On Interval Valued Fuzzy Semirings

By

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2009

بسم الله الرحمن الرحيم

IN THE NAME OF ALLAH,

THE MOST MERCIFUL, THE MOST COMPASSIONATE

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A Dissertation Submitted in the Partial Fulfillment of the requirements for the degree of

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CERTIFICATE

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We accept this dissertation as conforming to the required standard

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DEDICATED

TO

MY LOVING CHILDREN

(HAREEM

AND

HAMZA),

ALL FAMILY MEMBERS

RESPECTED TEACHERS AND

FRIENDS

For the encouragement, guidance and sacrifices

Throughout the years of my professional and

Intellectual development

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Nosheen Malik

ABSTRACT

The concept of fuzzy set which was first introduced by L. A. Zadeh in his classical paper [18] is very useful development and has been applied by many authors to generalize some basic notions of algebra. As a generalization of fuzzy set, Zadeh [19, 20, 21] introduced the concept of interval valued fuzzy set. Fuzzy semirings were first investigated in [2] and fuzzy h-ideals of hemirings in [23].

In this dissertation, we chaeacterized different classes of hemirings by the properties of their interval valued fuzzy ideals.

This dissertation consists of four chapters. First chapter is of introductory nature which provides some basic definitions and results of fuzzy sets and fuzzy hemirings. In chapter 2, we have reviewed the interval valued fuzzy sets and the partial order in them as defined in [22], and also introduced interval valued fuzzy ideals of hemirings. In chapter 3, we characterized fully idempotent hemirings by the properties of interval valued fuzzy ideals. Finally in chapter 4, we have investigated the hemirings by the properties of their interval valued fuzzy h-ideals.

Contents

Chapter 1

PRELIMINARIES

In this chapter we present a brief summary of basic definitions and preliminary results, used in the subsequent chapters of this dissertation. For undefined terms and notations we refer to [8] and [9].

1.1 Definitions and Examples

Definition 1 *Let S be a non-empty set and "$*$" be a binary operation on S. Then $(S, *)$ is called a semigroup if this operation is associative, that is*

$$a * (b * c) = (a * b) * c \qquad \text{for all } a, b, c \in S.$$

A semigroup $(S, *)$ is called commutative if

$$a * b = b * a \qquad \text{for all } a, b \in S.$$

Definition 2 *Let $(S, *)$ be a semigroup. If there exists an element $e \in S$ such that*

$$a * e = e * a = a \qquad \text{for all } a \in S$$

then e is called the identity element in S and $(S, *)$ is called a monoid.

Definition 3 *A non-empty set R together with two binary operations, named as addition "+"and multiplication "·", is said to be a hemiring if*

(i) $(R, +)$ *is a commutative monoid with identity '0'.*

(ii) $(R, \cdot)$ *is a semigroup.*

(iii) Multiplication distributes over addition from either side, that is for all $a, b, c \in R$

$$\begin{aligned} a(b+c) &= ab + ac \\ (a+b)\,c &= ac + bc \end{aligned}$$

(iv) $0 \cdot a = a \cdot 0 = 0$ *holds for all* $a \in R$.

A hemiring $(R, +, \cdot)$ *is called commutative if*

$$a \cdot b = b \cdot a \qquad \text{for all } a, b \in R.$$

Definition 4 *A subset S of a hemiring R is called a subhemiring of R if S is itself a hemiring under the operations inherited from R or* $S \subseteq R$ *is a subhemiring of R if* $0 \in S$ *and for all* $a, b \in S$ *we have* $a + b, ab \in S$.

Example 5 1. *All rings are hemirings with subrings as subhemirings.*

2. *Let* $R = [0, 1]$ *be the unit closed interval of real numbers. Define addition as*

$$a + b = \max\{a, b\} \qquad \text{for all } a, b \in R$$

and multiplication as the usual multiplication of real numbers. Then $(R, +, \cdot)$ *is a commutative hemiring. Furthermore, the subinterval* $[0, r]$, *where* $0 < r \leq 1$, *is subhemiring of the hemiring* $R = [0, 1]$.

3. The set of whole numbers as well as the set of non-negative rational numbers are commutative hemirings under usual addition and multiplication of real numbers. The set of whole numbers is a subhemiring of the set of non-negative rational numbers.

4. Let $(S, +)$ be a commutative monoid with identity 0. Define multiplication as

$$a \cdot b = 0 \qquad \text{for all } a, b \in S.$$

Then $(S, +, \cdot)$ is a commutative hemiring.

5. Let S be a non-empty set and $P(S)$ be the family of all subsets of S. Then $(P(S), \cup, \cap)$ is a commutative hemiring.

6. Let $(S, +)$ be a commutative monoid with identity 0. Then the set of all endomorphisms of S is a hemiring under the operations of pointwise addition and composition of mappings. The set of all automorphisms of S is a subhemiring of the set of all endomorphisms of S.

7. The set of all square matrices with entries from non-negative real numbers is a hemiring under the usual addition and multiplication of matrices.

Definition 6 *Let A and B be subsets of a hemiring $(R, +, \cdot)$. Then the sum and product of A and B are defined as*

$$A + B = \{a + b : a \in A,\ b \in B\}$$

and

$$AB = \left\{ \sum_{finite} a_i b_i : a_i \in A,\ b_i \in B \right\}$$

Definition 7 *Let* $(R, +, \cdot)$ *be a hemiring. An element* $a \in R$ *is called multiplicatively idempotent if* $a^2 = a$.

Definition 8 *A hemiring* $(R, +, \cdot)$ *is called multiplicatively idempotent if each element of* R *is multiplicatively idempotent.*

Definition 9 *A hemiring* $(R, +, \cdot)$ *is called von Neumann regular if for any* $a \in R$ *there exists* $x \in R$ *such that* $a = axa$ *or we have* $a \in aRa$ *for all* $a \in R$.

Definition 10 [3] *A hemiring* $(R, +, \cdot)$ *is called left (respectively right) weakly regular if we have* $a \in RaRa$ *(respectively* $a \in aRaR$*) for all* $a \in R$.

Remark 11 *Clearly if* R *is commutative then* R *is (right or left) weakly regular if and only if* R *is von-Neumann regular.*

1.2 Ideals in Hemirings

Ideals play an important role in the theory of rings and it is therefore natural to study them also in the theory of hemirings.

Definition 12 *A non-empty subset* I *of a hemiring* R *is called a left (respectively right) ideal of* R *if for all* $a, b \in I$ *and* $r \in R$

i) $a + b \in I$

ii) $ra \in I$ *(respectively* $ar \in I$*).*

Note that $0 \in I$.

Definition 13 *A non-empty subset* I *of a hemiring* R *is called an ideal of* R *if it is both a left and a right ideal of* R.

Remark 14 *If A and B are left (respectively right) ideals of a hemiring R then $A \cap B$ is a left (respectively right) ideal of R. If A is a subset of R, then intersection of all left (right) ideals of R which contain A is a left (right) ideal of R containing A. Of course this is the smallest left (right) ideal of R containing A and is called the left (right) ideal of R generated by A.*

Proposition 15 *If A and B are left (respectively right) ideals of a hemiring R then $A + B$ is the smallest left (respectively right) ideal of R containing both A and B.*

Proposition 16 *If A and B are ideals of a hemiring R then AB is an ideal of R contained in $A \cap B$.*

Theorem 17 *A hemiring R is von Neumann regular if and only if for any right ideal A and any left ideal B of R, $A \cap B = AB$.*

Theorem 18 [3] *The following assertions for a hemiring R are equivalent:*

1) *R is right weakly regular.*
2) *All right ideals of R are idempotent.*
3) *$IJ = I \cap J$ for each right ideal I and two-sided ideal J of R.*

Definition 19 *An ideal I of a hemiring R is called prime ideal of R if for any two ideals A and B of R*

$$AB \subseteq I \Rightarrow A \subseteq I \quad or \; B \subseteq I.$$

Definition 20 *An ideal I of a hemiring R is called irreducible ideal of R if for any ideals A and B of R*

$$A \cap B = I \Rightarrow A = I \quad or \quad B = I.$$

Definition 21 *An ideal I (left, right or two-sided) of a hemiring R is called a k-ideal of R if for any $y, z \in I$ and $x \in R$, from $x + y = z$ it follows that $x \in I$.*

Definition 22 *An ideal I (left, right or two-sided) of a hemiring R is called an h-ideal of R if for any $a, b \in I$ and $x, y \in R$, from $x + a + y = b + y$ it follows that $x \in I$.*

Remark 23 *Clearly a left (respectively right) h-ideal is always a left (respectively right) k-ideal but the converse is not true. Also note that in case of rings, k-ideals and h-ideals coinside due to the existance of additive inverses.*

Definition 24 *An ideal I of a hemiring R is called idempotent if $I^2 = I$.*

Definition 25 *A hemiring R is called fully idempotent if each two-sided ideal of R is idempotent, that is $I^2 = I$.*

Proposition 26 *If R is commutative hemiring then R is fully idempotent if and only if R is von Neumann regular.*

Proposition 27 *If R is commutative hemiring then R is fully idempotent if and only if R is weakly regular.*

1.3 Fuzzy Hemirings

The theory of fuzzy sets was introduced by L.A.Zadeh in [18] as a generalization of the abstract set theory. This concept has been applied by many authors to generalize some of the basic notions of algebra. In this section, we will give a review of some basic concepts of fuzzy sets and fuzzy ideals of hemirings.

Definition 28 *A fuzzy subset f of a non-empty set X is a function from X to the unit closed interval $[0,1]$, that is $f : X \longrightarrow [0,1]$.*

A fuzzy subset $f : X \longrightarrow [0,1]$ is non-empty if f is not a zero map.

Note that $f(x) \in [0,1]$ for all $x \in X$.

Definition 29 *Let f and g be two fuzzy subsets of a non-empty set X, then $f \subseteq g$ if and only if $f(x) \leq g(x)$ for all $x \in X$.*

Definition 30 *Let f and g be any two fuzzy subsets of a non-empty set X, then the union and intersection of f and g are defined as*

$$(f \cup g)(x) = f(x) \vee g(x) \qquad \text{for all } x \in X$$

$$(f \cap g)(x) = f(x) \wedge g(x) \qquad \text{for all } x \in X.$$

More generally if $\{f_i : i \in I\}$ is a family of fuzzy subsets of X then the union and intersection are defined as

$$(\cup_i f_i)(x) = \vee_i (f_i(x)) \qquad \text{for all } x \in X$$

$$(\cap_i f_i)(x) = \wedge_i (f_i(x)) \qquad \text{for all } x \in X.$$

Definition 31 *Let f and g be any two fuzzy subsets of a hemiring R, then the sum of f and g is defined as*

$$(f+g)(x) = \vee_{x=y+z} [f(y) \wedge g(z)] \qquad \text{for all } x \in R.$$

Definition 32 *Let f and g be any two fuzzy subsets of a hemiring R, then the product of f and g is defined as*

$$(fg)(x) = \vee_{x=\Sigma_{i=1}^{n} y_i z_i} [\wedge_i \{f(y_i) \wedge g(z_i)\}] \qquad \text{for all } x \in R.$$

Definition 33 *Let f and g be any two fuzzy subsets of a hemiring R, then the intrinsic product of f and g is defined as*

$$(f \odot_h g)(x) = \vee_{x+\Sigma_{i=1}^{n} a_i b_i + z = \Sigma_{j=1}^{m} c_j d_j + z} \{\wedge_{ij} [f(a_i) \wedge g(b_i) \wedge f(c_j) \wedge g(d_j)]\}$$

and $(f \odot_h g)(x) = 0$ if x cannot be expressed as $x + \Sigma_{i=1}^{n} a_i b_i + z = \Sigma_{j=1}^{m} c_j d_j + z$.

Definition 34 *Let f be a fuzzy subset of a hemiring R. Then f is said to be a fuzzy subhemiring of R if for all $x, y \in R$.*

(i) $f(x+y) \geq f(x) \wedge f(y)$

(ii) $f(xy) \geq f(x) \wedge f(y)$.

Definition 35 *Let R be a hemiring and let $A \subseteq R$. The characteristic function C_A of A is defined to be a function from R into $[0,1]$, that is $C_A : R \rightarrow [0,1]$ such that for all $x \in R$*

$$C_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Clearly the characteristic function of any subset of R is also a fuzzy subset of R.

Definition 36 [2] *A fuzzy subset λ of a hemiring R is said to be a fuzzy left (respectively right) ideal of the hemiring R if for all $x, y \in R$*

1) $\lambda(x+y) \geq \lambda(x) \wedge \lambda(y)$

2) $\lambda(xy) \geq \lambda(y)$ (respectively $\lambda(xy) \geq \lambda(x)$).

A fuzzy subset λ of a hemiring R is called a fuzzy ideal of hemiring R if it is both, fuzzy left and right ideal of R

Proposition 37 [2] *If λ and μ are fuzzy left (respectively right) ideals of a hemiring R then $\lambda \cap \mu$ is also a fuzzy left (respectively right) ideal of R.*

Proposition 38 [2] *If λ and μ are fuzzy left (respectively right) ideals of a hemiring R then their sum $\lambda + \mu$ is also a fuzzy left (respectively right) ideal of R.*

Proposition 39 [2] *If λ and μ are fuzzy left (respectively right) ideals of a hemiring R then their product $\lambda\mu$ is also a fuzzy left (respectively right) ideal of R.*

1.4 h-Hemiregular Hemirings

Definition 40 [17] *A hemiring R is said to be h-hemiregular if for each $a \in R$, there exist $x_1, x_2, z \in R$ such that*

$$a + ax_1a + z = ax_2a + z.$$

In the case of rings the h-hemiregularity becomes the classical regularity of rings.

Example 41 [23] *Let R be the set of all non-negative integers N_0 with an element ∞ such that $\infty \geq x$ for all $x \in N_0$. Consider two operations:*

$$a + b = max\{a, b\} \qquad \text{for all } a, b \in R$$

and

$$a \cdot b = min\{a, b\} \qquad \text{for all } a, b \in R$$

Then $(R, +, \cdot)$ is an h-hemiregular hemiring.

Definition 42 [23] *Let A be a subset of a hemiring R. Then the h-closure $\bar{A}$ of A in R is defined as*

$$\bar{A} = Cl\left(A\right) = \left\{x \in R : x + a_1 + z = a_2 + z \quad \text{for some } a_1, a_2 \in A, z \in R\right\}.$$

Note that

(*i*) *If A is a left ideal of R, then $\bar{A}$ is the smallest left h-ideal of R containing A.*

(*ii*) *If A is subset of R, then $\overline{(\bar{A})} = \bar{A}$.*

(*iii*) *If $A \subseteq B \subseteq R$ then $\bar{A} \subseteq \overline{B}$.*

Lemma 43 [23] *If A and B are subsets of a hemiring R, then $\overline{AB} = Cl\left(\bar{A}\,\overline{B}\right)$.*

Lemma 44 [23] *If A and B are, respectively, right and left h-ideals of a hemiring R, then $\overline{AB} \subseteq A \cap B$.*

Lemma 45 [23] *If A and B are, respectively, right and left h-ideals of a hemiring R, then $\overline{AB} = A \cap B$ if and only if R is h-hemiregular.*

Chapter 2

REGULAR AND WEAKLY REGULAR HEMIRINGS

In this chapter we present the basic definitions and operations of the interval valued fuzzy sets and then investigate the notion of the interval valued fuzzy ideals in hemirings and regular hemirings.

2.1 Interval Valued Fuzzy Subsets

Let $\mathcal{L}$ be the family of all closed subintervals of $[0,1]$. Then according to the partial order as in [22] that is, $[\alpha,\alpha'] \leq [\beta,\beta']$ if and only if $\alpha \leq \beta, \alpha' \leq \beta'$ defined on $\mathcal{L}$ for all $[\alpha,\alpha'], [\beta,\beta'] \in \mathcal{L}$, the minimal element of $\mathcal{L}$ is $\tilde{O} = [0,0]$ and the maximal element is $\tilde{I} = [1,1]$.

