Numerical solution of mixed convection boundary layer flow of a non-Newtonian fluid over a stretching cylinder in a thermally stratified medium

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بسم الله الرحمن الرحيم

In the Name of Allah, The Most beneficiary,
The Most Gracious, The Most Merciful

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Muhammad Awais

Abstract

The industrial applications of non-Newtonian fluids made it one of the most important phenomena during the last few decades. The main reason behind is the considerable amount of work that has been done in flow characteristics of non-Newtonian fluids and much more is also needed in different non-Newtonian models. Due to the diversity of non-Newtonian fluids one cannot express the behavior of all non-Newtonian fluids in a single constitutive equation. Therefore several models of non-Newtonian fluids have been proposed. Among these a particularly simple model namely the Sisko fluid model exists. The Sisko fluid model is the combination of Newtonian and non-Newtonian fluids (e.g. polymeric suspensions, biological fluids, drilling mud, paints, liquid crystals and lubricant greases etc.). The fluid model is capable of describing shear thinning and shear ticking phenomena. Sisko [1] was the first person who initiated the analyses of lubricating grease.

The present work is focused on two dimensional steady flow with mixed convective boundary layers in Sisko fluid model over a stretching cylinder in a thermally stratified medium. The governing partial differential equations (PDEs) are modeled. Moreover, these obtained PDEs are reduced into ordinary differential equations by using suitable similarity transformations. Numerical solutions are obtained by using Shooting method in conjunction with the Runge-Kutta-Fehlberg method. To visualize the behavior of velocity and temperature profiles after taking variation in physical parameters results are plotted through graphs. For further analysis Skin friction coefficient and Nusselts's number are computed.

Contents

Chapter 1

Basic Definitions

1.1 Preliminaries

This chapter includes some basic definitions and relevant governing equations are presented.

1.2 Basic definitions

1.2.1 Fluid

A fluid is a substance that deforms continuously under an applied shear stress regardless of how minor is the applied stress acting up on it.

1.2.2 Flow

It is a phenomenon in which the material deformation increases continuously without limit with respect to various forces.

1.2.3 Fluid mechanics

It is branch of mechanics which deals with nature and properties of fluids at stationary position as well as in moving state.

1.3 Some physical properties of fluids

1.3.1 Density

Density of a fluid is defined as the mass (m) per unit volume (V). Mathematically, the density ρ at a point may be defined as

$$\rho = \lim_{\delta V \to 0} \frac{\delta m}{\delta V}, \tag{1.1}$$

Dimension and unit of density are $\left[ML^{-3}\right]$ and kg/m^3 respectively.

1.3.2 Viscosity

A physical property which offers resistance to the flow is called viscosity. Mathematically, it is defined as the ratio of shear stress to the rate of shear strain i.e.

$$\text{Viscosity} = \mu = \frac{\text{shear stress}}{\text{rate of shear strain}}, \tag{1.2}$$

In the above definition, μ is called the coefficient of viscosity or absolute viscosity or dynamic viscosity or simply viscosity having dimensions $\left[ML^{-1}T^{-1}\right]$.

1.3.3 Kinematic viscosity

The ratio of dynamic viscosity to the density of a fluid is known as kinematic viscosity and is denoted by ν. Mathematically

$$\nu = \frac{\mu}{\rho}, \tag{1.3}$$

where ρ is the density of the fluid and dimension of kinematic viscosity $\left[L^2T^{-1}\right]$.

1.4 Classification of fluids

1.4.1 Ideal / Inviscid fluids

The fluids which have zero viscosity are called ideal fluid. But naturally no fluid with zero viscosity exists. However, in some engineering problems fluid with extremely low viscosity are considered as ideal fluid. Usually gases are treated as ideal fluid in many engineering problems.

Ideal fluids are also called the inviscid fluids..

1.4.2 Real fluids

Real fluids are those which have non-zero viscosity. These fluids may be compressible or incompressible. Depending upon the relationship between the shear stress and rate of shear strain, real fluids are further divided into two main categories namely **Newtonian** and **non-Newtonian** fluids.

Newtonian Fluids

The real fluids for which the shear stress is directly proportional to the linear deformation rate are called Newtonian fluids (e.g. water). In mathematical notation it will be

$$\tau_{yx} = \mu \frac{du}{dy}, \tag{1.4}$$

where τ_{yx} is the shear stress acting on a plane normal to $y-$axis, u is the velocity of fluid in the $x-$direction and μ is the constan of proportionality.

Non-Newtonian fluids

The real fluids that do not obey the Newtons law of viscosity are known as non-Newtonian fluids. For such fluids shear stress is not linearly proportional to the deformation rate. Mathematically

$$\tau_{yx} = \eta \cdot \frac{du}{dy}. \tag{1.5}$$

where $\eta = k \left| \frac{du}{dy} \right|^{n-1}$ is the apparent viscosity, n is flow behavior index and k is consistency index. For $n = 1$ with $k = \mu$, the above equation reduces to the Newton's law of viscosity. Examples of non-Newtonian fluids include ketchup, tooth paste, blood, paints, greases, biological fluids, polymer melts and so forth.

1.5 Types of flows

1.5.1 Laminar flow

A flow in which each fluid particle has a definite path and the path of individual particles do not cross each other and move along well defined paths is known as laminar flow.

1.5.2 Turbulent flow

A turbulent flow is one in which fluid particle does not have a definite path and the path of individual particles also cross each other, such types of flow cannot be handled easily.

1.5.3 Steady flow

A flow in which fluid properties at each point in flow field do not depend upon time is called steady flow. That is a flow in which the quantity of fluid flowing per second is constant. A steady flow may be uniform or non uniform. For such a flow, we can write

$$\frac{\partial \zeta}{\partial t} = 0. \tag{1.6}$$

where ζ represents any fluid property, may be velocity, density etc.

1.5.4 Unsteady flow

A flow in which fluid properties at each point in flow field also depend upon time is called unsteady flow. For such a flow, we can write

$$\frac{\partial \zeta}{\partial t} \neq 0. \tag{1.7}$$

where ζ represents any fluid property, may be velocity, density etc.

1.5.5 Incompressible flow

A flow in which density of the flowing fluid does not change during the flow. Mathematically, incompressible fluid is expressed by saying that the density ρ of a fluid particle does not change as it moves in the flow field, i.e.

$$\rho \neq \rho\left(x, y, z, t\right) \quad \text{or} \quad \rho = \text{constant}. \tag{1.8}$$

or

$$\frac{d\rho}{dt} = 0, \tag{1.9}$$

where d/dt is the total derivative, which is the sum of local and convective derivatives. All the liquids are generally exhibit incompressible flow.

1.5.6 Compressible flow

A flow in which density of the fluid varies during the flow is termed as compressible flow i.e.

$$\rho = \rho\left(x, y, z, t\right). \tag{1.10}$$

1.6 Relationship in Cylindrical Coordinates

The del operator in cylindrical coordinates be

$$\boldsymbol{\nabla} = \frac{1}{r}\frac{\partial\ \left(r\ (.)\right)}{\partial r} + \frac{1}{r}\frac{\partial\ (.)}{\partial \theta} + \frac{\partial\ (.)}{\partial z}, \tag{1.11}$$

the divergence of vector $\mathbf{V} = [\mathbf{v}_r, v_\theta, v_z]$ is defined as

$$\boldsymbol{\nabla}.\mathbf{V} = \frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial\ (v_\theta)}{\partial \theta} + \frac{\partial\ (v_z)}{\partial z}, \tag{1.12}$$

the divergence of stress tensor $\mathbf{T}$ is written as

$$\mathbf{T} = \left[\frac{1}{r}\frac{\partial\ \left(r\tau_{rr}\right)}{\partial r} + \frac{1}{r}\frac{\partial(\tau_{r\theta})}{\partial \theta} + \frac{\partial(\tau_{rz})}{\partial z} - \frac{(\tau_{\theta\theta})}{r}\right].\hat{r} \tag{1.13}$$

$$+ \left[\frac{1}{r}\frac{\partial\ \left(r\tau_{\theta r}\right)}{\partial r} + \frac{1}{r}\frac{\partial\left(\tau_{\theta\theta}\right)}{\partial \theta} + \frac{\partial(\tau_{\theta z})}{\partial z} + \frac{\tau_{r\theta}}{r}\right].\hat{\theta} \tag{1.14}$$

$$+ \left[\frac{1}{r}\frac{\partial(r\tau_{zr})}{\partial r} + \frac{1}{r}\frac{\partial\left(\tau_{z\theta}\right)}{\partial \theta} + \frac{\partial\left(\tau_{zz}\right)}{\partial z}\right].\hat{z},$$

The gradient of a vector produces a second rank tensor.

$$\nabla \mathbf{V} = \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right) & \frac{\partial v_r}{\partial z} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) & \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_z}{\partial r} & \frac{1}{r}\frac{\partial v_z}{\partial \theta} & \frac{\partial v_z}{\partial z} \end{bmatrix}. \tag{1.15}$$

