Magnetohydrodynamic flow of a Maxwell fluid between two side walls perpendicular to a plate

By

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A Dissertation Submitted in the Partial Fulfillment of the Requirements for the Degree of

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CERTIFICATE

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We accept this dissertation as conforming to the required standard

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2009

Dedicated To

My affectionate father and pious mother & especially my grand parents (late) whose prayers are accompaniment in the journey of my life

And

My brothers and sister Humaira whose constant encouragement through out my educational carrier proved to a great source of inspiration for me.

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Preface

Many substances of multi-phase nature and/or of high molecular weight encountered in industrial applications display shear-thinning or shear-thickening behaviour, stress relaxation, normal stress differences and yield stress etc. In particular, polymer melts, suspensions, liquid crystals or biological fluids exhibit such properties which lead to nonlinear viscoelastic behaviour that cannot be simply described by the classical Navier-Stokes' theory. Such fluids are called non-Newtonian fluids. Amongst the many models which have been used to describe the non-Newtonian behaviours, the fluids of differential type have received special attention as well as much controversy [1]. These fluids cannot predict the stress relaxation phenomena exhibited by some polymeric liquids. Amongst the non-Newtonian fluid models which are capable for describing such phenomena are the rate type fluids models. Some interesting studies regarding these fluids are presented in the references [2-10]. One of the simplest rate type models to account for stress relaxation phenomenon is the Maxwell model [11]. While there are several proper forms, the so-called `upper convected Maxwell model' is in the most common use in theoretical studies and may be derived from a molecular point of view. Rajagopal and Srinivasa [12] showed that the classical Maxwell model is a linearization of a model that stores energy like a solid and dissipates energy as a viscous fluid. The storing of energy characterizing the fluid's elastic response and the dissipation of energy characterizing its viscous nature.

The present study is of interest because the theoretical study of magnetohydrodynamic (MHD) channel flows has wide range of applications in designing cooling systems with liquid metals, the control of under ground spreading of chemical wastes and pollutants etc. Further, a great deal of interest has been focused on understanding the rheological effects occurring in the flow of non-Newtonian fluids through porous medium. Such problems are of special interest in oil reservoir engineering, where an increasing interest is being shown in the possibility of improving oil recovery efficiency from water flooding projects through mobility control with non-Newtonian displacing fluids. The analysis of MHD flows through porous medium has been the subject of several recent papers [13-15]. In view of above motivation, the aim of this dissertation is to study the MHD channel flows of a Maxwell through porous medium. The dissertation consists of three chapters.

Chapter one contains basic definitions and equations used in the next chapters.

In chapter two, we reviewed paper by Hayat et al. [16]. This work consists of exact solution of unsteady flow of a Maxwell fluid caused by a suddenly moved plate between two side walls perpendicular to the plate. The solutions have been obtained by using the Fourier sine transforms.

Chapter three presents the unsteady channel flows of a magnetohydrodynamic (MHD) Maxwell fluid through porous space. Three characteristic examples which are, flow due to impulsive motion of plate, flow due to constantly accelerating plate and flow due to variable accelerating plate between two side walls perpendicular to the plate are considered. Closed form solutions for the velocity field, the tangential stress and the volume flow rate are developed by applying the Fourier sine transforms. The influence of various parameters of interest on the velocity field and the resulting tangential stress at the bottom wall is also shown and discussed through several graphs.

Contents

Chapter 1

Basic definitions and equations

1.1 Introduction

In this chapter we present the basic definitions related to fluid mechanics, magnetohydrodynamics and porous medium. Further, this chapter includes the basic laws and equations which govern the flow.

1.2 Definitions

1.2.1 Fluid

A fluid is defined as a substance which deforms continuously the action of shear stress, regardless of its magnitude (a small shear stress, which may appear to be of negligible magnitude, will also cause deformation in the fluid).

1.2.2 Flow

A material goes under deformation when different forces acts upon it. If deformation continuously increases without limit then the phenomenon is known as flow.

1.2.3 Dynamic viscosity

The proportionality constant between the viscous stress τ_{xy} and the velocity gradient is called dynamic viscosity μ. Its dimensions are $[\mu] = ML^{-1}T^{-1}$ and its units in the SI, Pa s or kg/(m s).

1.3 Types of fluid

1.3.1 Inviscid fluids

All the fluids for which the viscosity is zero are inviscid fluids. Gases are usually treated as inviscid fluid for engineering purposes.

1.3.2 Viscous fluids

Fluids for which the viscosity is non-zero are viscous fluids. All real fluids are viscous fluids. They are further categorized as Newtonian and non-Newtonian fluids.

Newtonian fluids/ non-Newtonian fluids

For the most frequently encountered substances such as water and air, the viscosity depends only on thermodynamic variables, like temperature, and is independent of the velocity gradient or deformation rate. These fluids are called Newtonian and the constitutive relation between viscous stress and the deformation rate is linear.

However, this is not the case for all substances, where the viscosity may be a function of the deformation rate. In this case, the fluid is called non-Newtonian, and the constitutive equation is a nonlinear function of the deformation rate. Examples of this type of fluids are certain oils, paints, polymer solutions, etc.

1.4 Types of flow

1.4.1 Steady/unsteady flow

The flow is steady when the fluid variables do not depend on time. That is, for any variable the partial derivative with respect to time is zero,

$$\frac{\partial(\cdot)}{\partial t} = 0. \tag{1.1}$$

When this is not the case, the flow is unsteady.

1.4.2 Compressible/incompressible flow

All substances are compressible to a certain extent, i.e. of variable density. However, in many practical situations, the density variations are so small that they can be neglected and the density can be considered constant.

Flow of variable density are called compressible whereas flows that are modeled assuming constant density are called incompressible, and is typical of liquids. An incompressible flow satisfies

$$\nabla.\mathbf{V} = \mathrm{div}\,(\mathbf{V}) = 0. \tag{1.2}$$

1.4.3 One/two/three-dimensional flow

The flow is one-, two- or three-dimensional when the fluid variables (such as density, velocity, temperature, etc.) depend on one, two or three spatial coordinates.

1.4.4 Laminar/turbulent flow

The flow is laminar when the motion of the fluid particles is well-organized, as if layers of fluid slide over others. It is predictable and deterministic.

However, due to nonlinearity of the transport equations, a fluid flow can have a random component, so that the real flow is the sum of an average motion plus some chaotic fluctuations. This flow is called turbulent.

1.5 Magnetohydrodynamics

Magnetohydrodynamics (MHD) is the study of the electrically conducting flow of fluid in the presence of magnetic field. Magnetohydrodynamics studies the dynamics of electrically conducting fluids. Examples of such fluids are plasmas, liquid metals, and salt water. This was first introduce by Hanns Alfven. The study of MHD flows of non-Newtonian fluids has become the basis of many scientific and engineering applications.

1.5.1 Maxwell equations and generalized Ohm's law

Maxwell's equations had been introduced by James Clerk Maxwell in the nineteenth century. The set of equations that governs the behavior of electromagnetic waves in all practical situations are called Maxwell's equations. These equations are as follows :

Gauss law states that the total flux inside a surface is proportional to the total charge enclosed within the surface.

Mathematically,

$$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\varepsilon_0}. \tag{1.3}$$

Gauss law for magnetism states that the magnetic field has zero divergence.

Mathematically,

$$\nabla \cdot \mathbf{B} = 0. \tag{1.4}$$

Faraday's law of induction states that a changing magnetic field induces an electric field.

Mathematically,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{1.5}$$

Ampere's law states that magnetic fields can be produced either by current $\mathbf{J}$ or by changing electric field $(\partial \mathbf{E}/\partial t)$.

Mathematically,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{1.6}$$

In above equations, $\mathbf{E}$ is electric field ε_0 the permittivity of the free space, ρ_e is the electric charge density, $\mathbf{B}$ is the magnetic field, and μ_0 is the magnetic permeability.

The generalized Ohm's law states that current density is proportional to force per unit charge.

Mathematically,

$$\mathbf{J} = \sigma\left(\mathbf{E} + \mathbf{V} \times \mathbf{B}\right), \tag{1.7}$$

where σ is the electric conductivity of the fluid.

1.6 Porous medium

A solid that contains pores; normally, it refers to interconnected pores that can transmit the flow of fluids. Many natural substances such as rocks, soils biological tissues (e.g. bones) are examples of porous medium.

1.6.1 Porosity

Porosity is a measure of the void spaces in a material, and is measured as a fraction, between $0-1$ or as a percentage between $0-100\%$. In other words, the porosity of a porous medium (such as rock or sediment) describes the fraction of void space in the material, where the void may contain for example air or water. It is defined by the ratio

$$\phi = \frac{V_V}{V_T},$$

where V_V is the volume of void spaces (such as fluid) and V_T is the total or bulk volume of material, including the solid and void-components.

1.6.2 Permeability

Permeability is a measure of the ability of a material (typically, a rock or unconsolidated material) to transmit fluids.

1.6.3 Darcy's law

Darcy's law is a generalized relationship for flow in porous media. It shows that the volumetric flow rate is a function of the flow area, elevation, fluid pressure and a proportionality constant. In fluid dynamics, Darcy's law is a phenomenologically derived constitutive equations that describes the flow of a fluid through porous medium. The law was formulated by Henry Darcy based on the results of experiments on the flow of water through beds of sand. It also forms the scientific basis of fluid permeability used in the earth sciences.

1.7 Law of conservation of mass

The first principle to which we apply the relation between system and control volume formulation is conservation of mass. It is intuitive that mass can be neither created nor destroyed; if the flow rate of mass into a control volume exceeds the rate of flow out, mass will accumulate within the control volume. Let $\tilde{V}$ be the control volume. Assume that its surface remain fixed in space. The surface is permeable, so that the fluid can freely enter in and leave. Since the mass can neither be created nor destroyed, so mass in control volume $\tilde{V}$ is conserved at all time, that is,

$$\frac{d}{dt}\int_{\tilde{V}} \rho d\tilde{V} = 0, \tag{1.8}$$

where ρ is the density field at any time.

Reynold's transport theorem stated that if Φ be a field (scalar, vector or tensor) associated with the fluid, then

$$\begin{aligned} \frac{d}{dt}\int_{\tilde{V}} \Phi d\tilde{V} &= \int_{\tilde{V}} \left(\frac{d\Phi}{dt} + \Phi \nabla . \mathbf{V}\right) d\tilde{V}, \\ &= \int_{\tilde{V}} \left(\frac{\partial \Phi}{\partial t} + \operatorname{div}(\Phi \mathbf{V})\right) d\tilde{V}, \end{aligned} \tag{1.9}$$

where $\mathbf{V}$ is the velocity of the fluid and d/dt is the material derivative.

By setting $\Phi = \rho$, we get from Eqs. (1.8) and (1.9)

$$\int_{\tilde{V}} \left(\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{V})\right) d\tilde{V} = 0. \tag{1.10}$$

Since the control volume $\tilde{V}$ is arbitrary, a necessary and sufficient condition for conservation of mass is

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{V}) = 0. \tag{1.11}$$

When the fluid is considered to be incompressible then the density is constant, so Eq. (1.11) reduces to

$$\operatorname{div} \mathbf{V} = 0. \tag{1.12}$$

1.8 Law of conservation of momentum

Every particle of fluid at rest, steady or in accelerated motion obeys Newton's second law of motion. Newton's second law states that the sum of all external forces acting on the system is equal to the time rate of change of linear momentum of the system, i.e.

$$\frac{d}{dt}(m\mathbf{V}) = \mathbf{F}. \tag{1.13}$$

let $\tilde{V}$ be an arbitrary volume bounded by the surface $\tilde{S}$ occupying the fluid element at any time t. The rate of change of momentum within the volume $\tilde{V}$ will be

$$\frac{d}{dt}(m\mathbf{V}) = \frac{d}{dt}\int_{\tilde{V}} (\rho\mathbf{V})\, d\tilde{V}. \tag{1.14}$$

Let $\mathbf{b}$ be the vector which represents the resultant of body forces per unit mass, then the net body force acting on a mass of volume $\tilde{V}$ will be

$$\mathbf{F}_b = \int_{\tilde{V}} (\rho\mathbf{b})\, d\tilde{V}. \tag{1.15}$$

Let $\mathbf{T}$ is the stress tensor which represents the resultant of surface force per unit area, then the net surface force acting on the surface $\tilde{S}$ containing volume $\tilde{V}$ will be

$$\mathbf{F}_S = \int_{\tilde{S}} (\mathbf{n}\cdot\mathbf{T})\, d\tilde{S}, \tag{1.16}$$

where $\mathbf{n}$ is the unit normal to the surface.

Since time rate of change of momentum is equal to the sum of the resultant forces, so

$$\frac{d}{dt}\int_{\tilde{V}} (\rho\mathbf{V})\, d\tilde{V} = \int_{\tilde{V}} (\rho\mathbf{b})\, d\tilde{V} + \int_{\tilde{S}} (\mathbf{n}\cdot\mathbf{T})\, d\tilde{S}. \tag{1.17}$$

In view of Eq. (1.9), the above equation can be written as

$$\int_{\tilde{V}} \left[\frac{\partial(\rho\mathbf{V})}{\partial t} + \operatorname{div}(\rho\mathbf{V}\mathbf{V})\right] d\tilde{V} = \int_{\tilde{V}} (\rho\mathbf{b})\, d\tilde{V} + \int_{\tilde{S}} (\mathbf{n}\cdot\mathbf{T})\, d\tilde{S}. \tag{1.18}$$

The Gauss's theorem states that the volume integral of the divergence of a vector field over an arbitrary control volume $\tilde{V}$ is equal to the flow rate of the field (vector or tensor) across the surface $\tilde{S}$ bounding the domain

$$\int_{\tilde{V}} (\nabla \cdot \mathbf{T})\, d\tilde{V} = \int_{\tilde{S}} (\mathbf{n} \cdot \mathbf{T})\, d\tilde{S}. \tag{1.19}$$

Using Eq. (1.19) into Eq.(1.18) we get

$$\int_{\tilde{V}} \left[\frac{\partial}{\partial t} (\rho \mathbf{V}) + \nabla \cdot (\rho \mathbf{V}\mathbf{V}) \right] d\tilde{V} = \int_{\tilde{V}} \rho \mathbf{b} d\tilde{V} + \int_{\tilde{V}} (\nabla \cdot \mathbf{T})\, d\tilde{V}. \tag{1.20}$$

Since the volume $\tilde{V}$ is arbitrary, a necessary and sufficient condition for conservation of momentum is

$$\frac{\partial}{\partial t} (\rho \mathbf{V}) + \nabla \cdot (\rho \mathbf{V}\mathbf{V}) = \rho \mathbf{b} + \nabla \cdot \mathbf{T}. \tag{1.21}$$

Since fluid is incompressible, therefore ρ is constant and

$$\rho \left[\frac{\partial \mathbf{V}}{\partial t} + \nabla \cdot (\mathbf{V}\mathbf{V}) \right] = \rho \mathbf{b} + \nabla \cdot \mathbf{T}. \tag{1.22}$$

As

$$\nabla \cdot (\mathbf{V}\mathbf{V}) = \mathbf{V} \nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla \mathbf{V}, \tag{1.23}$$

For incompressible fluid

$$\nabla \cdot \mathbf{V} = 0, \tag{1.24}$$

so that

$$\nabla \cdot (\mathbf{V}\mathbf{V}) = \mathbf{V} \cdot \nabla \mathbf{V}. \tag{1.25}$$

In view of above equation, Eq.(1.22) becomes

$$\rho \left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} . \nabla) \mathbf{V} \right] = \rho \mathbf{b} + \nabla \cdot \mathbf{T}. \tag{1.26}$$

Equation.(1.26) represents the differential form of law of conservation of momentum in which ρ is the density, $\mathbf{T}$ the Cauchy stress tensor, $\mathbf{V}$ the velocity and $\mathbf{b}$ the body force per unit mass.

The Cauchy stress tensor can be written as

$$\mathrm{T} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}, \tag{1.27}$$

where σ_{xx}, σ_{yy}, and σ_{zz} are normal stresses and other are shear stresses.

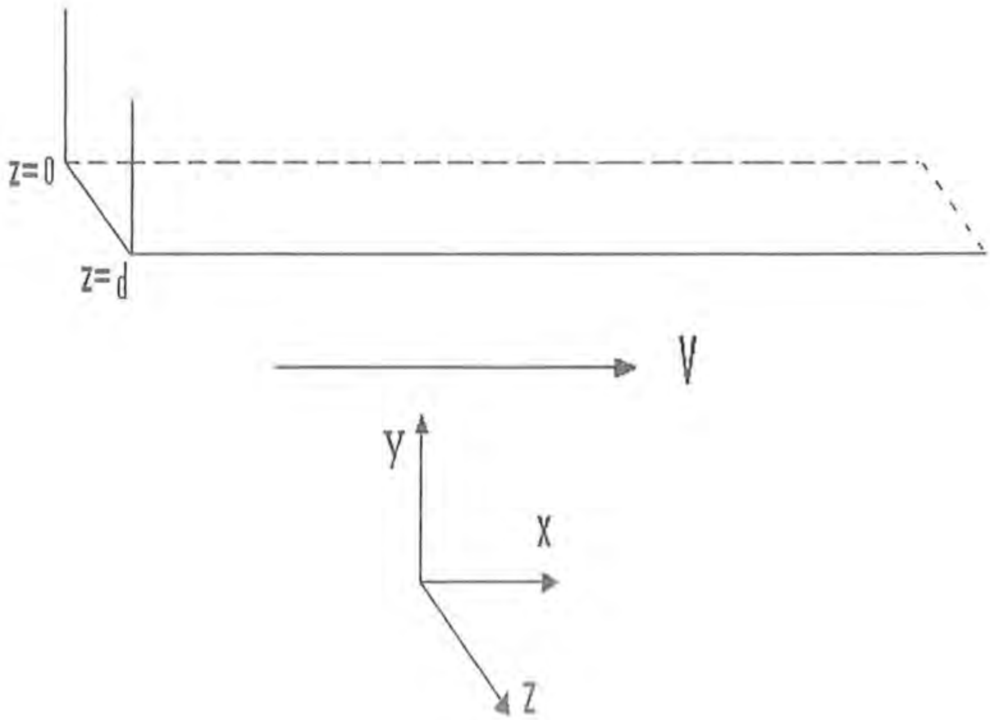
Chapter 2

Unsteady flow of a Maxwell fluid between two side walls due to a suddenly moved plate

2.1 Introduction

In this chapter the unsteady flow of a Maxwell fluid caused by a suddenly moved plane wall between two side walls perpendicular to the plane is considered. The expressions for the velocity, the shear stress and the volume flux are developed employing the Fourier sine transforms. The solutions obtained for large values of time are similar to the steady solutions of a Newtonian fluid. The unsteady solutions for Newtonian fluid have been recovered as limiting cases by choosing $\lambda \to 0$. The contents of this chapter provide a detailed review of a paper by Hayat et al. [16].

2.2 Geometry



Geometry of flow problem in Cartesian coordinate system.