Definition 46 *An interval valued fuzzy subset λ of a hemiring R is a function $\lambda : R \to \mathcal{L}$.*

We write $\lambda(x) = [\lambda^-(x), \lambda^+(x)] \subseteq [0,1]$ for all $x \in R$. Where $\lambda^-, \lambda^+ : R \to [0,1]$ are lower and upper fuzzy sets of λ, giving lower and upper limits of the image interval for each $x \in R$. Note that we have $0 \leq \lambda^-(x) \leq 1$ and $0 \leq \lambda^+(x) \leq 1$ for all $x \in R$. For simplicity we write $\lambda = [\lambda^-, \lambda^+]$.

Note that for any two interval valued fuzzy subsets λ and μ of a hemiring R, we have $\lambda \subseteq \mu$ if and only if $\lambda(x) \leq \mu(x)$ for all $x \in R$, that is $\lambda^-(x) \leq \mu^-(x)$ and $\lambda^+(x) \leq \mu^+(x)$ for all $x \in R$.

Definition 47 *For any two interval valued fuzzy subsets λ and μ of a hemiring R, union and intersection are defined, for all $x \in R$*

$$(\lambda \cup \mu)(x) = [\lambda^-(x) \vee \mu^-(x), \lambda^+(x) \vee \mu^+(x)]$$

$$(\lambda \cap \mu)(x) = [\lambda^-(x) \wedge \mu^-(x), \lambda^+(x) \wedge \mu^+(x)].$$

More generally if $\{\lambda_i : i \in I\}$ is a family of interval valued fuzzy subsets of R then for all $x \in R$,

$$(\cup_i \lambda_i)(x) = [\vee_i \lambda_i^-(x), \vee_i \lambda_i^+(x)]$$

$$(\cap_i \lambda_i)(x) = [\wedge_i \lambda_i^-(x), \wedge_i \lambda_i^+(x)].$$

Definition 48 *Let λ and μ be interval valued fuzzy subsets of a hemiring R then their sum is defined as*

$$(\lambda + \mu)(x) = \vee_{x=y+z} [\lambda^-(y) \wedge \mu^-(z), \lambda^+(y) \wedge \mu^+(z)] \qquad \textit{for all } x \in R.$$

Definition 49 *Let λ and μ be interval valued fuzzy subsets of a hemiring R then their product is defined as*

$$(\lambda\mu)(x) = \vee_{x=\Sigma_{i=1}^n y_i z_i} \{\wedge_i [\lambda^-(y_i) \wedge \mu^-(z_i), \lambda^+(y_i) \wedge \mu^+(z_i)]\}$$

if x can be expressed as $x = \sum_{i=1}^{n} y_i z_i$. Otherwise $(\lambda\mu)(x) = \tilde{O}$.

Definition 50 *Let A be a subset of a hemiring R. Then the interval valued characteristic function C_A of A is defined to be a function $C_A : R \to \mathcal{L}$ such that for all $x \in R$*

$$C_A(x) = \begin{cases} \tilde{I} = [1,1] & \text{if } x \in A \\ \tilde{O} = [0,0] & \text{if } x \notin A. \end{cases}$$

Clearly the interval valued characteristic function of any subset of R is also an interval valued fuzzy subset of R. The interval valued characteristic function can be used to indicate either membership or non-membership of any member of R in a subset A of R. Note that $C_R(x) = \tilde{I}$ for all $x \in R$.

Lemma 51 *Let λ, μ and ν be the interval valued fuzzy subsets of a hemiring R. If $\lambda \subseteq \mu$ then $\lambda\nu \subseteq \mu\nu$ and $\nu\lambda \subseteq \nu\mu$.*

Proof. Let $x \in R$. If x is not expressed as $x = \sum_{i=1}^{n} y_i z_i$, then

$$(\lambda\nu)(x) = \tilde{O} = (\mu\nu)(x)$$

Otherwise

$$\begin{aligned} (\lambda\nu)(x) &= \vee_{x=\Sigma_{i=1}^{n} y_i z_i} \left\{ \wedge_i \left[\lambda^-(y_i) \wedge \nu^-(z_i), \lambda^+(y_i) \wedge \nu^+(z_i) \right] \right\} \\ &\leq \vee_{x=\Sigma_{i=1}^{n} y_i z_i} \left\{ \wedge_i \left[\mu^-(y_i) \wedge \nu^-(z_i), \mu^+(y_i) \wedge \nu^+(z_i) \right] \right\} \qquad (\because \ \lambda \subseteq \mu) \\ &= (\mu\nu)(x) \end{aligned}$$

Hence

$$\lambda\nu \subseteq \mu\nu$$

Similarly we can prove that $\nu\lambda \subseteq \nu\mu$. ■

2.2 Interval Valued Fuzzy Ideals

Definition 52 *Let f be an interval valued fuzzy subset of a hemiring R. Then f is said to be an interval valued fuzzy subhemiring of R if for all $x, y \in R$*

$$(i) \quad f(x+y) \geq f(x) \wedge f(y)$$

$$(ii) \quad f(xy) \geq f(x) \wedge f(y).$$

Definition 53 *Let λ be an interval valued fuzzy subset of a hemiring R. Then λ is said to be an interval valued fuzzy left (respectively. right) ideal of R if and only if for all $x, y \in R$*

$$(i) \quad \lambda(x+y) \geq \lambda(x) \wedge \lambda(y)$$

$$(ii) \quad \lambda(xy) \geq \lambda(y) \text{ (respectively } \lambda(xy) \geq \lambda(x)).$$

An interval valued fuzzy subset $\lambda : R \to \mathcal{L}$ is called an interval valued fuzzy ideal of hemiring R if it is both, interval valued fuzzy left and right ideal of R.

Remark 54 *An interval valued fuzzy subset λ of a hemiring R is an interval valued fuzzy two sided ideal of R if for all $x, y \in R$*

$$(i) \quad \lambda(x+y) \geq \lambda(x) \wedge \lambda(y)$$

$$(ii) \quad \lambda(xy) \geq \lambda(x) \vee \lambda(y).$$

Remark 55 *Every interval valued fuzzy ideal of a hemiring R is also an interval valued fuzzy subhemiring of R but the converse is not true.*

Remark 56 *Note that if $\lambda = [\lambda^-, \lambda^+]$ is an interval valued fuzzy left ideal of R then for all $x, y \in R$*

$(i)\qquad \lambda(x+y) \geq \lambda(x) \wedge \lambda(y)$

$\Rightarrow \left[\lambda^{-}(x+y), \lambda^{+}(x+y)\right] \geq \left[\lambda^{-}(x), \lambda^{+}(x)\right] \wedge \left[\lambda^{-}(y), \lambda^{+}(y)\right]$

$\Rightarrow \left[\lambda^{-}(x+y), \lambda^{+}(x+y)\right] \geq \left[\lambda^{-}(x) \wedge \lambda^{-}(y), \lambda^{+}(y) \wedge \lambda^{+}(y)\right]$

$\Rightarrow \lambda^{-}(x+y) \geq \lambda^{-}(x) \wedge \lambda^{-}(y)$ and $\lambda^{+}(x+y) \geq \lambda^{+}(x) \wedge \lambda^{+}(y)$

$(ii)\qquad \lambda(xy) \geq \lambda(y)$

$\Rightarrow \left[\lambda^{-}(xy), \lambda^{+}(xy)\right] \geq \left[\lambda^{-}(y), \lambda^{+}(y)\right]$

$\Rightarrow \lambda^{-}(xy) \geq \lambda^{-}(y)$ and $\lambda^{+}(xy) \geq \lambda^{+}(y)$.

This shows that $\lambda^{-}$and $\lambda^{+}$are fuzzy left ideals of R. It's converse is also true and can be proved by reversing the above process. Similarly $\lambda = \left[\lambda^{-}, \lambda^{+}\right]$ is an interval valued fuzzy right (two sided) ideal of R if and only if $\lambda^{-}$and λ^{+} are fuzzy right (two sided) ideals of R. Similarly $\lambda = \left[\lambda^{-}, \lambda^{+}\right]$ is an interval valued fuzzy subhemiring of R if and only if $\lambda^{-}$and λ^{+} are fuzzy subhemirings of R.

Lemma 57 *An interval valued fuzzy subset λ of a hemiring R is an interval valued fuzzy subhemiring of R if and only if $\lambda + \lambda \subseteq \lambda$ and $\lambda^2 \subseteq \lambda$.*

Proof. Let λ be an an interval valued fuzzy subhemiring of R then λ^{-} and λ^{+} are fuzzy subhemiring of R, and for all $x \in R$

$$\begin{aligned}
(\lambda+\lambda)(x) &= \vee_{x=y+z}\left[\lambda^{-}(y) \wedge \lambda^{-}(z), \lambda^{+}(y) \wedge \lambda^{+}(z)\right] \\
&\leq \vee_{x=y+z}\left[\lambda^{-}(y+z), \lambda^{+}(y+z)\right] \\
&= \vee_{x=y+z}\left[\lambda^{-}(x), \lambda^{+}(x)\right] \\
&= \lambda(x).
\end{aligned}$$

Thus

$$\lambda + \lambda \subseteq \lambda.$$

And $\lambda^2(x) = \tilde{O} \leq \lambda(x)$ if x cannot be expressed as $x = \sum_{i=1}^{n} y_i z_i$, otherwise

$$\begin{aligned}
\lambda^2(x) &= (\lambda\lambda)(x) \\
&= \vee_{x=\Sigma_{i=1}^{n} y_i z_i} \left\{ \wedge_i \left[\lambda^-(y_i) \wedge \lambda^-(z_i), \lambda^+(y_i) \wedge \lambda^+(z_i) \right] \right\} \\
&\leq \vee_{x=\Sigma_{i=1}^{n} y_i z_i} \left[\wedge_i \lambda^-(y_i z_i), \wedge_i \lambda^+(y_i z_i) \right] \\
&\leq \vee_{x=\Sigma_{i=1}^{n} y_i z_i} \left[\lambda^-(\sum_{i=1}^{n} y_i z_i), \lambda^+(\sum_{i=1}^{n} y_i z_i) \right] \\
&= \vee_{x=\Sigma_{i=1}^{n} y_i z_i} \left[\lambda^-(x), \lambda^+(x) \right] \\
&= \lambda(x).
\end{aligned}$$

Thus $\lambda^2 \subseteq \lambda$.

Conversely, let λ be an interval valued fuzzy subset of R such that $\lambda + \lambda \subseteq \lambda$ and $\lambda^2 \subseteq \lambda$. Then for all $x, y \in R$

$$\begin{aligned}
\lambda(x+y) &\geq (\lambda + \lambda)(x+y) \\
&= \vee_{x+y=a+b} \left[\lambda^-(a) \wedge \lambda^-(b), \lambda^+(a) \wedge \lambda^+(b) \right] \\
&\geq \left[\lambda^-(x) \wedge \lambda^-(y), \lambda^+(x) \wedge \lambda^+(y) \right] \\
&= \left[\lambda^-(x), \lambda^+(x) \right] \wedge \left[\lambda^-(y), \lambda^+(y) \right] \\
&= \lambda(x) \wedge \lambda(y)
\end{aligned}$$

and

$$\begin{aligned}
\lambda(xy) &\geq \lambda^2(xy) = (\lambda\lambda)(xy) \\
&= \vee_{xy=\Sigma_{i=1}^n y_i z_i} \left\{ \wedge_i \left[\lambda^-(y_i) \wedge \lambda^-(z_i), \lambda^+(y_i) \wedge \lambda^+(z_i) \right] \right\} \\
&\geq \left[\lambda^-(x) \wedge \lambda^-(y), \lambda^+(x) \wedge \lambda^+(y) \right] \\
&= \left[\lambda^-(x), \lambda^+(x) \right] \wedge \left[\lambda^-(y), \lambda^+(y) \right] \\
&= \lambda(x) \wedge \lambda(y).
\end{aligned}$$

Thus

$$\lambda(x+y) \geq \lambda(x) \wedge \lambda(y) \qquad \text{for all } x, y \in R$$

and

$$\lambda(xy) \geq \lambda(x) \wedge \lambda(y) \qquad \text{for all } x, y \in R.$$

Hence λ is an interval valued fuzzy subhemiring of R. ■

Lemma 58 *An interval valued fuzzy subset λ of a hemiring R is an interval valued fuzzy left (respectively right) ideal of R if and only if $\lambda + \lambda \subseteq \lambda$ and $C_R\lambda \subseteq \lambda$ (respectively $\lambda C_R \subseteq \lambda$).*

Proof. Let λ be an an interval valued fuzzy left ideal of R then λ^- and λ^+ are fuzzy left ideals of R, and for all $x \in R$

$$\begin{aligned}
(\lambda + \lambda)(x) &= \vee_{x=y+z} \left[\lambda^-(y) \wedge \lambda^-(z), \lambda^+(y) \wedge \lambda^+(z) \right] \\
&\leq \vee_{x=y+z} \left[\lambda^-(y+z), \lambda^+(y+z) \right] \\
&= \vee_{x=y+z} \left[\lambda^-(x), \lambda^+(x) \right] \\
&= \lambda(x).
\end{aligned}$$

Thus

$$\lambda + \lambda \subseteq \lambda.$$

And $C_R\lambda(x) = \tilde{O} \leq \lambda(x)$ if x cannot be expressed as $x = \sum_{i=1}^{n} y_i z_i$, otherwise

$$\begin{aligned}
C_R\lambda(x) &= \vee_{x=\Sigma_{i=1}^n y_i z_i} \left\{ \wedge_i \left[C_R^-(y_i) \wedge \lambda^-(z_i), C_R^+(y_i) \wedge \lambda^+(z_i) \right] \right\} \\
&\leq \vee_{x=\Sigma_{i=1}^n y_i z_i} \left\{ \wedge_i \left[1 \wedge \lambda^-(z_i), 1 \wedge \lambda^+(z_i) \right] \right\} \\
&= \vee_{x=\Sigma_{i=1}^n y_i z_i} \left\{ \wedge_i \left[\lambda^-(z_i), \lambda^+(z_i) \right] \right\} \\
&\leq \vee_{x=\Sigma_{i=1}^n y_i z_i} \left\{ \wedge_i \left[\lambda^-(y_i z_i), \lambda^+(y_i z_i) \right] \right\} \\
&\leq \vee_{x=\Sigma_{i=1}^n y_i z_i} \left[\lambda^-(\sum_{i=1}^{n} y_i z_i), \lambda^+(\sum_{i=1}^{n} y_i z_i) \right] \\
&= \vee_{x=\Sigma_{i=1}^n y_i z_i} \left[\lambda^-(x), \lambda^+(x) \right] \\
&= \lambda(x).
\end{aligned}$$

Thus

$$C_R\lambda \subseteq \lambda$$

Conversely, let λ be an interval valued fuzzy subset of R such that $\lambda + \lambda \subseteq \lambda$ and $C_R\lambda \subseteq \lambda$. Then for all $x, y \in R$

$$\begin{aligned}
\lambda(x+y) &\geq (\lambda + \lambda)(x+y) \\
&= \vee_{x+y=a+b} \left[\lambda^-(a) \wedge \lambda^-(b), \lambda^+(a) \wedge \lambda^+(b) \right] \\
&\geq \left[\lambda^-(x) \wedge \lambda^-(y), \lambda^+(x) \wedge \lambda^+(y) \right] \\
&= \left[\lambda^-(x), \lambda^+(x) \right] \wedge \left[\lambda^-(y), \lambda^+(y) \right] \\
&= \lambda(x) \wedge \lambda(y)
\end{aligned}$$

and

$$\begin{aligned}
\lambda(xy) &\geq C_R\lambda(xy) \\
&= \vee_{xy=\Sigma_{i=1}^{n} y_i z_i}\left\{\wedge_i\left[C_R^-(y_i)\wedge\lambda^-(z_i), C_R^+(y_i)\wedge\lambda^+(z_i)\right]\right\} \\
&\geq \left[C_R^-(x)\wedge\lambda^-(y), C_R^+(x)\wedge\lambda^+(y)\right] \\
&= \left[1\wedge\lambda^-(y), 1\wedge\lambda^+(y)\right] \\
&= \left[\lambda^-(y), \lambda^+(y)\right] \\
&= \lambda(y).
\end{aligned}$$

Thus

$$\lambda(x+y) \geq \lambda(x)\wedge\lambda(y) \qquad \text{for all } x, y \in R$$

and

$$\lambda(xy) \geq \lambda(y) \qquad \text{for all } x, y \in R$$

Hence λ is an interval valued fuzzy left ideal of R. ■

Proposition 59 *A subset A of a hemiring R is a subhemiring of R if and only if the interval valued characteristic function C_A is an interval valued fuzzy subhemiring of R.*

Proof. Suppose that A is a subhemiring of R and let $x, y \in R$

Case I If $x, y \in A$ then $x+y, xy \in A$. Thus

$$C_A(x+y) = \tilde{I} = \tilde{I}\wedge\tilde{I} = C_A(x)\wedge C_A(y)$$

and

$$C_A(xy) = \tilde{I} = \tilde{I}\wedge\tilde{I} = C_A(x)\wedge C_A(y)$$

Case II If at least one, say $y \notin A$ then $C_A(y) = \tilde{O}$. Then

$$\begin{aligned} C_A(x+y) &\geq \tilde{O} = C_A(x) \wedge \tilde{O} \\ &= C_A(x) \wedge C_A(y) \end{aligned}$$

and

$$\begin{aligned} C_A(xy) &\geq \tilde{O} = C_A(x) \wedge \tilde{O} \\ &= C_A(x) \wedge C_A(y). \end{aligned}$$

Thus in both cases

$$C_A(x+y) \geq C_A(x) \wedge C_A(y) \qquad \text{for all } x, y \in R$$

and

$$C_A(xy) \geq C_A(x) \wedge C_A(y) \qquad \text{for all } x, y \in R$$

Hence C_A is an interval valued fuzzy subhemiring of R.

