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Rotating flows of non-Newtonian fluids in a porous space

By

Sher Baz Khan

DEPARTMENT OF MATHEMATICS
QUAID-I-AZAM UNIVERSITY
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A Thesis

Submitted in the Partial Fulfillment of the

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CERTIFICATE

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We accept this dissertation as conforming to the required standard

1. ______________________
Prof. Dr. Muhammad Ayub
(Chairman)

2. ______________________ 23/10/08
Dr. Tasawar Hayat
(Supervisor)

3. ______________________
Prof. Dr. Nawazish Ali Shah
(External Examiner)

4. ______________________
Prof. Dr. Muhammad Ozair Ahmad
(External Examiner)

DEPARTMENT OF MATHEMATICS
QUAID-I-AZAM UNIVERSITY
ISLAMABAD, PAKISTAN
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Preface

In recent years, the flows of non-Newtonian fluids have emerged as a research subject of great interest to many researchers. The traditional Newtonian fluids cannot precisely describe the characteristics of most of the fluids in industry and technology. The non-Newtonian fluids have a non-linear relationship between the stress and the rate of strain at a point. Such fluids have been modeled in a number of diverse manners with their constitutive equations varying greatly in complexity. Amongst the various types of non-Newtonian fluids, the rate and differential type fluids are most commonly attended. Therefore the consideration of Oldroyd-B, Burgers, generalized Burgers, third grade and fourth grade fluids is a motivation in the present thesis. An intensive research effort, both theoretical and experimental has been devoted in the last few decades to problems of non-Newtonian fluids. Particularly this is due to their applications in polymer processing, bio fluid dynamics, petroleum drilling and many other similar activities. Geophysical applications concerning ice and magna flows are also based on the constitutive equations of non-Newtonian fluids. Furthermore the analytic solutions of flows in non-Newtonian fluid mechanics are rare. This is because of the nonlinearities appearing in the boundary and initial value problems. Analytic solutions of such fluids are important not only in their own right as solutions of particular flows but also serve as accuracy checks for the numerical solutions. In view of the paucity of analytic solutions in non-Newtonian fluids, the closed form and series solutions have been constructed here.

Recent developments in engineering witness that there is an increasing research activity concerning the flow of non-Newtonian fluids through a porous space. Such flows are of special interest in the chemical industry separation processes, in transpiration cooling, in textile coating, and in pollutant dispersion along aquifers. Relevant production issues include not only petroleum drilling but also numerous activities such as the manufacturing of foods and paper. The boundary layer concept for such fluids is of specific interest due to its applications to many practical problems, among which we cite the possibility of reducing frictional drag on the hulls of ships and submarines. Much existing attempts of flows of non-Newtonian fluids in a porous space takes into account the classical Darcy's law which is not realistic. Therefore an important omission in the literature concerning the modified Darcy's law for non-Newtonian flows in a porous space is addressed in this thesis.

Rotating shear flows have ample applications in meteorology, geophysics, turbomachinery and so on. The relative directions between the vorticity associated with the system rotation and flow vorticity strongly influence the time evolution of the shear flow. Furthermore, the magnetohydrodynamic (MHD) rotating fluids are encountered frequently in geophysics and astrophysics. The involvement of such fluids in the sunspot development, the solar cycle and the structure of rotating magnetic stars is of great interest. The hydromagnetic flow of non-Newtonian fluid in a porous space has become the basis of many scientific and engineering applications. Particularly, the Hall effect is important when the Hall parameter, which is the ratio between the electron-cyclotron frequency and the electron-atom-collision frequency is high. This appears when the magnetic field is strong. In most previous attempts, the Hall term is ignored in applying Ohm's law as it has no marked effect for small magnetic field. However, the current trend is to analyze the effects of strong magnetic field, so that the influence of the electromagnetic force is noticeable. Keeping these facts in mind the MHD rotating flows with Hall effect are investigated in all the chapters of this thesis.

In view of the aforesaid observations, taking care of analytic solutions the flow problems are presented in this thesis to develop mathematical models that are competent to account the Hall effect, modified Darcy's law, MHD and non-Newtonian rotating fluids. The organization of this thesis is as follows.

Chapter one provides the literature survey relevant to hydrodynamic and MHD non-Newtonian fluids in rotating and non-rotating frames. Fundamental flow equations are also presented. Resulting equations for rotating generalized and fourth grade fluids in a non-porous space are also included.

The rotating flow of an Oldroyd-B fluid in a porous space is addressed in chapter two. Modified Darcy's law has been utilized in the mathematical modeling. In addition the effect of Hall current is taken into account. Two illustrative examples of oscillatory flows are considered. These flows are subjected to the general periodic oscillation and the elliptic harmonic oscillations of the plate. In the former case, the flows induced by some special oscillations are derived whereas in the three cases when oscillating frequency is smaller than, greater than or equal to the twice of the angular velocity are discussed in the later case. The closed form solutions of the governing equations are provided. The velocity is found to increase when Hall parameter increases. The variation of medium permeability parameter is similar to that of the Hall parameter. These results have been published in **Nonlinear Dynamics 47 (2007) 353-362.**

The aim of chapter three is to study the flow analysis of chapter two in a Burgers' fluid. Burgers' fluid model is important in characterizing various viscoelastic materials: food products such as cheese, soil and asphalt. This model is also popular in the modeling of high-temperature viscoelasticity of fine-grained polycrystalline olivine in calculating the transient creep properties of the earth's mantle and specifically related to the postglacial uplift. Modified Darcy's law in a Burgers' fluid has been first proposed and then used in the mathematical formulation. Through computations it is revealed that velocity in Burgers' fluid is greater than that of an Oldroyd-B fluid. The contents of this chapter are published in **J. Porous Media 11 (2008) 277-287.**

Chapter four extends the mathematical formulation and solutions of chapter three to a generalized Burgers' fluid. The corresponding modified Darcy's law has been proposed. It is noted that the velocity components have opposite behavior in generalized Burgers' fluid. The research material presented in this chapter is published in **Applied Mathematical Modeling 32 (2008)749-760.**

The rotating flow of an electrically conducting third grad fluid in a porous space is described in chapter five. The flow analysis is given in the presence of a Hall current. Modified Darcy's law valid for a third grade fluid is developed. The fluid motion is caused by a suddenly moved plate. Series solution of the arising nonlinear problem is constructed by a homotopic method namely a homotopy analysis method (HAM). Convergence of the obtained solution is properly analyzed. It is observed that the behaviors of velocity components are opposite when the material parameter of third grade parameter increases. This work has been published in **Nonlinear Dynamics 49 (2007) 83-91.**

Chapter six extends the flow analysis of chapter five for a fourth grade fluid. Note that the fourth grade fluid is able to predict the shear thinning and shear thickening effects. However this fluid model is unable to explain the relaxation and retardation phenomena.

Expression of corresponding modified Darcy's law is proposed. Recurrence formulas in the series solution are explicitly obtained. The influence of pertinent parameters on the velocity components is sketched and discussed. The contents of this chapter are accepted for publication in **J. Porous Media.**

Contents

Chapter 1

Preliminaries

1.1 Introduction

Any fluid that does not behave in accordance with the Newtonian constitutive relation is called non-Newtonian. For such fluid the shear rate is not directly and linearly proportional to the shear stress. A Newtonian fluid can be described by using a single constitutive relation. The well known Navier-Stokes equation can predict the behavior of Newtonian fluids. Examples of the Newtonian fluids include water, air, gasoline, benzene etc. Unlike the Newtonian fluids, the non-Newtonian fluid cannot be described by a single constitutive relationship. This is because of their complexity and variety. In recent years, non-Newtonian fluids have been appearing in an increasing number of applications. These applications include molten plastics, polymer solutions, dyes, varnishes, multigrade oils, paints and printing ink. Many industrial fluids are also non-Newtonian in their flow characteristics. Most particulate slurries (China clay and coal in water, sewage sludge, etc.), multiphase mixtures (gas-liquid dispersions, such as froths and foams, butter) are non-Newtonian fluids. Besides this pharmaceutical formulations, cosmetics and toiletries, synthetic lubricants, biological fluids (blood at low shear rate, synovial fluid, salvia) and food stuffs (jams, jellies, soups, marmalades) exhibit non-Newtonian characteristics. Non-Newtonian fluids have been the subject of many recent books by Astarita and Marrucci [1], Schowalter [2] and Slattery [3] and in the review papers by Irvine Jr. and Karni [4], Andersson and Irgens [5] and Rajagopal [6].

It is well established fact that constitutive equations of non-Newtonian fluids offer interesting challenges to the workers in the field through various quarters. The Navier-Stokes equations are inadequate for the flow dynamics of such fluids. The resulting equations of non-Newtonian fluids are much complicated than the Navier-Stokes equations. The arising equations of non-Newtonian fluids are also of higher order in comparison to the Navier-Stokes equations. Therefore, the adherence boundary/initial conditions are not sufficient to determine the solution completely. This issue has been discussed in detail by Rajagopal [7, 8], Rajagopal and Gupta [9], Rajagopal and Kaloni [10], Rajagopal and Na [11], Rajagopal et al. [12], Dunn and Rajagopal [13] and Fosdick and Rajagopal [14].

Due to variety of non-Newtonian fluids, several constitutive relationships are available in the literature. Amongst these there are differential and rate type fluids which have been accorded much attention by the scientific community. The simplest subclass of a differential type fluids is known as second grade. Many researchers are engaged in developing analytical/numerical solutions for second grade fluids. Rajagopal [15] has established closed form solutions for some unidirectional flows. In another paper, Rajagopal [16] discussed the creeping flow of a second grade fluid. Hayat et al. [17, 18] and Siddiqui et al. [19] have analyzed the periodic flows of a second grade fluids. Rajagopal and Gupta [20] examined the exact flow and stability of solution between rotating parallel plates. Bandelli and Rajagopal [21] studied a number of unidirectional transient flows of a second grade fluid in a domain with one finite dimension using integral transform method. Related studies are also given in Bandelli and Rajagopal [22] and Bandelli [23]. Hayat et al. [24] also discussed the transient flows of a second grade fluid. Erdogan [25] considered the unsteady flows of a second grade fluid. These flows are induced by a constant accelerating plane and flow imposed by flat plate that applies a constant tangential stress to the fluid. Erdogan and Imrak [26] studied the effect of side walls on the flow of a second grade fluid in a duct of uniform cross-section. An exact solution of the governing equation of a fluid of second grade fluid for three dimensional vortex flow is also studied by Erdogan and Imrak [27]. Fetecau and Fetecau [28] obtained the exact solutions corresponding to Couette flows of second grade fluids in heated cylindrical geometries. The flow of a suddenly moved plate in a second grade fluid is given by Fetecau and Zierep [29]. Fetecau and Fetecau [30] obtained starting solutions for some unsteady unidirectional flows of a second grade fluid. Very recently Fetecau and Fetecau [31] constructed starting solutions corresponding to the motion of a second grade fluid due to the longitudinal and torsional oscillations of a circular cylinder. In another study, Fetecau et al. [32] analyzed the decay of a potential vortex and propagation of a heat wave in an incompressible second grade fluid. By assuming a certain form of the stream function, solutions for such fluids for the steady planar case were investigated by Kaloni and Huschilt [33], Siddiqui and Kaloni [34], Siddiqui [35], Benharbit and Siddiqui [36] and Siddiqui et al. [37]. Viscometric flows of second grade fluid have been analyzed by Markovitz and Coleman [38] and solution to unsteady flows have been developed by Ting [39].

The boundary layer flows of non-Newtonian fluids through various aspects have been studied by many workers. Mention may be made in this direction to the works of Garg [40], Garg and Rajagopal [41], Gorla and Voss [42], Hsu et al. [43], Massoudi and Ramezan [44, 45], Massoudi [46, 47], Pakdemirli [48], Rajagopal et al. [49, 50], Rajagopal and Na [51], Rajeswari and Rathna [52], Riley and Weidman [53], Kumari et al. [54], Andersson et al. [55,56], Andersson and Dandapat [57], Chamkha [58], Kim et al. [59], Kumari and Nath [60], Liao [61], Xu et al. [62], Lok et al. [63, 64], Hayat and Sajid [65]. Vajravelu and Rollins [66], Rollins and Vajravelu [67] and Cheng et al. [68].

Literature survey indicates that much attention has not been focused to the flows of third and fourth grade fluids. Few attempts are available in this direction. Rajagopal and Na [11] discussed the Stokes problem for a third grade fluid. Fosdick and Straughan [69] studied the catastrophic instabilities in a third grade fluid. Rajagopal and Mollica [70] investigated the axial shearing flow of a third grade fluid between two eccentrically placed cylinders. Akyildiz [71] examined the thin film flow in a fluid of third grade. Asghar et al. [72] studied the flow of third grade fluid over a porous moved plate. Erdogan [73] obtained the solution for the Stokes first problem in a third grade fluid. Hayat et al. [74] obtained the travelling wave solution for the flow of a third grade fluid on a porous plate. Hayat et al. [75] also analyzed the fluctuating flow of a third order fluid past an infinite plate with variable suction. Kaloni and Siddiqui [76] made an interesting study for flow of a fourth grade fluid between eccentric disks. Hayat et al. [77] presented the numerical solution for MHD flow of a fourth grade fluid between two porous plates. Hayat and Kara [78] examined the Couette flow of a third grade fluid with variable magnetic field.

The rate type fluids are another subclass of non-Newtonian fluids. The simplest subclass of such fluids is called Maxwell fluid model. In an 1867 paper, Maxwell appears to have been the first to observe that some fluids, such as air, exhibit both viscous and elastic behaviors. Especially this fluid model has been applied to viscoelastic problems where the dimensionless relaxation time is small. However, in some situations as would be the case of more concentrated polymeric fluids it can be useful to large dimensionless times. Fetecau and Fetecau [79] developed exact solution to unsteady helical flows of a Maxwell fluid. Fetecau and Zierep [80] studied the Rayleigh Stokes problem for an edge in a Maxwell fluid. Fetecau and Fetecau [81] determined the expressions of velocity and tangential tension corresponding to a potential vortex in a Maxwell fluid. In [82, 83] Fetecau and Fetecau extended the analysis of refs. [80] and [81] to an Oldroyd-B fluid. Morer ecently Ravindran et al. [84] discussed a problem dealing with the steady flows of a Burgers' fluid in an orthogonal rheometer. In continuation, Hayat et al. [85] presented three periodic flows of a Burgers' fluid.

During the past few decades the study of electrically conducting fluids in a rotating frame of reference has increased substantially. This is due to their promising applications in geophysical and cosmical fluid dynamics. It is known through order analysis that the Coriolis force is more significant in comparison to the inertia and viscous forces appearing in the fundamental equation of motion. The Coriolis and magnetohydrodynamic (MHD) forces are of comparable magnitude. Particularly, the effects of rotation in MHD flows are of great value in the study of wind generated ocean currents on rotating earth. The rotating flows are also important in the solar physics involved in the sunspot development, the solar cycle and the structure of rotating magnetic stars. In this direction Lehnert [86, 87] discussed the magnetohydrodynamic (MHD) waves analysis in an incompressible viscous rotating fluid by taking into account small amplitude assumption. Here the role of Coriolis force on the properties of MHD waves in the sun is analyzed. The solution of many rotating flow problems hinges on the understanding of the behavior of the boundary layers. Some authors including Thornley [88], Loper [89], Loper and Benton [90], Debnath [91–93], Gupta [94], Soundalgekar and Pop [95], Ganapathy [96], Acheson [97], Murthy and Ram [98], Deka et al. [99], Mazumder [100] and Singh [101] have performed interesting flow analyses in a rotating system. However much progress is not made on rotating flows of non-Newtonian fluids. With this fact in view Hayat and Hutter [102] and Hayat at al. [103] examined the rotating flow of a second grade fluid. Hayat and Kara [104] presented variational analysis of a third grade fluid in a rotating system. Oscillating flow of a third grade fluid on a porous plate in a rotating frame has been analyzed by Hayat et al. [105]. Puri [106] investigated the rotating flow of an elastico-viscous fluid on an oscillating plate. Puri and Kulshrestha [107] discussed flows due to moved plate in a rotating frame.

1.2 Porous medium

A material which is composed of a solid matrix with interconnected voids (pores) is called a porous medium. These pores allow the flow of fluids through the material. The distribution of pores is irregular in nature. Beach sand, rye bread, human lungs, limestone, wood, etc. are examples of natural porous media.

1.2.1 Porosity

By porosity ϕ of a porous medium we mean the fraction of the connected void area to the total area of the material. It means that $1-\phi$ is the fraction that solid area occupies the material.

1.2.2 Volumetric flux density

The average of the fluid velocity, say, $\mathbf{V}=(u,v,w)$ over a volume element V_m of the medium (both solid and fluid material) is called volumetric flux density or seepage velocity. Whereas an average of the fluid velocity, say, $\mathbf{V}$ over a volume V_D is called the intrinsic average velocity which is related to $\mathbf{V}$ by the Dupuit-Forchheimer relationship $\mathbf{V}_D=\phi\mathbf{V}$.

1.2.3 Permeability

By Darcy experiment, for unidirectional flow there exists a relationship between flow rate and the applied pressure difference. Mathematically it is written as

$$u=-\frac{K}{\mu}\frac{\partial p}{\partial x}. \tag{1.1}$$

In above expression μ is the dynamic viscosity of the fluid, $\partial p/\partial x$ is the pressure gradient and u is the velocity parallel to the $x-$ axis. Here K is called the permeability of the porous medium. Note that the permeability coefficient depends on the geometry of the medium and independent upon the fluid nature. Expression (1.1) in the case of three dimensions can be expressed as follows

$$\mathbf{V}_D = -\frac{1}{\mu}\mathbf{K}\cdot\nabla P, \tag{1.2}$$

where in this case $\mathbf{K}$ is a second-order tensor. If the medium is an isotropic one then the permeability is a scalar. Thus the last equation can be written as

$$\nabla p = -\frac{\mu}{K}\mathbf{V}_D. \tag{1.3}$$

1.3 Equation of motion in a rotating system

In order to derive the equation of motion in a rotating frame of reference we consider two coordinate systems. The notations in a rotating system have been designated by primes.

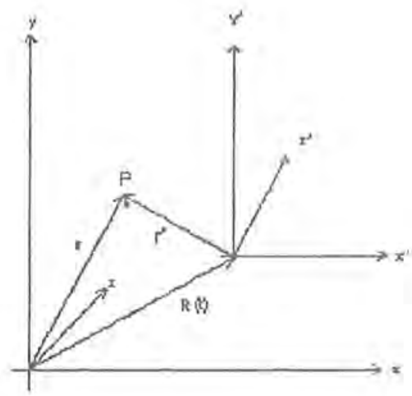


Figure 1.1. Representation of a fluid particle P in an inertial and rotating frames.

We choose r and r' the position vectors in the inertial and rotating systems respectively. If R(t) is the position vector of the inertial system as well as the origin of the rotating system then one may write

$$\mathbf{r} = \mathbf{R} + \mathbf{r}', \qquad \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{R}}{dt} + \frac{d\mathbf{r}'}{dt}, \tag{1.4}$$

$$\mathbf{r}' = x_{i'} e_{i'}, \qquad \frac{d\mathbf{r}'}{dt} = \frac{dx_{i'}}{dt} e_{i'} + x_{i'} \frac{de_{i'}}{dt}, \tag{1.5}$$

where $e_{i'}$ is a unit vector. Denoting Ω as the angular velocity of the primed system, one has

$$\frac{de_{i'}}{dt} = \Omega \times e_{i'}. \tag{1.6}$$

From Eqs. (1.4) − (1.6), we can write

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{R}}{dt} + e_{i'} \frac{dx_{i'}}{dt} + \Omega \times \mathbf{r}'. \tag{1.7}$$

On differentiating Eq.(1.7) one obtains

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2\mathbf{R}}{dt^2} + e_{i'} \frac{d^2 x_{i'}}{dt^2} + \frac{de_{i'}}{dt} \frac{dx_{i'}}{dt} + \frac{d\Omega}{dt} \times \mathbf{r}' + \Omega \times \frac{d\mathbf{r}'}{dt}. \tag{1.8}$$

Expression of a fluid particle in a primed system is defined as

$$\frac{d\mathbf{V}'}{dt} = e_{i'} \frac{d^2 x_{i'}}{d^2 t}. \tag{1.9}$$

Since

$$\frac{d\mathbf{r}'}{dt} = \mathbf{V}' + \Omega \times \mathbf{r}', \qquad \frac{dx_{i'}}{dt} \frac{de_{i'}}{dt} = \Omega \times \mathbf{V}', \tag{1.10}$$

therefore Eq.(1.9) takes the form

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2\mathbf{R}}{dt^2} + \frac{d\mathbf{V}'}{dt} + 2\Omega \times \mathbf{V}' + \frac{d\Omega}{dt} \times \mathbf{r}' + \Omega \times (\Omega \times \mathbf{r}'). \tag{1.11}$$

As the primed system is fixed with respect to rotating frame of reference and thus there is no translation and no translational acceleration within the rotating system. Also the rotation is constant. Thus the total acceleration of the inertial and rotating systems is

$$\frac{d\mathbf{V}}{dt} + 2\Omega \times \mathbf{V} + \Omega \times (\Omega \times \mathbf{r}), \tag{1.12}$$

where the primes have been suppressed and the first term in Eq.(1.12) is the fluid particle acceleration, the second term represents the Coriolis acceleration and the last term gives the

$$\mathbf{r}' = x_{i'} e_{i'}, \qquad \frac{d\mathbf{r}'}{dt} = \frac{dx_{i'}}{dt} e_{i'} + x_{i'} \frac{de_{i'}}{dt}, \tag{1.5}$$

where $e_{i'}$ is a unit vector. Denoting $\boldsymbol{\Omega}$ as the angular velocity of the primed system, one has

$$\frac{de_{i'}}{dt} = \boldsymbol{\Omega} \times e_{i'}. \tag{1.6}$$

From Eqs. (1.4) – (1.6), we can write

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{R}}{dt} + e_{i'} \frac{dx_{i'}}{dt} + \boldsymbol{\Omega} \times \mathbf{r}'. \tag{1.7}$$

On differentiating Eq.(1.7) one obtains

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2\mathbf{R}}{dt^2} + e_{i'} \frac{d^2 x_{i'}}{dt^2} + \frac{de_{i'}}{dt} \frac{dx_{i'}}{dt} + \frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r}' + \boldsymbol{\Omega} \times \frac{d\mathbf{r}'}{dt}. \tag{1.8}$$

Expression of a fluid particle in a primed system is defined as

$$\frac{d\mathbf{V}'}{dt} = e_{i'} \frac{d^2 x_{i'}}{d^2 t}. \tag{1.9}$$

Since

$$\frac{d\mathbf{r}'}{dt} = \mathbf{V}' + \boldsymbol{\Omega} \times \mathbf{r}', \qquad \frac{dx_{i'}}{dt} \frac{de_{i'}}{dt} = \boldsymbol{\Omega} \times \mathbf{V}', \tag{1.10}$$

therefore Eq.(1.9) takes the form

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2\mathbf{R}}{dt^2} + \frac{d\mathbf{V}'}{dt} + 2\boldsymbol{\Omega} \times \mathbf{V}' + \frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r}' + \boldsymbol{\Omega} \times \left(\boldsymbol{\Omega} \times \mathbf{r}' \right). \tag{1.11}$$

As the primed system is fixed with respect to rotating frame of reference and thus there is no translation and no translational acceleration within the rotating system. Also the rotation is constant. Thus the total acceleration of the inertial and rotating systems is

$$\frac{d\mathbf{V}}{dt} + 2\boldsymbol{\Omega} \times \mathbf{V} + \boldsymbol{\Omega} \times \left(\boldsymbol{\Omega} \times \mathbf{r} \right), \tag{1.12}$$

where the primes have been suppressed and the first term in Eq.(1.12) is the fluid particle acceleration, the second term represents the Coriolis acceleration and the last term gives the centrifugal acceleration. In view of expression (1.12), equation of motion in a rotating frame is

$$\rho\left[\frac{d\mathbf{V}}{dt}+2\boldsymbol{\Omega}\times\mathbf{V}+\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})\right]=\rho\mathbf{B}_1+\operatorname{div}\mathbf{T}, \tag{1.13}$$

where $\rho\mathbf{B}_1$ indicates the body forces.

1.4 Basic equations

Here we present some fundamental equations that will govern the flow problems in various situations considered in this thesis. In absence of displacement current we have

1.4.1 Maxwell's equations

In absence of displacement current

$$div\mathbf{B}=0, \qquad Curl\mathbf{B}=\mu_e\mathbf{J}, \qquad Curl\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}, \tag{1.14}$$

where $\mathbf{B}$ is the total magnetic field, $\mathbf{E}$ is the electric field, μ_e is the magnetic permeability and $\mathbf{J}$ is the current density. The Eqs. (1.14) are only valid in the absence of displacement current. The contents of this thesis involve such equations for the magnetohydrodynamic concept. Mathematically the electric field in the magnetohydrodynamic flow is taken to be zero throughout the thesis.

1.4.2 Generalized Ohm's law

Expression of the current density including the Hall term is

$$\mathbf{J}+\frac{\omega_e\tau_e}{B_0}(\mathbf{J}\times\mathbf{B})=\sigma\left[\mathbf{E}+\mathbf{V}\times\mathbf{B}+\frac{1}{en_e}\nabla p_e\right], \tag{1.15}$$

in which ω_e is the cyclotron frequency of electron, τ_e is the electron collision time, $\mathbf{B}=(0,0,\mathbf{B}_0)$ where $\mathbf{B}_0$ is an applied magnetic field, σ is the electrical conductivity, e is the electron charge, $1/en_e$ is the Hall factor, n_e is the number density of the electrons, and p_e is the electron pressure. Neglecting the ion-slip terms and applied voltage $(\mathbf{E}=\mathbf{0})$, Eq. (1.15) helps in writing the following relation

$$\mathbf{J} \times \mathbf{B} = -\frac{\sigma B_0^2}{1 - im_0}\mathbf{V}, \tag{1.16}$$

where $m_0 = \omega_e \tau_e$ is the Hall parameter.

1.4.3 The continuity equation

Let S be a fixed smooth surface over a volume element V_m of the medium. The mass flow rate in V_m is defined as

$$-\int_{\mathbf{S}} \rho \mathbf{V} \cdot d\mathbf{S}$$

in which minus sign indicates the surface element $d\mathbf{S}$ out of V_m. The rate of change of mass in V_m is

$$\frac{d}{dt}\int_{V_m} \rho dV_m.$$

Since $\mathbf{V}_m$ is assumed to be a fixed region, therefore the above expression reduces to

$$\int_{V_m} \frac{\partial \rho}{\partial t} dV_m.$$

By mass conservation one can write

$$\int_{V_m} \frac{\partial \rho}{\partial t} dV_m = -\int_{\mathbf{S}} \rho \mathbf{V} \cdot d\mathbf{S}, \tag{1.17}$$

which upon divergence theorem becomes

$$\int_{V_m} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) \right] dV_m = 0. \tag{1.18}$$

Since V_m is any arbitrary volume element of the medium, it follows from above equation that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \tag{1.19}$$

where ρ is the fluid density. The above expression for an incompressible fluid is

$$divV = 0. \tag{1.20}$$

1.4.4 The momentum equation

The equation of motion for an electrically conducting and rotating fluid occupying a porous space is

$$\rho\left[\frac{d\mathbf{V}}{dt} + 2\mathbf{\Omega}\times\mathbf{V} + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})\right] = div\mathbf{T} + \mathbf{J}\times\mathbf{B} + \mathbf{R}, \tag{1.21}$$

where $\mathbf{R}$ is the Darcy's resistance in the porous medium.

1.5 Components of the velocity gradient

The acceleration in Eulerian description is

$$\mathbf{a} = \frac{d\mathbf{V}}{d\tau} = \frac{\partial\mathbf{V}}{\partial\tau} + (\mathbf{V}\cdot\nabla)\mathbf{V}, \tag{1.22}$$

where first term on the right-hand side is local acceleration and the second term represents its convective part. Since $\mathbf{L} = \nabla\mathbf{V} = L_{ij}$ is a second rank tensor therefore

$$\mathbf{L} = \frac{1}{2}\left(\mathbf{L}+\mathbf{L}^{\mathsf{T}}\right) + \frac{1}{2}\left(\mathbf{L}-\mathbf{L}^{\mathsf{T}}\right) \tag{1.23}$$

$$= \mathbf{D}+\mathbf{W}, \tag{1.24}$$

where $\mathbf{D}$ is the symmetric and $\mathbf{W}$ is the antisymmetric components of the velocity tensor $\mathbf{L}$. Here $\mathbf{D}$ is called the rate of strain tensor and $\mathbf{W}$ is called the vorticity tensor.

1.6 Rivlin-Ericksen tensor

Expression of $nth-$Rivlin Ericksen tensor is written as

$$\mathbf{A}_n(t) = \left.\frac{d^n\mathbf{F}_t(\tau)}{d\tau^n}\right|_{\tau=t}, \quad n = 1, 2, ... \tag{1.25}$$

Keeping in mind of having no deformation at $\tau = t$ we have

$$\mathbf{A}_0 = \mathbf{F}_t(\tau)|_{\tau=t} = \mathbf{I}, \tag{1.26}$$

$$\mathbf{A}_1(t) = \left.\frac{d\mathbf{F}_t(\tau)}{d\tau}\right|_{\tau=t} = \frac{d}{d\tau}\left[\{\mathbf{C}_t(\tau)\}^\mathsf{T}\mathbf{C}_t(\tau)\right], \tag{1.27}$$

where $\mathbf{F} = \mathbf{C}^\mathsf{T}\mathbf{C}$ is the right Cauchy-Green tensor. Considering

$$\frac{d\mathbf{C}_t(\tau)}{d\tau} = \frac{d}{d\tau}\frac{\partial \eta_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{d\eta_i}{d\tau}\right) = \frac{\partial u_i}{\partial x_j} = \mathbf{L}, \tag{1.28}$$

in which η_i denotes the position of the particle one may write

$$\frac{d}{d\tau}\left[\mathbf{C}_t(\tau)\right]^\mathsf{T} = \left[\frac{d\mathbf{C}_t(\tau)}{d\tau}\right]^\mathsf{T} = \mathbf{L}^\mathsf{T}. \tag{1.29}$$

Now

$$\begin{aligned}
\frac{d\mathbf{F}_t(\tau)}{d\tau} &= \frac{d}{d\tau}\left[(\mathbf{C}_t(\tau))^\mathsf{T}\mathbf{C}_t(\tau)\right] \\
&= (\mathbf{C}_t(\tau))^\mathsf{T}\frac{d\mathbf{C}_t(\tau)}{d\tau} + \mathbf{C}_t(\tau)\frac{d(\mathbf{C}_t(\tau))^\mathsf{T}}{d\tau} \\
&= \mathbf{C}_t(\tau)^\mathsf{T}\mathbf{L} + \mathbf{L}^\mathsf{T}\mathbf{C}_t(\tau).
\end{aligned} \tag{1.30}$$

At $\tau = t$, $\mathbf{C}_t(\tau) = \mathbf{I}$ and therefore

$$\mathbf{A} = \mathbf{L} + \mathbf{L}^\mathsf{T}. \tag{1.31}$$

For $n = 2$, Eq. (1.25) yields

$$\begin{aligned}
\mathbf{A}_2(t) &= \left.\frac{d^2\mathbf{F}_t(\tau)}{d\tau^2}\right|_{\tau=t} \\
&= \frac{d}{d\tau}\left[\left.\frac{d\mathbf{F}_t(\tau)}{d\tau}\right|_{\tau=t}\right] = \frac{d}{d\tau}\left[\mathbf{C}_t(\tau)^\mathsf{T}\mathbf{L} + \mathbf{L}^\mathsf{T}\mathbf{C}_t(\tau)\right] \\
&= \frac{d}{d\tau}(\mathbf{C}_t(\tau))^\mathsf{T}\mathbf{L} + (\mathbf{C}_t(\tau))^\mathsf{T}\frac{d\mathbf{L}}{d\tau} + \frac{d\mathbf{L}^\mathsf{T}}{d\tau}\mathbf{C}_t(\tau) + \mathbf{L}^\mathsf{T}\frac{d\mathbf{C}_t(\tau)}{d\tau} \\
&= \mathbf{L}^\mathsf{T}\mathbf{L} + (\mathbf{C}_t(\tau))^\mathsf{T}\frac{d\mathbf{L}}{d\tau} + \frac{d\mathbf{L}^\mathsf{T}}{d\tau}\mathbf{C}_t(\tau) + \mathbf{L}^\mathsf{T}\mathbf{L}.
\end{aligned} \tag{1.32}$$

Writing

$$\begin{aligned}
\frac{d\mathbf{L}}{d\tau} &= \frac{d}{d\tau}\left(\frac{\partial V_i}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{dV_i}{d\tau}\right) \\
&= \frac{\partial}{\partial x_j}\left[\frac{\partial \mathbf{V}}{\partial \tau} + (\mathbf{V}\cdot\nabla)\,\mathbf{V}\right] \\
&= \frac{\partial}{\partial \tau}\frac{\partial V_i}{\partial x_j} + \frac{\partial}{\partial x_j}V_i\cdot\nabla V_i + V_i\cdot\frac{\partial}{\partial x_j}\nabla V_i \\
&= \frac{\partial \mathbf{L}}{\partial \tau} + \boldsymbol{\nabla}\mathbf{V}\cdot\boldsymbol{\nabla}\mathbf{V} + \mathbf{V}\cdot\boldsymbol{\nabla}\frac{\partial V_i}{\partial x_j} \\
&= \frac{\partial \mathbf{L}}{\partial \tau} + \mathbf{L}\cdot\mathbf{L} + \mathbf{V}\cdot\boldsymbol{\nabla}\mathbf{L},
\end{aligned} \tag{1.33}$$

we also have

$$\frac{d\mathbf{L}^{\mathsf{T}}}{d\tau} = \frac{\partial \mathbf{L}^{\mathsf{T}}}{\partial \tau} + \mathbf{L}^{\mathsf{T}}\cdot\mathbf{L}^{\mathsf{T}} + \mathbf{V}\cdot\boldsymbol{\nabla}\mathbf{L}^{\mathsf{T}} \tag{1.34}$$

and now Eq. (1.9) reduces to

$$\mathbf{A}_2(t) = \mathbf{L}^{\mathsf{T}}\mathbf{L} + (\mathbf{C}_t(\tau))^{\mathsf{T}}\left\{\frac{\partial \mathbf{L}}{\partial \tau} + \mathbf{L}\cdot\mathbf{L} + \mathbf{V}\cdot\boldsymbol{\nabla}\mathbf{L} + \left(\frac{\partial \mathbf{L}^{\mathsf{T}}}{\partial \tau} + \mathbf{L}^{\mathsf{T}}\cdot\mathbf{L}^{\mathsf{T}} + \mathbf{V}\cdot\boldsymbol{\nabla}\mathbf{L}^{\mathsf{T}}\right)\mathbf{C}_t(\tau) + \mathbf{L}^{\mathsf{T}}\mathbf{L}\right\}. \tag{1.35}$$

At $\tau = t$, $\mathbf{F}_t(\tau) = 1$ and above equation may be written as

$$\begin{aligned}
\mathbf{A}_2(t) &= \left(\frac{\partial}{\partial \tau} + \mathbf{V}\cdot\boldsymbol{\nabla}\right)\mathbf{A}_1 + \mathbf{A}_1\mathbf{L} + \mathbf{L}^{\mathsf{T}}\mathbf{A}_1 \\
&= \frac{d\mathbf{A}_1}{d\tau} + \mathbf{A}_1\mathbf{L} + \mathbf{L}^{\mathsf{T}}\mathbf{A}_1.
\end{aligned} \tag{1.36}$$

Similarly

$$\mathbf{A}_3(t) = \frac{d\mathbf{A}_2}{d\tau} + \mathbf{A}_2\mathbf{L} + \mathbf{L}^{\mathsf{T}}\mathbf{A}_2, \tag{1.37}$$

$$\mathbf{A}_{n+1}(t) = \frac{d\mathbf{A}_n}{d\tau} + \mathbf{A}_n\mathbf{L} + \mathbf{L}^{\mathsf{T}}\mathbf{A}_n. \tag{1.38}$$

1.7 Complex Fourier series of $f(t)$

If $f(t)$ is a periodic function with period T_0 and frequency $n_0(=\frac{2\pi}{T_0})$ then

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikn_0t}, \tag{1.39}$$

where the spectral coefficients or Fourier coefficients $\{c_k\}$ are given through

$$c_k = \frac{1}{T_0} \int\limits_{T_0} f(t) e^{-ikn_0 t} dt. \tag{1.40}$$

1.8 Fourier transform pair

Let f be a piecewise continuous function on a closed interval. Suppose that $\int\limits_{-\infty}^{\infty} |f(t)| \, dt$ exists then the temporal Fourier transform of f is defined by

$$F(\omega) = \int\limits_{-\infty}^{\infty} f(t) e^{-i\omega t} dt. \tag{1.41}$$

The inverse Fourier transform of $F(\omega)$ is the function $f(t)$ defined by

$$f(t) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega \tag{1.42}$$

for all values of t such that the above integral exists. The transformations (1.41) and (1.42) are called the Fourier transform pair.

1.9 Homotopy analysis method (HAM)

In order to solve various types of nonlinear problems analytically, Liao [108] proposed a general method namely a homotopy analysis method (HAM). To present the basic idea of HAM, we consider the following differential equation

$$\mathcal{N}[u(t)] = 0. \tag{1.43}$$

In above equation $\mathcal{N}$ is a non-linear operator, t indicates an independent variable and $u(t)$ signifies the unknown function. Liao constructs the following zero-order deformation equation

$$(1-q)\{\mathcal{L}[\Phi(t;q) - u(t)]\} = q\hbar H(t) \mathcal{N}[\Phi(t;q)], \tag{1.44}$$

where $q \in [0,1]$ and $\hbar(\neq 0)$ are the embedding and auxiliary parameters respectively, $H(t) \neq 0$ is an auxiliary function, $\mathcal{L}$ denotes an auxiliary linear operator, $u_0(t)$ is an initial guess and Φ is an unknown function. It is obvious that when $q = 0$ and $q = 1$, the following holds

$$\Phi(t;0) = u_0(t), \qquad \Phi(t;1) = u(t), \tag{1.45}$$

respectively. When q varies from 0 to 1, $\Phi(t;q)$ varies from $u_0(t)$ to $u(t)$. By Taylor series expansion

$$\Phi(t;q) = u_0(t) + \sum_{m=1}^{\infty} u_m(t) q^m, \tag{1.46}$$

$$u_m(t) = \frac{1}{m!} \left. \frac{\partial^m \Phi(t;q)}{\partial q^m} \right|_{q=0}. \tag{1.47}$$

If the values of $\mathcal{L}$, $u_0(t)$, $\hbar$ and $H(t)$ are so properly chosen that the series (1.46) converges at $q = 1$, one obtains

$$u(t) = u_0(t) + \sum_{m=1}^{\infty} u_m(t), \tag{1.48}$$

which must be one of the solutions of the original non-linear problem.

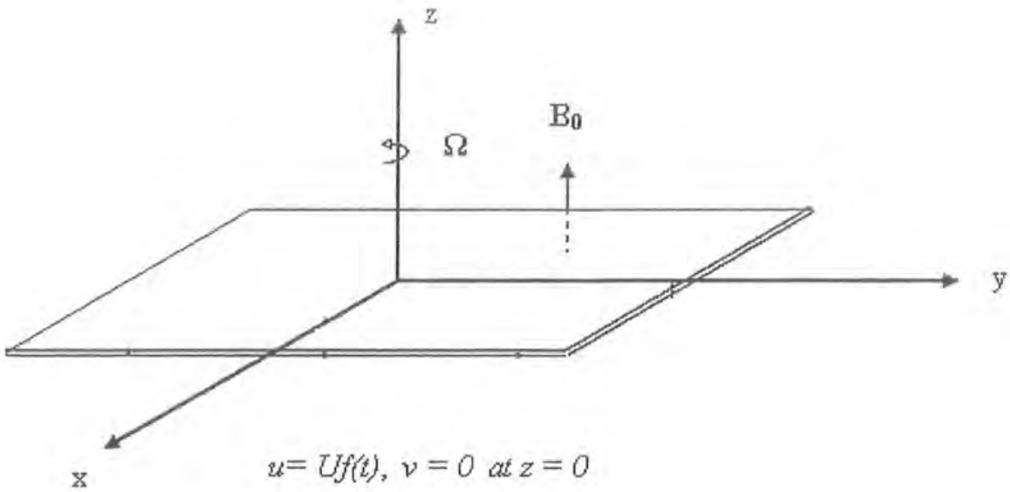
1.10 Mathematical modeling

In this section we derive the arising equations for the generalized Burgers' and fourth grade fluids in a rotating system. Such equations will be used later in this thesis.

1.10.1 Governing equation for generalized Burgers' fluid

We consider the time dependent flow of an incompressible electrically conducting generalized Burgers' fluid bounded by a rigid non-conducting plate at $z = 0$. A uniform magnetic field $\mathbf{B}_0$

is applied in the z−direction.



The plate at $z = 0$ performing periodic oscillations $f(t)$ with period T_0.

In the undisturbed state, both the fluid and the plate are in a state of solid body rotation with a constant angular velocity $\mathbf{\Omega}$ about the z−axis normal to the plate. The induced magnetic field, ion slip and thermoelectric effects are neglected. However the Hall effect is taken into account. The electric field is chosen zero. This corresponds to the fact that no energy is added/extracted from the fluid. Under these considerations the equations which govern the flow are written in usual notations as

$$\nabla \cdot \mathbf{V} = 0, \tag{1.49}$$

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\,\mathbf{V} + 2\mathbf{\Omega} \times \mathbf{V} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = \frac{1}{\rho}\nabla \cdot \mathbf{T} + \frac{1}{\rho}\mathbf{J} \times \mathbf{B}, \tag{1.50}$$

$$\mathbf{J} + \frac{\omega_e \tau_e}{B_0}(\mathbf{J} \times \mathbf{B}) = \sigma\left(\mathbf{E} + \mathbf{V} \times \mathbf{B} + \frac{1}{en_e}\nabla p_e\right), \tag{1.51}$$

$$\nabla \times \mathbf{B} = \mu_e \mathbf{J}, \tag{1.52}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.53}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{1.54}$$

The Cauchy stress $\mathbf{T}$ in a Burgers' fluid is related to the fluid motion in the following manner [109]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \tag{1.55}$$

$$\left(1 + \lambda_1 \frac{\delta}{\delta t} + \lambda_2 \frac{\delta^2}{\delta t^2}\right) \mathbf{S} = \mu \left(1 + \lambda_3 \frac{\delta}{\delta t}\right) \mathbf{A}_1, \tag{1.56}$$

where μ is the dynamic viscosity, λ_1 is the relaxation time and λ_3 ($< \lambda_1$ [110]) is the retardation time. A new material constant λ_2 multiplies the upper second order convected derivative

$$\frac{\delta^2 \mathbf{S}}{\delta t^2} = \frac{\delta}{\delta t}\left(\frac{\delta \mathbf{S}}{\delta t}\right) = \frac{\delta}{\delta t}\left(\frac{d\mathbf{S}}{dt} - \mathbf{LS} - \mathbf{SL}^{\mathsf{T}}\right). \tag{1.57}$$

Here d/dt signifies the material derivative, $\mathbf{L} = \text{grad}\mathbf{V}$ and $\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^{\mathsf{T}}$ ($\mathbf{L}^{\mathsf{T}}$ indicates the transpose).