1.7 Governing laws

1.7.1 Law of conservation of mass

This Law states that matter can change its form, mixtures can be separated or made, and pure substances can be decomposed, but the total amount of mass remains constant. In the vector form, it can be written as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \tag{1.16}$$

$$\nabla = \frac{1}{r}\frac{\partial\,(r\,(.))}{\partial r} + \frac{1}{r}\frac{\partial\,(.)}{\partial \theta} + \frac{\partial\,(.)}{\partial z}, \tag{1.17}$$

a three dimensional differential operator. For an incompressible fluid, the density is constant and thus Eq (1.16) becomes

$$\nabla \cdot \mathbf{V} = 0. \tag{1.18}$$

1.7.2 Law of conservation of momentum

Every particle of fluid at rest or in steady state or in accelerated motion obeys Newton's second law of motion which states that, the sum of all external forces acting on a system is equal to the time rate of change of linear momentum of the system. In vector form, it can be written as

$$\rho \frac{d\mathbf{V}}{dt} = \operatorname{div} \mathbf{T} + \rho \mathbf{b}, \tag{1.19}$$

For Navier-Stokes equations

$$\mathbf{T} = -p\mathbf{I} + \boldsymbol{\mu}\mathbf{A_1}, \tag{1.20}$$

$$\mathbf{A_1} = (\text{grad }\ \mathbf{V}) + (\text{grad }\ \mathbf{V})^t. \tag{1.21}$$

where

$\rho =$ density

$\mathbf{V} =$ vector field

$\mathbf{T} =$ Cauchy stress tensor

$\mathbf{b} =$ body forces

$p =$ pressure

$\mu =$ dynamic viscosity

$\mathbf{A}_1 =$ First rivlin-ericksen tensor

The Cauchy's stress tensor can be expressed in matrix form as

$$\mathbf{T} = \begin{bmatrix} \sigma_{rr} & T_{r\theta} & T_{rz} \\ T_{\theta r} & \sigma_{\theta\theta} & T_{\theta z} \\ T_{zr} & T_{z\theta} & \sigma_{zz} \end{bmatrix}, \tag{1.22}$$

where σ_{rr}, $\sigma_{\theta\theta}$ and σ_{zz} are normal stresses while all others are shear stresses. Eq (1.23) can be expressed in scalar form as

$$\begin{aligned} \rho\frac{dv_r}{dt} &= \frac{1}{r}\frac{\partial(r\sigma_{rr})}{\partial r} + \frac{1}{r}\frac{\partial(T_{r\theta})}{\partial\theta} + \frac{\partial\left(T_{rz}\right)}{\partial z} - \frac{\sigma_{\theta\theta}}{r} + \rho b_r, \\ \rho\frac{dv_\theta}{dt} &= \frac{1}{r}\frac{\partial\left(rT_{\theta r}\right)}{\partial r} + \frac{1}{r}\frac{\partial\left(\sigma_{\theta\theta}\right)}{\partial\theta} + \frac{\partial\left(T_{\theta z}\right)}{\partial z} + \frac{T_{r\theta}}{r} + \rho b_\theta, \\ \rho\frac{dv_z}{dt} &= \frac{1}{r}\frac{\partial\left(rT_{zr}\right)}{\partial r} + \frac{1}{r}\frac{\partial\left(\sigma_{z\theta}\right)}{\partial\theta} + \frac{\partial\left(\sigma_{zz}\right)}{\partial z} + \rho b_z. \end{aligned} \tag{1.23}$$

1.7.3 Law of conservation of energy

Like laws of conservation of mass and momentum, law of conservation of energy is essential to study the heat transfer phenomenon in fluid dynamics problems. It states that energy can be transfer from one form to another in an isolated system but it cannot be created or destroyed and the total energy of the system is conserved. Mathematically

$$\rho c_p\frac{dT}{dt} = k\boldsymbol{\nabla}^2T + \mathbf{T}.\mathbf{L}, \tag{1.24}$$

,where c_p is specific heat at constant pressure, θ is temperature of fluid, $\mathbf{T}.\mathbf{L}$ is viscous dissipation term and k is the thermal conductivity which describes that how fast a particular material conduct heat.

1.8 Heat transfer

It is discipline of thermal engineering that deals with the generation, use, conversion and exchange of thermal energy and heat between physical systems.

1.8.1 Conduction

The transfer of energy between objects that are in physical contact is called condition.

1.8.2 Convection

The transfer of energy between an object and its environment, due to fluid motion is called convection.

1.9 Boundary layer

Ludwig Prandtl a German astronomer revealed the idea of boundary layer, in 1904, on his paper which he presented in mathematical congress. Boundary layer is a layer adjacent to the solid surface, where the viscosity effects are dominant. In determining the flow field, the viscous effects are considering into account, which have significant role on fluid motion. Thus a fluid flow is retarded in the vicinity of the wall and a finite, slow moving boundary layer is formed. The thickness of the boundary layer is taken to be the distance from the wall to the point at which the velocity is 99% of the free-stream velocity. As the solution of the Navier-Stokes equation is expensive, so this approach helps us to reduce equations.

1.10 Mixed Convection

The convection which involves the combine effects of natural and forced convection is known as mixed convection. If the flow is generated by any external force in the presence of gravity then such a convection is known as mixed convection

1.11 Mathematical description of Boussinesq approximation

Let us assume a two dimensional steady flow over a smooth surface. Here we consider constant properties except that density is allowed to vary to produce buoyancy force, as

$$p_1 = p_n + p_\infty, \tag{1.25}$$

where p_∞ and p_n represent static pressure inside boundary layer and pressure variation due to buoyancy, respectively. Using this definition, we get the following results

$$p_n = p_1 - p_\infty \text{ and } p_n \to 0 \text{ as } y \to \infty, \tag{1.26}$$

The continuity and momentum equations inside the boundary layer are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1.27}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p_1}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} - \rho g_x, \tag{1.28}$$

While outside the boundary layer, we have

$$p_1 \to p_\infty,\ p_n \to 0,\ \rho \to \rho_\infty \text{ and } u \to U(x) \text{ as } y \to \infty. \tag{1.29}$$

Thus the momentum equations outside the boundary layer becomes

$$\rho_\infty U(x)\frac{\partial U(x)}{\partial x} = -\frac{\partial p_1}{\partial x} - \rho_\infty g_x, \tag{1.30}$$

or

$$\rho_\infty U(x)\frac{\partial U(x)}{\partial x} + \rho_\infty g_x = -\frac{\partial p_1}{\partial x}, \tag{1.31}$$

Substituting Eq. (1.31) in Eq. (1.28) we get

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \rho_\infty U(x)\frac{\partial U(x)}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} + (\rho_\infty - \rho)\, g_x, \tag{1.32}$$

In the above equation, the density difference, $\rho_\infty - \rho$, can be related to the temperature difference. Using the Taylor's expansion about (T_∞, C_∞) one has

$$\rho(T,C) = \rho(T_\infty, C_\infty) + (T - T_\infty)\left(\frac{\partial\rho}{\partial T}\right)_{p_\infty} + (C - C_\infty)\left(\frac{\partial\rho}{\partial C}\right)_{p_\infty} + ..., \tag{1.33}$$

or

$$\rho \approx \rho_\infty + (T - T_\infty)\left(\frac{\partial\rho}{\partial T}\right)_{p_\infty} + (C - C_\infty)\left(\frac{\partial\rho}{\partial C}\right)_{p_\infty}, \tag{1.34}$$

Since the coefficient of volumetric thermal expansion β and volumetric mass expansion β^* are defined as

$$\beta = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_{p_\infty}, \tag{1.35}$$

$$\beta^* = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial C}\right)_{p_\infty}, \tag{1.36}$$

so the above relationship reduces to

$$\rho \approx \rho_\infty - (\beta\rho)_\infty (T - T_\infty) - (\beta^*\rho)_\infty (C - C_\infty), \tag{1.37}$$

If $(\beta\rho)_\infty = \beta\rho =$ constant, which is true for moderate temperature difference, then we have

$$\rho_\infty - \rho = \rho\left[\beta (T - T_\infty) + \beta^* (C - C_\infty)\right]. \tag{1.38}$$

The above relationship (1.39) is called **Boussinesq approximation.**

1.12 Dimensionless numbers

1.12.1 Reynolds number

Reynolds number Re is the ratio of inertial forces $(V\rho)$ to viscous forces (μ/L) which is a dimensionless number. Mathematically

$$\mathrm{Re} = \frac{UL}{\nu}. \tag{1.39}$$

where L is characteristic length, U free stream velocity and ν is kinematic viscosity. Laminar flow occur at low Reynolds number where viscous forces are dominant, while turbulent flow occur at high Reynolds number and it is dominated by inertial forces.

1.12.2 Nusselt number

It is a dimensionless number, used in heat transfer which is a ratio of convective to conductive heat transfer across (normal) to the boundary introduced by German mathematician Nusselt. Mathematically

$$Nu = \frac{hL}{k}. \tag{1.40}$$

where h is convective heat transfer coefficient and k thermal conductivity of the fluid.