Conversely, suppose that C_A is an interval valued fuzzy subhemiring of R and let $x, y \in A$ then

$$\begin{aligned} C_A(x+y) &\geq C_A(x) \wedge C_A(y) \\ &= \tilde{I} \wedge \tilde{I} = \tilde{I} \end{aligned}$$

and

$$\begin{aligned} C_A(xy) &\geq C_A(x) \wedge C_A(y) \\ &= \tilde{I} \wedge \tilde{I} = \tilde{I}. \end{aligned}$$

$\Rightarrow$

$$C_A(x+y) = \tilde{I} = C_A(xy)$$

Thus

$$x+y, xy \in A \qquad \text{for all } x, y \in A$$

This shows that A is a subhemiring of R. ■

Proposition 60 *A subset A of a hemiring R is a left (respectively right) ideal of R if and only if the interval valued characteristic function C_A is an interval valued fuzzy left (respectively right) ideal of R.*

Proof. Let A be a left ideal of R and let $x, y \in R$. Then by the Proposition 59

$$C_A(x+y) \geq C_A(x) \wedge C_A(y) \qquad \text{for all } x, y \in R$$

Case I If $y \in A$ then $xy \in A$ because A is left ideal of R. Thus

$$C_A(xy) = \tilde{I} = C_A(y).$$

Case II If $y \notin A$ then$C_A(xy) \geq \tilde{O} = C_A(y)$. Thus $C_A(xy) \geq C_A(y)$ for all $x, y \in R$.

Hence C_A is an interval valued fuzzy left ideal of R

Conversely, let C_A be an interval valued fuzzy left ideal of R and let $x, y \in A$ then $C_A(x) = \tilde{I} = C_A(y)$ and therefore

$$\begin{aligned} C_A(x+y) &\geq C_A(x) \wedge C_A(y) \\ &= \tilde{I} \wedge \tilde{I} = \tilde{I} \end{aligned}$$

This shows that

$$x + y \in A \qquad \text{for all } x,\ y \in A$$

Let $x \in R$ and $y \in A$, then

$$C_A(xy) \geq C_A(y) = \tilde{I}$$

$\Rightarrow$

$$C_A(xy) = \tilde{I}$$

$\Rightarrow$

$$xy \in A \qquad \text{for all } x \in R \text{ and } y \in A$$

Thus A is a left ideal of R. ■

Proposition 61 *If λ and μ are interval valued fuzzy left (respectively right) ideals of R then their sum $\lambda + \mu$ is also an interval valued fuzzy left (respectively right) ideal of R.*

Proof. To show that $\lambda + \mu$ is an interval valued fuzzy left ideal of R, we will prove that

(i) $\quad (\lambda + \mu)(x + y) \geq (\lambda + \mu)(x) \wedge (\lambda + \mu)(y) \qquad \text{for all } x, y \in R$

(ii) $\quad (\lambda + \mu)(xy) \geq (\lambda + \mu)(y) \qquad \text{for all } x, y \in R$

Recall that if $\lambda = [\lambda^-, \lambda^+]$ and $\mu = [\mu^-, \mu^+]$ are interval valued fuzzy left ideals of R then $\lambda^-, \lambda^+, \mu^-, \mu^+$ are fuzzy left ideals of R. Let $x, y \in R$. Then

$$\begin{aligned}
(\lambda+\mu)(x) \wedge (\lambda+\mu)(y) &= \left\{\vee_{x=a+b}\left[\lambda^-(a)\wedge\mu^-(b), \lambda^+(a)\wedge\mu^+(b)\right]\right\} \wedge \\
&\quad \left\{\vee_{y=u+v}\left[\lambda^-(u)\wedge\mu^-(v), \lambda^+(u)\wedge\mu^+(v)\right]\right\} \\
&= \vee_{x=a+b}\left[\lambda^-(a)\wedge\mu^-(b), \lambda^+(a)\wedge\mu^+(b)\right] \wedge \\
&\quad \left\{\vee_{y=u+v}\left[\lambda^-(u)\wedge\mu^-(v), \lambda^+(u)\wedge\mu^+(v)\right]\right\} \\
&= \vee_{x=a+b,\ y=u+v}\left\{\left[\lambda^-(a)\wedge\mu^-(b), \lambda^+(a)\wedge\mu^+(b)\right] \wedge \right. \\
&\quad \left.\left[\lambda^-(u)\wedge\mu^-(v), \lambda^+(u)\wedge\mu^+(v)\right]\right\} \\
&= \vee_{x=a+b,\ y=u+v}[\lambda^-(a)\wedge\mu^-(b)\wedge\lambda^-(u)\wedge\mu^-(v), \\
&\quad \lambda^+(a)\wedge\mu^+(b)\wedge\lambda^+(u)\wedge\mu^+(v)] \\
&\leq \vee_{x+y=(a+u)+(b+v)}\left[\lambda^-(a+u)\wedge\mu^-(b+v), \lambda^+(a+u)\wedge\mu^+(b+v)\right] \\
&\leq \vee_{x+y=u'+v'}\left[\lambda^-(u')\wedge\mu^-(v'), \lambda^+(u')\wedge\mu^+(v')\right] \\
&= (\lambda+\mu)(x+y)
\end{aligned}$$

and

$$\begin{aligned}
(\lambda+\mu)(x) &= \vee_{x=u+v}[\lambda^-(u)\wedge\mu^-(v), \lambda^+(u)\wedge\mu^+(v)] \\
&\leq \vee_{yx=yu+yv}\left[\lambda^-(yu)\wedge\mu^-(yv), \lambda^+(yu)\wedge\mu^+(yv)\right] \\
&\leq \vee_{yx=u'+v'}\left[\lambda^-(u')\wedge\mu^-(v'), \lambda^+(u')\wedge\mu^+(v')\right] \\
&= (\lambda+\mu)(yx).
\end{aligned}$$

Thus

$$(\lambda+\mu)(x+y) \geq (\lambda+\mu)(x) \wedge (\lambda+\mu)(y) \qquad \text{for all } x, y \in R$$

and

$$(\lambda + \mu)(yx) \geq (\lambda + \mu)(x) \qquad \text{for all } x, y \in R$$

Hence $\lambda + \mu$ is an interval valued fuzzy left ideal of R. ∎

Proposition 62 *If λ is an interval valued fuzzy two-sided (respectively right) ideal and μ is an interval valued fuzzy left (respectively two-sided) ideal of a hemiring R then $\lambda\mu$ is an interval valued fuzzy left (respectively right) ideal of R contained in $\lambda \cap \mu$.*

Proof. Let λ be an interval valued fuzzy ideal and μ be an interval valued fuzzy left ideal of R. Then for any $x, y \in R$

$$(\lambda\mu)(x) = \vee_{x=\Sigma_{i=1}^{n} y_i z_i} \{\wedge_i \left[\lambda^-(y_i) \wedge \mu^-(z_i), \lambda^+(y_i) \wedge \mu^+(z_i)\right]\}$$

and

$$(\lambda\mu)(y) = \vee_{y=\Sigma_{i=1}^{n} y'_i z'_i} \{\wedge_i \left[\lambda^-(y'_i) \wedge \mu^-(z'_i), \lambda^+(y'_i) \wedge \mu^+(z'_i)\right]\}.$$

Now

$$\begin{aligned}
(\lambda\mu)(x)\wedge(\lambda\mu)(y) &= \left\{\vee_{x=\Sigma_{i=1}^{n}y_iz_i}\{\wedge_i\left[\lambda^-(y_i)\wedge\mu^-(z_i),\lambda^+(y_i)\wedge\mu^+(z_i)\right]\}\right\} \\
&\quad \wedge\left\{\vee_{y=\Sigma_{j=1}^{m}y'_jz'_j}\left\{\wedge_j\left[\lambda^-\left(y'_j\right)\wedge\mu^-\left(z'_j\right),\lambda^+\left(y'_j\right)\wedge\mu^+\left(z'_j\right)\right]\right\}\right\} \\
&= \vee_{x=\Sigma_{i=1}^{n}y_iz_i}\{\wedge_i\left[\lambda^-(y_i)\wedge\mu^-(z_i),\lambda^+(y_i)\wedge\mu^+(z_i)\right]\} \\
&\quad \wedge\left\{\vee_{y=\Sigma_{j=1}^{m}y'_jz'_j}\left\{\wedge_j\left[\lambda^-\left(y'_j\right)\wedge\mu^-\left(z'_j\right),\lambda^+\left(y'_j\right)\wedge\mu^+\left(z'_j\right)\right]\right\}\right\} \\
&= \vee_{x=\Sigma_{i=1}^{n}y_iz_i,\ y=\Sigma_{j=1}^{m}y'_jz'_j}\left\{\wedge_i\left[\lambda^-(y_i)\wedge\mu^-(z_i),\lambda^+(y_i)\wedge\mu^+(z_i)\right]\right\} \\
&\quad \wedge\left\{\wedge_j\left[\lambda^-\left(y'_j\right)\wedge\mu^-\left(z'_j\right),\lambda^+\left(y'_j\right)\wedge\mu^+\left(z'_j\right)\right]\right\} \\
&= \vee_{x+y=\Sigma_{i=1}^{n}y_iz_i+\Sigma_{j=1}^{m}y'_jz'_j}[\{\wedge_i[\lambda^-(y_i)\wedge\mu^-(z_i)]\}\wedge\{\wedge_j[\lambda^-\left(y'_j\right)\wedge\mu^-\left(z'_j\right)]\}, \\
&\quad \{\wedge_i[\lambda^+(y_i)\wedge\mu^+(z_i)]\}\wedge\{\wedge_j[\lambda^+\left(y'_j\right)\wedge\mu^+\left(z'_j\right)]\}] \\
&= \vee_{x+y=\Sigma_{i=1}^{n}y_iz_i+\Sigma_{j=1}^{m}y'_jz'_j}[\{\wedge_i\lambda^-(y_i)\}\wedge\{\wedge_j\lambda^-\left(y'_j\right)\}\wedge\{\wedge_i\mu^-(z_i)\}\wedge \\
&\quad \{\wedge_j\mu^-\left(z'_j\right)\},\{\wedge_i\lambda^+(y_i)\}\wedge\{\wedge_j\lambda^+\left(y'_j\right)\}\wedge\{\wedge_i\mu^+(z_i)\}\wedge\{\wedge_j\mu^+\left(z'_j\right)\}] \\
&\leq \vee_{x+y=\sum_{k=1}^{r}u_kv_k}\{\wedge_k[\lambda^-(u_k)\wedge\mu^-(v_k),\lambda^+(u_k)\wedge\mu^+(v_k)]\} \\
&= (\lambda\mu)(x+y).
\end{aligned}$$

For any $a, x \in R$

$$\begin{aligned}
(\lambda\mu)(x) &= \vee_{x=\Sigma_{i=1}^{n}y_iz_i}\{\wedge_i\left[\lambda^-(y_i)\wedge\mu^-(z_i),\lambda^+(y_i)\wedge\mu^+(z_i)\right]\} \\
&\leq \vee_{ax=a\Sigma_{i=1}^{n}y_iz_i}\{\wedge_i\left[\lambda^-(ay_i)\wedge\mu^-(z_i),\lambda^+(ay_i)\wedge\mu^+(z_i)\right]\} \\
&= \vee_{ax=\Sigma_{i=1}^{n}(ay_i)z_i}\{\wedge_i\left[\lambda^-(ay_i)\wedge\mu^-(z_i),\lambda^+(ay_i)\wedge\mu^+(z_i)\right]\} \\
&\leq \vee_{ax=\Sigma_{i=1}^{n}y'_iz'_i}\{\wedge_i\left[\lambda^-(y'_i)\wedge\mu^-(z'_i),\lambda^+(y'_i)\wedge\mu^+(z'_i)\right]\} \\
&= (\lambda\mu)(ax).
\end{aligned}$$

Thus

$$(\lambda\mu)(x+y) \geq (\lambda\mu)(x) \wedge (\lambda\mu)(y) \qquad \text{for all } x, y \in R$$

and

$$(\lambda\mu)(ax) \geq (\lambda\mu)(x) \qquad \text{for all } a, x \in R.$$

Hence $\lambda\mu$ is an interval valued fuzzy left ideal of R. Now for any $x \in R$

$$\begin{aligned}
(\lambda\mu)(x) &= \vee_{x=\Sigma_{i=1}^{n} y_i z_i} \left\{\wedge_i \left[\lambda^-(y_i) \wedge \mu^-(z_i), \lambda^+(y_i) \wedge \mu^+(z_i)\right]\right\} \\
&\leq \vee_{x=\Sigma_{i=1}^{n} y_i z_i} \left\{\wedge_i \left[\lambda^-(y_i z_i) \wedge \mu^-(y_i z_i), \lambda^+(y_i z_i) \wedge \mu^+(y_i z_i)\right]\right\} \\
&= \vee_{x=\Sigma_{i=1}^{n} y_i z_i} \left\{\left[\wedge_i \lambda(y_i z_i)\right] \wedge \left[\wedge_i \mu(y_i z_i)\right]\right\} \\
&\leq \vee_{x=\Sigma_{i=1}^{n} y_i z_i} \left\{\lambda\left(\sum_{i=1}^{n} y_i z_i\right) \wedge \mu\left(\sum_{i=1}^{n} y_i z_i\right)\right\} \\
&= \vee_{x=\Sigma_{i=1}^{n} y_i z_i} \left\{\lambda(x) \wedge \mu(x)\right\} \\
&= \lambda(x) \wedge \mu(x) \\
&= (\lambda \cap \mu)(x).
\end{aligned}$$

Thus $\lambda\mu \subseteq \lambda \cap \mu$. ■

Remark 63 *If λ and μ are interval valued fuzzy ideals of R then $\lambda\mu$ is also an interval valued fuzzy ideal of R contained in $\lambda \cap \mu$. While if λ and μ are interval valued fuzzy left (respectively right) ideals of R then $\lambda\mu$ is again an interval valued fuzzy left (respectively right) ideal of R, but in this case $\lambda\mu$ is not contained in $\lambda \cap \mu$. In general $\lambda\mu \neq \lambda \cap \mu$.*

Remark 64 *If λ and μ are interval valued fuzzy subhemirings of R then $\lambda \cap \mu$ is also an interval valued fuzzy subhemiring of R. Similarly, if λ and μ are interval valued* *fuzzy left (respectively right) ideals of R then* $\lambda \cap \mu$ *is also an interval valued fuzzy left (respectively right) ideal of* R.