For generalized Burgers' fluid, Eq. (1.57) is of the form [111]

$$\left(1 + \lambda_1 \frac{\delta}{\delta t} + \lambda_2 \frac{\delta^2}{\delta t^2}\right) \mathbf{S} = \mu \left(1 + \lambda_3 \frac{\delta}{\delta t} + \lambda_4 \frac{\delta^2}{\delta t^2}\right) \mathbf{A}_1. \tag{1.58}$$

It should be noted that in above equation there is second convected derivative of stress and strain tensors. It also contains two relaxation and two retardation times. Since the plate is infinite in extent, all the physical quantities except the pressure, depend on z and t only. Therefore the extra stress tensor $\mathbf{S}$ and $\mathbf{V}$ are

$$\mathbf{S} = \mathbf{S}(z, t) \tag{1.59}$$

$$\mathbf{V}(z, t) = (u, v, 0). \tag{1.60}$$

Invoking Eqs. (1.49), (1.59) and (1.60), equation of motion and (1.58) yield the following scalar equations

$$\rho\left(\frac{\partial u}{\partial t} - 2\Omega v\right) = -\frac{\partial \hat{p}}{\partial x} + \frac{\partial S_{xz}}{\partial z} - \frac{\sigma B_0^2 u}{1 - im_0}, \tag{1.61}$$

$$\rho\left(\frac{\partial v}{\partial t} + 2\Omega u\right) = -\frac{\partial \hat{p}}{\partial y} + \frac{\partial S_{yz}}{\partial z} - \frac{\sigma B_0^2 v}{1 - im_0}, \tag{1.62}$$

$$0 = -\frac{\partial \hat{p}}{\partial z} + \frac{\partial S_{zz}}{\partial z}, \tag{1.63}$$

$$\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)S_{xx}-2\left[\left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial z}S_{xz}+\lambda_2\frac{\partial u}{\partial z}\left(\frac{\partial S_{xz}}{\partial t}-\frac{\partial u}{\partial z}S_{zz}\right)\right]$$
$$= -2\mu\left(\lambda_3+\frac{3\lambda_4}{2}\frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial z}\right)^2, \tag{1.64}$$

$$\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)S_{xy}-\left[\begin{array}{c}\left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial z}S_{yz}+\frac{\partial v}{\partial z}S_{xz}\right)\\ +\lambda_2\frac{\partial u}{\partial z}\left(\frac{\partial S_{yz}}{\partial t}-\frac{\partial v}{\partial z}S_{zz}\right)+\lambda_2\frac{\partial v}{\partial z}\left(\frac{\partial S_{xz}}{\partial t}-\frac{\partial u}{\partial z}S_{zz}\right)\end{array}\right]$$
$$= -2\mu\left(\lambda_3+\frac{3\lambda_4}{2}\frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial z}\right), \tag{1.65}$$

$$\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)S_{xz}-\left[\left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial z}S_{zz}+\lambda_2\frac{\partial u}{\partial z}\frac{\partial S_{zz}}{\partial z}\right]$$
$$= \mu\left(1+\lambda_3\frac{\partial}{\partial t}+\lambda_4\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial u}{\partial z}\right), \tag{1.66}$$

$$\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)S_{yy}-2\left[\left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial z}S_{yz}+\lambda_2\frac{\partial v}{\partial z}\left(\frac{\partial S_{yz}}{\partial t}-\frac{\partial v}{\partial z}S_{zz}\right)\right]$$
$$= -2\mu\left(\lambda_3+\frac{3\lambda_4}{2}\frac{\partial}{\partial t}\right)\left(\frac{\partial v}{\partial z}\right)^2, \tag{1.67}$$

$$\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)S_{yz}-\left[\left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial z}S_{zz}+\lambda_2\frac{\partial v}{\partial z}\frac{\partial S_{zz}}{\partial z}\right]$$
$$= \mu\left(1+\lambda_3\frac{\partial}{\partial t}+\lambda_4\frac{\partial^2}{\partial t^2}\right)\frac{\partial v}{\partial z}, \tag{1.68}$$

$$\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)S_{zz}=0, \tag{1.69}$$

$$\hat{p}=p-\frac{1}{2}\rho\Omega^2r^2, \tag{1.70}$$

$$r^2=x^2+y^2. \tag{1.71}$$

In above expressions $m_0=\omega_e\tau_e$ denotes the Hall parameter. Equation (1.69) can be integrated easily. Employing $S_{zz}=0$ [112–114] in Eq. (1.63) we see that $\hat{p}$ is independent of z. Cross differentiation of Eqs.(1.61) and (1.62) yields

$$\rho\left(\frac{\partial^2 u}{\partial z\partial t}-2\Omega\frac{\partial v}{\partial z}\right)=\frac{\partial^2 S_{xz}}{\partial z^2}-\frac{\sigma B_0^2}{1-im_0}\frac{\partial u}{\partial z}, \tag{1.72}$$

$$\rho\left(\frac{\partial^2 v}{\partial z\partial t}+2\Omega\frac{\partial u}{\partial z}\right)=\frac{\partial^2 S_{yz}}{\partial z^2}-\frac{\sigma B_0^2}{1-im_0}\frac{\partial v}{\partial z}. \tag{1.73}$$

Upon making use of $S_{zz}=0$, Eqs. (1.64) to (1.68) become

$$\begin{aligned}&\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)S_{xx}-2\left[\left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial z}S_{xz}+\lambda_2\frac{\partial u}{\partial z}\frac{\partial S_{xz}}{\partial t}\right]\\ =&\;-2\mu\left(\lambda_3+\frac{3\lambda_4}{2}\frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial z}\right)^2,\end{aligned} \tag{1.74}$$

$$\begin{aligned}&\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)S_{xy}-\left[\begin{array}{c}\left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial z}S_{yz}+\frac{\partial v}{\partial z}S_{xz}\right)\\ +\lambda_2\frac{\partial u}{\partial z}\frac{\partial S_{yz}}{\partial t}+\lambda_2\frac{\partial v}{\partial z}\frac{\partial S_{xz}}{\partial t}\end{array}\right]\\ =&\;-2\mu\left(\lambda_3+\frac{3\lambda_4}{2}\frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial z}\right),\end{aligned} \tag{1.75}$$

$$\begin{aligned}&\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)S_{xz}\\ =&\;\mu\left(1+\lambda_3\frac{\partial}{\partial t}+\lambda_4\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial u}{\partial z}\right),\end{aligned} \tag{1.76}$$

$$\begin{aligned}&\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)S_{yy}-2\left[\left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial z}S_{yz}+\lambda_2\frac{\partial v}{\partial z}\frac{\partial S_{yz}}{\partial t}\right]\\ =&\;-2\mu\left(\lambda_3+\frac{3\lambda_4}{2}\frac{\partial}{\partial t}\right)\left(\frac{\partial v}{\partial z}\right)^2,\end{aligned} \tag{1.77}$$

$$\begin{aligned}&\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)S_{yz}\\ =&\;\mu\left(1+\lambda_3\frac{\partial}{\partial t}+\lambda_4\frac{\partial^2}{\partial t^2}\right)\frac{\partial v}{\partial z}.\end{aligned} \tag{1.78}$$

Multiplying Eqs.(1.72) and (1.73) by $\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)$ and then using Eqs. (1.76) and (1.78) in the resulting equations we arrive at

$$\begin{aligned}&\rho\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2 u}{\partial z\partial t}-2\Omega\frac{\partial v}{\partial z}\right)+\frac{\sigma B_0^2}{1-im_0}\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)\frac{\partial u}{\partial z}\\ =&\ \mu\left(1+\lambda_3\frac{\partial}{\partial t}+\lambda_4\frac{\partial^2}{\partial t^2}\right)\frac{\partial^3 u}{\partial z^3},\end{aligned}\tag{1.79}$$

$$\begin{aligned}&\rho\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2 v}{\partial z\partial t}+2\Omega\frac{\partial u}{\partial z}\right)+\frac{\sigma B_0^2}{1-im_0}\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)\frac{\partial v}{\partial z}\\ =&\ \mu\left(1+\lambda_3\frac{\partial}{\partial t}+\lambda_4\frac{\partial^2}{\partial t^2}\right)\frac{\partial^3 v}{\partial z^3}.\end{aligned}\tag{1.80}$$

The above equations can be combined as follows

$$\begin{aligned}&\rho\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2 F}{\partial z\partial t}+2i\Omega\frac{\partial F}{\partial z}\right)+\frac{\sigma B_0^2}{1-im_0}\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)\frac{\partial F}{\partial z}\\ &=\mu\left(1+\lambda_3\frac{\partial}{\partial t}+\lambda_4\frac{\partial^2}{\partial t^2}\right)\frac{\partial^3 F}{\partial z^3},\end{aligned}\tag{1.81}$$

where

$$F=u+iv.\tag{1.82}$$

1.10.2 Governing equation for fourth grade fluid

Here an infinite non-conducting and porous plate (located at $z=0$) and the fourth grade fluid (which is in contact with the plate and occupies the whole of the region $z\geq 0$) are in uniform rotation with angular velocity $\mathbf{\Omega}$ parallel to the $z-$axis. The fluid is assumed incompressible and electrically conducting in the presence of a constant magnetic field $\mathbf{B}_0$ acting in the $z-$direction. The Hall current is taken into account whereas the induced magnetic field, ion-slip and thermoelectric effects are neglected.

The Cauchy stress tensor of a fourth grade fluid is

$$\mathbf{T}=-p\mathbf{I}+\mathbf{S},\tag{1.83}$$

$$\mathbf{S}=\sum_{j=1}^{4} S_{j}, \tag{1.84}$$

whence

$$\mathbf{S}_1=\mu\mathbf{A}_1, \tag{1.85}$$

$$\mathbf{S}_2=\alpha_1\mathbf{A}_2+\alpha_2\mathbf{A}_1^2, \tag{1.86}$$

$$\mathbf{S}_3=\beta_1\mathbf{A}_3+\beta_2\left(\mathbf{A}_1\mathbf{A}_2+\mathbf{A}_2\mathbf{A}_1\right)+\beta_3\left(tr\mathbf{A}_2\right)\mathbf{A}_1, \tag{1.87}$$

$$\begin{aligned}\mathbf{S}_4 &= \gamma_1\mathbf{A}_4+\gamma_2\left(\mathbf{A}_3\mathbf{A}_1+\mathbf{A}_1\mathbf{A}_3\right)+\gamma_3\mathbf{A}_2^2+\gamma_4\left(\mathbf{A}_2\mathbf{A}_1^2+\mathbf{A}_1^2\mathbf{A}_2\right)\\ &\quad+\gamma_5\left(tr\mathbf{A}_2\right)\mathbf{A}_2+\gamma_6\left(tr\mathbf{A}_2\right)\mathbf{A}_1^2+\gamma_7\left(tr\mathbf{A}_3\right)\mathbf{A}_1+\gamma_8\left(tr\mathbf{A}_2\mathbf{A}_1\right)\mathbf{A}_1,\end{aligned} \tag{1.88}$$

$$\mathbf{A}_1=\mathbf{L}+\mathbf{L}^{\mathsf{T}}, \qquad \mathbf{L}=\nabla\mathbf{V}, \tag{1.89}$$

$$\mathbf{A}_i=\frac{\partial\mathbf{A}_{i-1}}{\partial t}+\left(\mathbf{V}\cdot\nabla\right)\mathbf{A}_{i-1}+\mathbf{A}_{i-1}\mathbf{L}+\mathbf{L}^{\mathsf{T}}\mathbf{A}_{i-1}, \qquad i>1. \tag{1.90}$$

In above equations μ is the dynamic viscosity, $\mathbf{A}_i$ are the Rivlin-Ericksen tensors, $\mathbf{L}^{\mathsf{T}}$ is the transpose of $\mathbf{L}$ and α_1, α_2, β_1- β_3 and $\gamma_1-\gamma_8$ are the material constants of the fourth grade fluid. The incompressibility condition is satisfied by the following definition of velocity field

$$\mathbf{V}\left(z\right)=\left(u(z),v(z),-w_0\right). \tag{1.91}$$

Here u and v are the velocity components in x and y directions, respectively and $w_0>0$ corresponds to suction velocity and $w_0<0$ indicates the injection (or blowing) velocity. The expression of $\mathbf{S}$ is

$$\mathbf{S}(z)=\begin{pmatrix} S_{xx} & S_{xy} & S_{xz}\\ S_{yx} & S_{yy} & S_{yz}\\ S_{zx} & S_{zy} & S_{zz}\end{pmatrix}. \tag{1.92}$$

Making use of Eqs. (1.91) and (1.92) , Eqs. (1.83) – (1.90) yield

$$\begin{aligned} T_{xx} &= -p+\alpha_2\left(\frac{du}{dz}\right)^2-2\beta_2 w_0\frac{du}{dz}\frac{d^2u}{dz^2}+2\gamma_2 w_0^2\frac{du}{dz}\frac{d^3u}{dz^3} \\ &\quad +\gamma_3 w_0^2\left(\frac{d^2u}{dz^2}\right)^2+2\gamma_6\left(\frac{du}{dz}\right)^2\left\{\left(\frac{du}{dz}\right)^2+\left(\frac{dv}{dz}\right)^2\right\}, \end{aligned}$$

$$\begin{aligned} T_{yy} &= -p+\alpha_2\left(\frac{dv}{dz}\right)^2-2\beta_2 w_0\frac{dv}{dz}\frac{d^2v}{dz^2}+2\gamma_2 w_0^2\frac{dv}{dz}\frac{d^3v}{dz^3} \\ &\quad +\gamma_3 w_0^2\left(\frac{d^2v}{dz^2}\right)^2+2\gamma_6\left(\frac{dv}{dz}\right)^2\left\{\left(\frac{du}{dz}\right)^2+\left(\frac{dv}{dz}\right)^2\right\}, \end{aligned}$$

$$\begin{aligned} T_{zz} &= -p+(2\alpha_1+\alpha_2)\left\{\left(\frac{du}{dz}\right)^2+\left(\frac{dv}{dz}\right)^2\right\} \\ &\quad -2\beta_1 w_0\frac{d}{dz}\left[\frac{d^2u}{dz^2}+\frac{d^2v}{dz^2}\right]-2\beta_2 w_0\left\{\frac{du}{dz}\frac{d^2u}{dz^2}+\frac{dv}{dz}\frac{d^2v}{dz^2}\right\} \\ &\quad +2\gamma_1 w_0^2\left[3\left\{\left(\frac{d^2u}{dz^2}\right)^2+\left(\frac{d^2v}{dz^2}\right)^2\right\}+4\left(\frac{du}{dz}\frac{d^3u}{dz^3}+\frac{dv}{dz}\frac{d^3v}{dz^3}\right)\right] \\ &\quad +2\gamma_2 w_0^2\left(\frac{du}{dz}\frac{d^3u}{dz^3}+\frac{dv}{dz}\frac{d^3v}{dz^3}\right)+\gamma_3 w_0^2\left\{\left(\frac{d^2u}{dz^2}\right)^2+\left(\frac{d^2v}{dz^2}\right)^2\right\} \\ &\quad +2\left(2\gamma_3+2\gamma_4+2\gamma_5+\gamma_6\right)\left\{\left(\frac{du}{dz}\right)^2+\left(\frac{dv}{dz}\right)^2\right\}, \end{aligned}$$

$$\begin{aligned} T_{xy} &= \alpha_2\frac{du}{dz}\frac{dv}{dz}-2\beta_2 w_0\left\{\frac{du}{dz}\frac{d^2v}{dz^2}+\frac{dv}{dz}\frac{d^2u}{dz^2}\right\} \\ &\quad +\gamma_2 w_0^2\left(\frac{dv}{dz}\frac{d^3u}{dz^3}+\frac{du}{dz}\frac{d^3v}{dz^3}\right)+\gamma_3 w_0^2\frac{d^2u}{dz^2}\frac{d^2v}{dz^2} \\ &\quad +2\gamma_6\left\{\left(\frac{du}{dz}\right)^2+\left(\frac{dv}{dz}\right)^2\right\}\frac{du}{dz}\frac{dv}{dz}, \end{aligned}$$

$$T_{xz} = \begin{array}{l} \mu\frac{du}{dz} - \alpha_1 w_0 \frac{d^2u}{dz^2} + \beta_1 w_0^2 \frac{d^3u}{dz^3} + 2(\beta_2+\beta_3)\frac{du}{dz}\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\} \\ -\gamma_1 w_0^3 \frac{d^4u}{dz^4} - 6\gamma_2 w_0^2 \left\{\frac{du}{dz}\left(\frac{du}{dz}\frac{d^2u}{dz^2} + \frac{dv}{dz}\frac{d^2v}{dz^2}\right)\right\} \\ -(2\gamma_3+\gamma_4+2\gamma_5) w_0 \frac{d^2u}{dz^2}\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\} \\ -(\gamma_4+6\gamma_7+2\gamma_8) w_0 \frac{du}{dz}\left(\frac{du}{dz}\frac{d^2u}{dz^2} + \frac{dv}{dz}\frac{d^2v}{dz^2}\right), \end{array}$$

$$T_{yz} = \begin{array}{l} \mu\frac{dv}{dz} - \alpha_1 w_0 \frac{d^2v}{dz^2} + \beta_1 w_0^2 \frac{d^3v}{dz^3} + 2(\beta_2+\beta_3)\frac{dv}{dz}\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\} \\ -\gamma_1 w_0^3 \frac{d^4v}{dz^4} - 6\gamma_2 w_0^2 \left\{\frac{dv}{dz}\left(\frac{du}{dz}\frac{d^2u}{dz^2} + \frac{dv}{dz}\frac{d^2v}{dz^2}\right)\right\} \\ -(2\gamma_3+\gamma_4+2\gamma_5) w_0 \frac{d^2v}{dz^2}\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\} \\ -(\gamma_4+6\gamma_7+2\gamma_8) w_0 \frac{dv}{dz}\left(\frac{du}{dz}\frac{d^2u}{dz^2} + \frac{dv}{dz}\frac{d^2v}{dz^2}\right), \end{array}$$

and $T_{xy} = T_{yx}$, $T_{xz} = T_{zx}$, $T_{yz} = T_{zy}$.

Invoking Eqs. (1.50) to (1.91) one can write

$$-2\Omega\rho\frac{dv}{dz} - \rho w_0\frac{d^2u}{dz^2} = \frac{d^2}{dz^2}\left[\begin{array}{l} \mu\frac{du}{dz} - \alpha_1 w_0 \frac{d^2u}{dz^2} + \beta_1 w_0^2 \frac{d^3u}{dz^3} + 2(\beta_2+\beta_3)\frac{du}{dz}\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\} \\ -\gamma_1 w_0^3 \frac{d^4u}{dz^4} - 6\gamma_2 w_0^2 \left\{\frac{du}{dz}\left(\frac{du}{dz}\frac{d^2u}{dz^2} + \frac{dv}{dz}\frac{d^2v}{dz^2}\right)\right\} \\ -(2\gamma_3+\gamma_4+2\gamma_5) w_0 \frac{d^2u}{dz^2}\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\} \\ -(\gamma_4+6\gamma_7+2\gamma_8) w_0 \frac{du}{dz}\left(\frac{du}{dz}\frac{d^2u}{dz^2} + \frac{dv}{dz}\frac{d^2v}{dz^2}\right) \end{array}\right] - \frac{\sigma B_0^2}{1-im_0}\frac{du}{dz}, \quad (1.93)$$

$$2\Omega\rho\frac{du}{dz} - \rho w_0\frac{d^2v}{dz^2} = \frac{d^2}{dz^2}\left[\begin{array}{l} \mu\frac{dv}{dz} - \alpha_1 w_0 \frac{d^2v}{dz^2} + \beta_1 w_0^2 \frac{d^3v}{dz^3} + 2(\beta_2+\beta_3)\frac{dv}{dz}\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\} \\ -\gamma_1 w_0^3 \frac{d^4v}{dz^4} - 6\gamma_2 w_0^2 \left\{\frac{dv}{dz}\left(\frac{du}{dz}\frac{d^2u}{dz^2} + \frac{dv}{dz}\frac{d^2v}{dz^2}\right)\right\} \\ -(2\gamma_3+\gamma_4+2\gamma_5) w_0 \frac{d^2v}{dz^2}\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\} \\ -(\gamma_4+6\gamma_7+2\gamma_8) w_0 \frac{dv}{dz}\left(\frac{du}{dz}\frac{d^2u}{dz^2} + \frac{dv}{dz}\frac{d^2v}{dz^2}\right) \end{array}\right] - \frac{\sigma B_0^2}{1-im_0}\frac{dv}{dz}. \quad (1.94)$$

Employing Eq. (1.82) one obtains

$$
2\Omega i\rho\frac{dF}{dz}-\rho w_0\frac{d^2F}{dz^2} = \frac{d^2}{dz^2}\left[\begin{array}{c}\mu\frac{dF}{dz}-\alpha_1 w_0\frac{d^2F}{dz^2}+\beta_1 w_0^2\frac{d^3F}{dz^3}+2\left(\beta_2+\beta_3\right)\frac{dF}{dz}\left(\frac{dF}{dz}\frac{d\overline{F}}{dz}\right)\\ -\gamma_1 w_0^3\frac{d^4F}{dz^4}-3\gamma_2 w_0^2\frac{dF}{dz}\left(\frac{dF}{dz}\frac{d^2\overline{F}}{dz^2}+\frac{d^2F}{dz^2}\frac{d\overline{F}}{dz}\right)\\ -\left(2\gamma_3+\gamma_4+2\gamma_5\right)w_0\frac{d^2F}{dz^2}\left(\frac{dF}{dz}\frac{d\overline{F}}{dz}\right)\\ -\left(\gamma_4+6\gamma_7+2\gamma_8\right)\frac{w_0}{2}\frac{dF}{dz}\left(\frac{dF}{dz}\frac{d^2\overline{F}}{dz^2}+\frac{d^2F}{dz^2}\frac{d\overline{F}}{dz}\right)\end{array}\right] -\frac{\sigma B_0^2}{1-im_0}\frac{dF}{dz}, \tag{1.95}
$$

$$
\overline{F}=u-iv. \tag{1.96}
$$

Chapter 2

Hall effect on the rotating oscillating flows of an Oldroyd-B fluid in a porous medium

This theoretical study explores the influence of Hall current on the rotating flow of a non-Newtonian fluid in a porous medium. The non-Newtonian fluid model is an Oldroyd-B fluid. Mathematical modeling is based on the modified Darcy's law. The general periodic and elliptic harmonic oscillations are imposed. The flow equations are solved analytically and closed form solutions are derived. The characterization of the flow is made via the velocity components. The present results are compared with the previous studies in the literature analyzed for the emerging parameters.

2.1 Basic equations

The constitutive equation for an extra stress tensor in an Oldroyd-B fluid is [115]

$$\mathbf{S}+\lambda_1\left(\frac{d\mathbf{S}}{dt}-\mathbf{LS}-\mathbf{SL}^{\top}\right)=\mu\left[\mathbf{A}_1+\lambda_2\left(\frac{d\mathbf{A}_1}{dt}-\mathbf{LA}_1-\mathbf{A}_1\mathbf{L}^{\top}\right)\right]. \tag{2.1}$$

In above equation t is the time, μ is the dynamic viscosity, λ_1, λ_2 are the relaxation and retardation times, d/dt is the material time derivative, $\mathbf{L}$ is the velocity gradient and $\mathbf{L}^{\top}$ is the transpose of $\mathbf{L}$. The first Rivlin-Ericksen tensor $\mathbf{A}_1$ and $\mathbf{L}$ are defined as

$$\mathbf{L} = \nabla \mathbf{V}, \qquad \mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T,$$

where $\mathbf{V}$ is the velocity.

On the basis of Oldroyd's model, the following filtration law for describing both relaxation and retardation phenomena is suggested [116]

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \nabla p = -\frac{\mu\phi}{k_1} \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) \mathbf{V}, \tag{2.2}$$

in which ϕ is porosity of the porous medium and k_1 is the permeability. It should be noted that for $\lambda_1 = \lambda_2 = 0$, Eq. (2.2) reduces into classical Darcy's law. Since the pressure gradient in Eq. (2.2) can also be interpreted as a measure of the resistance to flow in the bulk of the porous medium and $\mathbf{R}$ is a measure of the flow resistance offered by the solid matrix, thus $\mathbf{R}$ can be inferred from Eq. (2.2) to satisfy the following equation [117]:

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \mathbf{R} = -\frac{\mu\phi}{k_1} \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) \mathbf{V}. \tag{2.3}$$

2.2 Mathematical analysis

We consider the rotating flow of an electrically conducting and incompressible Oldroyd-B fluid bounded by a non-conducting rigid infinite plate at $z = 0$. The fluid as well as the plate are in a state of solid body rotation with constant angular velocity $\boldsymbol{\Omega} = \Omega\hat{\mathbf{k}}$ ($\hat{\mathbf{k}}$ is a unit vector in the z-direction). A uniform strong magnetic field of flux density $\mathbf{B} = B_0\hat{\mathbf{k}}$ is applied to the fluid system. We assume a valid assumption, on the laboratory scale, that the induced magnetic field is neglected, since the magnetic Reynolds number is assumed to be very small. In general, for an electrically conducting fluid, Hall current significantly effects the flow in the presence of a strong magnetic field. Since the plate is infinite in extent, the physical variables (except pressure) depend upon z (the distance from the plate) and the time t. Thus, the momentum equation, and velocity $\mathbf{V}$ for the present flow situation are of the following forms

$$\rho\left[\frac{d\mathbf{V}}{dt}+2\boldsymbol{\Omega}\times\mathbf{V}+\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})\right]=\operatorname{div}\mathbf{T}+\mathbf{J}\times\mathbf{B}+\mathbf{R}, \tag{2.4}$$

$$\mathbf{V}(z,t)=(u(z,t),v(z,t),0), \tag{2.5}$$

Note that the incompressibility condition (1.49) is satisfied with the definition of velocity in Eq.(2.5). Equation (2.4) can also be written as

$$\begin{aligned}\rho\left(1+\lambda_1\frac{\partial}{\partial t}\right)\left[\frac{d\mathbf{V}}{dt}+2\boldsymbol{\Omega}\times\mathbf{V}\right] &= -\left(1+\lambda_1\frac{\partial}{\partial t}\right)\nabla\hat{p}+\left(1+\lambda_1\frac{\partial}{\partial t}\right)div\mathbf{S}\\ &\quad+\left(1+\lambda_1\frac{\partial}{\partial t}\right)\mathbf{J}\times\mathbf{B}+\left(1+\lambda_1\frac{\partial}{\partial t}\right)\mathbf{R}, \end{aligned}\tag{2.6}$$

whence

$$\hat{p}=p-\frac{\rho}{2}\Omega^2\left(x^2+y^2\right). \tag{2.7}$$

Following the methodology of (2.3) the x and $y-$components of $\mathbf{R}$ satisfy

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)R_x=-\frac{\mu\phi}{k_1}\left(1+\lambda_2\frac{\partial}{\partial t}\right)u \tag{2.8}$$

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)R_y=-\frac{\mu\phi}{k_1}\left(1+\lambda_2\frac{\partial}{\partial t}\right)v \tag{2.9}$$

Upon making use of Eqs. (1.51) and (2.5) – (2.9) we arrive at

$$\begin{aligned}\rho\left(1+\lambda_1\frac{\partial}{\partial t}\right)\left[\frac{\partial u}{\partial t}-2\Omega v\right] &= -\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial\hat{p}}{\partial x}+\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial S_{xz}}{\partial z}\\ &\quad-\frac{\sigma B_0^2}{1-im_0}\left(1+\lambda_1\frac{\partial}{\partial t}\right)u-\frac{\mu\phi}{k_1}\left(1+\lambda_2\frac{\partial}{\partial t}\right)u, \end{aligned}\tag{2.10}$$

$$\begin{aligned}\rho\left(1+\lambda_1\frac{\partial}{\partial t}\right)\left[\frac{\partial v}{\partial t}+2\Omega u\right] &= -\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial\hat{p}}{\partial y}+\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial S_{yz}}{\partial z}\\ &\quad-\frac{\sigma B_0^2}{1-im_0}\left(1+\lambda_1\frac{\partial}{\partial t}\right)v-\frac{\mu\phi}{k_1}\left(1+\lambda_2\frac{\partial}{\partial t}\right)v, \end{aligned}\tag{2.11}$$

$$0=-\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial\hat{p}}{\partial z}+\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial S_{zz}}{\partial z}. \tag{2.12}$$

Equations (2.1) and (2.5) yield the following expressions

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)S_{xz}-\lambda_1 S_{zz}\frac{\partial u}{\partial z}=\mu\left(1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial z}, \tag{2.13}$$

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)S_{yz}-\lambda_1 S_{zz}\frac{\partial v}{\partial z}=\mu\left(1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial z}, \tag{2.14}$$

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)S_{zz}=0. \tag{2.15}$$

Integration of above equation gives

$$S_{zz}=f(z)e^{-\frac{t}{\lambda_1}} \tag{2.16}$$

in which the function $f(z)$ is arbitrary. We seek the solution when $f(z)=0$ [118] and thus

$$S_{zz}=0. \tag{2.17}$$

Invoking Eq. (2.17) into Eqs. (2.10)–(2.12) and then eliminating the pressure gradient we get

$$\rho\left(1+\lambda_1\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 u}{\partial z\partial t}-2\Omega\frac{\partial v}{\partial z}\right)+\frac{\sigma B_0^2}{1-im_0}\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial z}$$
$$+\frac{\mu\phi}{k_1}\left(1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial z}=\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial^2 S_{xz}}{\partial z^2}, \tag{2.18}$$

$$\rho\left(1+\lambda_1\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 v}{\partial z\partial t}+2\Omega\frac{\partial u}{\partial z}\right)+\frac{\sigma B_0^2}{1-im_0}\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial z}$$
$$+\frac{\mu\phi}{k_1}\left(1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial z}=\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial^2 S_{yz}}{\partial z^2}, \tag{2.19}$$

where from Eqs. (2.13) and (2.14) one obtains

$$\rho\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial^2 S_{xz}}{\partial z^2}=\mu\left(1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial^3 u}{\partial z^3}, \tag{2.20}$$

$$\rho\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial^2 S_{yz}}{\partial z^2}=\mu\left(1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial^3 v}{\partial z^3}. \tag{2.21}$$

Combination of Eqs. (2.18) – (2.21) results the following partial differential equation

$$\rho\left(1+\lambda_1\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 F}{\partial z\partial t}+2i\Omega\frac{\partial F}{\partial z}\right)+\frac{\sigma B_0^2}{1-im_0}\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial F}{\partial z}$$

$$+\frac{\mu\phi}{k_1}\left(1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial F}{\partial z}=\mu\left(1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial^3 F}{\partial z^3} \tag{2.22}$$

in which

$$F=u+iv. \tag{2.23}$$

In the next two sections Eq. (2.22) will be solved for the two boundary value problems.

2.3 The case of general periodic oscillation

For this case the plate at $z=0$ is performing periodic oscillations $f(t)$ with period T_0. The resulting boundary condition therefore is

$$u=Uf(t),\qquad v=0\qquad \text{at } z=0. \tag{2.24}$$

By Fourier series

$$f(t)=\sum_{k=-\infty}^{\infty}c_k e^{ikn_0 t} \tag{2.25}$$

and the Fourier series coefficients c_k are

$$c_k=\frac{1}{T_0}\int_{T_0}f(t)e^{-ikn_0 t}dt, \tag{2.26}$$

with non-zero frequency $n_0=2\pi/T_0$. Also far away from the plate the fluid is at rest. The boundary conditions in terms of F become

$$F(0,t)=U\sum_{k=-\infty}^{\infty}c_k e^{ikn_0 t},\qquad F(\infty,t)=0 \tag{2.27}$$

in which U is the characteristic velocity. To simplify the flow analysis, we introduce the following dimensionless quantities:

$$\begin{aligned} z^* &= \frac{zU}{\nu}, \quad F^* = \frac{F}{U}, \quad t^* = \frac{tU^2}{\nu}, \quad n_0^* = \frac{n_0 \nu}{U^2}, \quad \lambda_1^* = \frac{\lambda_1 U^2}{\nu}, \\ \lambda_2^* &= \frac{\lambda_2 U^2}{\nu}, \quad \Omega^* = \frac{\Omega \nu}{U^2}, \quad M^{*2} = \frac{\sigma B_0^2 \nu}{\rho U^2}, \quad \frac{1}{K} = \frac{\phi \nu^2}{k_1 U^2}. \end{aligned} \tag{2.28}$$

It follows that Eq. (2.22) and boundary conditions (2.27) may be written as

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 F}{\partial z \partial t} + 2i\Omega \frac{\partial F}{\partial z}\right) + \frac{M^2}{1 - i m_0}\left(1 + \lambda_1 \frac{\partial}{\partial t}\right)\frac{\partial F}{\partial z}$$

$$+\frac{1}{K}\left(1 + \lambda_2 \frac{\partial}{\partial t}\right)\frac{\partial F}{\partial z} = \left(1 + \lambda_2 \frac{\partial}{\partial t}\right)\frac{\partial^3 F}{\partial z^3}, \tag{2.29}$$

$$F(0,t) = \sum_{k=-\infty}^{\infty} c_k e^{ikn_0 t}, \qquad F(\infty, t) = 0, \tag{2.30}$$

where the asterisks have been dropped for brevity.

The Fourier transform of the function $F(z,t)$ is defined by

$$\bar{F}(z,\omega) = \int_{-\infty}^{\infty} F(z,t) e^{-i\omega t} dt, \tag{2.31}$$

in which ω is the temporal frequency. The problem in the transformed $\omega-$plane now reduces to

$$\frac{d^3 \bar{F}}{dz^3} - \frac{1}{1 + i\omega\lambda_2}\left[(w + 2\Omega)(i - \omega\lambda_1) + \frac{M^2}{1 - i m_0}(1 + i\omega\lambda_1) + \frac{1}{K}(1 + i\omega\lambda_2)\right]\frac{d\bar{F}}{dz} = 0, \tag{2.32}$$

$$\bar{F}(0,\omega) = 2\pi \sum_{k=-\infty}^{\infty} c_k \delta(\omega - k n_0), \qquad \bar{F}(\infty, \omega) = 0. \tag{2.33}$$

The solution of Eq. (2.32) subject to the boundary conditions (2.33) is

$$\bar{F}(z,\omega) = 2\pi \sum_{k=-\infty}^{\infty} c_k \delta(\omega - k n_0) e^{-(\xi + i\eta) z}, \tag{2.34}$$

where

$$(\xi + i\eta)^2 = \frac{1}{1+i\omega\lambda_2}\left[(\omega+2\Omega)(i-\omega\lambda_1) + \frac{M^2}{1-im_0}(1+i\omega\lambda_1) + \frac{1}{K}(1+i\omega\lambda_2)\right] \tag{2.35}$$

The inverse Fourier transform of Eq. (2.34) yields

$$F(z,t) = \sum_{k=-\infty}^{\infty} c_k e^{-\xi_k z + i(kn_0 t - \eta_k z)}, \tag{2.36}$$

where

$$\xi_k = \frac{1}{\sqrt{2}}\left[a_{1k} + \left(a_{1k}^2 + a_{2k}^2\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}, \tag{2.37}$$

$$\eta_k = \left[\frac{a_{2k}^2}{2\left\{a_{1k} + \left(a_{1k}^2 + a_{2k}^2\right)^{\frac{1}{2}}\right\}}\right]^{\frac{1}{2}}, \tag{2.38}$$

$$a_{1k} = \frac{1}{\left(1+k^2 n_0^2 \lambda_2^2\right)}\left[\begin{array}{c} kn_0(\lambda_2-\lambda_1)(kn_0+2\Omega) \\ +\frac{M^2}{1+m_0^2}\left\{\left(1+k^2n_0^2\lambda_1\lambda_2\right) + m_0 kn_0(\lambda_2-\lambda_1)\right\} \\ +\frac{1}{K}\left(1+k^2n_0^2\lambda_2^2\right)\end{array}\right], \tag{2.39}$$

$$a_{2k} = \frac{1}{\left(1+k^2 n_0^2 \lambda_2^2\right)}\left[\begin{array}{c} \left(1+k^2n_0^2\lambda_1\lambda_2\right)(kn_0+2\Omega) \\ -\frac{M^2}{1+m_0^2}\left\{kn_0(\lambda_2-\lambda_1) - m_0\left(1+k^2n_0^2\lambda_1\lambda_2\right)\right\}\end{array}\right] \tag{2.40}$$

and ξ_k and η_k are real. We note that the solution (2.36) is a result for the velocity due to general periodic oscillation of a rigid plate in its own plane. As the special cases of this oscillation, the velocities for different plate oscillations are obtained by an appropriate choice of the Fourier coefficients which give rise to different plate oscillations. The oscillations and their corresponding Fourier coefficients are given in the following table.

	Oscillations $f(t)$	Fourier coefficients c_k
(i)	$e^{in_0 t}$	$c_1 = 1$ and $c_k = 0$ $(\forall k \neq 1)$,
(ii)	$\cos n_0 t$	$c_1 = c_{-1} = \frac{1}{2}$ and $c_k = 0$, otherwise,
(iii)	$\sin n_0 t$	$c_1 = -c_{-1} = \frac{1}{2i}$ and $c_k = 0$, otherwise,
(iv)	$\begin{bmatrix} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert < T_0/2 \end{bmatrix}$	$c_0 = \frac{2T_1}{T_0}$, $c_k = \frac{\sin(kn_0T_1)}{k\pi}$, $(\forall k \neq 0)$,
(v)	$\sum\limits_{k=-\infty}^{\infty} \delta(t - kT_0)$	$c_k = \frac{1}{T_0}$, $\forall k$.

The resulting expressions of flow fields in the above five cases can be deduced by using successively the appropriate Fourier coefficients in Eq. (2.36). These are

$$F_1(z,t) = e^{-\xi_1 z + i(n_0 t - \eta_1 z)}, \tag{2.41}$$

$$F_2(z,t) = \frac{1}{2}\left[e^{-\xi_1 z + i(n_0 t - \eta_1 z)} + e^{-\xi_{-1} z - i\left(n_0 t + \eta_{-1} z\right)}\right], \tag{2.42}$$

$$F_3(z,t) = \frac{1}{2i}\left[e^{-\xi_1 z + i(n_0 t - \eta_1 z)} - e^{-\xi_{-1} z - i\left(n_0 t + \eta_{-1} z\right)}\right], \tag{2.43}$$

$$F_4(z,t) = \sum_{k=-\infty}^{\infty} \frac{\sin k n_0 T_1}{k\pi} e^{-\xi_k z + i(k n_0 t - \eta_k z)}, \quad k \neq 0, \tag{2.44}$$

$$F_5(z,t) = \frac{1}{T_0} \sum_{k=-\infty}^{\infty} e^{-\xi_k z + i(k n_0 t - \eta_k z)}. \tag{2.45}$$

2.4 The case of elliptic harmonic oscillation

Here, the flow is induced due to harmonic oscillations of a plate. The governing problem consists of Eq.(2.29) and the following dimensionless boundary conditions

$$F(0,t) = a e^{in_0 t} + b e^{-in_0 t}, \qquad F(\infty, t) = 0. \tag{2.46}$$

in which a and b are complex constants. In addition, we have the bounded solutions at infinity. For the present problem, let us consider the following solution

$$F = aF_1(z)e^{in_0t} + bF_2(z)e^{-in_0t}, \qquad n > 0. \tag{2.47}$$

Making use of Eq.(2.47) into Eqs. (2.29) and (2.46) and then solving the resulting ordinary differential systems for F_1 and F_2 we get for $n_0 < 2\Omega$ as

$$u = \begin{bmatrix} e^{-\alpha_1\Psi_1}\left\{a_1\cos\left(\beta_1\Psi_1 - n_0t\right) + a_2\sin\left(\beta_1\Psi_1 - n_0t\right)\right\} \\ +e^{-\alpha_2\Psi_2}\left\{b_1\cos\left(\beta_2\Psi_2 + n_0t\right) + b_2\sin\left(\beta_2\Psi_2 + n_0t\right)\right\} \end{bmatrix}, \tag{2.48}$$

$$v = \begin{bmatrix} e^{-\alpha_1\Psi_1}\left\{a_2\cos\left(\beta_1\Psi_1 - n_0t\right) - a_1\sin\left(\beta_1\Psi_1 - n_0t\right)\right\} \\ +e^{-\alpha_2\Psi_2}\left\{b_2\cos\left(\beta_2\Psi_2 + n_0t\right) - b_1\sin\left(\beta_2\Psi_2 + n_0t\right)\right\} \end{bmatrix}, \tag{2.49}$$

where $a = a_1 + ia_2$ and $b = b_1 + ib_2$. In above expressions

$$\alpha_j = \frac{1}{\sqrt{2}}\left[S_j + \left(S_j^2+1\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}, \quad \beta_j = \frac{1}{\sqrt{2}}\left[-S_j + \left(S_j^2+1\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}, \qquad j = 1, 2,$$

$$\Psi_1 = \left(\frac{B_1 - \lambda_2 n_0 A_1}{1 + \lambda_2^2 n_0^2}\right)^{\frac{1}{2}} z, \qquad \Psi_2 = \left(\frac{B_2 + \lambda_2 n_0 A_2}{1 + \lambda_2^2 n_0^2}\right)^{\frac{1}{2}} z,$$

$$S_1 = \frac{A_1 + \lambda_2 n_0 B_1}{B_1 - \lambda_2 n_0 A_1}, \qquad S_2 = \frac{A_2 - \lambda_2 n_0 B_2}{B_2 + \lambda_2 n_0 A_2},$$

$$A_1 = \frac{M^2}{1 + m_0^2}\left(1 - m_0\lambda_1 n_0\right) - \lambda_1 n_0\left(n_0 + 2\Omega\right) + \frac{1}{K},$$

$$B_1 = \frac{M^2}{1 + m_0^2}\left(m_0 + \lambda_1 n_0\right) + n_0 + 2\Omega + \frac{\lambda_2 n_0}{K},$$

$$A_2 = \frac{M^2}{1 + m_0^2}\left(1 + m_0\lambda_1 n_0\right) + \lambda_1 n_0\left(2\Omega - n_0\right) + \frac{1}{K},$$

$$B_2 = 2\Omega - n_0 - \frac{M^2}{1 + m_0^2}\left(\lambda_1 n_0 - m_0\right) - \frac{\lambda_2 n_0}{K}.$$

For $n_0 > 2\Omega$ we have

$$u = \begin{bmatrix} e^{-\alpha_1\Psi_1}\left\{a_1\cos\left(\beta_1\Psi_1 - n_0t\right) + a_2\sin\left(\beta_1\Psi_1 - n_0t\right)\right\} \\ +e^{-\alpha_3\Psi_3}\left\{b_1\cos\left(\beta_3\Psi_3 - n_0t\right) - b_2\sin\left(\beta_3\Psi_3 - n_0t\right)\right\} \end{bmatrix}, \tag{2.50}$$

$$v = \begin{bmatrix} e^{-\alpha_1\Psi_1}\left\{a_2\cos\left(\beta_1\Psi_1 - n_0t\right) - a_1\sin\left(\beta_1\Psi_1 - n_0t\right)\right\} \\ +e^{-\alpha_3\Psi_3}\left\{b_2\cos\left(\beta_3\Psi_3 - n_0t\right) + b_1\sin\left(\beta_3\Psi_3 - n_0t\right)\right\} \end{bmatrix}, \tag{2.51}$$

$$\alpha_3 = \frac{1}{\sqrt{2}}\left[S_3 + \left(S_3^2+1\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}, \quad \beta_3 = \frac{1}{\sqrt{2}}\left[-S_3 + \left(S_3^2+1\right)^{\frac{1}{2}}\right]^{\frac{1}{2}},$$

$$\Psi_3 = \left(\frac{A_3\lambda_2n_0 - B_3}{1+\lambda_2^2n_0^2}\right)^{\frac{1}{2}} z, \quad S_3 = \frac{A_3 + \lambda_2n_0B_3}{A_3\lambda_2n_0 - B_3},$$

$$A_3 = \frac{M^2}{1+m_0^2}\left(1 + m_0\lambda_1n_0\right) - \lambda_1n_0\left(n_0 - 2\Omega\right) + \frac{1}{K},$$

$$B_3 = n_0 - 2\Omega - \frac{M^2}{1+m_0^2}\left(m_0 - \lambda_1n_0\right) + \frac{\lambda_2n_0}{K}.$$

For resonant case ($n_0 = 2\Omega$), the expressions for u and v are

$$u = \begin{bmatrix} e^{-\alpha_1\Psi_1}\left\{a_1\cos\left(\beta_1\Psi_1 - n_0t\right) + a_2\sin\left(\beta_1\Psi_1 - n_0t\right)\right\} \\ +e^{-\alpha_0\Psi_0}\left\{b_1\cos\left(\beta_0\Psi_0 + n_0t\right) + b_2\sin\left(\beta_0\Psi_0 + n_0t\right)\right\} \end{bmatrix}, \tag{2.52}$$

$$v = \begin{bmatrix} e^{-\alpha_1\Psi_1}\left\{a_2\cos\left(\beta_1\Psi_1 - n_0t\right) - a_1\sin\left(\beta_1\Psi_1 - n_0t\right)\right\} \\ +e^{-\alpha_0\Psi_0}\left\{b_2\cos\left(\beta_0\Psi_0 + n_0t\right) - b_1\sin\left(\beta_0\Psi_0 + n_0t\right)\right\} \end{bmatrix}, \tag{2.53}$$

whence

$$\alpha_0 = \frac{1}{\sqrt{2}}\left[S_0 + \left(S_0^2+1\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}, \quad \beta_0 = \frac{1}{\sqrt{2}}\left[-S_0 + \left(S_0^2+1\right)^{\frac{1}{2}}\right]^{\frac{1}{2}},$$

$$\Psi_0 = \left(\frac{B_0 + A_0\lambda_2n_0}{1+\lambda_2^2n_0^2}\right)^{\frac{1}{2}} z, \quad S_0 = \frac{A_0 - \lambda_2n_0B_0}{B_0 + \lambda_2n_0A_0},$$

$$A_0 = \frac{M^2}{1+m_0^2}\left(1 + m_0\lambda_1n_0\right) + \frac{1}{K}, \quad B_0 = -\frac{M^2}{1+m_0^2}\left(\lambda_1n_0 - m_0\right) - \frac{\lambda_2n_0}{K}.$$

2.5 Results and discussion

This section presents various graphical results obtained for both the cases of periodic oscillation and elliptic harmonic oscillation. The effects of the various parameters on the velocity profiles, especially, Hall parameter m_0 and the parameter K have been studied through several graphs in all the three cases of elliptic harmonic oscillations and the periodic oscillation $F_l(z,t)$ when ($l = 5$).

In order to see the variations of the parameters m_0 and K, we made figures $2.1 - 2.6$ for elliptic harmonic oscillations. Figures 2.1 and 2.2 are prepared for the case $n_0 < 2\Omega$, 2.3 and 2.4 for $n_0 > 2\Omega$ and 2.5 and 2.6 for resonant case when $n_0 = 2\Omega$ keeping $a_1 = a_2 = b_1 = b_2 = 1$ fixed. Panels a and b show the real part u and imaginary part v of the velocity profile, respectively. These figures indicate that u decreases in all the three cases by increasing K when $m_0 = 0$ and $m_0 \neq 0$. Moreover, it is observed that with an increase in K, v decreases near the plate and increases far away from the plate for both $m_0 = 0$ and $m_0 \neq 0$ in all cases of frequencies.

Figure 2.1a

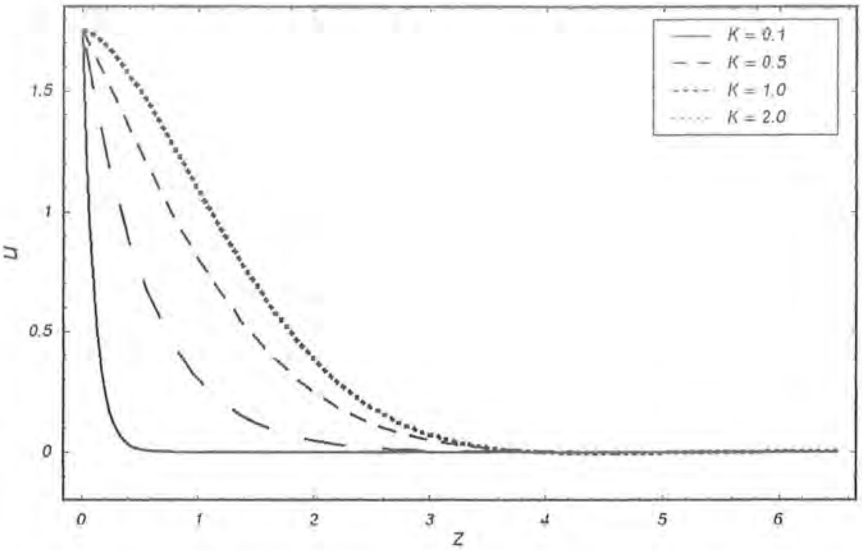
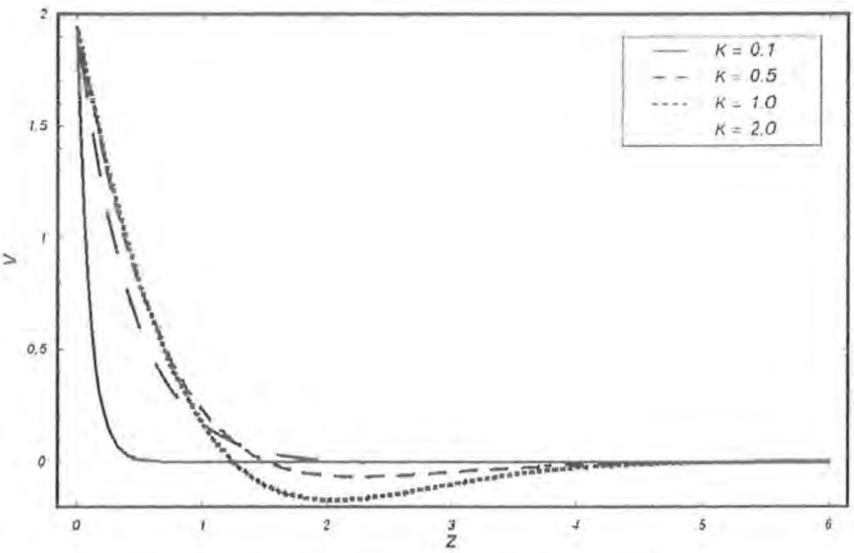


Figure 2.1b



Figures 2.1 : The variation of velocity parts for various values of K when $\lambda_1 = 2$, $\lambda_2 = 1$, $M = t = 0.5$, $m_0 = 0$ and $n_0/2\Omega = 0.5$ are fixed.

Figure 2.2a

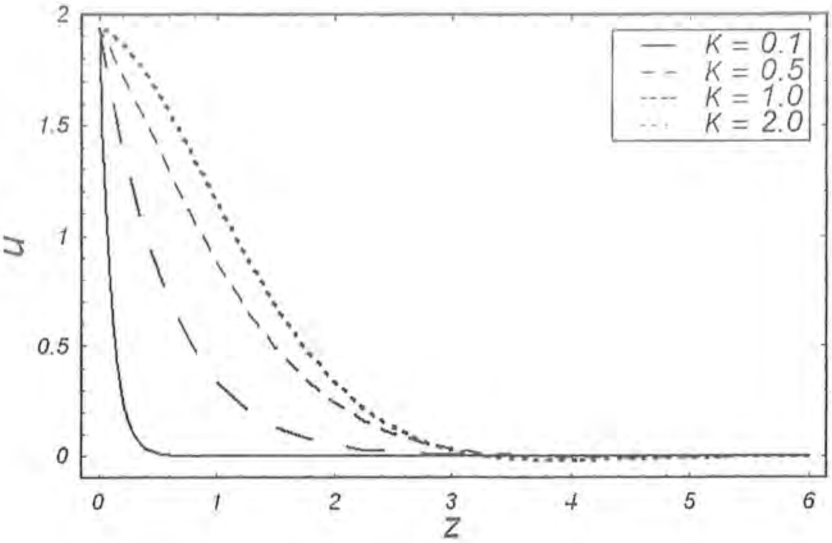
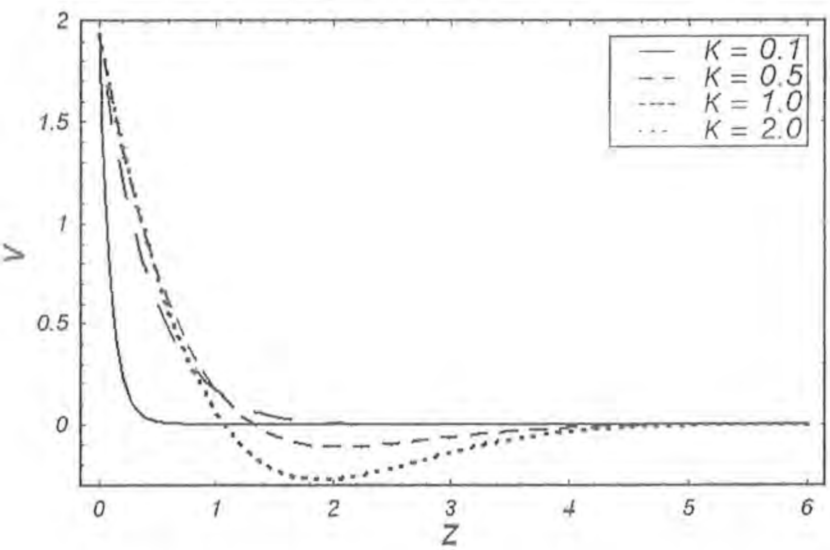


Figure 2.2b



Figures 2.2 : The variation of velocity parts for various values of K when $\lambda_1 = 2,\ \lambda_2 = 1,\ M = t = 0.5,\ m_0 = 1.5$ and $n_0/2\Omega = 0.5$ are fixed.

