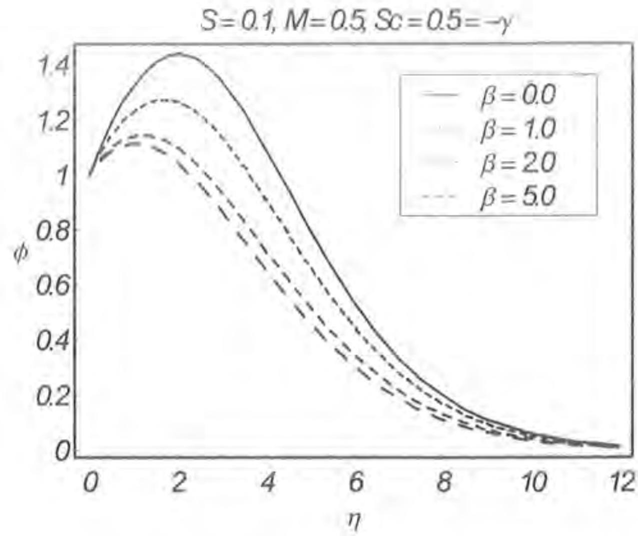

Fig. 10: The variation of Deborah number β on ϕ for generative chemical reaction

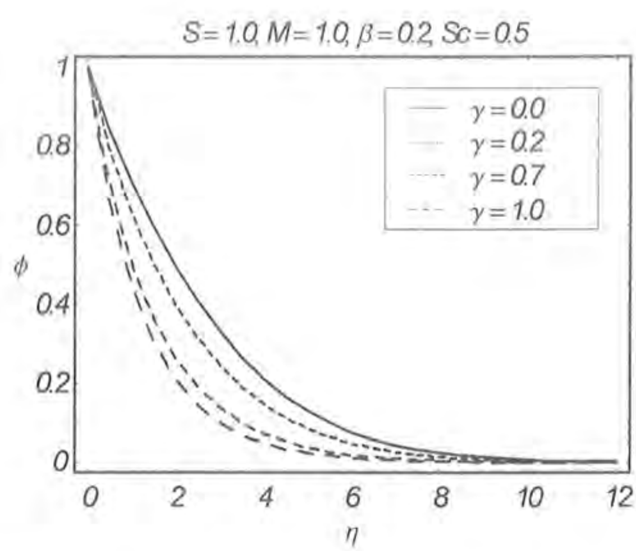

Fig. 11: The variation of destructive chemical reaction γ on the concentration field ϕ

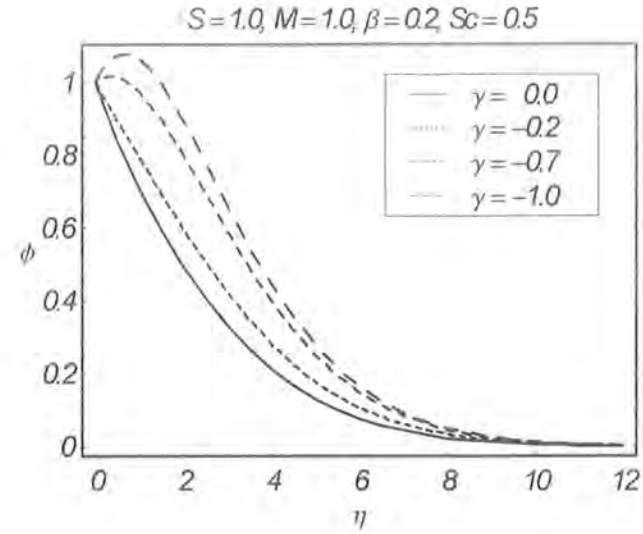

Fig. 12: The variation of generative chemical reaction γ on the concentration field ϕ

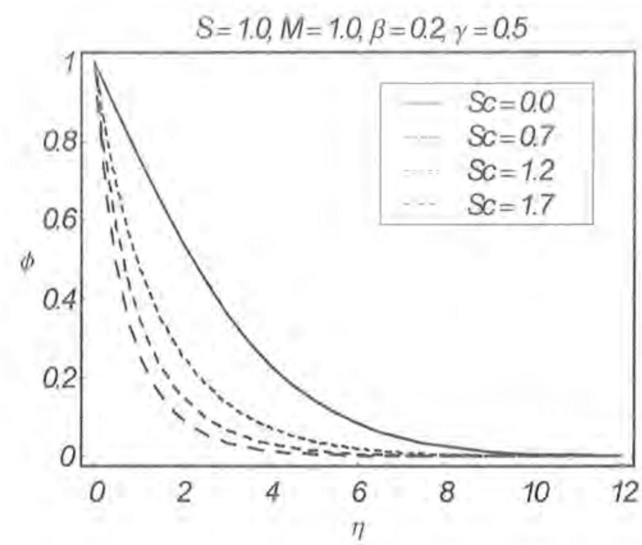

Fig. 13: The variation of Schmidt parameter Sc on the concentration field ϕ

2.5 Concluding remarks

In this study the mass transfer in the MHD flow of an upper-convected Maxwell (UCM) fluid over a porous shrinking sheet with chemical reaction species is investigated. Homotopy analysis method is employed in developing the series solutions. Expressions of velocity $f'$and the concentration field ϕ are determined. From the presented analysis we made the following conclusions

- The velocity field f' is increasing function of S, M, β.
- The concentration field ϕ for destructive/generative chemical reactions decreases when S, M, β and Sc increases.
- The concentration field ϕ has opposite results for destructive ($\gamma > 0$) and generative ($\gamma < 0$) chemical reactions.
- The magnitudes of the surface mass transfer $-\phi'(0)$ is increased by increasing S and M while it decreases when β increases.

Chapter 3

Effects of magnetic field and mass transfer on the flow of a Jeffery fluid over a porous shrinking sheet

3.1 Introduction

This chapter describes the mass transfer of the steady two-dimensional magnetohydrodynamic (MHD) boundary layer flow of a Jeffery fluid past a porous shrinking sheet in the presence of chemical reaction. The governing nonlinear partial differential systems are converted into a nonlinear ordinary differential system. Similar solutions of velocity and concentration fields are obtained using the homotopy analysis method (HAM). The convergence of the obtained series solutions is explicitly discussed. The values of the surface mass transfer and gradient of mass transfer for various parameters are presented. The variations of interesting flow parameters are discussed.

3.2 Mathematical formulation

Let us examine the steady, incompressible, MHD flow Jeffery fluid over a porous shrinking sheet with suction. The sheet coincides with the plane $y = 0$ and the flow occupies the region $y > 0$. The x and y axes are chosen along and perpendicular to the sheet, respectively. A constant magnetic field of strength $\mathbf{B}_0$ acts along the y-axis. The equations which can govern the flow are

$$\nabla.\mathbf{V} = 0, \tag{3.1}$$

$$\rho a_i = \nabla.\mathbf{T}, \tag{3.2}$$

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \tag{3.3}$$

$$\left(1 + \lambda_1 \frac{D}{Dt}\right) \mathbf{S} = \mu \left(1 + \lambda_2 \frac{D}{Dt}\right) \mathbf{A}_1, \tag{3.4}$$

where

$$\mathbf{V} = [u(x, y),\ v(x, y),\ 0], \tag{3.5}$$

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T. \tag{3.6}$$

In above equations $\mathbf{T}$ is Cauchy stress tensor, $\mathbf{S}$ is the extra stress tensor, $\mathbf{I}$ is identity tensor, $\mathbf{A}_1$ is Rivilin-Ericksen tensor, $\mathbf{L}$ is velocity gradient, μ is dynamic viscosity, λ_1 is relaxation time, λ_2 is retardation time and D/Dt is covariant derivative.