Definition 65 *An interval valued fuzzy left (respectively right) ideal* λ *of a hemiring* R *is called idempotent if* $\lambda^2 = \lambda$.

2.3 Characterizations of Hemirings by the Properties of their Interval Valued Fuzzy Ideals

In this section we characterize regular and weakly regular hemirings by the properties of their interval valued fuzzy ideals.

Theorem 66 *The following assertions for a hemiring* R *are equivalent:*

(1) R *is von Neumann regular.*

(2) *For any right ideal* A *and any left ideal* B *of* R, $A \cap B = AB$

(3) *For any interval valued fuzzy right ideal* λ *and interval valued fuzzy left ideal* μ *of* R, $\lambda \cap \mu = \lambda\mu$.

Proof. For (1) $\Leftrightarrow$ (2), we refer to Golan [8, Proposition 5.27, p. 63].

(1) $\Rightarrow$ (3)

Let λ be an interval valued fuzzy right ideal and μ be an interval valued fuzzy left ideal of R. Since R is regular so for any $x \in R$ there exist $a \in R$ such that $x = xax$. Now

$$\begin{aligned} (\lambda \cap \mu)(x) &= \lambda(x) \wedge \mu(x) \\ &\leq \lambda(x) \wedge \mu(ax). \end{aligned}$$

Thus

$$\begin{aligned}(\lambda \cap \mu)(x) &\leq \vee_{x=\Sigma_{i=1}^{n} y_i z_i} \left\{ \wedge_i \left[\lambda^- (y_i) \wedge \mu^- (z_i), \lambda^+ (y_i) \wedge \mu^+ (z_i) \right] \right\} \\ &= (\lambda\mu)(x).\end{aligned}$$

This implies

$$\lambda \cap \mu \subseteq \lambda\mu$$

And by Proposition 62

$$\lambda\mu \subseteq \lambda \cap \mu$$

Hence

$$\lambda\mu = \lambda \cap \mu.$$

(3) $\Rightarrow$ (2)

Let A be a right ideal and B be a left ideal of R. Then the interval valued characteristic functions C_A and C_B are interval valued fuzzy right and interval valued fuzzy left ideals of R respectively, and by hypothesis (3)

$$C_A.C_B = C_A \cap C_B$$

$\Rightarrow$

$$C_{AB} = C_{A \cap B}.$$

Thus $AB = A \cap B$. ■

Theorem 67 *The following assertions for a hemiring R with identity "1" are equivalent:*

1) *R is right weakly regular.*

2) *All right ideals of R are idempotent.*

3) $IJ = I \cap J$ *for each right ideal I and two-sided ideal J of R.*

4) *All interval valued fuzzy right ideals of R are idempotent.*

5) $\lambda\mu = \lambda \cap \mu$ *for each interval valued fuzzy right ideal λ and interval valued fuzzy two-sided ideal μ of R.*

If R is commutative then the above assertions are equivalent to

6) *R is von-Neumann regular.*

Proof. $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ follow by Theorem 18.

$(1) \Rightarrow (4)$

Let λ be an interval valued fuzzy right ideal of R and let $x \in R$. Then

$$\begin{aligned}
\lambda^2(x) &= (\lambda.\lambda)(x) \\
&= \vee_{x=\Sigma_{i=1}^n y_i z_i}\left\{\wedge_i\left[\lambda^-(y_i)\wedge\lambda^-(z_i),\lambda^+(y_i)\wedge\lambda^+(z_i)\right]\right\} \\
&\leq \vee_{x=\Sigma_{i=1}^n y_i z_i}\left\{\wedge_i\left[\lambda^-(y_i z_i)\wedge\lambda^-(z_i),\lambda^+(y_i z_i)\wedge\lambda^+(z_i)\right]\right\} \\
&= \vee_{x=\Sigma_{i=1}^n y_i z_i}\left[\left\{\wedge_i\lambda^-(y_i z_i)\right\}\wedge\lambda^-(z_i),\left\{\wedge_i\lambda^+(y_i z_i)\right\}\wedge\lambda^+(z_i)\right] \\
&\leq \vee_{x=\Sigma_{i=1}^n y_i z_i}\left[\lambda^-\left(\sum_{i=1}^n y_i z_i\right)\wedge\lambda^-(z_i),\lambda^+\left(\sum_{i=1}^n y_i z_i\right)\wedge\lambda^+(z_i)\right] \\
&\leq \vee_{x=\Sigma_{i=1}^n y_i z_i}\left[\lambda^-(x)\wedge\lambda^-(z_i),\lambda^+(x)\wedge\lambda^+(z_i)\right] \\
&\leq \vee_{x=\Sigma_{i=1}^n y_i z_i}\left[\lambda^-(x),\lambda^+(x)\right] \\
&= \lambda(x)
\end{aligned}$$

$\Rightarrow$

$$\lambda^2 \subseteq \lambda$$

Now since R is right weakly regular so $x \in xRxR$. Hence we can write $x = \sum_{i=1}^{n} xa_ixb_i$ where $a_i, b_i \in R$ and $n \in \mathbb{N}$

Now

$$\lambda(x) = \lambda(x) \wedge \lambda(x) \leq \lambda(xa_i) \wedge \lambda(xb_i) \qquad \text{for all } i$$

$\Rightarrow$

$$\begin{aligned}
\lambda(x) &\leq \wedge_i \{\lambda(xa_i) \wedge \lambda(xb_i)\} \\
&= \wedge_i \left[\lambda^-(xa_i) \wedge \lambda^-(xb_i), \lambda^+(xa_i) \wedge \lambda^+(xb_i)\right] \\
&\leq \vee_{x=\Sigma_{i=1}^{n} xa_ixb_i} \left\{\wedge_i \left[\lambda^-(xa_i) \wedge \lambda^-(xb_i), \lambda^+(xa_i) \wedge \lambda^+(xb_i)\right]\right\} \\
&\leq \vee_{x=\Sigma_{j=1}^{m} y_jz_j} \left\{\wedge_j \left[\lambda^-(y_j) \wedge \lambda^-(z_j), \lambda^+(y_j) \wedge \lambda^+(z_j)\right]\right\} \\
&= \lambda.\lambda(x) \\
&= \lambda^2(x)
\end{aligned}$$

$\Rightarrow$

$$\lambda \subseteq \lambda^2$$

Thus

$$\lambda = \lambda^2$$

and hence λ is idempotent.

(4) $\Rightarrow$ (1)

Let $x \in R$ and let $A = xR$ be a right ideal of R generated by x then $x \in A$ and the characteristic function C_A of A is interval valued fuzzy right ideal of R and by hypothesis (4)

$$C_A = C_A.C_A = C_{A^2}$$

$\Rightarrow$

$$A = A^2$$

$\Rightarrow$

$$x \in A^2 = (xR)^2 \qquad (\because x \in A)$$

$\Rightarrow$

$$x \in (xR)^2 = xRxR$$

Thus R is right weakly regular.

$(1) \Rightarrow (5)$

Let λ be an interval valued fuzzy right ideal and μ be an interval valued fuzzy two-sided ideal of R. Since R is right weakly regular so for any $x \in R$ we can write $x = \sum_{i=1}^{n} xa_ixb_i \quad , a_i, b_i \in R$, $n \in \mathbb{N}$. Now

$$\begin{aligned}(\lambda \cap \mu)(x) &= \lambda(x) \wedge \mu(x) \\ &\leq \lambda(xa_i) \wedge \mu(xb_i) \qquad \text{for all } i\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}(\lambda \cap \mu)(x) &\leq \wedge_i \{\lambda(xa_i) \wedge \mu(xb_i)\} \\ &= \wedge_i \left[\lambda^-(xa_i) \wedge \mu^-(xb_i), \lambda^+(xa_i) \wedge \mu^+(xb_i)\right] \\ &\leq \vee_{x=\Sigma_{i=1}^{n} xa_ixb_i} \left\{\wedge_i \left[\lambda^-(xa_i) \wedge \mu^-(xb_i), \lambda^+(xa_i) \wedge \mu^+(xb_i)\right]\right\} \\ &\leq \vee_{x=\Sigma_{j=1}^{m} y_jz_j} \left\{\wedge_i \left[\lambda^-(y_i) \wedge \mu^-(z_i), \lambda^+(y_i) \wedge \mu^+(z_i)\right]\right\} \\ &= (\lambda\mu)(x)\end{aligned}$$

This implies

$$\lambda \cap \mu \subseteq \lambda\mu$$

And by Proposition 62

$$\lambda\mu \subseteq \lambda \cap \mu$$

Hence

$$\lambda\mu = \lambda \cap \mu$$

$(5) \Rightarrow (3)$

Let I be a right ideal and J be a two-sided ideal of R. Then the interval valued characteristic functions C_I and C_J are interval valued fuzzy right and interval valued fuzzy two-sided ideals of R and by hypothesis (5)

$$C_I.C_J = C_I \cap C_J$$

$\Rightarrow$

$$C_{IJ} = C_{I \cap J}$$

$\Rightarrow$

$$IJ = I \cap J$$

$(1) \Leftrightarrow (6)$ is obvious. ■

Chapter 3

FULLY IDEMPOTENT HEMIRINGS

In this chapter we characterize those hemirings for which each interval valued fuzzy ideal is idempotent. The space of interval valued fuzzy prime ideals is topologized.

3.1 Idempotent Ideals

Recall that a hemiring R is fully idempotent if each of its ideal is idempotent, that is $I^2 = I$ for each ideal I of R.

Theorem 68 *The following assertions for a hemiring R with identity "1" are equivalent:*

1) *R is fully idempotent.*
2) *Each interval valued fuzzy ideal of R is idempotent.*

3) *If* λ *and* μ *are interval valued fuzzy ideals of* R *then* $\lambda \cap \mu = \lambda\mu$.

If R *is commutative then the above conditions are equivalent to*

4) R *is von-Neumann regular.*

Proof. (1) $\Rightarrow$ (2)

Let λ be an interval valued fuzzy ideal of R and $x \in R$. Then

$$\begin{aligned}
\lambda^2(x) &= (\lambda\lambda)(x) \\
&= \vee_{x=\Sigma_{i=1}^{n} y_i z_i} \left\{\wedge_i \left[\lambda^-(y_i) \wedge \lambda^-(z_i), \lambda^+(y_i) \wedge \lambda^+(z_i)\right]\right\} \\
&\leq \vee_{x=\Sigma_{i=1}^{n} y_i z_i} \left\{\wedge_i \left[\lambda^-(y_i z_i) \wedge \lambda^-(y_i z_i), \lambda^+(y_i z_i) \wedge \lambda^+(y_i z_i)\right]\right\} \\
&= \vee_{x=\Sigma_{i=1}^{n} y_i z_i} \left[\wedge_i \lambda^-(y_i z_i), \wedge_i \lambda^+(y_i z_i)\right] \\
&\leq \vee_{x=\Sigma_{i=1}^{n} y_i z_i} \left[\lambda^-\left(\sum_{i=1}^{n} y_i z_i\right), \lambda^+\left(\sum_{i=1}^{n} y_i z_i\right)\right] \\
&= \vee_{x=\Sigma_{i=1}^{n} y_i z_i} \left[\lambda^-(x), \lambda^+(x)\right] \\
&= \lambda(x).
\end{aligned}$$

Thus $\lambda^2 \subseteq \lambda$. For the reverse inclusion, since R is fully idempotent, so

$$x \in <x> = <x>^2 = RxRRxR$$

Thus $x = \sum_{i=1}^{n} a_i x a_i' b_i x b_i'$ for some $a_i, a_i', b_i, b_i' \in R$.

As λ is interval valued fuzzy ideal so

$$\begin{aligned}
\lambda(x) &= \lambda(x) \wedge \lambda(x) \\
&\leq \lambda(a_i x a_i') \wedge \lambda(b_i x b_i') \qquad \text{for all } a_i, a_i', b_i, b_i' \in R
\end{aligned}$$

This implies

$$\begin{aligned}\lambda(x) &\leq \wedge_i \left[\lambda^-(a_i x a_i') \wedge \lambda^-(b_i x b_i'), \lambda^+(a_i x a_i') \wedge \lambda^+(b_i x b_i')\right] \\ &\leq \vee_{x=\Sigma_{i=1}^n a_i x a_i' b_i x b_i'} \left\{\wedge_i \left[\lambda^-(a_i x a_i') \wedge \lambda^-(b_i x b_i'), \lambda^+(a_i x a_i') \wedge \lambda^+(b_i x b_i')\right]\right\} \\ &\leq \vee_{x=\Sigma_{j=1}^m y_j z_j} \left\{\wedge_j \left[\lambda^-(y_j) \wedge \lambda^-(z_j), \lambda^+(y_j) \wedge \lambda^+(z_j)\right]\right\} \\ &= \lambda\lambda(x) = \lambda^2(x)\end{aligned}$$

This shows $\lambda \subseteq \lambda^2$. Hence $\lambda = \lambda^2$.

$(2) \Rightarrow (1)$

Let I be an ideal of R and let C_I be the interval valued characteristic function of I. Then C_I is the interval valued fuzzy ideal of R which is, by hypothesis, idempotent, that is

$$(C_I)^2 = C_I C_I = C_I$$

This implies $C_{I^2} = C_I$. Hence $I^2 = I$. Thus R is fully idempotent.

$(1) \Rightarrow (3)$

Let λ and μ be interval valued fuzzy ideals of R. Then since R is fully idempotent so for any $x \in R$, we have $x \in <x> = <x>^2$. Thus as we argued in $(1) \Rightarrow (2)$

$$\begin{aligned}(\lambda \cap \mu)(x) &= \lambda(x) \wedge \mu(x) \\ &\leq \vee_{x=\Sigma_{j=1}^m y_j z_j} \left\{\wedge_j \left[\lambda^-(y_j) \wedge \mu^-(z_j), \lambda^+(y_j) \wedge \mu^+(z_j)\right]\right\} \\ &= (\lambda\mu)(x)\end{aligned}$$

This shows $\lambda \cap \mu \subseteq \lambda\mu$. And $\lambda\mu \subseteq \lambda \cap \mu$ by Proposition 62. Hence

$$\lambda \cap \mu = \lambda\mu.$$

(3) $\Rightarrow$ (2)

Let λ be an interval valued fuzzy ideal of R . Then by hypothesis (3)

$$\lambda^2 = \lambda\lambda = \lambda \cap \lambda = \lambda$$

Thus λ is idempotent.

(1) $\Rightarrow$ (4)

If R is a commutative fully idempotent hemiring and if $A = xR$ is an ideal of R generated by any $x \in R$, then $A = A^2$. So

$$x \in A^2 = (xR)^2 \qquad (\because x \in A)$$

$\Rightarrow$

$$x \in (xR)^2 = xRxR = xRRx \subseteq xRx$$

Thus $x \in xRx$, and R is von-Neumann regular.