Figure 2.3a

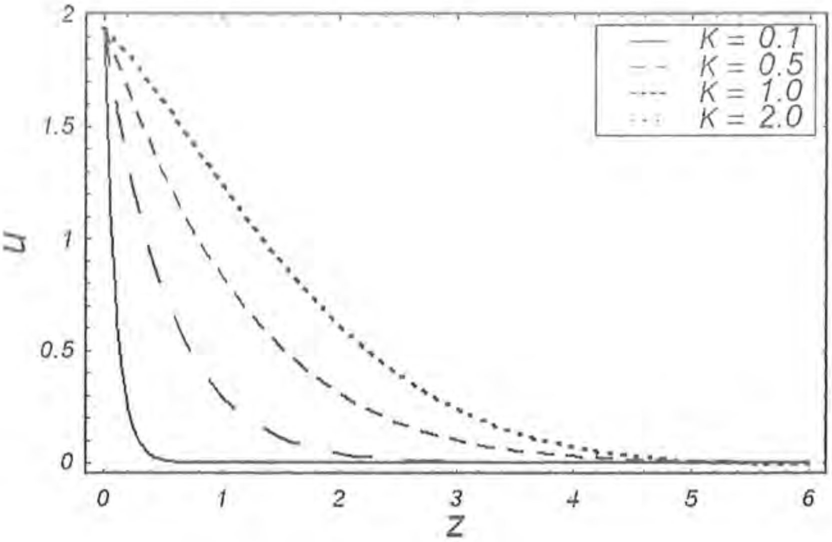
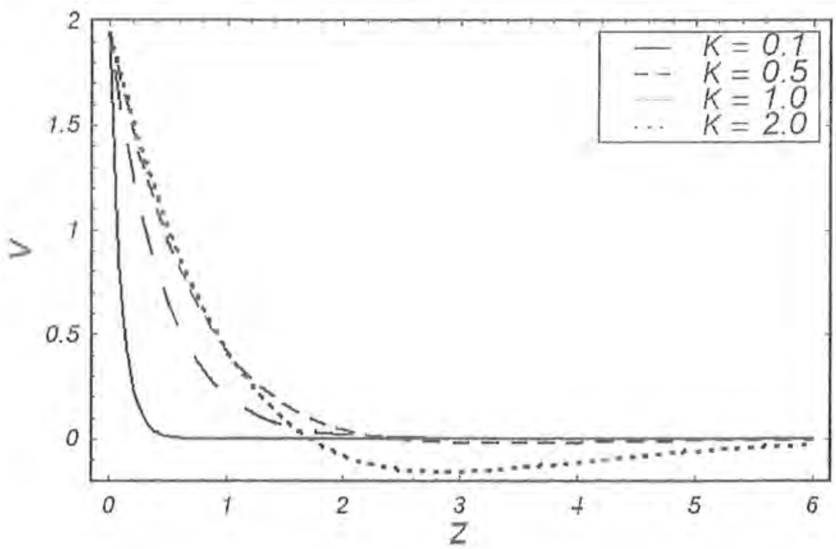


Figure 2.3b



Figures 2.3 : The variation of velocity parts for various values of K when $\lambda_1 = 2,\ \lambda_2 = 1,\ M = t = 0.5,\ m_0 = 0$ and $n_0/2\Omega = 2.5$ are fixed.

Figure 2.4a

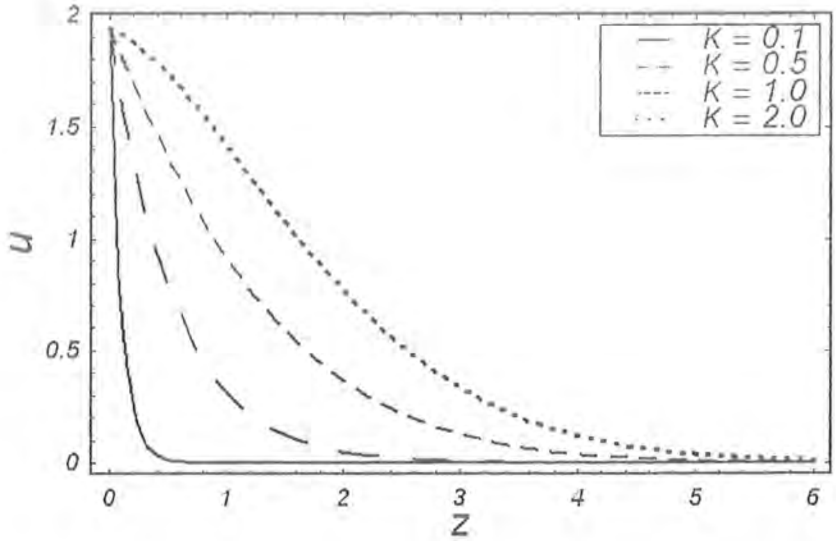
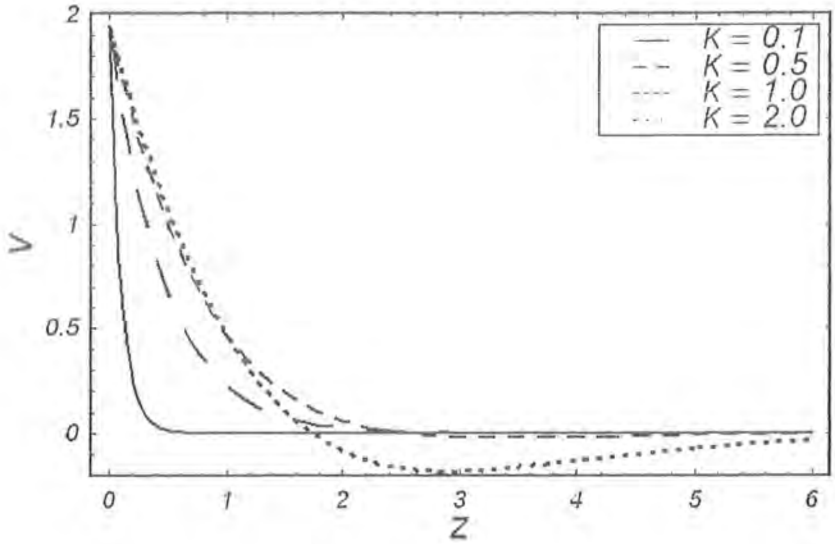


Figure 2.4b



Figures 2.4 : The variation of velocity parts for various values of K when $\lambda_1 = 2,\ \lambda_2 = 1,\ M = t = 0.5,\ m_0 = 1.5$ and $n_0/2\Omega = 2.5$ are fixed.

Figure 2.5a

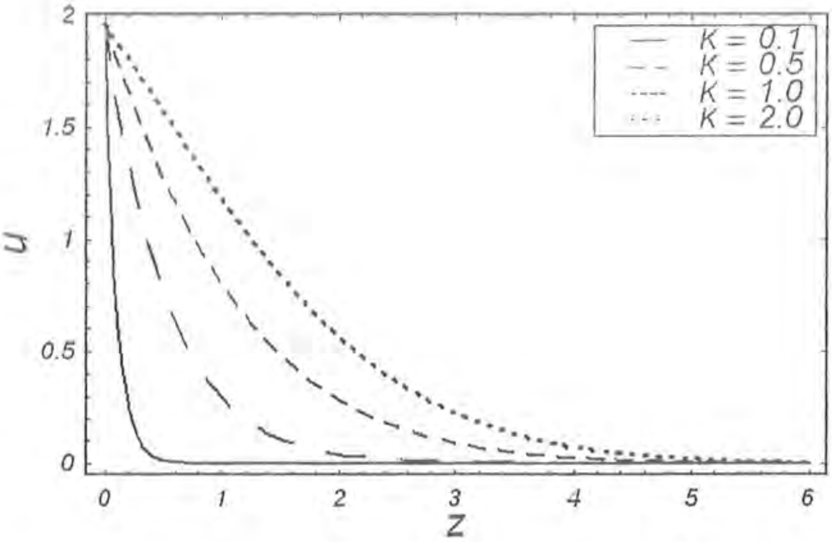
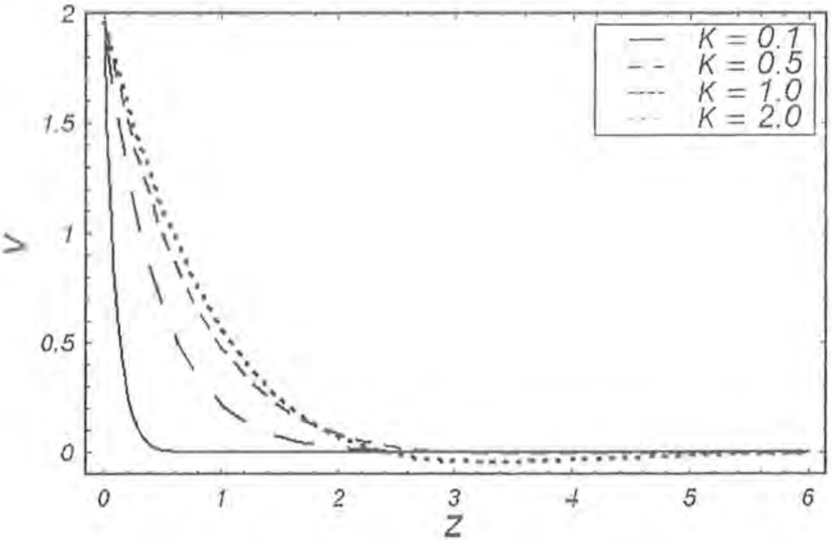


Figure 2.5b



Figures 2.5 : The variation of velocity parts for various values of K when $\lambda_1 = 2,\ \lambda_2 = 1,\ M = t = 0.5,\ m_0 = 0$ and $n_0/2\Omega = 1$ are fixed.

Figure 2.6a

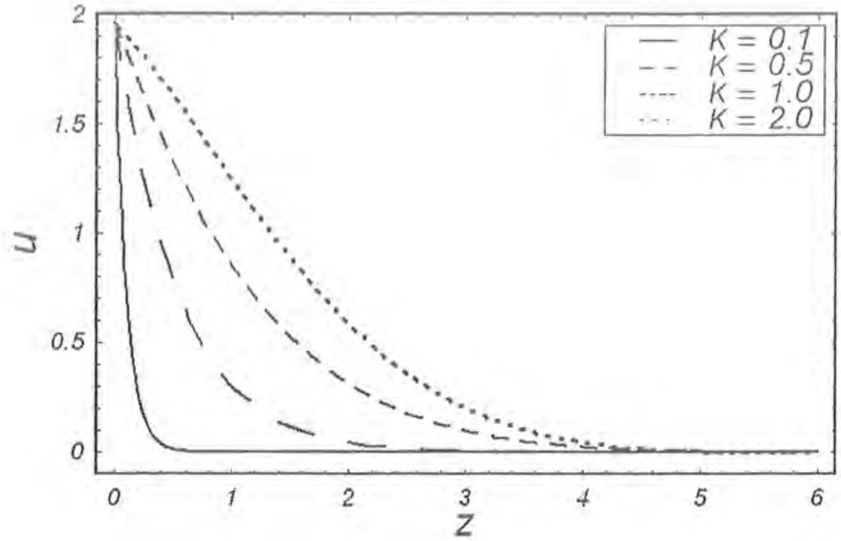
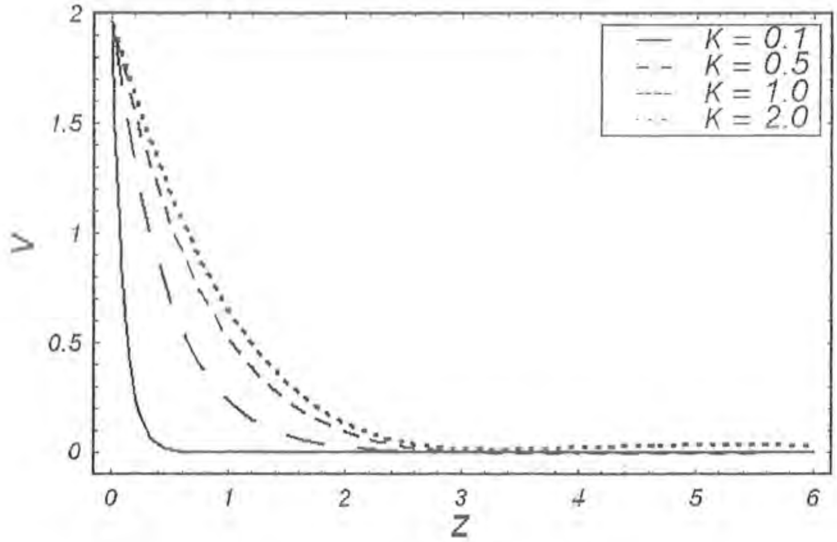


Figure 2.6b



Figures 2.6 : The variation of velocity parts for various values of K when $\lambda_1 = 2,\ \lambda_2 = 1,\ M = t = 0.5,\ m_0 = 1.5$ and $n_0/2\Omega = 1$ are fixed.

The effect of the Hall parameter m_0 on the velocity profiles with fixed values of other parameters is also illustrated in these figures. It is seen that with an increase of m_0, u increases whereas v decreases for all the three cases of elliptic harmonic oscillation but the behavior of the velocity profiles is quite opposite in the case of the periodic oscillation $F_l(z,t)$ when $(l=5)$ for both u and v. The velocity profile decreases with an increase of M. In fact, the effects of a transverse magnetic field on an electrically conducting fluid give rise to a resistive type force. This force has the tendency to slow down the motion of the fluid.

It is also apparent from the figures that the velocity profiles u for the case $n_0 > 2\Omega$ are the greatest when compared with the other cases. The velocity profiles u for the case $n_0 < 2\Omega$ are the smallest. However, the velocity profiles v are largest when $n_0 < 2\Omega$ and smallest for $n_0 > 2\Omega$.

2.6 Concluding remarks

The Hartmann flow of an Oldroyd-B fluid in a porous medium has been studied taking into account the Hall effect. The effects of the parameters m_0, M and K have been discussed. These parameters strongly effect the velocity profiles for all values of the frequencies. Meaningful analytic solutions are obtained for all values of the frequencies including the resonant frequency $(n=2\Omega)$. Actually, the Lorentz force is responsible to give a meaningful solution in the resonant case.

Chapter 3

The influence of Hall current on rotating flows of a Burgers' fluid through a porous medium

The main goal of this chapter is to extend the flow analysis of previous chapter to a Burgers' fluid. Two oscillatory flow problems in a porous medium are investigated. Modified Darcy's law corresponding to a rotating Burgers' fluid is first developed and then used in the flow modeling. The analytical solutions of the governing problems are obtained in the closed forms. The effects of key parameters are delineated by plotting graphs.

3.1 Governing equations

In a Burgers' fluid the constitutive relationship of an extra stress tensor $\mathbf{S}$ is given by

$$\mathbf{S}+\lambda\frac{\delta\mathbf{S}}{\delta t}+\beta\frac{\delta^2\mathbf{S}}{\delta t^2}=\mu\left(1+\lambda_r\frac{\delta}{\delta t}\right)\mathbf{A}_1, \tag{3.1}$$

$$\frac{\delta\mathbf{S}}{\delta t}=\frac{d\mathbf{S}}{dt}-\mathbf{LS}-\mathbf{SL}^{\top}. \tag{3.2}$$

Here t is the time, μ is the dynamic viscosity, $\mathbf{A}_1$ is the first Rivlin-Ericksen tensor, λ and β are the relaxation times, λ_r $(<\lambda)$ is the retardation time, $\mathbf{L}$ is the velocity gradient, $\mathbf{L}^{\top}$

is the transpose of L and $\delta/\delta t$ is the upper convected time derivative. It may be noted that the model (3.1) includes as special cases of an Oldroyd-B model (for $\beta = 0$), a Maxwell model (for $\beta = \lambda_r = 0$), a Newtonian fluid model ($\beta = \lambda_r = \lambda = 0$) and a second grade model (for $\beta = \lambda = 0$).

Having in mind Eq. (2.2), the pressure drop and velocity in a Burgers' fluid are related through the following expression

$$\left(1 + \lambda\frac{\partial}{\partial t} + \beta\frac{\partial^2}{\partial t^2}\right)\nabla p = -\frac{\mu\phi}{k_1}\left(1 + \lambda_r\frac{\partial}{\partial t}\right)\mathbf{V}, \tag{3.3}$$

where ϕ and k_1 respectively indicate the porosity and permeability of the porous space. Since the pressure gradient in above equation is a measure of the resistance to the flow in the bulk of porous space and the Darcy's resistance $\mathbf{R}$ is interpreted as the flow resistance offered by the solid matrix. Therefore, $\mathbf{R}$ through equation (3.3) satisfies

$$\left(1 + \lambda\frac{\partial}{\partial t} + \beta\frac{\partial^2}{\partial t^2}\right)\mathbf{R} = -\frac{\mu\phi}{k_1}\left(1 + \lambda_r\frac{\partial}{\partial t}\right)\mathbf{V}. \tag{3.4}$$

Let us now consider an incompressible electrically conducting rotating Burgers' fluid over an infinite rigid plate at $z = 0$. The fluid occupies the porous space $z > 0$. The fluid and the plate are in a state of rigid body rotation with uniform angular velocity $\mathbf{\Omega}$ about the z-axis. The plate is non-conducting and a uniform applied magnetic field $\mathbf{B}_0$ acts in the z-direction. The magnetic Reynolds number is taken small so that induced magnetic field is neglected. The effects of Hall current are included. For the problem under consideration, the extra stress and velocity are given in the following equations

$$\mathbf{S}(z,t) = \begin{pmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{pmatrix}, \tag{3.5}$$

$$\mathbf{V}(z,t) = (u(z,t), v(z,t), 0), \tag{3.6}$$

where u and v are the velocity components. Clearly Eq. (3.6) satisfies the incompressibility condition. Equation (2.4) can be written in the form as

$$\begin{aligned}\rho\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\left[\frac{d\mathbf{V}}{dt}+2\mathbf{\Omega}\times\mathbf{V}\right] &= -\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\boldsymbol{\nabla}\hat{p}+\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)div\mathbf{S}\\ &\quad -\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)(\mathbf{J}\times\mathbf{B})+\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\mathbf{R},\end{aligned} \tag{3.7}$$

where Eq. (3.4) helps in writing the following equations

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)R_x=-\frac{\mu\phi}{k_1}\left(1+\lambda_r\frac{\partial}{\partial t}\right)u, \tag{3.8}$$

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)R_y=-\frac{\mu\phi}{k_1}\left(1+\lambda_r\frac{\partial}{\partial t}\right)v \tag{3.9}$$

and the modified pressure $\hat{p}$ including the centrifugal term is

$$\hat{p}=p-\frac{\rho}{2}\Omega^2\left(x^2+y^2\right). \tag{3.10}$$

By using Eq. (1.51) one has

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)(\mathbf{J}\times\mathbf{B})=-\frac{\sigma B_0^2}{1-im_0}\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\mathbf{V}, \tag{3.11}$$

where σ designates the electrical conductivity of fluid and $m_0\,(=\omega_e\tau_e)$ is the Hall parameter. Adopping the similar procedure as for the derivation of Eq. (2.22) we have

$$\begin{aligned}&\rho\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2 F}{\partial z\partial t}+2i\Omega\frac{\partial F}{\partial z}\right)+\frac{\sigma B_0^2}{1-im_0}\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\frac{\partial F}{\partial z}\\ &\quad+\frac{\mu\phi}{k_1}\left(1+\lambda_r\frac{\partial}{\partial t}\right)\frac{\partial F}{\partial z}=\mu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\frac{\partial^3 F}{\partial z^3},\end{aligned} \tag{3.12}$$

where

$$F=u+iv.$$

In the next two sections, we will find the solutions of Eq. (3.12) for the two flow problems induced by the general periodic and elliptic harmonic oscillations.

3.2 Flow due to general periodic oscillation

Here, the disturbance in the fluid is induced due to periodic oscillation of a plate at $z = 0$. The fluid far away from the plate is stationary. Hence the boundary conditions corresponding the fluid motion are

$$F(0,t) = U \sum_{k=-\infty}^{\infty} c_k e^{ikn_0 t}, \qquad F(\infty, t) = 0, \tag{3.13}$$

where the Fourier series coefficients $\{c_k\}$ are

$$c_k = \frac{1}{T_0} \int_{T_0} f(t) e^{-ikn_0 t} dt \tag{3.14}$$

and $n_0 = 2\pi/T_0$ (T_0 being the time period) is the non-zero oscillating frequency.

Defining

$$\begin{aligned} z^* &= \frac{zU}{\nu}, \quad F^* = \frac{F}{U}, \quad t^* = \frac{tU^2}{\nu}, \quad n_0^* = \frac{n_0 \nu}{U^2}, \quad \lambda^* = \frac{\lambda U^2}{\nu}, \quad \Omega^* = \frac{\Omega v}{U^2}, \\ \lambda_r^* &= \frac{\lambda_r U^2}{\nu}, \quad M^2 = \frac{\sigma B_0^2 \nu}{\rho U^2}, \quad \frac{1}{K} = \frac{\phi \nu^2}{k_1 U^2}, \quad \beta^* = \frac{\beta U^4}{\nu^2} \end{aligned} \tag{3.15}$$

the problem reduces to

$$\begin{aligned} &\left(1 + \lambda \frac{\partial}{\partial t} + \beta \frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2 F}{\partial z \partial t} + 2i\Omega \frac{\partial F}{\partial z}\right) + \frac{M^2}{1 - im_0}\left(1 + \lambda \frac{\partial}{\partial t} + \beta \frac{\partial^2}{\partial t^2}\right)\frac{\partial F}{\partial z} \\ &\quad + \frac{1}{K}\left(1 + \lambda_r \frac{\partial}{\partial t}\right)\frac{\partial F}{\partial z} = \left(1 + \lambda_r \frac{\partial}{\partial t}\right)\frac{\partial^3 F}{\partial z^3}, \end{aligned} \tag{3.16}$$

$$F(0,t) = \sum_{k=-\infty}^{\infty} c_k e^{ikn_0 t}, \qquad F(\infty, t) = 0, \tag{3.17}$$

where the asterisks have been omitted for simplicity. The solution of the problem has been obtained using Fourier transform technique. Without giving details of algebra the solutions corresponding to Eq. (3.17) and five oscillations in the table of chapter two are directly written as

$$F(z,t) = \sum_{k=-\infty}^{\infty} c_k e^{-\xi_k z + i(kn_0 t - \eta_k z)}, \tag{3.18}$$

$$F_1(z,t) = e^{-\xi_1 z + i(n_0 t - \eta_1 z)}, \tag{3.19}$$

$$F_2(z,t) = \frac{1}{2}\left[e^{-\xi_1 z + i(n_0 t - \eta_1 z)} + e^{-\xi_{-1} z + i\left(n_0 t + \eta_{-1} z\right)}\right], \tag{3.20}$$

$$F_3(z,t) = \frac{1}{2i}\left[e^{-\xi_1 z + i(n_0 t - \eta_1 z)} - e^{-\xi_{-1} z - i\left(n_0 t + \eta_{-1} z\right)}\right], \tag{3.21}$$

$$F_4(z,t) = \sum_{k=-\infty}^{\infty} \frac{\sin k n_0 T_1}{k\pi} e^{-\xi_k z + i(k n_0 t - \eta_k z)}, \qquad k \neq 0, \tag{3.22}$$

$$F_5(z,t) = \frac{1}{T_0} \sum_{k=-\infty}^{\infty} e^{-\xi_k z + i(k n_0 t - \eta_k z)}, \tag{3.23}$$

where

$$\xi_k = \frac{1}{\sqrt{2}}\left[a_{1k} + \left(a_{1k}^2 + a_{2k}^2\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}, \tag{3.24}$$

$$\eta_k = \left[\frac{a_{2k}^2}{2\left\{a_{1k} + \left(a_{1k}^2 + a_{2k}^2\right)^{\frac{1}{2}}\right\}}\right]^{\frac{1}{2}}, \tag{3.25}$$

$$a_{1k} = \frac{1}{\left(1 + k^2 n_0^2 \lambda_r^2\right)}\left[\begin{array}{c} k n_0 \left(\lambda_r - \lambda - \beta \lambda_r k^2 n_0^2\right)\left(k n_0 + 2\Omega\right) + \frac{\left(1 + k^2 n_0^2 \lambda_r^2\right)}{K} \\ + \frac{M^2}{1 + m_0^2}\left\{1 + k^2 n_0^2 \lambda \lambda_r + m_0 k n_0 \left(\lambda_r - \lambda\right) - \beta k^2 n_0^2 \left(1 + m_0 \lambda_r\right)\right\} \end{array}\right], \tag{3.26}$$

$$a_{2k} = \frac{1}{\left(1 + k^2 n_0^2 \lambda_r^2\right)}\left[\begin{array}{c} \left(1 + k^2 n_0^2 \lambda \lambda_r - \beta k^2 n_0^2\right)\left(k n_0 + 2\Omega\right) \\ - \frac{M^2}{1 + m_0^2}\left\{k n_0 \left(\lambda_r - \lambda\right) - m_0 \left(1 + k^2 n_0^2 \lambda \lambda_r\right) - \beta k^2 n_0^2 \left(\lambda_r - m_0\right)\right\} \end{array}\right] \tag{3.27}$$

and ξ_k and η_k being real and positive.

3.3 Flow due to elliptic harmonic oscillation

Here, the problem which governs the flow consists of Eq. (3.16) and

$$F(0,t) = a e^{i n_0 t} + b e^{-i n_0 t}, \qquad F(\infty, t) = 0, \tag{3.28}$$

where a and b are complex constants. For $n_0 < 2\Omega$, the expressions for u and v are

$$u = \left[\begin{array}{c} e^{-\alpha_1 \Psi_1}\left\{a_1 \cos\left(\beta_1 \Psi_1 - n_0 t\right) + a_2 \sin\left(\beta_1 \Psi_1 - n_0 t\right)\right\} \\ + e^{-\alpha_2 \Psi_2}\left\{b_1 \cos\left(\beta_2 \Psi_2 + n_0 t\right) + b_2 \sin\left(\beta_2 \Psi_2 + n_0 t\right)\right\} \end{array}\right], \tag{3.29}$$

$$v = \left[\begin{array}{c} e^{-\alpha_1 \Psi_1} \left\{ a_2 \cos\left(\beta_1 \Psi_1 - n_0 t\right) - a_1 \sin\left(\beta_1 \Psi_1 - n_0 t\right) \right\} \\ + e^{-\alpha_2 \Psi_2} \left\{ b_2 \cos\left(\beta_2 \Psi_2 + n_0 t\right) - b_1 \sin\left(\beta_2 \Psi_2 + n_0 t\right) \right\} \end{array} \right], \tag{3.30}$$

where $a = a_1 + ia_2$, $b = b_1 + ib_2$ and

$$\alpha_j = \frac{1}{\sqrt{2}} \left[S_j + \left(S_j^2 + 1 \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}, \quad \beta_j = \frac{1}{\sqrt{2}} \left[-S_j + \left(S_j^2 + 1 \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}, \quad j = 1, 2,$$

$$\Psi_1 = \left(\frac{B_1 - \lambda_r n_0 A_1}{1 + \lambda_r^2 n_0^2} \right)^{\frac{1}{2}} z, \quad \Psi_2 = \left(\frac{B_2 + \lambda_r n_0 A_2}{1 + \lambda_r^2 n_0^2} \right)^{\frac{1}{2}} z,$$

$$S_1 = \frac{A_1 + \lambda_r n_0 B_1}{B_1 - \lambda_r n_0 A_1}, \quad S_2 = \frac{A_2 - \lambda_r n_0 B_2}{B_2 + \lambda_r n_0 A_2},$$

$$A_1 = \frac{M^2}{1 + m_0^2} \left(1 - m_0 \lambda n_0 - \beta n_0^2 \right) - \lambda n_0 \left(n_0 + 2\Omega \right) + \frac{1}{K},$$

$$B_1 = \frac{M^2}{1 + m_0^2} \left(m_0 + \lambda n_0 - \beta n_0^2 m_0 \right) + \left(n_0 + 2\Omega \right) \left(1 - \beta n_0^2 \right) + \frac{\lambda_r n_0}{K},$$

$$A_2 = \frac{M^2}{1 + m_0^2} \left(1 + m_0 \lambda_1 n_0 - \beta n_0^2 m_0 \right) + \lambda n_0 \left(2\Omega - n_0 \right) + \frac{1}{K},$$

$$B_2 = \left(2\Omega - n_0 \right) \left(1 - \beta n_0^2 \right) - \frac{M^2}{1 + m_0^2} \left(\lambda n_0 - m_0 - \beta n_0^2 \right) - \frac{\lambda_r n_0}{K}.$$

When $n_0 > 2\Omega$ then

$$u = \left[\begin{array}{c} e^{-\alpha_1 \Psi_1} \left\{ a_1 \cos\left(\beta_1 \Psi_1 - n_0 t\right) + a_2 \sin\left(\beta_1 \Psi_1 - n_0 t\right) \right\} \\ + e^{-\alpha_3 \Psi_3} \left\{ b_1 \cos\left(\beta_3 \Psi_3 - n_0 t\right) - b_2 \sin\left(\beta_3 \Psi_3 - n_0 t\right) \right\} \end{array} \right], \tag{3.31}$$

$$v = \left[\begin{array}{c} e^{-\alpha_1 \Psi_1} \left\{ a_2 \cos\left(\beta_1 \Psi_1 - n_0 t\right) - a_1 \sin\left(\beta_1 \Psi_1 - n_0 t\right) \right\} \\ + e^{-\alpha_3 \Psi_3} \left\{ b_2 \cos\left(\beta_3 \Psi_3 - n_0 t\right) + b_1 \sin\left(\beta_3 \Psi_3 - n_0 t\right) \right\} \end{array} \right], \tag{3.32}$$

$$\alpha_3 = \frac{1}{\sqrt{2}} \left[S_3 + \left(S_3^2 + 1 \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}, \quad \beta_3 = \frac{1}{\sqrt{2}} \left[-S_3 + \left(S_3^2 + 1 \right)^{\frac{1}{2}} \right]^{\frac{1}{2}},$$

$$\Psi_3 = \left(\frac{\lambda_r n_0 A_3 - B_3}{1 + \lambda_r^2 n_0^2} \right)^{\frac{1}{2}} z, \quad S_3 = \frac{A_3 + \lambda_r n_0 B_3}{\lambda_r n_0 A_3 - B_3},$$

$$A_3 = \frac{M^2}{1 + m_0^2} \left(1 + m_0 \lambda n_0 - \beta n_0^2 m_0 \right) - \lambda n_0 \left(n_0 - 2\Omega \right) + \frac{1}{K},$$

$$B_3 = (n_0 - 2\Omega)\left(1 - \beta n_0^2\right) - \frac{M^2}{1+m_0^2}\left(m_0 - \lambda n_0 + \beta n_0^2\right) + \frac{\lambda_r n_0}{K}.$$

For resonant case $(n_0 = 2\Omega)$, we have

$$u = \begin{bmatrix} e^{-\alpha_1 \Psi_1}\left\{a_1 \cos\left(\beta_1 \Psi_1 - n_0 t\right) + a_2 \sin\left(\beta_1 \Psi_1 - n_0 t\right)\right\} \\ +e^{-\alpha_0 \Psi_0}\left\{b_1 \cos\left(\beta_0 \Psi_0 + n_0 t\right) + b_2 \sin\left(\beta_0 \Psi_0 + n_0 t\right)\right\} \end{bmatrix}, \tag{3.33}$$

$$v = \begin{bmatrix} e^{-\alpha_1 \Psi_1}\left\{a_2 \cos\left(\beta_1 \Psi_1 - n_0 t\right) - a_1 \sin\left(\beta_1 \Psi_1 - n_0 t\right)\right\} \\ +e^{-\alpha_0 \Psi_0}\left\{b_2 \cos\left(\beta_0 \Psi_0 + n_0 t\right) - b_1 \sin\left(\beta_0 \Psi_0 + n_0 t\right)\right\} \end{bmatrix}, \tag{3.34}$$

whence

$$\alpha_0 = \frac{1}{\sqrt{2}}\left[S_0 + \left(S_0^2+1\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}, \qquad \beta_0 = \frac{1}{\sqrt{2}}\left[-S_0 + \left(S_0^2+1\right)^{\frac{1}{2}}\right]^{\frac{1}{2}},$$

$$\Psi_0 = \left(\frac{\lambda_r n_0 A_0 + B_0}{1+\lambda_r^2 n_0^2}\right)^{\frac{1}{2}} z, \quad S_0 = \frac{A_0 - \lambda_r n_0 B_0}{\lambda_r n_0 A_0 + B_0},$$

$$A_0 = \frac{M^2}{1+m_0^2}\left(1 + m_0 \lambda n_0 - \beta n_0^2 m_0\right) + \frac{1}{K}, \quad B_0 = -\frac{M^2}{1+m_0^2}\left(\lambda n_0 - m_0 - \beta n_0^2\right) - \frac{\lambda_r n_0}{K}.$$

3.4 Results and discussion

Our interest here is to see the variations of M, m_0 and β. For this purpose, the obtained analytical results of u and v have been plotted against z. Further, the influence of K is also shown in figures 3.1 – 3.12 for both kinds of fluids i.e. an Oldroyd-B fluid ($\lambda \neq 0$, $\lambda_r \neq 0$, $\beta = 0$) and a Burgers' fluid for all values of the frequencies including the resonant frequency with $m_0 = 0$ and $m_0 \neq 0$. It is found that u for an Oldroyd-B fluid is smaller when compared with the Burgers' fluid case. However, this result cannot be generalized with other chosen values of parameters because the behavior of the rheological parameter β of the Burgers' fluid is non-monotonous. The behavior of u and v are quite opposite. It is also apparent from these figures that u increases by increasing m_0. This observation holds for all values of the frequencies.

Figures 3.1 and 3.2 show the behavior of the Hall parameter ($m_0 = 0$ and 1.5) for an Oldroyd-B fluid whereas figures 3.3 and 3.4 show the behavior of Hall parameter ($m_0 = 0$ and 1.5) for a Burgers' fluid in the case $n_0 < 2\Omega$. Figures 3.5 and 3.6 depict the variation of the Hall parameter ($m_0 = 0$ and 1.5) for an Oldroyd-B fluid whereas figures 3.7 and 3.8 give the variation of the velocity parts for a Burgers' fluid when $n_0 > 2\Omega$. The next four figures namely $3.9 - 3.12$ have been made for $n_0 = 2\Omega$. Here figures 3.9 and 3.10 show the velocity profiles for an Oldroyd-B fluid and figures 3.11 and 3.12 depict the velocity profiles for a Burgers' fluid. All these figures indicate that u increases by increasing K for $m_0 = 0$ and $m_0 \neq 0$. However, v increases and then decreases.

Figure 3.1a

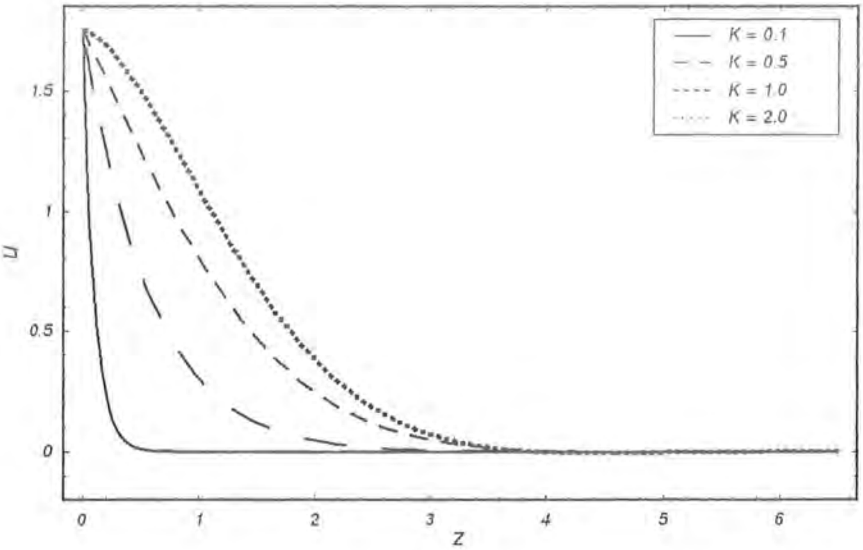
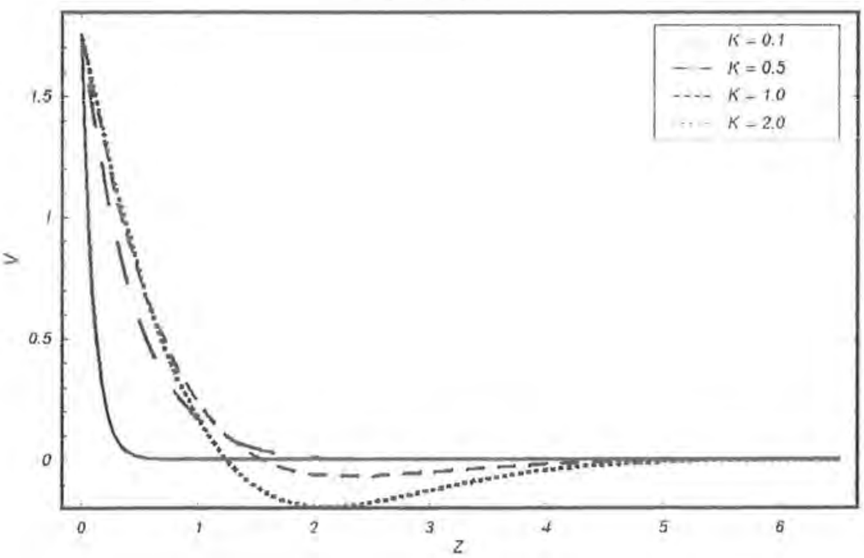


Figure 3.1b



Figures 3.1 : The variation of velocity parts for various values of K for an Oldroyd-B fluid ($\lambda = 2,\ \lambda_r = 1$ and $\beta = 0$) when $M = 0.5,\ t = 1,\ m_0 = 0$ and $n_0/2\Omega = 0.5$ are fixed.

Figure 3.2a

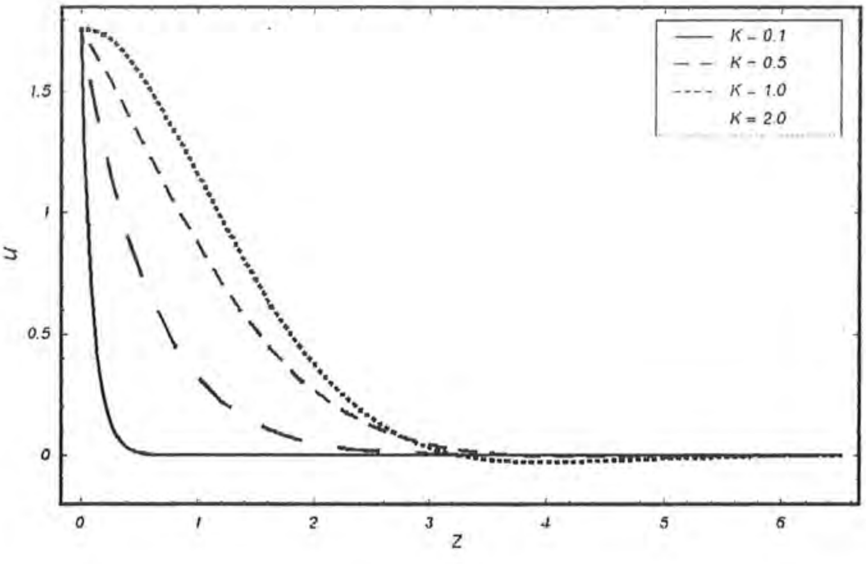
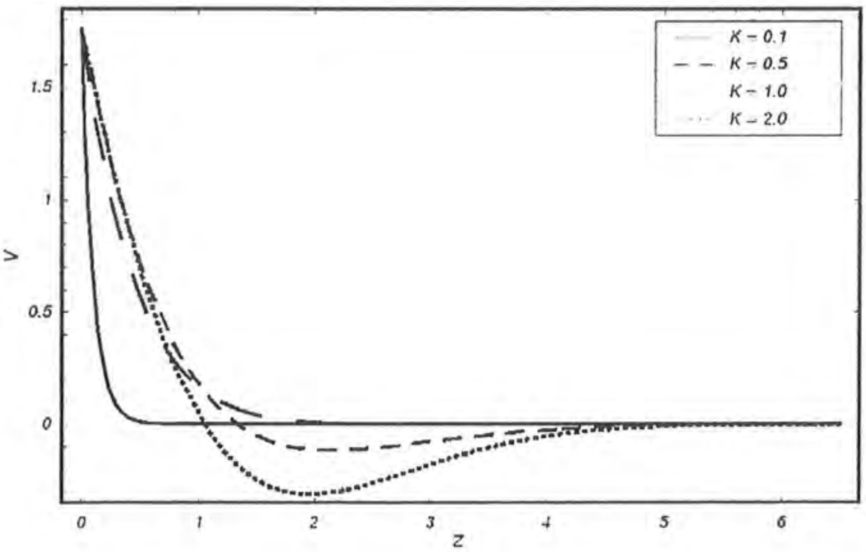


Figure 3.2b



Figures 3.2 : The variation of velocity parts for various values of K for an Oldroyd-B fluid ($\lambda = 2,\ \lambda_r = 1$ and $\beta = 0$) when $M = 0.5,\ t = 1,\ m_0 = 2$ and $n_0/2\Omega = 0.5$ are fixed.

Figure 3.3a

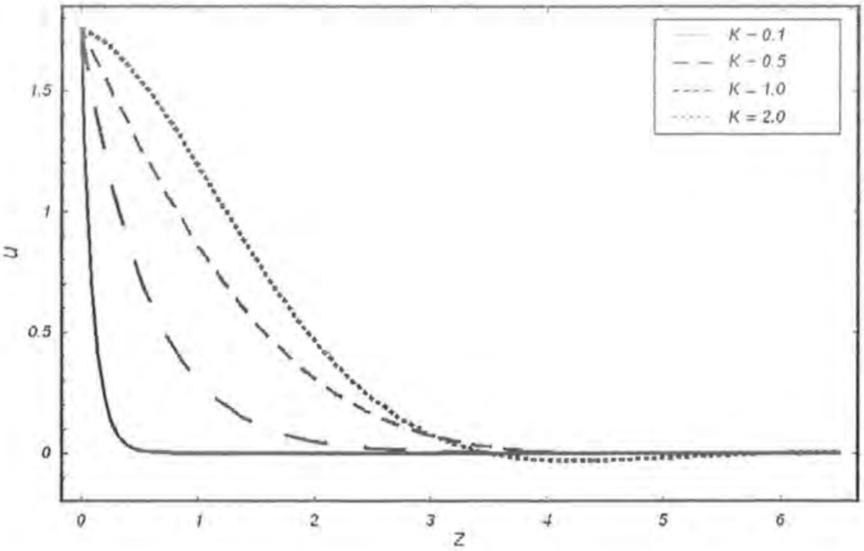
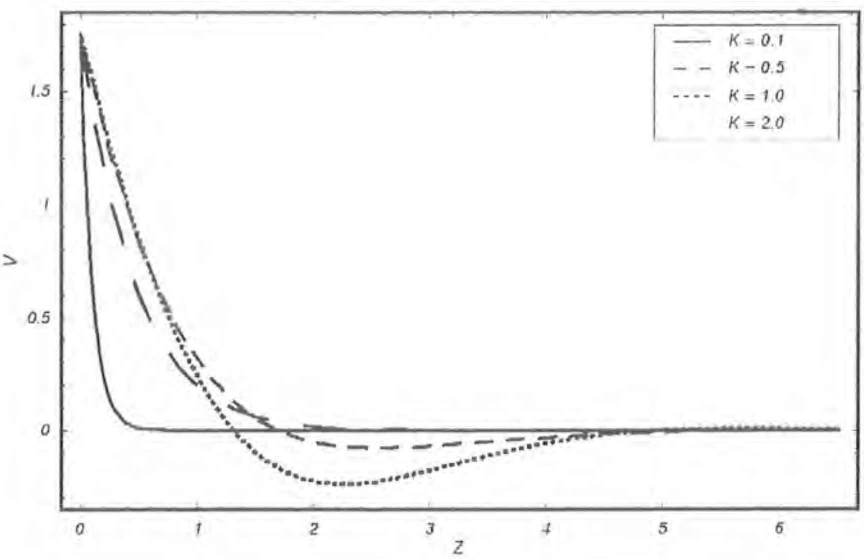


Figure 3.3b



Figures 3.3 : The variation of velocity parts for various values of K for a Burgers' fluid ($\lambda = 2,\ \lambda_r = 1$ and $\beta = 1.5$) when $M = 0.5,\ t = 1,\ m_0 = 0$ and $n_0/2\Omega = 0.5$ are fixed.