In the absence of pressure gradient, the flow is governed by the following equation

$$\rho \left(1 + \lambda_1 \frac{D}{Dt}\right) a_i = \mu \left(1 + \lambda_2 \frac{D}{Dt}\right) \nabla \cdot \mathbf{A}_1, \tag{3.7}$$

where D/Dt is

$$\frac{Da_i}{Dt} = \frac{\partial a_i}{\partial t} + u_r a_{i,r} - u_{i,r} a_r. \tag{3.8}$$

For $i = 1$

$$\left(1 + \lambda_1 \frac{D}{Dt}\right) a_1 = \nu \left(1 + \lambda_2 \frac{D}{Dt}\right) \nabla \cdot \mathbf{A}_1, \tag{3.9}$$

$$\frac{Da_1}{Dt} = \frac{\partial a_1}{\partial t} + u_1 a_{1,1} + u_2 a_{1,2} - u_{1,1} a_1 - u_{1,2} a_2, \tag{3.10}$$

$$a_1 = \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}, \tag{3.11}$$

$$a_2 = \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}, \tag{3.12}$$

$$\nabla \cdot \mathbf{A} = \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \end{bmatrix}. \tag{3.13}$$

Using Eqs. (3.10) – (3.13) in Eq. (3.9) we have

$$\begin{aligned} & u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \lambda_1 \begin{bmatrix} u\frac{\partial}{\partial x}\left\{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right\} + v\frac{\partial}{\partial y}\left\{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right\} \\ -\frac{\partial u}{\partial x}\left\{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right\} - \frac{\partial u}{\partial y}\left\{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right\} \end{bmatrix} \\ & = \nu \left[2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} + \lambda_2 \begin{bmatrix} u\frac{\partial}{\partial x}\left\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right\} + v\frac{\partial}{\partial y}\left\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right\} \\ -\frac{\partial u}{\partial x}\left\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right\} - \frac{\partial u}{\partial y}\left\{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right\} \end{bmatrix} \right], \end{aligned} \tag{3.14}$$

where ν is the kinematic viscosity.

Invoking continuity equation, above equation takes the form

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\lambda_1\left[u^2\frac{\partial^2 u}{\partial x^2}+v^2\frac{\partial^2 u}{\partial y^2}+2uv\frac{\partial^2 u}{\partial x\partial y}\right]$$
$$=\nu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\lambda_2\left[u\frac{\partial^3 u}{\partial x\partial y^2}+v\frac{\partial^3 u}{\partial y^3}-\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}\right]\right]. \tag{3.15}$$

For $i=2$ we obtained

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\lambda_1\left[u^2\frac{\partial^2 v}{\partial x^2}+v^2\frac{\partial^2 v}{\partial y^2}+2uv\frac{\partial^2 v}{\partial x\partial y}\right]$$
$$=\nu\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\lambda_2\left[\begin{array}{c} u\frac{\partial^3 v}{\partial x\partial y^2}+u\frac{\partial^3 v}{\partial x^3}+v\frac{\partial^3 v}{\partial x^2\partial y}+v\frac{\partial^3 v}{\partial y^3} \\ -\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x^2}-\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial y^2}-\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial x^2}-\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y^2}\end{array}\right]\right], \tag{3.16}$$

which by the following boundary layer approximations

$$u=O(1),\ x=O(1),\ v=O(\delta),\ y=O(\delta), \tag{3.17}$$

is identically satisfied. Furthermore, Eq. (3.15) in view of Eq. (3.17) reduced to

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\lambda_1\left[u^2\frac{\partial^2 u}{\partial x^2}+v^2\frac{\partial^2 u}{\partial y^2}+2uv\frac{\partial^2 u}{\partial x\partial y}\right]$$
$$=\nu\left[\frac{\partial^2 u}{\partial y^2}+\lambda_2\left[u\frac{\partial^3 u}{\partial x\partial y^2}+v\frac{\partial^3 u}{\partial y^3}-\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}\right]\right]. \tag{3.18}$$

For magnetohydrodynamic case, the above equation modifies in the following form

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\lambda_1\left[u^2\frac{\partial^2 u}{\partial x^2}+v^2\frac{\partial^2 u}{\partial y^2}+2uv\frac{\partial^2 u}{\partial x\partial y}\right]$$
$$=\nu\left[\frac{\partial^2 u}{\partial y^2}+\lambda_2\left[u\frac{\partial^3 u}{\partial x\partial y^2}+v\frac{\partial^3 u}{\partial y^3}-\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}\right]\right]-\frac{\sigma B_0^2}{\rho}u. \tag{3.19}$$

Furthermore mass transfer is the flow along a sheet that contains a species A slightly soluble in the fluid B. Let C_w be the concentration at the sheet surface and the solubility of A in B and concentration of A far away from the sheet is C_∞. Also the reaction of a species A with B be the first order homogeneous chemical reaction of rate constant K_1. The concentration of dissolved A is considered small enough. Through boundary layer approximations, the governing equation for concentration field is

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\frac{\partial^2 C}{\partial y^2} - K_1 C, \tag{3.20}$$

where C, D and δ are respectively the concentration of the species, the diffusion coefficient of the diffusing species in the fluid and the boundary layer thickness.

The subjected boundary conditions are

$$u(x,y) = -cx,\ v(x,y) = -V_0,\ C(x,y) = C_w \quad at\ \ y = 0, \tag{3.21}$$

$$u(x,y) \to 0,\ C(x,y) \to C_\infty \quad as\ \ y \to \infty, \tag{3.22}$$

where $c > 0$ is the rate of shrinking and $V_0 > 0$ is the suction velocity at the surface. Defining

$$\eta = \sqrt{\frac{c}{\nu}}y,\ u = cxf'(\eta),\ v = -\sqrt{c\nu}f(\eta), \phi = \frac{C - C_\infty}{C_w - C_\infty}, \tag{3.23}$$

equation (3.1) is identically satisfied and Eqs. (3.19) – (3.22) are

$$f''' - M^2 f' - f'^2 + f\, f'' + \beta_1 \left(2ff'f'' - f^2 f'''\right) + \beta_2 \left(f''^2 - ff''''\right) = 0, \tag{3.24}$$

$$\phi'' + Scf\phi' - Sc\gamma\phi = 0, \tag{3.25}$$

$$f = S,\ f' = -1,\ \phi = 1 \quad \text{at} \quad \eta = 0, \tag{3.26}$$

$$f' = 0,\ \phi = 0 \quad \text{at} \quad \eta = \infty, \tag{3.27}$$

where

$$S = \frac{V_0}{\sqrt{\nu c}},\ M^2 = \frac{\sigma B_0^2}{\rho c},\ \beta_1 = \lambda_1 c,\ \beta_2 = \lambda_2 c,\ Sc = \frac{\nu}{D},\ \gamma = \frac{K_1}{c}. \tag{3.28}$$

In above expressions S, M, Sc and γ are the suction, Hartman, Schmidt and chemical reaction parameters respectively. Moreover β_1 and β_2 are the Deborah numbers in terms of relaxation and retardation times respectively. It is noted that for destructive/generative chemical reaction $\gamma > 0/\gamma < 0$ respectively and $\gamma = 0$ corresponds to non-reactive species. The surface mass transfer is

$$\phi'(0) = \left(\frac{\partial\phi}{\partial\eta}\right)_{\eta=0}. \tag{3.29}$$

3.2.1 Solution by homotopy analysis method (HAM)

For the HAM solution, we select

$$f_0(\eta) = S - 1 + \exp(-\eta),\ g_0(\eta) = \exp(-\eta), \tag{3.30}$$

with the following operator

$$\mathcal{L}_f = f''' - f', \; \mathcal{L}_\phi = \phi'' - \phi, \tag{3.31}$$

which satisfy

$$\mathcal{L}_f \left(C_1 + C_2 e^{\eta} + C_3 e^{-\eta}\right) = 0, \;\; \mathcal{L}_\phi \left(C_4 e^{\eta} + C_5 e^{-\eta}\right) = 0, \tag{3.32}$$

where C_i $(i = 1 - 5)$ are the arbitrary constants.