2.3 Governing equations

The equations of motion of an incompressible fluid in the absence of body forces are

$$\operatorname{div} \mathbf{V} = 0, \tag{2.1}$$

$$\rho \frac{d\mathbf{V}}{dt} = \operatorname{div} \mathbf{T}, \tag{2.2}$$

where ρ is the density of fluid, $\mathbf{V}$ the velocity and d/dt represents the material time derivative. The Cauchy stress tensor $\mathbf{T}$ for a Maxwell fluid is given by

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad \mathbf{S} + \lambda \frac{D\mathbf{S}}{Dt} = \mu \mathbf{A}, \tag{2.3}$$

in which $-p\mathbf{I}$ is the constitutively indeterminate part of the stress due to the constraint of incompressibility, $\mathbf{S}$ the extra-stress tensor, $\mathbf{A}$ the first Rivlin-Ericksen tensor, μ the dynamic viscosity and λ the relaxation time. The contravariant convected derivative D/Dt is defined by

$$\frac{D\mathbf{S}}{Dt} = \frac{\partial \mathbf{S}}{\partial t} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T, \tag{2.4}$$

and

$$\mathbf{A} = \mathbf{L} + \mathbf{L}^T, \quad \mathbf{L} = \operatorname{grad} \mathbf{V}. \tag{2.5}$$

For the following problem under consideration, we assume the velocity field and the extra stress of the form

$$\mathbf{V} = [u(y, z, t), 0, 0], \quad \mathbf{S} = \mathbf{S}(y, z, t), \tag{2.6}$$

where u is the velocity component in x-coordinate direction. Using Eq. (2.6), the continuity Eq. (2.1) is identically satisfied and from Eqs. (2.4) and (2.5), one obtains

$$\mathbf{L} = \begin{bmatrix} 0 & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{L}^T = \begin{bmatrix} 0 & 0 & 0 \\ \frac{\partial u}{\partial y} & 0 & 0 \\ \frac{\partial u}{\partial z} & 0 & 0 \end{bmatrix},$$

$$\mathbf{LS} = \begin{bmatrix} S_{yx}\frac{\partial u}{\partial y} + S_{zx}\frac{\partial u}{\partial z} & S_{yy}\frac{\partial u}{\partial y} + S_{zy}\frac{\partial u}{\partial z} & S_{yz}\frac{\partial u}{\partial y} + S_{zz}\frac{\partial u}{\partial z} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{SL}^T = \begin{bmatrix} S_{xy}\frac{\partial u}{\partial y} + S_{xz}\frac{\partial u}{\partial z} & 0 & 0 \\ S_{yy}\frac{\partial u}{\partial y} + S_{yz}\frac{\partial u}{\partial z} & 0 & 0 \\ S_{zy}\frac{\partial u}{\partial y} + S_{zz}\frac{\partial u}{\partial z} & 0 & 0 \end{bmatrix},$$

and

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial u}{\partial y} & 0 & 0 \\ \frac{\partial u}{\partial z} & 0 & 0 \end{bmatrix}. \tag{2.7}$$

Thus from Eq. (2.4), we get

$$\frac{D\mathbf{S}}{Dt} = \begin{bmatrix} \frac{\partial S_{xx}}{\partial t} - 2S_{xy}\frac{\partial u}{\partial y} - 2S_{xz}\frac{\partial u}{\partial z} & \frac{\partial S_{xy}}{\partial t} - S_{yy}\frac{\partial u}{\partial y} - S_{zy}\frac{\partial u}{\partial z} & \frac{\partial S_{xz}}{\partial t} - S_{yz}\frac{\partial u}{\partial y} - S_{zz}\frac{\partial u}{\partial z} \\ \frac{\partial S_{yx}}{\partial t} - S_{yy}\frac{\partial u}{\partial y} - S_{yz}\frac{\partial u}{\partial z} & \frac{\partial S_{yy}}{\partial t} & \frac{\partial S_{yz}}{\partial t} \\ \frac{\partial S_{zx}}{\partial t} - S_{zy}\frac{\partial u}{\partial y} - S_{zz}\frac{\partial u}{\partial z} & \frac{\partial S_{zy}}{\partial t} & \frac{\partial S_{zz}}{\partial t} \end{bmatrix}. \tag{2.8}$$

Now using Eqs. (2.7) and (2.8) into Eq. (2.3) gives the following scalar equations:

$$\left(1 + \lambda\frac{\partial}{\partial t}\right) S_{xx} - 2\lambda\left(S_{xy}\frac{\partial u}{\partial y} + S_{xz}\frac{\partial u}{\partial z}\right) = 0, \tag{2.9}$$

$$\left(1 + \lambda\frac{\partial}{\partial t}\right) S_{xy} - \lambda\left(S_{yy}\frac{\partial u}{\partial y} + S_{zy}\frac{\partial u}{\partial z}\right) = \mu\frac{\partial u}{\partial y}, \tag{2.10}$$

$$\left(1 + \lambda\frac{\partial}{\partial t}\right) S_{xz} - \lambda\left(S_{yz}\frac{\partial u}{\partial y} + S_{zz}\frac{\partial u}{\partial z}\right) = \mu\frac{\partial u}{\partial z}, \tag{2.11}$$

$$\left(1 + \lambda\frac{\partial}{\partial t}\right) (S_{yy},\ S_{yz},\ S_{zz}) = 0. \tag{2.12}$$

If the fluid has been at rest up to the moment $t = 0$, *i.e.*,

$$\mathbf{S}(y, z, 0) = \frac{\partial \mathbf{S}(y, z, 0)}{\partial t} = 0, \tag{2.13}$$

then from Eqs. (2.10) to (2.12), it results that

$$S_{yy} = S_{yz} = S_{zz} = 0, \tag{2.14}$$

for all time and

$$\left(1 + \lambda\frac{\partial}{\partial t}\right) S_{xy} = \mu\frac{\partial u}{\partial y}, \tag{2.15}$$

$$\left(1 + \lambda\frac{\partial}{\partial t}\right) S_{xz} = \mu\frac{\partial u}{\partial z}, \tag{2.16}$$

where S_{xy} and S_{xz} are the tangential stresses.

With the help of Eq. (2.6), the equation of motion, in the absence of body forces, reduces to

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial S_{xy}}{\partial y} + \frac{\partial S_{xz}}{\partial z}, \tag{2.17}$$

$$0 = -\frac{\partial p}{\partial y}, \tag{2.18}$$

$$0 = -\frac{\partial p}{\partial z}. \tag{2.19}$$

Eliminating S_{xy} and S_{xz} between Eqs. (2.15) – (2.17), we find the governing equation under the form

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)\frac{\partial u(y,z,t)}{\partial t} = -\frac{1}{\rho}\left(1 + \lambda\frac{\partial}{\partial t}\right)\frac{\partial p}{\partial x} + \nu\left[\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]u(y,z,t), \tag{2.20}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid.

2.4 Formulation of the problem

We consider an incompressible Maxwell fluid at rest, occupying the space above an infinite flat plate perpendicular to the y-axis and between two side walls situated in the planes $z = 0$ and $z = d$ of a Cartesian coordinate system. At time $t = 0^+$, the plate is impulsively brought to the constant velocity V. Owing to the shear the fluid is gradually moved. Its velocity is of the form (2.6) and the governing equation (2.20), in the absence of a pressure gradient in the flow direction, takes the form

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial u(y,z,t)}{\partial t}=\nu\left[\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]u(y,z,t). \tag{2.21}$$

The appropriate initial and boundary conditions are

$$u(y,z,0)=\frac{\partial u(y,z,0)}{\partial t}=0 \quad \text{for } y>0 \text{ and } 0\le z\le d, \tag{2.22}$$

$$u(0,z,t)=V \quad \text{for } t>0 \text{ and } 0<z<d, \tag{2.23}$$

$$u(y,0,t)=u(y,d,t)=0 \quad \text{for } y,t>0, \tag{2.24}$$

and

$$u(y,z,t),\ \frac{\partial u(y,z,t)}{\partial y}\to 0 \quad \text{as } y\to\infty \text{ and } t>0. \tag{2.25}$$

2.5 Solution of the problem

Calculation of velocity field

In order to determine the analytical solution corresponding to the above problem we shall use the Fourier sine transform with respect to the spatial variable y and the finite Fourier with respect to z [17].

First, we define double Fourier sine transform pair as

$$u_{sn}(\xi,t)=\sqrt{\frac{2}{\pi}}\int_0^\infty\int_0^d u(y,z,t)\sin(y\xi)\sin(\lambda_n z)\,dz dy, \tag{2.26}$$

and

$$u(y,z,t)=\frac{4}{\pi d}\sum_{n=1}^{\infty}\frac{\sin(\lambda_n z)}{\lambda_n}\int_0^\infty u_{sn}(\xi,t)\sin(y\xi)\,d\xi. \tag{2.27}$$

Consequently, we multiply Eq. (2.21) by $\sqrt{2/\pi}\sin(y\xi)\sin(\lambda_n z)$, where $\lambda_n=n\pi/d$, and integrate the result with respect to y and z from 0 to infinity, respectively, from 0 to d, we get

$$\lambda\frac{\partial^2 u_{sn}(\xi,t)}{\partial t^2}+\frac{\partial u_{sn}(\xi,t)}{\partial t}+\nu\left(\xi^2+\lambda_n^2\right)u_{sn}(\xi,t)=\sqrt{\frac{2}{\pi}}\frac{\nu\xi}{\lambda_n}\left[1-(-1)^n\right]u(0,z,t). \tag{2.28}$$

Applying the boundary condition (2.23), the above equation reduces to

$$\lambda \frac{\partial^2 u_{sn}(\xi,t)}{\partial t^2} + \frac{\partial u_{sn}(\xi,t)}{\partial t} + \nu\left(\xi^2+\lambda_n^2\right) u_{sn}(\xi,t) = \sqrt{\frac{2}{\pi}} \frac{\nu \xi V}{\lambda_n}\left[1-(-1)^n\right]. \tag{2.29}$$

Also, the double Fourier sine transform of initial conditions (2.22) gives

$$u_{sn}(\xi,0) = \frac{\partial u_{sn}(\xi,0)}{\partial t} = 0; \quad n = 1,2,3... \tag{2.30}$$

The complementary solution of Eq. (2.29) is

$$u_{sn}^c(\xi,t) = C_1 e^{m_{1n}t} + C_2 e^{m_{2n}t}, \tag{2.31}$$

whereas the particular integral is given by

$$u_{sn}^p(\xi,t) = \sqrt{\frac{2}{\pi}} \frac{V\xi}{\lambda_n} \frac{\left[1-(-1)^n\right]}{\left(\xi^2+\lambda_n^2\right)}. \tag{2.32}$$

Thus the general solution is given by

$$u_{sn}(\xi,t) = C_1 e^{m_{1n}t} + C_2 e^{m_{2n}t} + \sqrt{\frac{2}{\pi}} \frac{V\xi}{\lambda_n} \frac{\left[1-(-1)^n\right]}{\left(\xi^2+\lambda_n^2\right)}, \tag{2.33}$$

where C_1 and C_2 are arbitrary constants to be determined and

$$m_{1n,2n} = \frac{-1 \pm \sqrt{1-4\nu\lambda\left(\xi^2+\lambda_n^2\right)}}{2\lambda}. \tag{2.34}$$

In view of Eq. (2.33) and initial conditions (2.30), one obtain

$$C_1 = \frac{-m_{1n}}{m_{2n}-m_{1n}} \sqrt{\frac{2}{\pi}} \frac{V\xi}{\lambda_n} \frac{\left[1-(-1)^n\right]}{\left(\xi^2+\lambda_n^2\right)},$$

and

$$C_2 = \frac{m_{2n}}{m_{2n}-m_{1n}} \sqrt{\frac{2}{\pi}} \frac{V\xi}{\lambda_n} \frac{\left[1-(-1)^n\right]}{\left(\xi^2+\lambda_n^2\right)}.$$

Substituting the values of C_1 and C_2, the solution (2.33) becomes

$$u_{sn}(\xi,t)=\sqrt{\frac{2}{\pi}}\frac{V\xi}{\lambda_n}\frac{[1-(-1)^n]}{(\xi^2+\lambda_n^2)}\left[1-\frac{m_{2n}e^{m_{1n}t}-m_{1n}e^{m_{2n}t}}{m_{2n}-m_{1n}}\right]. \tag{2.35}$$

We know that

$$\int_0^\infty \frac{\sin(y\xi)}{\xi^2+a^2}=\sqrt{2/\pi}e^{-ay}. \tag{2.36}$$

Inverting solution (2.35) by means of double Fourier sine transform and having in mind Eq. (2.36), we reach on the following expression for the velocity field

$$\begin{aligned}u(y,z,t) &= \frac{2}{d}V\sum_{n=1}^{\infty}[1-(-1)^n]\frac{\sin(\lambda_n z)}{\lambda_n}e^{-\lambda_n y}-\frac{4}{\pi d}V\sum_{n=1}^{\infty}[1-(-1)^n]\frac{\sin(\lambda_n z)}{\lambda_n}\\ &\times\int_0^\infty\frac{\xi\sin(y\xi)}{\xi^2+\lambda_n^2}\frac{m_{2n}e^{m_{1n}t}-m_{1n}e^{m_{2n}t}}{m_{2n}-m_{1n}}d\xi. \end{aligned} \tag{2.37}$$

Since

$$[1-(-1)^n]=\begin{cases}2, & \text{if } n \text{ is odd,}\\ 0, & \text{if } n \text{ is even.}\end{cases}$$

Consequently, we get the next expression for the velocity field under the form

$$u(y,z,t)=\frac{4}{d}V\sum_{n=1}^{\infty}\frac{\sin(\lambda_N z)}{\lambda_N}e^{-\lambda_N y}-\frac{8}{\pi d}V\sum_{n=1}^{\infty}\frac{\sin(\lambda_N z)}{\lambda_N}\int_0^\infty\frac{\xi\sin(y\xi)}{\xi^2+\lambda_N^2}\frac{m_{2N}e^{m_{1N}t}-m_{1N}e^{m_{2N}t}}{m_{2N}-m_{1N}}d\xi, \tag{2.38}$$

where $N=2n-1$.

Now setting $d=2h$ and changing the origin of the coordinate system by taking $z=z^*+h$ and dropping out the star notation for simplicity, the velocity field $u(y,z,t)$ can be written in a more suitable form

$$\begin{aligned}u(y,z,t) &= \frac{2}{h}V\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}e^{-\mu_N y}-\frac{4V}{\pi h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\\ &\times\int_0^\infty\frac{\xi\sin(y\xi)}{\xi^2+\mu_N^2}\frac{m_{2N}e^{m_{1N}t}-m_{1n}e^{m_{2N}t}}{m_{2N}-m_{1N}}d\xi, \end{aligned} \tag{2.39}$$

in which $\mu_N = (2n-1)\pi/(2h)$ and $m_{1N,2N} = \frac{-1\pm\sqrt{1-4\nu\lambda(\xi^2+\mu_N^2)}}{2\lambda}$.

Direct computation shows that $u(y,z,t)$ given by Eq. (2.39) satisfies the linear partial differential Eq. (2.21), the initial conditions (2.22) and the boundary conditions (2.23) to (2.25). When t goes to infinity, $u(y,z,t)$ tends to steady-state solution given by

$$u_s(y,z) = u(y,z,\infty) = \frac{4}{d}V\sum_{n=1}^{\infty}\frac{\sin(\lambda_N z)}{\lambda_N}e^{-\lambda_N y}. \tag{2.40}$$

Calculation of shear stress

In order to determine the shear stress at the bottom wall, i.e. $\tau_w(z,t) = \tau(0,z,t) = S_{xy}(0,z,t)$, we first integrate Eq. (2.15) with respect to t and take into consideration the initial conditions (2.13). For the solution of first order linear differential equation (2.15), we have an integrating factor (I.F.) of the form

$$I.F. = \exp\left(\frac{t}{\lambda}\right). \tag{2.41}$$

Multiplying Eq. (2.15) by integrating factor given in Eq. (2.41) to obtain

$$d\left[\exp\left(\frac{t}{\lambda}\right)\tau_{xy}(y,z,t)\right] = \left(\frac{\mu}{\lambda}\right)\exp\left(\frac{t}{\lambda}\right)\left(\frac{\partial u}{\partial y}\right). \tag{2.42}$$

Integration of above equation, with respect to t yields

$$\tau(y,z,t) = \left(\frac{\mu}{\lambda}\right)\exp\left(\frac{-t}{\lambda}\right)\int_0^t \exp\left(\frac{-\tau}{\lambda}\right)\frac{\partial u}{\partial y}d\tau + D_1, \tag{2.43}$$

where D_1 is a constant of integration.

Using initial condition (2.13), we get $D_1 = 0$. Hence, the Eq. (2.43) becomes

$$\tau(y,z,t) = \left(\frac{\mu}{\lambda}\right)\exp\left(\frac{-t}{\lambda}\right)\int_0^t \exp\left(\frac{\tau}{\lambda}\right)\frac{\partial u}{\partial y}d\tau. \tag{2.44}$$

As from Eq. (2.39), we find that

$$\begin{aligned}\frac{\partial u}{\partial y} &= -\frac{2}{h}V\sum_{n=1}^{\infty}(-1)^{n+1}\cos(\mu_N z)\,e^{-\mu_N y}-\frac{4V}{\pi h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\\ &\times\int_0^{\infty}\frac{\xi^2\cos(y\xi)}{\xi^2+\mu_N^2}\frac{m_{2N}e^{m_{1N}t}-m_{1N}e^{m_{2N}t}}{m_{2N}-m_{1N}}d\xi. \end{aligned} \tag{2.45}$$

Introduction of Eq. (2.45) into Eq. (2.44) results

$$\begin{aligned}\tau(y,z,t) &= \frac{-2\mu V}{\lambda h}\exp\left(\frac{-t}{\lambda}\right)\sum_{n=1}^{\infty}(-1)^{n+1}\cos(\mu_N z)\,e^{-\mu_N y}\int_0^t\exp\left(\frac{\tau}{\lambda}\right)d\tau\\ &-\frac{4\mu V}{\pi h\lambda}\exp\left(\frac{-t}{\lambda}\right)\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\\ &\times\int_0^{\infty}\frac{\xi^2\cos(y\xi)}{\xi^2+\mu_N^2}\int_0^t\exp\left(\frac{\tau}{\lambda}\right)\frac{m_{2N}e^{m_{1N}\tau}-m_{1N}e^{m_{2N}\tau}}{m_{2N}-m_{1N}}d\tau d\xi, \end{aligned} \tag{2.46}$$

or

$$\begin{aligned}\tau(y,z,t) &= \frac{-2\mu V}{\lambda h}\left(1-e^{\frac{-t}{\lambda}}\right)\sum_{n=1}^{\infty}(-1)^{n+1}\cos(\mu_N z)\,e^{-\mu_N y}\\ &-\frac{4\mu V}{\pi h\lambda}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\int_0^{\infty}\frac{\xi^2\cos(y\xi)}{\xi^2+\mu_N^2}\\ &\times\left[\begin{array}{c}\frac{\lambda m_{2N}}{(\lambda m_{1N}+1)(m_{2N}-m_{1N})}\left(e^{m_{1N}t}-e^{\frac{-t}{\lambda}}\right)\\ \\ -\frac{\lambda m_{1N}}{(\lambda m_{2N}+1)(m_{2N}-m_{1N})}\left(e^{m_{2N}t}-e^{\frac{-t}{\lambda}}\right)\end{array}\right]d\xi. \end{aligned} \tag{2.47}$$

Since

$$\lambda m_{1N}+1=\frac{1+\sqrt{1-4\nu\lambda\left(\xi^2+\mu_N^2\right)}}{2},$$

$$\lambda m_{2N}+1=\frac{1-\sqrt{1-4\nu\lambda\left(\xi^2+\mu_N^2\right)}}{2},$$

and

$$m_{2N}-m_{1N}=\frac{-\sqrt{1-4\nu\lambda\left(\xi^2+\mu_N^2\right)}}{\lambda}. \tag{2.48}$$

In view of Eq. (2.48) the above expression for the shear stress can be written in a more suitable form as

$$\begin{aligned}\tau(y,z,t) &= \frac{-2\mu V}{\lambda h}\left(1-e^{\frac{-t}{\lambda}}\right)\sum_{n=1}^{\infty}(-1)^{n+1}\cos(\mu_N z)\,e^{-\mu_N y} \\ &\quad -\frac{4\mu V}{\pi h\lambda}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\int_0^{\infty}\frac{\xi^2\cos(y\xi)}{\xi^2+\mu_N^2} \\ &\quad \times\frac{e^{m_{2N}t}-e^{m_{1N}t}}{m_{2N}-m_{1N}}d\xi. \qquad (2.49)\end{aligned}$$

Thus the shear stress at the bottom wall $\tau_w(z,t)=\tau(0,z,t)$ is given by

$$\begin{aligned}\tau_w(z,t) &= \frac{-2\mu V}{\lambda h}\left(1-e^{\frac{-t}{\lambda}}\right)\sum_{n=1}^{\infty}(-1)^{n+1}\cos(\mu_N z) \\ &\quad -\frac{4\mu V}{\pi h\lambda}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\int_0^{\infty}\frac{\xi^2}{\xi^2+\mu_N^2} \\ &\quad \times\frac{e^{m_{2N}t}-e^{m_{1N}t}}{m_{2N}-m_{1N}}d\xi. \qquad (2.50)\end{aligned}$$

Calculation of volume flux

The volume flux $Q(t)$ across a plane normal to the flow is given by

$$Q(t)=\int_0^{\infty}\int_{-h}^{h}u(y,z,t)\,dydz. \qquad (2.51)$$

Inserting $u(y,z,t)$ from Eq. (2.39) into Eq. (2.51), we find that

$$\begin{aligned}Q(t) &= \frac{2}{h}V\int_0^{\infty}\int_{-h}^{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}e^{-\mu_N y}dydz \\ &\quad -\frac{4}{\pi h}V\int_0^{\infty}\int_{-h}^{h}\left[\begin{array}{c}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\int_0^{\infty}\frac{\xi\sin(y\xi)}{\xi^2+\mu_N^2} \\ \times\frac{m_{2N}e^{m_{1N}t}-m_{1n}e^{m_{2N}t}}{m_{2N}-m_{1N}}d\xi\end{array}\right]dydz,\end{aligned}$$

or

$$
\begin{aligned}
Q(t) &= \frac{2}{h}V\int_0^\infty e^{-\mu_N y}dy\int_{-h}^{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}dz \\
&\quad -\frac{4V}{\pi h}\int_{-h}^{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}dz \\
&\quad \times\int_0^\infty\int_0^\infty\frac{\xi\sin(y\xi)}{\xi^2+\mu_N^2}\frac{m_{2N}e^{m_{1N}t}-m_{1N}e^{m_{2N}t}}{m_{2N}-m_{1N}}dyd\xi,
\end{aligned}
$$

or

$$
\begin{aligned}
Q(t) &= \frac{2}{h}V\left.\frac{e^{-\mu_N y}}{-\mu_N}\right|_0^\infty\sum_{n=1}^{\infty}(-1)^{n+1}\left.\frac{\sin(\mu_N z)}{\mu_N^2}\right|_{-h}^{h}-\frac{4V}{\pi h}\sum_{n=1}^{\infty}(-1)^{n+1}\left.\frac{\sin(\mu_N z)}{\mu_N^2}\right|_{-h}^{h} \\
&\times\int_0^\infty\int_0^\infty\frac{\xi\sin(y\xi)}{\xi^2+\mu_N^2}\frac{m_{2N}e^{m_{1N}t}-m_{1N}e^{m_{2N}t}}{m_{2N}-m_{1N}}dyd\xi,
\end{aligned}
$$

or

$$
Q(t)=\frac{4V}{h}\sum_{n=1}^{\infty}\frac{1}{\mu_N^3}-\frac{8V}{\pi h}\sum_{n=1}^{\infty}\frac{1}{\mu_N^2}\int_0^\infty\int_0^\infty\frac{\xi\sin(y\xi)}{\xi^2+\mu_N^2}\frac{m_{2N}e^{m_{1N}t}-m_{1N}e^{m_{2N}t}}{m_{2N}-m_{1N}}dyd\xi. \qquad (2.52)
$$

2.6 Limiting case $h \to \infty$ (flow over an infinite plate)

In the absence of side walls, namely when $h \to \infty$, the velocity field and shear stress are going to the known expressions.