Figure 3.4a

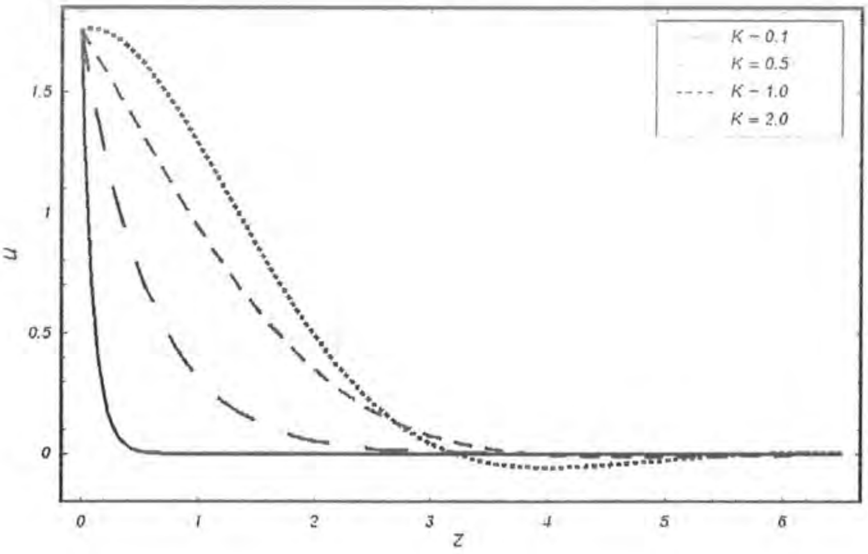
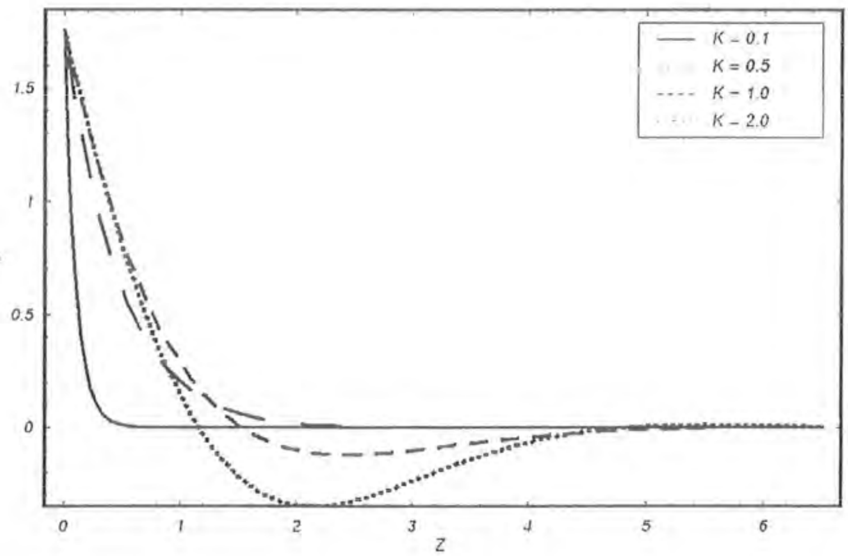


Figure 3.4b



Figures 3.4 : The variation of velocity parts for various values of K for a Burgers' fluid ($\lambda = 2,\ \lambda_r = 1$ and $\beta = 1.5$) when $M = 0.5,\ t = 1,\ m_0 = 2$ and $n_0/2\Omega = 0.5$ are fixed.

Figure 3.5a

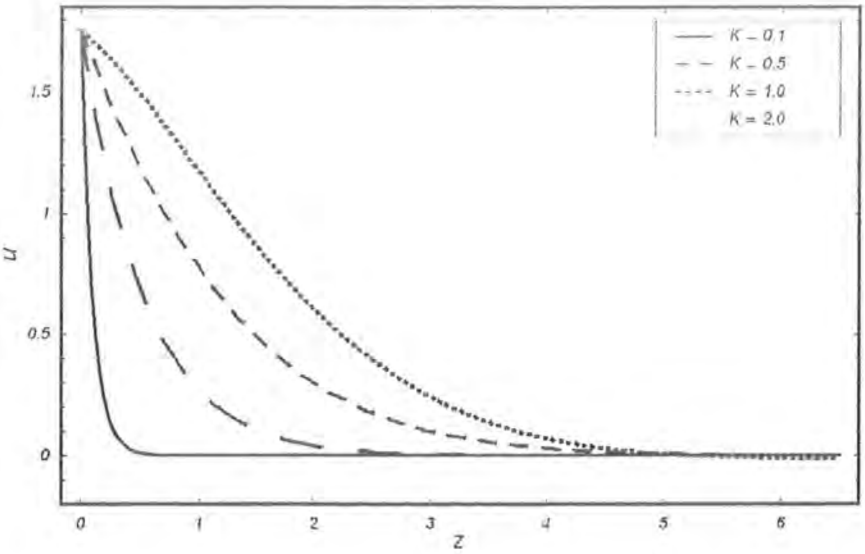
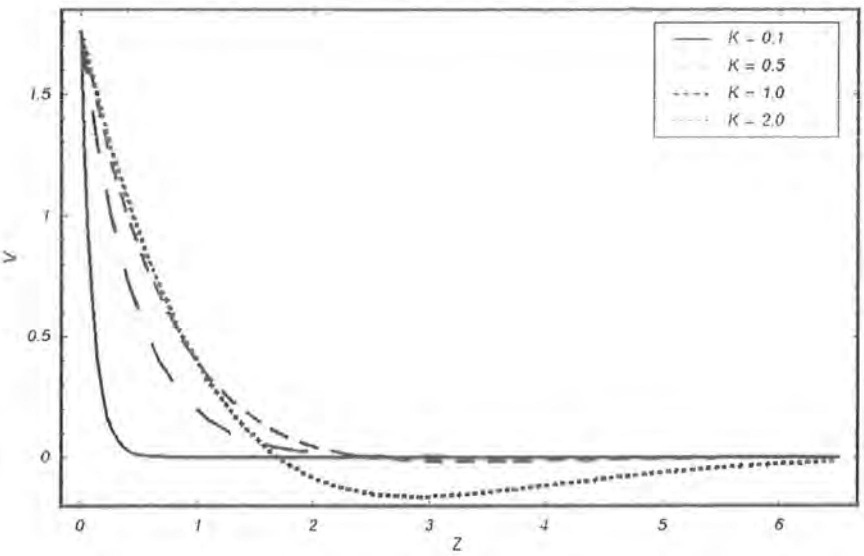


Figure 3.5b



Figures 3.5 : The variation of velocity parts for various values of K for an Oldroyd-B fluid ($\lambda = 2,\ \lambda_r = 1$ and $\beta = 0$) when $M = 0.5,\ t = 1,\ m_0 = 0$ and $n_0/2\Omega = 2.5$ are fixed.

Figure 3.6a

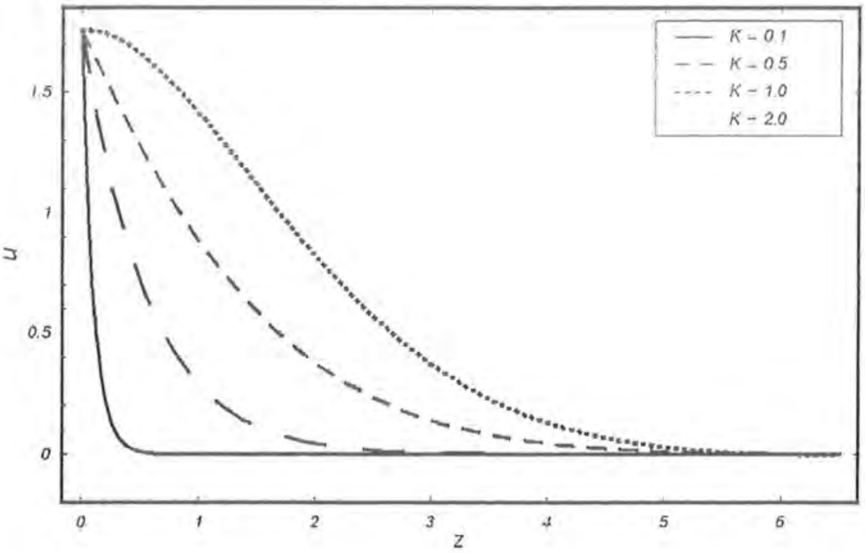
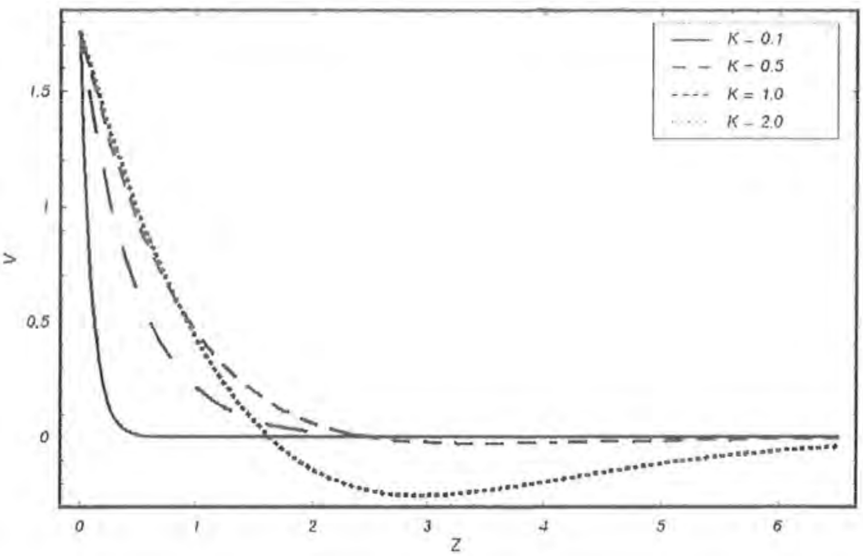


Figure 3.6b



Figures 3.6 : The variation of velocity parts for various values of K for an Oldroyd-B fluid ($\lambda = 2,\ \lambda_r = 1$ and $\beta = 0$) when $M = 0.5,\ t = 1,\ m_0 = 2$ and $n_0/2\Omega = 2.5$ are fixed.

Figure 3.7a

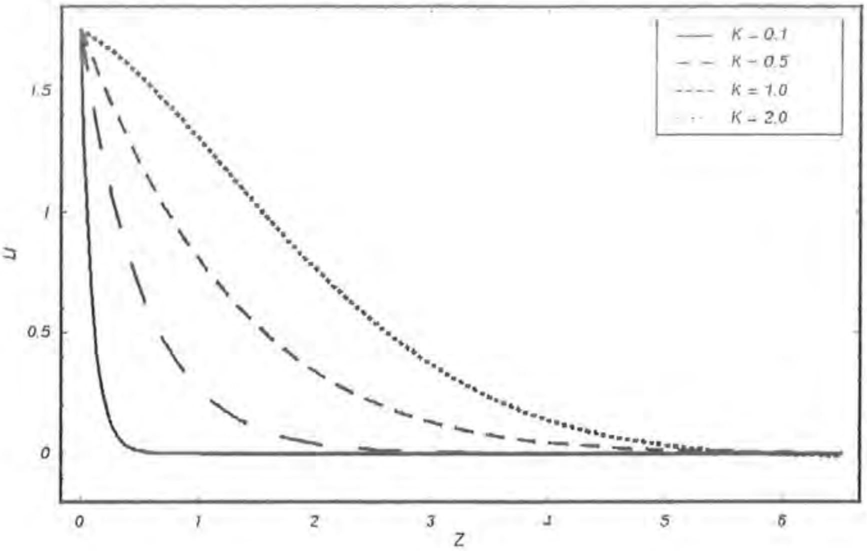
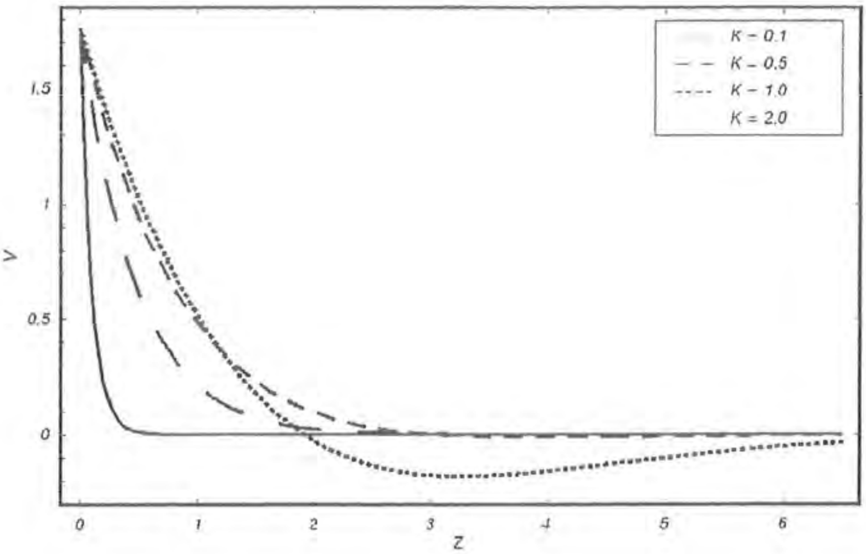


Figure 3.7b



Figures 3.7 : The variation of velocity parts for various values of K for a Burgers' fluid ($\lambda = 2,\ \lambda_r = 1$ and $\beta = 1.5$) when $M = 0.5,\ t = 1,\ m_0 = 0$ and $n_0/2\Omega = 2.5$ are fixed.

Figure 3.8a

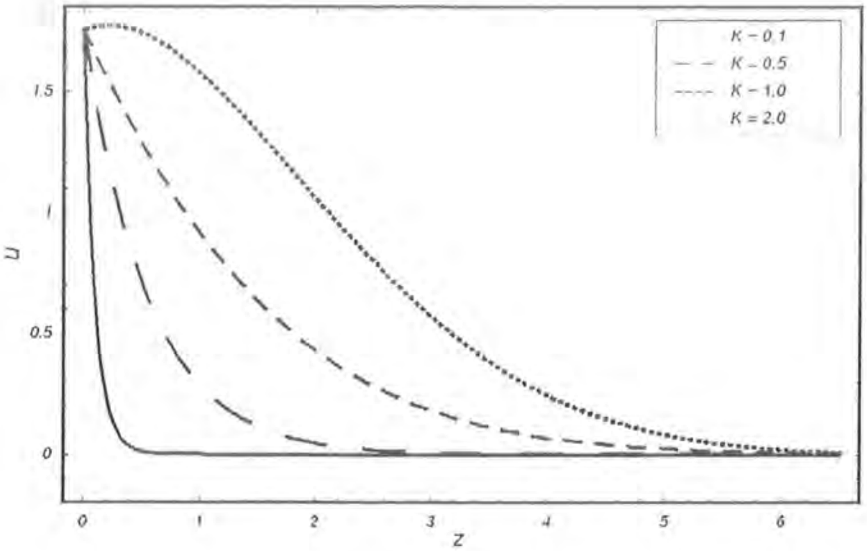
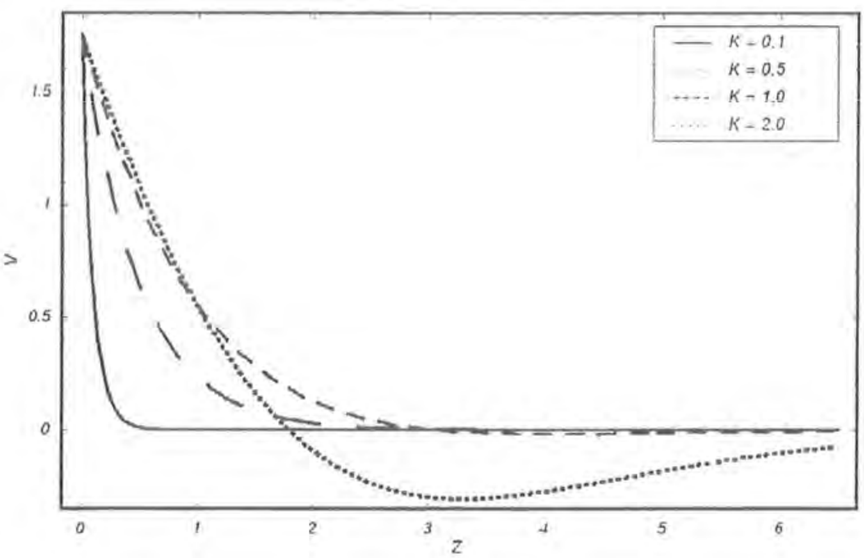


Figure 3.8b



Figures 3.8 : The variation of velocity parts for various values of K for a Burgers' fluid ($\lambda = 2,\ \lambda_r = 1$ and $\beta = 1.5$) when $M = 0.5,\ t = 1,\ m_0 = 2$ and $n_0/2\Omega = 2.5$ are fixed.

Figure 3.9a

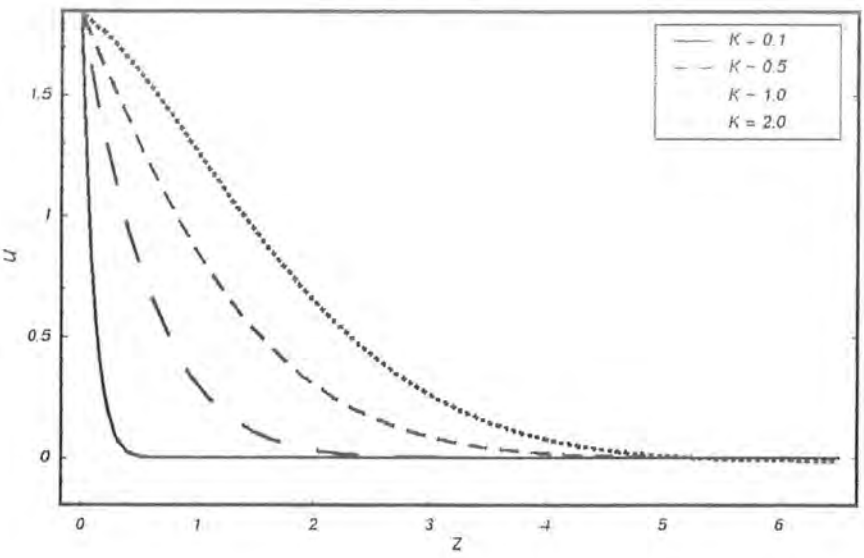
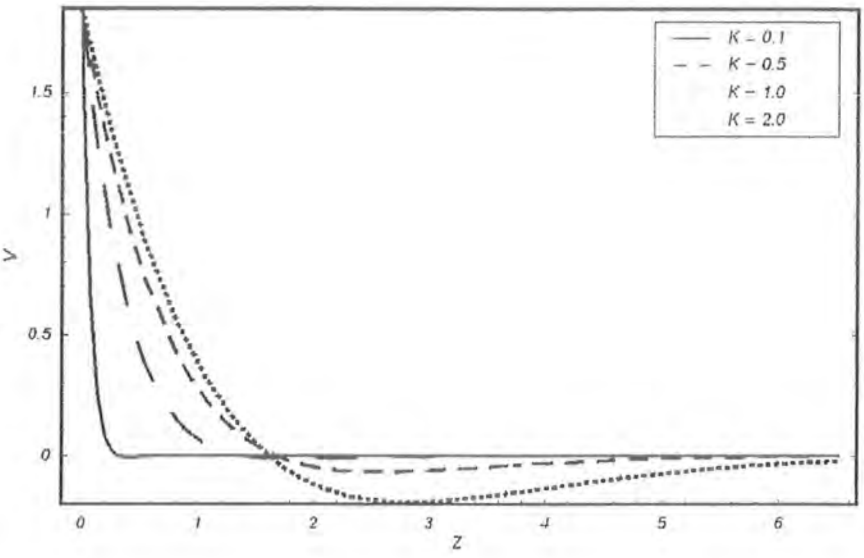


Figure 3.9b



Figures 3.9 : The variation of velocity parts for various values of K for an Oldroyd-B fluid ($\lambda = 2,\ \lambda_r = 1$ and $\beta = 0$) when $M = 0.5,\ t = 1,\ m_0 = 0$ and $n_0/2\Omega = 1$ are fixed.

Figure 3.10a

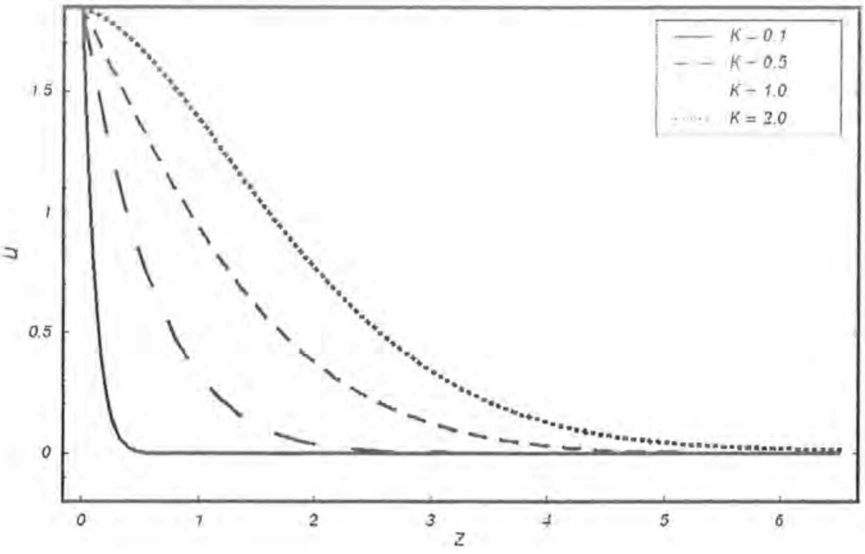
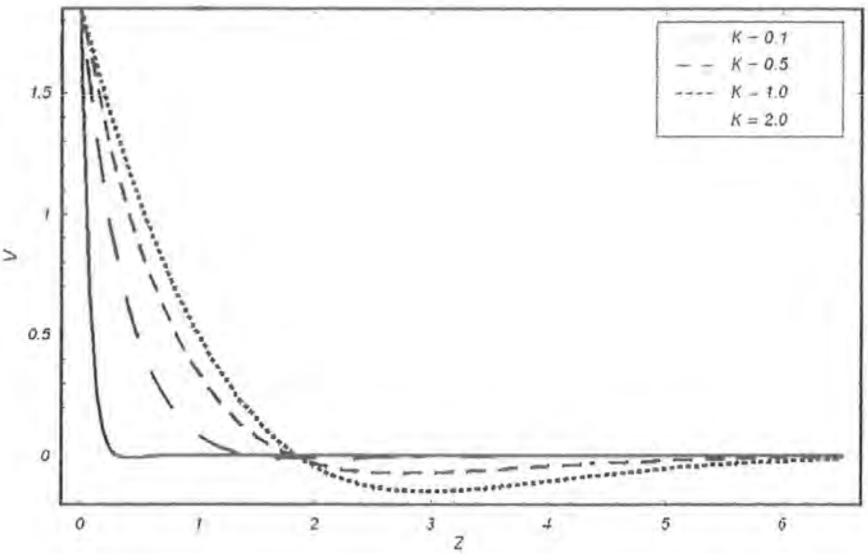


Figure 3.10b



Figures 3.10 : The variation of velocity parts for various values of K for an Oldroyd-B fluid ($\lambda = 2,\ \lambda_r = 1$ and $\beta = 0$) when $M = 0.5,\ t = 1,\ m_0 = 2$ and $n_0/2\Omega = 1$ are fixed.

Figure 3.11a

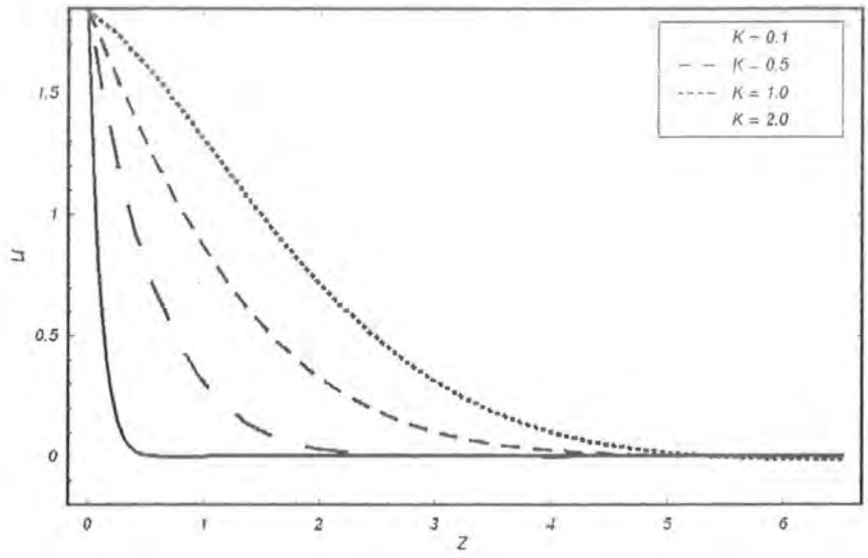
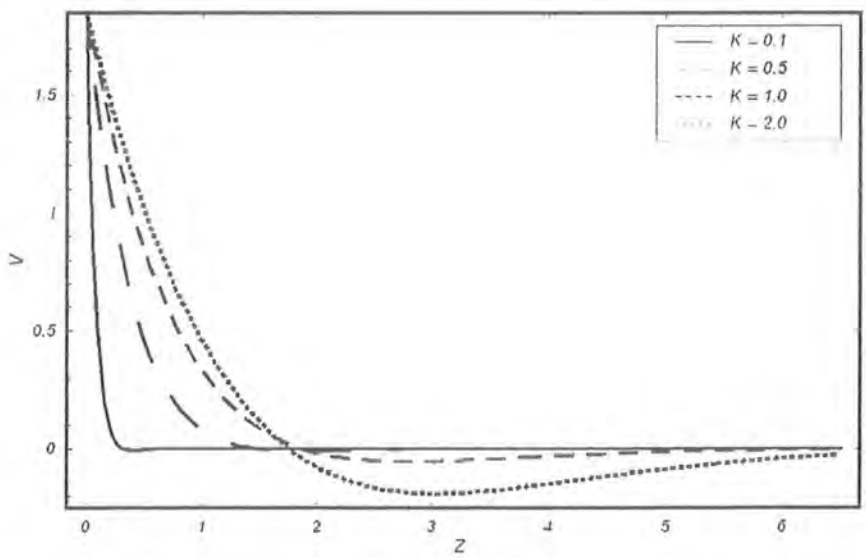


Figure 3.11b



Figures 3.11 : The variation of velocity parts for various values of K for a Burgers' fluid ($\lambda = 2,\ \lambda_r = 1$ and $\beta = 1.5$) when $M = 0.5,\ t = 1,\ m_0 = 0$ and $n_0/2\Omega = 1$ are fixed.

Figure 3.12a

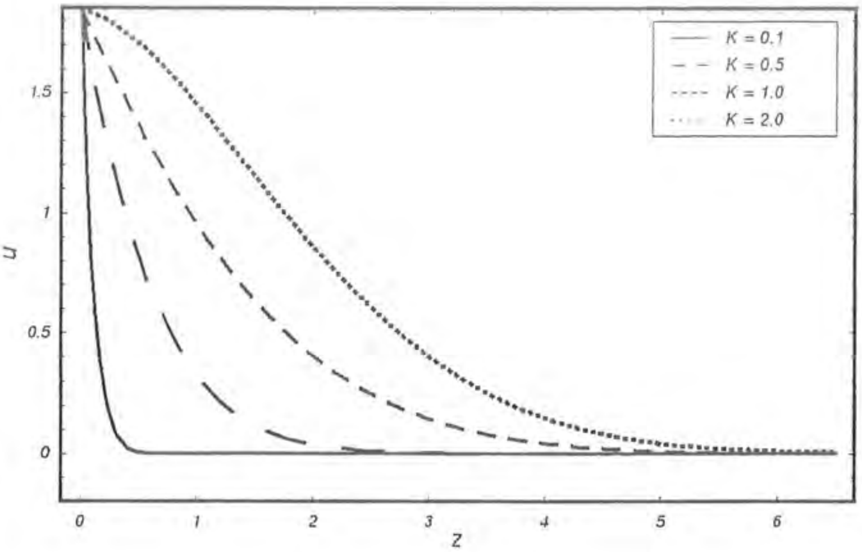
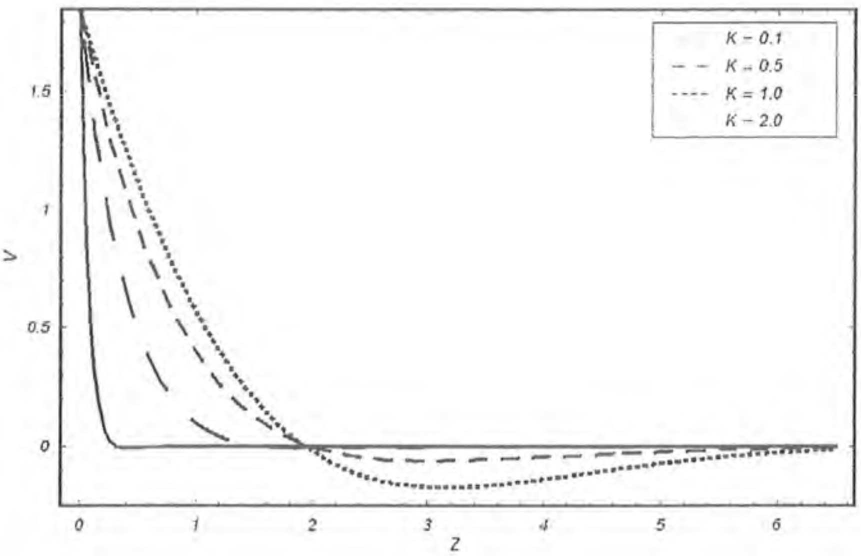


Figure 3.12b



Figures 3.12 : The variation of velocity parts for various values of K for a Burgers' fluid ($\lambda = 2,\ \lambda_r = 1$ and $\beta = 1.5$) when $M = 0.5,\ t = 1,\ m_0 = 2$ and $n_0/2\Omega = 1$ are fixed.

3.5 Closing remarks

In this work, the two problems of a Burgers' fluid have been discussed with Hall current. The modified Darcy's law has been taken into account for the presented flows. The various emerging parameters strongly effect the velocity profiles for all values of the frequencies. It is found that velocity profiles in case of Burgers' fluid are greater than that of an Oldroyd-B fluid. The presented analysis is more general and the results for several other fluid models can be taken as the limiting cases by choosing suitable parameters.

Chapter 4

Exact solutions for rotating flows of a generalized Burgers' fluid in a porous space

This chapter concentrates on the oscillatory flows of a generalized Burgers' fluid over an infinite insulating plate when the fluid is permeated by a transverse magnetic field. The influence of Hall current are taken into account. Modified Darcy's law for a generalized Burgers' fluid has been developed. The governing equations in a rotating frame are first developed and then solved for the two problems. To be more specific, the present work is a contribution towards extending the flow analysis of a previous chapter into two directions. (i) To assess and quantify the extent of material constants of generalized Burgers' fluid and (ii) to model the analysis in a porous medium using modified Darcy's law for generalized Burgers' fluid. Emphasis is given in determining the effects of pertinent dimensionless parameters on the velocity components. The influence of various emerging parameters is discussed through various graphs. The solutions for the Newtonian, second grade, Maxwell, Oldroyd-B and Burgers' fluids can be obtained as the limiting cases of the present closed form solutions.

4.1 Mathematical modelling

We consider the unsteady flow induced in a semi-infinite expanse of an incompressible, electrically conducting generalized Burgers' fluid bounded by an infinite plate at $z = 0$. The fluid and plate are in a state of rigid body rotation with a constant angular velocity $\mathbf{\Omega} = \Omega\hat{\mathbf{k}}$ ($\hat{\mathbf{k}}$ is a unit vector parallel to $z-$axis). The $z-$axis is taken normal to the plate. The fluid occupies the porous space ($z > 0$). A uniform magnetic field $\mathbf{B}_0$ is applied and the effects of Hall current are taken into consideration. The equations which can govern the present flow situation constitutes Eqs. (1.49) and (2.4)

The velocity and extra stress are defined as

$$\mathbf{V}(z,t) = (u(z,t), v(z,t), 0), \tag{4.1}$$

$$\mathbf{S}(z) = \begin{pmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{pmatrix}. \tag{4.2}$$

The extra stress tensor in a generalized Burgers' fluid is given as follows

$$\left(1 + \lambda_1 \frac{\delta}{\delta t} + \lambda_2 \frac{\delta^2}{\delta t^2}\right) \mathbf{S} = \mu \left(1 + \lambda_3 \frac{\delta}{\delta t} + \lambda_4 \frac{\delta^2}{\delta t^2}\right) \mathbf{A}_1, \tag{4.3}$$

in which μ is the dynamic viscosity, λ_1 and λ_2 are the relaxation times and λ_3 and λ_4 are the retardation times, $\mathbf{A}_1 \left(= \operatorname{grad}\mathbf{V} + (\operatorname{grad}\mathbf{V})^{\mathsf{T}}\right)$ is the first Rivlin-Ericksen tensor and the second order upper convected derivative is defined by

$$\frac{\delta^2 \mathbf{S}}{\delta t^2} = \frac{\delta}{\delta t}\left(\frac{\delta \mathbf{S}}{\delta t}\right) = \frac{\delta}{\delta t}\left(\frac{d\mathbf{S}}{dt} - \mathbf{LS} - \mathbf{SL}^{\mathsf{T}}\right), \tag{4.4}$$

where d/dt is the material derivative and $\mathbf{L} = \operatorname{grad}\mathbf{V}$. It should be noted that in expression (4.3) there is second convected derivative of stress and strain tensors. Also, Eq. (4.3) contains two relaxation and two retardation times. It is noteworthy that Eq.(4.3) includes the Newtonian, second grade, Maxwell, Oldroyd-B and Burgers' fluid models as the limiting cases. Furthermore the incompressibility condition is satisfied by Eq. (4.1).

In an unbounded porous medium, Darcy's law relates the pressure drop and Darcian velocity. This law holds for viscous fluids of low speed and neglects the boundary effects on the flow. There are several forms of Darcy's law for viscous fluids but only a few mathematical macroscopic filtration models have been proposed concerning viscoelastic fluid flow in a porous medium. By analogy with the constitutive equations of viscoelastic fluids, the following filtration laws have been suggested:

Maxwell's model [119]

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\nabla p=-\frac{\mu}{k_1}\mathbf{V}_D, \tag{4.5}$$

where $\mathbf{V}_D\,(=\phi\mathbf{V})$ is the Darcian velocity, ϕ the porosity and k_1 the permeability of the porous medium

Oldroyd-B model [120]

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\nabla p=-\frac{\mu}{k_1}\left(1+\lambda_3\frac{\partial}{\partial t}\right)\mathbf{V}_D. \tag{4.6}$$

Second grade model [121]

$$\nabla p=-\frac{\mu}{k_1}\left(1+\lambda_3\frac{\partial}{\partial t}\right)\mathbf{V}_D. \tag{4.7}$$

Note that Eqs. (4.5)–(4.7) reduce to the well known Darcy's law for a viscous fluid when $\lambda_1=\lambda_3=0$.

On the basis of Eqs. (4.5)–(4.7), the following filtration law in the generalized Burgers' fluid has been proposed:

$$\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)\nabla p=-\frac{\mu}{k_1}\left(1+\lambda_3\frac{\partial}{\partial t}+\lambda_4\frac{\partial^2}{\partial t^2}\right)\mathbf{V}_D. \tag{4.8}$$

For Burgers' fluid $\lambda_4=0$ and thus above expression reduces to

$$\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)\nabla p=-\frac{\mu}{k_1}\left(1+\lambda_3\frac{\partial}{\partial t}\right)\mathbf{V}_D. \tag{4.9}$$

It is remarked that the expressions (4.8) and (4.9) for modified Darcy's law are new and have not been available in the existing literature.

Since the pressure gradient in Eq. (4.8) is interpreted as a measure of the resistance to flow in the bulk of the porous medium and $\mathbf{R} = (R_x, R_y, 0)$ is a measure of the flow resistance offered by the solid matrix. Thus $\mathbf{R}$ can be inferred from Eq. (4.8) to satisfy the following equation

$$\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)\mathbf{R} = -\frac{\mu}{k_1}\left(1+\lambda_3\frac{\partial}{\partial t}+\lambda_4\frac{\partial^2}{\partial t^2}\right)\mathbf{V}_D. \tag{4.10}$$

Under the aforesaid assumptions, Eqs. (1.51), (2.4) and (4.3) give

$$\rho\left(\frac{\partial u}{\partial t}-2\Omega v\right) = -\frac{\partial \hat{p}}{\partial x}+\frac{\partial S_{xz}}{\partial z}-\frac{\sigma B_0^2 u}{1-im_0}+R_x, \tag{4.11}$$

$$\rho\left(\frac{\partial v}{\partial t}+2\Omega u\right) = -\frac{\partial \hat{p}}{\partial y}+\frac{\partial S_{yz}}{\partial z}-\frac{\sigma B_0^2 v}{1-im_0}+R_y, \tag{4.12}$$

$$0 = -\frac{\partial \hat{p}}{\partial z}+\frac{\partial S_{zz}}{\partial z}, \tag{4.13}$$

$$\begin{aligned}&\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)S_{xx}-2\left[\left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial z}S_{xz}+\lambda_2\frac{\partial u}{\partial z}\left(\frac{\partial S_{xz}}{\partial t}-\frac{\partial u}{\partial z}S_{zz}\right)\right]\\ =& -2\mu\left(\lambda_3+\frac{3\lambda_4}{2}\frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial z}\right)^2,\end{aligned} \tag{4.14}$$

$$\begin{aligned}&\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)S_{xy}-\left[\begin{array}{c}\left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial z}S_{yz}+\frac{\partial v}{\partial z}S_{xz}\right)\\ +\lambda_2\frac{\partial u}{\partial z}\left(\frac{\partial S_{yz}}{\partial t}-\frac{\partial v}{\partial z}S_{zz}\right)+\lambda_2\frac{\partial v}{\partial z}\left(\frac{\partial S_{xz}}{\partial t}-\frac{\partial u}{\partial z}S_{zz}\right)\end{array}\right]\\ =& -2\mu\left(\lambda_3+\frac{3\lambda_4}{2}\frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial z}\right),\end{aligned} \tag{4.15}$$

$$\begin{aligned}&\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)S_{xz}-\left[\left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial z}S_{zz}+\lambda_2\frac{\partial u}{\partial z}\frac{\partial S_{zz}}{\partial z}\right]\\ =& \mu\left(1+\lambda_3\frac{\partial}{\partial t}+\lambda_4\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial u}{\partial z}\right),\end{aligned} \tag{4.16}$$

$$\begin{aligned}&\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)S_{yy}-2\left[\left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial z}S_{yz}+\lambda_2\frac{\partial v}{\partial z}\left(\frac{\partial S_{yz}}{\partial t}-\frac{\partial v}{\partial z}S_{zz}\right)\right]\\ =& -2\mu\left(\lambda_3+\frac{3\lambda_4}{2}\frac{\partial}{\partial t}\right)\left(\frac{\partial v}{\partial z}\right)^2,\end{aligned} \tag{4.17}$$

$$\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)S_{yz}-\left[\left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial z}S_{zz}+\lambda_2\frac{\partial v}{\partial z}\frac{\partial S_{zz}}{\partial z}\right]$$
$$=\mu\left(1+\lambda_3\frac{\partial}{\partial t}+\lambda_4\frac{\partial^2}{\partial t^2}\right)\frac{\partial v}{\partial z}, \tag{4.18}$$

$$\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)S_{zz}=0, \tag{4.19}$$

$$\hat{p}=p-\frac{1}{2}\rho\Omega^2\left(x^2+y^2\right), \tag{4.20}$$

where $m_0=\omega_e\tau_e$ is the Hall parameter. Equation (4.19) can be integrated easily. We shall seek the possibility of a solution in which $S_{zz}=0$. Using $S_{zz}=0$ in above equations and then combining the resulting equations along with Eq. (4.10) we arrive at

$$\rho\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2 F}{\partial z\partial t}+2i\Omega\frac{\partial F}{\partial z}\right)+\frac{\sigma B_0^2}{1-im_0}\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)\frac{\partial F}{\partial z}$$
$$+\frac{\mu\phi}{k_1}\left(1+\lambda_3\frac{\partial}{\partial t}+\lambda_4\frac{\partial^2}{\partial t^2}\right)\frac{\partial F}{\partial z}=\mu\left(1+\lambda_3\frac{\partial}{\partial t}+\lambda_4\frac{\partial^2}{\partial t^2}\right)\frac{\partial^3 F}{\partial z^3}, \tag{4.21}$$

where

$$F=u+iv.$$

In the next two sections, we shall obtain the exact solutions of Eq. (4.21) for some oscillatory flows.

4.2 Flow due to general periodic oscillation

We shall begin with the case of rotating MHD flow above a plate. It is assumed that the flow is entirely driven by the general periodic plate oscillation. The disturbance far away from the plate is zero. The governing problem consists of Eq. (4.21) and the following boundary conditions

$$F(0,t)=U\sum_{k=-\infty}^{\infty}c_k e^{ikn_0t}, \qquad F(\infty,t)=0, \tag{4.22}$$

in which $n_0 = 2\pi/T_0$ (T_0 being the time period) is the oscillating imposed frequency and $\{c_k\}$ are the Fourier series coefficients defined by

$$c_k = \frac{1}{T_0}\int\limits_{T_0} f(t)e^{-ikn_0t}dt. \tag{4.23}$$

Now we nondimensionalize various quantities as follows:

$$z^* = \frac{zU}{\nu}, \quad F^* = \frac{F}{U}, \quad t^* = \frac{tU^2}{\nu}, \quad n_0^* = \frac{n_0\nu}{U^2}, \quad \lambda_1^* = \frac{\lambda_1 U^2}{\nu}, \quad \Omega^* = \frac{\Omega\nu}{U^2},$$

$$\lambda_3^* = \frac{\lambda_3 U^2}{\nu}, \quad M^2 = \frac{\sigma B_0^2\nu}{\rho U^2}, \quad \frac{1}{K} = \frac{\phi\nu^2}{k_1 U^2}, \quad \lambda_2^* = \frac{\lambda_2 U^4}{\nu^2}, \quad \lambda_4^* = \frac{\lambda_4 U^4}{\nu^2}. \tag{4.24}$$

Dropping the asterisks, the problem becomes

$$\left(1 + \lambda_1\frac{\partial}{\partial t} + \lambda_2\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2 F}{\partial z\partial t} + 2i\Omega\frac{\partial F}{\partial z}\right) + \frac{M^2}{1 - im_0}\left(1 + \lambda_1\frac{\partial}{\partial t} + \lambda_2\frac{\partial^2}{\partial t^2}\right)\frac{\partial F}{\partial z}$$

$$+\frac{1}{K}\left(1 + \lambda_3\frac{\delta}{\delta t} + \lambda_4\frac{\delta^2}{\delta t^2}\right)\frac{\partial F}{\partial z} = \left(1 + \lambda_3\frac{\delta}{\delta t} + \lambda_4\frac{\delta^2}{\delta t^2}\right)\frac{\partial^3 F}{\partial z^3}, \tag{4.25}$$

$$F(0,t) = \sum_{k=-\infty}^{\infty} c_k e^{ikn_0t}, \qquad F(\infty, t) = 0. \tag{4.26}$$

By means of Fourier transform, the exact solution is of the following form

$$F(z,t) = \sum_{k=-\infty}^{\infty} c_k e^{-\xi_k z + i(kn_0t - \eta_k z)}, \tag{4.27}$$

$$\xi_k = \frac{1}{\sqrt{2}}\left[a_{1k} + \left(a_{1k}^2 + a_{2k}^2\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}, \tag{4.28}$$

$$\eta_k = \left[\frac{a_{2k}^2}{2\left\{a_{1k} + \left(a_{1k}^2 + a_{2k}^2\right)^{\frac{1}{2}}\right\}}\right]^{\frac{1}{2}}, \tag{4.29}$$

$$a_{1k}=\frac{1}{\left\{\left(1-\lambda_4k^2n_0^2\right)^2+k^2n_0^2\lambda_3^2\right\}}\left[\begin{array}{c}kn_0\left\{\lambda_3-\lambda_1-\left(\lambda_2\lambda_3-\lambda_1\lambda_4\right)k^2n_0^2\right\}\left(kn_0+2\Omega\right)\\+\frac{1}{K}\left\{1+k^2n_0^2\lambda_3^2+\lambda_4k^2n_0^2\left(\lambda_4k^2n_0^2-2\right)\right\}\\+\frac{M^2}{1+m_0^2}\left\{\begin{array}{c}1+k^2n_0^2\lambda_1\lambda_3+m_0kn_0\left(\lambda_3-\lambda_1\right)\\-\lambda_2k^2n_0^2\left(1+m_0\lambda_3kn_0\right)\\-\lambda_4k^2n_0^2\left(1-\lambda_2-m_0\lambda_1kn_0\right)\end{array}\right\}\end{array}\right], \tag{4.30}$$

$$a_{2k}=\frac{1}{\left\{\left(1-\lambda_4k^2n_0^2\right)^2+k^2n_0^2\lambda_3^2\right\}}\left[\begin{array}{c}\left\{\begin{array}{c}1+k^2n_0^2\lambda_1\lambda_3-\lambda_2k^2n_0^2\\-\lambda_4k^2n_0^2\left(1-\lambda_2k^2n_0^2\right)\end{array}\right\}\left(kn_0+2\Omega\right)\\-\frac{M^2}{1+m_0^2}\left\{\begin{array}{c}kn_0\left(\lambda_3-\lambda_1\right)-m_0\left(1+k^2n_0^2\lambda_1\lambda_3\right)\\-\lambda_2k^2n_0^2\left(\lambda_3kn_0-m_0\right)\\-\lambda_4k^2n_0^2\left(m_0-\lambda_1kn_0-m_0\lambda_2k^2n_0^2\right)\end{array}\right\}\end{array}\right], \tag{4.31}$$

where ξ_k and η_k are real and positive. Equation (4.27) is the exact solution for the velocity field induced by general periodic plate oscillation. The flow fields for different plate oscillations can be obtained from Eq. (4.27) through an appropriate choice of the Fourier coefficients. For example, the respective velocity fields due to the oscillations e^{in_0t}, $\cos n_0t$, $\sin n_0t$, $\left[\begin{array}{ll}1, & |\,t\,|<\frac{T_1}{2}\\ 0, T_1<|\,t\,|<\frac{T_0}{2}\end{array}\right]$, $\sum\limits_{k=-\infty}^{\infty}\delta(t-kT_0)$ are given by

$$F_1(z,t)=e^{-\xi_1z+i(n_0t-\eta_1z)}, \tag{4.32}$$

$$F_2(z,t)=\frac{1}{2}\left[e^{-\xi_1z+i(n_0t-\eta_1z)}+e^{-\xi_{-1}z+i\left(n_0t+\eta_{-1}z\right)}\right], \tag{4.33}$$

$$F_3(z,t)=\frac{1}{2i}\left[e^{-\xi_1z+i(n_0t-\eta_1z)}-e^{-\xi_{-1}z-i\left(n_0t+\eta_{-1}z\right)}\right], \tag{4.34}$$

$$F_4(z,t)=\sum_{k=-\infty}^{\infty}\frac{\sin kn_0T_1}{k\pi}e^{-\xi_kz+i(kn_0t-\eta_kz)},\qquad k\neq 0, \tag{4.35}$$

$$F_5(z,t)=\frac{1}{T_0}\sum_{k=-\infty}^{\infty}e^{-\xi_kz+i(kn_0t-\eta_kz)}. \tag{4.36}$$

4.3 Flow due to elliptic harmonic oscillation

In this section we consider the rotating MHD flow of a generalized Burgers' fluid induced by the elliptic oscillations. The nondimensional boundary conditions are

$$F(0,t) = ae^{in_0 t} + be^{-in_0 t}, \qquad F(\infty, t) = 0, \tag{4.37}$$

where $a = a_1 + ia_2$ and $b = b_1 + ib_2$. Here we look the solutions for $n_0 < 2\Omega$, $n_0 > 2\Omega$ and $n_0 = 2\Omega$. For $n_0 < 2\Omega$ we have

$$u = \begin{bmatrix} e^{-\alpha_1 \Psi_1} \{a_1 \cos(\beta_1 \Psi_1 - n_0 t) + a_2 \sin(\beta_1 \Psi_1 - n_0 t)\} \\ +e^{-\alpha_2 \Psi_2} \{b_1 \cos(\beta_2 \Psi_2 + n_0 t) + b_2 \sin(\beta_2 \Psi_2 + n_0 t)\} \end{bmatrix}, \tag{4.38}$$

$$v = \begin{bmatrix} e^{-\alpha_1 \Psi_1} \{a_2 \cos(\beta_1 \Psi_1 - n_0 t) - a_1 \sin(\beta_1 \Psi_1 - n_0 t)\} \\ +e^{-\alpha_2 \Psi_2} \{b_2 \cos(\beta_2 \Psi_2 + n_0 t) - b_1 \sin(\beta_2 \Psi_2 + n_0 t)\} \end{bmatrix}. \tag{4.39}$$