1.12.3 Prandtl number

It is a dimensionless number defined as the ratio of momentum diffusivity (ν) to thermal diffusivity (α). It control the relative thickness of momentum and thermal boundary layer. Mathematically

$$\Pr = \frac{\nu}{\alpha}. \tag{1.41}$$

Chapter 2

Introduction

2.1 Introduction

During last two decades flow of non-Newtonion fluids has been involved in various physical phenomenon such as polymer processing, ink-jet printing, geological flows in the earth mantle, liquid crystals, additive suspensions, animal blood, turbulent shear flows and many others. In view of litrature available regarding flow of non-Newtonion fluids and its extensive use in industrial and technological applications, special attention has been paid to these fluids. Therefore several fundamental equations are suggested to predict the physical behavior and structure of such fluids. Among these, comparatively simple model, named Sisko fluids exists. The Sisko fluid model is the combination of Newtonian and non-Newtonian fluids. The fluid model is capable of describing shear thinning and shear thickening phenomenon, which represent the decrease and increase in viscosity with increasing shear rate respectively. This type of fluids exists commonly in nature. Such fluids are well known and have many industrial applications. For the flow of greases it is the most relevant model. Sisko [1] was the first person who presented and analyzed the lubricating grease. After that many researchers have work on this model. M. Khan et al. [$2-5$], S. Nadeem et al. [6], N. S. Akbar [7], F. Talay Akyildiz et al. [8] and many others investigated the Sisko fluid model in different geometries with pertinent physical properties of fluid.

Convection is a mode of heat transfer that plays an important role in practical models. Many convection processes take place in our surroundings, such as in atmospheres, oceans, planetary mantles, and it also specify the mechanism of heat transfer for a large fraction of the outermost interiors of our sun and all stars. Fluid movement during convection may be invisibly slow, or it may be obvious and rapid, as in a hurricane. On astronomical scales, convection of gas and dust is thought to occur in the accretion disks of black holes, at speeds which may closely approach that of light. Due to abundant use in nature mixed convection is studied by many scientists. M. Swati et al.[9] studied mixed convection along a stretching cylinder in a thermally stratified medium. N. Bachok et al.[10] analyzed mixed convection boundary layer flow over a permeable vertical cylinder with prescribed surface heat flux. J.J. Heckel et al.[11] also examined mixed convection along slender vertical cylinders with variable surface temperature. K.L. Hsiao[12] explored MHD mixed convection for viscoelastic fluid past a porous wedge. C.H. Chen [13] discussed laminar mixed convection adjacent to vertical, continuously stretching sheet. S. Nadeem et al.[14] analyzed unsteady mixed convection flow of nanofluid on a rotating cone with magnetic field. J.M. Buchlin [15] inspect natural and forced convective heat transfer on slender cylinders.

Stratification effects in any fluid may cause due to temperature variation or concentration differences or the presence of different fluids in any medium and combination of these. As fluid heats and cools, it expands and contracts, causing change in density. This is called thermal stratification and it is generally occures when thermal energy transforms from heated bodies and thermal sources into the medium. Stratification may also arise due to concentration differences such as transport processes in the sea where stratification exists due to salinity variation. Third type of stratification occures when fluids having different densities are present and stable situation arises such that fluid having less density overlies the havier fluid. Stratification may double in practical situations, where the heat and mass transfer mechanisms run parallel Stratification has abundant applications in our real world. Applications of stratification include heat rejection into the environment such as from lakes, rivers and seas. Thermal energy storage systems such as solar ponds and heat transfer from thermal sources such as the condensers of power plants are also examples of stratification. Due to the huge implementations in fluid mechanics many researchers have worked on stratification phenomenon. The flow due to a heated surface immersed in a stable stratified medium has been investigated experimentally and analytically in several studies such as Yang et al.[16], Jaluria et al.[17], Chen et al.[18], and

Ishak et al.[19] Swati Mukhopadhyay et al.[20], N. Kishan et al. [21], M. A. Mansour1[22] etc.

The boundary layer flow and heat transfer due to stretching cylinders have remarkable importance in fiber technology and extrusion process. There are many examples in metallurgical and engineering such as hot rolling, metal and polymer sheet extrusion, drawing, annealing and tinning of copper wires, crystal growth, glass fiber production etc. The steady two-dimensional boundary layer flow due to contineous solid surface was first studied by B. C. Sakiadis [23]. After this J.N. Kapoor et al. [24] found the similarity solution of boundary layer equations for power law fluids and then Crane [23] studied it. After Crane [23], Gupta and Gupta [26], Chen et al.[27], R. R. Rangi et al.[28], Datta et al. [29] extended the work including the effect of heat and mass transfer analysis under different physical situations. S. Nadeem et al. [30] studied boundary layer flow of nanofluid over an exponentially stretching surface. S. Nadeem et al.[31] presented HAM solution for boundary layer flow in the region of the stagnation point towards a stretching sheet.

Now a days stretching of any surface to produce disturbance in any fluid is one of the leading feature in fluid mechanics. For a long period scientists didn't considered stretching of cylinder, but when Lin et al. [$32-33$] considered the laminar boundary layer and heat transfer along cylinders moving horizontally and vertically with constant velocity and found no similarity solutions due to the curvature effect of the cylinder. After that A. Ishak et al.[34] showed that the similarity solutions could be obtained by assuming that the cylinder is stretched with a linear velocity in the axial direction. In fact, the study by A. Ishak et al.[34] is an extension of the problem considered by Grubka et al.[35] and Ali [36], i.e. from a stretching sheet to a stretching cylinder. The stretching problems [$37-38$] for steady and unsteady flows have been studied extensively in various aspects, such as for non-Newtonian fluids, MHD flows, porous plates, porous medium, with and without heat transfer analysis.

The main objective of this theses is to examine the behavior of mixed convection boundary layer flow of Sisko fluid over the stretching cylinder in a stratified medium which is not discussed so far. The final non linear differential equations are solved numerically by Shooting method. The influence of curvature parameter M, material parameters A, mixed convection parameter λ, stratification parameter S and Prandtl number Pr on velocity and temperature profile is discussed.

2.2 Objectives of dissertation

The main objectives of present investigation are

- To study the mixed convection flow along a stretching cylinder in a thermally stratified medium
- To investigate the mixed convection boundary layer flow of Sisko fluid along a stretching cylinder in a thermally stratified medium

2.3 Method of Solution

As we know that mathematical modelling of many physical phenomenon take place in nature give non-linear system of equations, such as governing equations of fluid velocity and temperature. To solve governing equations of fluid different analytical as well as numerical techniques have been used. Perturbation method, Adomian decomposition method and homotopy analysis method are mostly used analytical techniques. For computation of numerical solutions researchers have used techniques like Shooting methods, finite difference method, finite volume method etc. In present investigation shooting method is used to find the numerical solution. Reason behind to prefer shooting method is that it transforms boundary value problem into initial value problem. Also it is comparatively more rapid and accurate numerical method.

2.4 Outlines of dissertation

This dissertation consist four chapters. Governing laws and basic definitions are explained in first chapter. Chapter two includes Introduction and objective of this thesis. In chapter three the problem of mixed convection flow along a stretching cylinder in a thermally stratified medium is reviewed. In this chapter governing equations are modeled and effects of physical parameters are discussed.

In chapter four problem of chapter three is extended and considered mixed convection boundary layer flow of Sisko fluid along a stretching cylinder in a thermally stratified medium. Numerical solution is calculated with the help of shooting method. Graphs present behavior of parameters on velocity and temperature profiles while tables show the effect of parameters on skin-friction coefficient and local Nusselt number.

Chapter 3

Mixed Convection Flow along a Stretching Cylinder in a Thermally Stratified Medium

3.1 Introduction

In this chapter we studied the axisymmetric, boundary layer mixed convection flow of a viscous and incompressible fluid over a stretching cylinder in a thermally stratified medium. It is review of Swati Mukhopadhyay and Anuar Ishak [7] paper. First of all modelled partial differential equations are transformed to highly nonlinear ordinary differential equations by using similarity transformations. Numerical solutions of these equations are obtained by shooting method in cunjuction with Runge- Kutta- Fehlberg method. The effects of different physical parameters on the velocity and temperature profiles are examined in detail. Influence of these parameters on skin friction coefficient and local Nusselt number are discussed through tables.

3.1.1 Problem formulation

We consider the axisymmetric, steady, mixed convection flow of an incompressible viscous fluid along a stretching cylinder embedded in a thermally stratified fluid-saturated medium. The basic governing continuity equation, momentum equation and the energy equation are

$$\begin{aligned} \text{div } \mathbf{V} &= 0, && (3.1) \\ \rho\frac{D\mathbf{V}}{Dt} &= \boldsymbol{\nabla}\cdot\mathbf{T}+\rho\mathbf{b}, && (3.2) \end{aligned}$$

$$\rho c_p\frac{DT}{Dt} = K\ \boldsymbol{\nabla}^2 T. \qquad (3.3)$$

where $\mathbf{V}$ is the velocity field, ρ is the density, $\mathbf{b}$ represents body forces, $\frac{D}{Dt}$ is the material time derivative, $\boldsymbol{\nabla}$ is differential operator, $\mathbf{T}$ is cauchy stress tensor, c_p is the specific heat at constant pressure, k is the thermal conductivity of the fluid and T denotes the temperature of the fluid. Some of these terms are describe below.

$$\mathbf{V} = [v(x,r), 0, u(x,r)], \qquad (3.4)$$

$$\boldsymbol{\nabla} = \frac{1}{r}\frac{\partial\ (r\ (.))}{\partial r} + \frac{1}{r}\frac{\partial\ (.)}{\partial\theta} + \frac{\partial\ (.)}{\partial x}, \qquad (3.5)$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V}.\boldsymbol{\nabla}, \qquad (3.6)$$