3.2.2 Zeroth order deformation equation

The problems at the zeroth order are

$$(1-p)\,\mathcal{L}_f \left[\widehat{f}\,(\eta;p) - f_0(\eta)\right] = p h_f N_f \left[\widehat{f}(\eta;p),\; \widehat{\phi}(\eta;p)\right], \tag{3.33}$$

$$(1-p)\mathcal{L}_\phi \left[\widehat{\phi}(\eta;p) - \phi_0(\eta)\right] = p h_\phi N_\phi \left[\widehat{f}(\eta;p),\; \widehat{\phi}(\eta;p)\right], \tag{3.34}$$

$$\widehat{f}(0;p) = S,\; \widehat{f}'(0;p) = -1,\; \widehat{f}'(\infty,p) = 0, \quad \widehat{\phi}(0;p) = 1,\; \widehat{\phi}(\infty;p) = 0, \tag{3.35}$$

where p is an embedding parameter and h_f, h_ϕ are the non-zero auxiliary parameters. Furthermore the non-linear operators are

$$N_f[f(\eta;p)] = \frac{\partial^3 f(\eta;p)}{\partial \eta^3} - M^2 \frac{\partial f(\eta;p)}{\partial \eta} + f(\eta;p) \frac{\partial^2 f(\eta;p)}{\partial \eta^2} - \left(\frac{\partial f(\eta;p)}{\partial \eta}\right)^2$$
$$+ \beta_1 \left(2 f(\eta;p) \frac{\partial f(\eta;p)}{\partial \eta} \frac{\partial^2 f(\eta;p)}{\partial \eta^2} - f^{\,2}(\eta;p) \frac{\partial^3 f(\eta;p)}{\partial \eta^3}\right)$$
$$+ \beta_2 \left(\frac{\partial^2 f(\eta;p)}{\partial \eta^2} \frac{\partial^2 f(\eta;p)}{\partial \eta^2} - f\,(\eta;p) \frac{\partial^4 f(\eta;p)}{\partial \eta^4}\right), \qquad (3.36)$$

$$N_\phi[\phi(\eta;p), f(\eta;p)] = \frac{\partial^2 \phi(\eta;p)}{\partial \eta^2} + Scf(\eta;p) \frac{\partial \phi(\eta;p)}{\partial \eta} - Sc\gamma\phi(\eta;p). \qquad (3.37)$$

For $p = 0$ and $p = 1$ we have

$$\widehat{f}\,(\eta;0) = f_0\,(\eta)\,,\ \widehat{\phi}\,(\eta;0) = \phi_0\,(\eta) \text{ and } \widehat{f}\,(\eta;1) = f\,(\eta)\,,\ \widehat{\phi}\,(\eta;1) = \phi\,(\eta)\,. \qquad (3.38)$$

Note that when p increases from 0 to 1 then $f(\eta;p)$and $\phi\,(\eta;p)$ varies from the initial guesses $f_0\,(\eta)$, $\phi_0\,(\eta)$ to the final solutions $f\,(\eta)$ and $\phi\,(\eta)$. Using Taylor series we may write

$$f(\eta;p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta) p^m, \qquad (3.39)$$

$$\phi(\eta;p) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta) p^m, \qquad (3.40)$$

$$f_m(\eta) = \frac{1}{m!} \frac{\partial^m f(\eta;p)}{\partial \eta^m}\bigg|_{p=0}, \qquad \phi_m(\eta) = \frac{1}{m!} \frac{\partial^m \phi(\eta;p)}{\partial \eta^m}\bigg|_{p=0}. \qquad (3.41)$$

Obviously Eqs. (3.36) and (3.37) have two non-zero auxiliary parameters h_f and h_ϕ. The convergence of the series (3.42) and (3.43) is dependent upon h_f and h_ϕ. The values of $h_i(i = f,$ $\phi)$ are chosen properly so that Eqs. (3.39) and (3.40) are convergent at $p = 1$. In view of Eq. (3.38) we have

$$
\begin{aligned}
f(\eta) &= f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), && (3.42)\\
\phi(\eta) &= \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta). && (3.43)
\end{aligned}
$$

3.2.3 mth order deformation equations

The m^{th} order deformation problems are obtained by differentiating Eqs. (3.33) and (3.34) m times with respect to p and then setting $p = 0$. These are given by

$$
\mathcal{L}_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = h_f R_m^f(\eta), \qquad (3.44)
$$

$$
\mathcal{L}_f[\phi_m(\eta) - \chi_m \phi_{m-1}(\eta)] = h_\phi R_m^\phi(\eta), \qquad (3.45)
$$

$$
f_m(0) = f_m'(0) = f_m'(\infty) = 0 \quad \text{and} \quad \phi_m(0) = \phi_m(\infty) = 0, \qquad (3.46)
$$

$$
\begin{aligned}
R_m^f(\eta) = {} & f_{m-1}''' - M^2 f_{m-1}' + \sum_{k=0}^{m-1} [f_{m-1-k} f_k'' - f_{m-1-k}' f_k'] \\
& + \beta_1 \sum_{k=0}^{m-1} f_{m-1-k} \sum_{l=0}^{k} [2 f_{k-l}' f_l'' - f_{k-l} f_l'''] \\
& + \beta_2 \sum_{k=0}^{m-1} f_{m-1-k}'' f_k'' - f_{m-1-k} f_k''''], \qquad (3.47)
\end{aligned}
$$

$$
R_m^\phi(\eta) = \phi_{m-1}''(\eta) - Sc\gamma\phi_{m-1} + Sc \sum_{k=0}^{m-1} \phi_{m-1-k}' f_k, \qquad (3.48)
$$

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{3.49}$$

Denoting $f_m^*(\eta)$ and $\phi_m^*(\eta)$ as the special solutions, we have the following general solutions

$$f_m(\eta) = f_m^*(\eta) + C_1 + C_2 \exp(\eta) + C_3 \exp(-\eta), \tag{3.50}$$

$$\phi_m(\eta) = \phi_m^*(\eta) + C_4 \exp(\eta) + C_5 \exp(-\eta), \tag{3.51}$$

where

$$\begin{aligned} C_2 &= C_4 = 0,\ C_3 = \left.\frac{\partial f_m^*(\eta)}{\partial \eta}\right|_{\eta=0}, \\ C_1 &= -C_3 - f_m^*(0),\ C_5 = -\phi_m^*(0). \end{aligned} \tag{3.52}$$

The symbolic computation software MATHEMATICA is employed to obtain series solutions upto first few order of approximations. The relevant series solutions can be written in the form

$$f_m(\eta) = \sum_{n=0}^{2m+1} \sum_{q=0}^{2m+1-n} a_{m,n}^q \eta^q \exp(-n\eta), \quad m \geq 0, \tag{3.53}$$

$$\phi_m(\eta) = \sum_{n=0}^{2m+1} \sum_{q=0}^{2m+1-n} b_{m,n}^q \eta^q \exp(-n\eta), \tag{3.54}$$

in which the coefficients $a_{m,n}^q$ and $b_{m,n}^q$ can be determined in the next section.

3.2.4 Derivation of the coefficients

First of all we calculate the derivatives appearing in the Eqs. (3.47) and (3.48). From Eqs (3.53) and (3.54)we have the coefficients of the form

$$a1^q_{m,n} = \left[(q+1)a^{q+1}_{m,n} - na^q_{m,n}\right], \tag{3.55}$$

$$b1^q_{m,n} = \left[(q+1)b^{q+1}_{m,n} - nb^q_{m,n}\right] \tag{3.56}$$

$$a2^q_{m,n} = \left[(q+1)a1^{q+1}_{m,n} - na1^q_{m,n}\right], \tag{3.57}$$

$$a3^q_{m,n} = \left[(q+1)a2^{q+1}_{m,n} - na2^q_{m,n}\right], \tag{3.58}$$

$$a4^q_{m,n} = \left[(q+1)a3^{q+1}_{m,n} - na3^q_{m,n}\right], \tag{3.59}$$

$$b2^q_{m,n} = \left[(q+1)b1^{q+1}_{m,n} - nb1^q_{m,n}\right]. \tag{3.60}$$

For the product terms in Eqs. (3.47) and (3.48), the coefficients are

$$\alpha^q_{m,n} = \sum_{k=0}^{m-1} \sum_{i=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+1\}} \sum_{j=\max\{0,q-2m+2k+1+n-i\}}^{\min\{q,2k+1-i\}} a^{q-j}_{m-1-k,n-i} a2^j_{k,i}, \tag{3.61}$$

$$\alpha1^q_{m,n} = \sum_{k=0}^{m-1} \sum_{i=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+1\}} \sum_{j=\max\{0,q-2m+2k+1+n-i\}}^{\min\{q,2k+1-i\}} a1^{q-j}_{m-1-k,n-i} a1^j_{k,i}, \tag{3.62}$$

$$\alpha2^q_{m,n} = \sum_{k=0}^{m-1} \sum_{i=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+1\}} \sum_{j=\max\{0,q-2m+2k+1+n-i\}}^{\min\{q,2k+1-i\}} a^{q-j}_{m-1-k,n-i} a4^j_{k,i}, \tag{3.63}$$

$$\alpha3^q_{m,n} = \sum_{k=0}^{m-1} \sum_{i=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+1\}} \sum_{j=\max\{0,q-2m+2k+1+n-i\}}^{\min\{q,2k+1-i\}} a2^{q-j}_{m-1-k,n-i} a2^j_{k,i}, \tag{3.64}$$

$$\alpha4^q_{m,n} = \sum_{k=0}^{m-1} \sum_{i=\max\{0,n-2m+2k+2\}}^{\min\{n,2k+1\}} \sum_{j=\max\{0,q-2m+2k+2+n-i\}}^{\min\{q,2k+1-i\}} b1^{q-j}_{m-1-k,n-i} a^j_{k,i}. \tag{3.65}$$