The governing problem for flow over an infinite plate is

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2} \quad \text{for} \quad y,t>0, \tag{2.53}$$

$$u(y,0)=\frac{\partial u(y,0)}{\partial t}=0 \quad \text{for} \quad y>0, \tag{2.54}$$

$$u(0,t)=V \quad \text{for} \quad t>0, \tag{2.55}$$

$$u(y,t),\ \frac{\partial u(y,t)}{\partial y}\to 0 \ \text{ as } \ y\to\infty \text{ and } t>0. \tag{2.56}$$

Consequently, multiplying Eq. (2.53) by $\sqrt{\frac{2}{\pi}}\sin(y\xi)$ and integrate with respect to y from 0 to ∞, and having in mind the boundary conditions (2.55) and (2.56) we get

$$\lambda\frac{\partial^2 u_{sn}(\xi,t)}{\partial^2 t}+\frac{\partial u_{sn}(\xi,t)}{\partial t}+\nu\xi^2 u_{sn}(\xi,t)=\sqrt{\frac{2}{\pi}}\nu\xi V. \tag{2.57}$$

The Fourier sine transform of initial conditions (2.54) gives

$$u_{sn}(\xi,0)=\frac{\partial u_{sn}(\xi,0)}{\partial t}=0; \quad n=1,2,3... \tag{2.58}$$

The complementary solution of Eq. (2.57) is given

$$u_{sn}^{c}(\xi,t)=C_3e^{m_3t}+C_4e^{m_4t}; \quad 0<\xi<\frac{1}{2\sqrt{\lambda\nu}}, \tag{2.59}$$

and

$$u_{sn}^{c}(\xi,t)=\exp\left(\frac{-t}{2\lambda}\right)\left[C_5\cos\left(\frac{\beta t}{2\lambda}\right)+C_6\sin\left(\frac{\beta t}{2\lambda}\right)\right]; \quad \xi>\frac{1}{2\sqrt{\lambda\nu}}. \tag{2.60}$$

where

$$m_{3,4}=\frac{-1\pm\sqrt{1-4\nu\lambda\xi^2}}{2\lambda}; \quad 0<\xi<\frac{1}{2\sqrt{\lambda\nu}},$$

$$m_{3,4}=\frac{-1\pm i\sqrt{4\nu\lambda\xi^2-1}}{2\lambda}; \quad \xi>\frac{1}{2\sqrt{\lambda\nu}},$$

and

$$\beta = \sqrt{4\nu\lambda\xi^2 - 1}.$$

The particular solution is given by

$$u_{sn}^{p}(\xi, t) = \sqrt{\frac{2}{\pi}}\frac{V}{\xi}. \tag{2.61}$$

Hence the general solution is given by

$$u_{sn}(\xi, t) = \begin{cases} C_3 e^{m_3 t} + C_4 e^{m_4 t} + \sqrt{\frac{2}{\pi}}\frac{V}{\xi} & ; \quad 0 < \xi < \frac{1}{2\sqrt{\lambda\nu}} \\ \\ \exp\left(\frac{-t}{2\lambda}\right)\left[C_5 \cos\left(\frac{\beta t}{2\lambda}\right) + C_6 \sin\left(\frac{\beta t}{2\lambda}\right)\right] + \sqrt{\frac{2}{\pi}}\frac{V}{\xi} & ; \quad \xi > \frac{1}{2\sqrt{\lambda\nu}} \end{cases} \tag{2.62}$$

where C_i $(i = 3 - 6)$ are arbitrary constants to be determined.

Making use of initial conditions (2.58) into the above solution, we get

$$\begin{aligned} C_3 &= \frac{-m_4}{m_4 - m_3}\sqrt{\frac{2}{\pi}}\frac{V}{\xi}, \quad C_4 = \frac{m_3}{m_4 - m_3}\sqrt{\frac{2}{\pi}}\frac{V}{\xi}, \\ C_5 &= -\sqrt{\frac{2}{\pi}}\frac{V}{\xi} \quad , \ C_6 = -\sqrt{\frac{2}{\pi}}\frac{V}{\xi\beta}. \end{aligned}$$

Thus, the general solution (2.62) becomes

$$u_{sn}(\xi, t) = \begin{cases} \sqrt{\frac{2}{\pi}}\frac{V}{\xi}\left(1 - \frac{m_4 e^{m_3 t} - m_3 e^{m_4 t}}{m_4 - m_3}\right); \quad 0 < \xi < \frac{1}{2\sqrt{\lambda\nu}}, \\ \\ \sqrt{\frac{2}{\pi}}\frac{V}{\xi}\left[1 - \exp\left(\frac{-t}{2\lambda}\right)\left\{\cos\left(\frac{\beta t}{2\lambda}\right) + \frac{1}{\beta}\sin\left(\frac{\beta t}{2\lambda}\right)\right\}\right]; \quad \xi > \frac{1}{2\sqrt{\lambda\nu}}. \end{cases} \tag{2.63}$$

Inverting solution (2.63) by means of Fourier sine transform [17] we find that the velocity field is given by

$$
\begin{aligned}
u(y,t) &= \frac{2}{\pi}V\int\limits_{0}^{\frac{1}{2\sqrt{\lambda\nu}}}\frac{\sin(y\xi)}{\xi}d\xi-\frac{2}{\pi}V\int\limits_{0}^{\frac{1}{2\sqrt{\lambda\nu}}}\frac{m_4e^{m_3t}-m_{3N}e^{m_4t}}{m_4-m_3}\frac{\sin(y\xi)}{\xi}d\xi \\
&+\frac{2}{\pi}V\int\limits_{\frac{1}{2\sqrt{\lambda\nu}}}^{\infty}\frac{\sin(y\xi)}{\xi}d\xi-\frac{2}{\pi}V\int\limits_{\frac{1}{2\sqrt{\lambda\nu}}}^{\infty}\exp\left(\frac{-t}{2\lambda}\right)\left[\cos\left(\frac{\beta t}{2\lambda}\right)+\frac{1}{\beta}\sin\left(\frac{\beta t}{2\lambda}\right)\right] \\
&\times\frac{\sin(y\xi)}{\xi}d\xi, \qquad (2.64)
\end{aligned}
$$

or

$$
\begin{aligned}
u(y,t) &= \frac{2}{\pi}V\int\limits_{0}^{\infty}\frac{\sin(y\xi)}{\xi}d\xi-\frac{2}{\pi}V\int\limits_{0}^{\frac{1}{2\sqrt{\lambda\nu}}}\frac{m_4e^{m_3t}-m_3e^{m_4t}}{m_4-m_3}\frac{\sin(y\xi)}{\xi}d\xi \\
&-\frac{2}{\pi}V\int\limits_{\frac{1}{2\sqrt{\lambda\nu}}}^{\infty}\exp\left(\frac{-t}{2\lambda}\right)\left[\cos\left(\frac{\beta t}{2\lambda}\right)+\frac{1}{\beta}\sin\left(\frac{\beta t}{2\lambda}\right)\right]\frac{\sin(y\xi)}{\xi}d\xi. \qquad (2.65)
\end{aligned}
$$

As

$$
\int\limits_{0}^{\infty}\frac{\sin(y\xi)}{\xi}d\xi=\frac{\pi}{2}, \qquad (2.66)
$$

so Eq. (2.65) in view of Eq. (2.66) becomes

$$
\begin{aligned}
u(y,t) &= V-\frac{2}{\pi}V\int\limits_{0}^{\frac{1}{2\sqrt{\lambda\nu}}}\frac{m_4e^{m_3t}-m_3e^{m_4t}}{m_4-m_3}\frac{\sin(y\xi)}{\xi}d\xi \\
&-\frac{2}{\pi}V\int\limits_{\frac{1}{2\sqrt{\lambda\nu}}}^{\infty}\exp\left(\frac{-t}{2\lambda}\right)\left[\cos\left(\frac{\beta t}{2\lambda}\right)+\frac{1}{\beta}\sin\left(\frac{\beta t}{2\lambda}\right)\right]\frac{\sin(y\xi)}{\xi}d\xi, \qquad (2.67)
\end{aligned}
$$

giving the required expression for velocity field over an infinite plate.

To find the shear stress corresponding to flow over an infinite plate, we use the relation

$$
\tau(y,t)=\left(\frac{\mu}{\lambda}\right)\exp\left(-\frac{t}{\lambda}\right)\int_0^t\exp\left(\frac{\tau}{\lambda}\right)\frac{du}{dy}d\tau. \qquad (2.68)
$$

Putting Eq. (2.67) into Eq. (2.68) to give

$$\tau(y,t) = \frac{-2V\mu}{\lambda\pi}\exp\left(\frac{-t}{\lambda}\right)\int_0^{\frac{1}{2\sqrt{\lambda\nu}}}\cos(y\xi)\left[\int_0^t \exp\left(\frac{\tau}{\lambda}\right)\frac{m_4 e^{m_3\tau}-m_3 e^{m_4\tau}}{m_4-m_3}d\tau\right]d\xi$$
$$-\frac{2V\mu}{\lambda\pi}\exp\left(\frac{-t}{\lambda}\right)\int_{\frac{1}{2\sqrt{\lambda\nu}}}^{\infty}\cos(y\xi)\left[\int_0^t \exp\left(\frac{\tau}{2\lambda}\right)\left\{\begin{array}{c}\cos\left(\frac{\beta\tau}{2\lambda}\right)\\ +\frac{1}{\beta}\sin\left(\frac{\beta\tau}{2\lambda}\right)\end{array}\right\}d\tau\right]d\xi, \quad (2.69)$$

or

$$\tau(y,t) = \frac{-2V\mu}{\lambda\pi}\exp\left(\frac{-t}{\lambda}\right)\int_0^{\frac{1}{2\sqrt{\lambda\nu}}}\cos(y\xi)\left[\int_0^t \frac{m_4 e^{\left(m_3+\frac{1}{\lambda}\right)\tau}-m_3 e^{\left(m_4+\frac{1}{\lambda}\right)\tau}}{m_4-m_3}d\tau\right]d\xi$$
$$-\frac{2V\mu}{\lambda\pi}\exp\left(\frac{-t}{\lambda}\right)\int_{\frac{1}{2\sqrt{\lambda\nu}}}^{\infty}\cos(y\xi)\left[\begin{array}{c}\int_0^t \exp\left(\frac{\tau}{2\lambda}\right)\cos\left(\frac{\beta\tau}{2\lambda}\right)d\tau\\ +\int_0^t \exp\left(\frac{\tau}{2\lambda}\right)\frac{1}{\beta}\sin\left(\frac{\beta\tau}{2\lambda}\right)d\tau\end{array}\right]d\xi. \quad (2.70)$$

Since

$$\int_0^t \frac{m_4 e^{\left(m_3+\frac{1}{\lambda}\right)\tau}-m_3 e^{\left(m_4+\frac{1}{\lambda}\right)\tau}}{m_4-m_3}d\tau = \exp\left(\frac{t}{\lambda}\right)\frac{e^{m_4 t}-e^{m_3 t}}{m_4-m_3}, \quad (2.71)$$

and

$$\int_0^t \exp\left(\frac{\tau}{2\lambda}\right)\cos\left(\frac{\beta\tau}{2\lambda}\right)d\tau + \int_0^t \exp\left(\frac{\tau}{2\lambda}\right)\frac{1}{\beta}\sin\left(\frac{\beta\tau}{2\lambda}\right)d\tau = \frac{2}{\beta}\exp\left(\frac{-t}{2\lambda}\right)\sin\left(\frac{\beta t}{2\lambda}\right), \quad (2.72)$$

so the expression (2.70) for shear stress takes the more suitable form as

$$\tau(y,t) = \frac{-2\mu V}{\lambda\pi}\int_0^{\frac{1}{2\sqrt{\lambda\nu}}}\frac{e^{m_4 t}-e^{m_3 t}}{m_4-m_3}\cos(y\xi)\,d\xi$$
$$-\frac{4\mu V}{\lambda\pi}\exp\left(\frac{-t}{2\lambda}\right)\int_{\frac{1}{2\sqrt{\lambda\nu}}}^{\infty}\frac{1}{\beta}\sin\left(\frac{\beta t}{2\lambda}\right)\cos(y\xi)\,d\xi, \quad (2.73)$$

giving the required expression for shear stress for flow over an infinite plate.

2.7 Special case $\lambda \to 0$ (Newtonian fluid)

It is worthy pointing out that by letting $\lambda \to 0$ into Eq. (2.39), we get the similar solutions for a Newtonian fluid.

Equation (2.39) can be written as

$$\begin{aligned} u(y,z,t) &= \frac{2V}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}e^{-\mu_N y}-\frac{4V}{\pi h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N} \\ &\times\int_0^{\infty}\frac{\xi\sin(y\xi)}{\xi^2+\mu_N^2}e^{m_{1N}t}\left(\frac{1-\frac{m_{1N}}{m_{2N}}e^{(m_{2N}-m_{1N})}}{1-\frac{m_{1N}}{m_{2N}}}\right)d\xi, \end{aligned} \tag{2.74}$$

Taking limit as $\lambda \to 0$ in above equation, we get

$$\begin{aligned} u(y,z,t) &= \frac{2V}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}e^{-\mu_N y}-\frac{4V}{\pi h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N} \\ &\times\int_0^{\infty}\frac{\xi\sin(y\xi)}{\xi^2+\mu_N^2}e^{-\nu(\xi^2+\mu_N^2)t}d\xi. \end{aligned} \tag{2.75}$$

Since

$$\begin{aligned} \int_0^{\infty}\frac{\xi\sin(y\xi)}{\xi^2+b^2}e^{-a\xi^2}d\xi &= \frac{\pi}{4}e^{ab^2}\left[\begin{array}{c} e^{-by}\operatorname{erf c}\left(b\sqrt{a}-\frac{y}{2\sqrt{a}}\right) \\ -e^{by}\operatorname{erf c}\left(b\sqrt{a}+\frac{y}{2\sqrt{a}}\right)\end{array}\right]; \\ \operatorname{Re}(a) &\geq 0, \ \operatorname{Re}(b)\geq 0. \end{aligned} \tag{2.76}$$

In view of Eq. (2.76), the solution (2.75) becomes

$$\begin{aligned} u(y,z,t) &= \frac{2V}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}e^{-\mu_N y}-\frac{V}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N} \\ &\times\left[\begin{array}{c} e^{-\mu_N y}\operatorname{erf c}\left(\mu_N\sqrt{\nu t}-\frac{y}{2\sqrt{\nu t}}\right) \\ -e^{\mu_N y}\operatorname{erf c}\left(\mu_N\sqrt{\nu t}+\frac{y}{2\sqrt{\nu t}}\right)\end{array}\right], \end{aligned} \tag{2.77}$$

and this represent the exact solution for a Newtonian fluid performing the same motion.

For Newtonian fluid the expression for shear stress is

$$\tau_{xy} = \mu \frac{\partial u}{\partial y}. \tag{2.78}$$

Taking derivative of Eq. (2.75) with respect to y and substituting into Eq. (2.78) to get

$$\begin{aligned} \tau_{xy} &= \frac{-2\mu V}{h} \sum_{n=1}^{\infty} (-1)^{n+1} \cos(\mu_N z)\, e^{-\mu_N y} - \frac{4V\mu}{\pi h} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\cos(\mu_N z)}{\mu_N} \\ &\quad \times \int_0^{\infty} \frac{\xi^2 \cos(y\xi)}{\xi^2 + \mu_N^2} e^{-\nu(\xi^2+\mu_N^2)t} d\xi, \end{aligned} \tag{2.79}$$

or

$$\begin{aligned} \tau_{xy} &= \frac{-2\mu V}{h} \sum_{n=1}^{\infty} (-1)^{n+1} \cos(\mu_N z)\, e^{-\mu_N y} - \frac{4V\mu}{\pi h} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\cos(\mu_N z)}{\mu_N} \\ &\quad \times \int_0^{\infty} \frac{\xi^2 + \mu_N^2 - \mu_N^2}{\xi^2 + \mu_N^2} \cos(y\xi)\, e^{-\nu(\xi^2+\mu_N^2)t} d\xi, \end{aligned} \tag{2.80}$$

or

$$\begin{aligned} \tau_{xy} &= \frac{-2\mu V}{h} \sum_{n=1}^{\infty} (-1)^{n+1} \cos(\mu_N z)\, e^{-\mu_N y} - \frac{4V\mu}{\pi h} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\cos(\mu_N z)}{\mu_N} \\ &\quad \times \int_0^{\infty} \cos(y\xi)\, e^{-\nu(\xi^2+\mu_N^2)t} d\xi + \frac{4V\mu}{\pi h} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\cos(\mu_N z)}{\mu_N} \\ &\quad \times \int_0^{\infty} \frac{\mu_N^2}{\xi^2 + \mu_N^2} \cos(y\xi)\, e^{-\nu(\xi^2+\mu_N^2)t} d\xi. \end{aligned} \tag{2.81}$$

By using the formulae

$$\int_0^{\infty} e^{-a\xi^2} \cos(y\xi)\, d\xi = \frac{1}{2}\sqrt{\frac{\pi}{a}} \exp\left(\frac{-y^2}{4a}\right), \tag{2.82}$$

and

$$\int_0^{\infty} \frac{\cos(y\xi)}{\xi^2 + b^2} e^{-a\xi^2} d\xi = \frac{\pi}{4} \frac{e^{ab^2}}{b} \left[\begin{array}{c} e^{-by} \operatorname{erf} c\left(b\sqrt{a} - \frac{y}{2\sqrt{a}}\right) \\ +e^{by} \operatorname{erf} c\left(b\sqrt{a} + \frac{y}{2\sqrt{a}}\right) \end{array} \right], \tag{2.83}$$

we obtain from Eq. (2.81)

$$\begin{aligned}\tau_{xy} &= \frac{-2\mu V}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\cos(\mu_N z)\,e^{-\mu_N y} - \frac{2V\mu}{h\sqrt{\pi\nu t}}\exp\left(\frac{-y^2}{4\nu t}\right) \\ &\quad\times\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}e^{-\nu t\mu_N^2} + \frac{\mu V}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\cos(\mu_N z) \\ &\quad\times\left[\begin{array}{c} e^{-\mu_N y}\,\mathrm{erf}\,c\left(\mu_N\sqrt{\nu t}-\frac{y}{2\sqrt{\nu t}}\right) \\ +e^{\mu_N y}\,\mathrm{erf}\,c\left(\mu_N\sqrt{\nu t}+\frac{y}{2\sqrt{\nu t}}\right)\end{array}\right]. \end{aligned} \tag{2.84}$$

Similarly, the volume flux for Newtonian fluid is given by

$$Q(t) = \frac{4V}{h}\sum_{n=1}^{\infty}\frac{1}{\mu_N^3} - \frac{2V}{h}\sum_{n=1}^{\infty}\frac{1}{\mu_N^2}\int_0^{\infty}\left[\begin{array}{c} e^{-\mu_N y}\,\mathrm{erf}\,c\left(\mu_N\sqrt{\nu t}-\frac{y}{2\sqrt{\nu t}}\right) \\ -e^{\mu_N y}\,\mathrm{erf}\,c\left(\mu_N\sqrt{\nu t}+\frac{y}{2\sqrt{\nu t}}\right)\end{array}\right]dy. \tag{2.85}$$

2.8 Numerical results and conclusions

This chapter provides exact solutions for the unsteady flows of a Maxwell fluid between two side walls perpendicular to a plate. The motion is produced by the infinite plate that at time $t^{+}=0$ begins to slide into its plane with a constant velocity V. The obtained solutions satisfy all imposed initial and boundary conditions and in the absence of the side walls reduce to the solutions corresponding to the flow over an infinite flat plate. In the special case when $\lambda \to 0$, all solutions tend to those for Newtonian fluid performing the same motions. The steady-state solutions

$$u(y,z)=\frac{2}{h}V\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}e^{-\mu_N y}, \tag{2.86}$$

$$\tau(y,z)=\frac{-2\mu V}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\cos(\mu_N z)\,e^{-\mu_N y}, \tag{2.87}$$

and

$$Q=\frac{4V}{h}\sum_{n=1}^{\infty}\frac{1}{\mu_N^3}, \tag{2.88}$$

are the same for both types of fluids (Newtonian and non-Newtonian fluids) and also obtained as limiting cases for $t\to\infty$. The series that gives the steady velocity (2.86), is a convergent series. Unfortunately, the corresponding series that gives the shear stress (2.87) is not a convergent series. Consequently, the shear stress at the bottom wall cannot be calculated by means of solutions (2.50) and (2.84). For this reason, another expression for the steady velocity can be established.