When $n_0 > 2\Omega$ we obtain

$$u = \begin{bmatrix} e^{-\alpha_1 \Psi_1} \{a_1 \cos(\beta_1 \Psi_1 - n_0 t) + a_2 \sin(\beta_1 \Psi_1 - n_0 t)\} \\ +e^{-\alpha_3 \Psi_3} \{b_1 \cos(\beta_3 \Psi_3 - n_0 t) - b_2 \sin(\beta_3 \Psi_3 - n_0 t)\} \end{bmatrix}, \tag{4.40}$$

$$v = \begin{bmatrix} e^{-\alpha_1 \Psi_1} \{a_2 \cos(\beta_1 \Psi_1 - n_0 t) - a_1 \sin(\beta_1 \Psi_1 - n_0 t)\} \\ +e^{-\alpha_3 \Psi_3} \{b_2 \cos(\beta_3 \Psi_3 - n_0 t) + b_1 \sin(\beta_3 \Psi_3 - n_0 t)\} \end{bmatrix}. \tag{4.41}$$

For resonant case $n_0 = 2\Omega$ we get

$$u = \begin{bmatrix} e^{-\alpha_1 \Psi_1} \{a_1 \cos(\beta_1 \Psi_1 - n_0 t) + a_2 \sin(\beta_1 \Psi_1 - n_0 t)\} \\ +e^{-\alpha_0 \Psi_0} \{b_1 \cos(\beta_0 \Psi_0 + n_0 t) + b_2 \sin(\beta_0 \Psi_0 + n_0 t)\} \end{bmatrix}, \tag{4.42}$$

$$v = \begin{bmatrix} e^{-\alpha_1 \Psi_1} \{a_2 \cos(\beta_1 \Psi_1 - n_0 t) - a_1 \sin(\beta_1 \Psi_1 - n_0 t)\} \\ +e^{-\alpha_0 \Psi_0} \{b_2 \cos(\beta_0 \Psi_0 + n_0 t) - b_1 \sin(\beta_0 \Psi_0 + n_0 t)\} \end{bmatrix}, \tag{4.43}$$

where

$$\alpha_j = \frac{1}{\sqrt{2}} \left[S_j + (S_j^2 + 1)^{\frac{1}{2}} \right]^{\frac{1}{2}}, \quad \beta_j = \frac{1}{\sqrt{2}} \left[-S_j + (S_j^2 + 1)^{\frac{1}{2}} \right]^{\frac{1}{2}}, \quad j = 0, 1, 2, 3,$$

$$\Psi_1 = \left[\frac{\left(1-\lambda_4 n_0^2\right) B_1 - \lambda_3 n_0 A_1}{\left(1-\lambda_4 n_0^2\right)^2 + \lambda_3^2 n_0^2}\right]^{\frac{1}{2}} z, \quad \Psi_2 = \left[\frac{\left(1-\lambda_4 n_0^2\right) B_2 + \lambda_3 n_0 A_2}{\left(1-\lambda_4 n_0^2\right)^2 + \lambda_3^2 n_0^2}\right]^{\frac{1}{2}} z,$$

$$\Psi_3 = \left[\frac{\lambda_3 n_0 A_3 - \left(1-\lambda_4 n_0^2\right) B_3}{\left(1-\lambda_4 n_0^2\right)^2 + \lambda_3^2 n_0^2}\right]^{\frac{1}{2}} z, \quad \Psi_0 = \left[\frac{\lambda_3 n_0 A_0 + \left(1-\lambda_4 n_0^2\right) B_0}{\left(1-\lambda_4 n_0^2\right)^2 + \lambda_3^2 n_0^2}\right]^{\frac{1}{2}} z,$$

$$S_1 = \frac{\left(1-\lambda_4 n_0^2\right) A_1 + \lambda_3 n_0 B_1}{\left(1-\lambda_4 n_0^2\right) B_1 - \lambda_3 n_0 A_1}, \quad S_2 = \frac{\left(1-\lambda_4 n_0^2\right) A_2 - \lambda_3 n_0 B_2}{\left(1-\lambda_4 n_0^2\right) B_2 + \lambda_3 n_0 A_2},$$

$$S_3 = \frac{\left(1-\lambda_4 n_0^2\right) A_3 + \lambda_3 n_0 B_3}{\lambda_3 n_0 A_3 - \left(1-\lambda_4 n_0^2\right) B_3}, \quad S_0 = \frac{\left(1-\lambda_4 n_0^2\right) A_0 - \lambda_3 n_0 B_0}{\lambda_3 n_0 A_0 + \left(1-\lambda_4 n_0^2\right) B_0},$$

$$A_1 = \frac{M^2}{1+m_0^2}\left(1 - m_0\lambda_1 n_0 - \lambda_2 n_0^2\right) - \lambda_1 n_0 \left(n_0 + 2\Omega\right) + \frac{1}{K}\left(1-\lambda_4 n_0^2\right),$$

$$B_1 = \frac{M^2}{1+m_0^2}\left(m_0 + \lambda_1 n_0 - \lambda_2 n_0^2 m_0\right) + \left(n_0 + 2\Omega\right)\left(1-\lambda_2 n_0^2\right) + \frac{\lambda_3 n_0}{K},$$

$$A_2 = \frac{M^2}{1+m_0^2}\left(1 + m_0\lambda_1 n_0 - \lambda_2 n_0^2\right) + \lambda_1 n_0 \left(2\Omega - n_0\right) + \frac{1}{K}\left(1-\lambda_4 n_0^2\right),$$

$$B_2 = \left(2\Omega - n_0\right)\left(1-\lambda_2 n_0^2\right) - \frac{M^2}{1+m_0^2}\left(\lambda_1 n_0 - m_0 - \lambda_2 n_0^2 m_0\right) - \frac{\lambda_3 n_0}{K},$$

$$A_3 = \frac{M^2}{1+m_0^2}\left(1 + m_0\lambda_1 n_0 - \lambda_2 n_0^2\right) - \lambda_1 n_0 \left(n_0 - 2\Omega\right) + \frac{1}{K}\left(1-\lambda_4 n_0^2\right),$$

$$B_3 = \left(n_0 - 2\Omega\right)\left(1-\lambda_2 n_0^2\right) - \frac{M^2}{1+m_0^2}\left(m_0 - \lambda_1 n_0 + \lambda_2 n_0^2 m_0\right) + \frac{\lambda_3 n_0}{K},$$

$$A_0 = \frac{M^2}{1+m_0^2}\left(1 + m_0\lambda_1 n_0 - \lambda_2 n_0^2\right) + \frac{1}{K}\left(1-\lambda_4 n_0^2\right),$$

$$B_0 = -\frac{M^2}{1+m_0^2}\left(\lambda_1 n_0 - m_0 + \lambda_2 n_0^2 m_0\right) - \frac{\lambda_3 n_0}{K}.$$

4.4 Results and discussion

This section includes discussion regarding the variation of velocity profiles in various fluids. Specifically, the analysis is made for six cases of fluids: a Newtonian fluid ($\lambda_i = 0$, for $i = 1-4$), a Maxwell fluid ($\lambda_1 \neq 0,\ \lambda_2 = \lambda_3 = \lambda_4 = 0$), a second grade fluid ($\lambda_3 \neq 0,\ \lambda_1 = \lambda_2 = \lambda_4 = 0$), an Oldroyd-B fluid ($\lambda_1 \neq 0,\ \lambda_3 \neq 0,\ \lambda_2 = \lambda_4 = 0$), a Burgers' fluid ($\lambda_1 \neq 0,\ \lambda_2 \neq 0,\ \lambda_3 \neq 0,\ \lambda_4 = 0$)

and a generalized Burgers' fluid ($\lambda_i \neq 0$, for $i = 1 - 4$). Graphs are plotted and included only for the case $n < 2\Omega$. The effects of various parameters especially M, K, m_0 and the respective rheological parameters λ_2 and λ_4 of the Burgers' and the generalized Burgers' fluids on the velocity profiles u and v have been studied.

Figures 4.1 to 4.6 are prepared for the flow due to general periodic oscillation when the oscillation is of type $\cos nt$ and figures 4.7 to 4.10 for flow due to elliptic harmonic oscillation when $a_1 = a_2 = b_1 = b_2 = 1$. The figures 4.1 and 4.7 depict that the velocity profile u decreases by increasing the magnetic parameter M for both generalized Burgers' and Oldroyd-B fluids. Figures 4.2 and 4.8 show that the velocity profile v decreases by increasing M. In fact an increase in the magnetic parameter tend to reduce the velocity profiles monotonically due to the effect of magnetic force against the flow direction of flow for both fluids.

The influence of the permeability of the porous medium K on the velocity profiles is illustrated in figures 4.3, 4.4, 4.9 and 4.10 for both fluids. As expected, the increase of the permeability of the porous medium reduces the drag force and hence causes the velocity profiles to increase for both fluids. Moreover, it is noted from figures 4.5 and 4.6 that the velocity profiles increase by increasing Hall parameter m_0 for both fluids. This is due to the fact that the effective conductivity decreases with increasing m_0 which reduces the magnetic damping force on the velocity.

Further, from these figures, it appears that the velocity is an increasing function of the rheological parameter λ_2 of the Burgers' fluid. However, this result cannot be generalized for other chosen value of λ_2 as the behavior of λ_2 is non-monotonous. Similarly, it is also observed that the velocity is also an increasing function of the rheological parameter λ_4 of generalized Burgers' fluid (Tables 1 and 2).

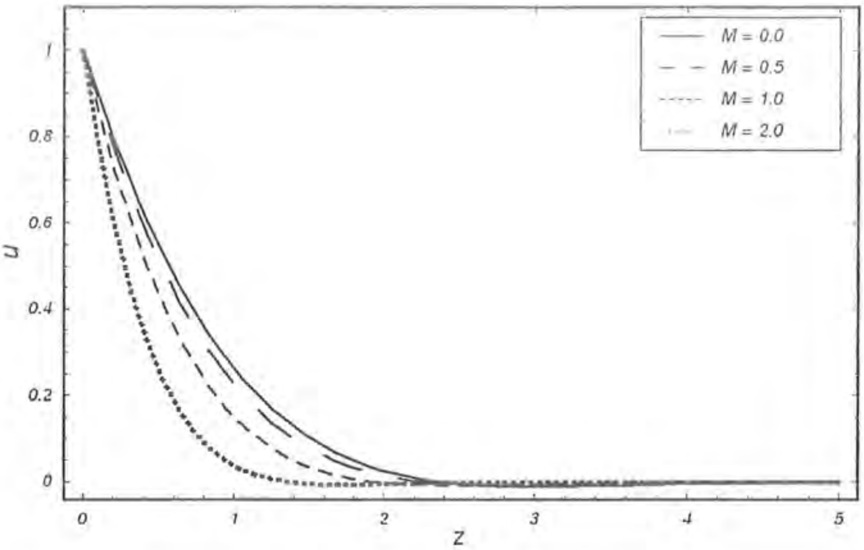
Table 1 : A comparison of velocity parts (flow due to general periodic oscillation) for various kinds of fluids when $M = K = m_0 = 1$, $n = \Omega = 0.5$, $t = 0.1$ at $z = 0.5$ when oscillation is of type $\cos nt$

Type of fluid	Rheological parameters	u	v
Newtonian	$\lambda_i = 0$ for $i = 1 - 4$	0.490418	−0.0100216
Second grade	$\lambda_1 = \lambda_2 = \lambda_4 = 0,\ \lambda_3 = 1$	0.504373	0.0388267
Maxwell	$\lambda_2 = \lambda_3 = \lambda_4 = 0,\ \lambda_1 = 2$	0.478217	−0.133002
Oldroyd-B	$\lambda_2 = \lambda_4 = 0, \lambda_1 = 2, \lambda_3 = 1$	0.468636	−0.565648
Burgers'	$\lambda_1 = 2, \lambda_2 = 1.5, \lambda_3 = 1, \lambda_4 = 0$	0.503968	−0.077967
G. Burgers'	$\lambda_1 = 2, \lambda_2 = 1.5, \lambda_3 = 1, \lambda_4 = 1.3$	0.429581	−0.0590261

Table 2 : A comparison of velocity parts (flow due to elliptic harmonic oscillation) for various kinds of fluids when $M = K = m_0 = 1$, $n = \Omega = 0.5$, $t = 0.1$ at $z = 0.5$

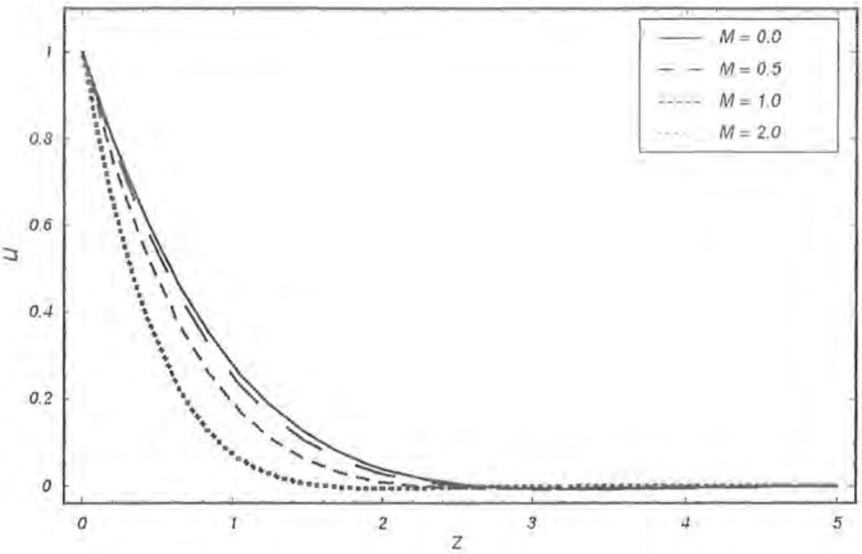
Type of fluid	Rheological parameters	u	v
Newtonian	$\lambda_i = 0$ for $i = 1 - 4$	1.255560	0.706111
Second grade	$\lambda_1 = \lambda_2 = \lambda_4 = 0,\ \lambda_3 = 1$	1.247460	0.770027
Maxwell	$\lambda_2 = \lambda_3 = \lambda_4 = 0,\ \lambda_1 = 2$	1.338390	0.574481
Oldroyd-B	$\lambda_2 = \lambda_4 = 0, \lambda_1 = 2, \lambda_3 = 1$	1.262900	0.611641
Burgers'	$\lambda_1 = 2, \lambda_2 = 1.5, \lambda_3 = 1, \lambda_4 = 0$	1.318890	0.696984
G. Burgers'	$\lambda_1 = 2, \lambda_2 = 1.5, \lambda_3 = 1, \lambda_4 = 1.3$	1.300370	0.638440

Figure4.1a



(a) G. Burgers' fluid ($\lambda_2 = 1.5,\ \lambda_4 = 1.3$)

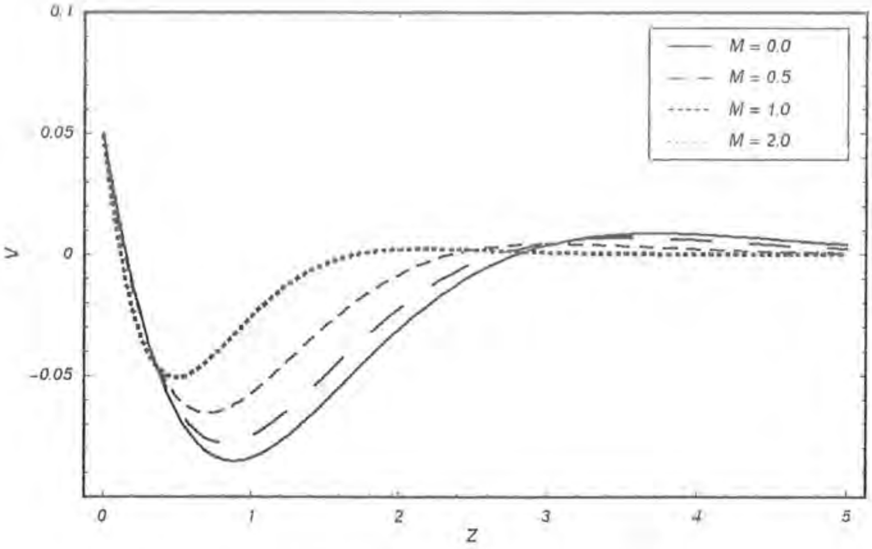
Figure 4.1b



(b) Oldroyd-B fluid ($\lambda_2 = \lambda_4 = 0$)

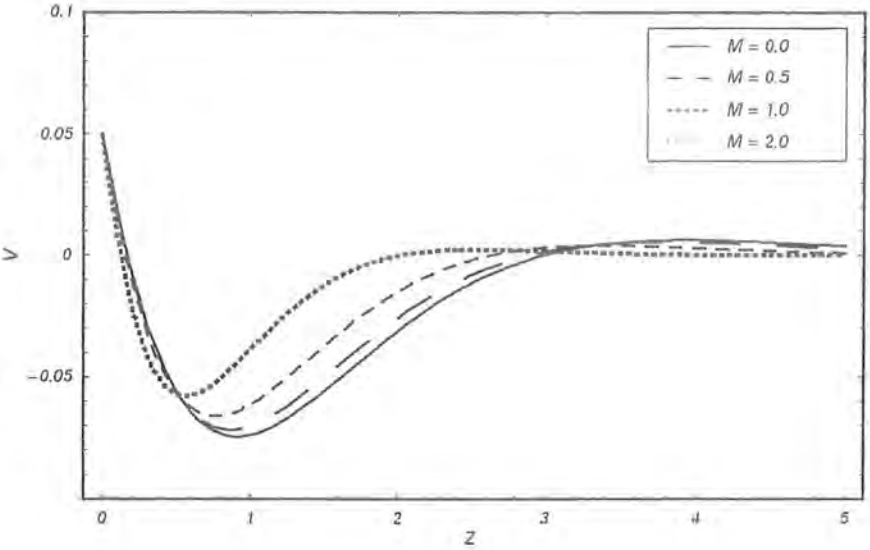
Figures 4.1 : The variation of velocity profile u for various values of magnetic parameter M when $\lambda_1 = 2,\ \lambda_3 = 1,\ K = m_0 = 1,\ t = 0.1$ and $n/2\Omega = 0.5$ are fixed.

Figure 4.2a



(a) G. Burgers' fluid ($\lambda_2 = 1.5,\ \lambda_4 = 1.3$)

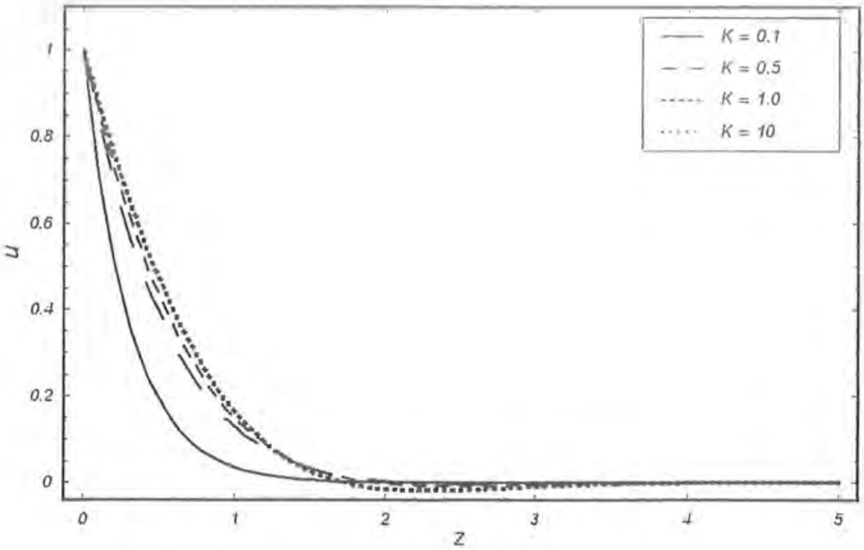
Figure 4.2b



(b) Oldroyd-B fluid ($\lambda_2 = \lambda_4 = 0$)

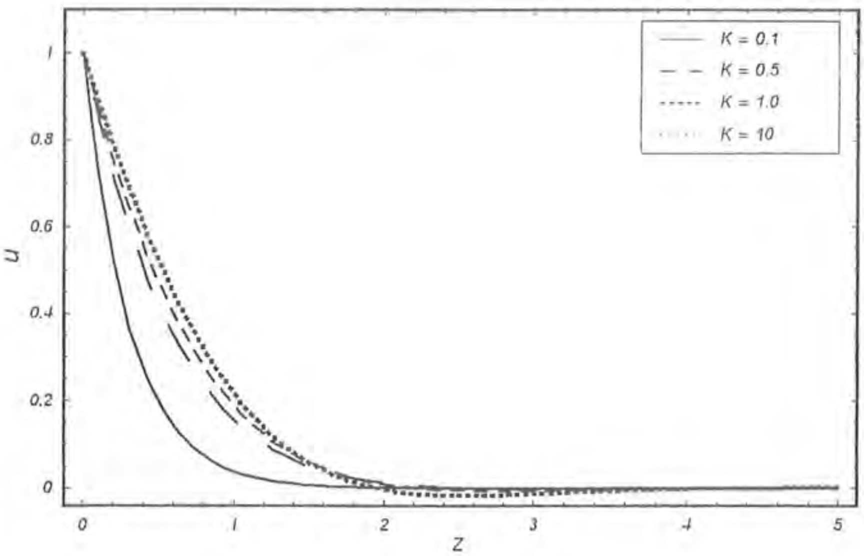
Figures 4.2 : The variation of velocity profile v for various values of magnetic parameter M when $\lambda_1 = 2,\ \lambda_3 = 1,\ K = m_0 = 1,\ t = 0.1$ and $n/2\Omega = 0.5$ are fixed.

Figure 4.3a



(a) G. Burgers' fluid ($\lambda_2 = 1.5,\ \lambda_4 = 1.3$)

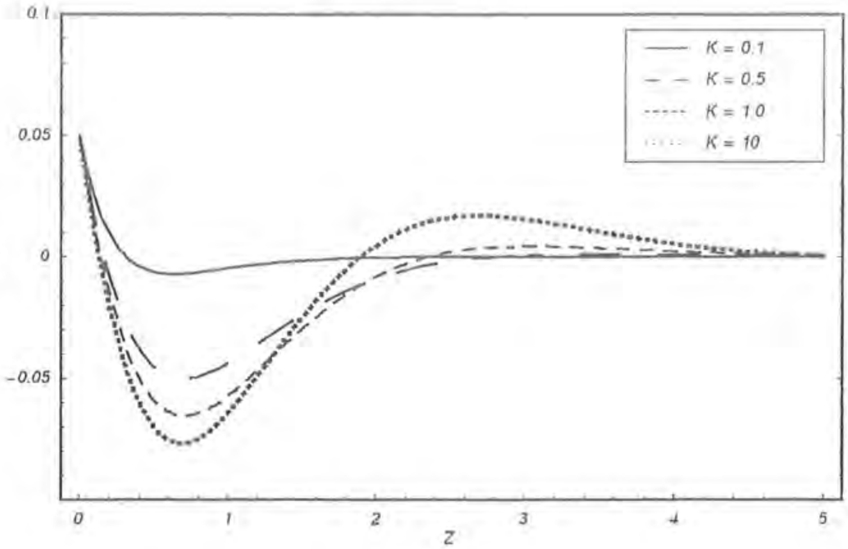
Figure 4.3b



(b) Oldroyd-B fluid ($\lambda_2 = \lambda_4 = 0$)

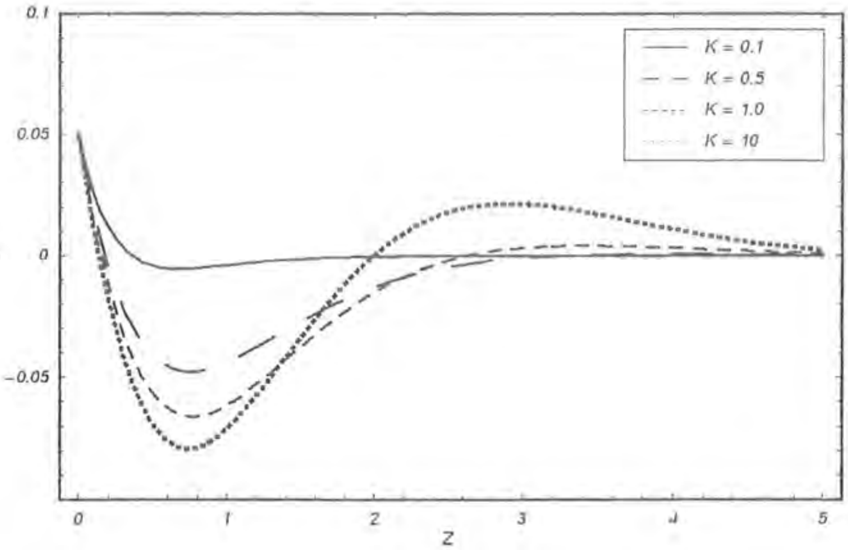
Figures 4.3 : The variation of velocity profile u for various values of permeability parameter K when $\lambda_1 = 2,\ \lambda_3 = 1,\ M = m_0 = 1,\ t = 0.1$ and $n/2\Omega = 0.5$ are fixed.

Figure 4.4a



(a) G. Burgers' fluid ($\lambda_2 = 1.5,\ \lambda_4 = 1.3$)

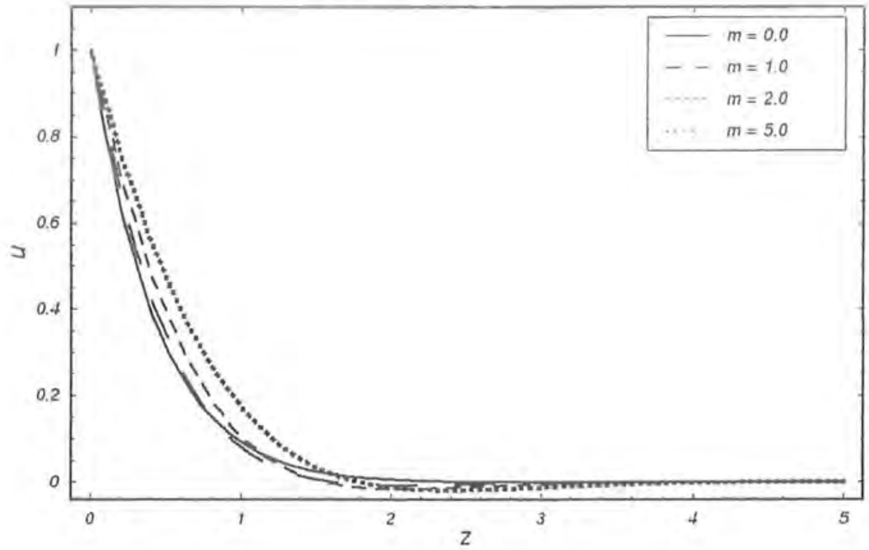
Figure 4.4b



(b) Oldroyd-B fluid ($\lambda_2 = \lambda_4 = 0$)

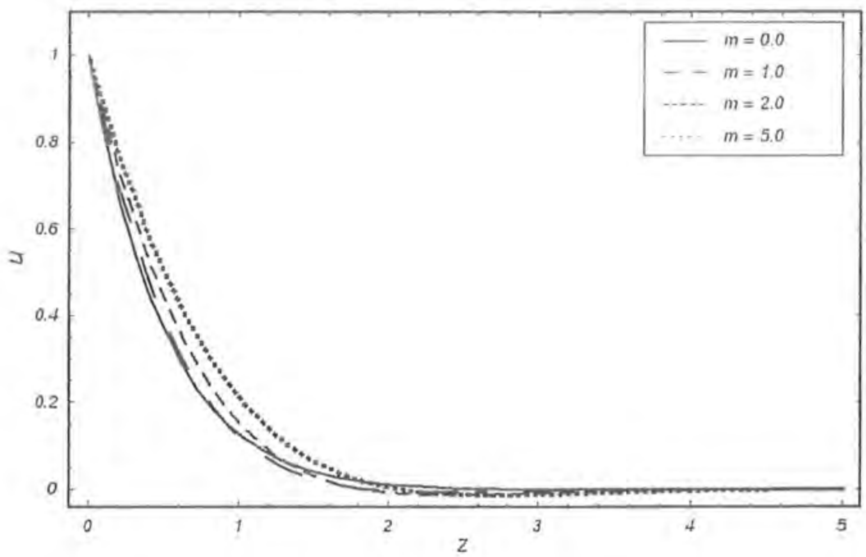
Figures 4.4 : The variation of velocity profile v for various values of permeability parameter K when $\lambda_1 = 2,\ \lambda_3 = 1,\ M = m_0 = 1,\ t = 0.1$ and $n/2\Omega = 0.5$ are fixed.

Figure 4.5a



(a) G. Burgers' fluid ($\lambda_2 = 1.5,\ \lambda_4 = 1.3$)

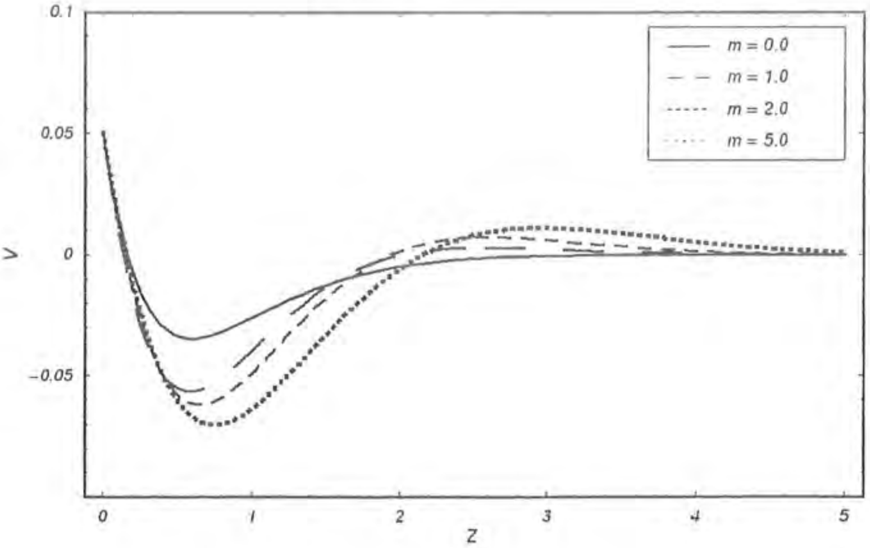
Figure 4.5b



(b) Oldroyd-B fluid ($\lambda_2 = \lambda_4 = 0$)

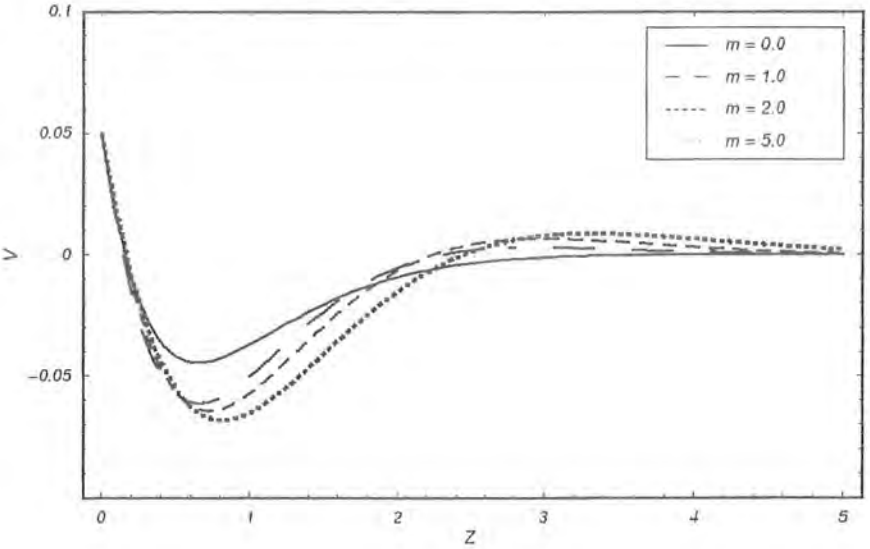
Figures 4.5 : The variation of velocity profile u for various values of Hall parameter m_0 when $\lambda_1 = 2,\ \lambda_3 = 1$, $M = K = 1,\ t = 0.1$ and $n/2\Omega = 0.5$ are fixed.

Figure 4.6a



(a) G. Burgers' fluid ($\lambda_2 = 1.5, \ \lambda_4 = 1.3$)

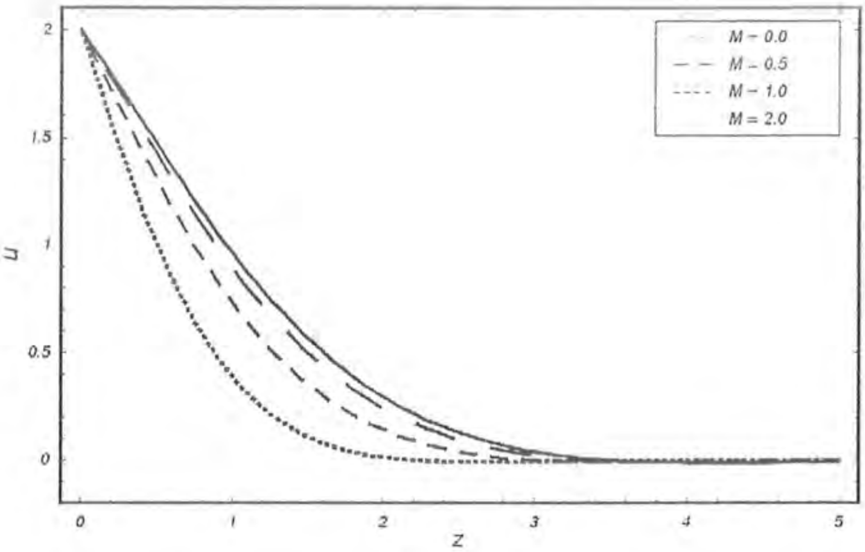
Figure 4.6b



(b) Oldroyd-B fluid ($\lambda_2 = \lambda_4 = 0$)

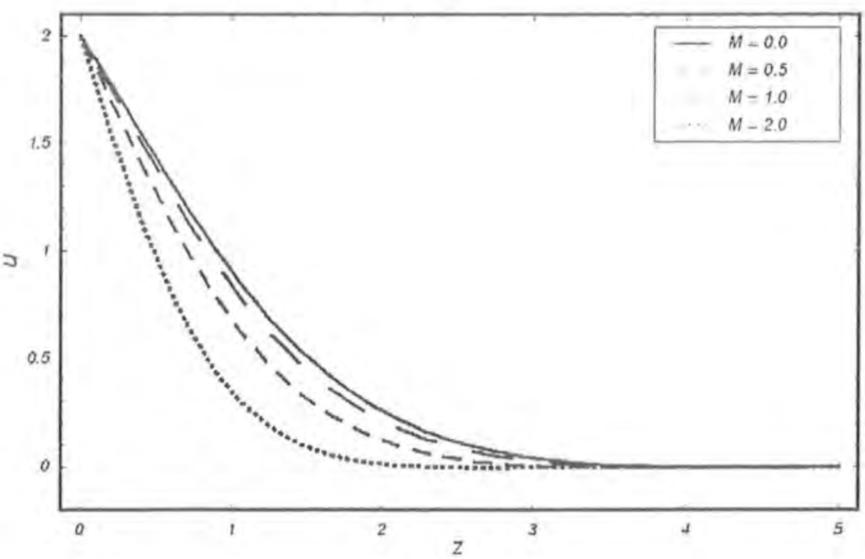
Figures 4.6 : The variation of velocity profile v for various values of Hall parameter m_0 when $\lambda_1 = 2, \ \lambda_3 = 1, M = K = 1, \ t = 0.1$ and $n/2\Omega = 0.5$ are fixed.

Figure 4.7a



(a) G. Burgers' fluid ($\lambda_2 = 1.5,\ \lambda_4 = 1.3$)

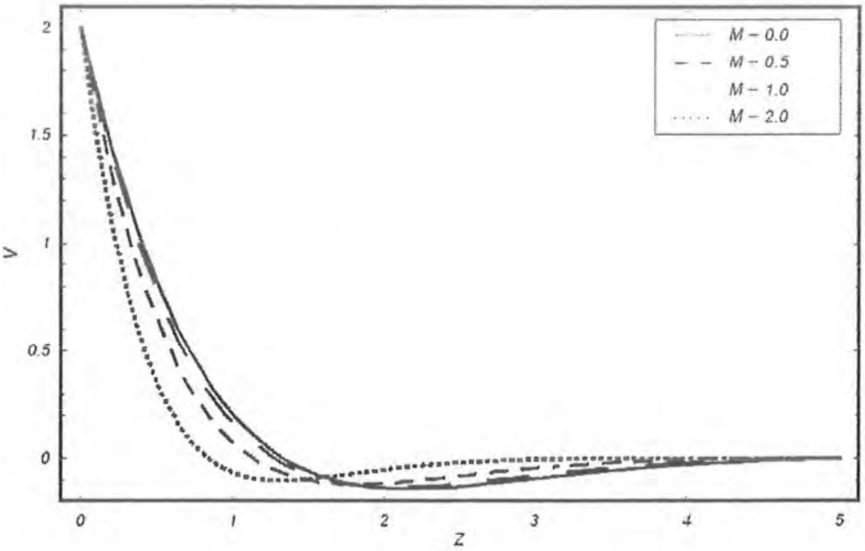
Figure 4.7b



(b) Oldroyd-B fluid ($\lambda_2 = \lambda_4 = 0$)

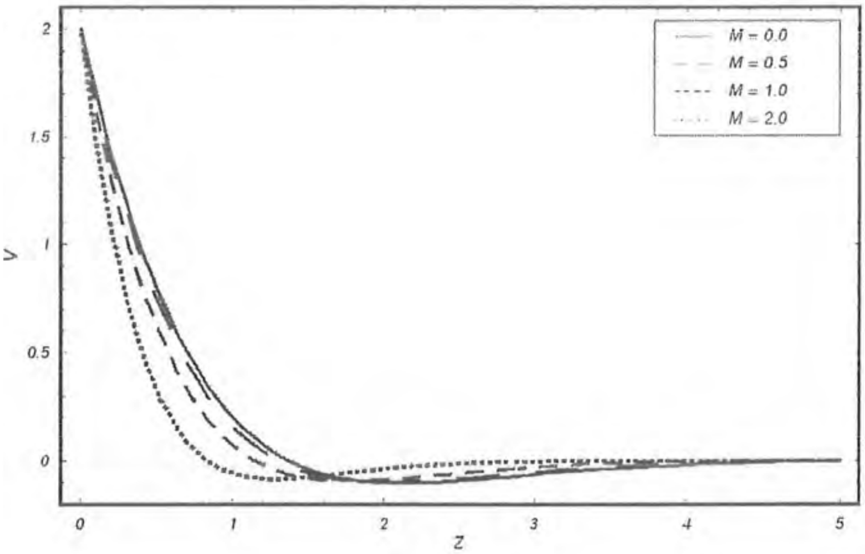
Figures 4.7 : The variation of velocity profile u for various values of magnetic parameter M when $\lambda_1 = 2,\ \lambda_3 = 1,\ K = m_0 = 1,\ t = 0.1$ and $n/2\Omega = 0.5$ are fixed.

Figure 4.8a



(a) G. Burgers' fluid ($\lambda_2 = 1.5,\ \lambda_4 = 1.3$)

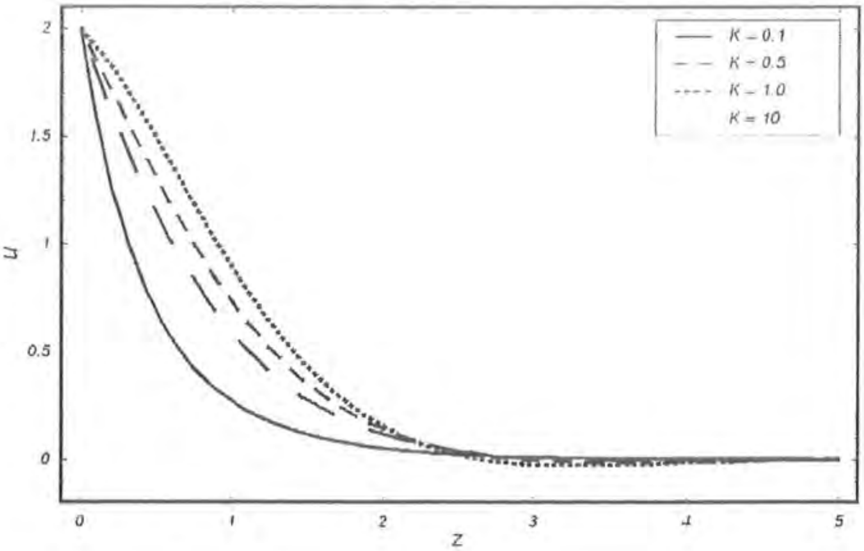
Figure 4.8b



(b) Oldroyd-B fluid ($\lambda_2 = \lambda_4 = 0$)

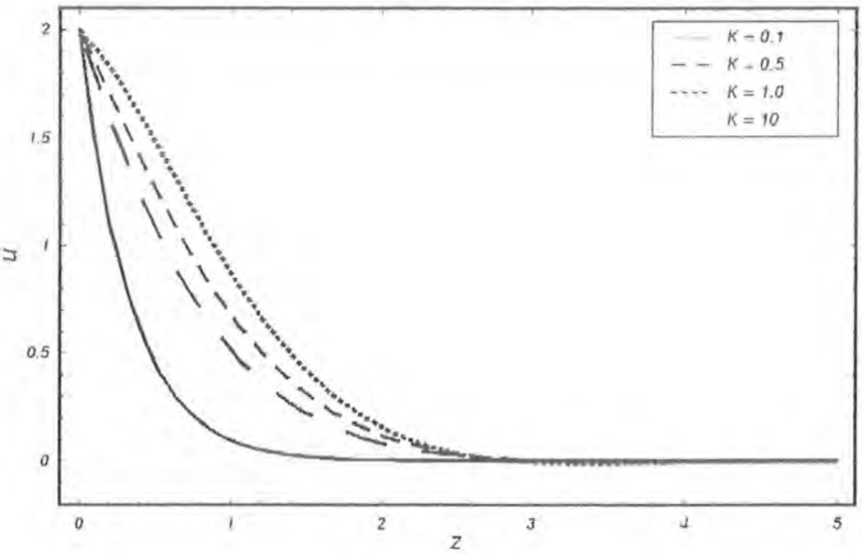
Figures 4.8 : The variation of velocity profile v for various values of magnetic parameter M when $\lambda_1 = 2,\ \lambda_3 = 1,\ K = m_0 = 1,\ t = 0.1$ and $n/2\Omega = 0.5$ are fixed.

Figure 4.9a



(a) G. Burgers' fluid ($\lambda_2 = 1.5,\ \lambda_4 = 1.3$)

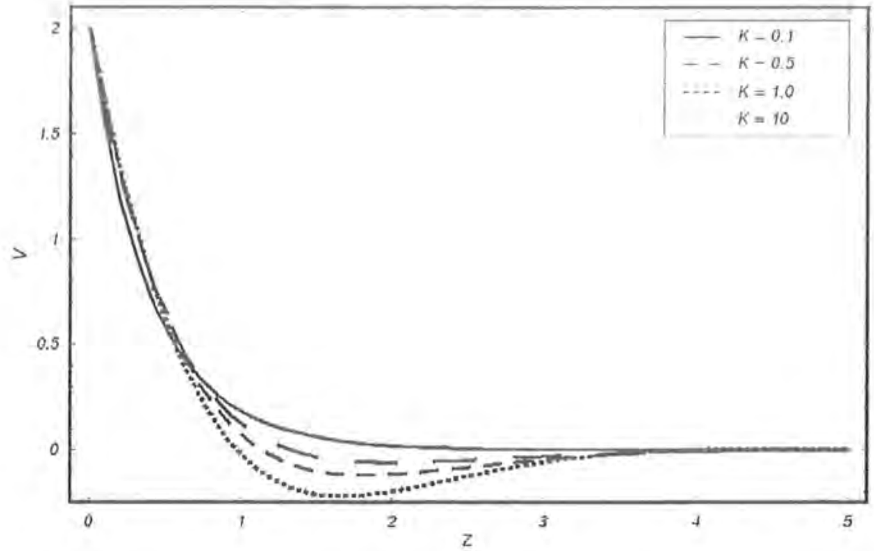
Figure 4.9b



(b) Oldroyd-B fluid ($\lambda_2 = \lambda_4 = 0$)

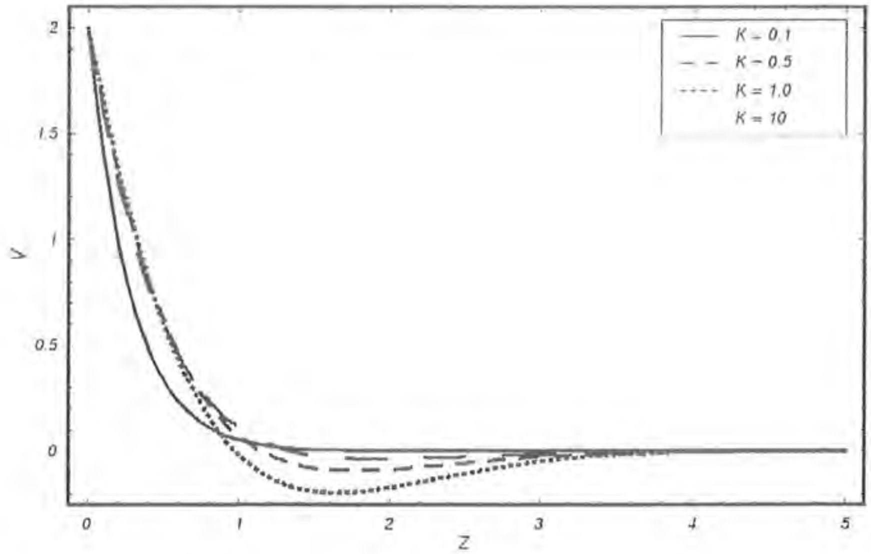
Figures 4.9 : The variation of velocity profile u for various values of permeability parameter K when $\lambda_1 = 2,\ \lambda_3 = 1,\ M = m_0 = 1,\ t = 0.1$ and $n/2\Omega = 0.5$ are fixed.

Figure 4.10a



(a) G. Burgers' fluid ($\lambda_2 = 1.5,\ \lambda_4 = 1.3$)

Figure 4.10b



(b) Oldroyd-B fluid ($\lambda_2 = \lambda_4 = 0$)

Figures 4.10 : The variation of velocity profile v for various values of permeability parameter K when $\lambda_1 = 2,\ \lambda_3 = 1,\ M = m_0 = 1,\ t = 0.1$ and $n/2\Omega = 0.5$ are fixed.

4.5 Concluding remarks

In this chapter the two problems of generalized Burgers' fluid have been discussed with Hall current. Modified Darcy's law for a generalized Burgers' fluid is first proposed and then used in the modelling of the equations. The result for the flow induced by general periodic oscillation of a plate is constructed. The solutions for all the frequencies including the resonant frequency are given in second problem. It is found that velocity profiles u and v have reverse behavior in case of generalized Burgers' and Burgers' fluids. The graphs for $n > 2\Omega$ and $n = 2\Omega$ are not included but it is seen that variation of various parameters in these cases are similar to that of $n < 2\Omega$. Also, the comparison among the several fluid models is made. It is very interesting to note that asymptotic solutions are possible for all values of the frequencies. The results of [112] can be recovered by taking $\lambda_2 = \lambda_4 = \phi = 0$.

Chapter 5

Effect of Hall current on the rotating flow of a third grade fluid in a porous space

This chapter looks at the steady rotating MHD flow of an incompressible third grade fluid in a porous space. Modified Darcy's law is utilized for the flow modeling. The Hall effects are present. The equations governing the flow lead to a non-linear ordinary differential equation. The arising problem is solved analytically by a homotopy analysis method (HAM). Graphs are displayed to gain a rudimentary understanding of the various interesting parameters on the velocity distribution.