For the product of three functions term in Eqs. (3.47) the coefficients are

$$\alpha5^t_{k,p} = \sum_{k=0}^{m-1} \sum_{i=\max\{0,p-2k+2l-1\}}^{\min\{p,2l+1\}} \sum_{j=\max\{0,t-2k+2l-1+p-i\}}^{\min\{t,2l+1-i\}} a1^{t-j}_{k-l,p-i} a2^j_{l,i}, \tag{3.66}$$

$$\alpha6^t_{k,p} = \sum_{k=0}^{m-1} \sum_{i=\max\{0,p-2k+2l-1\}}^{\min\{p,2l+1\}} \sum_{j=\max\{0,t-2k+2l-1+p-i\}}^{\min\{t,2l+1-i\}} a^{t-j}_{k-l,p-i} a3^j_{l,i}, \tag{3.67}$$

$$\alpha 7_{k,p}^{t} = \sum_{k=0}^{m-1} \sum_{p=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+2\}} \sum_{t=\max\{0,q-2m+2k+1+n-p\}}^{\min\{q,2k+2-p\}} a_{m-1-k,n-p}^{q-t} \alpha 3_{l,i}^{j}, \quad (3.68)$$

$$\alpha 8_{k,p}^{t} = \sum_{k=0}^{m-1} \sum_{p=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+2\}} \sum_{t=\max\{0,q-2m+2k+1+n-p\}}^{\min\{q,2k+2-p\}} a_{m-1-k,n-p}^{q-t} \alpha 4_{l,i}^{j}. \quad (3.69)$$

Using the above relations in Eqs. (3.47) and (3.48) we get

$$\begin{aligned} h_f R_m^f(\eta) = {} & h_f \sum_{n=0}^{2m-1} \sum_{q=0}^{2m-1-n} \left[a3_{m-1,n}^{q} - M^2 a1_{m-1,n}^{q} \right] \eta^q e^{-n\eta} \\ & + h_f \sum_{n=0}^{2m} \sum_{q=0}^{2m-n} \left[\alpha_{m,n}^{q} - \alpha 1_{m,n}^{q} \right] \eta^q e^{-n\eta} \\ & + h_f \sum_{n=0}^{2m+1} \sum_{q=0}^{2m+1-n} \left[2\,(\beta)\, \alpha 7_{m,n}^{q} - (\beta) \alpha 8_{m,n}^{q} \right] \eta^q e^{-n\eta} \\ & + h_f \sum_{n=0}^{2m} \sum_{q=0}^{2m-n} \left[\alpha 3_{m,n}^{q} - \alpha 2_{m,n}^{q} \right] \eta^q e^{-n\eta}, \end{aligned} \quad (3.70)$$

$$\begin{aligned} h_\phi R_m^\phi(\eta) = {} & h_\phi \sum_{n=0}^{2m-1} \sum_{q=0}^{2m-1-n} \left[b2_{m-1,n}^{q} - Sc\gamma b_{m-1,n}^{q} \right] \eta^q e^{-n\eta} \\ & + h_\phi \sum_{n=0}^{2m} \sum_{q=0}^{2m-n} \left[(Sc)\, \alpha 4_{m,n}^{q} \right] \eta^q e^{-n\eta}, \end{aligned} \quad (3.71)$$

$$h_f R_m^f(\eta) = \sum_{n=0}^{2m+1} \sum_{q=0}^{2m+1-n} h_f \left[\begin{array}{c} \chi_{2m+1-n-q}\{a3_{m-1,n}^{q} - M^2 a1_{m-1,n}^{q}\} \\ +\chi_{2m+1-n-q}\{\alpha_{m,n}^{q} - \alpha 1_{m,n}^{q}\} \\ +2(\beta)\alpha 5_{m,n}^{q} - (\beta)\alpha 6_{m,n}^{q} \\ +\chi_{2m+1-n-q}\{\alpha 3_{m,n}^{q} - \alpha 21_{m,n}^{q}\} \end{array} \right] \eta^q e^{-n\eta}, \quad (3.72)$$

$$h_\phi R_m^\phi(\eta) = \sum_{n=0}^{2m} \sum_{q=0}^{2m-n} h_\phi \left[\begin{array}{c} \chi_{2m+1-n-q}\{b2_{m-1,n}^{q} - Sc\gamma b_{m-1,n}^{q}\} \\ +(Sc)\, \alpha 2_{m,n}^{q} \end{array} \right] \eta^q e^{-n\eta}, \quad (3.73)$$

or we can write

$$h_f R_m^f(\eta) = \sum_{n=0}^{2m+1}\sum_{q=0}^{2m+1-n} \Delta_{m,n}^q \eta^q e^{-n\eta}, \tag{3.74}$$

$$h_\phi R_m^\phi(\eta) = \sum_{n=0}^{2m}\sum_{q=0}^{2m-n} \Gamma_{m,n}^q \eta^q e^{-n\eta}, \tag{3.75}$$

in which $\Delta_{m,n}^q$ and $\Gamma_{m,n}^q$ are given by

$$\Delta_{m,n}^q = h_f \begin{bmatrix} \chi_{2m+1-n-q}\{a3_{m-1,n}^q - M^2 a1_{m-1,n}^q\} \\ +\chi_{2m+1-n-q}\{\alpha_{m,n}^q - \alpha 1_{m,n}^q\} \\ +2(\beta)\alpha 7_{m,n}^q - (\beta)\alpha 8_{m,n}^q \\ +\chi_{2m+1-n-q}\{\alpha 3_{m,n}^q - \alpha 2_{m,n}^q\} \end{bmatrix}, \tag{3.76}$$

$$\Gamma_{m,n}^q = h_\varphi \begin{bmatrix} \chi_{2m+1-n-q}\{b2_{m-1,n}^q - Sc\gamma b_{m-1,n}^q\} \\ +(Sc)\,\alpha 4_{m,n}^q \end{bmatrix}. \tag{3.77}$$

From Eqs. (3.47), (3.48), (3.74) and (3.75) we arrive at

$$\mathcal{L}_f\left[f_m(\eta) - \chi_m f_{m-1}(\eta)\right] = \sum_{n=0}^{2m+1}\sum_{q=0}^{2m+1-n} \Delta_{m,n}^q \eta^q e^{-n\eta}, \tag{3.78}$$

$$\mathcal{L}_\varphi\left[\phi_m(\eta) - \chi_m \phi_{m-1}(\eta)\right] = \sum_{n=0}^{2m}\sum_{q=0}^{2m-n} \Gamma_{m,n}^q \eta^q e^{-n\eta}. \tag{3.79}$$

Through inverse of linear operators on both sides, we have

$$f_m(\eta) - \chi_m f_{m-1}(\eta) = \sum_{n=0}^{2m+1}\sum_{q=0}^{2m+1-n}\sum_{k=1}^{q+1} \Delta_{m,n}^q \mu 1_{n,k}^q \eta^q e^{-n\eta} + C_1^m + C_2^m e^\eta + C_3^m e^{-\eta} \tag{3.80}$$

$$\phi_m(\eta) - \chi_m \phi_{m-1}(\eta) = \sum_{n=0}^{2m}\sum_{q=0}^{2m-n}\sum_{k=1}^{q+1} \Gamma_{m,n}^q \mu 2_{n,k}^q \eta^q e^{-n\eta} + C_4^m e^\eta + C_5^m e^{-\eta}, \tag{3.81}$$

where C_i $(i = 1-5)$ are the constants of integration. Employing boundary conditions one has

$$C_2^m = 0, \; C_4^m = 0. \tag{3.82}$$

$$C_3^m = \sum_{q=0}^{2m+1} \Delta_{m,0}^q \mu 1_{0,1}^q + \sum_{q=0}^{2m} \Delta_{m,1}^q (\mu 1_{1,1}^q - \mu 1_{1,0}^q) + \sum_{n=2}^{2m+1} \sum_{q=0}^{2m+1-n} \Delta_{m,n}^q (\mu 1_{n,1}^q - \mu 1_{n,0}^q), \tag{3.83}$$

$$C_1^m = -\sum_{q=0}^{2m+1} \Delta_{m,0}^q \mu 1_{0,1}^q + \sum_{q=0}^{2m+1} \Delta_{m,1}^q \mu 1_{1,1}^q + \sum_{n=2}^{2m+1} \sum_{q=0}^{2m+1-n} \Delta_{m,n}^q \{(\mu 1_{n,1}^q - (n-1)\, \mu 1_{n,0}^q)\}, \tag{3.84}$$

$$C_5^m = -\left[\sum_{q=0}^{2m+1} \Gamma_{m,0}^q \mu 2_{0,0}^q + \sum_{q=0}^{2m} \Gamma_{m,1}^q \mu 2_{1,0}^q + \sum_{n=2}^{2m+1} \sum_{q=0}^{2m+1-n} \Gamma_{m,n}^q \mu 2_{n,0}^q\right], \tag{3.85}$$

$$\mu 1_{n,k}^q = \sum_{r=0}^{q-k} \sum_{p=0}^{q-k-r} \frac{-q!}{k!\,(n-1)^{q+1-k-r-p}\, n^{r+1}\,(n+1)^{p+1}}, \tag{3.86}$$

$$\mu 2_{n,k}^q = \sum_{p=0}^{q-k} \frac{q!}{k!\,(n-1)^{q+1-k-p}\,(n+1)^{p+1}}. \tag{3.87}$$