(4) $\Rightarrow$ (1)

If R is a commutative von-Neumann regular hemiring and if I is an ideals of R, then by Theorem 66

$$I \cap I = II$$

$\Rightarrow$

$$I = I^2$$

Thus R is fully idempotent. ∎

Definition 69 *An interval valued fuzzy ideal ξ of a hemiring R is called interval valued fuzzy prime ideal of R if for any interval valued fuzzy ideals λ and μ of R*

$$\lambda\mu \subseteq \xi \Rightarrow \lambda \subseteq \xi \ or \ \mu \subseteq \xi$$

Definition 70 *An interval valued fuzzy ideal ξ of a hemiring R is called interval valued fuzzy irreducible ideal of R if for any interval valued fuzzy ideals λ and μ of R*

$$\lambda \cap \mu = \xi \Rightarrow \lambda = \xi \text{ or } \mu = \xi$$

Remark 71 $\lambda = [\lambda^-, \lambda^+]$ *is an interval valued fuzzy prime ideal of R if and only if $\lambda^-$and λ^+ are fuzzy prime ideals of R.*

Remark 72 $\lambda = [\lambda^-, \lambda^+]$ *is an interval valued fuzzy irreducible ideal of R if and only if λ^- and λ^+ are fuzzy irreducible ideals of R.*

Remark 73 *Each interval valued fuzzy prime ideal of a hemiring R is an interval valued fuzzy irreducible ideal of R but the converse is not true.*

Lemma 74 *Let R be a fully idempotent hemiring. If λ is an interval valued fuzzy ideal of R with $\lambda(x) = [\alpha, \beta] \in \mathcal{L}$ where $x \in R$, then there exists an interval valued fuzzy prime ideal ξ of R such that $\lambda \subseteq \xi$ and $\xi(x) = [\alpha, \beta]$.*

Proof. Let $X = \{\mu : \mu \text{ is an interval valued fuzzy ideal of } R \text{ and } \mu(x) = [\alpha, \beta] \text{ and } \lambda \subseteq \mu\}$.

Note that $X \neq \phi$ as $\lambda \in X$. Let $\mathcal{F} = \{\lambda_i : i \in I\}$ be a totally ordered subset of X, then we claim that $\cup_i \lambda_i$ is an interval valued fuzzy ideal of R. For this, let $x, y \in R$ then

$$\begin{aligned} (\cup_i \lambda_i)(x) &= \vee_i (\lambda_i(x)) \\ &\leq \vee_i (\lambda_i(xy)) = (\cup_i \lambda_i)(xy) \end{aligned}$$

And

$$\begin{aligned} (\cup_i \lambda_i)(x) &= \vee_i (\lambda_i(x)) \\ &\leq \vee_i (\lambda_i(yx)) = (\cup_i \lambda_i)(yx) \end{aligned}$$

Also

$$
\begin{aligned}
(\cup_i\lambda_i)(x) \wedge (\cup_j\lambda_j)(y) &= (\cup_i\lambda_i)(x) \wedge (\vee_j\lambda_j(y)) \\
&= \vee_j [[\vee_i\lambda_i(x)] \wedge \lambda_j(y)] \\
&= \vee_j [\vee_i [\lambda_i(x) \wedge \lambda_j(y)]] \\
&\leq \vee_j \left[\vee_i \left[\lambda_i^j(x) \wedge \lambda_i^j(y)\right]\right] \qquad \text{where } \lambda_i \vee \lambda_j = \lambda_i^j \in \mathcal{F} \\
&\leq \vee_j \left[\vee_i \left[\lambda_i^j(x+y)\right]\right] \\
&= \vee_{i,j} \left[\lambda_i^j(x+y)\right] \\
&\leq \vee_i [\lambda_i(x+y)] \\
&= (\cup_i\lambda_i)(x+y).
\end{aligned}
$$

Thus $\cup_i\lambda_i$ is an interval valued fuzzy ideal of R. Clearly $\lambda \subseteq \cup_i\lambda_i$ and

$$
\begin{aligned}
(\cup_i\lambda_i)(x) &= \vee_i(\lambda_i(x)) \\
&= \vee_i[\alpha,\beta] = [\alpha,\beta]
\end{aligned}
$$

This shows that $\cup_i\lambda_i \in \mathcal{F}$. Thus $\cup_i\lambda_i$ is least upper bound of $\mathcal{F}$. Hence by Zorn's lemma, there exists an interval valued fuzzy ideal ξ of R which is maximal with respect to the property that $\lambda \subseteq \xi$ and $\xi(x) = [\alpha,\beta]$. Let Ψ and θ be any interval valued fuzzy ideals of R such that $\theta \cap \Psi = \xi$. Then $\xi \subseteq \theta$ and $\xi \subseteq \Psi$ and if $\xi \neq \theta$ and $\xi \neq \Psi$. Then, since ξ is maximal with respect to the property that $\xi(x) = [\alpha,\beta]$ for $x \in R$. So $\theta, \Psi \notin X$ and

$$\theta(x) \neq [\alpha,\beta] \neq \Psi(x)$$

Hence

$$\begin{aligned}[\alpha,\beta] &= \xi(x) = (\theta \cap \Psi)(x) \\ &= \theta(x) \wedge \Psi(x) \neq [\alpha,\beta].\end{aligned}$$

Which is impossible. Thus $\xi = \theta$ or $\xi = \Psi$. And since every interval valued fuzzy irreducible ideal of fully idempotent hemiring R is also interval valued fuzzy prime ideal, so ξ is the required interval valued fuzzy prime ideal of R. ∎

Theorem 75 *Let R be a hemiring with identity "1" then the following assertions are equivalent:*

1) *R is fully idempotent.*

2) *The set $\mathcal{F}_R$ of all interval valued fuzzy ideals of R (ordered by $\subseteq$) is distributive lattice under the sum and intersection of interval valued fuzzy ideals with $\lambda \cap \mu = \lambda\mu$ for each pair of interval valued fuzzy ideals λ, μ of R.*

3) *Each interval valued fuzzy ideal is intersection of all those interval valued fuzzy prime ideals of R which contain it.*

If R is commutative then the above three assertions are equivalent to

4) *R is von-Neumann regular.*

Proof. $(1) \Rightarrow (2)$

The set $\mathcal{F}_R$ of all interval valued fuzzy ideals of R (ordered by "$\subseteq$" i.e. $\lambda \subseteq \mu$ iff $\lambda^-(x) \leq \mu^-(x)$ and $\lambda^+(x) \leq \mu^+(x)$ for all $x \in R$) is clearly a lattice under the sum and intersection of interval valued fuzzy ideals. Moreover, since R is fully idempotent so by Theorem 68, $\lambda \cap \mu = \lambda\mu$ for each pair λ and μ of interval valued fuzzy ideals of R. For distributive lattice, we have to show that for all $\lambda, \mu, \xi \in \mathcal{F}_R$ $(\lambda \cap \mu) + \xi = (\lambda + \xi) \cap (\mu + \xi)$.

Let $x \in R$. Then

$$\begin{aligned}
[(\lambda \cap \mu) + \xi](x) &= \vee_{x=y+z}\left[\left(\lambda^- \wedge \mu^-\right)(y) \wedge \xi^-(z), \left(\lambda^+ \wedge \mu^+\right)(y) \wedge \xi^+(z)\right] \\
&= \vee_{x=y+z}\left[\lambda^-(y) \wedge \xi^-(z) \wedge \mu^-(y) \wedge \xi^-(z), \lambda^+(y) \wedge \xi^+(z) \wedge \mu^+(y) \wedge \xi^+(z)\right] \\
&= \vee_{x=y+z}\left[\lambda^-(y) \wedge \xi^-(z), \lambda^+(y) \wedge \xi^+(z)\right] \wedge \left[\mu^-(y) \wedge \xi^-(z), \mu^+(y) \wedge \xi^+(z)\right] \\
&\leq \vee_{x=y+z}\left[\left(\lambda^- + \xi^-\right)(x), \left(\lambda^+ + \xi^+\right)(x)\right] \wedge \left[\left(\mu^- + \xi^-\right)(x), \left(\mu^+ + \xi^+\right)(x)\right] \\
&= \vee_{x=y+z}\left[(\lambda + \xi)(x) \wedge (\mu + \xi)(x)\right] \\
&= \left[(\lambda + \xi) \cap (\mu + \xi)\right](x).
\end{aligned}$$

This implies $(\lambda \cap \mu) + \xi \subseteq (\lambda + \xi) \cap (\mu + \xi)$. For reverse containment

$$\begin{aligned}
&[(\lambda + \xi) \cap (\mu + \xi)](x) \\
&= [(\lambda + \xi)(\mu + \xi)](x) \\
&= \vee_{x=\Sigma_{i=1}^n y_i z_i}\{\wedge_i[\left(\lambda^- + \xi^-\right)(y_i) \wedge \left(\mu^- + \xi^-\right)(z_i), \left(\lambda^+ + \xi^+\right)(y_j) \wedge \left(\mu^+ + \xi^+\right)(z_i)]\} \\
&= \vee_{x=\Sigma_{i=1}^n y_i z_i}[\wedge_i\{(\vee_{y_i=r_i+s_i}\left[\lambda^-(r_i) \wedge \xi^-(s_i), \lambda^+(r_i) \wedge \xi^+(s_i)\right]) \\
&\qquad \wedge \left(\vee_{z_i=t_i+\mu_i}\left[\mu^-(t_i) \wedge \xi^-(\mu_i), \mu^+(t_i) \wedge \xi^+(\mu_i)\right]\right)\}] \\
&= \vee_{x=\Sigma_{i=1}^n y_i z_i}[\wedge_i\{\vee_{\substack{y_i=r_i+s_i \\ z_i=t_i+\mu_i}}[\lambda^-(r_i) \wedge \xi^-(s_i) \wedge \mu^-(t_i) \wedge \xi^-(\mu_i), \\
&\qquad \lambda^+(r_i) \wedge \xi^+(s_i) \wedge \mu^+(t_i) \wedge \xi^+(\mu_i)]\}] \\
&= \vee_{x=\Sigma_{i=1}^n y_i z_i}[\wedge_i\{\vee_{\substack{y_i=r_i+s_i \\ z_i=t_i+\mu_i}}[\lambda^-(r_i) \wedge \xi^-(s_i) \wedge \xi^-(s_i) \wedge \mu^-(t_i) \wedge \xi^-(\mu_i), \\
&\qquad \lambda^+(r_i) \wedge \xi^+(s_i) \wedge \xi^+(s_i) \wedge \mu^+(t_i) \wedge \xi^+(\mu_i)]\}] \\
&\leq \vee_{x=\Sigma_{i=1}^n y_i z_i}[\wedge_i\{\vee_{\substack{y_i=r_i+s_i \\ z_i=t_i+\mu_i}}[\lambda^-(r_i t_i) \wedge \mu^-(r_i t_i) \wedge \xi^-(s_i t_i) \wedge \xi^-(s_i u_i) \wedge \xi^-(r_i \mu_i), \\
&\qquad \lambda^+(r_i t_i) \wedge \mu^+(r_i t_i) \wedge \xi^+(s_i t_i) \wedge \xi^+(s_i u_i) \wedge \xi^+(r_i \mu_i)]\}] \\
&\leq \vee_{x=\Sigma_{i=1}^n y_i z_i}[\wedge_i\{\vee_{y_i z_i=r_i t_i+(s_i t_i+r_i u_i+s_i u_i)}[\left(\lambda^- \wedge \mu^-\right)(r_i t_i) \wedge \xi^-(s_i t_i + s_i u_i + r_i u_i),
\end{aligned}$$

$$\left(\lambda^+ \wedge \mu^+\right)(r_i t_i) \wedge \xi^+ (s_i t_i + s_i u_i + r_i u_i)]\}]$$

$$\leq \vee_{x=\Sigma_{i=1}^n y_i z_i} [\wedge_i \{\vee_{y_i z_i = r_i t_i + (s_i t_i + r_i u_i + s_i u_i)} [\left(\left(\lambda^- \wedge \mu^-\right) + \xi^-\right)(y_i z_i),$$

$$\left(\left(\lambda^+ \wedge \mu^+\right) + \xi^+\right)(y_i z_i)]\}]$$

$$= \vee_{x=\Sigma_{i=1}^n y_i z_i} \{\wedge_i \left[\left(\left(\lambda^- \wedge \mu^-\right) + \xi^-\right)(y_i z_i), \left(\left(\lambda^+ \wedge \mu^+\right) + \xi^+\right)(y_i z_i)\right]\}$$

$$\leq \vee_{x=\Sigma_{i=1}^n y_i z_i} \left[\left(\left(\lambda^- \wedge \mu^-\right) + \xi^-\right)(\Sigma_{i=1}^n y_i z_i), \left(\left(\lambda^+ \wedge \mu^+\right) + \xi^+\right)(\Sigma_{i=1}^n y_i z_i)\right]$$

$$= \vee_{x=\Sigma_{i=1}^n y_i z_i} \left[\left(\left(\lambda^- \wedge \mu^-\right) + \xi^-\right)(x), \left(\left(\lambda^+ \wedge \mu^+\right) + \xi^+\right)(x)\right]$$

$$= ((\lambda \cap \mu) + \xi)(x)$$

This shows

$$(\lambda + \xi) \cap (\mu + \xi) \subseteq (\lambda \cap \mu) + \xi$$

Thus

$$(\lambda \cap \mu) + \xi = (\lambda + \xi) \cap (\mu + \xi)$$

Hence $\mathcal{F}_R$ is a distributive lattice.

(2) $\Rightarrow$ (1)

Let $\mathcal{F}_R$ be a distributive lattice under the sum and intersection of interval valued fuzzy ideals of R and let $\lambda \cap \mu = \lambda\mu$ for each pair of interval valued fuzzy ideals λ and μ of R. Then for any interval valued fuzzy ideal λ of R, we have

$$\lambda^2 = \lambda\lambda = \lambda \cap \lambda = \lambda$$

Thus R is fully idempotent.

(1) $\Rightarrow$ (3)

Suppose that R is fully idempotent hemiring. Let λ be an interval valued fuzzy ideal of R and $\{\lambda_i : i \in I\}$ be the family of all interval valued fuzzy prime ideals of R

which contain λ. Then obviously

$$\lambda \subseteq \cap_{i\in I}\lambda_i$$

For reverse containment, let $x \in R$ then by Lemma 74, there exists an interval valued fuzzy prime ideal ξ of R such that $\lambda \subseteq \xi$ and $\lambda(x) = \xi(x)$. Then $\xi \in \{\lambda_i : i \in I\}$. Hence $\cap_{i\in I}\lambda_i \subseteq \xi$. Thus

$$\cap_{i\in I}\lambda_i(x) \leq \xi(x) = \lambda(x)$$

This shows $\cap_{i\in I}\lambda_i \subseteq \lambda$. Thus $\lambda = \cap_{i\in I}\lambda_i$.

(3) $\Rightarrow$ (1)

Let λ be an interval valued fuzzy ideal of R then λ^2 is also an interval valued fuzzy ideal of R and by hypothesis (3), λ^2 can be written as

$$\lambda^2 = \cap_{i\in I}\lambda_i$$

Where $\{\lambda_i : i \in I\}$ is a family of interval valued fuzzy prime ideals of R which contain λ^2. Now since $\lambda^2 \subseteq \lambda_i$ for all i and since λ_i are interval valued fuzzy prime ideals so $\lambda \subseteq \lambda_i$ for all i. Hence $\lambda \subseteq \cap_{i\in I}\lambda_i = \lambda^2 \subseteq \lambda$. Thus $\lambda = \lambda^2$. Hence R is fully idempotent.

(1) $\Leftrightarrow$ (4) follows by Theorem 68. ■

Theorem 76 *Let R be a fully idempotent hemiring. An interval valued fuzzy ideal ξ of R is interval valued fuzzy prime if and only if it is interval valued fuzzy irreducible.*

Proof. Every interval valued fuzzy prime ideal of R is also interval valued fuzzy irreducible.

Conversely, let ξ be an interval valued fuzzy irreducible ideal of R and λ, μ be any two interval valued fuzzy ideals of R such that

$$\lambda\mu \subseteq \xi.$$

Then since R is fully idempotent so by Theorem 68

$$\lambda\mu = \lambda \cap \mu.$$

Thus we have

$$\lambda \cap \mu \subseteq \xi.$$

This implies

$$(\lambda \cap \mu) + \xi = \xi.$$

Again since R is fully idempotent so by Theorem 75, the set of all interval valued fuzzy ideals of R is a distributive lattice under sum and intersection of interval valued fuzzy ideals, hence

$$(\lambda + \xi) \cap (\mu + \xi) = \xi.$$

Thus $\lambda + \xi = \xi$ or $\mu + \xi = \xi$ $\quad$ ($\because$ ξ is irreducible).