Since the flow field is considered steady, so the term $\frac{\partial}{\partial t}$ is neglected throughout the problem. So Eq (3.6) reduces to

$$\frac{D}{Dt} = \mathbf{V}.\boldsymbol{\nabla}. \qquad (3.7)$$

$$\rho\mathbf{b} = g\beta\left(T - T_\infty\right), \qquad (3.8)$$

where Eq (3.8) comes through Bossiness approximation

$$\mathbf{T} = -p\mathbf{I}+\mu\mathbf{A}_1. \qquad (3.9)$$

In Eqs (3.4) – (3.9), v and u are the components of the velocity field in r and x directions respectively, $\mathbf{I}$ is identity tensor, p is the pressure, μ is dynamic viscosity g is gravity, β volumetric thermal expansion and $\mathbf{A}_1$ is the first Rivilin - Ericksen tensor defined as

$$\mathbf{A}_1 = \boldsymbol{\nabla}\mathbf{V}+\left(\boldsymbol{\nabla}\mathbf{V}\right)^T, \qquad (3.10)$$

where

$$\boldsymbol{\nabla}\mathbf{V}=\begin{bmatrix} \frac{\partial v}{\partial r} & \frac{1}{r}\left(\frac{\partial v}{\partial \theta}-w\right) & \frac{\partial v}{\partial x} \\ \frac{\partial w}{\partial r} & \frac{1}{r}\left(\frac{\partial w}{\partial \theta}+v\right) & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial r} & \frac{1}{r}\frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial x} \end{bmatrix}, \tag{3.11}$$

using Eq (3.11) in Eq (3.10) we have

$$\mathbf{A}_1=\begin{bmatrix} 2\,\frac{\partial v}{\partial r} & \frac{1}{r}\frac{\partial v}{\partial \theta}-\frac{w}{r}+\frac{\partial w}{\partial r} & \frac{\partial v}{\partial x}+\frac{\partial u}{\partial r} \\ \frac{\partial w}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial \theta}-\frac{w}{r} & \frac{2}{r}\left(\frac{\partial w}{\partial \theta}+v\right) & \frac{\partial w}{\partial x}+\frac{1}{r}\frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r}+\frac{\partial u}{\partial x} & \frac{1}{r}\frac{\partial u}{\partial \theta}+\frac{\partial w}{\partial x} & 2\,\frac{\partial u}{\partial x} \end{bmatrix}, \tag{3.12}$$

As $\mathbf{V}=[v(x,r),0,u(x,r)]$ applying in above equation we have

$$\mathbf{A}_1=\begin{bmatrix} 2\frac{\partial v}{\partial r} & 0 & \frac{\partial v}{\partial x}+\frac{\partial u}{\partial r} \\ 0 & \frac{2}{r}\left(v\right) & 0 \\ \frac{\partial v}{\partial r}+\frac{\partial u}{\partial x} & 0 & 2\frac{\partial u}{\partial x} \end{bmatrix}, \tag{3.13}$$

Using Eq (3.4) and Eq (3.7) in continuity equation i.e. in Eq (3.1) we have

$$\frac{\partial}{\partial x}\left(r\ v\right)+\frac{\partial}{\partial x}\left(r\ u\right)=0, \tag{3.14}$$

Using the $Eq.$(3.8)and Eq (3.9) in $Eq.$(3.2) i.e. in momentum equation we obtain

$$\rho\frac{D\mathbf{V}}{Dt}=-\boldsymbol{\nabla}p+\mu\boldsymbol{\nabla}\cdot\mathbf{A}_1+g\beta\left(T-T_\infty\right). \tag{3.15}$$

Since there is no pressure gradient so the $Eq.$(3.15) reduces to

$$\frac{D\mathbf{V}}{Dt}=\nu\boldsymbol{\nabla}\cdot\mathbf{A}_{1,} \tag{3.16}$$

where ν is the kinematic viscosity $\left(\upsilon = \dfrac{\mu}{\rho}\right)$ and $(\boldsymbol{\nabla}.\mathbf{A}_1)$ in component form is as follows

$$(\boldsymbol{\nabla}\cdot\mathbf{A}_1)_x = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right)+\frac{\partial^2 u}{\partial x^2}, \tag{3.17}$$

$$(\boldsymbol{\nabla}\cdot\mathbf{A}_1)_r = \frac{2}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial v}{\partial r}\right)+\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial r}-\frac{2v}{r^2}, \tag{3.18}$$

$$(\boldsymbol{\nabla}\cdot\mathbf{A}_1)_\theta = 0. \tag{3.19}$$

Material time derivative $\frac{\mathbf{D}}{\mathbf{Dt}}$ for velocity field $\mathbf{V}$ in component form is given as

$$\left(\frac{D\mathbf{V}}{Dt}\right)_x = u\,\frac{\partial u}{\partial x}+v\,\frac{\partial u}{\partial r}, \tag{3.20}$$

$$\left(\frac{D\mathbf{V}}{Dt}\right)_r = u\,\frac{\partial v}{\partial x}+v\,\frac{\partial v}{\partial r}, \tag{3.21}$$

$$\left(\frac{D\mathbf{V}}{Dt}\right)_\theta = 0. \tag{3.22}$$

Using *Eqs* (3.17) − (3.22) in momentum equation i.e. *Eq* (3.16) , θ component satisfies identically, while x and r component are respectively describe below

$$u\,\frac{\partial u}{\partial x}+v\,\frac{\partial u}{\partial r} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right)+\nu\frac{\partial^2 u}{\partial x^2}, \tag{3.23}$$

$$u\,\frac{\partial v}{\partial x}+v\,\frac{\partial v}{\partial r} = \frac{2\nu}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial v}{\partial r}\right)+\frac{\nu\partial^2 v}{\partial x^2}+\frac{\nu\partial^2 u}{\partial x\partial r}-\frac{2\nu v}{r^2}. \tag{3.24}$$

Now we have to calculate the terms of energy equation i.e. *Eq* (3.3) . L.H.S of *Eq* (3.3) is simply material time derivative and R.H .S of *Eq* (3.3) is a $k\,\mathrm{div}(grad\ T)$ respectively given below

$$\frac{DT}{Dt} = u\,\frac{\partial T}{\partial x}+v\,\frac{\partial T}{\partial r}, \tag{3.25}$$

$$\boldsymbol{\nabla}\,(k.\boldsymbol{\nabla}T) = \frac{k}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial T}{\partial r}\right)+k\frac{\partial^2 T}{\partial x^2}, \tag{3.26}$$

incorporating these above two equations in *Eq* (3.3) we have

$$\rho\,c_p\,\left(u\,\frac{\partial T}{\partial x}+v\,\frac{\partial T}{\partial r}\right) = \frac{k}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial T}{\partial r}\right)+k\,\frac{\partial^2 T}{\partial x^2}, \tag{3.27}$$

$$u\,\frac{\partial T}{\partial x}+v\,\frac{\partial T}{\partial r}=\frac{K}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+K\,\frac{\partial^2 T}{\partial x^2}. \tag{3.28}$$

where K is the thermal diffusivity of the fluid i.e. $K=\dfrac{k}{\rho c_p}$. According to the boundary layer theory we assume that

$$u=O(1),\ x=O(1),\ v=O(\delta),\ r=O(\delta),\ T=O(1),\ \nu=O(\delta^2),\ K=O(\delta^2). \tag{3.29}$$

where δ is small psitive number. By using above boundary layer approximation the governing continuity, momentum and energy equations take the form

$$\frac{\partial}{\partial x}\,(r\ v)+\frac{\partial}{\partial x}\,(r\ u)=0, \tag{3.30}$$

$$u\,\frac{\partial u}{\partial x}+v\,\frac{\partial u}{\partial r}=\frac{\nu}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right)+g\beta\,(T-T_\infty)\,, \tag{3.31}$$

where the term $g\beta\,(T-T_\infty)$ is due to mixed convection.

$$u\,\frac{\partial T}{\partial x}+v\,\frac{\partial T}{\partial r}=\frac{K}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right), \tag{3.32}$$

It is assumed that the convecting fluid and the medium are in local thermodynamic equilibrium. The boundary conditions for the problem are given by

$$u\,(x,r)=U(x),v(x,r)=0,\ T\,(x,r)=T_w(x),at\ r=R\ where\ T_w\,(x)=T_0+\frac{b\ x}{L} \tag{3.33}$$

$$u\,(x,r)\to 0,\ T\,(x,r)\to T_\infty(x),\ as\ r\to\infty\ \ where\ T_\infty\,(x)=T_0+\frac{c\ x}{L}$$