Substitution of values of constants into Eqs. (3.80) and (3.81) one may write

$$\begin{aligned}
&\sum_{n=0}^{2m+1} \sum_{q=0}^{2m+1-n} \left[a_{m,n}^q - \chi_{m+2-n}\chi_{m+2-n-q} a_{m-1,n}^q\right] \eta^q e^{-n\eta} = \sum_{n=0}^{2m+1} \sum_{q=0}^{2m+1-n} \sum_{k=1}^{q+1} \Delta_{m,n}^q \mu 1_{n,k}^q \eta^q e^{-n\eta} \\
&+ \left[\sum_{q=0}^{2m+1} \Delta_{m,0}^q \mu 1_{0,1}^q + \sum_{q=0}^{2m} \Delta_{m,1}^q (\mu 1_{1,1}^q - \mu 1_{1,0}^q) + \sum_{n=2}^{2m+1} \sum_{q=0}^{2m+1-n} \Delta_{m,n}^q (\mu 1_{n,1}^q - \mu 1_{n,0}^q)\right] e^{-\eta} \\
&- \left[+ \sum_{q=0}^{2m+1} \Delta_{m,0}^q \mu 1_{0,1}^q - \sum_{q=0}^{2m} \Delta_{m,1}^q \mu 1_{1,1}^q - \sum_{n=2}^{2m+1} \sum_{q=0}^{2m+1-n} \Delta_{m,n}^q \{(\mu 1_{n,1}^q - (n-1)\, \mu 1_{n,0}^q)\}\right]
\end{aligned} \tag{3.88}$$

$$\sum_{n=0}^{2m+1}\sum_{q=0}^{2m+1-n}\left[b^q_{m,n}-\chi_{m+2-n}\chi_{m+2-n-q}b^q_{m-1,n}\right]\eta^q e^{-n\eta}=\sum_{n=0}^{2m+1}\sum_{q=0}^{2m+1-n}\sum_{k=1}^{q+1}\Gamma^q_{m,n}\mu 2^q_{n,k}\eta^q e^{-n\eta}$$
$$-\left[\sum_{q=0}^{2m+1}\Gamma^q_{m,0}\mu 2^q_{0,0}+\sum_{q=0}^{2m}\Gamma^q_{m,1}\mu 2^q_{1,0}+\sum_{n=2}^{2m+1}\sum_{q=0}^{2m+1-n}\Gamma^q_{m,n}\mu 2^q_{n,0}\right]e^{-\eta}. \tag{3.89}$$

Comparing like powers of η in the above equations we arrive at

$$a^0_{m,0} = \chi_m\chi_{m+2}\chi_{m+2-q}a^0_{m-1,0}-\sum_{q=0}^{2m}\Delta^q_{m,0}\mu 1^q_{0,1}-\sum_{n=2}^{2m+1}\sum_{q=0}^{2m+1-n}\Delta^q_{m,n}(\mu 1^q_{n,1}-(n-1)\,\mu 1^q_{n,0}), \tag{3.90}$$

$$a^0_{m,1} = \chi_m\chi_{m+1}\chi_{m+1-q}a^0_{m-1,0}-\sum_{q=0}^{2m}\Delta^q_{m,1}\mu 1^q_{1,1}-\sum_{n=2}^{2m+1}\sum_{q=0}^{2m+1-n}\Delta^q_{m,n}(\mu 1^q_{n,1}-n\mu 1^q_{n,0}), \tag{3.91}$$

$$a^k_{m,n}=\chi_m\chi_{m+2-n}\chi_{m+2-n-q}a^k_{m-1,n}-\sum_{q=0}^{m+1-n}\Delta^q_{m,n}\mu 1^q_{n,k}-\sum_{n=2}^{2m+1}\sum_{q=k}^{2m+1-n}\Delta^q_{m,n}(\mu 1^q_{n,1}-(n-1)\,\mu 1^q_{n,0}),\quad n\geq 2. \tag{3.92}$$

Similarly

$$b^0_{m,1}=\chi_m\chi_{m+1}\chi_{m+1-q}b^0_{m-1,1}-\sum_{n=2}^{2m+1}\sum_{q=0}^{2m+1-n}\Gamma^q_{m,n}\mu 2^q_{n,0}, \tag{3.93}$$

$$b^k_{m,n}=\chi_m\chi_{m+1}\chi_{m+1-q}b^k_{m-1,n}-\sum_{q=k}^{2m+1-n}\Gamma^q_{m,n}\mu 2^q_{n,k},\quad n\geq 2. \tag{3.94}$$

The coefficients $a^q_{m,n}$ and $b^q_{m,n}$ are calculated by using the above recurrence relations. These have been presented by the initial guess in Eq. (3.30). The values of coefficients are

$$a^0_{0,0} = S-1,\ a^0_{0,1}=1,\ a^1_{0,0}=1, \tag{3.95}$$

$$b^0_{0,0} = 0,\ b^1_{0,0}=0,\ b^0_{0,1}=1 \tag{3.96}$$

Therefore the analytic solutions are

$$\begin{aligned}
f(\eta) &= \sum_{m=0}^{\infty} f_m(\eta) \\
&= \lim_{M\to\infty}\left[\sum_{m=0}^{M} a_{m,0}^{0} + \sum_{n=1}^{M+1} e^{-n\eta}\left(\sum_{m=n-1}^{M}\sum_{q=0}^{2m+1-n} a_{m,n}^{q}\eta^{q}\right)\right], \quad (3.97)
\end{aligned}$$

$$\begin{aligned}
\phi(\eta) &= \sum_{m=0}^{\infty} \phi_m(\eta) \\
&= \lim_{M\to\infty}\left[\sum_{m=0}^{M} b_{m,0}^{0} + \sum_{n=1}^{M+1} e^{-n\eta}\left(\sum_{m=n-1}^{M}\sum_{q=0}^{2m+1-n} b_{m,n}^{q}\eta^{q}\right)\right]. \quad (3.98)
\end{aligned}$$

3.3 Convergence of HAM solution

The series solutions of the considered problems are given in Eqs. (3.97) and (3.96). The convergence of these solutions depend upon the values of the parameters h_f and h_ϕ. To see the admissible values of h_f and h_ϕ, the h curves are displayed for 15th order of approximation in Fig. 1. From Fig. 1 it is evident that admissible values of h_f is $-1.1 \leq h_f \leq -0.25$ and h_ϕ is $-1.5 \leq h_\phi \leq -0.3$. Our computations depict that the solutions series (3.97) and (3.96) converge in the whole region of η when $h_f = h_\phi = -0.4$. Table 1 is made just to decide that how much order of approximations is necessary for a convergent solution. It is found that 25th order approximations are sufficient in the present problem.

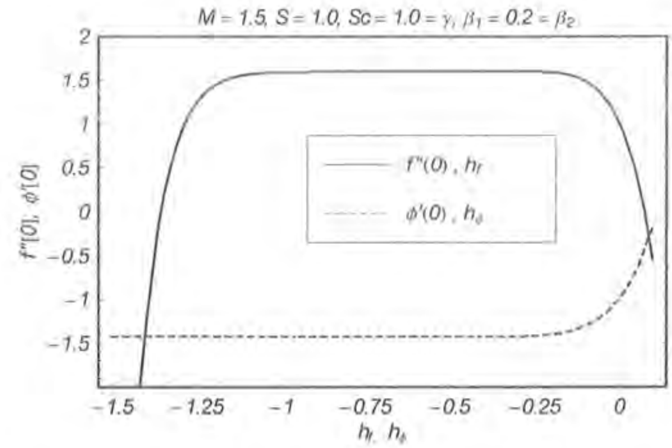

Fig. 1: h curves of $f''(0)$ and $\phi'(0)$ at 15th order of approximation.