Hence $\lambda \subseteq \xi$ or $\mu \subseteq \xi$. Thus ξ is an interval valued fuzzy prime ideal of R. ∎

Example 77 *Let S be a non-empty set. Define a binary operation "$*$" on S as follows*

$x * y = y$ $\quad$ for all $x, y \in S$. Then $(S, *)$ is a semigroup.

Now let $R = S \cup \{\infty\} \cup \{0\}$ where $\{\infty\}$ is a ring with a single element "∞", and "0" is absorbing zero i.e. $x * 0 = 0 * x = \infty * 0 = 0 * \infty = 0$ $\quad$ for all $x \in S \cup \{0\}$

and $x * \infty = \infty * x = \infty$ for all $x \in S \cup \{\infty\}$.

Now define another binary operation " $+$ " on R as

$0 + x = x + 0 = x$ for all $x \in S \cup \{0\}$

and $x + y = \infty$ for all $x, y \in S$

and $x + \infty = \infty + x = \infty$ for all $x \in R$.

Then $(R, +, *)$ is a hemiring.

FACT 1:

Every element of R is multiplicatively idempotent.

FACT 2:

R is a regular hemiring.

FACT 3:

An interval valued fuzzy subset λ of R is interval valued fuzzy right ideal of R if and only if

i) $\lambda(0) \geq \lambda(x)$ *for all* $x \in R$

ii) $\lambda(\infty) \geq \lambda(x)$ *for all* $x \in S \cup \{\infty\}$

iii)$\lambda(x) = \lambda(y)$ *for all* $x, y \in S$

Proof. Suppose that $(i), (ii)$ and (iii) hold. Then

(a)

Case I

When $x, y \in S \cup \{\infty\}$ then $x \neq 0 \neq y$.

$\lambda(x + y) = \lambda(\infty) \geq \lambda(x) \wedge \lambda(y)$ by (i).

Case II

When at least one of x and y, say $x = 0$ and $y \in R$.

Then $x + y = 0 + y = y$

$\Rightarrow \lambda(x + y) = \lambda(y) = \lambda(x) \wedge \lambda(y)$

because $\lambda(x) = \lambda(0) \geq \lambda(a)$ for all $a \in R$.

(b)

Case I When $x, y \in S$.

Then $\lambda(x * y) = \lambda(y) = \lambda(x)$ *by (iii)*.

Case II

When any one, say $x = 0$ and $y \in R$.

$\lambda(x * y) = \lambda(0 * y) = \lambda(0) = \lambda(x)$.

$\lambda(y * x) = \lambda(y * 0) = \lambda(0) \geq \lambda(y)$.

Case III

When any one, say $x = \infty$ and $y \in R$.

$\lambda(x * y) = \lambda(\infty * y) = \lambda(\infty) = \lambda(x)$

$\lambda(y * x) = \lambda(y * \infty) = \lambda(\infty) \geq \lambda(y)$ *by (ii)*.

Thus in any case $\lambda(x * y) \geq \lambda(x)$ for all $x, y \in R$.

Hence λ is an interval valued fuzzy right ideal of R.

Conversely, suppose that λ is an interval valued fuzzy right ideal of R. If $x \in R$ then

$$\lambda(0) = \lambda(x * 0) \geq \lambda(x).$$

Similarly if $x \in S \cup \{\infty\}$ then

$$\lambda(\infty) = \lambda(x * \infty) \geq \lambda(x).$$

Also if $x, y \in S$ then

$$\begin{aligned} \lambda(x) &= \lambda(y * x) \geq \lambda(y) \\ \lambda(y) &= \lambda(x * y) \geq \lambda(x) \end{aligned}$$

Thus $\lambda(x) = \lambda(y)$ for all $x, y \in S$. ■

FACT 4:

The (crisp) right ideals of R are $\{0\}, \{0, \infty\}$ and R itself which are all idempotent.

FACT 5:

All interval valued fuzzy right ideals of R are idempotent.

Proof. Let $\lambda : R \to \mathcal{L}$ be an interval valued fuzzy right ideal of $R = S \cup \{0, \infty\}$, then

$$\begin{aligned} \lambda^2(0) &= \vee_{0=\Sigma_{i=1}^n y_i z_i} \left\{ \wedge_i \left[\lambda^-(y_i) \wedge \lambda^-(z_i), \lambda^+(y_i) \wedge \lambda^+(z_i) \right] \right\} \\ &\geq \left[\lambda^-(0) \wedge \lambda^-(0), \lambda^+(0) \wedge \lambda^+(0) \right] \\ &= \lambda(0) \geq \lambda^2(0). \end{aligned}$$

Thus $\lambda^2(0) = \lambda(0)$.

Now for $x \in S$, no expression of the form $x = \sum_{i=1}^n y_i z_i$ involves only 0 and ∞

Thus $\wedge_i \left[\lambda^-(y_i) \wedge \lambda^-(z_i), \lambda^+(y_i) \wedge \lambda^+(z_i) \right] \neq \lambda(0)$

and $\wedge_i \left[\lambda^-(y_i) \wedge \lambda^-(z_i), \lambda^+(y_i) \wedge \lambda^+(z_i) \right] \neq \lambda(\infty)$

Note that $x = x * x$ is among the possible expressions of x and since

$$\lambda(x) = \lambda(y) \qquad \text{for all } x, y \in S.$$

So

$$\begin{aligned}\lambda^2(x) &= \vee_{x=\Sigma_{i=1}^n y_i z_i}\left\{\wedge_i\left[\lambda^-(y_i)\wedge\lambda^-(z_i),\lambda^+(y_i)\wedge\lambda^+(z_i)\right]\right\}\\ &\geq \vee_{x=x*x}\left[\lambda^-(x)\wedge\lambda^-(x),\lambda^+(x)\wedge\lambda^+(x)\right]\\ &= \lambda(x)\geq\lambda^2(x) \qquad \text{for all } x\in S.\end{aligned}$$

Hence

$$\lambda^2(x)=\lambda(x) \qquad \text{for all } x\in S.$$

Now we calculate $\lambda^2(\infty)$. Clearly, no expression for ∞ contains only 0 and one expression for ∞ is $\infty=\infty*\infty$.

Thus for $\infty=\sum_{i=1}^n y_i z_i$

$$\wedge_i\left[\lambda^-(y_i)\wedge\lambda^-(z_i),\lambda^+(y_i)\wedge\lambda^+(z_i)\right]\neq\lambda(0)$$

And

$$\wedge_i\left[\lambda^-(y_i)\wedge\lambda^-(z_i),\lambda^+(y_i)\wedge\lambda^+(z_i)\right]=\lambda(\infty)$$

Thus

$$\vee_{0=\Sigma_{i=1}^n y_i z_i}\left\{\wedge_i\left[\lambda^-(y_i)\wedge\lambda^-(z_i),\lambda^+(y_i)\wedge\lambda^+(z_i)\right]\right\}=\lambda(\infty)$$

Hence

$$\lambda^2(\infty)=\lambda(\infty).$$

Thus the interval valued fuzzy right ideal λ is idempotent. ∎

Example 78 *Consider the hemiring $R=\{0,a,b,c,d\}$ defined by the following operations*

$+$	0	a	b	c	d
0	0	a	b	c	d
a	a	c	a	b	a
b	b	a	b	c	b
c	c	b	c	a	c
d	d	a	b	c	d

$\cdot$	0	a	b	c	d
0	0	0	0	0	0
a	0	a	b	c	d
b	0	b	b	b	d
c	0	c	b	a	d
d	0	d	d	d	0

FACT 1:

The (crisp) ideals of R are $\{0\}, \{0,d\}, \{0,b,d\}$ and R itself. Now

$\{0\}^2 = \{0\}$

$\{0,d\}^2 = \{0\} \neq \{0,d\}$

$\{0,b,d\}^2 = \{0,b,d\}$

Thus the ideal $\{0,d\}$ is not idempotent and $\{0\}, \{0,b,d\}$ and R are idempotent.

FACT 2:

If we define two interval valued fuzzy subsets μ and λ of R as

$$\mu(x) = \begin{cases} \bar{O} \text{ if } & x = a,b,c \\ \bar{I} \text{ if } & x = 0,d \end{cases}$$

and $$\lambda(x) = \begin{cases} \bar{O} \text{ if } & x = a,c \\ \bar{I} \text{ if } & x = 0,b,d \end{cases}$$

Then both, μ and λ are interval valued fuzzy ideals of R but λ is idempotent while μ is not, as $\mu^2(d) = [0,0] = \bar{O} \neq \mu(d)$.

Note that μ and λ are interval valued characteristic functions of (crisp) ideals $\{0,d\}$ and $\{0,b,d\}$ of R respectively.

Example 79 *Consider the set $R = \{0,x\}$ with binary operations defined as*

+	0	x
0	0	x
x	x	x

$\cdot$	0	x
0	0	0
x	0	x

Then R is clearly a hemiring with an absorbing element '0'. Its only proper (crisp) ideal is zero ideal $\{0\}$. Since each ideal $\{0\}$and R are idempotent, so R is fully idempotent hemiring. Since R is also commutative so it is von Neumann regular. Then by Theorem 75, the lattice of all interval valued fuzzy ideals of R (ordered by $\subseteq$) is distributive under the sum and intersection of interval valued fuzzy ideals.

3.2 Fuzzy Prime Spectrum of a Fully Idempotent Hemiring

An interval valued fuzzy ideal λ of R is called normal if $\lambda(0) = [1,1]$.

Let R be a fully idempotent hemiring, $\mathcal{L}_R$ be the lattice of all normal interval valued fuzzy ideals of R and $\mathcal{F}_P$ be the set of all proper normal interval valued fuzzy prime ideals of R. For any interval valued fuzzy ideal λ of R, we define

$$\theta_\lambda = \{\mu \in \mathcal{F}_P : \lambda \nsubseteq \mu\}$$

and

$$\Im = \{\theta_\lambda : \lambda \in \mathcal{L}_R\}.$$

Theorem 80 *The set $\Im$ forms a topology on the set $\mathcal{F}_P$. The assignment $\lambda \to \theta_\lambda$ is an isomorphism between the lattice $\mathcal{L}_R$ and the lattice of open subsets of $\mathcal{F}_P$.*

Proof. First we show that $\Im$ forms a topology on the set $\mathcal{F}_P$.

(i) Let ψ be the interval valued fuzzy ideal of R defined by

$$\psi(x) = \begin{cases} \tilde{O} & \text{if } x \neq 0 \\ \tilde{I} & \text{if } x = 0. \end{cases}$$

then $\theta_\psi = \{\mu \in \mathcal{F}_P : \psi \nsubseteq \mu\} = \varphi$.

If C_R is the interval valued characteristic function of R then

$$C_R(x) = [1,1] = \tilde{I} \qquad \text{for all } x \in R$$

and by definition of $\mathcal{F}_P$, we have $\mu \subset C_R$ for all $\mu \in \mathcal{F}_P$. Thus

$$\theta_{C_R} = \{\mu \in \mathcal{F}_P : C_R \nsubseteq \mu\} = \mathcal{F}_P$$

and hence $\theta_\psi = \varphi$ and $\theta_{C_R} = \mathcal{F}_P$ are elements of $\Im$.

(ii) Now let $\theta_{\lambda_1}, \theta_{\lambda_2} \in \Im$ with $\lambda_1, \lambda_2 \in \mathcal{L}_R$. Then

$$\theta_{\lambda_1} \cap \theta_{\lambda_2} = \{\mu \in \mathcal{F}_P : \lambda_1 \nsubseteq \mu \text{ and } \lambda_2 \nsubseteq \mu\}$$

Since R is fully idempotent hemiring so $\lambda_1 \cap \lambda_2 = \lambda_1\lambda_2$. Now if $\lambda_1 \cap \lambda_2 \subseteq \mu$ then $\lambda_1\lambda_2 \subseteq \mu$. But μ is an interval valued fuzzy prime ideal of R so $\lambda_1 \subseteq \mu$ or $\lambda_2 \subseteq \mu$, which is a contradiction. Therefore $\lambda_1 \cap \lambda_2 \nsubseteq \mu$.

Conversely, if $\lambda_1 \cap \lambda_2 \nsubseteq \mu$ then $\lambda_1 \nsubseteq \mu$ and $\lambda_2 \nsubseteq \mu$. Thus

$$\begin{aligned} \theta_{\lambda_1} \cap \theta_{\lambda_2} &= \{\mu \in \mathcal{F}_P : \lambda_1 \nsubseteq \mu \text{ and } \lambda_2 \nsubseteq \mu\} \\ &= \{\mu \in \mathcal{F}_P : \lambda_1 \cap \lambda_2 \nsubseteq \mu\} \\ &= \theta_{\lambda_1 \cap \lambda_2}. \end{aligned}$$

And surely $\lambda_1 \cap \lambda_2 \in \mathcal{L}_R$. Thus $\theta_{\lambda_1} \cap \theta_{\lambda_2} \in \Im$ for all $\lambda_1, \lambda_2 \in \mathcal{L}_R$.

(iii) Consider a family $\{\theta_{\lambda_i} : i \in I\}$ of elements of $\Im$. Then

$$\begin{aligned}
\cup_{i\in I}\theta_{\lambda_i} &= \cup_{i\in I}\{\mu \in \mathcal{F}_P : \lambda_i \nsubseteq \mu\} \\
&= \{\mu \in \mathcal{F}_P : \lambda_k \nsubseteq \mu \text{ for some } k \in I\} \\
&= \{\mu \in \mathcal{F}_P : \Sigma_i\lambda_i \nsubseteq \mu\} \\
&= \theta_{\Sigma_i\lambda_i} \in \Im \qquad\qquad (\because \Sigma_i\lambda_i \in \mathcal{L}_R).
\end{aligned}$$

Thus $\Im$ is a topology on $\mathcal{F}_P$.

Define a map $\Psi : \mathcal{L}_R \to \Im$ by

$$\Psi(\lambda) = \theta_\lambda \qquad \text{for all } \lambda \in \mathcal{L}_R$$

Then by definition of θ_λ

$$\lambda_1 = \lambda_2 \Rightarrow \theta_{\lambda_1} = \theta_{\lambda_2} \qquad \text{for all } \lambda_1, \lambda_2 \in \mathcal{L}_R$$

This implies

$$\Psi(\lambda_1) = \Psi(\lambda_2) \qquad \text{for all } \lambda_1, \lambda_2 \in \mathcal{L}_R$$

Thus Ψ is a well-defined map. And from (ii) and (iii) above, Ψ preserves the finite intersection and arbitrary union. Thus Ψ is a lattice homomorphism.

Now for isomorphism, we will show that Ψ is a bijection. Ψ is clearly onto, and for $\lambda_1, \lambda_2 \in \mathcal{L}_R$ let

$$\Psi(\lambda_1) = \Psi(\lambda_2)$$

$\Rightarrow$

$$\theta_{\lambda_1} = \theta_{\lambda_2}$$

⇒

$$\lambda_1 = \lambda_2$$

As if $\lambda_1 \neq \lambda_2$ then there exists $x \in R$ such that

$$\lambda_1(x) \neq \lambda_2(x)$$

Therefore any one, say, $\lambda_1(x)$ is greater than $\lambda_2(x)$. Then for λ_2, by Lemma 74, there exists an interval valued fuzzy prime ideal μ of R such that $\lambda_2 \subseteq \mu$ and

$$\lambda_2(x) = \mu(x)$$

⇒

$$\lambda_1 \nleq \mu \qquad (\because \lambda_1(x) > \lambda_2(x) = \mu(x))$$

This implies

$$\mu \in \theta_{\lambda_1}$$

⇒

$$\mu \in \theta_{\lambda_2} \qquad (\because \ \theta_{\lambda_1} = \theta_{\lambda_2})$$

⇒

$$\lambda_2 \nsubseteq \mu$$

Which is a contradiction. Thus Ψ is an injection, and hence an isomorphism. ∎

Chapter 4

INTERVAL VALUED FUZZY h-IDEALS OF HEMIRINGS

In this chapter we define interval valued fuzzy left (right) h-ideals of a hemiring and study some properties of these ideals.