5.1 Development of the flow

Consider the hydromagnetic flow of an incompressible third grade fluid bounded by an insulated plate at $z = 0$. The fluid occupying the space $z > 0$ fills the porous space. The fluid has uniform properties and porous medium is isotropic and homogeneous. A uniform strong magnetic field $\mathbf{B}_0$ is applied in the $z-$direction. In the undisturbed state, both fluid and the plate are in a state of rigid body rotation with constant angular velocity $\mathbf{\Omega} = \Omega\hat{k}$ ($\hat{k}$ is a unit vector parallel to the $z-$axis.). The flow is driven by a sudden motion of the plate. The magnetic Reynolds number is taken small and hence the induced magnetic field is negligible. However, the Hall current effects are retained. In a porous space, the equations governing the MHD steady flows may be written in a rotating coordinate system as

$$\rho\left[(\mathbf{V}\cdot\nabla)\mathbf{V}+2\boldsymbol{\Omega}\times\mathbf{V}+\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})\right]=div\mathbf{T}+\mathbf{J}\times\mathbf{B}+\mathbf{R}, \tag{5.1}$$

$$div\mathbf{V}=0, \tag{5.2}$$

$$div\mathbf{B}=0, \qquad Curl\mathbf{B}=\mu_e\mathbf{J}, \qquad Curl\mathbf{E}=\mathbf{0}, \tag{5.3}$$

$$\mathbf{J}+\frac{\omega_e\tau_e}{B_0}(\mathbf{J}\times\mathbf{B})=\sigma\left[\mathbf{E}+\mathbf{V}\times\mathbf{B}+\frac{1}{en_e}\nabla p_e\right]. \tag{5.4}$$

In above equations $\mathbf{T}$ is the Cauchy stress tensor, ρ is the fluid density, $\mathbf{V}$ is the velocity, $\mathbf{r}$ is the radial coordinate and $\mathbf{R}$ is the Darcy's resistance in the porous space. In Maxwell's equations (5.3) the total magnetic field $\mathbf{B}=\mathbf{B}_0+\mathbf{b}$ where $\mathbf{B}_0$ is the applied magnetic field and $\mathbf{b}$ is the induced magnetic field. $\mathbf{E}$ is the total electric field, μ_e is the magnetic permeability and $\mathbf{J}$ is the current density. Note that in the generalized Ohm's law (5.4), ω_e is the cyclotron frequency of electrons, τ_e is the electron collision time, σ is the electrical conductivity, e is the electron charge, $1/en_e$ is the Hall factor, n_e is the number density of the electrons and p_e is the electron pressure. The ion-slip and applied voltage are neglected.

The constitutive equations for a third grade fluid are

$$\mathbf{T}=-p\mathbf{I}+\mathbf{S}, \tag{5.5}$$

$$\mathbf{S}=\mu\mathbf{A}_1+\alpha_1\mathbf{A}_2+\alpha_2\mathbf{A}_1^2+\beta_1\mathbf{A}_3+\beta_2(\mathbf{A}_1\mathbf{A}_2+\mathbf{A}_2\mathbf{A}_1)+\beta_3(tr\mathbf{A}_2)\mathbf{A}_1. \tag{5.6}$$

Considering the thermodynamic conditions [122]

$$\mu\geq 0, \quad \alpha_1\geq 0, \quad \beta_1=\beta_2=0, \quad \beta_3\geq 0, \quad |\alpha_1+\alpha_2|\leq\sqrt{24\mu\beta_3}, \tag{5.7}$$

we have

$$\mathbf{S}=\mu\mathbf{A}_1+\alpha_1\mathbf{A}_2+\alpha_2\mathbf{A}_1^2+\beta_3(tr\mathbf{A}_2)\mathbf{A}_1, \tag{5.8}$$

where

$$\mathbf{A}_1=\mathbf{L}+\mathbf{L}^{\mathsf{T}}, \qquad \mathbf{L}=\nabla\mathbf{V}, \tag{5.9}$$

$$\mathbf{A}_i = (\mathbf{V}\cdot\nabla)\,\mathbf{A}_{i-1} + \mathbf{A}_{i-1}\mathbf{L} + \mathbf{L}^{\mathsf{T}}\mathbf{A}_{i-1}, \qquad i>1. \tag{5.10}$$

In above equations p is the pressure, $\mathbf{I}$ and $\mathbf{S}$ are the identity and extra stress tensors, respectively, μ is the dynamic viscosity, $\mathbf{A}_i\,(i=1,2,..)$ are the Rivlin-Ericksen tensors, $\mathbf{L}$ is the velocity gradient and $\mathbf{L}^{\mathsf{T}}$ is the transpose of $\mathbf{L}$.

In porous space the relationship between pressure drop and velocity is

$$\nabla p = -\frac{\phi}{k_1}\left[\mu + 2\beta_3\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\}\right]\mathbf{V}, \tag{5.11}$$

where ϕ and k_1 respectively indicate the porosity and permeability of the porous space and u and v are the velocity components. As the pressure gradient in Eq. (5.11) is a measure of the resistance to the flow in the bulk of porous space and $\mathbf{R}$ in Eq. (5.1) is interpreted as the flow resistance offered by the solid matrix. Therefore, $\mathbf{R}$ through Eq. (5.11) satisfies the following expression

$$\mathbf{R} = -\frac{\phi}{k_1}\left[\mu + 2\beta_3\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\}\right]\mathbf{V}. \tag{5.12}$$

For the problem under consideration the extra stress tensor and velocity are defined as follows:

$$\mathbf{S}(z) = \begin{pmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{pmatrix}, \tag{5.13}$$

$$\mathbf{V}(z) = (u(z), v(z), 0)\,. \tag{5.14}$$

Clearly Eq. (5.14) satisfies the incompressibility condition (5.2) and Eq. (5.1) in scalar form yields

$$\begin{aligned} -2\Omega\rho\frac{dv}{dz} &= \mu\frac{d^3u}{dz^3} + 2\beta_3\frac{d^2}{dz^2}\left[\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\}\frac{du}{dz}\right] \\ &\quad -\frac{\phi}{k_1}\frac{d}{dz}\left[\mu u + 2\beta_3\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\}u\right] - \frac{\sigma B_0^2}{1-im_0}\frac{du}{dz}, \end{aligned} \tag{5.15}$$

$$
\begin{aligned}
2\Omega\rho\frac{du}{dz} &= \mu\frac{d^3v}{dz^3}+2\beta_3\frac{d^2}{dz^2}\left[\left\{\left(\frac{du}{dz}\right)^2+\left(\frac{dv}{dz}\right)^2\right\}\frac{dv}{dz}\right] \\
&\quad -\frac{\phi}{k_1}\frac{d}{dz}\left[\mu v+2\beta_3\left\{\left(\frac{du}{dz}\right)^2+\left(\frac{dv}{dz}\right)^2\right\}v\right]-\frac{\sigma B_0^2}{1-im_0}\frac{dv}{dz},
\end{aligned}
\tag{5.16}
$$

where $m_0 = w_e\tau_e$ is the Hall parameter. The boundary conditions are

$$
\begin{aligned}
u &= U, \quad v=0 \ \text{ at } \ z=0 \\
u &\to 0, \quad v\to 0 \ \text{ as } z\to\infty.
\end{aligned}
\tag{5.17}
$$

Equations (5.15) and (5.16) can be combined in the following form

$$
\begin{aligned}
2\Omega i\rho\frac{dF}{dz} &= \mu\frac{d^3F}{dz^3}+2\beta_3\frac{d^2}{dz^2}\left\{\left(\frac{dF}{dz}\right)^2\frac{d\overline{F}}{dz}\right\} \\
&\quad -\frac{\phi}{k_1}\frac{d}{dz}\left\{\mu F+2\beta_3\frac{dF}{dz}\frac{d\overline{F}}{dz}F\right\}-\frac{\sigma B_0^2}{1-im_0}\frac{dF}{dz},
\end{aligned}
\tag{5.18}
$$

where

$$
F = u+iv, \qquad \overline{F} = u-iv. \tag{5.19}
$$

The boundary conditions (5.17) can be written as

$$
F(0) = U, \qquad F(\infty) = 0. \tag{5.20}
$$

We introduce the following dimensionless variables

$$
\begin{aligned}
z^* &= \frac{\rho U}{\mu}z, \quad F^*=\frac{F}{U}, \quad \overline{F}^*=\frac{\overline{F}}{U}, \quad \Omega^*=\frac{\mu}{\rho U^2}\Omega, \\
\beta &= \frac{\beta_3\rho^2U^4}{\mu^3}, \quad M^2=\frac{\sigma B_0^2}{\rho\Omega}, \quad K=\frac{\Omega\rho k_1}{\phi\mu}.
\end{aligned}
$$

The dimensionless problem after omitting the asterisks becomes

$$
\begin{aligned}
\frac{d^3F}{dz^3} &= \Omega\left\{\frac{M^2}{1+m_0^2}+\left(2-\frac{M^2m_0}{1+m_0^2}\right)i\right\}\frac{dF}{dz}-2\beta\frac{d^2}{dz^2}\left\{\left(\frac{dF}{dz}\right)^2\frac{d\overline{F}}{dz}\right\} \\
&\quad +\frac{\Omega}{K}\left\{\frac{dF}{dz}+2\beta\frac{d}{dz}\left(\frac{dF}{dz}\frac{d\overline{F}}{dz}F\right)\right\},
\end{aligned}
\tag{5.21}
$$

$$F(0)=1, \qquad F(\infty)=0. \tag{5.22}$$

Taking

$$\eta=e^{-z} \tag{5.23}$$

the problem reduces to

$$\begin{aligned}\eta^3\frac{d^3F}{d\eta^3}+3\eta^2\frac{d^2F}{d\eta^2}+\eta\frac{dF}{d\eta} &= \Omega\eta\left\{\frac{M^2}{1+m_0^2}+\left(2-\frac{M^2m_0}{1+m_0^2}\right)i\right\}\frac{dF}{d\eta} \\ &\quad -2\beta\eta^3\left[\begin{array}{c} 9\left(\frac{dF}{d\eta}\right)^2\frac{d\overline{F}}{d\eta}+7\eta\frac{d}{d\eta}\left\{\left(\frac{dF}{d\eta}\right)^2\frac{d\overline{F}}{d\eta}\right\} \\ +\eta^2\frac{d^2}{d\eta^2}\left\{\left(\frac{dF}{d\eta}\right)^2\frac{d\overline{F}}{d\eta}\right\}\end{array}\right] \\ &\quad +\frac{\Omega}{K}\eta\left[\begin{array}{c}\frac{dF}{d\eta}+2\beta\eta^2\frac{d}{d\eta}\left(\frac{dF}{d\eta}\frac{d\overline{F}}{d\eta}F\right) \\ +4\beta\eta\frac{dF}{d\eta}\frac{d\overline{F}}{d\eta}F\end{array}\right],\end{aligned} \tag{5.24}$$

$$F(1)=1, \qquad F(0)=0. \tag{5.25}$$

It is worth mentioning to note that by setting $\phi=0$ or $k_1\to\infty$ we get the governing problem for rotating flow of a third grade fluid in a non-porous space. Equation (5.24) is highly non-linear and its HAM solution subject to the boundary conditions (5.25) will be sought in the next section.

5.2 HAM solution

For the HAM solution we choose

$$F_0(\eta)=\eta, \tag{5.26}$$

and the auxiliary linear operator

$$\mathcal{L}(f)=f'', \tag{5.27}$$

satisfying

$$\mathcal{L}(C_1+C_2\eta)=0, \tag{5.28}$$

in which C_1 and C_2 are arbitrary constants.

The deformation problem at the zeroth order satisfies

$$(1-q)\,\mathcal{L}\left[\hat{F}(\eta,q)-F_0(\eta)\right]=q\hbar\mathcal{N}\left[\hat{F}(\eta,q)\right], \tag{5.29}$$

$$\hat{F}(0,q)=0, \qquad \hat{F}(1,q)=1, \tag{5.30}$$

where $\hbar$ and $q\in[0,1]$ are respectively the auxiliary and embedding parameters and

$$\begin{aligned}
\mathcal{N}\left[\hat{F}(\eta,q)\right] \;=\;& \eta^3\frac{\partial^3\hat{F}(\eta,q)}{\partial\eta^3}+3\eta^2\frac{\partial^2\hat{F}(\eta,q)}{\partial\eta^2}+\eta\frac{\partial\hat{F}(\eta,q)}{\partial\eta} \\
& -\Omega\eta\left[\frac{M^2}{1+m_0^2}+\left(2-\frac{M^2m_0}{1+m_0^2}\right)i\right]\frac{\partial\hat{F}(\eta,q)}{\partial\eta} \\
& +2\beta\eta^3\left[\begin{array}{c} 9\left(\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\right)^2\frac{\partial\bar{\hat{F}}(\eta,q)}{\partial\eta} \\ +7\eta\frac{\partial}{\partial\eta}\left\{\left(\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\right)^2\frac{\partial\bar{\hat{F}}(\eta,q)}{\partial\eta}\right\} \\ +\eta^2\frac{\partial^2}{\partial\eta^2}\left\{\left(\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\right)^2\frac{\partial\bar{\hat{F}}(\eta,q)}{\partial\eta}\right\}\end{array}\right] \\
& -\frac{\Omega}{K}\eta\left[\frac{\partial\hat{F}(\eta,q)}{\partial\eta}+2\beta\left\{\begin{array}{c}\eta^2\frac{\partial}{\partial\eta}\left(\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\frac{\partial\bar{\hat{F}}(\eta,q)}{\partial\eta}F(\eta,q)\right) \\ +2\eta\left(\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\frac{\partial\bar{\hat{F}}(\eta,q)}{\partial\eta}F(\eta,q)\right)\end{array}\right\}\right]
\end{aligned} \tag{5.31}$$

is the non-linear differential operator. For $q=0$ and $q=1$ we have

$$\hat{F}(\eta,0)=F_0(\eta), \qquad \hat{F}(\eta,1)=F(\eta). \tag{5.32}$$

We note from above equation that the derivation of q from 0 to 1 is continuous variation of $\hat{F}(\eta,q)$ from $F_0(\eta)$ to $F(\eta)$. Due to Taylor's theorem and Eq. (5.32) we can write

$$\hat{F}(\eta,q)=F_0(\eta)+\sum_{m=1}^{\infty}F_m(\eta)\;q^m \tag{5.33}$$

in which

$$F_m(\eta)=\frac{1}{m!}\frac{\partial^m\hat{F}(\eta,q)}{\partial q^m}\bigg|_{q=0}.$$

Clearly the convergence of the series (5.33) depends on the auxiliary parameter $\hbar$. Assume that

$\hbar$ is selected such that the series (5.32) is convergent at $q = 1$, then due to Eq. (5.33) we have

$$\hat{F}(\eta, q) = F_0(\eta) + \sum_{m=1}^{\infty} F_m(\eta) \ . \tag{5.34}$$

Differentiating m-times the zeroth order deformation (5.29) with respect to q and then dividing them by $m!$ and finally setting $q = 0$ we have the mth order deformation problem

$$\mathcal{L}\left[F_m(\eta) - \chi_m F_{m-1}(\eta)\right] = \hbar \mathcal{R}_m(\eta), \tag{5.35}$$

$$F_m(0) = F_m(1) = 0, \tag{5.36}$$

$$
\begin{aligned}
\mathcal{R}_m(\eta) &= \eta^3 F'''_{m-1}(\eta) + 3\eta^2 F''_{m-1}(\eta) + \eta F'_{m-1}(\eta) \\
&\quad -\Omega\eta\left\{\frac{M^2}{1+m_0^2} + \left(2 - \frac{M^2 m_0}{1+m_0^2}\right) i\right\} F'_{m-1}(\eta) \\
&\quad +2\beta\eta^3 \left[\begin{array}{c}
9\sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} F'_{k-l}(\eta)\bar{F}'_l(\eta) \\
+7\eta \left\{\begin{array}{c}
2\sum_{k=0}^{m-1} F'_{m-1-k}(\eta)\sum_{l=0}^{k} F''_{k-l}(\eta)\bar{F}'_l(\eta) \\
+\sum_{k=0}^{m-1} F'_{m-1-k}(\eta)\sum_{l=0}^{k} F'_{k-l}(\eta)\bar{F}''_l(\eta)
\end{array}\right\} \\
+\eta^2 \left\{\begin{array}{c}
2\sum_{k=0}^{m-1} F''_{m-1-k}(\eta)\sum_{l=0}^{k} F''_{k-l}(\eta)\bar{F}'_l(\eta) \\
+2\sum_{k=0}^{m-1} F'_{m-1-k}(\eta)\sum_{l=0}^{k} F'''_{k-l}(\eta)\bar{F}'_l(\eta) \\
+4\sum_{k=0}^{m-1} F'_{m-1-k}(\eta)\sum_{l=0}^{k} F''_{k-l}(\eta)\bar{F}''_l(\eta) \\
+\sum_{k=0}^{m-1} F'_{m-1-k}(\eta)\sum_{l=0}^{k} F'_{k-l}(\eta)\bar{F}'''_l(\eta)
\end{array}\right\}
\end{array}\right] \\
&\quad -\frac{\Omega}{K}\eta \left[\begin{array}{c}
F'_{m-1}(\eta) + 2\beta\eta^2 \left\{\begin{array}{c}
\sum_{k=0}^{m-1} F''_{m-1-k}(\eta)\sum_{l=0}^{k} \bar{F}'_{k-l}(\eta) F_l(\eta) \\
+\sum_{k=0}^{m-1} F'_{m-1-k}(\eta)\sum_{l=0}^{k} \bar{F}''_{k-l}(\eta) F_l(\eta) \\
+\sum_{k=0}^{m-1} F'_{m-1-k}(\eta)\sum_{l=0}^{k} F'_{k-l}(\eta)\bar{F}'_l(\eta)
\end{array}\right\} \\
+4\beta\eta \sum_{k=0}^{m-1} F'_{m-1-k}(\eta)\sum_{l=0}^{k} \bar{F}'_{k-l}(\eta) F_l(\eta)
\end{array}\right] \qquad (5.37)
\end{aligned}
$$

and

$$
\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \qquad (5.38)
$$

For the solution of the mth order problem we use the symbolic computation software MATHEMATICA up to first few order of approximations and found that the solution of this problem is given by the following series

$$F_m(\eta) = \sum_{n=0}^{4m+2} a_{m,n}\eta^n, \qquad m \geq 0. \tag{5.39}$$

In order to obtain the recurrence formulae for the coefficients $a_{m,n}$ of $F_m(\eta)$ we substitute $Eq.(5.39)$ in $Eq.(5.35)$ and obtain for $m \geq 1$ and $0 \leq n \leq 4m+2$:

$$a_{m,1} = \chi_m \chi_{4m-1} a_{m-1,1} - \sum_{n=0}^{4m+2} \frac{\Psi_{m,n}}{(n+1)(n+2)}, \tag{5.40}$$

$$a_{m,n} = \chi_m \chi_{4m-n} a_{m-1,n} + \frac{\Psi_{m,n-2}}{n(n-1)}, \quad 2 \leq n \leq 4m+2, \tag{5.41}$$

where the coefficient $\Psi_{m,n}$ is define by

$$\Psi_{m,n} = \hbar \left[\begin{array}{c} \chi_{4m-n} \left\{ \begin{array}{c} \chi_{n-1} d_{m-1,n-3} + 3\chi_n c_{m-1,n-2} \\ +\chi_{n+1} b_{m-1,n-1} \left(1 + \frac{\Omega}{K} - \Omega \left(\frac{M^2}{1+m_0^2} + \left(2 - \frac{M^2 m_0}{1+m_0^2}\right) i\right)\right) \end{array} \right\} \\ +2\beta \left\{ \begin{array}{c} \chi_{n-1} \left(9\Delta 1_{m,n-3} - \frac{\Omega}{K}\left(\Delta 8_{m,n-3} + \Delta 9_{m,n-3} + \Delta 1_{m,n-3}\right)\right) \\ +7\chi_{n-2}\left(2\Delta 2_{m,n-4} + \Delta 3_{m,n-4}\right) - \frac{2\Omega}{K}\chi_n \Delta 10_{m,n-2} \\ +\chi_{n-3}\left(2\Delta 4_{m,n-5} + 2\Delta 5_{m,n-5} + 4\Delta 6_{m,n-5} + \Delta 7_{m,n-5}\right) \end{array} \right\} \end{array} \right], \tag{5.42}$$

in which the coefficients $\Delta 1_{m,n}$ to $\Delta 10_{m,n}$ for $m \geq 1$ and $0 \leq n \leq 4m+2$ are

$$\Delta 1_{m,n} = \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{q=\max\{0,n-4m+4k-2\}}^{\min\{n,4k+4\}} \sum_{s=\max\{0,q-4k+4l-2\}}^{\min\{q,4l+2\}} \bar{b}_{l,s} b_{k-l,q-s} b_{m-1-k,n-q},$$

$$\Delta 2_{m,n} = \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{q=\max\{0,n-4m+4k-2\}}^{\min\{n,4k+4\}} \sum_{s=\max\{0,q-4k+4l-2\}}^{\min\{q,4l+2\}} \bar{b}_{l,s} c_{k-l,q-s} b_{m-1-k,n-q},$$

$$\Delta 3_{m,n} = \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{q=\max\{0,n-4m+4k-2\}}^{\min\{n,4k+4\}} \sum_{s=\max\{0,q-4k+4l-2\}}^{\min\{q,4l+2\}} \bar{c}_{l,s} c_{k-l,q-s} b_{m-1-k,n-q},$$

$$\Delta 4_{m,n} = \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{q=\max\{0,n-4m+4k-2\}}^{\min\{n,4k+4\}} \sum_{s=\max\{0,q-4k+4l-2\}}^{\min\{q,4l+2\}} \bar{b}_{l,s} c_{k-l,q-s} c_{m-1-k,n-q},$$

$$\Delta 5_{m,n} = \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{q=\max\{0,n-4m+4k-2\}}^{\min\{n,4k+4\}} \sum_{s=\max\{0,q-4k+4l-2\}}^{\min\{q,4l+2\}} \bar{b}_{l,s} d_{k-l,q-s} b_{m-1-k,n-q},$$

$$\Delta 6_{m,n} = \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{q=\max\{0,n-4m+4k-2\}}^{\min\{n,4k+4\}} \sum_{s=\max\{0,q-4k+4l-2\}}^{\min\{q,4l+2\}} \bar{c}_{l,s} c_{k-l,q-s} b_{m-1-k,n-q},$$

$$\Delta 7_{m,n} = \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{q=\max\{0,n-4m+4k-2\}}^{\min\{n,4k+4\}} \sum_{s=\max\{0,q-4k+4l-2\}}^{\min\{q,4l+2\}} \bar{d}_{l,s} b_{k-l,q-s} b_{m-1-k,n-q},$$

$$\Delta 8_{m,n} = \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{q=\max\{0,n-4m+4k-2\}}^{\min\{n,4k+4\}} \sum_{s=\max\{0,q-4k+4l-2\}}^{\min\{q,4l+2\}} \bar{b}_{l,s} a_{k-l,q-s} c_{m-1-k,n-q},$$

$$\Delta 9_{m,n} = \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{q=\max\{0,n-4m+4k-2\}}^{\min\{n,4k+4\}} \sum_{s=\max\{0,q-4k+4l-2\}}^{\min\{q,4l+2\}} \bar{c}_{l,s} a_{k-l,q-s} b_{m-1-k,n-q},$$

$$\Delta 10_{m,n} = \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{q=\max\{0,n-4m+4k-2\}}^{\min\{n,4k+4\}} \sum_{s=\max\{0,q-4k+4l-2\}}^{\min\{q,4l+2\}} \bar{b}_{l,s} a_{k-l,q-s} b_{m-1-k,n-q}$$

and the coefficients $b_{m,n}$, $c_{m,n}$ and $d_{m,n}$ are [123]

$$b_{m,n} = (n+1)\, a_{m,n+1}, \tag{5.43}$$

$$c_{m,n} = (n+1)\, b_{m,n+1}, \tag{5.44}$$

$$d_{m,n} = (n+1)\, c_{m,n+1}. \tag{5.45}$$

With the above recurrence formulae we can calculate all coefficients $a_{m,n}$ using only the first three

$$a_{0,0} = 0, \quad a_{0,1} = 1, \quad a_{0,2} = 0 \tag{5.46}$$

given by the initial guess approximation for the function $F(\eta)$ in *Eq.*(5.26). The corresponding M-order approximation of *Eqs.*(5.24) and (5.25) is then given by

$$\sum_{m=0}^{M} F_m(\eta) = \sum_{n=1}^{4M+2} \sum_{m=n-1}^{4M+1} a_{m,n} \eta^n. \tag{5.47}$$

Therefore explicit, totally analytic solution of the present flow is

$$F(\eta) = \sum_{m=0}^{\infty} F_m(\eta) = \lim_{M \longrightarrow \infty} \left[\sum_{n=1}^{4M+2} \sum_{m=n-1}^{4M+1} a_{m,n} \eta^n \right]. \tag{5.48}$$

5.3 Convergence of the analytic solution

The expression given in Eq. (5.48) contains the auxiliary parameter $\hbar$ which gives the convergence region and rate of approximation for the homotopy analysis method [123]. In Fig. 5.1 (a, b) the $\hbar$-curves are plotted for different order of approximations for the non-dimensional velocity fields u and v. It is clear from Fig. 5.1 (a, b) that the range for the admissible values for $\hbar$ is $-0.2 \leq h < 0$. Our calculations indicate that the real and imaginary parts of the series given by Eq. (5.48) converge in the whole region of z when $\hbar = -0.1$.

Figure 5.1a

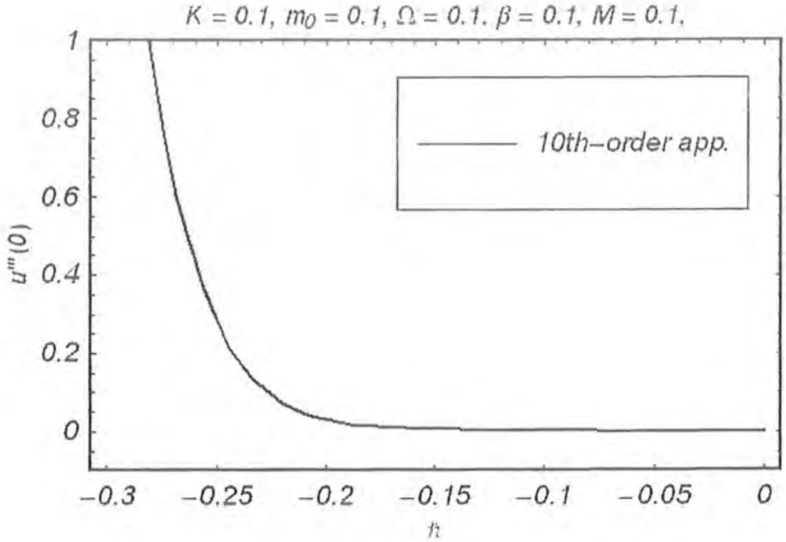
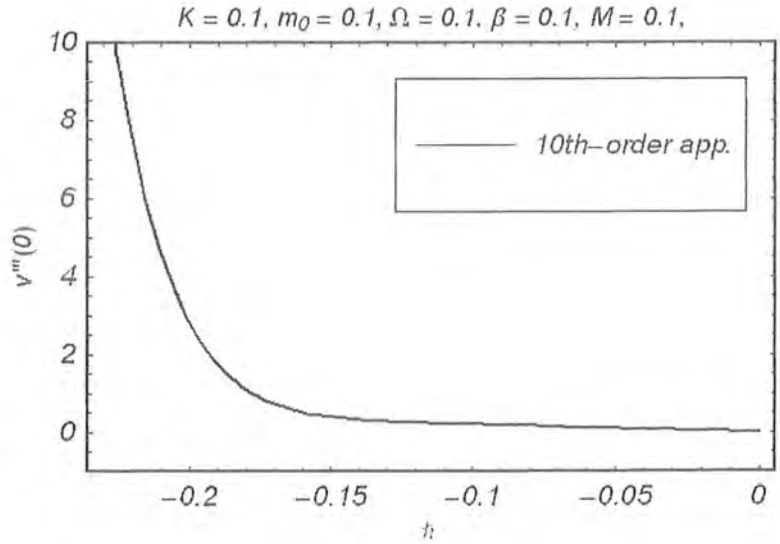


Figure 5.1b



5.4 Results and discussion

Figures 5.2 (a, b) show the variation of the velocity profiles u and v for different values of β. These figures show that with increase in β the velocity profile u decreases whereas the velocity profile v increases and then decreases by increasing β when other parameters $M = K = m_0 = \Omega = 1$ are fixed. It is observed from these figures that an increase in the velocity profile v is smaller in magnitude when compared with the velocity profile u. Figure 5.3 (a, b) depicts the variation of the velocity profiles u and v for various values of M when $\beta = K = m_0 = \Omega = 1$. These figures indicate that increasing M the velocity profile u increases whereas the velocity profile v decreases showing the reverse behavior as observed in the previous case. However, the decrease in the magnitude of the velocity profile v is smaller than that in the increase of the velocity profile u. Figure 5.4 (a, b) shows the variation of K on the velocity profiles u and v for $M = \beta = m_0 = \Omega = 1$. The velocity profile u decreases with increase in K whereas the velocity profile v increases with increase in K but with a small change when compared with the velocity profile u. In figure 5.5 (a, b) it is observed that the velocity profiles u and v increase for large values of m_0 when $M = K = \beta = \Omega = 1.5$. The magnitude of decrease in the velocity profile v is smaller than that of decrease of u. Figures 5.6 (a, b) depict the variation in the velocity profiles u and v for different values of Ω. It is clearly seen from these figures that the velocity profile u decreases by increasing Ω. However the velocity profile v increases by increasing Ω. The increase is smaller for the velocity profile v when compared with that of the velocity profile u when $M = K = m_0 = \beta = 1$.

Figure 5.2a

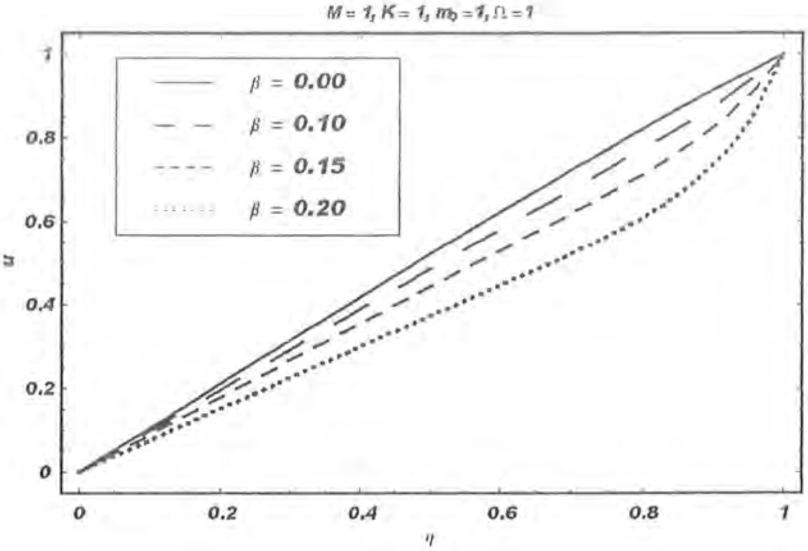


Figure 5.2b

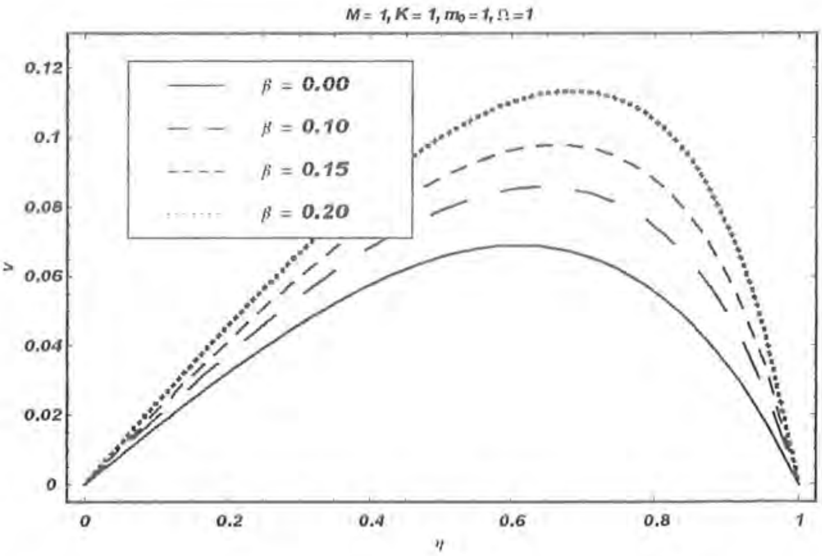


Figure 5.2. Variation of velocity profiles u and v with the change in parameter β for 10th-order approximation at $\hbar = -0.1$.

Figure 5.3a

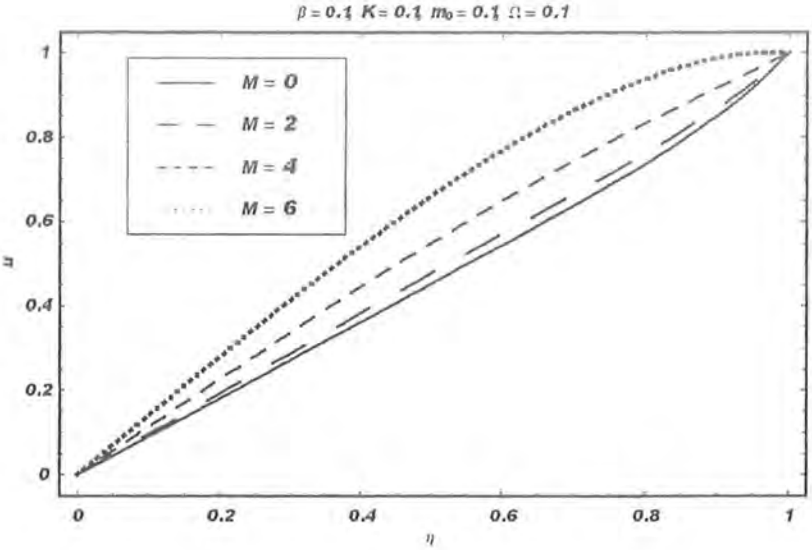


Figure 5.3b

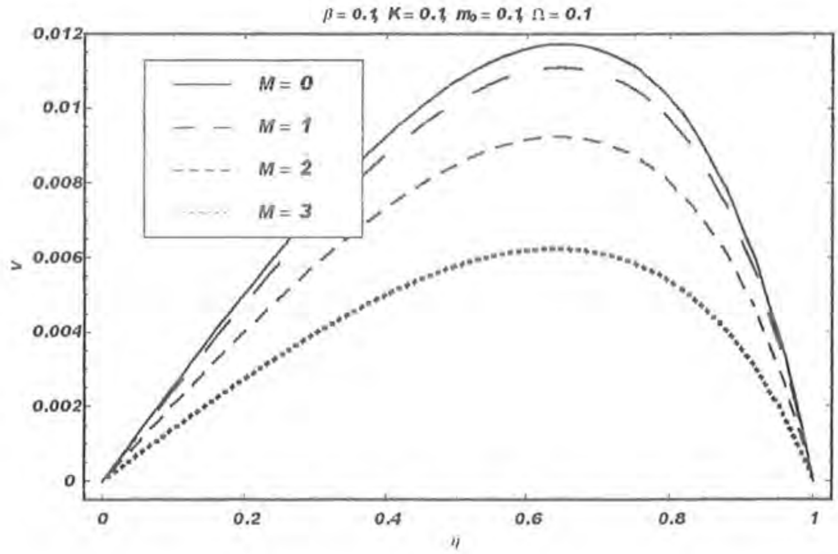


Figure 5.3. Variation of velocity profiles u and v with the change in parameter M for 10th-order approximation at $\hbar = -0.1$.

Figure 5.4a

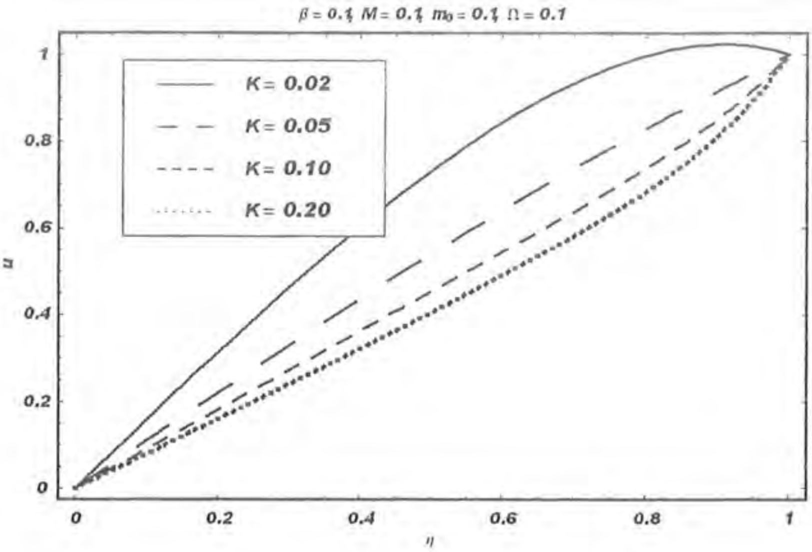


Figure 5.4b

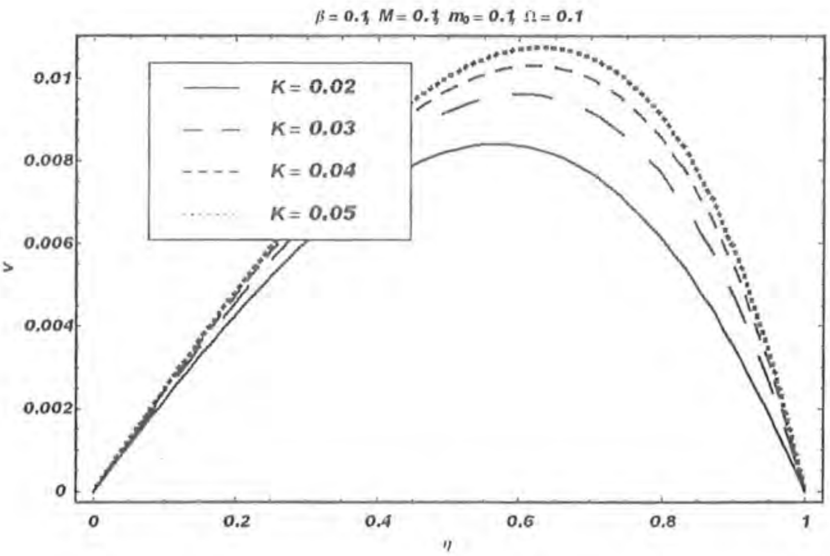


Figure 5.4. Variation of velocity fields u and v with the change in parameter K for 10th-order approximation at $\hbar = -0.1$.

Figure 5.5a

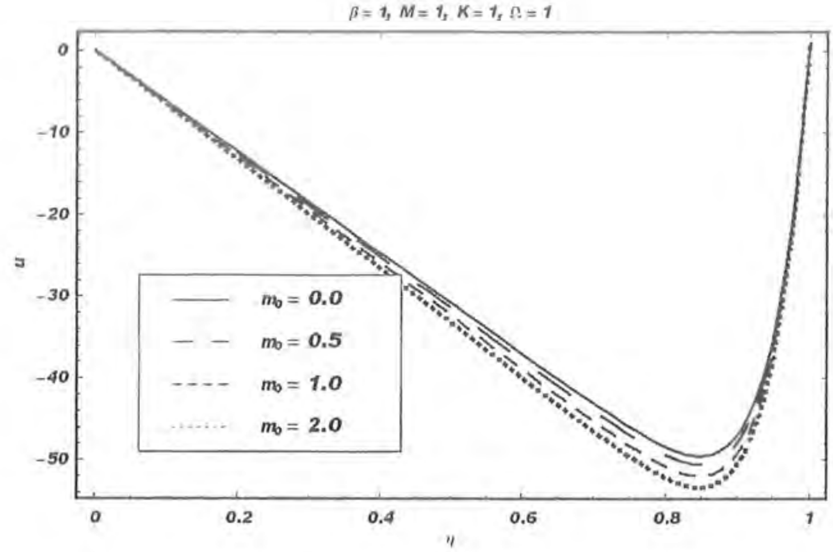


Figure 5.5b

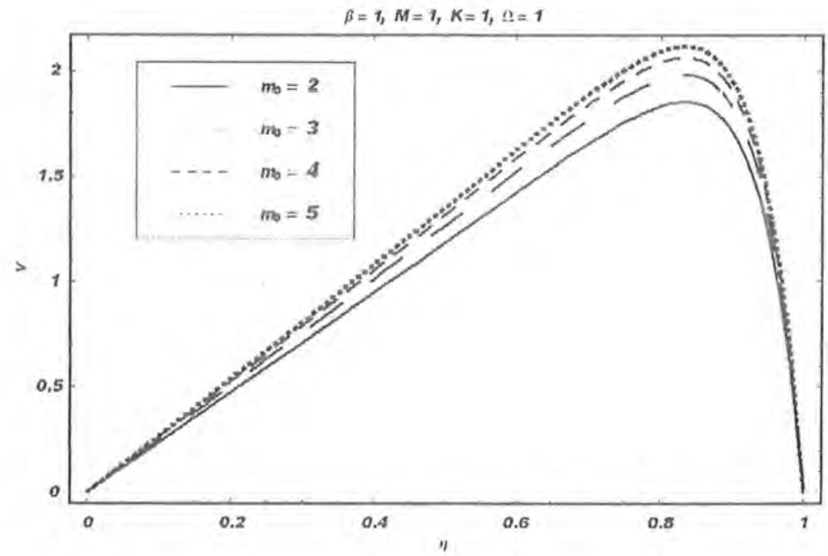


Figure 5.5. Variation of velocity profiles u and v with the change in parameter m_0 for 10th-order approximation at $\hbar = -0.1$.

Figure 5.6a

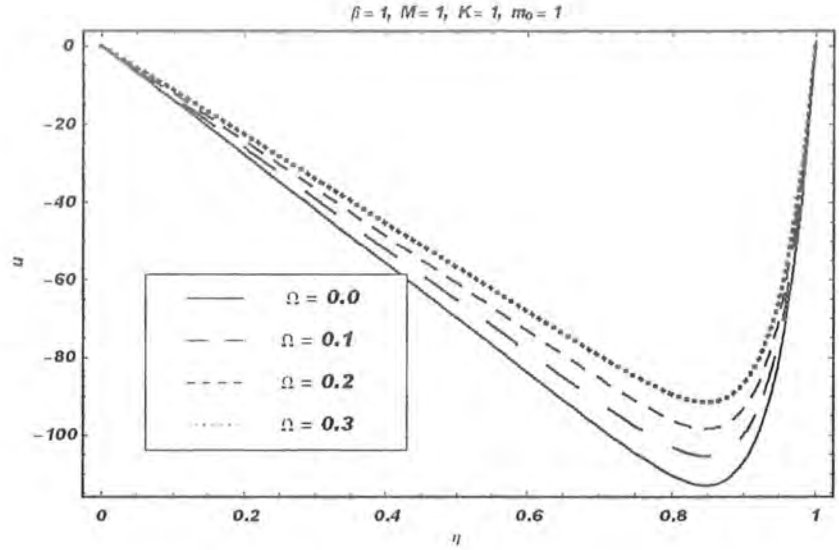


Figure 5.6b

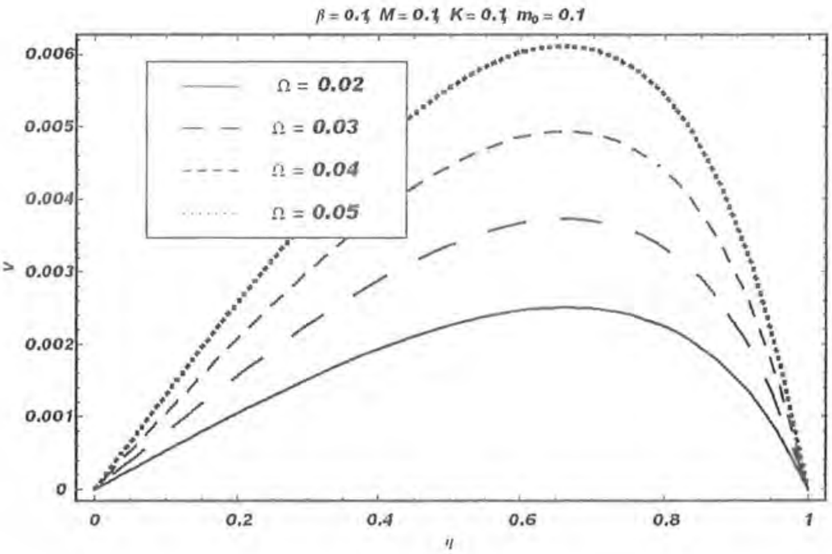


Figure 5.6. Variation of velocity profiles u and v with the change in parameter Ω for 10th-order approximation at $\hbar = -0.1$.

5.5 Concluding remarks

In this work the flow of a third grade fluid over a moved plate is investigated in the presence of Hall current. Based on modified Darcy's law for a third grade fluid the governing equation is modeled. The resulting non-linear problem has been solved analytically using HAM. It is noted that the velocity profiles u and v have reversed behavior for physical parameters β, M, K and Ω but the velocity profiles u and v show the same behavior for the physical parameter m_0. It is further found that the boundary layer thickness for v is large when compared to u.

Chapter 6

Series solution for rotating flow of a fourth grade fluid

The main objective of this chapter is to venture further in the regime of rotating non-Newtonian flows in a porous media. Therefore, we present an analytical study for a fourth grade fluid. This chapter investigates the steady magnetohydrodynamic (MHD) rotating flow of a fourth grade fluid over a suddenly moved porous plate. The fluid fills the porous half space and the resulting flow is analyzed on the basis of a modified Darcy's law. The homotopy analysis method (HAM) is employed to develop an explicit analytic solution of the highly non-linear problem. Recurrence formulas are developed and the convergence of the solution is carefully checked. With the determined velocity components, the effects of various pertinent parameters are discussed. The results for limiting cases are found to be in good agreement with those available in the open literature.