Table 1: Convergence of the HAM solutions for different order of approximations when $S = 1$, $M = 1.5$, $Sc = \gamma = 1$ and $\beta_1 = \beta_2 = 0.2$.

Order of approximation	$f''(0)$	$-\phi'(0)$
1	1.230000	1.133333
2	1.382294	1.227511
5	1.571364	1.369794
10	1.600672	1.423341
15	1.598050	1.428984
20	1.597998	1.429452
25	1.598038	1.429492
27	1.598035	1.429492
30	1.598035	1.429492
35	1.598035	1.429492
40	1.598035	1.429492

3.4 Results and discussion

In this section Figs. $2-16$ are displayed for the influence of parameters S, M, β_1, β_2 and Sc and the chemical reaction parameter γ on the velocity f' and the concentration field ϕ. The surface mass transfer $\phi'(0)$ and the gradient of mass transfer $-\phi'(\eta)$ for several values of the emerging parameters are also given in Tables (2 and 3), respectively. Figs. $(2-5)$ depict the effects of S, M, β_1 and β_2 on the velocity component f'. It is worth mentioning that the magnitude of the velocity decreases when S increases. The boundary layer thickness also decreases by increasing S. We noticed from Fig. 3 that the magnitude of the velocity f' decreases for large values of M. It further needs to be mentioned that the boundary layer thickness is decreasing function of M. The effects of the Deborah numbers β_1 and β_2 are similar to S and M on f' (Figs. 4 and 5). It is seen that f' has the similar results for the large values of β_1 and β_2. This change in the velocity is larger in case of suction. The variations of emerging parameters on the concentration field ϕ are shown in Figs. $6-16$. It can be seen from Fig. 6 that ϕ is a decreasing function of S and the concentration boundary layer also decreases when S increases. From Fig. 7 we observe that ϕ decreases for large values of M. The concentration boundary layer also decreases when M increases. Figs. (8 and 9) depict the variations of β_1 and β_2 on the concentration field ϕ for a non-reactive species $\gamma = 0$. We can see that without reactive species the concentration field ϕ is decreased as β_1 and β_2 increases. Figs. $10-13$ plot the distributions of the concentration field ϕ for various values of the Deborah number β_1 and β_2 in the case of destructive $(\gamma > 0)$ and generative $(\gamma < 0)$ chemical reactions, respectively. These Figs. display that ϕ is a decreasing function of β_1 and β_2.The concentration field ϕ also decreases for large values of β_1 and β_2 in case of generative chemical reaction $(\gamma < 0)$. The magnitude of ϕ is larger in case of $(\gamma < 0)$ when compared with the case of destructive chemical reaction $(\gamma > 0)$. We found that the concentration field is decreased for several values of β_1 and β_2 in all cases $(\gamma = 0, \gamma > 0$ and $\gamma < 0)$. The influence of the generative chemical reaction parameter $(\gamma < 0)$ on ϕ is illustrated in Fig.14. Obviously fluid concentration increases with an increase in the generative chemical reaction parameter. The variations of the destructive chemical reaction parameter $(\gamma > 0)$ on the concentration field ϕ are sketched in Fig.15. The fluid concentration ϕ has the opposite behavior for $(\gamma > 0)$ in comparison to the case of generative chemical reaction parameter $(\gamma < 0)$. However the change in concentration field is larger for the generative chemical reaction. The concentration boundary layer is decreased in case of destructive chemical reaction. The variations of the Schmidt number Sc on ϕ is shown in Fig. 16. It is clear that boundary layer and concentration field ϕ is decreased by increasing Sc.

Table 2: Values of the surface mass transfer $-\phi'(0)$ for some values of S , M , β with $Sc = \gamma = 1$

S	M	β_1	$-\phi'(0)$
0	1.5	0.2	0.87702
0.2			0.96225
0.4			1.05930
0.6			1.16853
0.8			1.28985
1			1.42949
0.6	1		1.13104
	1.2		1.14998
	1.4		1.16312
	1.6		1.17330
	1.8		1.18133
	2		1.18832
	1.5	0	1.16941
		0.5	1.16806
		1	1.16658
		1.5	1.16498
		2	1.16325

Table 3: Values of the surface mass transfer $-\phi'(0)$ and the gradient of mass transfer for some values of Sc and γ with $S = 1$, $M = 1.5$ and $\beta_1 = \beta_2 = 0.2$.

Sc	γ	$-\phi'(0)$	η	Sc	γ	$-\phi'(\eta)$
0	1	026126	0.2	0	1	0.23374
1		1.40889		0.5		0.71820
2		2.37912		1		1.01645
3		3.32871	0.6	0		0.20477
4		4.29063		0.5		0.49168
5		5.23907		1		0.56475
1	0.5	1.09454	0.2	1	0.2	0.64435
	1	1.40842			0.6	0.87646
	1.5	1.64540			1	1.01979
	2	1.84287	0.6		0.2	0.43512
	2.5	2.01559			0.6	0.52897
	3	2.17098			1	0.56471

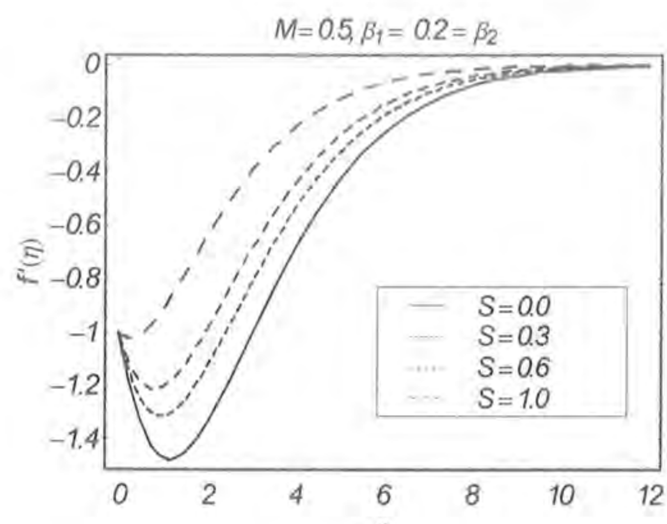

Fig. 2: The variation of suction parameter S on the velocity field f'

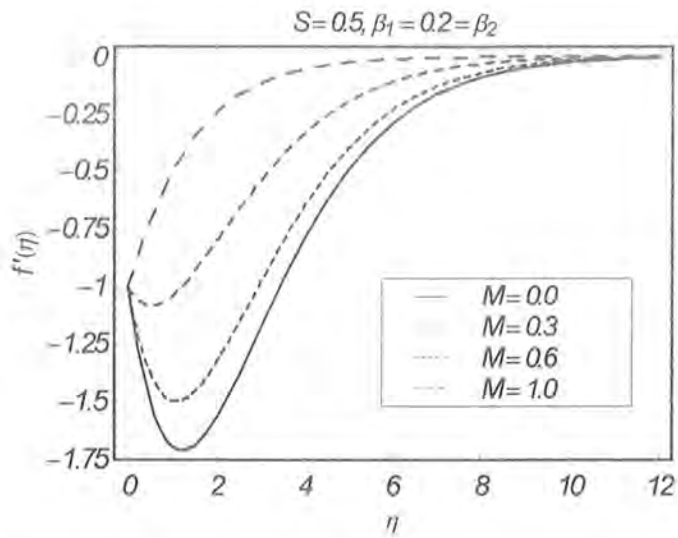

Fig. 3: The variation of Hartman number M on the velocity field f'