In the above expressions R is the radius of the cylinder, $U(x)=U_0\dfrac{x}{L}$ is the stretching velocity, $T_w=T_0+\dfrac{b\ x}{L}$ is the prescribed surface temperature, and $T_\infty=T_0+\dfrac{c\ x}{L}$ is the variable ambient temperature. U_0 is the reference velocity, T_0 the reference temperature, L the characteristic length, b and c are positive constants. To get the similarity solution of Eq (3.31) and Eq (3.32) subject to the boundry conditions Eq (3.33), we introduce the following similarity transformations

$$\eta = \frac{r^2 - R^2}{2R}\left(\frac{U}{\nu x}\right)^{\frac{1}{2}}, \quad \psi = (U\nu x)^{\frac{1}{2}}\, R\, f(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_0}. \tag{3.34}$$

where ψ is the stream function, which identically satisfies the continuity Eq (3.30) and it is define as

$$u = \frac{1}{r}\left(\frac{\partial\psi}{\partial r}\right) \quad , \quad v = \frac{-1}{r}\left(\frac{\partial\psi}{\partial x}\right). \tag{3.35}$$

substituting Eq (3.34) in Eq (3.30) and (3.32), we get the following highly nonlinear ordinary differential equations

$$(1 + 2M\eta)\, f^{'''} + 2Mf^{''} + ff^{''} - f^{'\,2} + \lambda\theta = 0, \tag{3.36}$$

$$(1 + 2M\eta)\, \theta^{''} + 2M\theta^{'} + \Pr\left(f\ \theta^{'} - f^{'}\theta - f^{'}S\right) = 0. \tag{3.37}$$

subject to the boundary conditions

$$f^{'} = 1 \quad , \quad f = 0 \quad , \quad \theta = 1 - S \quad , \quad at \quad \eta \to 0, \tag{3.38}$$

$$f^{'} \to 0 \quad , \quad \theta \to 0 \quad , \quad as \quad \eta \to \infty.$$

where prime denotes the differentiation with respect to η, S represents the stratification parameter, λ denotes mixed convection parameter , M for curvature parameter, Pr denotes Prandtl's number. These parameters are describe below

$$\begin{aligned} S &= \frac{c}{b} \quad , \quad M = \left(\frac{\nu L}{U_0 R^2}\right)^{\frac{1}{2}}, \\ \lambda &= \frac{g\beta Lb}{U_0^2} \quad , \quad \Pr = \frac{\nu}{k}. \end{aligned} \tag{3.39}$$

3.2 Skin friction coefficient

The formula for the calculation of surface shear stress at the surface of the cylinder is

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2}, \tag{3.40}$$

where τ_w is the shear stress at the surface of the cylinder and defined as

$$\tau_w = \mu \left(\frac{\partial u}{\partial r} \right)_{r\,=\,R}, \tag{3.41}$$

Applying similarity transformations

$$\tau_w = \mu \left(U \frac{r}{R} \sqrt{\frac{U}{\nu x}} f^{''}(\eta) \right)_{r\,=\,R\;,\;\eta\,=\,0}, \tag{3.42}$$

after imposing $\eta = 0$ and $r = R$ we have

$$\tau_w = \mu U \sqrt{\frac{U}{\nu x}} f^{''}(0), \tag{3.43}$$

using Eq (3.43) in Eq (3.40) we have

$$C_f = 2\sqrt{\frac{\nu}{Ux}} f^{''}(0). \tag{3.44}$$

as we know that

$$\operatorname{Re} x = \frac{Ux}{\nu}, \tag{3.45}$$

so Eq (3.44) takes the form

$$f^{''}(0) = \frac{1}{2} C_f (\operatorname{Re} x)^{\frac{1}{2}}. \tag{3.46}$$

3.3 Local Nusselt number

The local Nusselt's number of temperature distribution is defined as

$$Nu_x = \frac{x q_w}{k (T_w - T_0)}, \tag{3.47}$$

where q_w is the rate of heat transfer at the surface and it is defined as

$$q_w = -k \left(\frac{\partial T}{\partial r} \right)_{r\,=\,R}, \tag{3.48}$$

By usnig similarity transformation we have

$$q_w = -k\left((T_w - T_0)\,\frac{r}{R}\sqrt{\frac{U}{\nu x}}\theta^{'}(\eta)\right)_{r\,=\,R\;,\;\eta\,=\,0}, \tag{3.49}$$

$$q_w = -k(T_w - T_0)\sqrt{\frac{U}{\nu x}}\theta^{'}(0). \tag{3.50}$$

After using Eq (3.50) in Eq (3.47) we have

$$Nu_x\,\mathrm{Re}_x^{\frac{-1}{2}} = -\theta^{'}(0). \tag{3.51}$$

where $\mathrm{Re} = \dfrac{Ux}{\nu}$ is Reynolds number

3.4 Method of solution

As we know that the non-linear momentum $Eq.$(3.36) is of order third in f and non-linear energy $Eq.$(3.37) is second order in θ, so total order of both equations is five, which can be diminished to a system of five first order ordinary differential equations with five unknowns. Numerical solution of the system of equations is find by shooting method in conjunction with Runge- Kutta- Fehlberg method. For this we must have five initial conditions to solve system of five ordinary differential equations but as we know that we have only two initial conditions in f and one initial condition in θ i.e. one initial condition on f and one on θ is missing. However, the values of $f\ ^{'}$ and θ are known at $\eta \to \infty$. Thus, these two end conditions are exploit to produce two unknowns. The most important step of this method is to choose the appropriate finite value of η_∞. Thus to estimate the value of η_∞, we start with some initial guess and solve the boundary value problem consisting of $Eqs.$(3.36) $-$ (3.37) to obtain $f^{''}(0)$ and $\theta'(0)$. The solution process is repeated with another larger value of η_∞ until two successive values of $f^{''}(0)$ and $\theta'(0)$ differ only after desired number of significant digits. The last value of η_∞ is taken as the finite value of the limit η_∞ for the particular set of physical parameters to determine velocity $f^{'}(\eta)$ and $\theta(\eta)$ in the boundary layer. After getting all the five initial conditions we can solve this system nuerically.

3.5 Results and discussion

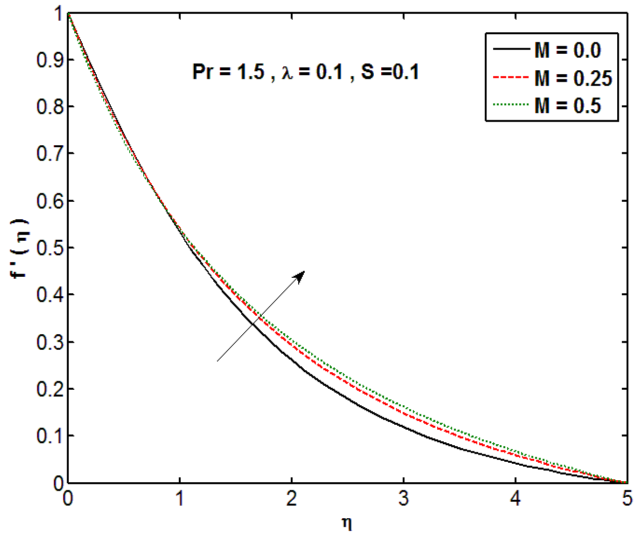

Figure 3.1: Effect of curvature parameter M on velocity profile $f^{'}(\eta)$.

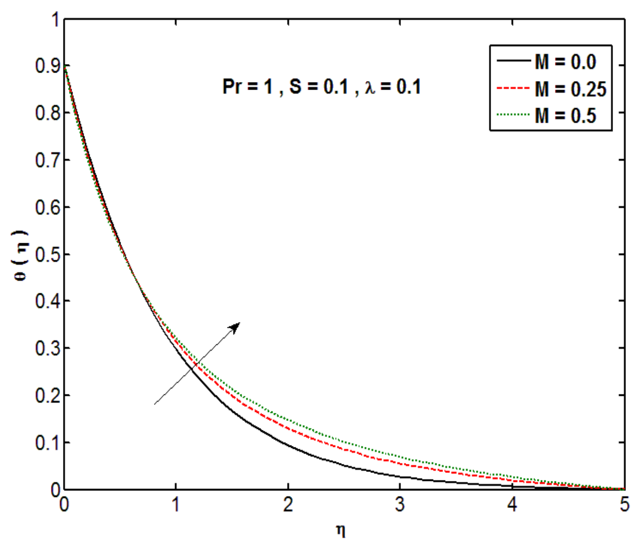

Figure 3.2: Influence of curvature parameter M on temperature profile $\theta(\eta)$.

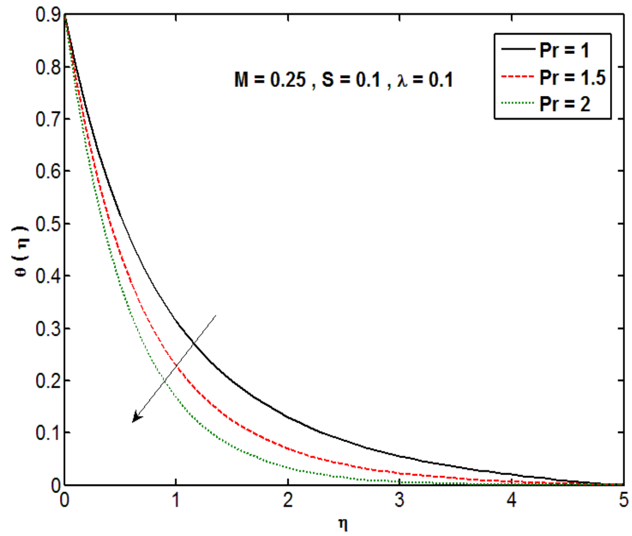

Figure 3.3: Effect of Prandlt number Pr on temperature profile $\theta(\eta)$.