6.1 Governing problem

The Cauchy stress tensor of a fourth grade fluid is [77]

$$\mathrm{T} = -p\mathrm{I} + \mathrm{S}. \tag{6.1}$$

The model may exhibits normal stresses, shear thinning or shear thickening behaviors. Here p is the pressure. $\mathbf{I}$ is the identity tensor and the extra stress tensor $\mathbf{S}$ for a fourth grade fluid is given by

$$\mathbf{S} = \sum_{j=1}^{4} S_j, \tag{6.2}$$

$$\mathbf{S}_1 = \mu \mathbf{A}_1, \tag{6.3}$$

$$\mathbf{S}_2 = \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2, \tag{6.4}$$

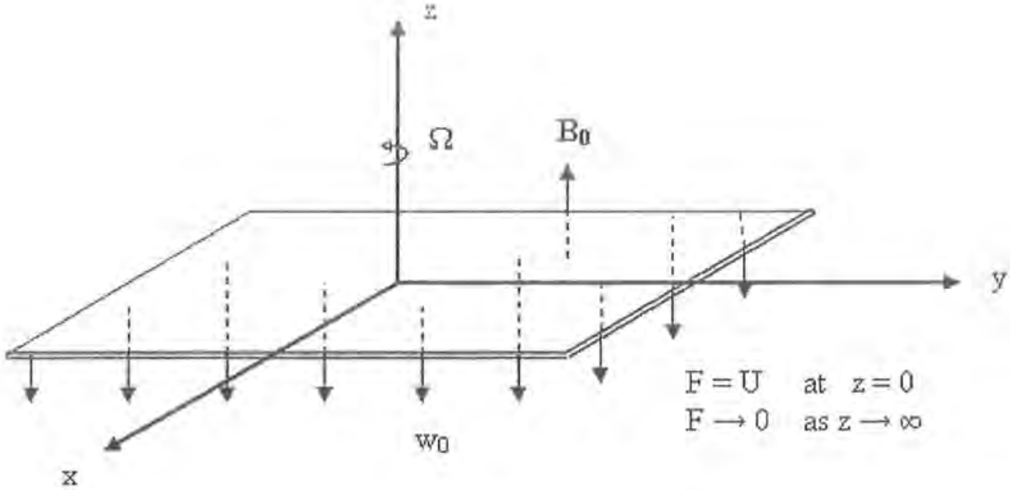
$$\mathbf{S}_3 = \beta_1 \mathbf{A}_3 + \beta_2 \left(\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1\right) + \beta_3 \left(tr \mathbf{A}_2\right) \mathbf{A}_1, \tag{6.5}$$

$$\begin{aligned} \mathbf{S}_4 \;=\;& \gamma_1 \mathbf{A}_4 + \gamma_2 \left(\mathbf{A}_3 \mathbf{A}_1 + \mathbf{A}_1 \mathbf{A}_3\right) + \gamma_3 \mathbf{A}_2^2 + \gamma_4 \left(\mathbf{A}_2 \mathbf{A}_1^2 + \mathbf{A}_1^2 \mathbf{A}_2\right) \\ & + \gamma_5 \left(tr \mathbf{A}_2\right) \mathbf{A}_2 + \gamma_6 \left(tr \mathbf{A}_2\right) \mathbf{A}_1^2 + \gamma_7 \left(tr \mathbf{A}_3\right) \mathbf{A}_1 + \gamma_8 \left(tr \mathbf{A}_2 \mathbf{A}_1\right) \mathbf{A}_1, \end{aligned} \tag{6.6}$$

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^{\mathsf{T}}, \qquad \mathbf{L} = \nabla \mathbf{V}, \tag{6.7}$$

$$\mathbf{A}_i = \frac{\partial \mathbf{A}_{i-1}}{\partial t} + \left(\mathbf{V} \cdot \nabla\right) \mathbf{A}_{i-1} + \mathbf{A}_{i-1} \mathbf{L} + \mathbf{L}^{\mathsf{T}} \mathbf{A}_{i-1}, \quad i > 1, \tag{6.8}$$

where μ is the dynamic viscosity, $\mathbf{A}_i$ are the Rivlin-Ericksen tensors, $\mathbf{L}^{\mathsf{T}}$ is the transpose of $\mathbf{L}$ and α_1, α_2, $\beta_1 -$ β_3 and $\gamma_1 - \gamma_8$ are the material constants of the fourth grade fluid. It is worth mentioning to note that Eq.(6.2) reduces to the third order fluid model when $\gamma_1 - \gamma_8$ are equal to zero; second order fluid when $\beta_1 -$ β_3 and $\gamma_1 - \gamma_8$ are zero and to the Navier-Stokes fluid when α_1, α_2, $\beta_1 - \beta_3$ and $\gamma_1 - \gamma_8$ are zero.



A schematic figure showing the fourth grade fluid model domain

Here we consider an incompressible flow of a fourth grade fluid bounded by a uniformly porous plate (at $z = 0$). The fluid occupies the half space $z > 0$. The flow is driven by a suddenly moved plate. In the undisturbed state, the fluid and the plate rotate as a rigid body rotation with constant angular velocity $\mathbf{\Omega}$ (= $\Omega\mathbf{k}$, $\mathbf{k}$ is a unit vector parallel to the $z-$axis). The flow is subjected to a strong magnetic field $\mathbf{B}_0$ applied in the $z-$direction. For small magnetic Reynolds number, the induced magnetic field is neglected which is a valid assumption on the laboratory scale. In general, for MHD fluid, the Hall current significantly effects the flow in the presence of a strong magnetic field. For uniformly porous plate, the incompressibility condition for steady flow suggests the velocity field of the following form

$$\mathbf{V}(z) = (u(z), v(z), -w_0), \tag{6.9}$$

where u and v are the velocity components parallel to the x and y-axes, respectively and $w_0 > 0$ corresponds to suction velocity and $w_0 < 0$ indicates the injection (or blowing) velocity. The extra stress tensor is

$$\mathbf{S}(z) = \begin{pmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{pmatrix}. \tag{6.10}$$

Employing the procedure of previous chapter, we write down the resulting differential system for the velocity components u and v in the form

$$\begin{aligned} -2\Omega\rho\frac{dv}{dz} - \rho w_0\frac{d^2u}{dz^2} &= \frac{d^2}{dz^2}\left[\begin{array}{c} \mu\frac{du}{dz} - \alpha_1 w_0\frac{d^2u}{dz^2} + \beta_1 w_0^2\frac{d^3u}{dz^3} + 2(\beta_2+\beta_3)\left[\frac{du}{dz}\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\}\right] \\ -\gamma_1 w_0^3\frac{d^4u}{dz^4} - 6\gamma_2 w_0^2\left\{\frac{du}{dz}\left(\frac{du}{dz}\frac{d^2u}{dz^2} + \frac{dv}{dz}\frac{d^2v}{dz^2}\right)\right\} \\ -(2\gamma_3+\gamma_4+2\gamma_5)\, w_0\frac{d^2u}{dz^2}\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\} \\ -(\gamma_4+6\gamma_7+2\gamma_8)\, w_0\frac{du}{dz}\left(\frac{du}{dz}\frac{d^2u}{dz^2} + \frac{dv}{dz}\frac{d^2v}{dz^2}\right) \end{array}\right] \\ &\quad -\frac{\phi}{k_1}\frac{d}{dz}\left[\begin{array}{c} \mu u - \alpha_1 w_0\frac{du}{dz} + \beta_1 w_0^2\frac{d^2u}{dz^2} + 2(\beta_2+\beta_3)\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\}u \\ -\gamma_1 w_0^3\frac{d^3u}{dz^3} - 6\gamma_2 w_0^2\left(\frac{du}{dz}\frac{d^2u}{dz^2} + \frac{dv}{dz}\frac{d^2v}{dz^2}\right)u \\ -(2\gamma_3+\gamma_4+2\gamma_5)\, w_0\frac{du}{dz}\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\}u \\ -(\gamma_4+6\gamma_7+2\gamma_8)\, w_0\left(\frac{du}{dz}\frac{d^2u}{dz^2} + \frac{dv}{dz}\frac{d^2v}{dz^2}\right)u \end{array}\right] \\ &\quad -\frac{\sigma B_0^2}{1-im_0}\frac{du}{dz}, \end{aligned} \tag{6.11}$$

$$\begin{aligned} 2\Omega\rho\frac{du}{dz} - \rho w_0\frac{d^2v}{dz^2} &= \frac{d^2}{dz^2}\left[\begin{array}{c} \mu\frac{dv}{dz} - \alpha_1 w_0\frac{d^2v}{dz^2} + \beta_1 w_0^2\frac{d^3v}{dz^3} + 2(\beta_2+\beta_3)\left[\frac{dv}{dz}\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\}\right] \\ -\gamma_1 w_0^3\frac{d^4v}{dz^4} - 6\gamma_2 w_0^2\left\{\frac{dv}{dz}\left(\frac{du}{dz}\frac{d^2u}{dz^2} + \frac{dv}{dz}\frac{d^2v}{dz^2}\right)\right\} \\ -(2\gamma_3+\gamma_4+2\gamma_5)\, w_0\frac{d^2v}{dz^2}\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\} \\ -(\gamma_4+6\gamma_7+2\gamma_8)\, w_0\frac{dv}{dz}\left(\frac{du}{dz}\frac{d^2u}{dz^2} + \frac{dv}{dz}\frac{d^2v}{dz^2}\right) \end{array}\right] \\ &\quad -\frac{\phi}{k_1}\frac{d}{dz}\left[\begin{array}{c} \mu v - \alpha_1 w_0\frac{dv}{dz} + \beta_1 w_0^2\frac{d^2v}{dz^2} + 2(\beta_2+\beta_3)\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\}v \\ -\gamma_1 w_0^3\frac{d^3v}{dz^3} - 6\gamma_2 w_0^2\left(\frac{du}{dz}\frac{d^2u}{dz^2} + \frac{dv}{dz}\frac{d^2v}{dz^2}\right)v \\ -(2\gamma_3+\gamma_4+2\gamma_5)\, w_0\frac{du}{dz}\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\}v \\ -(\gamma_4+6\gamma_7+2\gamma_8)\, w_0\left(\frac{du}{dz}\frac{d^2u}{dz^2} + \frac{dv}{dz}\frac{d^2v}{dz^2}\right)v \end{array}\right] \\ &\quad -\frac{\sigma B_0^2}{1-im_0}\frac{dv}{dz}, \end{aligned} \tag{6.12}$$

$$
\begin{aligned}
u &= U, \quad v = 0 \text{ at } z = 0 \\
u &\to 0, \quad v \to 0 \text{ as } z \to \infty,
\end{aligned}
\tag{6.13}
$$

in which m_0 is the Hall parameter, σ is the electrical conductivity of fluid, ϕ is the porosity of the medium and k_1 is the permeability of the porous medium. It should be pointed out that in obtaining Eqs. (6.11) and (6.12), the following expressions for pressure drop (∇p) and flow resistance ($\mathbf{R}$) have been used:

$$
\nabla p = -\frac{\phi}{k_1}\left[\begin{array}{c}
\mu - \alpha_1 w_0 \frac{d}{dz} + \beta_1 w_0^2 \frac{d^2}{dz^2} + 2\left(\beta_2+\beta_3\right)\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\} \\
-\gamma_1 w_0^3 \frac{d^3}{dz^3} - 6\gamma_2 w_0^2 \left(\frac{du}{dz}\frac{d^2u}{dz^2} + \frac{dv}{dz}\frac{d^2v}{dz^2}\right) \\
-\left(2\gamma_3+\gamma_4+2\gamma_5\right) w_0 \frac{du}{dz}\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\} \\
-\left(\gamma_4+6\gamma_7+2\gamma_8\right) w_0 \left(\frac{du}{dz}\frac{d^2u}{dz^2} + \frac{dv}{dz}\frac{d^2v}{dz^2}\right)
\end{array}\right]\mathbf{V},
\tag{6.14}
$$

$$
\mathbf{R} = -\frac{\phi}{k_1}\left[\begin{array}{c}
\mu - \alpha_1 w_0 \frac{d}{dz} + \beta_1 w_0^2 \frac{d^2}{dz^2} + 2\left(\beta_2+\beta_3\right)\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\} \\
-\gamma_1 w_0^3 \frac{d^3}{dz^3} - 6\gamma_2 w_0^2 \left(\frac{du}{dz}\frac{d^2u}{dz^2} + \frac{dv}{dz}\frac{d^2v}{dz^2}\right) \\
-\left(2\gamma_3+\gamma_4+2\gamma_5\right) w_0 \frac{du}{dz}\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right\} \\
-\left(\gamma_4+6\gamma_7+2\gamma_8\right) w_0 \left(\frac{du}{dz}\frac{d^2u}{dz^2} + \frac{dv}{dz}\frac{d^2v}{dz^2}\right)
\end{array}\right]\mathbf{V}.
\tag{6.15}
$$

Equations (6.11)−(6.13) can be combined as

$$
\begin{aligned}
2\Omega i\rho\frac{dF}{dz}-\rho w_0\frac{d^2F}{dz^2} &= \frac{d^2}{dz^2}\left[\begin{array}{c}
\mu\frac{dF}{dz}-\alpha_1 w_0\frac{d^2F}{dz^2}+\beta_1 w_0^2\frac{d^3F}{dz^3}+2\left(\beta_2+\beta_3\right)\frac{dF}{dz}\left(\frac{dF}{dz}\frac{d\overline{F}}{dz}\right)\\
-\gamma_1 w_0^3\frac{d^4F}{dz^4}-3\gamma_2 w_0^2\frac{dF}{dz}\left(\frac{dF}{dz}\frac{d^2\overline{F}}{dz^2}+\frac{d^2F}{dz^2}\frac{d\overline{F}}{dz}\right)\\
-\left(2\gamma_3+\gamma_4+2\gamma_5\right)w_0\frac{d^2F}{dz^2}\left(\frac{dF}{dz}\frac{d\overline{F}}{dz}\right)\\
-\left(\gamma_4+6\gamma_7+2\gamma_8\right)\frac{w_0}{2}\frac{dF}{dz}\left(\frac{dF}{dz}\frac{d^2\overline{F}}{dz^2}+\frac{d^2F}{dz^2}\frac{d\overline{F}}{dz}\right)
\end{array}\right]\\
&\quad -\frac{\phi}{k}\frac{d}{dz}\left[\begin{array}{c}
\mu F-\alpha_1 w_0\frac{dF}{dz}+\beta_1 w_0^2\frac{d^2F}{dz^2}+2\left(\beta_2+\beta_3\right)\left(\frac{dF}{dz}\frac{d\overline{F}}{dz}\right)F\\
-\gamma_1 w_0^3\frac{d^3F}{dz^3}-3\gamma_2 w_0^2\left(\frac{dF}{dz}\frac{d^2\overline{F}}{dz^2}+\frac{d^2F}{dz^2}\frac{d\overline{F}}{dz}\right)F\\
-\left(2\gamma_3+\gamma_4+2\gamma_5\right)w_0\frac{dF}{dz}\left(\frac{dF}{dz}\frac{d\overline{F}}{dz}\right)\\
-\left(\gamma_4+6\gamma_7+2\gamma_8\right)\frac{w_0}{2}\left(\frac{dF}{dz}\frac{d^2\overline{F}}{dz^2}+\frac{d^2F}{dz^2}\frac{d\overline{F}}{dz}\right)F
\end{array}\right]\\
&\quad -\frac{\sigma B_0^2}{1-im_0}\frac{dF}{dz},
\end{aligned}
\tag{6.16}
$$

$$F\left(0\right)=U,\qquad F\left(\infty\right)=0, \tag{6.17}$$

$$F=u+iv,\qquad \overline{F}=u-iv. \tag{6.18}$$

In order to scale the resulting non-linear problem, it is convenient to introduce the following dimensionless quantities

$$
\begin{aligned}
z^* &= \frac{\rho U}{\mu}z,\quad F^*=\frac{F}{U},\quad \overline{F}^*=\frac{\overline{F}}{U},\quad \Omega^*=\frac{\mu}{\rho U^2}\Omega,\quad w_0^*=\frac{w_0}{U},\quad \alpha_1^*=\frac{\alpha_1\rho U^2}{\mu^2}\\
\beta_i^* &= \frac{\beta_i\rho^2U^4}{\mu^3}\ \ (i=1,2,3),\quad \gamma_i^*=\frac{\gamma_i\rho^3U^6}{\mu^4}\ \ (i=1-8),\quad M^2=\frac{\sigma B_0^2}{\rho\Omega},\quad K=\frac{\Omega\rho k_1}{\phi\mu}.
\end{aligned}
$$

In terms of above dimensionless quantities, the Eqs. (6.16) and (6.17) after suppressing the asterisks give

$$
\begin{aligned}
\frac{d^3F}{dz^3} &= \Omega\left\{\frac{M^2}{1+m_0^2}+\left(2-\frac{M^2m_0}{1+m_0^2}\right)i\right\}\frac{dF}{dz}-w_0\frac{d^2F}{dz^2}+\alpha_1 w_0\frac{d^4F}{dz^4}\\
&-\beta_1 w_0^2\frac{d^5F}{dz^5}-2\left(\beta_2+\beta_3\right)\frac{d^2}{dz^2}\left\{\left(\frac{dF}{dz}\right)^2\frac{d\overline{F}}{dz}\right\}+\gamma_1 w_0^3\frac{d^6F}{dz^6}\\
&+3\gamma_2 w_0^2\frac{d^2}{dz^2}\left[\frac{dF}{dz}\left(\frac{dF}{dz}\frac{d^2\overline{F}}{dz^2}+\frac{d^2F}{dz^2}\frac{d\overline{F}}{dz}\right)\right]\\
&+\left(2\gamma_3+\gamma_4+2\gamma_5\right)w_0\frac{d^2}{dz^2}\left[\frac{d^2F}{dz^2}\left(\frac{dF}{dz}\frac{d\overline{F}}{dz}\right)\right]\\
&+\left(\gamma_4+6\gamma_7+2\gamma_8\right)\frac{w_0}{2}\frac{d^2}{dz^2}\left[\frac{dF}{dz}\left(\frac{dF}{dz}\frac{d^2\overline{F}}{dz^2}+\frac{d^2F}{dz^2}\frac{d\overline{F}}{dz}\right)\right]\\
&+\frac{\Omega}{K}\left[\begin{array}{c}
\frac{dF}{dz}-\alpha_1 w_0\frac{d^2F}{dz^2}+\beta_1 w_0^2\frac{d^3F}{dz^3}+2\left(\beta_2+\beta_3\right)\frac{d}{dz}\left\{\frac{dF}{dz}\frac{d\overline{F}}{dz}F\right\}\\
-\gamma_1 w_0^3\frac{d^4F}{dz^4}-3\gamma_2 w_0^2\frac{d}{dz}\left\{\left(\frac{dF}{dz}\frac{d^2\overline{F}}{dz^2}+\frac{d^2F}{dz^2}\frac{d\overline{F}}{dz}\right)F\right\}\\
-\left(2\gamma_3+\gamma_4+2\gamma_5\right)w_0\frac{d}{dz}\left\{\frac{d^2F}{dz^2}\left(\frac{dF}{dz}\frac{d\overline{F}}{dz}\right)\right\}\\
-\left(\gamma_4+6\gamma_7+2\gamma_8\right)\frac{w_0}{2}\frac{d}{dz}\left\{\left(\frac{dF}{dz}\frac{d^2\overline{F}}{dz^2}+\frac{d^2F}{dz^2}\frac{d\overline{F}}{dz}\right)F\right\}
\end{array}\right],
\end{aligned}
\tag{6.19}
$$

$$
F(0)=1, \qquad F(\infty)=0. \tag{6.20}
$$

Substituting

$$
\eta = e^{-z} \tag{6.21}
$$

the differential system takes the form

$$\begin{aligned}
\eta^3\frac{d^3F}{d\eta^3}+3\eta^2\frac{d^2F}{d\eta^2}+\eta\frac{dF}{d\eta} &= \Omega\eta\left\{\frac{M^2}{1+m_0^2}+\left(2-\frac{M^2m_0}{1+m_0^2}\right)i\right\}\frac{dF}{d\eta}+w_0\left(\eta\frac{dF}{d\eta}+\eta^2\frac{d^2F}{d\eta^2}\right)\\
&-\alpha_1w_0\left(\eta\frac{dF}{d\eta}+7\eta^2\frac{d^2F}{d\eta^2}+6\eta^3\frac{d^3F}{d\eta^3}+\eta^4\frac{d^4F}{d\eta^4}\right)\\
&-\beta_1w_0^2\left(\eta\frac{dF}{d\eta}+15\eta^2\frac{d^2F}{d\eta^2}+25\eta^3\frac{d^3F}{d\eta^3}+10\eta^4\frac{d^4F}{d\eta^4}+\eta^5\frac{d^5F}{d\eta^5}\right)\\
&-2\beta\left[9\eta^3\left(\frac{dF}{d\eta}\right)^2\frac{d\overline{F}}{d\eta}+7\eta^4\frac{d}{d\eta}\left\{\left(\frac{dF}{d\eta}\right)^2\frac{d\overline{F}}{d\eta}\right\}+\eta^5\frac{d^2}{d\eta^2}\left\{\left(\frac{dF}{d\eta}\right)^2\frac{d\overline{F}}{d\eta}\right\}\right]\\
&-\gamma_1w_0^3\left(\eta\frac{dF}{d\eta}+31\eta^2\frac{d^2F}{d\eta^2}+90\eta^3\frac{d^3F}{d\eta^3}+65\eta^4\frac{d^4F}{d\eta^4}+15\eta^5\frac{d^5F}{d\eta^5}+\eta^6\frac{d^6F}{d\eta^6}\right)\\
&+3\gamma_2w_0^2\left[\begin{array}{c}
\eta\left\{\frac{dF}{d\eta}\left(2\eta^2\frac{dF}{d\eta}\frac{d\overline{F}}{d\eta}+\eta^3\frac{dF}{d\eta}\frac{d^2\overline{F}}{d\eta^2}+\eta^3\frac{d^2F}{d\eta^2}\frac{d\overline{F}}{d\eta}\right)\right\}\\
+3\eta^2\frac{d}{d\eta}\left\{\frac{dF}{d\eta}\left(2\eta^2\frac{dF}{d\eta}\frac{d\overline{F}}{d\eta}+\eta^3\frac{dF}{d\eta}\frac{d^2\overline{F}}{d\eta^2}+\eta^3\frac{d^2F}{d\eta^2}\frac{d\overline{F}}{d\eta}\right)\right\}\\
+\eta^3\frac{d^2}{d\eta^2}\left\{\frac{dF}{d\eta}\left(2\eta^2\frac{dF}{d\eta}\frac{d\overline{F}}{d\eta}+\eta^3\frac{dF}{d\eta}\frac{d^2\overline{F}}{d\eta^2}+\eta^3\frac{d^2F}{d\eta^2}\frac{d\overline{F}}{d\eta}\right)\right\}
\end{array}\right]
\end{aligned}$$

$$\begin{aligned}
&-\Gamma_1w_0\left[\begin{array}{c}
\eta\frac{d}{d\eta}\left\{\eta^3\left(\left(\frac{dF}{d\eta}\right)^2\frac{d\overline{F}}{d\eta}+\eta\frac{d^2F}{d\eta^2}\frac{dF}{d\eta}\frac{d\overline{F}}{d\eta}\right)\right\}\\
+\eta^2\frac{d^2}{d\eta^2}\left\{\eta^3\left(\left(\frac{dF}{d\eta}\right)^2\frac{d\overline{F}}{d\eta}+\eta\frac{d^2F}{d\eta^2}\frac{dF}{d\eta}\frac{d\overline{F}}{d\eta}\right)\right\}
\end{array}\right]\\
&+\frac{1}{2}\Gamma_2w_0\left[\begin{array}{c}
\eta\left\{\frac{dF}{d\eta}\left(2\eta^2\frac{dF}{d\eta}\frac{d\overline{F}}{d\eta}+\eta^3\frac{dF}{d\eta}\frac{d^2\overline{F}}{d\eta^2}+\eta^3\frac{d^2F}{d\eta^2}\frac{d\overline{F}}{d\eta}\right)\right\}\\
+3\eta^2\frac{d}{d\eta}\left\{\frac{dF}{d\eta}\left(2\eta^2\frac{dF}{d\eta}\frac{d\overline{F}}{d\eta}+\eta^3\frac{dF}{d\eta}\frac{d^2\overline{F}}{d\eta^2}+\eta^3\frac{d^2F}{d\eta^2}\frac{d\overline{F}}{d\eta}\right)\right\}\\
+\eta^3\frac{d^2}{d\eta^2}\left\{\frac{dF}{d\eta}\left(2\eta^2\frac{dF}{d\eta}\frac{d\overline{F}}{d\eta}+\eta^3\frac{dF}{d\eta}\frac{d^2\overline{F}}{d\eta^2}+\eta^3\frac{d^2F}{d\eta^2}\frac{d\overline{F}}{d\eta}\right)\right\}
\end{array}\right]\\
&+\frac{\Omega}{K}\left[\begin{array}{c}
\eta\frac{dF}{d\eta}-\alpha_1w_0\left(\eta\frac{dF}{d\eta}+\eta^2\frac{d^2F}{d\eta^2}\right)\\
-\beta_1w_0^2\left(\eta\frac{dF}{d\eta}+3\eta^2\frac{d^2F}{d\eta^2}+\eta^3\frac{d^3F}{d\eta^3}\right)-2\beta\frac{d}{d\eta}\left(\eta^2\frac{dF}{d\eta}\frac{d\overline{F}}{d\eta}F\right)\\
+\gamma_1w_0^3\left(\eta\frac{dF}{d\eta}+7\eta^2\frac{d^2F}{d\eta^2}+6\eta^3\frac{d^3F}{d\eta^3}+\eta^4\frac{d^4F}{d\eta^4}\right)\\
-3\gamma_2w_0^2\eta\frac{d}{d\eta}\left\{\eta^2\left(2\frac{dF}{d\eta}\frac{d\overline{F}}{d\eta}+\eta\frac{dF}{d\eta}\frac{d^2\overline{F}}{d\eta^2}+\eta^3\frac{d^2F}{d\eta^2}\frac{d\overline{F}}{d\eta}\right)F\right\}\\
+\Gamma_1w_0\eta\left[\frac{d}{d\eta}\left\{\eta^3\left(\left(\frac{dF}{d\eta}\right)^2\frac{d\overline{F}}{d\eta}+\eta\frac{d^2F}{d\eta^2}\frac{dF}{d\eta}\frac{d\overline{F}}{d\eta}\right)\right\}\right]\\
-\Gamma_2\frac{w_0}{2}\eta\frac{d}{d\eta}\left\{\eta^2\left(2\frac{dF}{d\eta}\frac{d\overline{F}}{d\eta}+\eta\frac{dF}{d\eta}\frac{d^2\overline{F}}{d\eta^2}+\eta^3\frac{d^2F}{d\eta^2}\frac{d\overline{F}}{d\eta}\right)F\right\}
\end{array}\right], \qquad (6.22)
\end{aligned}$$

$$F(1)=1, \qquad F(0)=0. \qquad (6.23)$$

where

$$\beta = \beta_2 + \beta_3,$$

$$\Gamma_1 = (2\gamma_3 + \gamma_4 + 2\gamma_5),$$

$$\Gamma_2 = (\gamma_4 + 6\gamma_7 + 2\gamma_8).$$

We note that by choosing $\phi = 0$ or $k_1 \to \infty$ one obtains the governing problem for rotating flow of a fourth grade fluid in a non-porous space. In the next section, the HAM solution of the arising highly non-linear problem will be determined.

6.2 Solution by the homotopy analysis method

The velocity distribution $F(\eta)$ can be expressed by the set of base functions

$$\{\eta^n, n \geq 0\} \tag{6.24}$$

in the form

$$F(\eta) = \sum_{n=0}^{\infty} a_{m,n}\eta^n, \tag{6.25}$$

where $a_{m,n}$ is the coefficients. Based on the *Rule of solution expression* by (6.25), it is straightforward to choose the initial guess F_0 of F and an auxiliary linear operator $\mathcal{L}_F$ as

$$F_0(\eta) = \eta, \tag{6.26}$$

$$\mathcal{L}_F(f) = \frac{d^2 f}{d\eta^2}, \tag{6.27}$$

where

$$\mathcal{L}_F[C_1 + C_2\eta] = 0, \tag{6.28}$$

and $C_1$and C_2 are arbitrary constants. The zeroth-order deformation problem satisfies

$$(1-q)\mathcal{L}_F\left[\widehat{F}(\eta, q) - F_0(\eta)\right] = q\hbar\mathcal{N}_F\left[\widehat{F}(\eta, q)\right], \tag{6.29}$$

$$\widehat{F}(0, q) = 0, \quad \widehat{F}(1, q) = 1, \tag{6.30}$$

in which $\hbar$ and q ($\in [0,1]$) are respectively the non-zero auxiliary parameter and an embedding parameter and

$$
\begin{aligned}
\mathcal{N}_F\left[\hat{F}(\eta,q)\right] = {} & \eta^3\frac{\partial^3\hat{F}(\eta,q)}{\partial\eta^3} + 3\eta^2\frac{\partial^2\hat{F}(\eta,q)}{\partial\eta^2} + \eta\frac{\partial\hat{F}(\eta,q)}{\partial\eta} \\
& -\Omega\eta\left\{\frac{M^2}{1+m_0^2} + \left(2-\frac{M^2 m_0}{1+m_0^2}\right)i\right\}\frac{\partial\hat{F}(\eta,q)}{\partial\eta} \\
& -w_0\left\{\eta\frac{\partial\hat{F}(\eta,q)}{\partial\eta} + \eta^2\frac{\partial^2\hat{F}(\eta,q)}{\partial\eta^2}\right\} \\
& +\alpha_1 w_0\left\{\eta\frac{\partial\hat{F}(\eta,q)}{\partial\eta} + 7\eta^2\frac{\partial^2\hat{F}(\eta,q)}{\partial\eta^2} + 6\eta^3\frac{\partial^3\hat{F}(\eta,q)}{\partial\eta^3} + \eta^4\frac{\partial^4\hat{F}(\eta,q)}{\partial\eta^4}\right\} \\
& +\beta_1 w_0^2\left\{\begin{array}{c} \eta\frac{\partial\hat{F}(\eta,q)}{\partial\eta} + 15\eta^2\frac{\partial^2\hat{F}(\eta,q)}{\partial\eta^2} + 25\eta^3\frac{\partial^3\hat{F}(\eta,q)}{\partial\eta^3} \\ +10\eta^4\frac{\partial^4\hat{F}(\eta,q)}{\partial\eta^4} + \eta^5\frac{\partial^5\hat{F}(\eta,q)}{\partial\eta^5} \end{array}\right\} \\
& +2\beta\left[\begin{array}{c} 9\eta^3\left(\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\right)^2\frac{\partial\overline{\hat{F}}(\eta,q)}{\partial\eta} + 7\eta^4\frac{\partial}{\partial\eta}\left\{\left(\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\right)^2\frac{\partial\overline{\hat{F}}(\eta,q)}{\partial\eta}\right\} \\ +\eta^5\frac{\partial^2}{\partial\eta^2}\left\{\left(\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\right)^2\frac{\partial\overline{\hat{F}}(\eta,q)}{\partial\eta}\right\} \end{array}\right] \\
& +\gamma_1 w_0^3\left\{\begin{array}{c} \eta\frac{\partial\hat{F}(\eta,q)}{\partial\eta} + 31\eta^2\frac{\partial^2\hat{F}(\eta,q)}{\partial\eta^2} + 90\eta^3\frac{\partial^3\hat{F}(\eta,q)}{\partial\eta^3} \\ +65\eta^4\frac{\partial^4\hat{F}(\eta,q)}{\partial\eta^4} + 15\eta^5\frac{\partial^5\hat{F}(\eta,q)}{\partial\eta^5} + \eta^6\frac{\partial^6\hat{F}(\eta,q)}{\partial\eta^6} \end{array}\right\} \\
& -3\gamma_2 w_0^2\left[\begin{array}{c} \eta\left\{\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\left(\begin{array}{c} 2\eta^2\frac{\partial\hat{F}(\eta,q)}{d\eta}\frac{\partial\overline{\hat{F}}(\eta,q)}{\partial\eta} \\ +\eta^3\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\frac{\partial^2\overline{\hat{F}}(\eta,q)}{\partial\eta^2} + \eta^3\frac{\partial^2\hat{F}(\eta,q)}{\partial\eta^2}\frac{\partial\overline{\hat{F}}(\eta,q)}{\partial\eta} \end{array}\right)\right\} \\ +3\eta^2\frac{\partial}{\partial\eta}\left\{\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\left(\begin{array}{c} 2\eta^2\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\frac{\partial\overline{\hat{F}}(\eta,q)}{\partial\eta} \\ +\eta^3\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\frac{\partial^2\overline{\hat{F}}(\eta,q)}{\partial\eta^2} + \eta^3\frac{\partial^2\hat{F}(\eta,q)}{\partial\eta^2}\frac{\partial\overline{\hat{F}}(\eta,q)}{\partial\eta} \end{array}\right)\right\} \\ +\eta^3\frac{\partial^2}{\partial\eta^2}\left\{\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\left(\begin{array}{c} 2\eta^2\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\frac{\partial\overline{\hat{F}}(\eta,q)}{\partial\eta} \\ +\eta^3\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\frac{\partial^2\overline{\hat{F}}(\eta,q)}{\partial\eta^2} + \eta^3\frac{\partial^2\hat{F}(\eta,q)}{\partial\eta^2}\frac{\partial\overline{\hat{F}}(\eta,q)}{\partial\eta} \end{array}\right)\right\} \end{array}\right] \\
& +\Gamma_1 w_0\left[\begin{array}{c} \eta\frac{\partial}{\partial\eta}\left\{\eta^3\left(\left(\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\right)^2\frac{\partial\overline{\hat{F}}(\eta,q)}{\partial\eta} + \eta\frac{\partial^2\hat{F}(\eta,q)}{\partial\eta^2}\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\frac{\partial\overline{\hat{F}}(\eta,q)}{\partial\eta}\right)\right\} \\ +\eta^2\frac{\partial^2}{\partial\eta^2}\left\{\eta^3\left(\left(\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\right)^2\frac{\partial\overline{\hat{F}}(\eta,q)}{\partial\eta} + \eta\frac{\partial^2\hat{F}(\eta,q)}{\partial\eta^2}\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\frac{\partial\overline{\hat{F}}(\eta,q)}{\partial\eta}\right)\right\} \end{array}\right] \\
& -\frac{1}{2}\Gamma_2 w_0\left[\begin{array}{c} \eta\left\{\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\left(\begin{array}{c} 2\eta^2\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\frac{\partial\overline{\hat{F}}(\eta,q)}{\partial\eta} + \eta^3\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\frac{\partial^2\overline{\hat{F}}(\eta,q)}{\partial\eta^2} \\ +\eta^3\frac{\partial^2\hat{F}(\eta,q)}{\partial\eta^2}\frac{\partial\overline{\hat{F}}(\eta,q)}{\partial\eta} \end{array}\right)\right\} \\ +3\eta^2\frac{\partial}{\partial\eta}\left\{\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\left(\begin{array}{c} 2\eta^2\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\frac{\partial\overline{\hat{F}}(\eta,q)}{\partial\eta} + \eta^3\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\frac{\partial^2\hat{F}(\eta,q)}{\partial\eta^2} \\ +\eta^3\frac{\partial^2\hat{F}(\eta,q)}{\partial\eta^2}\frac{\partial\overline{\hat{F}}(\eta,q)}{\partial\eta} \end{array}\right)\right\} \\ +\eta^3\frac{\partial^2}{\partial\eta^2}\left\{\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\left(\begin{array}{c} 2\eta^2\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\frac{\partial\overline{\hat{F}}(\eta,q)}{\partial\eta} + \eta^3\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\frac{\partial^2\overline{\hat{F}}(\eta,q)}{\partial\eta^2} \\ +\eta^3\frac{\partial^2\hat{F}(\eta,q)}{\partial\eta^2}\frac{\partial\overline{\hat{F}}(\eta,q)}{\partial\eta} \end{array}\right)\right\} \end{array}\right]
\end{aligned}
$$

$$
-\frac{\Omega}{K}\left[\begin{array}{c}
\eta\frac{\partial\hat{F}(\eta,q)}{\partial\eta}-\alpha_1 w_0\left(\eta\frac{\partial\hat{F}(\eta,q)}{\partial\eta}+\eta^2\frac{\partial^2\hat{F}(\eta,q)}{\partial\eta^2}\right)\\
-\beta_1 w_0^2\left(\eta\frac{\partial\hat{F}(\eta,q)}{\partial\eta}+3\eta^2\frac{\partial^2\hat{F}(\eta,q)}{\partial\eta^2}+\eta^3\frac{\partial^3\hat{F}(\eta,q)}{\partial\eta^3}\right)\\
-2\beta\frac{\partial}{\partial\eta}\left(\eta^2\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\frac{d\overline{F}}{d\eta}F(\eta,q)\right)\\
+\gamma_1 w_0^3\left(\eta\frac{\partial\hat{F}(\eta,q)}{\partial\eta}+7\eta^2\frac{\partial^2\hat{F}(\eta,q)}{\partial\eta^2}+6\eta^3\frac{\partial^3\hat{F}(\eta,q)}{\partial\eta^3}+\eta^4\frac{\partial^4\hat{F}(\eta,q)}{\partial\eta^4}\right)\\
-3\gamma_2 w_0^2\eta\frac{\partial}{\partial\eta}\left\{\eta^2\left(\begin{array}{c}2\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\frac{\partial\overline{\hat{F}}(\eta,q)}{\partial\eta}+\eta\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\frac{\partial^2\overline{\hat{F}}(\eta,q)}{\partial\eta^2}\\+\eta^3\frac{\partial^2\hat{F}(\eta,q)}{\partial\eta^2}\frac{\partial\overline{\hat{F}}(\eta,q)}{\partial\eta}\end{array}\right)F(\eta,q)\right\}\\
+\Gamma_1 w_0\eta\left[\frac{d}{d\eta}\left\{\eta^3\left(\begin{array}{c}\left(\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\right)^2\frac{\partial\overline{\hat{F}}(\eta,q)}{\partial\eta}\\+\eta\frac{\partial^2\hat{F}(\eta,q)}{\partial\eta^2}\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\frac{\partial\overline{\hat{F}}(\eta,q)}{\partial\eta}\end{array}\right)\right\}\right]\\
-\Gamma_2\frac{w_0}{2}\eta\frac{\partial}{\partial\eta}\left\{\eta^2\left(\begin{array}{c}2\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\frac{\partial\overline{\hat{F}}(\eta,q)}{\partial\eta}+\eta\frac{\partial\hat{F}(\eta,q)}{\partial\eta}\frac{d^2\overline{F}}{d\eta^2}\\+\eta^3\frac{\partial^2\hat{F}(\eta,q)}{\partial\eta^2}\frac{\partial\overline{\hat{F}}(\eta,q)}{\partial\eta}\end{array}\right)F(\eta,q)\right\}
\end{array}\right] \tag{6.31}
$$

is the non-linear operator. Clearly, when $q=0$ and $q=1$, we have

$$
\widehat{F}(\eta,0)=F_0(\eta), \qquad \widehat{F}(\eta,1)=F(\eta), \tag{6.32}
$$

We note from above equation that the derivation of q from 0 to 1 is continuous variation of $\widehat{F}(\eta,q)$ from $F_0(\eta)$ to $F(\eta)$. By Taylor's theorem and Eq. (6.32), one has

$$
\widehat{F}(\eta,q)=F_0(\eta)+\sum_{m=1}^{\infty}F_m(\eta)\,q^m, \tag{6.33}
$$

in which

$$
F_m(\eta)=\frac{1}{m!}\frac{\partial^m\widehat{F}(\eta,q)}{\partial q^m}\bigg|_{q=0}. \tag{6.34}
$$

Obviously the convergence of the series (6.33) depends upon $\hbar$.The values of $\hbar$ are selected in such a manner that the series (6.33) is convergent at $q=1$. Then due to Eq.(6.32) we have

$$
F(\eta)=F_0(\eta)+\sum_{m=1}^{\infty}F_m(\eta), \tag{6.35}
$$

Now differentiating Eq. (6.29) m-times with respect to q, and then dividing by $m!$ and finally setting $q=0$ we have the m-th order deformation problem

$$
\mathcal{L}_F\left[F_m(\eta)-\chi_m F_{m-1}(\eta)\right]=\hbar\mathcal{R}_F(\eta), \tag{6.36}
$$

$$F_m(0) = F_m(1) = 0, \tag{6.37}$$

where

$$\begin{aligned}
\mathcal{R}_m(\eta) &= \eta^3 F'''_{m-1}(\eta) + 3\eta^2 F''_{m-1}(\eta) + \eta F'_{m-1}(\eta) \\
&\quad -\Omega\eta\left\{\frac{M^2}{1+m_0^2} + \left(2 - \frac{M^2 m_0}{1+m_0^2}\right) i\right\} F'_{m-1}(\eta) - w_0\left\{\eta F'_{m-1}(\eta) + \eta^2 F''_{m-1}(\eta)\right\} \\
&\quad +\alpha_1 w_0\left\{\eta F'_{m-1}(\eta) + 7\eta^2 F''_{m-1}(\eta) + 6\eta^3 F'''_{m-1}(\eta) + \eta^4 F^{(4)}_{m-1}(\eta)\right\} \\
&\quad +\beta_1 w_0^2\left\{\eta F'_{m-1}(\eta) + 15\eta^2 F''_{m-1}(\eta) + 25\eta^3 F'''_{m-1}(\eta) + 10\eta^4 F^{(4)}_{m-1}(\eta) + \eta^5 F^{(5)}_{m-1}(\eta)\right\} \\
&\quad +2\beta\eta^3\left[\begin{array}{c}
9\sum_{k=0}^{m-1} F'_{m-1-k}(\eta)\sum_{l=0}^{k} F'_{k-l}(\eta)\bar{F}'_l(\eta) \\
+7\eta\left\{\begin{array}{c}
2\sum_{k=0}^{m-1} F'_{m-1-k}(\eta)\sum_{l=0}^{k} F''_{k-l}(\eta)\bar{F}'_l(\eta) \\
+\sum_{k=0}^{m-1} F'_{m-1-k}(\eta)\sum_{l=0}^{k} F'_{k-l}(\eta)\bar{F}''_l(\eta)
\end{array}\right\} \\
+\eta^2\left\{\begin{array}{c}
2\sum_{k=0}^{m-1} F''_{m-1-k}(\eta)\sum_{l=0}^{k} F''_{k-l}(\eta)\bar{F}'_l(\eta) \\
+2\sum_{k=0}^{m-1} F'_{m-1-k}(\eta)\sum_{l=0}^{k} F'''_{k-l}(\eta)\bar{F}'_l(\eta) \\
+4\sum_{k=0}^{m-1} F'_{m-1-k}(\eta)\sum_{l=0}^{k} F''_{k-l}(\eta)\bar{F}''_l(\eta) \\
+\sum_{k=0}^{m-1} F'_{m-1-k}(\eta)\sum_{l=0}^{k} F'_{k-l}(\eta)\bar{F}'''_l(\eta)
\end{array}\right\}
\end{array}\right] \\
&\quad +\gamma_1 w_0^3\left\{\begin{array}{c}
\eta F'_{m-1}(\eta) + 31\eta^2 F''_{m-1}(\eta) + 90\eta^3 F'''_{m-1}(\eta) \\
+65\eta^4 F^{(4)}_{m-1}(\eta) + 15\eta^5 F^{(5)}_{m-1}(\eta) + \eta^6 F^{(6)}_{m-1}(\eta)
\end{array}\right\}
\end{aligned}$$

$$
-3\gamma_2 w_0^2 \left[\begin{array}{c}
18\eta^3 \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} F'_{k-l}(\eta)\, \bar{F}'_l(\eta) + 44\eta^4 \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} F''_{k-l}(\eta)\, \bar{F}'_l(\eta) \\
+13\eta^5 \sum_{k=0}^{m-1} F''_{m-1-k}(\eta) \sum_{l=0}^{k} F''_{k-l}(\eta)\, \bar{F}'_l(\eta) + 30\eta^4 \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} F'_{k-l}(\eta)\, \bar{F}''_l(\eta) \\
+35\eta^5 \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} F''_{k-l}(\eta)\, \bar{F}''_l(\eta) + 4\eta^6 \sum_{k=0}^{m-1} F''_{m-1-k}(\eta) \sum_{l=0}^{k} F''_{k-l}(\eta)\, \bar{F}''_l(\eta) \\
+13\eta^5 \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} F'''_{k-l}(\eta)\, \bar{F}'_l(\eta) + 3\eta^6 \sum_{k=0}^{m-1} F''_{m-1-k}(\eta) \sum_{l=0}^{k} F'''_{k-l}(\eta)\, \bar{F}'_l(\eta) \\
+4\eta^6 \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} F'''_{k-l}(\eta)\, \bar{F}''_l(\eta) + 11\eta^5 \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} F'_{k-l}(\eta)\, \bar{F}'''_l(\eta) \\
+5\eta^6 \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} F''_{k-l}(\eta)\, \bar{F}'''_l(\eta) + \eta^6 \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} F''''_{k-l}(\eta)\, \bar{F}'_l(\eta) \\
+\eta^6 \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} F'_{k-l}(\eta)\, \bar{F}''''_l(\eta)
\end{array} \right]
$$

$$
+w_0 \Gamma_1 \left[\begin{array}{c}
9\eta^3 \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} F'_{k-l}(\eta)\, \bar{F}'_l(\eta) + 30\eta^4 \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} F''_{k-l}(\eta)\, \bar{F}'_l(\eta) \\
+11\eta^5 \sum_{k=0}^{m-1} F''_{m-1-k}(\eta) \sum_{l=0}^{k} F''_{k-l}(\eta)\, \bar{F}'_l(\eta) + 7\eta^4 \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} F'_{k-l}(\eta)\, \bar{F}''_l(\eta) \\
+13\eta^5 \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} F''_{k-l}(\eta)\, \bar{F}''_l(\eta) + 2\eta^6 \sum_{k=0}^{m-1} F''_{m-1-k}(\eta) \sum_{l=0}^{k} F''_{k-l}(\eta)\, \bar{F}''_l(\eta) \\
+11\eta^5 \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} F'''_{k-l}(\eta)\, \bar{F}'_l(\eta) + 3\eta^6 \sum_{k=0}^{m-1} F''_{m-1-k}(\eta) \sum_{l=0}^{k} F'''_{k-l}(\eta)\, \bar{F}'_l(\eta) \\
+2\eta^6 \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} F'''_{k-l}(\eta)\, \bar{F}''_l(\eta) + 2\eta^5 \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} F'_{k-l}(\eta)\, \bar{F}'''_l(\eta) \\
+\eta^6 \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} F''_{k-l}(\eta)\, \bar{F}'''_l(\eta) + \eta^6 \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} F''''_{k-l}(\eta)\, \bar{F}'_l(\eta)
\end{array} \right]
$$

$$
-\frac{w_0\Gamma_2}{2}\left[\begin{array}{c}
18\eta^3\sum_{k=0}^{m-1}F'_{m-1-k}(\eta)\sum_{l=0}^{k}F'_{k-l}(\eta)\bar{F}'_l(\eta)+44\eta^4\sum_{k=0}^{m-1}F'_{m-1-k}(\eta)\sum_{l=0}^{k}F''_{k-l}(\eta)\bar{F}'_l(\eta)\\
+13\eta^5\sum_{k=0}^{m-1}F''_{m-1-k}(\eta)\sum_{l=0}^{k}F''_{k-l}(\eta)\bar{F}'_l(\eta)+30\eta^4\sum_{k=0}^{m-1}F'_{m-1-k}(\eta)\sum_{l=0}^{k}F'_{k-l}(\eta)\bar{F}''_l(\eta)\\
+35\eta^5\sum_{k=0}^{m-1}F'_{m-1-k}(\eta)\sum_{l=0}^{k}F''_{k-l}(\eta)\bar{F}''_l(\eta)+4\eta^6\sum_{k=0}^{m-1}F''_{m-1-k}(\eta)\sum_{l=0}^{k}F''_{k-l}(\eta)\bar{F}''_l(\eta)\\
+13\eta^5\sum_{k=0}^{m-1}F'_{m-1-k}(\eta)\sum_{l=0}^{k}F'''_{k-l}(\eta)\bar{F}'_l(\eta)+3\eta^6\sum_{k=0}^{m-1}F''_{m-1-k}(\eta)\sum_{l=0}^{k}F'''_{k-l}(\eta)\bar{F}'_l(\eta)\\
+4\eta^6\sum_{k=0}^{m-1}F'_{m-1-k}(\eta)\sum_{l=0}^{k}F'''_{k-l}(\eta)\bar{F}''_l(\eta)+11\eta^5\sum_{k=0}^{m-1}F'_{m-1-k}(\eta)\sum_{l=0}^{k}F'_{k-l}(\eta)\bar{F}'''_l(\eta)\\
+5\eta^6\sum_{k=0}^{m-1}F'_{m-1-k}(\eta)\sum_{l=0}^{k}F''_{k-l}(\eta)\bar{F}'''_l(\eta)+\eta^6\sum_{k=0}^{m-1}F'_{m-1-k}(\eta)\sum_{l=0}^{k}F''''_{k-l}(\eta)\bar{F}'_l(\eta)\\
+\eta^6\sum_{k=0}^{m-1}F'_{m-1-k}(\eta)\sum_{l=0}^{k}F'_{k-l}(\eta)\bar{F}''''_l(\eta)
\end{array}\right]
$$

$$
-\frac{\Omega}{K}\left[\eta F'_{m-1}(\eta)-\alpha_1 w_0\left\{\eta F'_{m-1}(\eta)+\eta^2F''_{m-1}(\eta)\right\}\right]
$$

$$
-\frac{\Omega}{K}\left[-\beta_1 w_0^2\left\{\eta^3F'''_{m-1}(\eta)+3\eta^2F''_{m-1}(\eta)+\eta F'_{m-1}(\eta)\right\}\right]
$$

$$
+\frac{2\beta\Omega}{K}\left[\begin{array}{c}
2\eta^2\sum_{k=0}^{m-1}F'_{m-1-k}(\eta)\sum_{l=0}^{k}\bar{F}'_{k-l}(\eta)F_l(\eta)+\eta^3\sum_{k=0}^{m-1}F'_{m-1-k}(\eta)\sum_{l=0}^{k}\bar{F}'_{k-l}(\eta)F'_l(\eta)\\
+\eta^3\sum_{k=0}^{m-1}F''_{m-1-k}(\eta)\sum_{l=0}^{k}\bar{F}'_{k-l}(\eta)F_l(\eta)+\eta^3\sum_{k=0}^{m-1}F'_{m-1-k}(\eta)\sum_{l=0}^{k}\bar{F}''_{k-l}(\eta)F_l(\eta)
\end{array}\right]
$$

$$
-\tfrac{\Omega}{K}\left[-\gamma_1 w_0^3\left\{\eta F'_{m-1}(\eta)+7\eta^2F''_{m-1}(\eta)+6\eta^3F'''_{m-1}(\eta)+\eta^4F^{(4)}_{m-1}(\eta)\right\}\right]
$$

$$
+\tfrac{3\gamma_2\Omega w_0^2}{K}\left[\begin{array}{c}
4\eta^2\sum_{k=0}^{m-1}F'_{m-1-k}(\eta)\sum_{l=0}^{k}\bar{F}'_{k-l}(\eta)F_l(\eta)+2\eta^3\sum_{k=0}^{m-1}F'_{m-1-k}(\eta)\sum_{l=0}^{k}\bar{F}'_{k-l}(\eta)F'_l(\eta)\\
+5\eta^3\sum_{k=0}^{m-1}F''_{m-1-k}(\eta)\sum_{l=0}^{k}\bar{F}'_{k-l}(\eta)F_l(\eta)+\eta^4\sum_{k=0}^{m-1}F''_{m-1-k}(\eta)\sum_{l=0}^{k}\bar{F}'_{k-l}(\eta)F'_l(\eta)\\
+5\eta^3\sum_{k=0}^{m-1}F_{m-1-k}(\eta)\sum_{l=0}^{k}\bar{F}''_{k-l}(\eta)F'_l(\eta)+\eta^4\sum_{k=0}^{m-1}F'_{m-1-k}(\eta)\sum_{l=0}^{k}\bar{F}''_{k-l}(\eta)F'_l(\eta)\\
+2\eta^4\sum_{k=0}^{m-1}F''_{m-1-k}(\eta)\sum_{l=0}^{k}\bar{F}''_{k-l}(\eta)F_l(\eta)+\eta^4\sum_{k=0}^{m-1}F'''_{m-1-k}(\eta)\sum_{l=0}^{k}\bar{F}'_{k-l}(\eta)F_l(\eta)\\
+\eta^4\sum_{k=0}^{m-1}F'_{m-1-k}(\eta)\sum_{l=0}^{k}\bar{F}'''_{k-l}(\eta)F_l(\eta)
\end{array}\right]
$$

$$-\frac{\Gamma_1 \Omega w_0}{K}\left[\begin{array}{l} 3\eta^3 \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} \bar{F}'_{k-l}(\eta) F'_l(\eta) + 6\eta^4 \sum_{k=0}^{m-1} F''_{m-1-k}(\eta) \sum_{l=0}^{k} \bar{F}'_{k-l}(\eta) F'_l(\eta) \\ +\eta^5 \sum_{k=0}^{m-1} F''_{m-1-k}(\eta) \sum_{l=0}^{k} \bar{F}'_{k-l}(\eta) F''_l(\eta) + \eta^4 \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} \bar{F}''_{k-l}(\eta) F'_l(\eta) \\ +\eta^5 \sum_{k=0}^{m-1} F''_{m-1-k}(\eta) \sum_{l=0}^{k} \bar{F}''_{k-l}(\eta) F'_l(\eta) + \eta^5 \sum_{k=0}^{m-1} F'''_{m-1-k}(\eta) \sum_{l=0}^{k} \bar{F}'_{k-l}(\eta) F'_l(\eta) \end{array}\right]$$

$$+\frac{\Gamma_2 \Omega w_0}{2K}\left[\begin{array}{l} 4\eta^2 \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} \bar{F}'_{k-l}(\eta) F_l(\eta) + 2\eta^3 \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} \bar{F}'_{k-l}(\eta) F'_l(\eta) \\ +5\eta^3 \sum_{k=0}^{m-1} F''_{m-1-k}(\eta) \sum_{l=0}^{k} \bar{F}'_{k-l}(\eta) F_l(\eta) + \eta^4 \sum_{k=0}^{m-1} F''_{m-1-k}(\eta) \sum_{l=0}^{k} \bar{F}'_{k-l}(\eta) F'_l(\eta) \\ +5\eta^3 \sum_{k=0}^{m-1} F_{m-1-k}(\eta) \sum_{l=0}^{k} \bar{F}''_{k-l}(\eta) F'_l(\eta) + \eta^4 \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} \bar{F}''_{k-l}(\eta) F'_l(\eta) \\ +2\eta^4 \sum_{k=0}^{m-1} F''_{m-1-k}(\eta) \sum_{l=0}^{k} \bar{F}''_{k-l}(\eta) F_l(\eta) + \eta^4 \sum_{k=0}^{m-1} F'''_{m-1-k}(\eta) \sum_{l=0}^{k} \bar{F}'_{k-l}(\eta) F_l(\eta) \\ \qquad +\eta^4 \sum_{k=0}^{m-1} F'_{m-1-k}(\eta) \sum_{l=0}^{k} \bar{F}'''_{k-l}(\eta) F_l(\eta) \end{array}\right], \tag{6.38}$$

$$\chi_m = \left| \begin{array}{ll} 0, & m \leq 1 \\ 1, & m > 1 \end{array} \right. . \tag{6.39}$$