Consider Eq. (2.21) in steady state form

$$\left[\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]u(y,z)=0, \tag{2.89}$$

and the boundary conditions

$$\left.\begin{aligned} u(0,z)=V;\quad 0\le z\le d,\\ u(y,0)=u(y,d)=0\quad y>0\\ u(y,z),\ \tfrac{\partial u(y,z)}{\partial y}\to 0\ \text{ as }\ y\to\infty \end{aligned}\right\} \tag{2.90}$$

Applying the Fourier sine transform with respect to y and having in mind the conditions (2.90), we get

$$\frac{\partial^2 u_{sn}}{\partial z^2} - \xi^2 u_{sn} = -\sqrt{\frac{2}{\pi}} V \xi.$$

Solving above equation, we find that

$$u_{sn}(\xi, z) = V\left[\sqrt{\frac{2}{\pi}}\frac{1}{\xi} - \sqrt{\frac{2}{\pi}}\frac{\cosh(\xi z)}{\xi\cosh(\xi d)}\right]. \tag{2.91}$$

Now inverting the above by means of Eq. (2.27), one obtain

$$u(y, z) = V\left[\frac{2}{\pi}\int_0^\infty \frac{\sin(y\xi)}{\xi}d\xi - \frac{2}{\pi}\int_0^\infty \frac{\cosh(\xi z)}{\xi\cosh(\xi d)}\sin(y\xi)\,d\xi\right]. \tag{2.92}$$

In view of Eq. (2.66), the above solution reduces to

$$u(y, z) = V\left[1 - \frac{2}{\pi}\int_0^\infty \frac{\cosh(\xi z)}{\xi\cosh(\xi d)}\sin(y\xi)\,d\xi\right]. \tag{2.93}$$

Thus, comparing Eq. (2.86) and (2.93), we find

$$\frac{2}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}e^{-\mu_N y} = 1 - \frac{2}{\pi}\int_0^\infty \frac{\cosh(\xi z)}{\xi\cosh(\xi d)}\sin(y\xi)\,d\xi. \tag{2.94}$$

With the help of above expression, Eqs. (2.39), (2.49), (2.77) and (2.84), can be written as

$$\begin{aligned}\frac{u(y,z,t)}{V} &= 1 - \frac{2}{\pi}\int_0^\infty \frac{\cosh(\xi z)}{\xi\cosh(\xi h)}\sin(y\xi)\,d\xi \\ &\quad - \frac{4}{\pi h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\int_0^\infty \frac{\xi\sin(y\xi)}{\xi^2+\mu_N^2} \\ &\quad \times \frac{m_{2N}e^{m_{1N}t} - m_{1N}e^{m_{2N}t}}{m_{2N}-m_{1N}}d\xi,\end{aligned} \tag{2.95}$$

$$\begin{aligned}\tau(y,z,t) &= -\frac{2\mu V}{\pi}\left[1 - \exp\left(\frac{-t}{\lambda}\right)\right]\int_0^\infty \frac{\cosh(\xi z)}{\xi\cosh(\xi h)}\cos(y\xi)\,d\xi \\ &\quad - \frac{4\mu V}{\pi h\lambda}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\int_0^\infty \frac{\xi^2\cos(y\xi)}{\xi^2+\mu_N^2}\frac{e^{m_{2N}t}-e^{m_{1N}t}}{m_{2N}-m_{1N}}d\xi,\end{aligned} \tag{2.96}$$

$$\frac{u(y,z,t)}{V} = 1-\frac{2}{\pi}\int_0^\infty \frac{\cosh(\xi z)}{\xi\cosh(\xi d)}\sin(y\xi)\,d\xi - \frac{1}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\left\{\begin{array}{l} e^{-\mu_N y}\,\mathrm{erf\,c}\left(\mu_N\sqrt{\nu t}-\frac{y}{2\sqrt{\nu t}}\right) \\ -e^{\mu_N y}\,\mathrm{erf\,c}\left(\mu_N\sqrt{\nu t}+\frac{y}{2\sqrt{\nu t}}\right)\end{array}\right\}, \quad (2.97)$$

and

$$\tau(y,z,t) = -\frac{2\mu V}{\pi}\int_0^\infty \frac{\cosh(\xi z)}{\xi\cosh(\xi h)}\cos(y\xi)\,d\xi - \frac{-2V\mu}{h\sqrt{\pi\nu t}}\exp\left(\frac{-y^2}{4\nu t}\right) \times\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}e^{-\nu t\mu_N^2} + \frac{\nu V}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\cos(\mu_N z) \times\left[\begin{array}{l} e^{-\mu_N y}\,\mathrm{erf\,c}\left(\mu_N\sqrt{\nu t}-\frac{y}{2\sqrt{\nu t}}\right) \\ +e^{\mu_N y}\,\mathrm{erf\,c}\left(\mu_N\sqrt{\nu t}+\frac{y}{2\sqrt{\nu t}}\right)\end{array}\right], \quad (2.98)$$

respectively.

We know that

$$\int_0^\infty \frac{\cosh(a\xi)}{\cosh(b\xi)}d\xi = \frac{\pi}{2b}\left(\cos\frac{\pi a}{2b}\right)^{-1}, \quad (2.99)$$

and

$$\int_0^\infty \frac{\xi^2 e^{-c(\xi^2+b^2)}}{\xi^2+b^2}d\xi = \frac{\pi}{2\sqrt{c}}i\,\mathrm{erf\,c}\left(b\sqrt{c}\right), \quad (2.100)$$

In view of Eqs. (2.99) and (2.100), the shear stress at the bottom wall can be written in the simple form

$$\tau_w(z,t) = \tau(0,z,t) = \frac{-\mu V}{h}\left(1-e^{\frac{-t}{\lambda}}\right)\frac{1}{\cos(\pi z/2h)} - \frac{4\mu V}{\pi\lambda h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\int_0^\infty \frac{\xi^2}{\xi^2+\mu_N^2}\frac{e^{m_{2N}t}-e^{m_{1N}t}}{m_{2N}-m_{1N}}d\xi, \quad (2.101)$$

and

$$\tau_w(z,t) = \frac{-\mu V}{h}\left[\frac{1}{\cos(\pi z/2h)} + 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N\sqrt{\nu t}}i\,\mathrm{erf\,c}\left(\mu_N\sqrt{\nu t}\right)\right]. \quad (2.102)$$

where $i\,\mathrm{erf\,c}\,(.)$ is the integral of the complementary error function.

Figure 2.1 shows the velocity profile $u(y,0,t)$ at middle of the channel for various values of time t. From this figure it can be seen that velocity is an increasing function of time until to reach steady state.

Tables 2.1 and 2.2 have been made to illustrate the effects of relaxation time λ on the velocity $u(y,0,t)$ and the shear stress $\tau(y,0,t) = \tau_w(z,t)$, respectively. It can be seen from tables 2.1 and 2.2 that velocity decreases by increasing λ. However, the magnitude of shear stress increases by increasing λ.

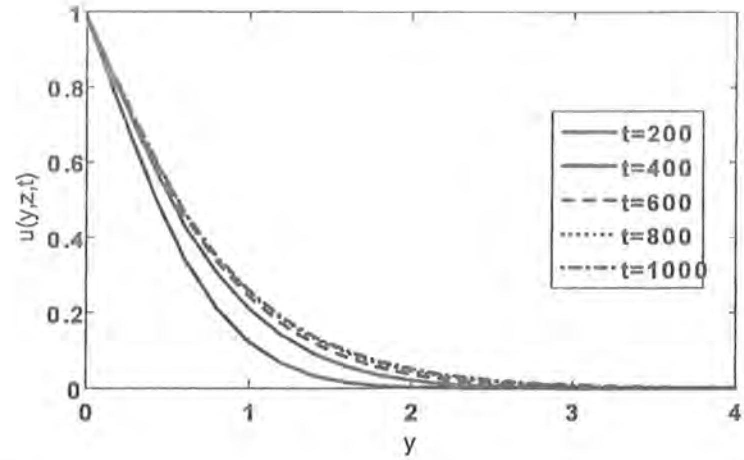


Figure 2.1: Profiles of the velocity $u(y, 0, t)$ for various values of time t when $\lambda = 0.02$ is fixed ($\mu = 1.48,\ \nu = 0.0011457$).

Table 2.1. Variation of velocity u for different values of λ. The other parameters chosen are $V = 1$, $\mu = 1.48$, $t = 15$ and $\nu = 0.0011457$.

y	$\lambda = 0$	$\lambda = 0.01$	$\lambda = 0.1$	$\lambda = 0.3$	$\lambda = 0.5$
0.0	0.968248	0.968248	0.968248	0.968248	0.968248
0.2	0.280629	0.280628	0.2805	0.280254	0.280029
0.4	0.0309616	0.39513	0.029927	0.277822	0.0255224
0.6	0.0012107	0.00120853	0.00099909	0.00061136	0.00028431
0.8	0.00001595	0.0000158595	0.00000827	0.00000080	0.00000191
1.0	0	0	0	0	0

Table 2.2: Variation of shear stress $\tau_w(z, t)$ for different values of λ. The other parameters chosen are $V = 1$, $\mu = 1.48$, $t = 15$ and $\nu = 0.0011457$.

z	$\lambda = 0$	$\lambda = 0.001$	$\lambda = 0.1$	$\lambda = 0.3$	$\lambda = 0.5$
−1.0	0	0	0	0	0
−0.8	−2.42501	−2.42503	−2.42755	−2.43269	−2.43802
−0.6	−3.88411	−3.8842	−3.89269	−3.91034	−3.9288
−0.4	−4.54076	−4.54087	−4.55131	−4.57296	−4.59554
−0.2	−4.81335	−4.81345	−4.82404	−4.84594	−4.86868
0.0	−4.88951	−4.88961	−4.9002	−4.9221	−4.94485
0.2	−4.81335	−4.81345	−4.82404	−4.84594	−4.86868
0.4	−4.54076	−4.54087	−4.55131	−4.57296	−4.59554
0.6	−3.88411	−3.8842	−3.89269	−3.91034	−3.9288
0.8	−2.42501	−2.42503	−2.42755	−2.43269	−2.43802
1.0	0	0	0	0	0

Chapter 3

Unsteady flows of a MHD Maxwell fluid between two side walls perpendicular to a plate

This chapter presents the unsteady flows of a magnetohydrodynamic (MHD) Maxwell fluid through porous space. An external uniform magnetic field normal to the plate is applied. Based on modified Darcy's law of Maxwell fluid, the governing equations are modelled. Three characteristic examples which are flow due to impulsive motion of plate, flow due to constantly accelerating plate and flow due to variable accelerating plate between two side walls perpendicular to the plate are considered. Closed form solutions for the velocity field, the tangential stress and the volume flow rate are developed by applying the Fourier sine transforms. The corresponding solutions for hydrodynamic flows as well as those in the absence of porous medium appear as limiting cases of the presented solutions. In the absence of side walls, all solutions that have been determined reduce to those corresponding to the motion over an infinite plate. The influence of various parameters of interest on the velocity field and the resulting tangential stress at the bottom wall is also shown and discussed through several graphs.

3.1 Governing equations

The equations which govern the unsteady incompressible flow of a MHD fluid through porous medium include the continuity and the momentum equation, as given below

$$\operatorname{div}\mathbf{V} = 0, \tag{3.1}$$

$$\rho\frac{d\mathbf{V}}{dt} = \operatorname{div}\mathbf{T} + \mathbf{J}\times\mathbf{B} + \mathbf{R}, \tag{3.2}$$

where $\mathbf{V}$ is the velocity vector, ρ the density of the fluid, $\mathbf{T}$ the Cauchy stress tensor, $\mathbf{J}$ the current density, $\mathbf{B}$ $(=\mathbf{B}_0 + \mathbf{b})$ the total magnetic field with $\mathbf{B}_0$ and $\mathbf{b}$ as the applied and the induced magnetic fields, respectively, $\mathbf{R}$ the Darcy's resistance and d/dt the material time derivative.

For electrically conducting fluid the Maxwell's equations and generalized Ohm's law are

$$\operatorname{div}\mathbf{B} = 0, \quad \operatorname{curl}\mathbf{B} = \mu_e\mathbf{J}, \quad \operatorname{curl}\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{3.3}$$

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{V}\times\mathbf{B}), \tag{3.4}$$

in which $\mathbf{E}$ is the electric field, μ_e the magnetic permeability and σ the electric conductivity. In the present analysis, the external electric field and the induced magnetic field are assumed to be negligible [13, 18]. Thus, the solution of Eq. (3.4) for $\mathbf{J}$ yields

$$\mathbf{J}\times\mathbf{B} = -\sigma B_0^2\mathbf{V}. \tag{3.5}$$

The constitutive equations for a Maxwell fluid are

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad \mathbf{S} + \lambda\frac{D\mathbf{S}}{Dt} = \mu\mathbf{A}, \tag{3.6}$$

where p is the pressure, $\mathbf{I}$ the identity tensor, $\mathbf{S}$ the extra stress tensor, $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ the first Rivlin-Ericksen tensor, $\mathbf{L}$ the velocity gradient, μ is the dynamic viscosity of the fluid, λ the relaxation time and

$$\frac{D\mathbf{S}}{Dt} = \frac{d\mathbf{S}}{dt} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T. \tag{3.7}$$

Through the use of the reference [19], the Darcy's resistance for a Maxwell fluid satisfies the following relation

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\mathbf{R}=-\frac{\mu\phi}{k}\mathbf{V}, \tag{3.8}$$

where $k\,(>0)$ and $\phi\ (0<\phi<1)$ are the permeability and porosity, respectively.

In the following we shall assume a velocity field and an extra stress of the form

$$\mathbf{V}=u\left(y,z,t\right)\mathbf{i},\quad \mathbf{S}=\mathbf{S}\left(y,z,t\right), \tag{3.9}$$

where $\mathbf{i}$ is the unit vector in the x-direction of the Cartesian coordinate system. For this velocity field the constraint of incompressibility Eq. (3.1) is automatically satisfied.

Substituting Eq. (3.9) into Eq. $(3.6)_2$ and taking into account the initial condition

$$\mathbf{S}\left(y,z,0\right)=\frac{\partial\mathbf{S}\left(y,z,0\right)}{\partial t}=\mathbf{0}, \tag{3.10}$$

we obtain S_{yy}=S_{yz}=S_{zz}=0 [20] and

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\tau_1=\mu\frac{\partial u}{\partial y},\quad \left(1+\lambda\frac{\partial}{\partial t}\right)\tau_2=\mu\frac{\partial u}{\partial z}, \tag{3.11}$$

where $\tau_1=S_{xy}$ and $\tau_2=S_{xz}$ are tangential stresses.

In view of Eqs. (3.5), (3.9) and (3.10), the equation of motion reduces to the following governing equation

$$\begin{aligned}\rho\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial u\left(y,z,t\right)}{\partial t} &= -\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial p}{\partial x}+\mu\left[\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]u\left(y,z,t\right)\\ &\quad -\left(1+\lambda\frac{\partial}{\partial t}\right)\sigma B_0^2 u\left(y,z,t\right)-\frac{\mu\phi}{k}u\left(y,z,t\right). \end{aligned} \tag{3.12}$$

In the next sections, we will solve Eq. (3.12) for three characteristic flow problems.

3.2 Flow due to impulsive motion of plate

Let us consider an incompressible electrically conducting Maxwell fluid occupying the porous space over a flat plate perpendicular to the y-axis and between two side walls perpendicular to the plate. The side walls are extended to infinity in the $x-$ and $y-$directions and are located at $z = 0$ and $z = d$. Initially, the fluid is at rest and after time $t = 0$ it is suddenly set into motion by translating the bottom plate in its plane with a constant velocity V. Consequently, the governing equation (3.12), in the absence of a pressure gradient in the flow direction, reduces to

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial u\left(y,z,t\right)}{\partial t}=\nu\left[\frac{\partial^{2}}{\partial y^{2}}+\frac{\partial^{2}}{\partial z^{2}}\right]u\left(y,z,t\right)-\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\sigma B_{0}^{2}}{\rho}u\left(y,z,t\right)-\frac{\nu\phi}{k}u\left(y,z,t\right), \tag{3.13}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid.

The problem now reduces to solve Eq. (3.13) subject to the following initial and boundary conditions

$$u\left(y,z,0\right)=\frac{\partial u\left(y,z,0\right)}{\partial t}=0 \text{ for } y>0 \text{ and } 0\leq z\leq d, \tag{3.14}$$

$$u\left(0,z,t\right)=V \text{ for } t>0 \text{ and } 0<z<d, \tag{3.15}$$

$$u\left(y,0,t\right)=u\left(y,d,t\right)=0 \text{ for } y>0 \text{ and } t\geq 0, \tag{3.16}$$

and

$$u\left(y,z,t\right),\frac{\partial u\left(y,z,t\right)}{\partial y}\to 0 \text{ as } y\to\infty \text{ and } t\geq 0, \tag{3.17}$$

3.2.1 Solution of the problem

Calculation of velocity field

In order to solve the above problem we shall use the Fourier sine transform with respect to the spatial variable y and the finite Fourier sine transform with respect to z. Consequently, multiplying Eq. (3.13) by $\sqrt{2/\pi}\sin(y\xi)\sin(\lambda_n z)$ with $\lambda_n = n\pi/d$, integrating the result with respect to y from 0 to ∞ and with respect to z from 0 to d and taking into account the boundary conditions (3.15) – (3.17), we obtain

$$\lambda\frac{\partial^{2}u_{sn}\left(\xi,t\right)}{\partial t^{2}}+\left(1+\lambda\frac{\sigma B_{0}^{2}}{\rho}\right)\frac{\partial u_{sn}\left(\xi,t\right)}{\partial t}+\nu(\xi^{2}+\lambda_{n}^{2}+a_{k})u_{sn}\left(\xi,t\right)=\nu\sqrt{\frac{2}{\pi}}\xi V\left[\frac{1-(-1)^{n}}{\lambda_{n}}\right], \tag{3.18}$$

where

$$a_k = \frac{\phi}{k} + \frac{\sigma B_0^2}{\mu},$$

and the double Fourier sine transforms

$$u_{sn}(\xi, t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \int_0^{d} u(y, z, t) \sin(y\xi) \sin(\lambda_n z)\, dz dy \; ; \qquad n = 1, 2, 3 \cdots \tag{3.19}$$

of $u(y, z, t)$ have to satisfy the initial conditions

$$u_{sn}(\xi, 0) = \frac{\partial u_{sn}(\xi, 0)}{\partial t} = 0 \;\text{ for }\; \xi > 0; \qquad n = 1, 2, 3 \cdots \tag{3.20}$$

The solution of homogenous part of Eq. (3.18) is given by

$$u_{sn}^{c}(\xi, t) = A_1 e^{r_{1n} t} + A_2 e^{r_{2n} t}, \tag{3.21}$$

whereas particular solution is given by

$$u_{sn}^{p}(\xi, t) = \frac{\nu \xi V \sqrt{2/\pi} \left[\frac{1-(-1)^n}{\lambda_n}\right]}{\lambda D^2 + \left(1 - \frac{\lambda \sigma \beta^2}{\rho}\right) D + \nu \left(\xi^2 + \lambda_n^2 + a_k\right)}, \tag{3.22}$$

$$u_{sn}^{p}(\xi, t) = \frac{\xi V \sqrt{2/\pi} \left[\frac{1-(-1)^n}{\lambda_n}\right]}{\left(\xi^2 + \lambda_n^2 + a_k\right)}, \tag{3.23}$$

with

$$r_{1n,2n} = \frac{-1 \pm \sqrt{\left(1 - \frac{\lambda \sigma \beta^2}{\rho}\right)^2 - 4\nu\lambda \left(\xi^2 + \lambda_n^2 + a_k\right)}}{2\lambda}. \tag{3.24}$$

Hence the complete solution is then given by

$$u_{sn}(\xi, t) = A_1 e^{r_{1n} t} + A_2 e^{r_{2n} t} + \frac{\xi V \sqrt{2/\pi} \left[\frac{1-(-1)^n}{\lambda_n}\right]}{\left(\xi^2 + \lambda_n^2 + a_k\right)}, \tag{3.25}$$

where A_1 and A_2 are arbitrary constants.