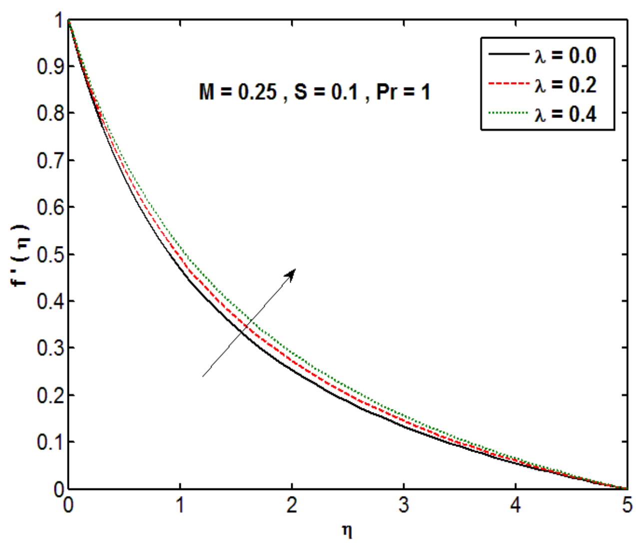

Figure 3.4: Influence of mixed convection parameter λ on velocity profile $f'(\eta)$.

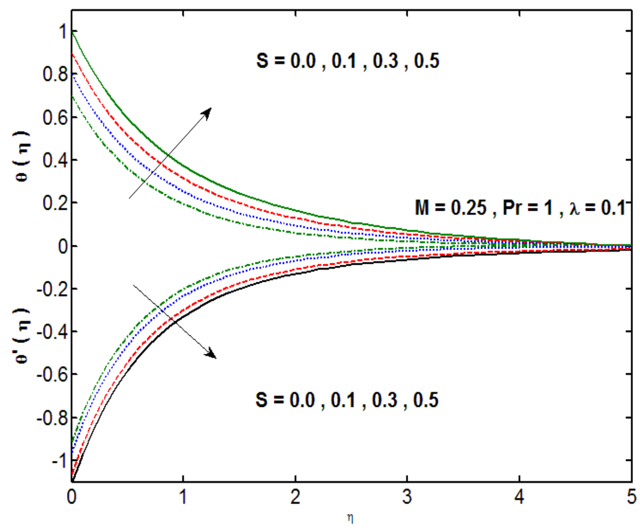

Figure 3.5 : Influence of stratification parameter S on temperature $\theta(\eta)$ and temperature gradient $\theta'(\eta)$.

Table 3.1: Influence of curvature parameter M and mixed convection parameter λ on skin friction coefficient.

M	λ	$f''(0)$
0	0.1	-0.9597
0.25		-1.0608
0.5		-1.1639
0.5	0.1	-1.1639
	0.2	-1.1217
	0.3	-1.0803

Table 3.2: Effect of flow parameters Pr, M and S on the local Nusselt number.

Pr	M	S	$-\theta^{'}(0)$
1	0.5	0.1	1.1537
2			1.6401
3			2.0333
1	0.0		0.9725
	0.25		1.0633
	0.5		1.1537
	0.5	0.0	1.2161
		0.1	1.1537
		0.2	1.0912

Figure 3.1 is plotted to study the behovior of velocity field due to variation in curvature parameter M. As curvature parameter M increases the radius of curvature decreases. This reduce the surface area of the cylinder, so it offers less resistance to fluid motion. Hence it is observed that with the increase of curvature parameter M velocity field increases.

Figure 3.2 shows the attribute of curvature M on temperature field. It is observed that temperature increases with an increase in curvature parameter as surface area decreases so, the transfer of thermal energy increases. Moreover as the curvature parameter increases the viscous forces become weaker. So this enhances the rate of heat transfer which causes increases in the temperature

Figure 3.3 represents the influence of Prandtl's number on temperature field. It is depicted that temperature field decreases with an increase in Pr, because increase in Pr causes decrease in thermal diffusivity.

Figure 3.4 demonstrate the behavior of velocity field on varying mixed convection parameter λ. It is perceived that velocity of the fluid increases with an increase in mixed convection parameter λ. Because λ is ratio of buoyancy to inertial forces, so by increasing mixed convection parameter λ buoyancy forces increases as a result velocity increases.

In **Figure 3.5** the influence of the stratification parameter S on the temperature $\theta(\eta)$ and the temperature gradient $\theta'(\eta)$ are exhibited. As stratification S decreases the temperature in the boundary layer, which results in a decreasing manner of the temperature gradient in absolute sense. The thermal boundary layer thickness also decreases with an increase in the stratification parameter S. With the increase in the stratification parameter, the buoyancy factor $(T_w - T_\infty)$ reduces within the boundary layer. Ambient thermal stratification causes a significant decrease in the local buoyancy level, which reduces the velocities in the boundary layer. All temperature profiles decay from the maximum value at the wall to zero in the free stream, that is, temperature converges at the outer edge of the boundary layer.

Table 3.1: Present the values of Skin-friction coeffieint for different values of physical parameters. It is noted that the Skin-friction coefficient increases with increasing the physical parameters M.and decreases by the increase of λ.

Table 3.2: Display the result of Local Nusselt's number for the different values of parameters M, S and $\Pr$. It can be shown form the table that as we increase the values of curvature parameter M and the Prandtl's number Pr the values of Nusselt's number increases whereaas by increasing the values of S, Local Nusselt's number decreases.

3.6 Concluding remarks

The main findings of present analysis are listed below

- Increase of curvature parameter M causes increase in velocity and temperature profiles.
- By increase of mixed convection parameter λ velocity increases.
- With an increase in Prandtl number Pr temperature decreases.
- Temperature profile decreases as stratification parameter S increases.

Chapter 4

Mixed Convection Boundary layer Flow of Sisko fluid along a Stretching Cylinder in a Thermally Stratified Medium

4.1 Introduction

The ambition in this chapter is to figure out the flow and heat problem of two dimensional steady axisymmetric laminar boundary layer mixed convection flow of Sisko fluid model along a stretching cylinder in a thermally stratified medium. The similarity transformations are used to reduced coupled partial differential equations into ordinary differential equations. To solve these equations a numerical approach called Shooting method has been used for the computation of velocity profile and temperature field for different values of physical parameters such as curvature parameter, mixed convection parameters,stratification parameter and Prandlt number. The dependence of Skin-friction coefficient and Nusselt's number has been analyzed in detail through tables.

4.1.1 Mathematically formulation

Consider the two-dimensional steady axisymmetric flow of an incompressible mixed convection boundary layer flow of Sisco fluid over a stretching cylinder in a thermally stratified medium. The continuity equation, linear momentum equation and energy equation are

$$\boldsymbol{\nabla}.\mathbf{V} = 0, \tag{4.1}$$

$$\rho\frac{D\mathbf{V}}{Dt} = \boldsymbol{\nabla}.\mathbf{T}+\rho b, \tag{4.2}$$

$$\frac{DT}{Dt} = K\ \boldsymbol{\nabla}^2 T. \tag{4.3}$$

where $\mathbf{V}$ denotes the flow velocity, ρ is the density, K thermal diffusivity of the fluid is defined as $K = \dfrac{k}{\rho c_p}$. T is the fluid temperature, $\mathbf{T}$ Cauchy stress tensor , $\boldsymbol{\nabla}$ is differential operator , $\dfrac{\mathbf{D}}{\mathbf{Dt}}$ is the material time derivative. Some of these are defined below

$$\mathbf{V} = [v(x,r), 0, u(x,r)]\,, \tag{4.4}$$

$$\boldsymbol{\nabla} = \frac{1}{r}\frac{\partial\ (r\ (.))}{\partial r} + \frac{1}{r}\frac{\partial\ (.)}{\partial \theta} + \frac{\partial\ (.)}{\partial x}, \tag{4.5}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V}.\boldsymbol{\nabla}, \tag{4.6}$$

Since the flow field is considered steady, so the term $\frac{\partial}{\partial t}$ is neglected throughout the problem. So Eq (4.6) reduces to

$$\frac{D}{Dt} = \mathbf{V}.\boldsymbol{\nabla}. \tag{4.7}$$

$$\rho b = \mathbf{g}\beta\left(T - T_\infty\right), \tag{4.8}$$

where Eq (4.8) comes through Bossiness approximation

$$\mathbf{T} = -p\mathbf{I}+\mu\mathbf{S}_1, \tag{4.9}$$

where $\mathbf{S}_1$ is the extra stress tensor and defined as

$$\mathbf{S}_1 = \left(a + b \left| \sqrt{\frac{1}{2} tr \left(\mathbf{A}_1^2 \right)} \right|^{n-1} \right) \mathbf{A}_1. \tag{4.10}$$

In $Eqs.$ (4.4) – (4.10) u and v are the axial and radial components of the velocity of fluid, a viscosity at heigh shear rate, b is consistancy index and n are the material parameter, μ is dynamic viscosity, $\mathbf{I}$ is the identity tensor, p is the pressure, g is gravity, β volumetric thermal expansion and $\mathbf{A}_1$ is the first Rivilin-Ericksen tensor defined below

$$\mathbf{A}_1 = \boldsymbol{\nabla}\mathbf{V} + \left(\boldsymbol{\nabla}\mathbf{V} \right)^T, \tag{4.11}$$

gradient of velocity i.e ($\boldsymbol{\nabla}\mathbf{V}$) is defined as

$$\boldsymbol{\nabla}\mathbf{V} = \begin{bmatrix} \frac{\partial v}{\partial r} & \frac{1}{r}\left(\frac{\partial v}{\partial \theta} - w\right) & \frac{\partial v}{\partial x} \\ \frac{\partial w}{\partial r} & \frac{1}{r}\left(\frac{\partial w}{\partial \theta} + v\right) & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial r} & \frac{1}{r}\frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial x} \end{bmatrix}, \tag{4.12}$$