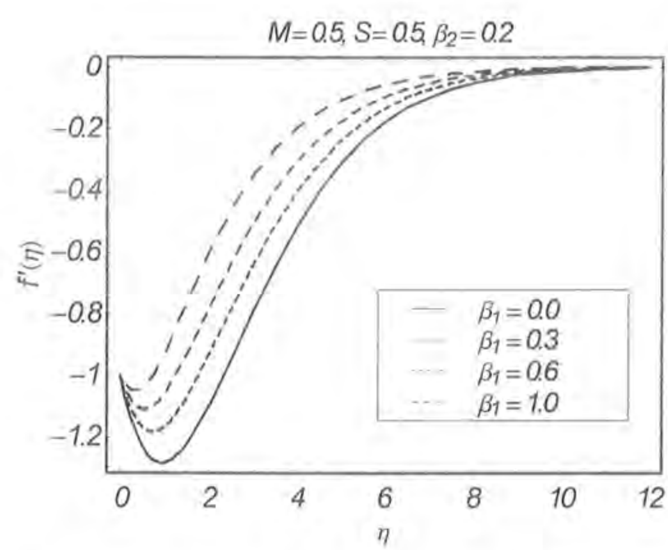

Fig. 4: The variation of Deborah number β_1 on the velocity field f'

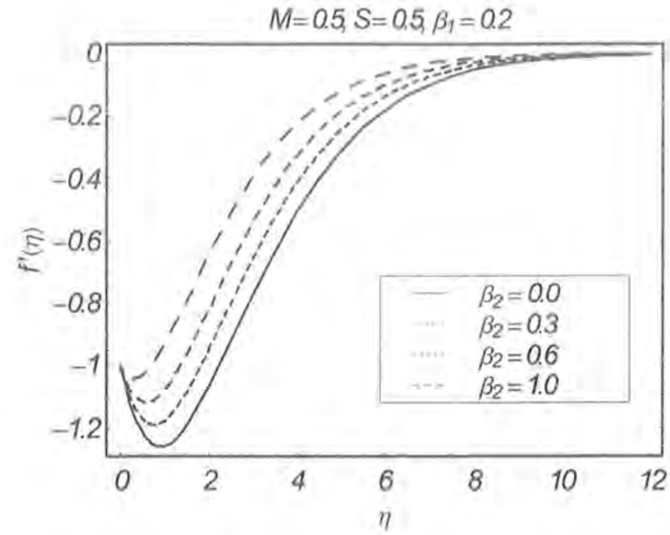

Fig. 5: The variation of Deborah number β_2 on the velocity field f'

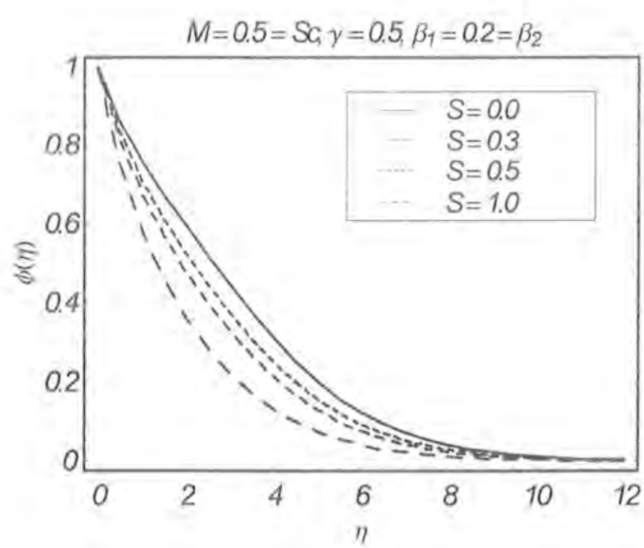

Fig. 6: The variation of suction parameter S on the concentration field ϕ

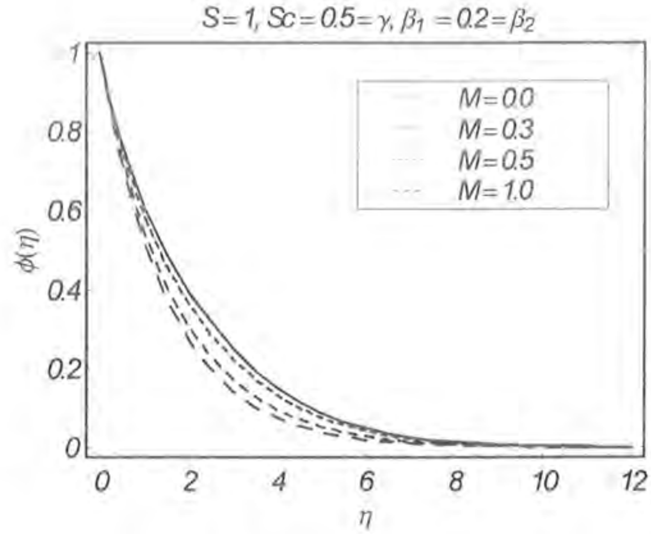

Fig. 7: The variation of Hartman number M on the concentration field ϕ

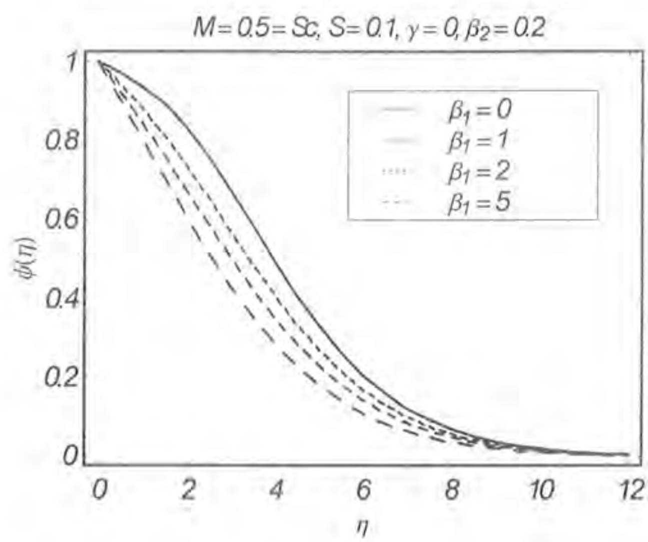

Fig. 8: The variation of Deborah number β_1 on the concentration field ϕ

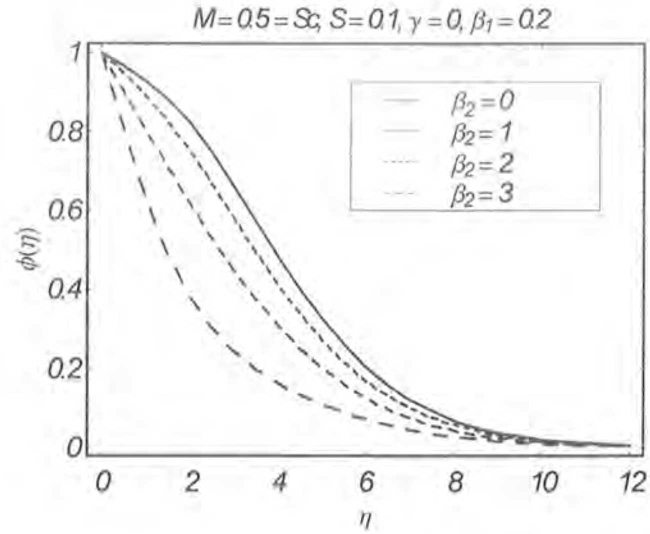

Fig. 9: The variation of Deborah number β_2 on the concentration field ϕ

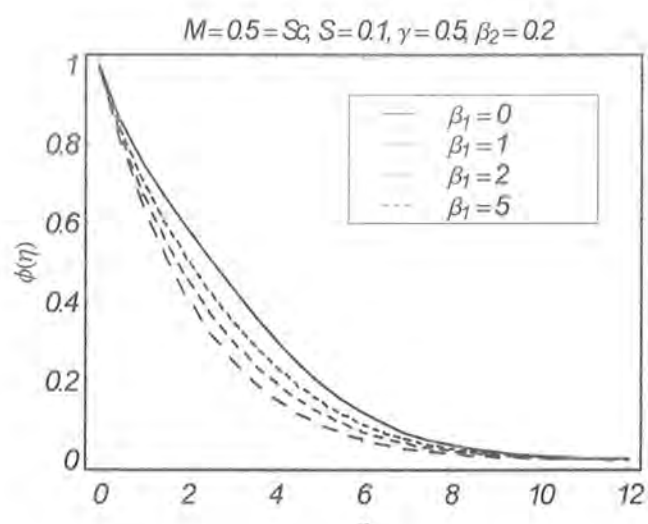

Fig. 10: The variation of Deborah number β_1 on ϕ for destructive chemical reaction