The general solution of Eqs. (6.36) and (6.37) is given by

$$F_m(\eta) = F_m^{\star}(\eta) + C_1 + C_2\eta, \tag{6.40}$$

in which $F_m^{\star}(\eta)$ denotes the special solutions of Eq. (6.36) and the integral constants C_1 and C_2 are determined by the boundary conditions (6.37)

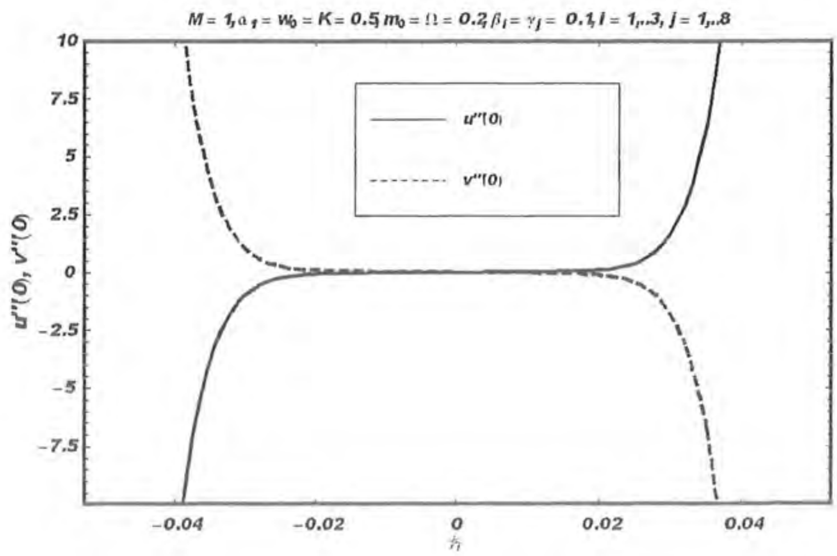
$$C_1 = -C_2 - F_m^{\star}(1), \quad C_2 = -F_m^{\star}(0). \tag{6.41}$$

In this way, it is easy to solve the linear non-homogeneous Eq. (6.36) by using Mathematica one after the other in the order $m = 1, 2, 3....$

6.3 Convergence of the series solution

The expression given in Eq. (6.36) depends upon the auxiliary parameter $\hbar$ that ensures the convergence region and rate of approximation for the homotopy analysis method. In figure 6.1 the $\hbar$-curves are plotted for different order of approximations for u and v. It is obvious from figure 6.1 that the range for the admissible values of $\hbar$ is $-0.05 \leq h < 0.05$. The real and imaginary parts of the series given by Eq. (6.35) converge in the whole region of η when $\hbar = -0.01$.

Figure 6.1



6.4 Results and discussion

Here figure 6.2 (a, b) have been made just to describe the variation of the velocity profiles u and v for different values of w_0. These figures explain that the velocity profiles u and v increase when w_0 is increased and $M = \alpha_1 = \beta_1 = \beta = \gamma_1 = \gamma_2 = \Gamma_1 = \Gamma_2 = K = m_0 = \Omega = 0.01$. It is observed from these figures that an increase in the velocity profile v is smaller in magnitude when compared with the velocity profile u. Figures 6.3 (a, b) depict the variation γ_1 on the velocity profiles u and v when $M = \alpha_1 = \beta_1 = \beta = \gamma_2 = \Gamma_1 = \Gamma_2 = K = m_0 = \Omega = 0.01$, and $w_0 = 1$. These figures show that the velocity profiles u and v increase by increasing γ_1. The variation of M on the velocity profiles u and v when $\alpha_1 = \beta_1 = \beta = \gamma_1 = \gamma_2 = \Gamma_1 = \Gamma_2 = K = m_0 = \Omega = 0.1$ is shown in figures 6.4 (a, b). It is noted that u increases and v decreases when M is increased. In figures 6.5 (a, b) it is observed that the velocity profiles u and v increase by increasing K and $M = \alpha_1 = \beta_1 = \beta = \gamma_1 = \gamma_2 = \Gamma_1 = \Gamma_2 = m_0 = \Omega = 0.1$ and $w_0 = 1$. An increase in the magnitude of the velocity profile v is however smaller than that of u. Figures 6.6 (a, b) depict the variation in the velocity profiles u and v for various values of Ω. It is clearly seen from these figures that the velocity profiles u and v decrease by increasing Ω when $M = \alpha_1 = \beta_1 = \beta = \gamma_1 = \gamma_2 = \Gamma_1 = \Gamma_2 = K = m_0 = 0.1$ and $w_0 = 1$. The decrease is smaller for the velocity profile v when compared with that of u when $M = \alpha_1 = \beta_1 = \beta = \gamma_1 = \gamma_2 = \Gamma_1 = \Gamma_2 = K = m_0 = 0.1$ and $w_0 = 1$.

Figure 6.2a

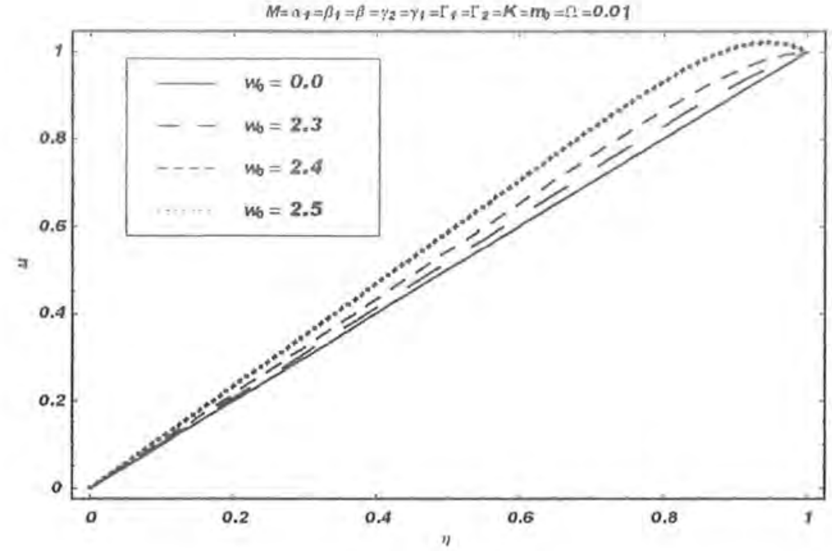


Figure 6.2b

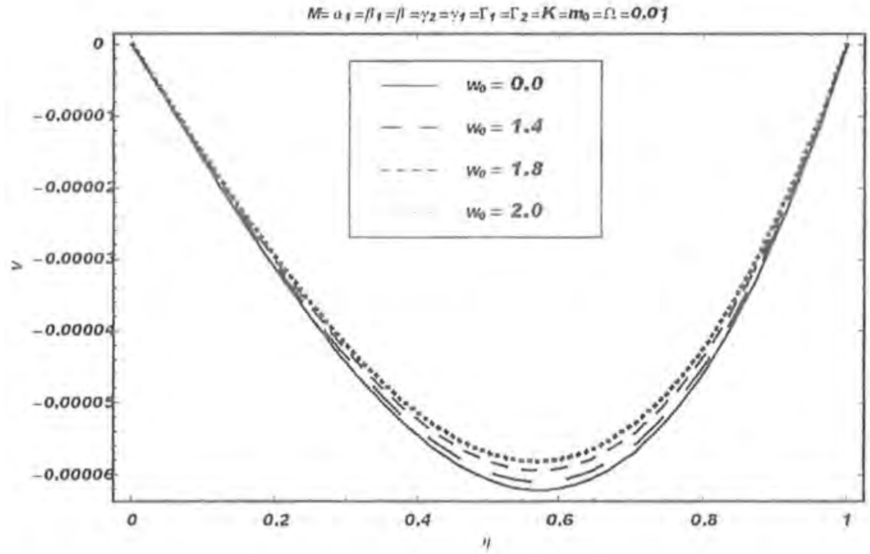


Figure 6.2. Variation of velocity profiles u and v with the change in parameter w_0 at $\hbar = -0.01$.

Figure 6.3a

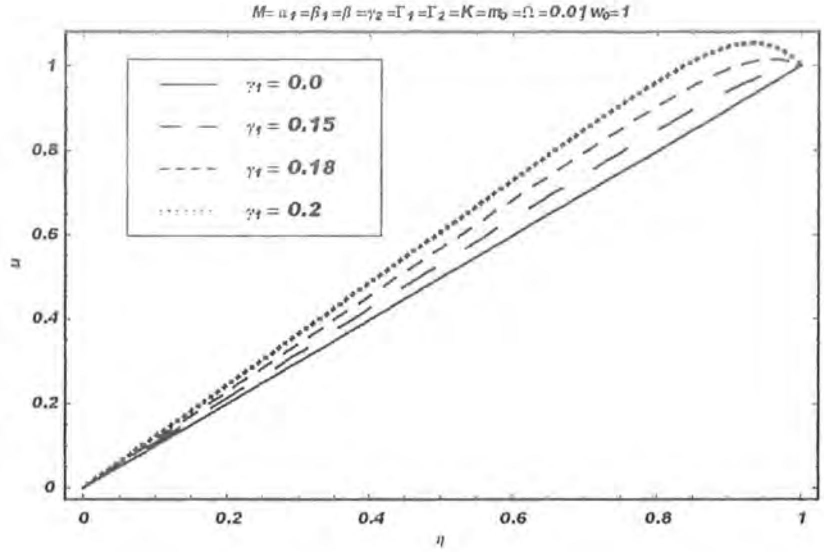


Figure 6.3b

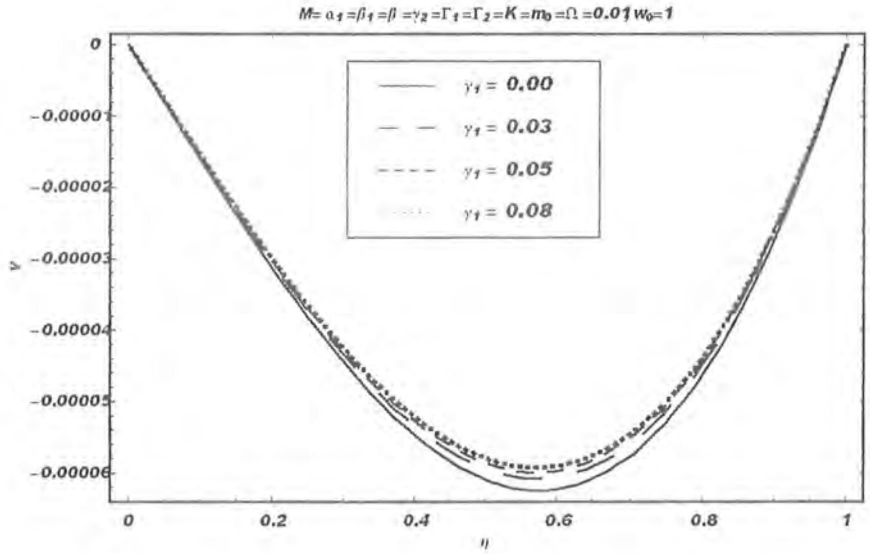


Figure 6.3. Variation of velocity profiles u and v with the change in parameter γ_1 at $\hbar = -0.01$.

Figure 6.4a

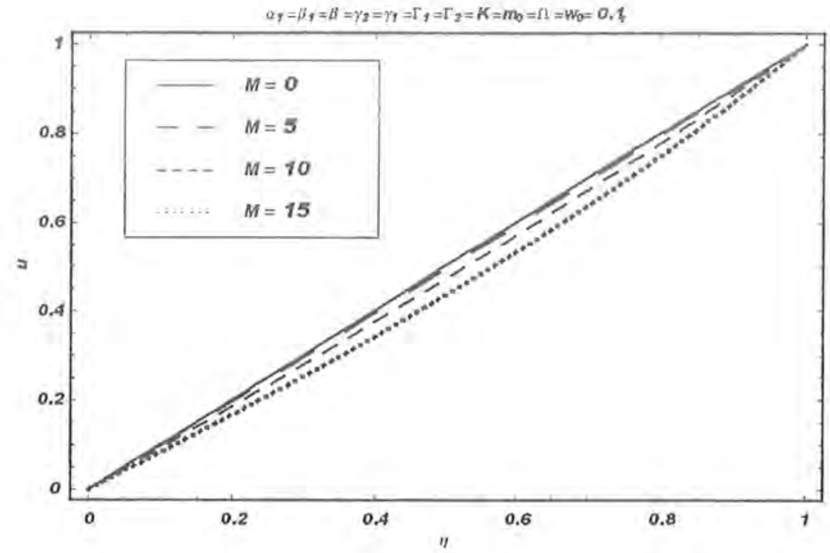


Figure 6.4b

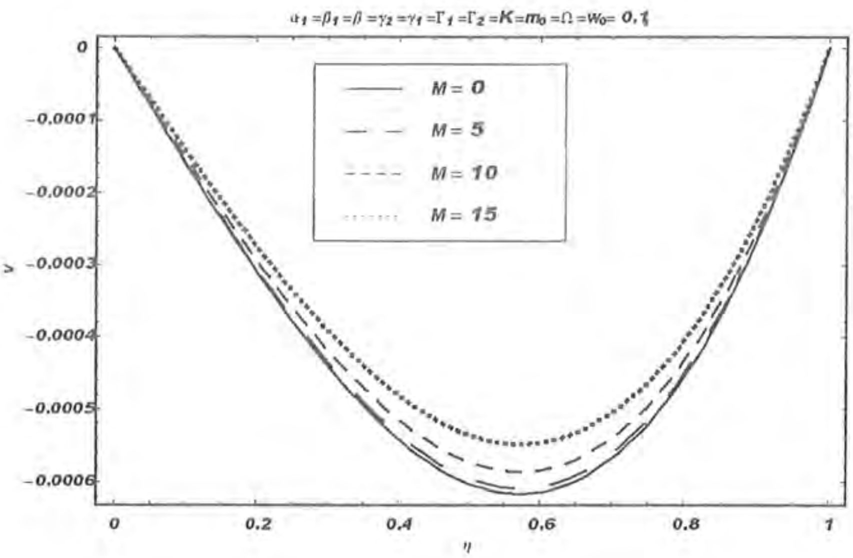


Figure 6.4. Variation of velocity profiles u and v with the change in parameter M at $\hbar = -0.01$.

Figure 6.5a

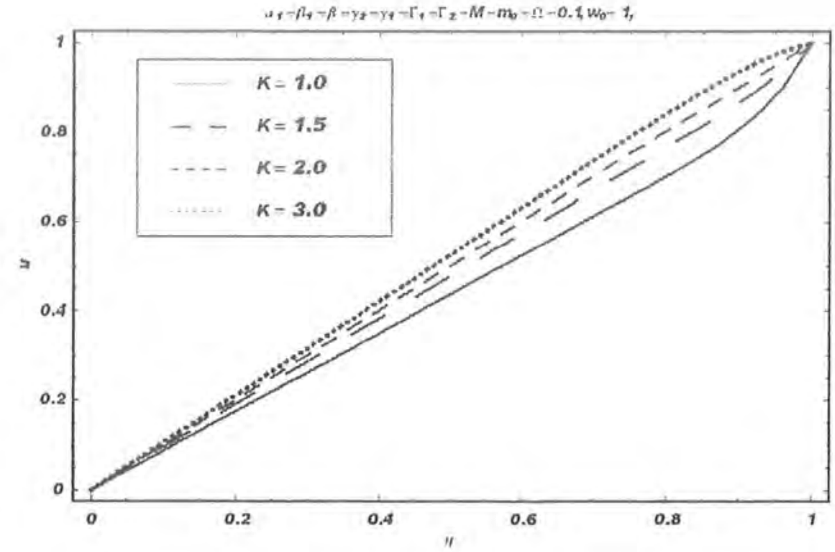


Figure 6.5b

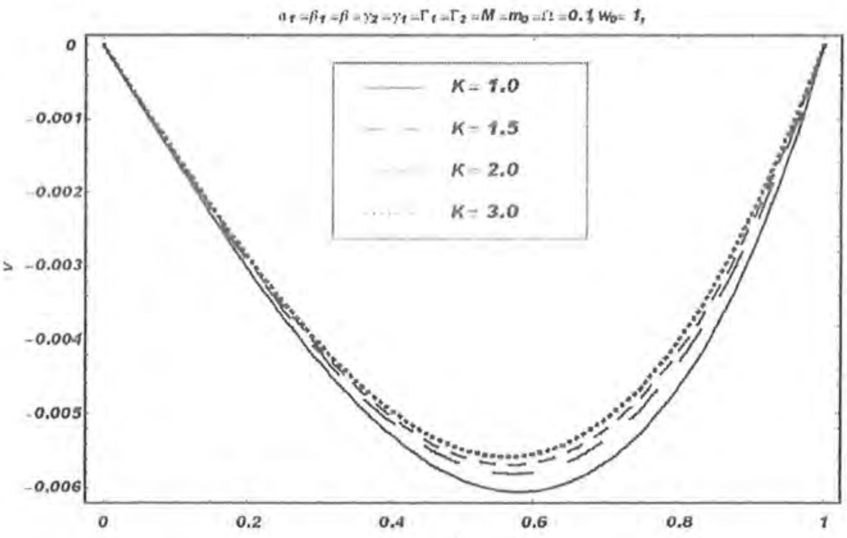


Figure 6.5. Variation of velocity profiles u and v with the change in parameter K at $\hbar = -0.01$.

Figure 6.6a

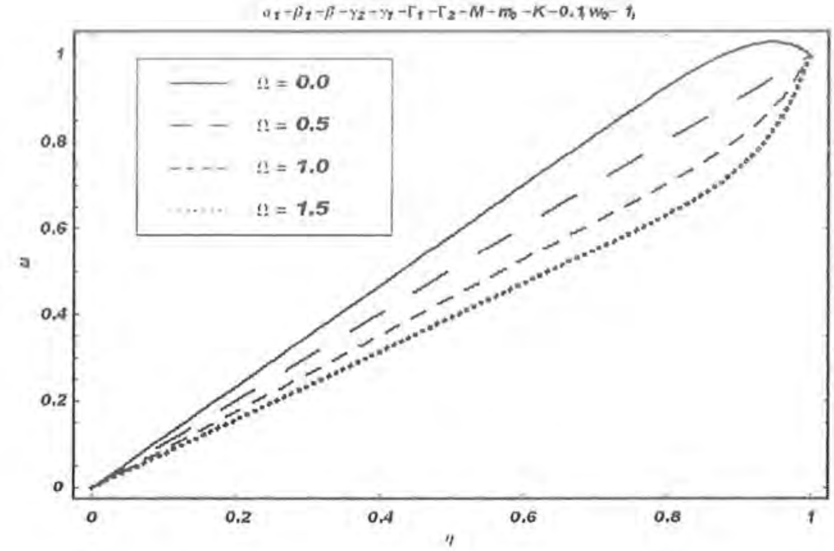


Figure 6.6b

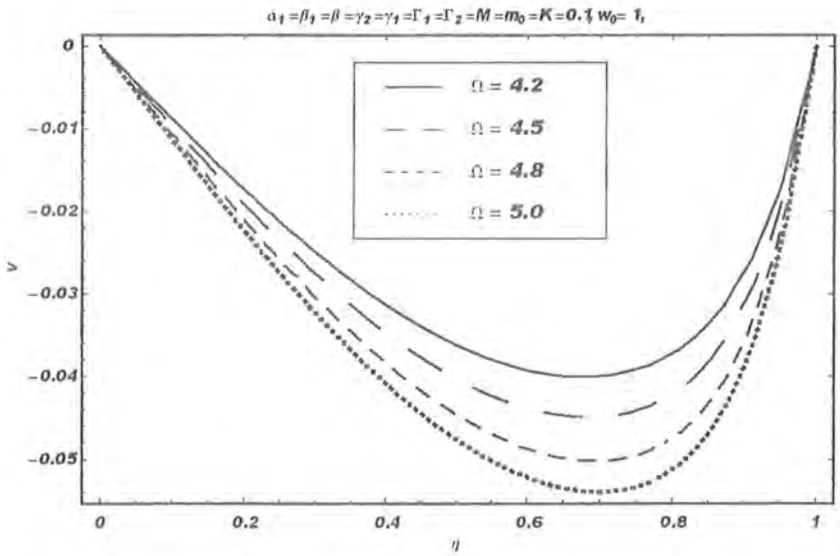


Figure 6.6. Variation of velocity profiles u and v with the change in parameter Ω at $\hbar = -0.01$.

6.5 Concluding remarks

In this chapter a new analytic approach based on the homotopy analysis method is used to investigate the rotating flow of a fourth grade fluid with Hall current. This analytic approach provides us with a new way to obtain explicit convergence series solution. Different from the conventional Darcy's law, the present analysis involves the modified Darcy's law. It is noted that u and v have the same qualitative behavior for the physical parameters w_0, γ_1, K and Ω. However u and v show opposite behaviors for M. It is further noted that the results of third grade fluid can be recovered as the special cases when $\gamma_1 = \gamma_2 = \Gamma_1 = \Gamma_2 = 0$.

Bibliography

[1] Astarita, G. and Marrucci, G., Principles of non-Newtonian fluid mechanics, McGraw-Hill, London (1974).

[2] Schowalter, W. R., Mechanics of non-Newtonian fluids, Pergamon, Oxford (1978).

[3] Slattery, J.C., Advanced transport phenomena, Cambridge University Press (1999).

[4] Irvine Jr., T.F. and Karni, J., Non-Newtonian fluid flow and heat transfer, Handbook of single-phase Convective heat transfer (S. Kakac, R.K. Shah and W. Aung, eds.). Wiley, New York (1987) 20.1.

[5] Andersson, H.I. and Irgens, F., Film flow of power law fluids, Encyclopedia of fluid mechanics, Polymer flow engineering (N.P. Cheremisinoff, ed.). Gulf Publishing Company, Houston, Texas **9** (1990) 617.

[6] Rajagopal, K.R., Flow of viscoelastic fluids between rotating disks, Theoret. Comput. Fluid Dynamics, **3** (1992) 185.

[7] Rajagopal, K. R., Boundedness and uniqueness of fluids of the differential type, Acta Cienca Indica 18(1)(1982)1.

[8] Rajagopal, K.R., On boundary conditions for fluids of the differential type, in: Sequira (Ed.), Navier-Stokes equations and related nonlinear problems, Plenum Press, New York, (1995) pp. 273.

[9] Rajagopal, K. R. and Gupta, A. S. , An exact solution for the flow of a non-Newtonian fluid past an infinite plate, Meccanica 19(1984)158.

[10] Rajagopal, K. R. and Kaloni, P. N., Some remarks on the boundary conditions for fluids of differential type, in: G. A. C. Graham, S. K. Malik(Eds), Continuum Mechanics and its applications, Hemisphere, Newyork, (1985)935.

[11] Rajagopal, K. R. and Na, T. Y., On Stokes' problem for a non-Newtonian fluid, Acta Mech. 48(1983)233.

[12] Rajagopal, K.R., Szeri, A.Z. and Troy, W., An existence theorem for the flow of a non-Newtonian fluid past an infinite porous plate, Int. J. Non-Linear Mech. 21 (1986)279.

[13] Dunn, J. E. and Rajagopal, K. R., Fluids of differential type-critical-review and thermodynamic analysis, Int. J. Eng. Sci. 33(1995)689.

[14] Fosdick, R. L. and Rajagopal, K. R., Thermodynamics and stability of fluids of third grade, Proc. R. Soc. Lond. A 339(1980)351.

[15] Rajagopal, K.R., A note on unsteady unidirectional flows of a non-Newtonian fluid, Int. J. Nonlinear Mech. **17** (1982) 369.

[16] Rajagopal, K.R., On the creeping flow of the second grade fluid, J. Non-Newtonian Fluid Mech. **15** (1984) 239.

[17] Hayat, T., Asghar, S. and Siddiqui, A.M., Periodic unsteady flows of a non-Newtonian fluid, Acta Mech. **131** (1998) 169.

[18] Hayat, T., Asghar, S. and Siddiqui, A.M., Some unsteady unidirectional flows of a non-Newtonian fluid, Int. J. Eng. Sci. **38** (2000) 337.

[19] Siddiqui, A.M., Hayat, T. and Asghar, S., Periodic flows of a non-Newtonian fluid between two parallel plates, Int. J. Nonlinear Mech. **34** (1999) 895.

[20] Rajagopal, K.R. and Gupta, A.S., Flow and stability of a second grade fluid between two parallel plates rotating about noncoincident axes, Int. J. Eng. Sci. **19** (1981) 1401.

[21] Bandelli, R. and Rajagopal, K.R., Starting up flows of second grade fluids in domains with one finite dimension, Int. J. Nonlinear Mech. **30** (6) (1995) 817.

[22] Bandelli, R. and Rajagopal, K.R., On the falling of objects in non-Newtonian fluids, Ann. Ferrara **39** (1996) 1.

[23] Bandelli, R., Unsteady unidirectional flows of second grade fluids in domains with heated boundaries, Int. J. Nonlinear Mech. **30** (1995) 263.

[24] Hayat, T., Khan, M., Siddiqui, A.M. and Asghar, S., Transient flows of a second grade fluid, Int. J. Nonlinear Mech. **39** (2004) 1621.

[25] Erdogan, M. E., On unsteady motion of a second grade fluid over a plane wall, Int. J. Nonlinear Mech. 38(2003)1045.

[26] Erdogan, M. E. and Imrak, C. E., Effects of side walls on the unsteady flow of a second grade fluid in a duct of uniform cross-section, Int. J. Nonlinear Mech. 39(2004)1379.

[27] Erdogan, M. E. and Imrak, C. E., An exact solution of the governing equation of a second grade fluid for three dimensional voxtex flow, Int. J. Eng. Sci. 43(2005)721.

[28] Fetecau, C. and Fetecau, C., Unsteady flows of second grade fluids in heated cylindrical domains, Acta Mech. 150(2001)127.

[29] Fetecau, C. and Zierep, J.,On a class of exact solutions of the equations of motion of a second grade fluid, Acta Mech. 150(2001)135.

[30] Fetecau, C. and Fetecau, C., Starting solutions for some unsteady unidirectional flows of a second grade fluid, Int. J. Eng. Sci. 43(2005)781.

[31] Fetecau, C. and Fetecau, C., Starting solutions for the motion of a second grade fluid due to longitudinal and torsional oscillations of a circular cylinder, Int. J. Eng. Sci. 44(2006)788.

[32] Fetecau, C., Fetecau, C. and Zierep, J., Decay of a potential vortex and propogation of a heat wave in a second grade fluid, Int. J. Nonlinear Mech. 37(2002)1051.

[33] Kaloni, P. N. and Huschilit, K., Semi inverse solutions of non-Newtonian fluid, Int. J. Nonlinear Mech. 19(1984) 373.

[34] Siddiqui, A. M.and Kaloni, P. N., Certain inverse solutions of a non-Newtonian fluids, Int. J. Nonlinear Mech. 21(1986)459.

[35] Siddiqui, A. M. , Some more inverse solutions of non-Newtonian fluid, Mech. Res. Comm. 17(1990)157.

[36] Benharbit, A. M. and Siddiqui, A. M., Certain solutions of the equations of the planar motion of a second grade fluid for steady and unsteady cases, Acta Mech.94(1992) 85.

[37] Siddiqui, A. M., Mohyuddin, M. R., Hayat. T. and Asghar, S., Some more inverse solutions for steady flows of a second grade fluid, Arch. Mech. 55(2003)373.

[38] Markovitz, H. and Coleman, B. D., Incompressible second order fluids, Adv. Appl. Mech. 8 (1964) 69.

[39] Ting, T. W., Certain non-steady flows of second order fluids, Arch. Rat. Mech. Anal. 14 (1963)1.

[40] Garg, V. K., Non-Newtonian flow over a wedge with suction, Int. J. Numer. Methods Fluids 15(1992)37.

[41] Garg, V. K. and Rajagopal, K. R., Flow of a non-Newtonian fluid past a wedge, Acta Mech. 88 (1991)113.

[42] Gorla, R. S. R. and Voss, D., Heat transfer in a second grade order viscoelastic boundary layer flow at a stagnation point, Int. J. Eng. Sci.18(1980)1301.

[43] Hsu, C. H. , Hsiao, K. L. and Teng, J. T., Conjuate heat transfer of a triangular fin in the flow of a second grade fluid, Int. J. Heat Fluid flow 19(1998)282.

[44] Massoudi, M. and Ramezan, M., Effect of injection or suction on the Falkner-Skan flows of second grade fluids, Int. J. Nonlinear Mech. 24(1989)221.

[45] Massoudi, M. and Ramezan, M., Heat transfer analysis of a viscoelastic fluid at a stagnation point, Mech. Res. Comm. 19(1992)129.

[46] Massoudi, M., Local similarity solutions for the flow of a non-Newtonian fluid over a wedge, Int. J. Nonlinear Mech. 36(2001)961.

[47] Massoudi, M. , Boundary layer flow of a second grade fluid with variable heat flux at the wall, Appl. Math. Computation 143(2003)201.

[48] Pakdemirli, M., Boundary layer flow of power-law fluid past arbitrary profiles, IMA J. Appl. Math. Computation 50(1992)133.

[49] Rajagopal, K. R., Gupta, A. S. and Wineman, A. S., On a boundary layer theory for non-Newtonian fluids, Int. J. Eng. Sci. 18 (1980)875.

[50] Rajagopal, K. R., Gupta, A. S. and Na, T. Y., A note on Falkner-Skan flows of a non-Newtonian fluid. Int. J. Nonlinear Mech. 18(1983)313.

[51] Rajagopal, K. R. ,and Na, T. Y. Natural convection flow of a non-Newtonian fluid between two vertical flat plates, Acta Mech. 54(1985)239.

[52] Rajeswari, G. and Rathan, S. L., Flow of a particular class of non-Newtonian visco elastic and visco inelastic fluids near a stagnation point, Z. Angew. Math. Phys. 13(1962)43.

[53] Riely, N. and Weidman, P. D., Multiple solution of the Falkner-Skan equation for flow past a stretching boundary, SIAM J. Appl. Math. 49(1989)1350.

[54] Kumari, M., Grasan, T. and Pop, I., Rotating flow of power-law fluid over a stretching surface, Technishe Mech. Band 26, Heft 1 (2006) 11.

[55] Andersson, H. I., Aarseth, J. B., Braud, N. and Dandapat, B. S., Flow of a power -law fluid film on an unsteady stretching surface, J. Non-Newtonian fluid Mech. 62 (1996) 1.

[56] Andersson, H. I., Bech, K. H. and Dandapat, B. S., Magnetohydrodynamic flow of a power-law fluid over a stretching sheet, Int. J. Nonlinear Mech. 27 (1992) 929.

[57] Andersson, H. I. and Dandapat, B. S., Flow of a power-law fluid over a stretching sheet, Stability, Appl. Anal. Continuous Media, 1(1991) 339.

[58] Chamkha, A. J. , Similarity solution for thermal boundary layer on a stretched surface of a non-Newtonian fluid, Int. Comm. Heat Mass Transfer,24(1997)643.

[59] Kim, H. W., Jeng, D. R. and Dewitt, K. J., Momemtum and heat transfer in a power-law flow over two dimensional or axisymmetric bodies, Int. J. Heat Mass Transfer,26(1983)245.

[60] Kumari, M. and Nath, G., MHD boundary-layer flow of a non-Newtonian fluid over a continuous moving surface with a parallel free stream, Acta Mech.46(2001)139.

[61] Liao, S. J., On the analytic solution of magnetohydrodynamic flows of non-Newtonian fluids over a stretched sheet, J. Fluid Mech., 488(2003)189.

[62] Xu, H., Liao, S. J. and Pop, I., Series solution of unsteady boundary layer flows of non-Newtonian fluids near a forward stagnation point, J. Non-Newtonian Fluid Mech.139(2006)31.

[63] Lok, Y.Y., Amin, N. and Pop, I., Unsteady mixed convection flow of a micropolar fluid near the stagnation point on a vertical surface, Int. J. Thermal Sciences 45(2006)1149.

[64] Lok, Y.Y., Phang, P., Amin, N. and Pop, I., Unsteady boundary layer flow of a micropolar fluid near the forward stagnation point of a plane surface, Int. J. Sci.41(2003)173.

[65] Hayat, T. and Sajid, M. , Analytic solution for axisymmetric flow and heat transfer of a second grade fluid past a stretching sheet, Int. J. Heat and Mass Transfer 50(2007)75.

[66] Vajravelu, K. and Rollins, D., Hydromagnetic flow of a second grade fluid over a stretching sheet, Appl. Math. Computation 148(2004)783-791.

[67] Rollins, D. and Vajravelu, K, Heat transfer in a second order fluid over a continuous stretching surface, Acta Mech. 93(1992)53.

[68] Cheng, J., Liao,S.J , Mohapatra, R . N. and Vajravelu, K., Series solution of nano boundary layer flows by means of the homoyopy analysis method, J. Math. Anal. Appl.(in press).

[69] Fosdick, R. L. and Straughan, B., Catastropic instabilities and related results in a fluid of third-grade, Int. J. Nonlinear Mech. 20 (1981) 91.

[70] Rajagopal, K.R.and Mollica, F., Secondary flows due to axial shearing of a third-grade fluid between two eccentrically placed cylinders., Int. J. Eng. Sci. 37 (1999) 411.

[71] Akyildiz, F.T., A note on the flow of a non-Newtonian film, Int. J. Nonlinear Mech. 33 (1998) 1061.

[72] Asghar, S., Mohyuddin, M.R. and Hayat, T., Unsteady flow of third grade fluid in the case of suction. Math. Computer Modelling 38 (2003) 201.

[73] Erdogan, M.E., Plane surface wave suddenly set in motion in a non-Newtonian fluid, Acta Mech. 108 (1995) 179.

[74] Hayat, T., Kara, A.H. and Momoniat, E., Exact flow of a third grade fluid on a porous wall, Int. J. Nonlinear Mech. 38 (2003) 1533.

[75] Hayat, T., Nadeem, S., Pudasaini, S. P. and Asghar, S., Fluctuating flow of a third order fluid past an infinite plate with variable suction, Arch. Mech. 55(2003)305.

[76] Kaloni, P. N. and Siddiqui, A. M., A note on the flow of a viscoelastic fluid between eccentric disks, J. Non-Newtonian Fluid Mech.26(1987)125.

[77] Hayat, T., Wang, Y. and Hutter, K., Flow of a fourth grade fluid, Math. Models and Methods in Appl. Sci. 12(2002)797.

[78] Hayat, T. and Kara, A.H., Couette flow of a third grade fluid with variable magnetic field, Math. Computer Modeling **43** (2006) 132.

[79] Fetecau, C. and Fetecau, C., Unsteady helical flows of a Maxwell fluid, Proc. Romanian Academy, Series A **5** (2004) 1.

[80] Fetecau, C. and Zierep, J., The Rayleigh Stokes problem for a Maxwell fluid, Z. Angew. Math. Phys. 54 (2003) 1086.

[81] Fetecau, C. and Fetecau, C., Decay of a potential vortex in a Maxwell fluid, Int. J. Nonlinear Mech. 38 (2003) 985.

[82] Fetecau, C. and Fetecau, C., The first problem of Stokes for an Oldroyd-B fluid, Int. J. Nonlinear Mech. 38 (2003) 1539.

[83] Fetecau, C. and Fetecau, C., Decay of a potential vortex in an Oldroyd-B fluid, Int. J. Eng. Sci. 43 (2005) 340.

[84] Ravindran, J., Krishnan, J. M. and Rajagopal, K. R., A note on the flow of a Burgers' fluid in an orthogonal rheometer, Int. J. Eng. Sci. 42 (2004) 1973.

[85] Hayat, T., Fetecau, C. and Asghar, S., Some simple flows of a Burgers' fluid, Int. J. Eng. Sci. 44 (2006) 1423.

[86] Lehnert, B., Magnetohydrodynamic waves under the action of the Coriolis force. J. Astrophysics. 119 (1954) 647.

[87] Lehnert, B., Magnetohydrodynamic waves under the action of the Coriolis force II. J. Astrophysics. 121 (1955) 481.

[88] Thornley, C., On Stokes and Rayleigh layers in a rotating system, Quart. J. Mech. Appl. Math., 21 (1968) 451.

[89] Loper, D.E., Steady hydromagnetic boundary layer near a rotating electrically conducting plate, Phys. Fluids, 13 (1970) 2999.

[90] Loper, D.E. and Benton, E.R., On the spin-up of an electrically conducting fluid. Part2. Hydromagnetic spin-up between infinite flat insulating plates, J. Fluid Mech.13 (1970) 785.

[91] Debnath, L., On unsteady magnetohydrodynamic boundary layers in a rotating flow, Z. Angew. Math. Mech.52 (1972) 623.

[92] Debnath, L., On the hydromagnetic spin-up flows in a rotating fluid, Lett. Appl. Eng. Sci., 1 (1973) 451.

[93] Debnath, L., On the unsteady hydromagnetic boundary layer flow induced by torsional oscillations of a disk, Plasma Phys.16 (1974) 1121.

[94] Gupta, A.S., Ekman layer on a porous plate, Phys. Fluids, 15 (1972) 930.

[95] Soundalgekar, V.M. and Pop, I., On hydromagnetic flow in a rotating fluid past an infinite porous wall, Z. Angew. Math. Mech., 58 (1973) 718.

[96] Ganapathy, R., A note on oscillatory Couette flow in a rotating system, ASME J. Appl. Mech., 61 (1994) 208.

[97] Acheson, D.J., Forced hydromagnetic oscillations of a rapidly rotating fluid, Phys. Fluids 18 (1975) 961.

[98] Murthy, S.N. and Ram, R.K.P., MHD Ekman layer on a porous plate, IL Nuovo Cimento, 29 (1975) 296.

[99] Deka, R.K., Gupta, A.S. and Takhar, H.S., Flow past an accelerated horizental plate in a rotating fluid, Acta Mech.138 (1999) 13.

[100] Mazumder, B.S., An exact solution of oscillatory Couette flow in a rotating system, ASME J. Appl. Mech.58 (1999) 1104.

[101] Singh, K.D., An oscillatory hydromagnetic flow in a rotating system, Z. Angew. Math. Mech., 80 (2000) 429.

[102] Hayat, T. and Hutter, K., Rotating flow of a second-order fluid on a porous plate, Int. J. Non-linear Mech., 39 (2004) 767.

[103] Hayat, T., Wang, Y. and Hutter, K., Hall effects on the unsteady hydromagnetic oscillatory flow of a second grade fluid, Int. J. Nonlinear Mech., 39 (2004) 1027.

[104] Hayat, T. and Kara, A.H., A variational analysis of non-Newtonian flow in a rotating system, Int. J. Compt. Fluid Mech. 20 (2006) 157.

[105] Hayat, T., Nadeem, S., Asghar, S. and Siddiqui, A.M., MHD rotating flow of a third grade fluid on an oscillating porous plate, Acta Mech. 152 (2001) 177.

[106] Puri, P., Rotating flow of an elastico-viscous fluid on an oscillating plate, Z. Angew. Math. Mech. (ZAMM) . 54 (1974) 743.

[107] Puri, P. and Kulshrestha, P.K., Rotating flow of non-Newtonian fluids, Applicable Analysis 4 (1974) 131.

[108] Liao, S.J., Beyond perturbation: introduction to homotopy analysis method, *Boca Raton: Chapman & Hall/CRC Press;* 2003.

[109] Chen, I.C., Hayat, T. and Chen.J.L., Exact solutions for the unsteady flow of a Burgers' fluid in a duct induced by time-dependent prescribed volume flow rate, Heat Mass Transfer 43 (2006) 85.

[110] Rajagopal, K.R. and Srinivasa, A.R., A thermodynamic frame work for rate type fluid models, J. Non-Newtonian Fluid Mech. 88 (2000) 207.

[111] Fetecau, C., Hayat, T. and Fetecau, C., Steady-state solutions for some simple flows of generalized Burgers fluids, Int. J. Nonlinear Mech. 41 (2006) 880.

[112] Asghar, S., Parveen, S., Hanif, S., Siddiqui, A.M. and Hayat, T., Hall effects on the unsteady hydromagnetic flow of an Oldroyd-B fluid, Int. J. Eng. Sci. 41 (2003) 609.

[113] Rajagopal, K.R. and Bhatnagar, R.K., Exact solutions for some simple flows of an Oldroyd-B fluid, Acta Mech. 113 (1995) 233.

[114] Hayat, T., Siddiqui, A.M. and Asghar, S., Some simple flows of an Oldroyd-B fluid, Int. J. Eng. Sci. 39 (2001) 135.

[115] Oldroyd, J.G., On the formulation of rheological equations of state, Proc. R. Soc. Lond. Ser A 200 (1950) 523.

[116] Kim, M.C., Lee, S.B., Kim, S. and Chung, B.J., Thermal instability of viscoelastic fluids in porous media, Int. J. Heat Mass Transfer 46 (2003) 5065.

[117] Vafai, K. and Tien, C.L. Boundary and inertia effects on flow and heat transfer in porous media, Int. J. Heat Mass Transfer 24 (1981) 195.

[118] Chen, C.I., Chen, C.K. and Yang, Y.T., Unsteady unidirectional flow of an Oldroyd-B fluid in a circular duct with different given volume flow rate conditions, Int. J. Heat Mass Transfer 40 (2004) 203.

[119] Khuzhayorov, B., Auriault, J.L. and Royer, P., Derivation of macroscopic filtration law for transient linear viscoelastic fluid flow in porous media, Int. J. Eng. Sci. 38 (2000) 487.

[120] Alishayev, M.G. and Mirzadzhanzadeh, A. Kh., About retardation phenomena in filtration theory (in Russian), Nieft i gaz. 6 (1975) 71.

[121] Tan, W.C. and Masuoka, T., Stokes' first problem for a second grade fluid in a porous half space with heated boundary, Int. J. Nonlinear Mech. 40 (2005) 515.

[122] Fosdick, R.L. and Rajagopal, K.R., Thermodynamics and stability of fluids of third grade, *Proc. Roy. Soc. Lond. A* 339 (1980) 351.

[123] Liao, S.J., A uniformly valid analytic solution of 2D viscous flow past a semi-infinite flat plate, *J. Fluid Mech.*, 385 (1999) 101.