Using Eq. (3.20) we find the values of constants A_1 and A_2 to be

$$A_1 = \frac{-r_{2n}}{r_{2n} - r_{1n}} \sqrt{\frac{2}{\pi}} V \xi \frac{\left[1 - (-1)^n\right]}{\lambda_n \left(\xi^2 + \lambda_n^2 + a_k\right)}, \tag{3.26}$$

$$A_2 = \frac{r_{1n}}{r_{2n} - r_{1n}} \sqrt{\frac{2}{\pi}} V \xi \frac{\left[1 - (-1)^n\right]}{\lambda_n \left(\xi^2 + \lambda_n^2 + a_k\right)}. \tag{3.27}$$

Now putting the values of A_1 and A_2 into Eq. (3.25), we reach at the following expression

$$u_{sn}(\xi, t) = \sqrt{\frac{2}{\pi}} V \xi \frac{\left[1 - (-1)^n\right]}{\lambda_n \left(\xi^2 + \lambda_n^2 + a_k\right)} \left[\frac{-r_{2n}}{r_{2n} - r_{1n}} e^{r_{1n} t} + \frac{r_{1n}}{r_{2n} - r_{1n}} e^{r_{2n} t} + 1\right], \tag{3.28}$$

or the above can be written as

$$u_{sn}(\xi, t) = \sqrt{\frac{2}{\pi}} V \xi \frac{\left[1 - (-1)^n\right]}{\lambda_n \left(\xi^2 + \lambda_n^2 + a_k\right)} \left[1 - \frac{r_{2n} e^{r_{1n} t} - r_{1n} e^{r_{2n} t}}{r_{2n} - r_{1n}}\right]. \tag{3.29}$$

Inverting the result (3.29) by means of the Fourier sine formulae [17], and having in mind Eq. (2.27) we get the following expression for the velocity field

$$\begin{aligned} u(y, z, t) &= \frac{2}{d} V \sum_{n=1}^{\infty} \left[1 - (-1)^n\right] \frac{\sin(\lambda_n z)}{\lambda_n} \exp\left[-\sqrt{(\lambda_N^2 + a_k)} y\right] - \frac{4V}{\pi d} \sum_{n=1}^{\infty} \left[1 - (-1)^n\right] \frac{\sin(\lambda_n z)}{\lambda_n} \\ &\times \int_0^{\infty} \frac{\xi \sin(y\xi)}{\xi^2 + \lambda_n^2 + a_k} \frac{r_{2n} e^{r_{1n} t} - r_{1n} e^{r_{2n} t}}{r_{2n} - r_{1n}} d\xi, \end{aligned} \tag{3.30}$$

or

$$\begin{aligned} u(y, z, t) &= \frac{4}{d} V \sum_{n=1}^{\infty} \frac{\sin(\lambda_N z)}{\lambda_N} \exp\left[-\sqrt{(\lambda_N^2 + a_k)} y\right] - \frac{8V}{\pi d} \sum_{n=1}^{\infty} \frac{\sin(\lambda_N z)}{\lambda_N} \\ &\times \int_0^{\infty} \frac{\xi \sin(y\xi)}{\xi^2 + \lambda_N^2 + a_k} \frac{r_{2N} e^{r_{1N} t} - r_{1N} e^{r_{2N} t}}{r_{2N} - r_{1N}} d\xi, \end{aligned} \tag{3.31}$$

where $N = 2n - 1$.

For later use, let us make $d = 2h$ and change the origin of the coordinate system at the middle of the channel. Consequently by putting $z = z^* + h$ and dropping the asterisks notation, we get the next expression for the velocity under the form

$$u(y,z,t) = \frac{2V}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\exp\left[-\sqrt{(\mu_N^2+a_k)}y\right] - \frac{4V}{\pi h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N} \times \int_0^{\infty}\frac{\xi\sin(y\xi)}{\xi^2+\mu_N^2+a_k}\frac{r_{2N}e^{r_{1N}t}-r_{1N}e^{r_{2N}t}}{r_{2N}-r_{1N}}d\xi, \tag{3.32}$$

in which

$$\mu_N = (2N-1)\pi/2h,$$

$$r_{1N,2N} = \frac{-\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)\pm\sqrt{\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)^2-4\lambda\nu\left(\xi^2+\mu_N^2+a_k\right)}}{2\lambda}.$$

Direct computation reveals that $u(y,z,t)$ given by Eq. (3.32) satisfies the linear partial differential Eq. (3.13), the initial condition (3.14) and the boundary conditions (3.15) and (3.16).

When t goes to infinity, $u(y,z,t)$ tends to steady state solution

$$u(y,z) = u(y,z,\infty) = \frac{2V}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\exp\left[-\sqrt{(\mu_N^2+a_k)}y\right]. \tag{3.33}$$

Calculation of the shear stress

To obtain the expression for the shear stress at the bottom wall,*i.e.* $\tau_w(z,t) = \tau_1(0,z,t) = S_{xy}(0,z,t)$, we use the relation (2.44). From Eqs. (2.44) and (3.32) one can immediately obtain the shear stress

Now from Eq. (3.32), we obtain

$$\frac{\partial u}{\partial y} = -\frac{2V}{h}\sum_{n=1}^{\infty}\sqrt{\mu_N^2+a_k}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\exp\left[-\sqrt{(\mu_N^2+a_k)}y\right] - \frac{4V}{\pi h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\int_0^{\infty}\frac{\xi^2\cos(y\xi)}{\xi^2+\mu_N^2+a_k}\frac{r_{2N}e^{r_{1N}t}-r_{1N}e^{r_{2N}t}}{r_{2N}-r_{1N}}d\xi. \tag{3.34}$$

Substituting Eq. (3.34) into Eq. (2.44), we have

$$\begin{aligned}\tau_1(y,z,t) &= \frac{-2V\mu}{\lambda h}\exp\left(\frac{-t}{\lambda}\right)\sum_{n=1}^{\infty}\sqrt{\mu_N^2+a_k}\,(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\exp\left[-\sqrt{(\mu_N^2+a_k)}y\right]\int_0^t\exp\left[\frac{\tau}{\lambda}\right]d\tau \\ &+\frac{4V\mu}{\lambda\pi h}\exp\left(\frac{-t}{\lambda}\right)\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\int_0^{\infty}\frac{\xi\sin(y\xi)}{\xi^2+\mu_N^2+a_k} \\ &\times\left[\int_0^t\exp\left(\frac{\tau}{\lambda}\right)\frac{r_{2N}e^{r_{1N}\tau}-r_{1N}e^{r_{2N}\tau}}{r_{2N}-r_{1N}}d\tau\right]d\xi, \qquad (3.35)\end{aligned}$$

or

$$\begin{aligned}\tau_1(y,z,t) &= \frac{-2\mu V}{h}\left[1-\exp\left(\frac{-t}{\lambda}\right)\right]\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\left(\sqrt{\mu_N^2+a_k}\right)\exp\left[-\sqrt{(\mu_N^2+a_k)}y\right] \\ &+\frac{4\mu V}{\lambda\pi h}\exp\left(\frac{-t}{\lambda}\right)\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\int_0^{\infty}\frac{\xi^2\cos(y\xi)}{\xi^2+\mu_N^2+a_k} \\ &\times\left[\frac{r_{2N}r_{4N}e^{r_{3N}t}-r_{1N}r_{3N}e^{r_{4N}t}}{r_{3N}r_{4N}(r_{2N}-r_{1N})}-\frac{r_{2N}r_{4N}-r_{1N}r_{3N}}{r_{3N}r_{4N}(r_{2N}-r_{1N})}\right]d\xi, \qquad (3.36)\end{aligned}$$

where

$$\begin{aligned}r_{3N} &= r_{1N}+1/\lambda, \quad r_{4N}=r_{2N}+1/\lambda \\ r_{3N}.r_{4N} &= \frac{\nu\left(\xi^2+\mu_N^2+\phi/k\right)}{\lambda}.\end{aligned}$$

The shear stress at the bottom wall $\tau_w(z,t)=\tau(0,z,t)$ is then given by

$$\begin{aligned}\tau_1(y,z,t) &= \frac{-2\mu V}{h}\left[1-\exp\left(\frac{-t}{\lambda}\right)\right]\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\left(\sqrt{\mu_N^2+a_k}\right)\exp\left[-\sqrt{(\mu_N^2+a_k)}y\right] \\ &+\frac{4\mu V}{\lambda\pi h}\exp\left(\frac{-t}{\lambda}\right)\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\int_0^{\infty}\frac{\xi^2\cos(y\xi)}{\xi^2+\mu_N^2+a_k} \\ &\times\left[\frac{r_{2N}r_{4N}e^{r_{3N}t}-r_{1N}r_{3N}e^{r_{4N}t}}{r_{3N}r_{4N}(r_{2N}-r_{1N})}-\frac{r_{2N}r_{4N}-r_{1N}r_{3N}}{r_{3N}r_{4N}(r_{2N}-r_{1N})}\right]d\xi. \qquad (3.37)\end{aligned}$$

As regards $\tau_2(y,z,t)$, giving the shear stress on the side walls, it can be immediately obtained from Eqs. (3.11) and (3.13).

Calculation of volume flux

The volume flux $Q(t)$ across a plane normal to flow is given by Eq. (2.51). Putting $u(y,z,t)$ from Eq. (3.32) into Eq. (2.51), one find

$$\begin{aligned}Q(t) &= \int_0^\infty \exp\left[-\sqrt{(\mu_N^2+a_k)}y\right]dy\int_{-h}^{h}\frac{2V}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}dz \\ &-\int_{-h}^{h}\frac{4V}{\pi h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}dz\int_0^\infty\int_0^\infty\frac{\xi\sin(y\xi)}{\xi^2+\mu_N^2+a_k} \\ &\times\frac{r_{2N}e^{r_{1N}t}-r_{1N}e^{r_{2N}t}}{r_{2N}-r_{1N}}dyd\xi, \end{aligned} \tag{3.38}$$

or the above expression for the volume flux can be written in more simple form as

$$\begin{aligned}Q(t) &= \frac{4V}{h}\sum_{n=1}^{\infty}\frac{1}{\mu_N^2\sqrt{\mu_N^2+a_k}}-\frac{8V}{\pi h}\sum_{n=1}^{\infty}\frac{1}{\mu_N^2}\int_0^\infty\int_0^\infty\frac{\xi\sin(y\xi)}{\xi^2+\mu_N^2+a_k} \\ &\times\frac{r_{2N}e^{r_{1N}t}-r_{1N}e^{r_{2N}t}}{r_{2N}-r_{1N}}dyd\xi. \end{aligned} \tag{3.39}$$

3.3 Flow due to constantly accelerating plate

Here, we consider a flow problem for which an incompressible electrically conducting fluid occupying the porous space between two parallel walls perpendicular to a flat plate is initially at rest. After time $t=0$, the plate begins to slide with a velocity At in the $x-$direction and induces the motion into the fluid. The governing equations, the initial and a part of boundary conditions are the same. Instead of the boundary condition (3.15) we use the boundary condition

$$u(0,z,t)=At;\quad t>0 \text{ and } 0<z<d. \tag{3.40}$$

Employing the methodology of the previous section, the expressions for the velocity field, the shear stress and the volume flux are, respectively, given by

$$\begin{aligned}
u(y,z,t) &= \frac{2At}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\exp\left[-\sqrt{(\mu_N^2+a_k)}y\right]\frac{\cos(\mu_N z)}{\mu_N}-\frac{4A}{\nu\pi h}\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)\\
&\times\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\int_0^{\infty}\left(1-\frac{r_{2N}e^{r_{1N}t}-r_{1N}e^{r_{2N}t}}{r_{2N}-r_{1N}}\right)\frac{\xi\sin(y\xi)}{(\xi^2+\mu_N^2+a_k)^2}d\xi\\
&+\frac{4A}{\pi h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\int_0^{\infty}\frac{e^{r_{1N}t}-e^{r_{2N}t}}{r_{2N}-r_{1N}}\frac{\xi\sin(y\xi)}{\xi^2+\mu_N^2+a_k}d\xi, \qquad (3.41)
\end{aligned}$$

$$\begin{aligned}
\tau_1(y,z,t) &= \frac{-2\mu A}{h}\left[t-\lambda\left\{1-\exp\left(\frac{-t}{\lambda}\right)\right\})\right]\sum_{n=1}^{\infty}(-1)^{n+1}\left(\sqrt{\mu_N^2+a_k}\right)\frac{\cos(\mu_N z)}{\mu_N}\\
&\times\exp\left[-\sqrt{(\mu_N^2+a_k)}y\right]-\frac{4\rho A}{\pi h}\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)\left[1-\exp\left(\frac{-t}{\lambda}\right)\right]\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\\
&\times\int_0^{\infty}\frac{\xi^2\cos(y\xi)}{(\xi^2+\mu_N^2+a_k)^2}d\xi+\frac{4\rho A}{\nu\pi h}\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\\
&\times\int_0^{\infty}\left[\begin{array}{c}\frac{r_{2N}r_{4N}e^{r_{1N}t}-r_{1N}r_{3N}e^{r_{2N}t}}{r_{2N}-r_{1N}}\\ \\ -\left(\frac{r_{2N}r_{4N}-r_{1N}r_{3N}}{r_{2N}-r_{1N}}\right)\exp\left(\frac{-t}{\lambda}\right)\end{array}\right]\frac{\xi^2\cos(y\xi)}{(\xi^2+\mu_N^2+\phi/k)(\xi^2+\mu_N^2+a_k)^2}d\xi\\
&+\frac{4\rho A}{\pi h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\int_0^{\infty}\left[\frac{r_{4N}e^{r_{1N}t}-r_{3N}e^{r_{2N}t}-\exp\left(\frac{-t}{\lambda}\right)}{r_{2N}-r_{1N}}\right]\\
&\times\frac{\xi^2\cos(y\xi)}{(\xi^2+\mu_N^2+\phi/k)(\xi^2+\mu_N^2+a_k)}d\xi, \qquad (3.42)
\end{aligned}$$

and

$$\begin{aligned}
Q(t) &= \frac{4At}{h}\sum_{n=1}^{\infty}\frac{1}{\mu_N^2\sqrt{\mu_N^2+a_k}}-\frac{8A}{\nu\pi h}\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)\\
&\times\sum_{n=1}^{\infty}\frac{1}{\mu_N^2}\int_0^{\infty}\int_0^{\infty}\left(1-\frac{r_{2N}e^{r_{1N}t}-r_{1N}e^{r_{2N}t}}{r_{2N}-r_{1N}}\right)\frac{\xi\sin(y\xi)}{(\xi^2+\mu_N^2+a_k)^2}dyd\xi\\
&+\frac{8A}{\pi h}\sum_{n=1}^{\infty}\frac{1}{\mu_N^2}\int_0^{\infty}\int_0^{\infty}\left(\frac{e^{r_{1N}t}-e^{r_{2N}t}}{r_{2N}-r_{1N}}\right)\frac{\xi\sin(y\xi)}{\xi^2+\mu_N^2+a_k}dyd\xi. \qquad (3.43)
\end{aligned}$$

3.4 Flow due to variable accelerating plate

In this section, we consider the flow induced by variable accelerating plate. Thus, the governing equations, the initial conditions and a part of boundary conditions are the same. For variable acceleration, instead of the condition (3.40), we consider the condition

$$u(0,z,t) = Bt^2; \quad t > 0 \text{ and } 0 < z < d. \tag{3.44}$$

Adopting a similar procedure as before, we find the corresponding expressions for the velocity field, the shear stress and the volume flux under the forms

$$\begin{aligned}
u(y,z,t) &= \frac{2Bt^2}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos\left(\mu_N z\right)}{\mu_N}\exp\left[-\sqrt{(\mu_N^2+a_k)}y\right] \\
&- \frac{8B\lambda}{\pi h\nu}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos\left(\mu_N z\right)}{\mu_N}\int_0^{\infty}\left(1-\frac{r_{2N}e^{r_{1N}t}-r_{1N}e^{r_{2N}t}}{r_{2N}-r_{1N}}\right)\frac{\xi\sin\left(y\xi\right)}{\left(\xi^2+\mu_N^2+a_k\right)^2}d\xi \\
&- \frac{8B}{\pi h\nu}\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos\left(\mu_N z\right)}{\mu_N}\int_0^{\infty}\left(t-\frac{e^{r_{2N}t}-e^{r_{1N}t}}{r_{2N}-r_{1N}}\right) \\
&\times \frac{\xi\sin\left(y\xi\right)}{\left(\xi^2+\mu_N^2+a_k\right)^2}d\xi + \frac{8B}{\pi h\nu^2}\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)^2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos\left(\mu_N z\right)}{\mu_N} \\
&\times \int_0^{\infty}\left(1-\frac{r_{2N}e^{r_{1N}t}-r_{1N}e^{r_{2N}t}}{r_{2N}-r_{1N}}\right)\frac{\xi\sin\left(y\xi\right)}{\left(\xi^2+\mu_N^2+a_k\right)^3}d\xi,
\end{aligned} \tag{3.45}$$

$$
\begin{aligned}
\tau_1(y,z,t) &= \frac{-2\mu B}{h}\left[t^2-2\lambda\left(t-\lambda\left\{1-\exp\left(\frac{-t}{\lambda}\right)\right\}\right)\right] \\
&\times\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos\left(\mu_N z\right)}{\mu_N}\left(\sqrt{\mu_N^2+a_k}\right)\exp\left[-\sqrt{(\mu_N^2+a_k)}y\right] \\
&-\frac{8\rho B}{\pi h}\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)\left[t-\lambda\left\{1-\exp\left(\frac{-t}{\lambda}\right)\right\}\right]\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos\left(\mu_N z\right)}{\mu_N} \\
&\times\int_0^\infty\frac{\xi^2\cos(y\xi)}{\left(\xi^2+\mu_N^2+a_k\right)^2}d\xi+\frac{8\lambda\rho B}{\pi h}\left[1-\exp\left(\frac{-t}{\lambda}\right)\right]\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos\left(\mu_N z\right)}{\mu_N} \\
&\times\int_0^\infty\frac{\xi^2\cos(y\xi)}{\left(\xi^2+\mu_N^2+a_k\right)^2}d\xi+\frac{8\lambda\rho B}{\nu\pi h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos\left(\mu_N z\right)}{\mu_N} \\
&\times\int_0^\infty\left[\frac{r_{2N}r_{4N}e^{r_{1N}t}-r_{1N}r_{3N}e^{r_{2N}t}}{r_{2N}-r_{1N}}-\frac{r_{2N}r_{4N}-r_{1N}r_{3N}}{r_{2N}-r_{1N}}\exp\left(\frac{-t}{\lambda}\right)\right] \\
&\times\frac{\xi^2\cos(y\xi)}{\left(\xi^2+\mu_N^2+a_k\right)^2\left(\xi^2+\mu_N^2+\phi/k\right)}d\xi+\frac{8\rho B}{\nu\pi h}\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right) \\
&\times\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos\left(\mu_N z\right)}{\mu_N}\int_0^\infty\left[\frac{r_{3N}e^{r_{2N}t}-r_{4N}e^{r_{1N}t}}{r_{2N}-r_{1N}}+\exp\left(\frac{-t}{\lambda}\right)\right] \\
&\times\frac{\xi^2\cos(y\xi)}{\left(\xi^2+\mu_N^2+a_k\right)^2\left(\xi^2+\mu_N^2+\phi/k\right)}d\xi-\frac{8\rho B}{\nu^2\pi h}\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)^2\sum_{n=1}^{\infty}(-1)^{n+1} \\
&\times\frac{\cos\left(\mu_N z\right)}{\mu_N}\int_0^\infty\left[\frac{r_{2N}r_{4N}e^{r_{1N}t}-r_{1N}r_{3N}e^{r_{2N}t}}{r_{2N}-r_{1N}}-\frac{r_{2N}r_{4N}-r_{1N}r_{3N}}{r_{2N}-r_{1N}}\exp\left(\frac{-t}{\lambda}\right)\right] \\
&\times\frac{\xi^2\cos(y\xi)}{\left(\xi^2+\mu_N^2+a_k\right)^3\left(\xi^2+\mu_N^2+\phi/k\right)}d\xi, \qquad (3.46)
\end{aligned}
$$

and

$$
\begin{aligned}
Q(t) &= \frac{4Bt^2}{h}\sum_{n=1}^{\infty}\frac{1}{\mu_N^2\sqrt{\mu_N^2+a_k}}-\frac{16B\lambda}{\nu\pi h}\sum_{n=1}^{\infty}\frac{1}{\mu_N^2}\int_0^\infty\int_0^\infty\left(1-\frac{r_{2N}e^{r_{1N}t}-r_{1N}e^{r_{2N}t}}{r_{2N}-r_{1N}}\right) \\
&\times\frac{\xi\sin(y\xi)}{\left(\xi^2+\mu_N^2+a_k\right)^2}dyd\xi-\frac{16B}{\pi h\nu}\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)\sum_{n=1}^{\infty}\frac{1}{\mu_N^2}\int_0^\infty\int_0^\infty\left(t-\frac{e^{r_{2N}t}-e^{r_{1N}t}}{r_{2N}-r_{1N}}\right) \\
&\times\frac{\xi\sin(y\xi)}{\left(\xi^2+\mu_N^2+a_k\right)^2}dyd\xi+\frac{16B}{\pi h\nu^2}\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)^2\sum_{n=1}^{\infty}\frac{1}{\mu_N^2}\int_0^\infty\int_0^\infty\left(t-\frac{r_{2N}e^{r_{1N}t}-r_{1N}e^{r_{2N}t}}{r_{2N}-r_{1N}}\right) \\
&\times\frac{\xi\sin(y\xi)}{\left(\xi^2+\mu_N^2+a_k\right)^3}dyd\xi. \qquad (3.47)
\end{aligned}
$$