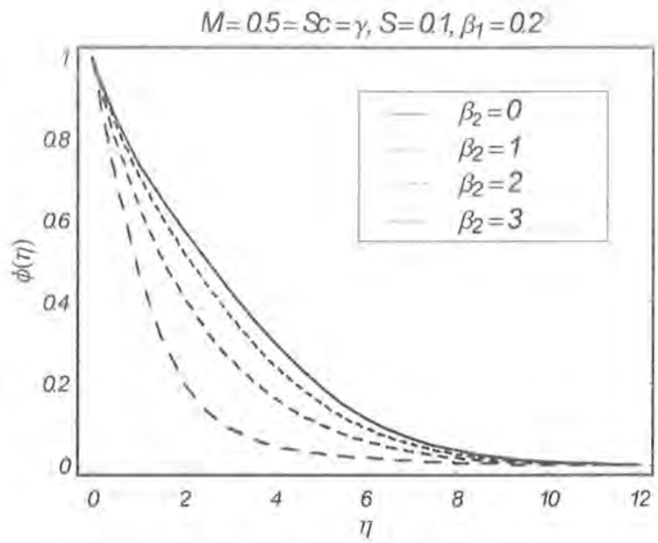

Fig. 11: The variation of Deborah number β_2 on ϕ for destructive chemical reaction

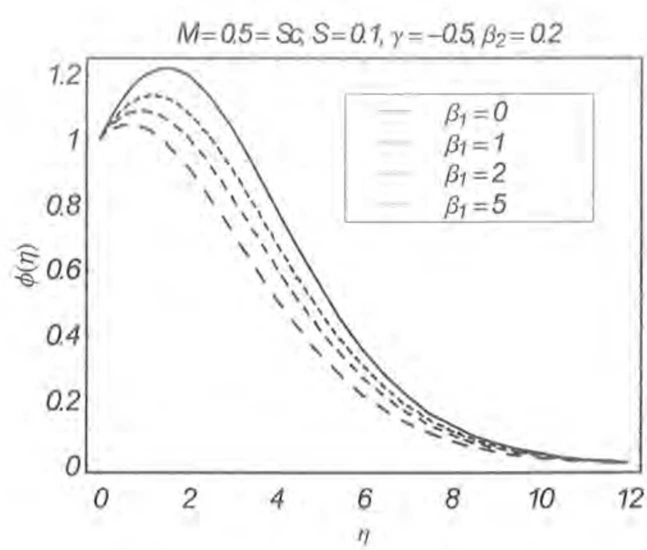

Fig. 12: The variation of Deborah number β_1 on ϕ for generative chemical reaction

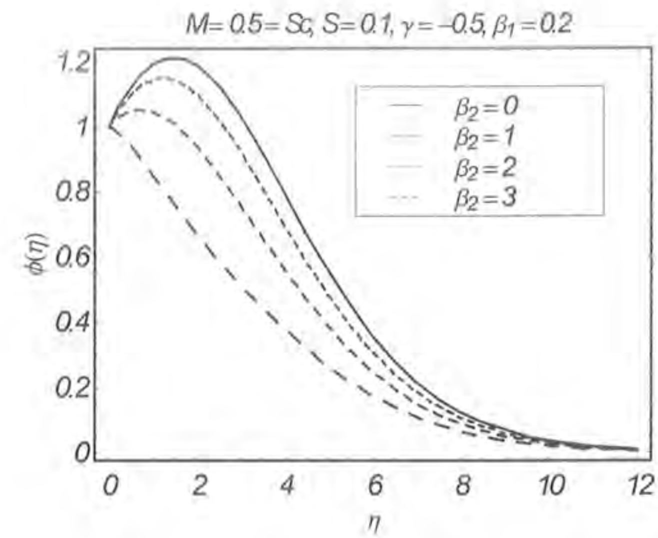

Fig. 13: The variation of Deborah number β_2 on ϕ for generative chemical reaction

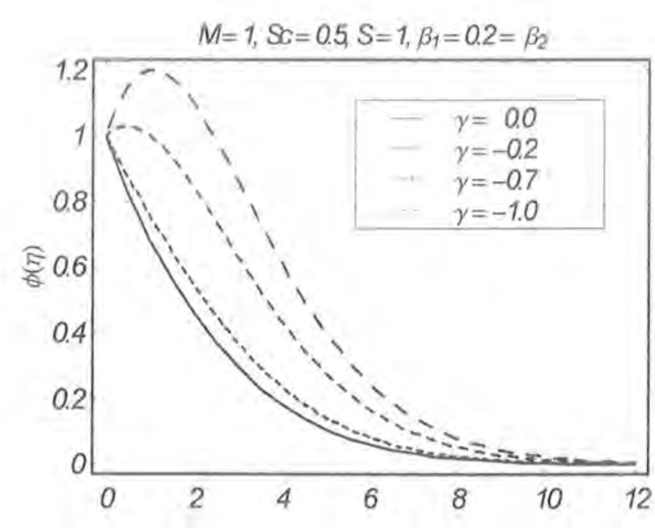

Fig. 14: The variation of generative chemical reaction ($\gamma < 0$) on the ϕ

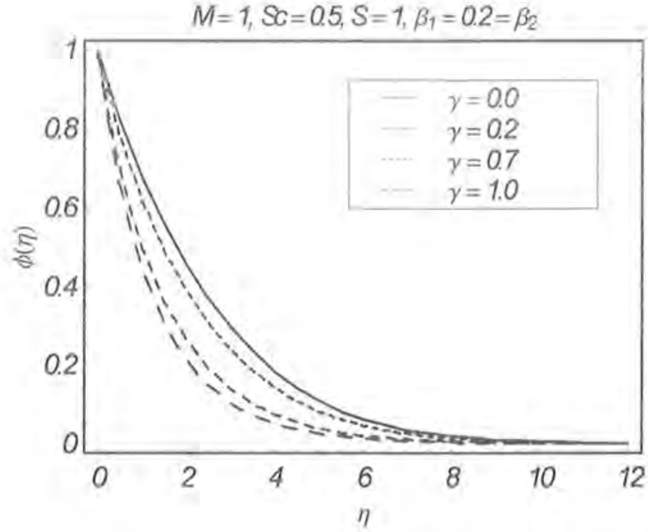

Fig. 15: The variation of destructive chemical reaction ($\gamma > 0$) on the ϕ

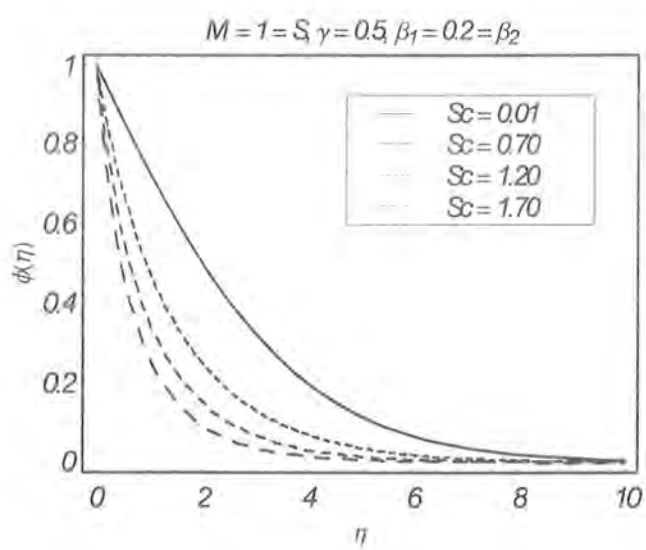

Fig. 16: The variation of Schmidt parameter Sc on the concentration field ϕ

3.5 Concluding remarks

This investigation concentrates on the mass transfer in the MHD flow of Jeffrey fluid over a porous shrinking sheet with chemical reaction species. The non-linear system of ordinary differential equations is solved analytically using homotopy analysis method (HAM). The velocity $f'$and the concentration field ϕ are obtained and discussed. The surface mass transfer and the gradient of mass transfer are also computed in tabular form. As a summary we can conclude that

- The velocity field f' increases by increasing S, M, β_1 and β_2.
- The concentration field ϕ decreases when Sc increases in both cases of destructive/generative chemical reactions.
- In destructive/generative chemical reactions, the effects of S, M, β_1 and β_2 on ϕ are quite opposite to that of f'.
- The concentration field ϕ has opposite results for destructive ($\gamma > 0$) and generative ($\gamma < 0$) chemical reactions.
- The magnitudes of the surface mass transfer $-\phi'(0)$ and the gradient of mass transfer $-\phi'(\eta)$ are increased by increasing S.

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