So Eq (4.11) takes the form

$$\mathbf{A}_1 = \begin{bmatrix} 2\,\frac{\partial v}{\partial r} & \frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{w}{r} + \frac{\partial w}{\partial r} & \frac{\partial v}{\partial x} + \frac{\partial u}{\partial r} \\ \frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{w}{r} & \frac{2}{r}\left(\frac{\partial w}{\partial \theta} + v\right) & \frac{\partial w}{\partial x} + \frac{1}{r}\frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} + \frac{\partial u}{\partial x} & \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial x} & 2\,\frac{\partial u}{\partial x} \end{bmatrix}, \tag{4.13}$$

Here u, v and w are component of velocity. As we consider two dimensional flow i.e. $\mathbf{V} = [v(x,r), 0, u(x,r)]$ so

$$\mathbf{A}_1 = \begin{bmatrix} 2\frac{\partial v}{\partial r} & 0 & \frac{\partial v}{\partial x} + \frac{\partial u}{\partial r} \\ 0 & \frac{2}{r}(v) & 0 \\ \frac{\partial v}{\partial r} + \frac{\partial u}{\partial x} & 0 & 2\frac{\partial u}{\partial x} \end{bmatrix}, \tag{4.14}$$

and

$$\mathbf{A}_1^2 = \begin{bmatrix} 4\left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial x}\right)^2 & 0 & 2\frac{\partial v}{\partial r}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial r}\right) + 2\frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial r}\right) \\ 0 & 4\left(\frac{v}{r}\right)^2 & 0 \\ 2\frac{\partial v}{\partial r}\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial x}\right) + 2\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial x}\right) & 0 & \left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial x}\right)^2 + 4\left(\frac{\partial u}{\partial x}\right)^2 \end{bmatrix}, \quad (4.15)$$

The trace of $\mathbf{A}_1^2$ is also determined, which is represented in the equation below

$$tr(\mathbf{A}_1^2) = 4(\frac{\partial u}{\partial x})^2 + 4(\frac{\partial v}{\partial r})^2 + 4\frac{v^2}{r^2} + 2(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial r})^2, \quad (4.16)$$

Using Eq (4.8) and Eq (4.9) in Eq (4.2) we have

$$\rho\frac{D\mathbf{V}}{Dt} = -\boldsymbol{\nabla} p + \mu\boldsymbol{\nabla}\cdot\mathbf{S}_1 + g\beta\left(T - T_\infty\right), \quad (4.17)$$

We have assumed that the flow is caused due to stretching of the cylinder therefore the pressure gradient is neglected.

$$\rho\frac{D\mathbf{V}}{Dt} = \mu\boldsymbol{\nabla}\cdot\mathbf{S}_1 + \mathbf{g}\beta\left(T - T_\infty\right). \quad (4.18)$$

Material time derivative $\frac{\mathbf{D}}{\mathbf{Dt}}$ for the velocity field $\mathbf{V}$ in component form is given as

$$\left(\frac{D\mathbf{V}}{Dt}\right)_x = u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial r}, \quad (4.19)$$

$$\left(\frac{D\mathbf{V}}{Dt}\right)_r = u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial r}, \quad (4.20)$$

$$\left(\frac{D\mathbf{V}}{Dt}\right)_\theta = 0. \quad (4.21)$$

The components of divergence of extra stress tensor $\mathbf{S}_1$ defined as

$$(\mathrm{div}\,\mathbf{S})_x = a\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial x^2}\right) + b\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial x^2}\right) \times \left|2\left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial x}\right)^2 + 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{v}{r}\right)^2\right|^{\frac{n-1}{2}}, \quad (4.22)$$

$$(\mathrm{div}\,S)_\theta = 0, \quad (4.23)$$

$$
\begin{aligned}
(\operatorname{div} S)_r \;=\;& a\left(\frac{2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)+\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial r}-2\frac{v}{r^2}\right)+\\
& b\left(\frac{2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)+\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial r}-2\frac{v}{r^2}\right)\times \qquad (4.24)\\
& \left|2\left(\frac{\partial v}{\partial r}\right)^2+\left(\frac{\partial u}{\partial r}+\frac{\partial v}{\partial x}\right)^2+2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{v}{r}\right)^2\right|^{\frac{n-1}{2}}.
\end{aligned}
$$

using Eqs (4.19) $-$ (4.24) in Eq (4.18) we have

$$
\begin{aligned}
u\,\frac{\partial u}{\partial x}+v\,\frac{\partial u}{\partial r} \;=\;& \frac{a}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{\partial^2 u}{\partial x^2}\right)+\frac{b}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{\partial^2 u}{\partial x^2}\right)\times\\
& \left|2\left(\frac{\partial v}{\partial r}\right)^2+\left(\frac{\partial u}{\partial r}+\frac{\partial v}{\partial x}\right)^2+2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{v}{r}\right)^2\right|^{\frac{n-1}{2}}, \qquad (4.25)
\end{aligned}
$$

$$
\begin{aligned}
u\,\frac{\partial v}{\partial x}+v\,\frac{\partial v}{\partial r} \;=\;& \frac{a}{\rho}\left(\frac{2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)+\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial r}-2\frac{v}{r^2}\right)+\\
& \frac{b}{\rho}\left(\frac{2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial r}-2\frac{v}{r^2}\right)\times \qquad (4.26)\\
& \left|2\left(\frac{\partial v}{\partial r}\right)^2+\left(\frac{\partial u}{\partial r}+\frac{\partial v}{\partial x}\right)^2+2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{v}{r}\right)^2\right|^{\frac{n-1}{2}}.
\end{aligned}
$$

L.H.S of Eq (4.3) is simply material time derivative and R.H.S of Eq (4.3) is a $k\,\operatorname{div}(grad\ T)$ respectively given below

$$
\frac{\mathbf{D}T}{\mathbf{Dt}} = u\,\frac{\partial T}{\partial x}+v\,\frac{\partial T}{\partial r}, \tag{4.27}
$$

$$
K\,\boldsymbol{\nabla}^2 T = \frac{K}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial T}{\partial r}\right)+K\,\frac{\partial^2 T}{\partial x^2}, \tag{4.28}
$$

incorporating these above two equations in Eq (4.3) we have

$$
u\,\frac{\partial T}{\partial x}+v\,\frac{\partial T}{\partial r} = \frac{K}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial T}{\partial r}\right)+K\,\frac{\partial^2 T}{\partial x^2}, \tag{4.29}
$$

Using the boundary layer theory we assume that

$$\begin{aligned} u &= O(1)\ ,\ x=O(1)\ ,\ v=O(\delta)\ ,\ r=O(\delta)\ ,\ T=O(1), \\ \nu &= O(\delta^2)\ ,\ K=O(\delta^2)\ ,\ \frac{a}{\rho}=O\left(\delta^2\right)\ ,\ \frac{b}{\rho}=O\left(\delta^{n+1}\right). \end{aligned} \tag{4.30}$$

Where δ is a small positive number. By using above boundary layer approximation the continuity, momentum and energy equations takes the form

$$\frac{\partial}{\partial x}(rv)+\frac{\partial}{\partial x}(ru)=0, \tag{4.31}$$

$$u\,\frac{\partial u}{\partial x}+v\,\frac{\partial u}{\partial r}=\frac{a}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right)+\frac{b}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right)\left|\left(\frac{\partial u}{\partial r}\right)\right|^{n-1}+g\beta\left(T-T_\infty\right), \tag{4.32}$$

And after simplification, above equation takes the form

$$u\,\frac{\partial u}{\partial x}+v\,\frac{\partial u}{\partial r}=\frac{a}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right)-\frac{b}{r\rho}\left(-\frac{\partial u}{\partial r}\right)^{n}+\frac{nb}{\rho}\left(-\frac{\partial u}{\partial r}\right)^{n-1}\frac{\partial^2 u}{\partial r^2}+g\beta\left(T-T_\infty\right), \tag{4.33}$$

$$u\,\frac{\partial T}{\partial x}+v\,\frac{\partial T}{\partial r}=\frac{K}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right). \tag{4.34}$$

subject to the boundary conditions

$$u\left(x,r\right)=U(x),\ v(x,r)=0,\ T\left(x,r\right)=T_w(x),\ at\ r=r_0\ \ where\ T_w\left(x\right)=T_0+\frac{c\ x}{L} \tag{4.35}$$

$$u\left(x,r\right)\rightarrow 0,\ T\left(x,r\right)\rightarrow T_\infty(x),\ as\ r\rightarrow\infty\ \ where\ T_\infty\left(x\right)=T_0+\frac{d\ x}{L}$$

In the above boundary conditions r_0 is the radius of the cylinder, $U(x)=U_0\frac{x}{L}$ is the stretching velocity, $T_w=T_0+\frac{c\ x}{L}$ is the prescribed surface temperature, $T_\infty=T_0+\frac{d\ x}{L}$ is the variable ambient temperature. U_0 is the reference velocity, T_0 the reference temperature, L the characteristic length, c and d are positive constants. To get the similarity solution of Eq (4.33) and Eq (4.34) subject to the boundary conditions Eq (4.35), following similarity transformations are use

$$\eta=\frac{r^2-r_0^2}{2r_0x}\,\mathrm{Re}_b^{\frac{1}{n+1}}\ ,\ \psi=xr_0U\,\mathrm{Re}_b^{\frac{-1}{n+1}}f(\eta)\ ,\ \theta\left(\eta\right)=\frac{T-T_\infty}{T_w-T_0}\ ,\ \mathrm{Re}_b=\frac{\rho x^n U^{\ 2-n}}{b}. \tag{4.36}$$

where ψ is the stream function, which identically satisfies the continuity equation Eq (4.31) and define as

$$u = \frac{1}{r}\left(\frac{\partial\psi}{\partial r}\right) \quad , \quad v = \frac{-1}{r}\left(\frac{\partial\psi}{\partial x}\right). \tag{4.37}$$