3.5 Limiting case $h \to \infty$ (flow over an infinite plate)

In this section, we consider the limiting case when h goes to infinity, that is, in the absence of side walls. Applying the limit when $h \to \infty$, the solutions corresponding to the motion over an infinite flat plate in case of impulsive motion of the plate become

$$u(y,t) = V\exp[-\sqrt{a_k}y] - \frac{2V}{\pi}\int_0^\infty \frac{\xi\sin(y\xi)}{\xi^2+a_k}\frac{r_6e^{r_5t}-r_5e^{r_6t}}{r_6-r_5}d\xi, \tag{3.48}$$

$$\tau_1(y,t) = -2\mu V\sqrt{a_k}\exp[-\sqrt{a_k}y]\left[1-\exp\left(\frac{-t}{\lambda}\right)\right] + \frac{4\mu V}{\lambda\pi}\exp\left(\frac{-t}{\lambda}\right)\int_0^\infty \frac{\xi^2\cos(y\xi)}{\xi^2+a_k}\left[\begin{array}{c}\frac{r_6r_8e^{r_7t}-r_5r_7e^{r_8t}}{r_7r_8(r_6-r_5)} \\ \\ -\frac{r_6r_8-r_5r_7}{r_7r_8(r_6-r_5)}\end{array}\right]d\xi. \tag{3.49}$$

where

$$r_{5,6} = \frac{-\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)\pm\sqrt{\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)^2-4\lambda\nu\left(\xi^2+a_k\right)}}{2\lambda},$$

$$r_7 = r_5+1/\lambda, \quad r_8 = r_6+1/\lambda,$$
$$r_7.r_8 = \frac{\nu\left(\xi^2+\frac{\phi}{k}\right)}{\lambda}.$$

For the flow induced by a constantly accelerating plate, the corresponding expressions for the velocity field and shear stress are

$$u(y,t) = At\exp[-\sqrt{a_k}y] - \frac{2A}{\nu\pi}\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right) \times\int_0^\infty\left(1-\frac{r_5e^{r_6t}-r_6e^{r_5t}}{r_5-r_6}\right)\frac{\xi\sin(y\xi)}{\left(\xi^2+a_k^2\right)^2}d\xi + \frac{2A}{\pi}\int_0^\infty\left(\frac{e^{r_6t}-e^{r_5t}}{r_5-r_6}\right)\frac{\xi\sin(y\xi)}{\xi^2+a_k}d\xi, \tag{3.50}$$

$$
\begin{aligned}
\tau_1(y,t) &= -\mu A\sqrt{a_k}\exp[-\sqrt{a_k}y]\left[t-\lambda(1-e^{\frac{-t}{\lambda}})\right] \\
&-\frac{2\rho A}{\pi}\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)\left[1-\exp\left(\frac{-t}{\lambda}\right)\right]\int_0^\infty \frac{\xi^2\cos(y\xi)}{(\xi^2+a_k)^2}d\xi \\
&+\frac{2\rho A}{\nu\pi}\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)\int_0^\infty \begin{bmatrix} \frac{r_6 r_8 e^{r_5 t}-r_5 r_7 e^{r_6 t}}{r_6-r_5} \\ \\ -\left(\frac{r_5 r_7-r_6 r_8}{r_6-r_5}\right)\exp\left(\frac{-t}{\lambda}\right)\end{bmatrix} \\
&\times\frac{\xi^2\cos(y\xi)}{(\xi^2+\phi/k)(\xi^2+a_k)^2}d\xi+\frac{2\rho A}{\pi h} \\
&\times\int_0^\infty\left(\frac{r_8 e^{r_5 t}-r_7 e^{r_6 t}-\exp\left(\frac{-t}{\lambda}\right)}{r_6-r_5}\right)\frac{\xi^2\cos(y\xi)}{(\xi^2+\phi/k)(\xi^2+a_k)}d\xi. \qquad (3.51)
\end{aligned}
$$

The corresponding expressions for the velocity field and the shear stress for variable accelerated plate are given by

$$
\begin{aligned}
u(y,t) &= Bt^2\exp[-\sqrt{a_k}y] \\
&-\frac{4B\lambda}{\pi\nu}\int_0^\infty\left(1-\frac{r_6 e^{r_5 t}-r_5 e^{r_6 t}}{r_6-r_5}\right)\frac{\xi\sin(y\xi)}{(\xi^2+a_k)^2}d\xi \\
&-\frac{4B}{\pi\nu}\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)\int_0^\infty\left(t-\frac{e^{r_6 t}-e^{r_5 t}}{r_6-r_5}\right)\frac{\xi\sin(y\xi)}{(\xi^2+a_k)^2}d\xi \\
&+\frac{4B}{\pi h\nu^2}\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)^2\int_0^\infty\left(1-\frac{r_6 e^{r_5 t}-r_5 e^{r_6 t}}{r_6-r_5}\right) \\
&\times\frac{\xi\sin(y\xi)}{(\xi^2+a_k)^3}d\xi, \qquad (3.52)
\end{aligned}
$$

and

$$\begin{aligned}
\tau_1(y,t) &= -\mu B\left[t^2-2\lambda\left(t-\lambda\left\{1-\exp\left(\frac{-t}{\lambda}\right)\right\}\right)\right]\sqrt{a_k}\exp\left[-\sqrt{a_k}y\right] \\
&\quad -\frac{4\rho B}{\pi}\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)\left[t-\lambda\left\{1-\exp\left(\frac{-t}{\lambda}\right)\right\}\right]\int_0^\infty\frac{\xi^2\cos(y\xi)}{\left(\xi^2+a_k\right)^2}d\xi \\
&\quad +\frac{4\lambda\rho B}{\pi}\left[1-\exp\left(\frac{-t}{\lambda}\right)\right]\int_0^\infty\frac{\xi^2\cos(y\xi)}{\left(\xi^2+a_k\right)^2}d\xi+\frac{4\rho B}{\nu\pi} \\
&\quad \times\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)\int_0^\infty\left(\frac{r_3e^{r_2t}-r_4e^{r_1t}}{r_2-r_1}+\exp\left(\frac{-t}{\lambda}\right)\right) \\
&\quad \times\frac{\xi^2\cos(y\xi)}{\left(\xi^2+a_k\right)^2\left(\xi^2+\phi/k\right)}d\xi+\frac{4\lambda\rho B}{\nu\pi} \\
&\quad \times\int_0^\infty\left[\frac{r_6r_8e^{r_5t}-r_5r_7e^{r_6t}}{r_6-r_5}-\frac{r_6r_8-r_5r_7}{r_6-r_5}\exp\left(\frac{-t}{\lambda}\right)\right] \\
&\quad \times\frac{\xi^2\cos(y\xi)}{\left(\xi^2+a_k\right)^2\left(\xi^2+\phi/k\right)}d\xi-\frac{4\rho B}{\nu^2\pi}\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)^2 \\
&\quad \times\int_0^\infty\left[\frac{r_6r_8e^{r_5t}-r_5r_7e^{r_6t}}{r_6-r_5}-\frac{r_6r_8-r_5r_7}{r_6-r_5}\exp\left(\frac{-t}{\lambda}\right)\right] \\
&\quad \times\frac{\xi^2\cos(y\xi)}{\left(\xi^2+a_k\right)^3\left(\xi^2+\phi/k\right)}d\xi.
\end{aligned} \tag{3.53}$$

3.6 Limiting case $\lambda \to 0$ (Newtonian fluid)

When $\lambda \to 0$, all previous solutions are going to the similar solutions for a Newtonian fluid. The velocity fields, shear stress and volume flux for impulsive motion of the plate are

$$\begin{aligned} u(y,z,t) &= \frac{2V}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\exp\left[-\sqrt{(\mu_N^2+a_k)}y\right] - \frac{V}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N} \\ &\times\left\{\begin{array}{c} e^{-\sqrt{(\mu_N^2+a_k)}y}\operatorname{erf}c\left(\sqrt{\nu\left(\mu_N^2+a_k\right)t}-\frac{y}{2\sqrt{\nu t}}\right) \\ -e^{\sqrt{(\mu_N^2+a_k)}y}\operatorname{erf}c\left(\sqrt{\nu\left(\mu_N^2+a_k\right)t}+\frac{y}{2\sqrt{\nu t}}\right)\end{array}\right\}, \end{aligned} \tag{3.54}$$

$$\begin{aligned} \tau_1(y,z,t) &= \frac{-2\mu V}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\sqrt{(\mu_N^2+a_k)}\exp\left[-\sqrt{(\mu_N^2+a_k)}y\right] \\ &-\frac{2V\mu}{h\sqrt{\pi\nu t}}\exp\left(\frac{-y^2}{4\nu t}\right)\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\exp\left[-\left(\mu_N^2+a_k\right)\nu t\right] \\ &+\frac{V\mu}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\sqrt{(\mu_N^2+a_k)} \\ &\times\left[\begin{array}{c} e^{-\sqrt{(\mu_N^2+a_k)}y}\operatorname{erf}c\left(\sqrt{\nu\left(\mu_N^2+a_k\right)t}-\frac{y}{2\sqrt{\nu t}}\right) \\ +e^{\sqrt{(\mu_N^2+a_k)}y}\operatorname{erf}c\left(\sqrt{\nu\left(\mu_N^2+a_k\right)t}+\frac{y}{2\sqrt{\nu t}}\right)\end{array}\right], \end{aligned} \tag{3.55}$$

and

$$Q(t) = \frac{4V}{h}\sum_{n=1}^{\infty}\frac{1}{\mu_N^2\sqrt{(\mu_N^2+a_k)}}\left[1-\operatorname{erf}c\left(\sqrt{(\mu_N^2+a_k)\nu t}\right)\right], \tag{3.56}$$

and those for the flow induced by a constantly accelerating plate are

$$\begin{aligned} u(y,z,t) &= \frac{2At}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\exp\left[-\sqrt{(\mu_N^2+a_k)}y\right] - \frac{A}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N} \\ &\times\int_0^t\left\{\begin{array}{c} e^{-\sqrt{(\mu_N^2+a_k)}y}\operatorname{erf}c\left[\sqrt{\nu\left(\mu_N^2+a_k\right)(t-\tau)}-\frac{y}{2\sqrt{\nu(t-\tau)}}\right] \\ -e^{\sqrt{(\mu_N^2+a_k)}y}\operatorname{erf}c\left[\sqrt{\nu\left(\mu_N^2+a_k\right)(t-\tau)}+\frac{y}{2\sqrt{\nu(t-\tau)}}\right]\end{array}\right\}d\tau, \end{aligned} \tag{3.57}$$

$$
\begin{aligned}
\tau_1(y,z,t) &= \frac{2A\mu t}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos\left(\mu_N z\right)}{\mu_N}\sqrt{\left(\mu_N^2+a_k\right)}\exp\left[-\sqrt{\left(\mu_N^2+a_k\right)}y\right] \\
&\quad -\frac{2A\mu}{h\sqrt{\pi\nu}}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos\left(\mu_N z\right)}{\mu_N}\int_0^t\frac{1}{\sqrt{(t-\tau)}}\exp\left(\frac{-y^2}{4\nu\,(t-\tau)}\right) \\
&\quad \times\exp\left[-\sqrt{\left(\mu_N^2+a_k\right)}y\right]d\tau+\frac{A\mu}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos\left(\mu_N z\right)}{\mu_N}\sqrt{\left(\mu_N^2+a_k\right)} \\
&\quad \times\int_0^t\left[\begin{array}{c} e^{-\sqrt{\left(\mu_N^2+a_k\right)}y}\operatorname{erf}c\left(\sqrt{\nu\left(\mu_N^2+a_k\right)(t-\tau)}-\frac{y}{2\sqrt{\nu(t-\tau)}}\right) \\ \\ -e^{\sqrt{\left(\mu_N^2+a_k\right)}y}\operatorname{erf}c\left(\sqrt{\nu\left(\mu_N^2+a_k\right)(t-\tau)}+\frac{y}{2\sqrt{\nu(t-\tau)}}\right)\end{array}\right]d\tau, \quad (3.58)
\end{aligned}
$$

and

$$
Q\left(t\right)=\frac{4A}{h}\sum_{n=1}^{\infty}\frac{1}{\mu_N^2\sqrt{\left(\mu_N^2+a_k\right)}}\left[t-\int_0^t\operatorname{erf}c\left(\sqrt{\nu\left(\mu_N^2+a_k\right)(t-\tau)}\right)d\tau\right]. \quad (3.59)
$$

Finally, the expressions for velocity field, shear stress and volume flux in case of variable accelerating plate are given by

$$
\begin{aligned}
u(y,z,t) &= \frac{2Bt^2}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos\left(\mu_N z\right)}{\mu_N}\exp\left[-\sqrt{\left(\mu_N^2+a_k\right)}y\right]-\frac{2B}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos\left(\mu_N z\right)}{\mu_N} \\
&\quad \times\int_0^t\tau\left[\begin{array}{c} e^{-\sqrt{\left(\mu_N^2+a_k\right)}y}\operatorname{erf}c\left(\sqrt{\nu\left(\mu_N^2+a_k\right)(t-\tau)}-\frac{y}{2\sqrt{\nu(t-\tau)}}\right) \\ \\ -e^{\sqrt{\left(\mu_N^2+a_k\right)}y}\operatorname{erf}c\left(\sqrt{\nu\left(\mu_N^2+a_k\right)(t-\tau)}+\frac{y}{2\sqrt{\nu(t-\tau)}}\right)\end{array}\right]d\tau, \quad (3.60)
\end{aligned}
$$

$$
\begin{aligned}
\tau_1(y,z,t) &= \frac{-2B\mu t^2}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos\left(\mu_N z\right)}{\mu_N}\sqrt{\left(\mu_N^2+a_k\right)}\exp\left[-\sqrt{\left(\mu_N^2+a_k\right)}y\right] \\
&\quad -\frac{4B\mu}{h\sqrt{\pi\nu}}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos\left(\mu_N z\right)}{\mu_N}\int_0^t\frac{\tau}{\sqrt{(t-\tau)}}\exp\left(\frac{-y^2}{4\nu\left(t-\tau\right)}\right) \\
&\quad \times\exp\left[-\nu\left(\mu_N^2+a_k\right)\left(t-\tau\right)\right]d\tau+\frac{2B\mu}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos\left(\mu_N z\right)}{\mu_N}\sqrt{\left(\mu_N^2+a_k\right)} \\
&\quad \times\int_0^t\tau\left[\begin{array}{c} e^{-\sqrt{\left(\mu_N^2+a_k\right)}y}\operatorname{erf}c\left(\sqrt{\nu\left(\mu_N^2+a_k\right)\left(t-\tau\right)}-\frac{y}{2\sqrt{\nu(t-\tau)}}\right) \\ +e^{\sqrt{\left(\mu_N^2+a_k\right)}y}\operatorname{erf}c\left(\sqrt{\nu\left(\mu_N^2+a_k\right)\left(t-\tau\right)}+\frac{y}{2\sqrt{\nu(t-\tau)}}\right)\end{array}\right]d\tau, \quad (3.61)
\end{aligned}
$$

and

$$
Q\left(t\right)=\frac{4B}{h}\sum_{n=1}^{\infty}\frac{1}{\mu_N^2\sqrt{\left(\mu_N^2+a_k\right)}}\left[t^2-2\int_0^t\tau\operatorname{erf}c\left(\sqrt{\nu\left(\mu_N^2+a_k\right)\left(t-\tau\right)}\right)\right]d\tau, \quad (3.62)
$$

respectively.

3.7 Numerical results and discussion

It clearly results from Eqs. (3.41) and (3.45) that the unsteady flows due to time-dependent motions of plate remain unsteady, while the flow due to impulsive motion of the plate becomes a steady motion. Thus, making limit as $t \to \infty$ into Eqs. (3.32) and (3.36), one obtain the velocity field and the shear stress corresponding to the steady motion as

$$u_s(y,z) = \frac{2V}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos\left(\mu_N z\right)}{\mu_N}\exp\left[-\sqrt{(\mu_N^2+a_k)}y\right], \tag{3.63}$$

$$\tau_{1s}(y,z) = \frac{-2\mu V}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos\left(\mu_N z\right)}{\mu_N}\left(\sqrt{\mu_N^2+a_k}\right)\exp\left[-\sqrt{(\mu_N^2+a_k)}y\right]. \tag{3.64}$$

These steady state solutions are the same for both Newtonian and non-Newtonian fluids. As the series (3.63) which gives the velocity field for steady flow is a convergent series. Unfortunately, the corresponding series (3.64) which gives the shear stress in not convergent. Consequently, the shear stress at the bottom wall $\tau_1(0,z,t)$ can not be calculated from Eqs. (3.36) and (3.55). For this reason, another expression for the steady velocity will be used. This expression, which has been determined in the reference [18] is

$$u_s(y,z) = V\left[\exp[-\sqrt{a_k}y] - \frac{2}{\pi}\int_0^{\infty}\frac{\cosh\sqrt{(\xi^2+a_k)}z}{\cosh\left(2h\sqrt{\xi^2+a_k}\right)}\frac{\xi\sin(y\xi)}{\sqrt{\xi^2+a_k}}d\xi\right]. \tag{3.65}$$

This last equality, together with the Eq. (3.63), tells us that

$$\frac{2}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos\left(\mu_N z\right)}{\mu_N}\exp\left[-\sqrt{(\mu_N^2+a_k)y}\right] = \exp[-\sqrt{a_k}y] - \frac{2}{\pi}\int_0^{\infty}\frac{\cosh\sqrt{(\xi^2+a_k)}z}{\cosh\left(2h\sqrt{\xi^2+a_k}\right)} \times\frac{\xi\sin(y\xi)}{\sqrt{\xi^2+a_k}}d\xi. \tag{3.66}$$