4.1 Interval Valued Fuzzy h-Ideals

Definition 81 *An interval valued fuzzy left (respectively. right) ideal λ of a hemiring R is called an interval valued fuzzy left (respectively. right) k-ideal of R if for all $x, y, z \in R$*

$$x + y = z \Rightarrow \lambda(x) \geq \lambda(y) \wedge \lambda(z).$$

Definition 82 *An interval valued fuzzy left (respectively. right) ideal λ of a hemiring R is called an interval valued fuzzy left (respectively. right) h-ideal of R if for all $a, b, x, y \in R$, $x + a + y = b + y \Rightarrow \lambda(x) \geq \lambda(a) \wedge \lambda(b)$.*

Remark 83 *Every interval valued fuzzy left (respectively. right) h-ideal of a hemiring R is an interval valued fuzzy left (respectively. right) k-ideal of R but the converse is not true. In case of rings, the h-ideals and k-ideals coincide.*

Theorem 84 *An interval valued fuzzy subset $\lambda = \left[\lambda^-, \lambda^+\right]$ of a hemiring R is an interval valued fuzzy left (respectively. right) k-ideal of R if and only if λ^- and λ^+ are fuzzy left (respectively. right) k-ideals of R.*

Theorem 85 *An interval valued fuzzy subset $\lambda = \left[\lambda^-, \lambda^+\right]$ of a hemiring R is an interval valued fuzzy left (respectively. right) h-ideal of R if and only if λ^- and λ^+ are fuzzy left (respectively. right) h-ideals of R.*

Theorem 86 *Let λ and μ be interval valued fuzzy left (respectively. right) h-ideals of a hemiring R, then $\lambda \cap \mu$ is also an interval valued fuzzy left (respectively. right) h-ideal of R.*

Proof. Let λ and μ be interval valued fuzzy left h-ideals of R, then $\lambda \cap \mu$ is an interval valued fuzzy left ideal of R.

Let $a, b, x, y \in R$ such that $x + a + y = b + y$ then since λ and μ are interval valued fuzzy left h-ideals, therefore

$$\begin{aligned}
(\lambda \cap \mu)(x) &= \lambda(x) \wedge \mu(x) \\
&\geq \{\lambda(a) \wedge \lambda(b)\} \wedge \{\mu(a) \wedge \mu(b)\} \\
&= \{\lambda(a) \wedge \mu(a)\} \wedge \{\lambda(b) \wedge \mu(b)\} \\
&= (\lambda \cap \mu)(a) \wedge (\lambda \cap \mu)(b).
\end{aligned}$$

Thus $\lambda \cap \mu$ is an interval valued fuzzy left h-ideal of R. ∎

Proposition 87 *A subset A of a hemiring R is left (respectively. right) h-ideal of R if and only if the interval valued characteristic function C_A is interval valued fuzzy left (respectively. right) h-ideal of R.*

Proof. Let A be a left h-ideal of a hemiring R, then by Proposition 60, C_A is an interval valued fuzzy left ideal of R. Let $x, y, z, a \in R$ such that

$$x + y + a = z + a$$

then we have the following cases

CASE 1 When at least one of y and z is not in A. Then

$$C_A(y) \wedge C_A(z) = \tilde{O}.$$

Thus

$$C_A(x) \geq \tilde{O} = C_A(y) \wedge C_A(z).$$

CASE 2 When $y, z \in A$, then $x \in A$ and so

$$C_A(x) = \tilde{I} = C_A(y) = C_A(z).$$

So

$$C_A(x) \geq C_A(y) \wedge C_A(z).$$

Thus C_A is an interval valued fuzzy left h-ideal of R.

Conversely, suppose that C_A is an interval valued fuzzy left h-ideal of R. Then we will show that A is a left h-ideal of R. Obviously A is a left ideal of R by Proposition 60. Now let $a, x \in R$ and $y, z \in A$ such that

$$x + y + a = z + a$$

Then

$$C_A(y) = C_A(z) = \tilde{I}.$$

Since C_A is an interval valued fuzzy left h-ideal of R, so

$$C_A(x) \geq C_A(y) \wedge C_A(z) = \tilde{I} \wedge \tilde{I} = \tilde{I}.$$

Thus $C_A(x) = \tilde{I}$ which implies $x \in A$. Hence A is a left h-ideal of R. ∎

Definition 88 *Let λ be an interval valued fuzzy subset of R and $[\alpha, \beta] \in \mathcal{L}$ then the level subset $U(\lambda, [\alpha, \beta])$ of R is defined as*

$$U(\lambda, [\alpha, \beta]) = \{x \in R : \lambda(x) \geq [\alpha, \beta]\}.$$

Lemma 89 *An interval valued fuzzy subset λ of a hemiring R is an interval valued fuzzy left (respectively. right) h-ideal of R if and only if each non-empty level subset of R defined by λ is a left (respectively. right) h-ideal of R.*

Proof. Let λ be an interval valued fuzzy left h-ideal of R and let $[\alpha, \beta] \in \mathcal{L}$. Then the level subset of R defined by λ is

$$U = U(\lambda, [\alpha, \beta]) = \{x \in R : \lambda(x) \geq [\alpha, \beta]\}.$$

Let $x, y \in U$. Then $\lambda(x) \geq [\alpha, \beta]$ and $\lambda(y) \geq [\alpha, \beta]$. Now since λ is an interval valued fuzzy left h-ideal of R therefore

$$\begin{aligned} \lambda(x+y) &\geq \lambda(x) \wedge \lambda(y) \\ &\geq [\alpha, \beta] \wedge [\alpha, \beta] = [\alpha, \beta]. \end{aligned}$$

Thus $x + y \in U$.

(*ii*) Now let $x \in R$ and and $y \in U$. Then

$$\lambda(xy) \geq \lambda(y) \geq [\alpha, \beta].$$

This implies $xy \in U$.

(*iii*) Let $a, x \in R$ and $y, z \in U$ such that

$$x + y + a = z + a.$$

Then since λ is interval valued fuzzy left h-ideal of R, so

$$\begin{aligned} \lambda(x) &\geq \lambda(y) \wedge \lambda(z) \\ &\geq [\alpha, \beta] \wedge [\alpha, \beta] = [\alpha, \beta]. \end{aligned}$$

Thus $x \in U$ and hence U is a left h-ideal of R.

Conversely, let λ be an interval valued fuzzy subset of R and each non-empty level subset defined by λ be a left h-ideal of R. Then for any $x, y \in R$, we have

$$\lambda(x) = [\alpha_1, \beta_1] \quad \text{and} \quad \lambda(y) = [\alpha_2, \beta_2]$$

for some $[\alpha_1, \beta_1], [\alpha_2, \beta_2] \in \mathcal{L}$. Let $\alpha' = \alpha_1 \wedge \alpha_2$ and $\beta' = \beta_1 \wedge \beta_2$. Then

$$\lambda(x) = [\alpha_1, \beta_1] \geq [\alpha_1, \beta_1] \wedge [\alpha_2, \beta_2] = [\alpha', \beta']$$

And hence

$$x, y \in U(\lambda, [\alpha', \beta']) \neq \phi.$$

Now since $U(\lambda, [\alpha', \beta'])$ is a left h-ideal of R, so

$$x + y \in U(\lambda, [\alpha', \beta'])$$

$\Rightarrow$

$$\lambda(x+y) \geq [\alpha', \beta'] = \lambda(x) \wedge \lambda(y).$$

Since $U(\lambda, [\alpha_2, \beta_2])$ is also a left h-ideal of R, so for all $r \in R$,

$$ry \in U(\lambda, [\alpha_2, \beta_2]).$$

Thus

$$\lambda(ry) \geq [\alpha_2, \beta_2] = \lambda(y).$$

Hence λ is an interval valued fuzzy left ideal of R.

Now let $x, y, z, a \in R$ such that

$$x + y + a = z + a$$

And let

$$\lambda(y) = [\alpha, \beta] \qquad \lambda(z) = [\alpha_0, \beta_0].$$

Let

$$\alpha'' = \alpha \wedge \alpha_0 \qquad \beta'' = \beta \wedge \beta_0.$$

Then $y, z \in U(\lambda, [\alpha'', \beta''])$.But by hypothesis $U(\lambda, [\alpha'', \beta''])$ is a left h-ideal of R, so

$$x \in U(\lambda, [\alpha'', \beta'']).$$

Thus

$$\lambda(x) \geq [\alpha'', \beta''] = \lambda(y) \wedge \lambda(z).$$

Hence λ is an interval valued fuzzy left h-ideal of R. ■

Proposition 90 *Let A be a non-empty subset of a hemiring R. Then the interval valued fuzzy subset λ defined by*

$$\lambda(x)=\begin{cases}[\alpha,\beta] & \text{if } x\in A\\ [\alpha_o,\beta_o] & \text{if } x\notin A\end{cases}$$

Where $\bar{O}\leq[\alpha_0,\beta_0]\leq[\alpha,\beta]\leq\bar{I}$ is an interval valued fuzzy left h-ideal of R if and only if A is a left h-ideal of R.

Proof. Let $\phi\neq A\subseteq R$ and let λ defined above be an interval valued fuzzy left h-ideal of R. Then for $x,y\in A$ we have

$$\begin{aligned}\lambda(x+y) &\geq \lambda(x)\wedge\lambda(y)\\ &= [\alpha,\beta]\wedge[\alpha,\beta]=[\alpha,\beta].\end{aligned}$$

But since λ assumes only two values, that is, $[\alpha,\beta]$ and $[\alpha_o,\beta_o]$, and also $[\alpha_o,\beta_o]\leq[\alpha,\beta]$. So $\lambda(x+y)=[\alpha,\beta]$. Hence $x+y\in A$.

Now let $x\in R$ and $y\in A$. Then

$$\lambda(xy)\geq\lambda(y)=[\alpha,\beta]$$

Hence $\lambda(xy)=[\alpha,\beta]$. Thus $xy\in A$.

Let $a,x\in R$ and $y,z\in A$ such that $x+y+a=z+a$.

Then since λ is an interval valued fuzzy left h-ideal of R, therefore

$$\begin{aligned}\lambda(x) &\geq \lambda(y)\wedge\lambda(z)\\ &= [\alpha,\beta]\wedge[\alpha,\beta]=[\alpha,\beta].\end{aligned}$$

Hence $\lambda(x)=[\alpha,\beta]$. Thus $x\in A$, so A is left h-ideal of R.

Conversely, let A be a left h-ideal of R and λ be an interval valued fuzzy subset of R, as defined in hypothesis. Then λ is an interval valued fuzzy left ideal of R. Now let $a,x,y,z\in R$ such that $x+y+a=z+a$. Then

CASE I Whenat least one of y and z is not in A. Then

$$\lambda(y) \wedge \lambda(z) = [\alpha_0, \beta_0]$$

Thus

$$\lambda(x) \geq \lambda(y) \wedge \lambda(z).$$

CASE II When $y, z \in A$ then $x \in A$, as A is left h-ideal. Thus

$$\lambda(x) = [\alpha, \beta] = \lambda(y) \wedge \lambda(z).$$

Thus in both cases

$$\lambda(x) \geq \lambda(y) \wedge \lambda(z).$$

Hence λ is an interval valued fuzzy left h-ideal of R. ■

Definition 91 *Let λ and μ be interval valued fuzzy subsets of a hemiring R then the intrinsic product "$\odot_h$" of λ and μ is defined as*

$$(\lambda \odot_h \mu)(x) = \vee_{x+\Sigma_{i=1}^{n} a_i b_i + z = \Sigma_{j=1}^{m} c_j d_j + z} \{\wedge_{i,j}[\lambda^-(a_i) \wedge \lambda^-(c_j) \wedge \mu^-(b_i) \wedge \mu^-(d_j), \\ \lambda^+(a_i) \wedge \lambda^+(c_j) \wedge \mu^+(b_i) \wedge \mu^+(d_j)]\}$$

and $(\lambda \odot_h \mu)(x) = [0,0] = \tilde{O}$ if x cannot be expressed as

$$x + \Sigma_{i=1}^{n} a_i b_i + z = \Sigma_{j=1}^{m} c_j d_j + z$$

Proposition 92 *Let R be a hemiring and λ, μ, ν, ϖ be any interval valued fuzzy subsets of R such that $\lambda \subseteq \nu$ and $\mu \subseteq \varpi$ then*

$$\lambda \odot_h \mu \subseteq \nu \odot_h \varpi.$$

Proof. If x can not be written in the form $x+\sum_{i=1}^{n} a_i b_i + z = \sum_{j=1}^{m} c_j d_j + z$ for any $a_i, b_i, c_j, d_j, z \in R$ then

$$(\lambda \odot_h \mu)(x) = \bar{O} = (\nu \odot_h \varpi)(x).$$

Otherwise since $\lambda \subseteq \nu$ and $\mu \subseteq \varpi$, so

$$\begin{aligned} \lambda^- &\subseteq \nu^- \quad \lambda^+ \subseteq \nu^+ \\ \mu^- &\subseteq \varpi^- \quad \mu^+ \subseteq \varpi^+ \end{aligned}$$

And hence for all $x \in R$

$$\begin{aligned} (\lambda \odot_h \mu)(x) &= \vee_{x+\Sigma_{i=1}^{n} a_i b_i + z = \Sigma_{j=1}^{m} c_j d_j + z} \{\wedge_{i,j} [\lambda^-(a_i) \wedge \lambda^-(c_j) \wedge \mu^-(b_i) \wedge \mu^-(d_j), \\ &\qquad \lambda^+(a_i) \wedge \lambda^+(c_j) \wedge \mu^+(b_i) \wedge \mu^+(d_j)]\} \\ &\leq \vee_{x+\Sigma_{i=1}^{n} a_i b_i + z = \Sigma_{j=1}^{m} c_j d_j + z} \{\wedge_{i,j} [\nu^-(a_i) \wedge \nu^-(c_j) \wedge \varpi^-(b_i) \wedge \varpi^-(d_j), \\ &\qquad \nu^+(a_i) \wedge \nu^+(c_j) \wedge \varpi^+(b_i) \wedge \varpi^+(d_j)]\} \\ &= (\nu \odot_h \varpi)(x) \end{aligned}$$

$\Rightarrow$

$$\lambda \odot_h \mu \subseteq \nu \odot_h \varpi.$$

■

Lemma 93 *If λ and μ are interval valued fuzzy right h-ideal and interval valued fuzzy left h-ideal of a hemiring R, respectively. Then*

$$\lambda \odot_h \mu \subseteq \lambda \cap \mu.$$

Proof. Let $x \in R$, If x cannot be expressed as $x + \Sigma_{i=1}^{n} a_i b_i + z = \Sigma_{j=1}^{m} c_j d_j + z$ for any $a_i, b_i, c_j, d_j, z \in R$ then

$$(\lambda \odot_h \mu)(x) = \tilde{O} \leq \lambda(x) \wedge \mu(x) = (\lambda \cap \mu)(x).$$

Otherwise, since λ and μ are interval valued fuzzy right and left h-ideals respectively so

$$\lambda(x) \geq \lambda(\Sigma_{i=1}^{n} a_i b_i) \wedge \lambda\left(\Sigma_{j=1}^{m} c_j d_j\right)$$

and

$$\mu(x) \geq \mu(\Sigma_{i=1}^{n} a_i b_i) \wedge \mu\left(\Sigma_{j=1}^{m} c_j d_j\right)$$