As a result, instead of the expressions (3.32) and (3.36) we can use the velocity field and the corresponding shear stress

$$\begin{aligned} u(y,z,t) &= V\left[\exp[-\sqrt{a_k}y] - \frac{2}{\pi}\int_0^\infty \frac{\cosh\sqrt{(\xi^2+a_k)}z}{\cosh\left(2h\sqrt{\xi^2+a_k}\right)}\frac{\xi\sin(y\xi)}{\sqrt{\xi^2+a_k}}d\xi\right] \\ &-\frac{4V}{\pi h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\int_0^\infty \frac{\xi\sin(y\xi)}{\xi^2+\mu_N^2+a_k} \\ &\times\frac{r_{2N}e^{r_{1N}t}-r_{1N}e^{r_{2N}t}}{r_{2N}-r_{1N}}d\xi, \end{aligned} \tag{3.67}$$

$$\begin{aligned} \tau_1(y,z,t) &= -\mu V\left[\sqrt{a_k}\exp[-\sqrt{a_k}y] - \frac{2}{\pi}\int_0^\infty \frac{\cosh\sqrt{(\xi^2+a_k)}z}{\cosh\left(2h\sqrt{\xi^2+a_k}\right)}\frac{\xi\cos(y\xi)}{\sqrt{\xi^2+a_k}}d\xi\right] \\ &+\frac{2\mu V}{h}e^{\frac{-t}{\lambda}}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\left(\sqrt{\mu_N^2+a_k}\right)\exp\left[-\sqrt{(\mu_N^2+a_k)}y\right] \\ &+\frac{4\mu V}{\pi\lambda h}e^{\frac{-t}{\lambda}}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\int_0^\infty \frac{\xi^2\cos(y\xi)}{\xi^2+\mu_N^2+a_k} \\ &\times\left[\frac{r_{2N}r_{4N}e^{r_{3N}t}-r_{1N}r_{3N}e^{r_{4N}t}}{r_{3N}r_{4N}(r_{2N}-r_{1N})} - \frac{r_{2N}r_{4N}-r_{1N}r_{3N}}{r_{3N}r_{4N}(r_{2N}-r_{1N})}\right]d\xi, \end{aligned} \tag{3.68}$$

for the flow due to impulsive motion of the plate,

$$\begin{aligned} u(y,z,t) &= At\left[\exp[-\sqrt{a_k}y] - \frac{2}{\pi}\int_0^\infty \frac{\cosh\sqrt{(\xi^2+a_k)}z}{\cosh\left(2h\sqrt{\xi^2+a_k}\right)}\frac{\xi\sin(y\xi)}{\sqrt{\xi^2+a_k}}d\xi\right] \\ &-\frac{4A}{\pi h\nu}\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\int_0^\infty\left(1-\frac{r_{2N}e^{r_{1N}t}-r_{1N}e^{r_{2N}t}}{r_{2N}-r_{1N}}\right) \\ &\times\frac{\xi\sin(y\xi)}{\left(\xi^2+\mu_N^2+a_k\right)^2}d\xi + \frac{4A}{\pi h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\int_0^\infty\left(\frac{e^{r_{1N}t}-e^{r_{2N}t}}{r_{2N}-r_{1N}}\right) \\ &\times\frac{\xi\sin(y\xi)}{\xi^2+\mu_N^2+a_k}d\xi, \end{aligned} \tag{3.69}$$

$$\begin{aligned}
\tau_1(y,z,t) &= -A\mu t\left[\sqrt{a_k}\exp[-\sqrt{a_k}y]-\frac{2}{\pi}\int_0^\infty \frac{\cosh\sqrt{(\xi^2+a_k)}z}{\cosh\left(2h\sqrt{\xi^2+a_k}\right)}\frac{\xi\cos(y\xi)}{\sqrt{\xi^2+a_k}}d\xi\right] \\
&+\frac{2A\mu\lambda}{h}\left[1-\exp\left(\frac{-t}{\lambda}\right)\right]\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\left(\sqrt{\mu_N^2+a_k}\right)\exp\left[-\sqrt{(\mu_N^2+a_k)}y\right] \\
&-\frac{4\rho A}{\pi h}\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)\left[1-\exp\left(\frac{-t}{\lambda}\right)\right]\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\int_0^\infty\frac{\xi^2\cos(y\xi)}{(\xi^2+\mu_N^2+a_k)^2}d\xi \\
&+\frac{4\rho A}{\nu\pi h}\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\int_0^\infty\left(\begin{array}{c}\frac{r_{2N}r_{4N}e^{r_{1N}t}-r_{1N}r_{3N}e^{r_{2N}t}}{r_{2N}-r_{1N}} \\ \\ -\frac{r_{1N}r_{3N}-r_{2N}r_{4N}}{r_{2N}-r_{1N}}\end{array}\right) \\
&\times\frac{\xi^2\cos(y\xi)}{\left(\xi^2+\mu_N^2+\frac{\phi}{k}\right)(\xi^2+\mu_N^2+a_k)^2}d\xi+\frac{4\rho A}{\pi h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N} \\
&\times\int_0^\infty\left[\frac{r_{4N}e^{r_{1N}t}-r_{3N}e^{r_{2N}t}}{r_{2N}-r_{1N}}-\exp\left(\frac{-t}{\lambda}\right)\right]\frac{\xi^2\cos(y\xi)}{(\xi^2+\mu_N^2+\phi/k)(\xi^2+\mu_N^2+a_k)}d\xi, \quad (3.70)
\end{aligned}$$

for the flow due to constantly accelerating plate and

$$\begin{aligned}
u(y,z,t) &= Bt^2\left[\exp[-\sqrt{a_k}y]-\frac{2}{\pi}\int_0^\infty\frac{\cosh\sqrt{(\xi^2+a_k)}z}{\cosh\left(2h\sqrt{\xi^2+a_k}\right)}\frac{\xi\sin(y\xi)}{\sqrt{\xi^2+a_k}}d\xi\right] \\
&-\frac{8B\lambda}{\pi h\nu}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\int_0^\infty\left(1-\frac{r_{2N}e^{r_{1N}t}-r_{1N}e^{r_{2N}t}}{r_{2N}-r_{1N}}\right)\frac{\xi\sin(y\xi)}{(\xi^2+\mu_N^2+a_k)^2}d\xi \\
&-\frac{8B}{\pi h\nu}\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\int_0^\infty\left(t-\frac{e^{r_{2N}t}-e^{r_{1N}t}}{r_{2N}-r_{1N}}\right) \\
&\times\frac{\xi\sin(y\xi)}{(\xi^2+\mu_N^2+a_k)^2}d\xi+\frac{8B}{\pi h\nu^2}\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)^2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N} \\
&\times\int_0^\infty\left(1-\frac{r_{2N}e^{r_{1N}t}-r_{1N}e^{r_{2N}t}}{r_{2N}-r_{1N}}\right)\frac{\xi\sin(y\xi)}{(\xi^2+\mu_N^2+a_k)^2}d\xi, \quad (3.71)
\end{aligned}$$

$$
\begin{aligned}
\tau_1(y,z,t) &= -B\mu t^2\left[\sqrt{a_k}\exp\left[-\sqrt{a_k}y\right]-\frac{2}{\pi}\int_0^\infty \frac{\cosh\sqrt{(\xi^2+a_k)}z}{\cosh\left(2h\sqrt{\xi^2+a_k}\right)}\frac{\xi\cos(y\xi)}{\sqrt{\xi^2+a_k}}d\xi\right] \\
&+\frac{4\lambda\mu B}{h}\left[t-\lambda\left\{1-\exp\left(\frac{-t}{\lambda}\right)\right\}\right]\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\left(\sqrt{\mu_N^2+a_k}\right) \\
&\times\exp\left[-\sqrt{(\mu_N^2+a_k)}y\right]-\frac{8\rho B}{\pi h}\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)\left[t-\lambda\left\{1-\exp\left(\frac{-t}{\lambda}\right)\right\}\right] \\
&\times\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\int_0^\infty\frac{\xi^2\cos(y\xi)}{\left(\xi^2+\mu_N^2+a_k\right)^2}d\xi+\frac{8\lambda\rho B}{\pi h}\left[1-\exp\left(\frac{-t}{\lambda}\right)\right] \\
&\times\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\int_0^\infty\frac{\xi^2\cos(y\xi)}{\left(\xi^2+\mu_N^2+a_k\right)^2}d\xi+\frac{8\rho B}{\nu\pi h}\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right) \\
&\times\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N}\int_0^\infty\left[\frac{r_{3N}e^{r_{2N}t}-r_{4N}e^{r_{1N}t}}{r_{2N}-r_{1N}}+\exp\left(\frac{-t}{\lambda}\right)\right] \\
&\times\frac{\xi^2\cos(y\xi)}{\left(\xi^2+\mu_N^2+a_k\right)^2\left(\xi^2+\mu_N^2+\phi/k\right)}d\xi+\frac{8\lambda\rho B}{\nu\pi h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N} \\
&\times\int_0^\infty\left[\frac{r_{2N}r_{4N}e^{r_{1N}t}-r_{1N}r_{3N}e^{r_{2N}t}}{r_{2N}-r_{1N}}-\frac{r_{2N}r_{4N}-r_{1N}r_{3N}}{r_{2N}-r_{1N}}\exp\left(\frac{-t}{\lambda}\right)\right] \\
&\times\frac{\xi^2\cos(y\xi)}{\left(\xi^2+\mu_N^2+a_k\right)^2\left(\xi^2+\mu_N^2+\phi/k\right)}d\xi-\frac{8\rho B}{\nu^2\pi h}\left(1+\lambda\frac{\sigma B_0^2}{\rho}\right)^2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(\mu_N z)}{\mu_N} \\
&\times\int_0^\infty\left[\frac{r_{2N}r_{4N}e^{r_{1N}t}-r_{1N}r_{3N}e^{r_{2N}t}}{r_{2N}-r_{1N}}-\frac{r_{2N}r_{4N}-r_{1N}r_{3N}}{r_{2N}-r_{1N}}\exp\left(\frac{-t}{\lambda}\right)\right] \\
&\times\frac{\xi^2\cos(y\xi)}{\left(\xi^2+\mu_N^2+a_k\right)^3\left(\xi^2+\mu_N^2+\phi/k\right)}d\xi. \qquad (3.72)
\end{aligned}
$$

for variable accelerating plate.

In order to reveal relevant physical effects of the obtained results, the graphs of the velocity $u(y,t)$ corresponding to the motion over an infinite plate as well as those of $u(y,0,t)$, giving the velocity profiles at the middle of the channel and $\tau_1(0,z,t)$ giving the tangential stress at the bottom wall have been drawn for different values of time t and magnetic parameter M. The graphs have been plotted by introducing dimensionless variables

$$
\begin{aligned}
u^* &= \frac{u}{V},\ t^*=\frac{\nu t}{h^2},\ y^*=\frac{y}{h},\ z^*=\frac{z}{h},\ \lambda^*=\frac{\lambda\nu}{h^2},\ \frac{1}{K}=\frac{\phi h^2}{k}, \\
M^2 &= \frac{\sigma\beta_0^2}{\mu}h^2,\ A^*=\frac{Ah^2}{\nu V},\ B^*=\frac{Bh^2}{\nu^2 V}, \qquad (3.73)
\end{aligned}
$$

and the asterisks have been omitted.

The velocity and the shear stress profiles for three different flows are portrayed in figures 3.1 − 3.12. In order to show the effects of side walls, a comparison of velocity for flow over an infinite plate $u(y,t)$ with that for flow between two side walls perpendicular to a plate $u(y,0,t)$ is made.

In figures 3.1 − 3.4, the velocity distribution and shear stress are plotted. These figures include the case of an impulsive motion of the plate. In figures 3.1 and 3.2, the velocity distribution is plotted against y for various value of time t and magnetic parameter M. As expected, with the increase of time the flow velocity increases. From these figures, it is observed that the velocity distribution decreases with an increase in the magnetic parameter in the presence as well as in the absence of side walls. Such an effect may also be expected because under the conditions considered the magnetic force is a resistance to the flow and hence the amplitude of the flow velocity decreases. Further, these figures also display a dependence of the velocity on the side walls. It is readily seen that in the presence of side walls the velocity distribution is much smaller in magnitude when compared with that in the absence of side walls. This indicates that there are significant effects of the side walls on the flow. Moreover, it can be expected from the governing equation (3.12) that increasing permeability of the porous medium K yields an effect opposite to that of M.

In figures 3.3 and 3.4, the variations of shear stress $\tau_1(0,z,t)$ at the bottom wall are plotted against z for various values of time t and the magnetic parameter M. Since the shear stress is an important physical quantity which provides useful informations about the nature of dissipation at the boundary. It is clearly seen that for large values of time t, the shear stress profiles tend to coincide. Further, from these figures, it is noted that the shear stress at the bottom wall is minimum at the middle of the channel and increases toward the side walls. As it was expected that the strongest shear stress occurs near the side walls. Moreover, it is clear from these figures that shear stress decreases substantially with an increase of the magnetic parameter.

The above mentioned effects of physical parameters on the velocity distribution and the shear stress at the middle of the channel for accelerated flows are indicated graphically through figures 3.5 − 3.12. The figures 3.5 − 3.8 and 3.9 − 3.12 include the case of constantly accelerating plate and variable accelerating plate, respectively. Qualitatively, the observations for accelerated flows are similar to that of impulsive motion of the plate. However, these observations are not similar quantitatively. A comparison reveals that the velocity and the shear stress profiles in case of variable accelerated flow are much greater (smaller) when compared to those of constantly accelerated flow for t greater (smaller) than unity.

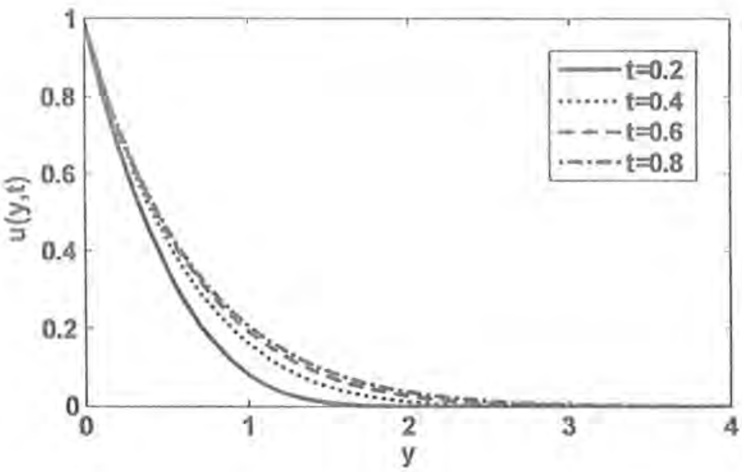


Figure 3.1 (a) : Profiles of the velocity $u(y,t)$ for different time t for impulsive motion of plate by keeping $\lambda = 0.01$, $K = 0.5$ and $M = 0.5$ fixed.

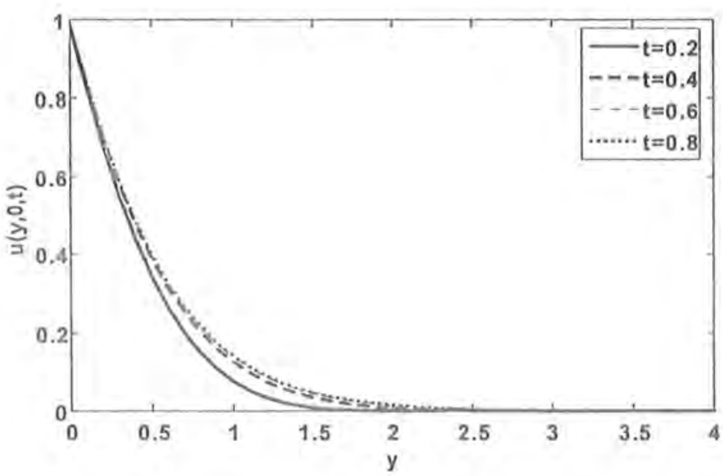


Figure 3.1 (b) : Profiles of the velocity $u(y,0,t)$ for different time t for impulsive motion of plate by keeping $\lambda = 0.01$, $K = 0.5$ and $M = 0.5$ fixed.

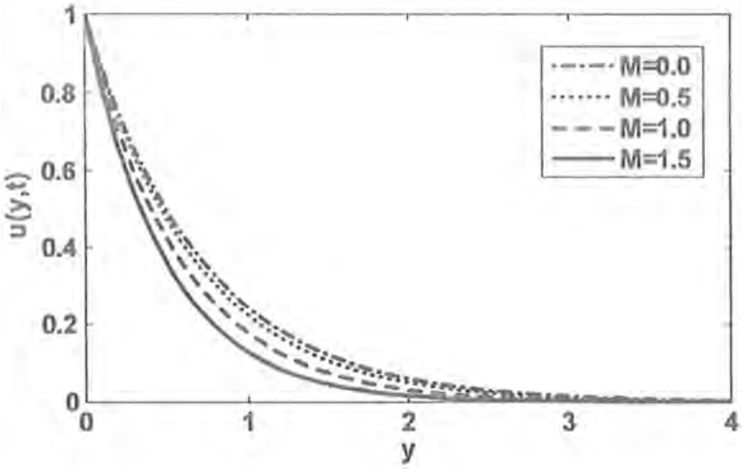


Figure 3.2 (a) : Profiles of the velocity $u(y,t)$ for different values of magnetic parameter M for impulsive motion of plate by keeping $\lambda = 0.01,\ K = 0.5$ and $t = 2$ fixed.

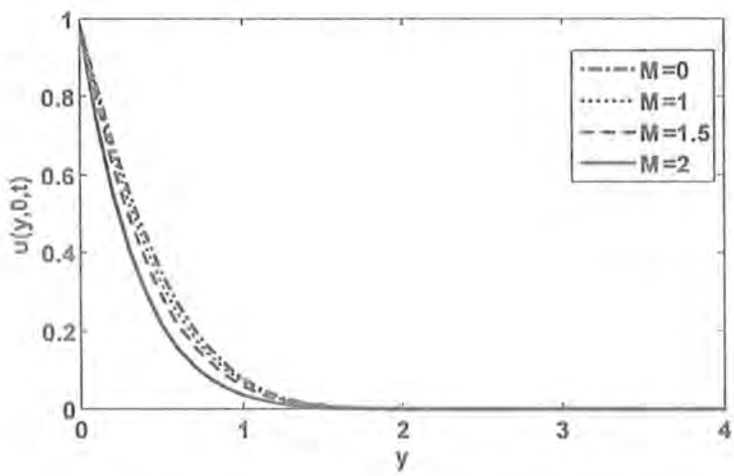


Figure 3.2 (b) : Profiles of the velocity $u(y,0,t)$ for different values of magnetic parameter M for impulsive motion of plate by keeping $\lambda = 0.01,\ K = 0.5$ and $t = 2$ fixed.

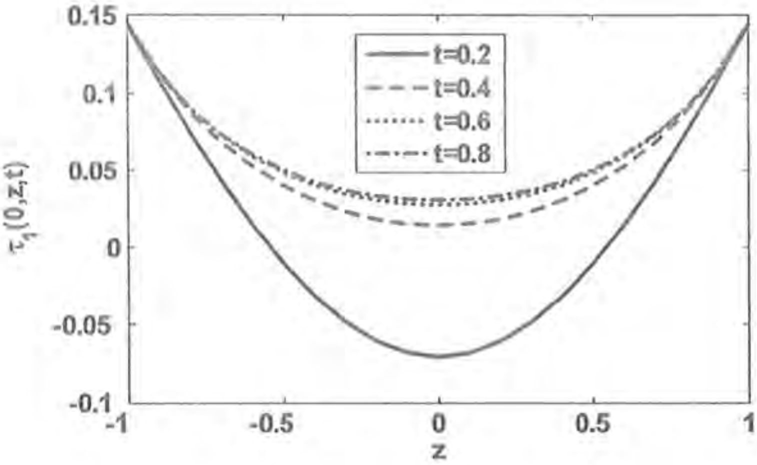


Figure 3.3 (a) : Profiles of the tangential stress $\tau_1\ (0, z, t)$ for various values of time t for impulsive motion of plate by keeping $\lambda = 0.01,\ K = 0.5$ and $M = 0.5$ fixed.

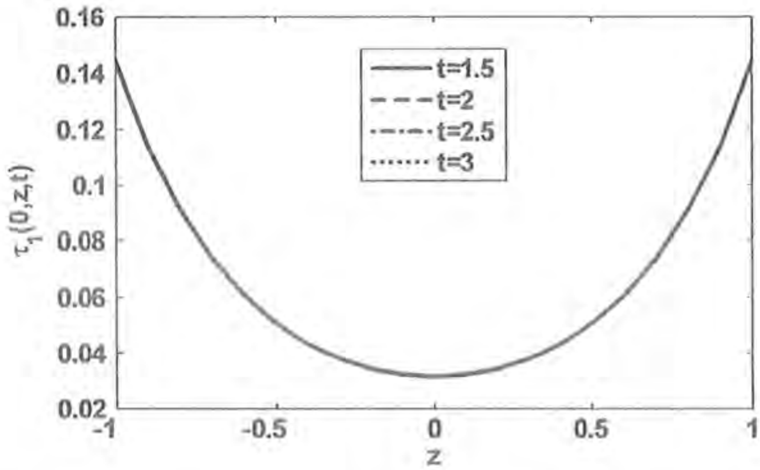


Figure 3.3 (b) : Profiles of the tangential stress $\tau_1\ (0, z, t)$ for various values of time t for impulsive motion of plate by keeping $\lambda = 0.01,\ K = 0.5$ and $M = 0.5$ fixed.

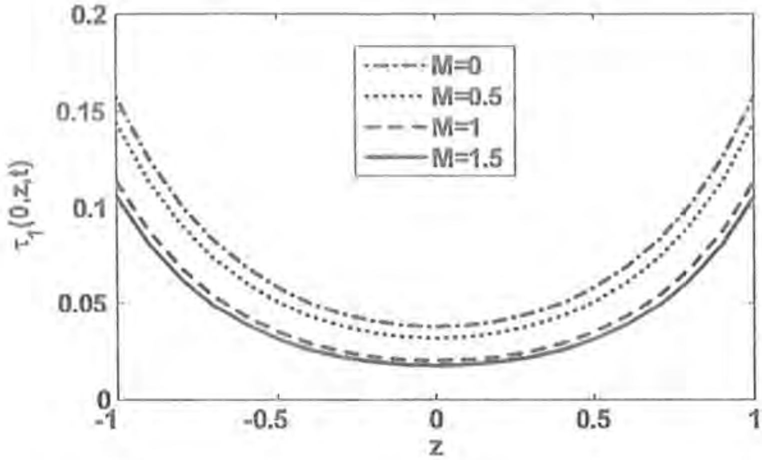


Figure 3.4 : Profiles of the tangential stress $\tau_1(0, z, t)$ for various values of magnetic parameter M for impulsive motion of plate by keeping $\lambda = 0.01$, $K = 0.5$ and $t = 2$ fixed.

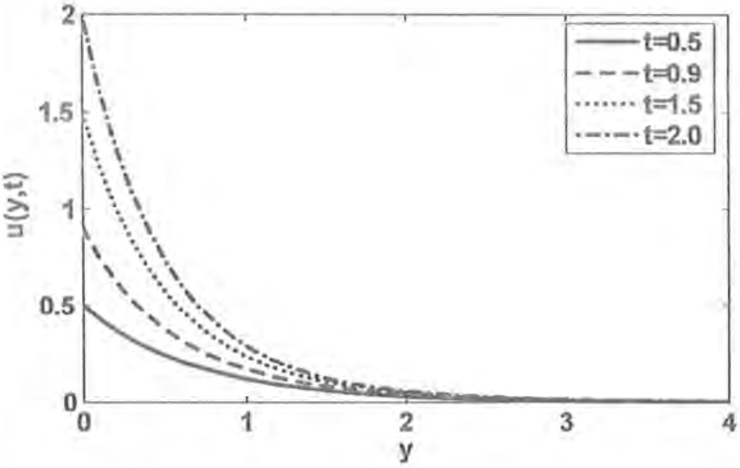


Figure 3.5 (a) : Profiles of the velocity $u(y,t)$ for various values of time t for constantly accelerating plate by keeping $\lambda = 0.01$, $K = 0.5$ and $M = 0.5$ fixed.

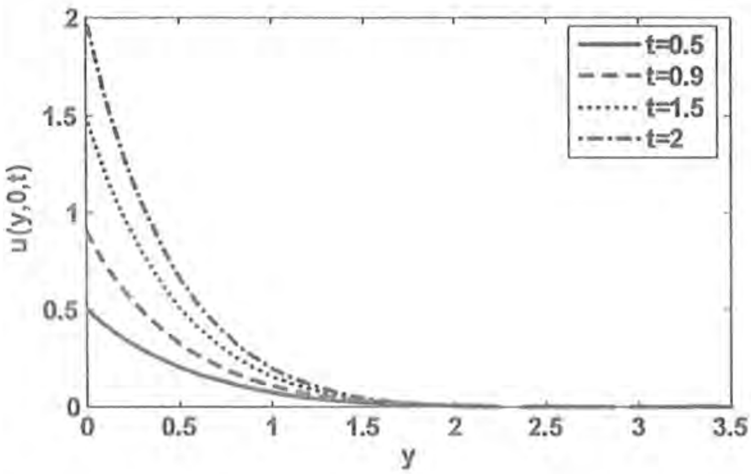


Figure 3.5 (b) : Profiles of the velocity $u(y,0,t)$ for various values of time t for constantly accelerating plate by keeping $\lambda = 0.01$, $K = 0.5$ and $M = 0.5$ fixed.

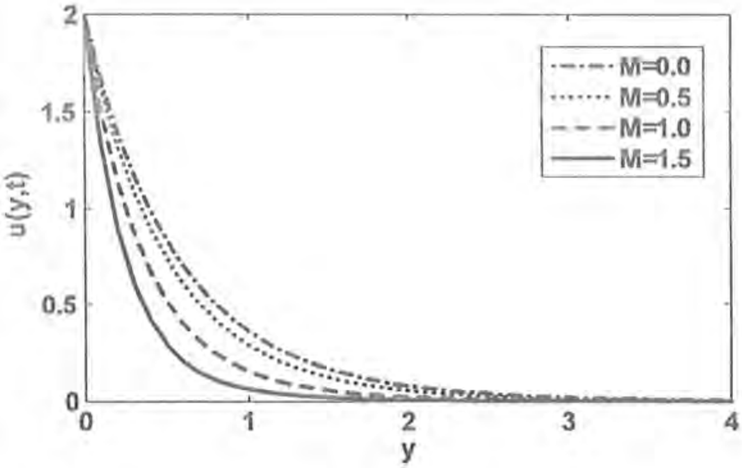


Figure 3.6 (a) : Profiles of the velocity $u(y,t)$ for various values of magnetic parameter M for constantly accelerating plate by keeping $\lambda = 0.01$, $K = 0.5$ and $t = 2$ fixed.

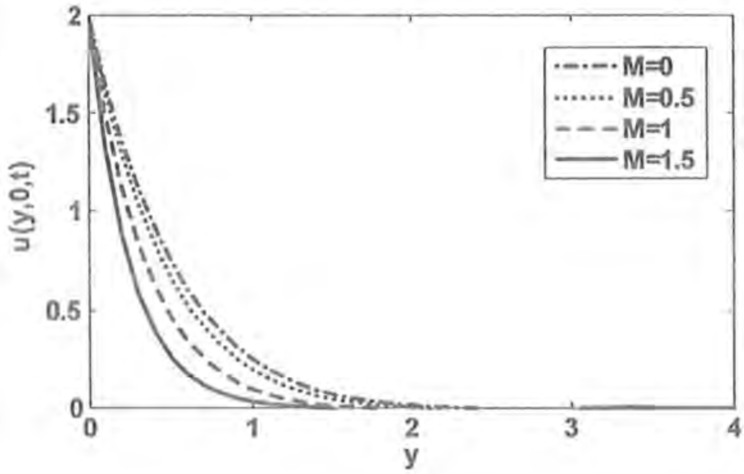


Figure 3.6 (b) : Profiles of the velocity $u(y,0,t)$ for various values of magnetic parameter M for constantly accelerating plate by keeping $\lambda = 0.01$, $K = 0.5$ and $t = 2$ fixed.

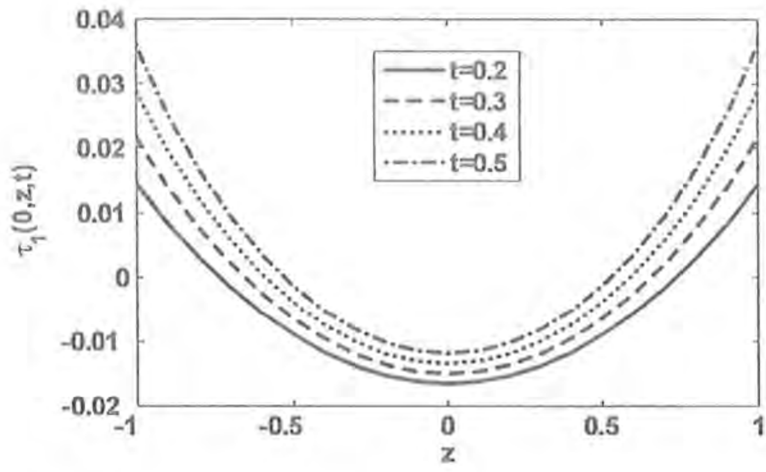


Figure 3.7 (a) : Profiles of the tangential stress $\tau_1(0,z,t)$ for various values of time t for constantly accelerating plate by keeping $\lambda = 0.01$, $K = 0.5$ and $M = 0.5$ fixed.

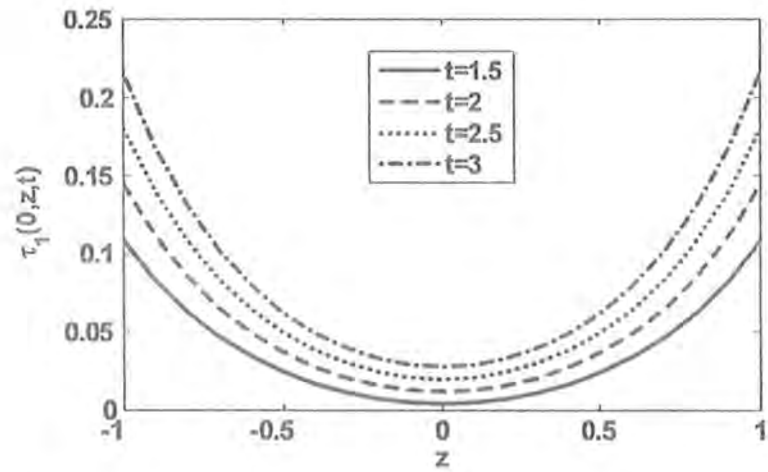


Figure 3.7 (b) : Profiles of the tangential stress $\tau_1(0,z,t)$ for various values of time t for constantly accelerating plate by keeping $\lambda = 0.01$, $K = 0.5$ and $M = 0.5$ fixed.

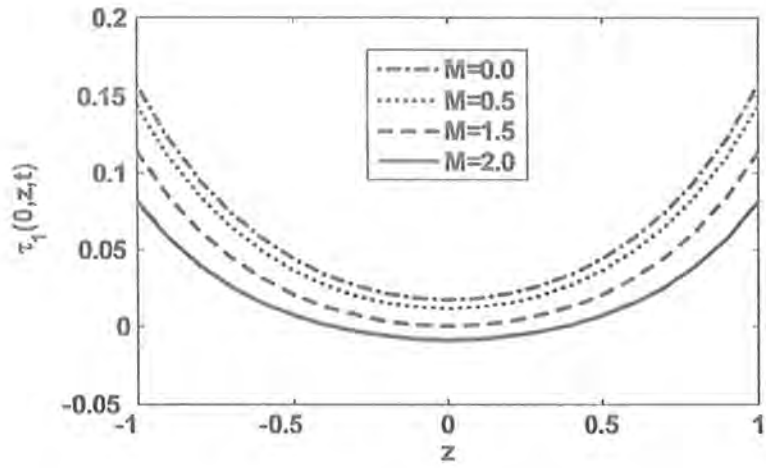


Figure 3.8 : Profiles of the tangential stress $\tau_1\,(0, z, t)$ for various values of magnetic parameter M for constantly accelerating plate by keeping $\lambda = 0.01,\ K = 0.5$ and $t = 2$ fixed.

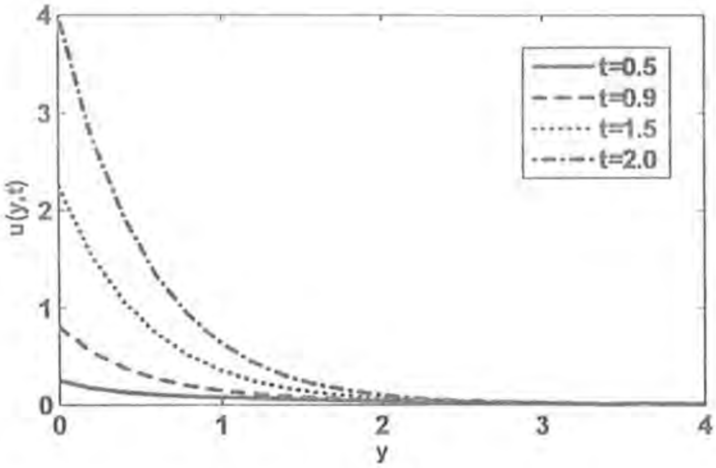


Figure 3.9 (a) : Profiles of the velocity $u(y,t)$ for various values of time t for variable accelerating plate by keeping $\lambda = 0.01$, $K = 0.5$ and $M = 0.5$ fixed.

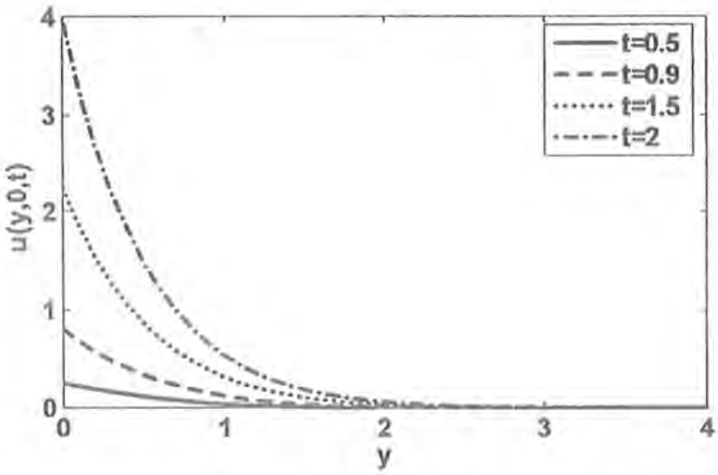


Figure 3.9 (b) : Profiles of the velocity $u(y,0,t)$ for various values of time t for variable accelerating plate by keeping $\lambda = 0.01$, $K = 0.5$ and $M = 0.5$ fixed.

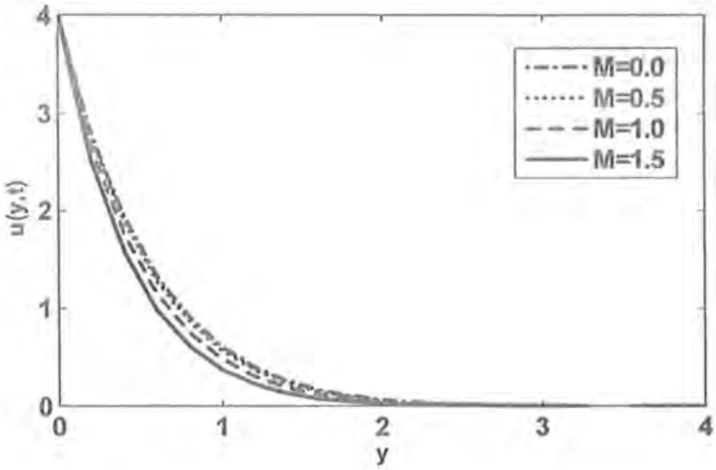


Figure 3.10 (a) : Profiles of the velocity $u(y,t)$ for various values of magnetic parameter M for variable accelerating plate by keeping $\lambda = 0.01$, $K = 0.5$ and $t = 2$ fixed.

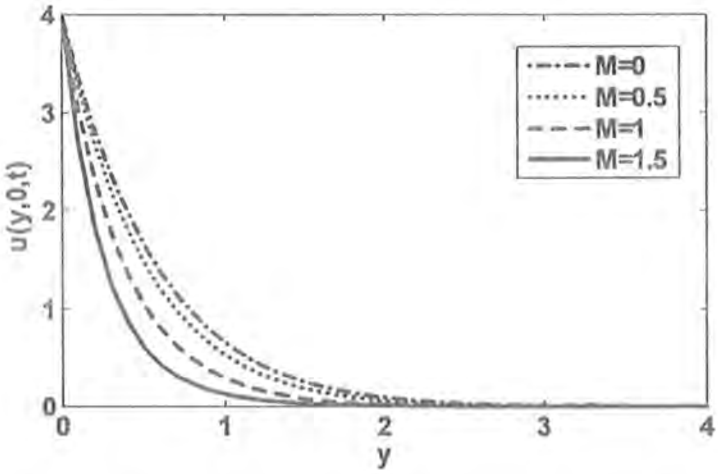


Figure 3.10 (b) : Profiles of the velocity $u(y,0,t)$ for various values of magnetic parameter M for variable accelerating plate by keeping $\lambda = 0.01$, $K = 0.5$ and $t = 2$ fixed.

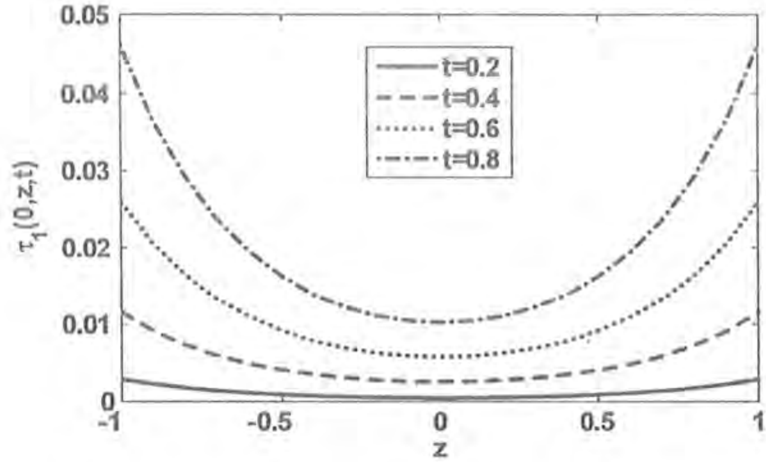


Figure 3.11 (a) : Profiles of the tangential stress $\tau_1(0,z,t)$ for various values of time t for variable accelerating plate by keeping $\lambda = 0.01,\ K = 0.5$ and $M = 5$ fixed.

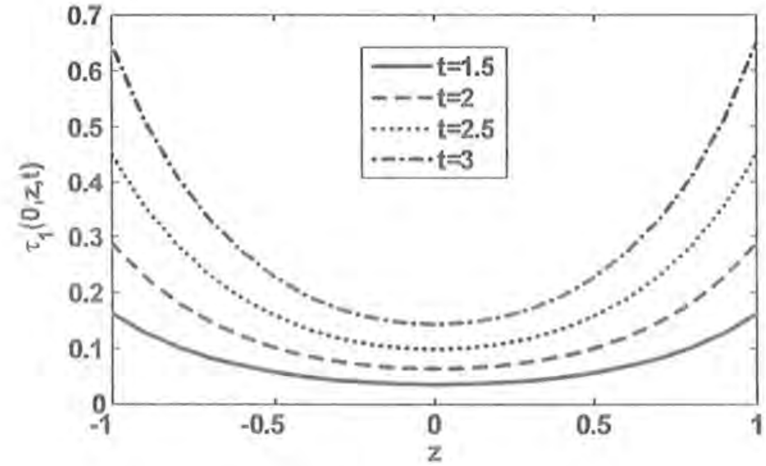


Figure 3.11 (b) : Profiles of the tangential stress $\tau_1(0,z,t)$ for various values of time t for variable accelerating plate by keeping $\lambda = 0.01,\ K = 0.5$ and $M = 5$ fixed.

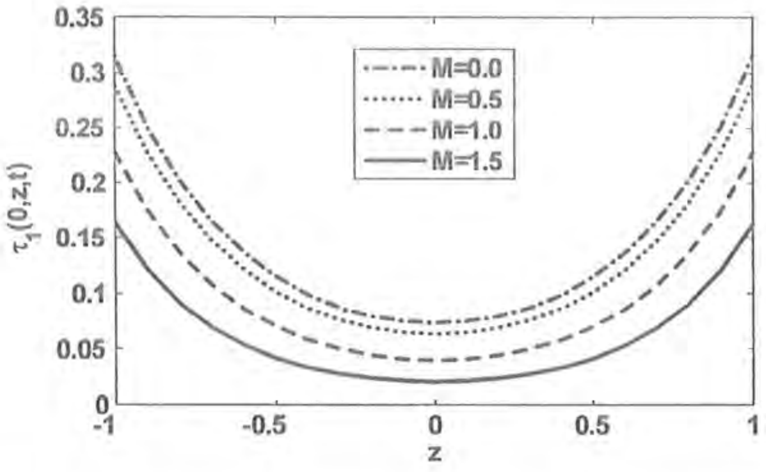


Figure 3.12 : Profiles of the tangential stress $\tau_1\,(0, z, t)$ for various values of magnetic parameter M for variable accelerating plate by keeping $\lambda = 0.01,\ K = 0.5$ and $t = 2$ fixed.

3.8 Closing remarks

In the present chapter, the exact analytic solutions have been obtained for three different problems of the unsteady flow of an electrically conducting incompressible Maxwell fluid through a porous medium between two side walls perpendicular to a plate. The system is stressed by a uniform transverse magnetic field. The expressions for the velocity field, the shear stress and the volume flow rate are constructed using Fourier sine transforms. The influence of the exerted magnetic field and the effects of the side walls on the velocity field and the shear stress at the bottom plate are graphically presented. The results categorically indicate the following findings:

- It is noted that the velocity and the shear stress profiles decrease monotonically by increasing magnetic parameter.
- It is seen that in the presence of side walls the velocity is much smaller in magnitude when compared with that in absence of side walls.
- It is observed that the shear stress at the bottom wall is minimum at the middle of the channel.
- As it was to be expected that the strongest shear stress occurs near the side walls.
- Qualitatively, the observations for accelerated flows are similar to that of impulsive motion of the plate. However, these observations are not similar quantitatively.

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