Now

$$\begin{aligned}
(\lambda \cap \mu)(x) &= \lambda(x) \wedge \mu(x) \\
&\geq \lambda(\Sigma_{i=1}^{n} a_i b_i) \wedge \lambda\left(\Sigma_{j=1}^{m} c_j d_j\right) \wedge \mu(\Sigma_{i=1}^{n} a_i b_i) \wedge \mu\left(\Sigma_{j=1}^{m} c_j d_j\right) \\
&\geq \wedge_{i,j} \{\lambda(a_i b_i) \wedge \lambda(c_j d_j) \wedge \mu(a_i b_i) \wedge \mu(c_j d_j)\} \\
&\geq \wedge_{i,j} \{\lambda(a_i) \wedge \lambda(c_j) \wedge \mu(b_i) \wedge \mu(d_j)\} \\
&= \wedge_{i,j}[\lambda^-(a_i) \wedge \lambda^-(c_j) \wedge \mu^-(b_i) \wedge \mu^-(d_j), \\
&\qquad \lambda^+(a_i) \wedge \lambda^+(c_j) \wedge \mu^+(b_i) \wedge \mu^+(d_j)].
\end{aligned}$$

Since above expression holds for any $a_i, b_i, c_j, d_j \in R$ and for all i, j therefore

$$\begin{aligned}
(\lambda \cap \mu)(x) &\geq \vee_{x+\Sigma_{i=1}^{n} a_i b_i + z = \Sigma_{j=1}^{m} c_j d_j + z} \{\wedge_{i,j}[\lambda^-(a_i) \wedge \lambda^-(c_j) \wedge \mu^-(b_i) \wedge \mu^-(d_j), \\
&\qquad \lambda^+(a_i) \wedge \lambda^+(c_j) \wedge \mu^+(b_i) \wedge \mu^+(d_j)]\} \\
&= (\lambda \odot_h \mu)(x)
\end{aligned}$$

$\Rightarrow$

$$\lambda \odot_h \mu \subseteq \lambda \cap \mu.$$

■

Lemma 94 *An interval valued fuzzy subset λ of a hemiring R is an interval valued fuzzy left (respectively right) h-ideal of R if and only if for all $x, y, a, b \in R$, we have*

(i) $\lambda(x+y) \geq \lambda(x) \wedge \lambda(y)$

(ii) $C_R \odot_h \lambda \subseteq \lambda$ (*respectively* $\lambda \odot_h C_R \subseteq \lambda$)

(iii) $x+a+y=b+y \Rightarrow \lambda(x) \geq \lambda(a) \wedge \lambda(b)$

Proof. Let λ be an interval valued fuzzy left h-ideal of R. Then by definition, (i) and (iii) are true. Now let $x \in R$. If x cannot be written as $x+\sum\limits_{i=1}^{n} a_i b_i + y = \sum\limits_{j=1}^{m} a'_j b'_j + y$ for any $a_i, a'_j, b_i, b'_j, y \in R$, then

$$\left(C_R \odot_h \lambda\right)(x) = \bar{O} \leq \lambda(x).$$

Otherwise since $C_R(x) = \bar{I}$ for all $x \in R$, so

$$\begin{aligned}
\left(C_R \odot_h \lambda\right)(x) &= \vee_{x+\Sigma_{i=1}^{n} a_i b_i + y = \Sigma_{j=1}^{m} a'_j b'_j + y} \left\{\wedge_{i,j}\left[\lambda^-(b_i) \wedge \lambda^-(b'_j), \lambda^+(b_i) \wedge \lambda^+(b'_j)\right]\right\} \\
&\leq \vee_{x+\Sigma_{i=1}^{n} a_i b_i + y = \Sigma_{j=1}^{m} a'_j b'_j + y} \left\{\wedge_{i,j}\left[\lambda^-(a_i b_i) \wedge \lambda^-(a'_j b'_j), \lambda^+(a_i b_i) \wedge \lambda^+(a'_j b'_j)\right]\right\} \\
&\leq \vee_{x+\Sigma_{i=1}^{n} a_i b_i + y = \Sigma_{j=1}^{m} a'_j b'_j + y} \left[\lambda^-(\sum_{i=1}^{n} a_i b_i) \wedge \lambda^-(\sum_{j=1}^{m} a'_j b'_j), \lambda^+(\sum_{i=1}^{n} a_i b_i) \wedge \lambda^+(\sum_{j=1}^{m} a'_j b'_j)\right] \\
&= \vee_{x+\Sigma_{i=1}^{n} a_i b_i + y = \Sigma_{j=1}^{m} a'_j b'_j + y} \left(\lambda(\sum_{i=1}^{n} a_i b_i) \wedge \lambda(\sum_{j=1}^{m} a'_j b'_j)\right) \\
&\leq \vee_{x+\Sigma_{i=1}^{n} a_i b_i + y = \Sigma_{j=1}^{m} a'_j b'_j + y} \left(\lambda(x)\right) \qquad (\because \lambda \text{ is interval valued fuzzy left } h-\text{ideal}) \\
&= \lambda(x).
\end{aligned}$$

Hence $C_R \odot_h \lambda \subseteq \lambda$.

Conversely, assume that $(i), (ii), (iii)$ hold for an interval valued fuzzy subset λ of R. Then to prove that λ is an interval valued fuzzy left h-ideal of R,we only have to show that $\lambda(xy) \geq \lambda(y)$ for all $x, y \in R$. So let $x, y \in R$, then by (ii)

$$\begin{aligned}
\lambda(xy) &\geq (C_R \odot_h \lambda)(xy) \\
&= \vee_{xy+\Sigma_{i=1}^n a_i b_i + z = \Sigma_{j=1}^m a'_j b'_j + z} \left\{ \wedge_{i,j} \left[\lambda^-(b_i) \wedge \lambda^-(b'_j), \lambda^+(b_i) \wedge \lambda^+(b'_j) \right] \right\} \\
&\geq \left[\lambda^-(y) \wedge \lambda^-(y), \lambda^+(y) \wedge \lambda^+(y) \right] \qquad (\because xy + 0y + 0 = xy + 0) \\
&= \lambda(y).
\end{aligned}$$

Therefore $\lambda(xy) \geq \lambda(y)$ for all $x, y \in R$. ■

4.2 h-Hemiregular Hemirings

Recall the following two definitions given in chapter 1.

Definition 95 [17] *A hemiring R is said to be h-hemiregular if for all $a \in R$ there exist $x_1, x_2, z \in R$ such that*

$$a + ax_1a + z = ax_2a + z$$

Definition 96 [23] *Let A be a subset of a hemiring R then the h-closure $\bar{A}$ of A in R is defined as*

$$\bar{A} = \{x \in R : x + a_1 + z = a_2 + z \text{ for some } a_1, a_2 \in A \text{ and } z \in R\}.$$

Lemma 97 *Let R be a hemiring and $A, B \subseteq R$ then*

$$C_A \odot_h C_B = C_{\overline{AB}}$$

Proof. Let $x \in R$. If $x \in \overline{AB}$ then $C_{\overline{AB}}(x) = \tilde{I}$ and $x + \sum_{i=1}^{n} a_i b_i + z = \sum_{j=1}^{m} a'_j b'_j + z$ for some $a_i, a'_j \in A$ and $b_i, b'_j \in B$ and $z \in R$. Thus for all i and j

$$C_A(a_i) = C_A(a'_j) = C_B(b_i) = C_B(b'_j) = \tilde{I}$$

And hence

$$(C_A \odot_h C_B)(x)$$

$$= \vee_{x+\Sigma_{i=1}^{n} a_i b_i + z = \Sigma_{j=1}^{m} a'_j b'_j + z} \{\wedge_{i,j} [C_A^-(a_i) \wedge C_A^-(a'_j) \wedge C_B^-(b_i) \wedge C_B^-(b'_j),$$
$$C_A^+(a_i) \wedge C_A^+(a'_j) \wedge C_B^+(b_i) \wedge C_B^+(b'_j)]\}$$

$$= \vee_{x+\Sigma_{i=1}^{n} a_i b_i + z = \Sigma_{j=1}^{m} a'_j b'_j + z} \{\wedge_{i,j} \tilde{I}\} = \tilde{I}$$

Therefore whenever $x \in \overline{AB}$ then

$$(C_A \odot_h C_B)(x) = \tilde{I} = (C_{\overline{AB}})(x)$$

And if $x \notin \overline{AB}$ then $C_{\overline{AB}}(x) = \tilde{O}$.

If possible, let $(C_A \odot_h C_B)(x) \neq \tilde{O}$ then

$\vee_{x+\Sigma_{i=1}^{n} a_i b_i + z = \Sigma_{j=1}^{m} a'_j b'_j + z} \{\wedge_{i,j} [C_A^-(a_i) \wedge C_A^-(a'_j) \wedge C_B^-(b_i) \wedge C_B^-(b'_j), C_A^+(a_i) \wedge C_A^+(a'_j) \wedge C_B^+(b_i) \wedge C_B^+(b'_j)]\} \neq [0, 0]$

Therefore there exist $p_i, q_i, p'_j, q'_j \in R$ such that

$$x + \Sigma_{i=1}^{n} p_i q_i + z = \Sigma_{j=1}^{m} p'_j q'_j + z$$

And

$\wedge_{i,j} [C_A^-(p_i) \wedge C_A^-(p'_j) \wedge C_B^-(q_i) \wedge C_B^-(q'_j), C_A^+(p_i) \wedge C_A^+(p'_j) \wedge C_B^+(q_i) \wedge C_B^+(q'_j)] \neq [0, 0]$

Then obviously for all i and j

$$C_A^-(p_i) = C_A^-(p'_j) = C_B^-(q_i) = C_B^-(q'_j) = 1$$

and

$$C_A^+(p_i) = C_A^+(p_j') = C_B^+(q_i) = C_B^+(q_j') = 1$$

Thus for all i

$$C_A(p_i) = C_B(q_i) = \tilde{I}$$

and for all j

$$C_A(p_j') = C_B(q_j') = \tilde{I}$$

$\Rightarrow$

$$p_i \in A, q_i \in B \qquad \text{for all } i$$

and

$$p_j' \in A, q_j' \in B \qquad \text{for all } j$$

$\Rightarrow$

$$x \in \overline{AB}$$

which contradicts $C_{\overline{AB}}(x) = \tilde{O}$. Therefore whenever $x \notin \overline{AB}$ then again we have

$$(C_A \odot_h C_B)(x) = \tilde{O} = C_{\overline{AB}}(x)$$

Hence proved that

$$C_A \odot_h C_B = C_{\overline{AB}}$$

■

Theorem 98 *A hemiring R is h-hemiregular if and only if for any interval valued fuzzy right h-ideal λ and interval valued fuzzy left h-ideal μ of R, we have*

$$\lambda \odot_h \mu = \lambda \cap \mu$$

Proof. Let R be an h-hemiregular hemiring then by Lemma 93, $\lambda \odot_h \mu \subseteq \lambda \cap \mu$. For reverse containment, since R is h-hemiregular so for all $a \in R$, there exist $x_1, x_2, y \in R$ such that

$$a + ax_1a + y = ax_2a + y.$$

Now $(\lambda \odot_h \mu)(a)$

$$= \vee_{a+\Sigma_{i=1}^{n} a_i b_i + y = \Sigma_{j=1}^{m} c_j d_j + y} \{\wedge_{ij}[\lambda^-(a_i) \wedge \lambda^-(c_j) \wedge \mu^-(b_i) \wedge \mu^-(d_j), \lambda^+(a_i) \wedge \lambda^+(c_j) \wedge \mu^+(b_i) \wedge \mu^+(d_j)]\}$$

$$\geq \left[\lambda^-(ax_1) \wedge \lambda^-(ax_2) \wedge \mu^-(a) \wedge \mu^-(a), \mu^+(a) \wedge \lambda^+(ax_1) \wedge \lambda^+(ax_2) \wedge \mu^+(a)\right]$$

$$\geq [\lambda^-(a) \wedge \lambda^-(a) \wedge \mu^-(a) \wedge \mu^-(a), \lambda^+(a) \wedge \lambda^+(a) \wedge \mu^+(a) \wedge \mu^+(a)]$$

$$= \left[\lambda^-(a) \wedge \mu^-(a), \lambda^+(a) \wedge \mu^+(a)\right]$$

$$= \lambda(a) \wedge \mu(a) = (\lambda \cap \mu)(a).$$

Thus

$$\lambda \cap \mu \subseteq \lambda \odot_h \mu.$$

Hence

$$\lambda \odot_h \mu = \lambda \cap \mu.$$

Conversely, let A and B be right and left h-ideals of R respectively, then their characteristic functions C_A and C_B are also interval valued fuzzy right and interval valued fuzzy left h-ideals of R respectively. Then by hypothesis

$$C_{\overline{AB}} = C_A \odot_h C_B = C_A \cap C_B = C_{A \cap B}.$$

Thus $\overline{AB} = A \cap B$. Hence, by Lemma 45, R is h-hemiregular. ∎

References

[1] A. W. Aho, J. D. Ullman, Introduction to Automata Theory, Languages and Computation, Addison-Wesley, Reading, MA, 1979.

[2] J. Ahsan, K. Saifullah, M. Farid Khan, Fuzzy Semirings, Fuzzy Sets and Systems, 60 (1993) 309-320.

[3] J. Ahsan, Semirings Characterized by Their Fuzzy Ideals, The Journal of Fuzzy Mathematics Vol.6, No.1, 1998.

[4] S. I. Baik, H. S. Kim, On fuzzy k-ideals in semirings, Kangweon-Kyunki Math. J. 8 (2000) 147ñ154.

[5] R. Biswas, Rosenfeldís fuzzy subgroups with interval valued membership function, Fuzzy Sets and Systems 63 (1994) 87-90.

[6] S. Ghosh, Fuzzy k-ideals of semirings, Fuzzy Sets and Syst. 95 (1998) 103ñ 108.

[7] J. A. Goguen, L-fuzzy sets, J. Math. Anal. Appl. 18 (1967) 145-174.

[8] J. S. Golan, The Theory of Semirings with Applications in Mathematics and Theoretical Computer Sciences, Pitman Monographs and Surveys in Pure and App. Maths., no. 54 (Longman, New York, 1992).

[9] U. Hebisch and H.J. Weinert, Halbringe, Algebraische Theorie und Anwendungen in der Informatik, Taubner Stuttgart, (1993).

[10] M. Henriksen, Ideals in semirings with commutative addition, Amer. Math. Soc. Notices 6 (1958) 321.

[11] K. Iizuka, On the Jacobson radical of a semiring, Tohuku Math. J. 11 (2) (1959)409ñ421.

[12] Y. B. Jun, M.A. O® ztu® rk, S.Z. Song, On fuzzy h-ideals in hemirings, Inform. Sci. 162 (2004) 211ñ226.

[13] M. Kondo, W.A. Dudek, On the transfer principle in fuzzy theory, Mathware Soft Comput. 12 (2005) 41ñ55.

[14] D. R. La Torre, On h-ideals and k-ideals in hemirings, Publ. Math. Debrecen 12 (1965) 219ñ226.

[15] X. P. Li, G. J. Wang, The S_H interval valued fuzzy subgroups, Fuzzy Sets and Systems 112 (2000) 319-325.

[16] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971) 512-517.

[17] Y. Yin, H. Li, The characterization of h-hemiregular hemirings and h-intra-hemiregular hemirings, Information Sciences (2008); doi:10.1016/j.ins.2008.04.002.

[18] L. A. Zadeh, Fuzzy Sets, Inform. Control 8 (1965) 267-274.

[19] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning, Part 1, Information Sciences 8 (1975) 199-249.

[20] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning, Part 2, Information Sciences 8 (1975) 301-357.

[21] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning, Part 3, Information Sciences 9 (1975) 43-80.

[22] W. Zeng and Y. Shi, Note on Interval-valued Fuzzy Set, FSKD 2005, LNA1 3613, PP.20-25, 2005. c Springer-Verlag Berlin Heidelberg 2005.

[23] J. Zhan, Wieslaw A.Dudek, Fuzzy h-ideals of hemirings, Information Sciences, 177 (2007) 876-886.