Using the similarity transformations of Eq (4.36) in Eq (4.33) and Eq (4.34), we will get the following highly nonlinear ordinary differential equations.

$$A(1+2M\eta)f^{'''} + n(1+2M\eta)^{\frac{n+1}{2}}\left(-f^{''}\right)^{n-1} f^{'''} + 2MAf^{''} -$$

$$(1+n)\, M\, (1+2M\eta)^{\frac{n-1}{2}}\left(-f^{''}\right)^{n} + \frac{2n}{n+1} f f^{''} - f^{'2} + \lambda\theta = 0. \tag{4.38}$$

$$(1+2M\eta)\,\theta^{''} + 2M\theta^{'} + \Pr\left(\frac{2n}{1+n} f\ \theta^{'} - f^{'}\theta - f^{'}S\right) = 0. \tag{4.39}$$

subject to the boundary conditions

$$f^{'} = 1 \quad , \quad f = 0 \quad , \quad \theta = 1 - S \quad , \quad at \quad \eta \to 0, \tag{4.40}$$

$$f^{'} \to 0 \quad , \quad \theta \to 0 \quad , \quad as \quad \eta \to \infty.$$

where prime denotes the derivative with respect to η, curvature parameter M, mixed convection parameter λ ,material parameter $\mathbf{A}$, Prandtl number Pr ,and stratification parameter S are define below

$$M = \frac{x}{r_o R_b^{\frac{1}{1+n}}} \quad , \quad \mathrm{Re}_a = \frac{\rho U x}{a} \quad , \quad \Pr = \frac{xU}{k}\mathrm{Re}_b^{\frac{-2}{1+n}}, \tag{4.41}$$

$$A = \frac{\mathrm{Re}_b^{\frac{2}{1+n}}}{\mathrm{Re}_a} \quad , \quad \lambda = \frac{g\beta x\,(T_p - T_\infty)}{U^2} \quad , \quad S = \frac{c}{d}.$$

4.2 Skin-friction Coefficient

Coefficient of skin friction for this problem is calculated as.

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^{\ 2}}, \tag{4.42}$$

where τ_w represents shear stress at the surface of cylinder and calculated as

$$\tau_w = a\left(\frac{\partial u}{\partial r}\right)_{r\,=\,r_0} - b\left(-\frac{\partial u}{\partial r}\right)_{r\,=\,r_0}, \tag{4.43}$$

After using similarity transformation and then simplification, Eq (4.43) takes the form

$$\tau_w = \frac{aU}{x}\,\mathrm{Re}_b^{\frac{1}{1+n}}\,f''(0) - b\left[-\frac{U}{x}\,\mathrm{Re}_b^{\frac{1}{1+n}}\,f''(0)\right]^n, \tag{4.44}$$

Substituting Eq (4.44) in Eq (4.44) and after simplification we have

$$\frac{1}{2}C_f\,\mathrm{Re}_b^{\frac{1}{1+n}} = \mathbf{A}f''(0) - \left[-f''(0)\right]^n, \tag{4.45}$$

4.3 Local Nusselt's number

The local Nusselt's number of temperature distribution is defined as

$$Nu_x = \frac{xq_w}{k\left(T_w - T_0\right)}, \tag{4.46}$$

where q_w is the rate of heat transfer at the surface and it is defined as

$$q_w = -k\left(\frac{\partial T}{\partial r}\right)_{r\,=\,r_0}, \tag{4.47}$$

After usnig similarity transformation we have

$$q_w = -k\left(\frac{T_w - T_0}{x}\right)\mathrm{Re}_b^{\frac{1}{1+n}}\,\theta'(0), \tag{4.48}$$

Using Eq (4.48) in Eq (4.46) we have

$$Nu_x\,\mathrm{Re}_b^{\frac{-1}{1+n}} = -\theta'(0). \tag{4.49}$$

4.4 Numerical solution.

As we know that the non-linear momentum $Eq.(4.38)$ is of order three in f and non-linear energy $Eq.(4.39)$ is of second order in θ, so both equations is of order five, which can be diminished to a system of five first order ordinary differential equations giving on solving five unknowns. In order to solve these equations we are using a numerical technique "Runge–Kutta-Fehlberg" method. We need five initial conditions, but as we know that only two initial conditions in f and one initial condition in θ are known i.e. one initial condition of f and one of θ is missing. However, the values of $f\,'$and θ are known at $\eta \to \infty$. Thus, these two end conditions was exploiting to produce two unknowns. The most important step of this method is to choose the appropriate finite value of η_∞. Thus to estimate the value of η_∞, we start with some initial guess and solve the boundary value problem consisting of $Eqs.(4.38) - (4.39)$ to obtain $f''(0)$ and $\theta'(0)$. The solution process is repeated with another larger value of η_∞ until two successive values of $f''(0)$ and $\theta'(0)$ differ only after desired number of significant digits. The last value of η_∞ is taken as the finite value of the limit η_∞ for the particular set of physical parameters to determine velocity $f(\eta)$ and temperature $\theta(\eta)$ in the boundary layer. After getting all the five initial conditions we solve this system of simultaneous equations using Runge–Kutta-Fehlberg integration scheme.

4.5 Results and discussions

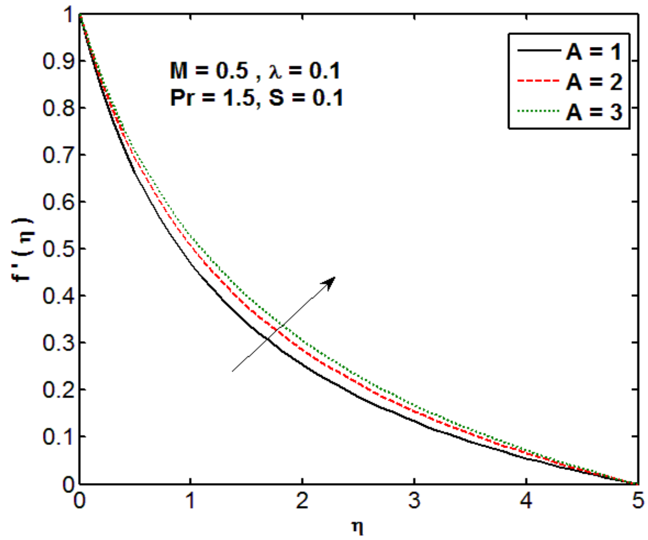

Figure 4.1: Effect of material parameter A on velocity profile $f^{'}(\eta)$ for $n = 0.2$.

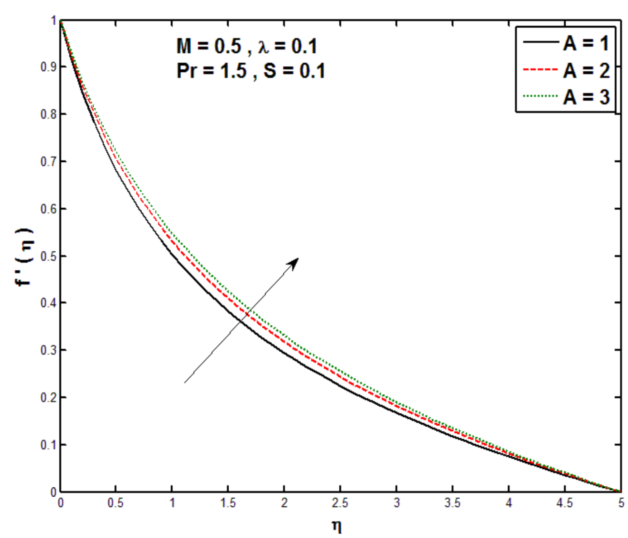

Figure 4.2: Influence of material parameter A on velocity profile $f^{'}(\eta)$for $n = 1$.

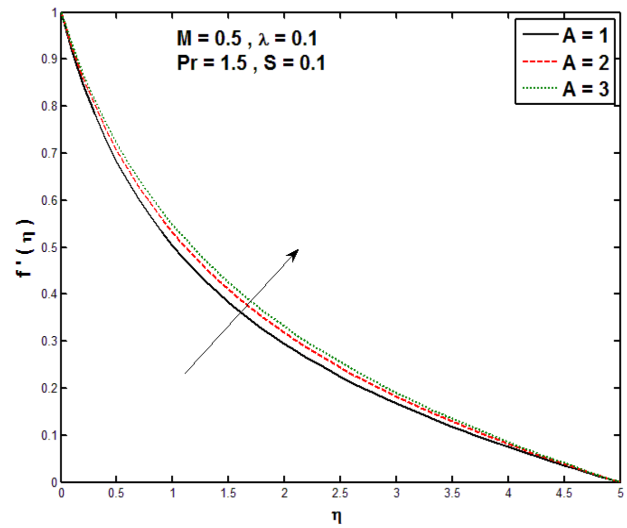

Figure 4.3: Influence of material parameter A on velocity profile $f^{'}(\eta)$ for $n = 2$.

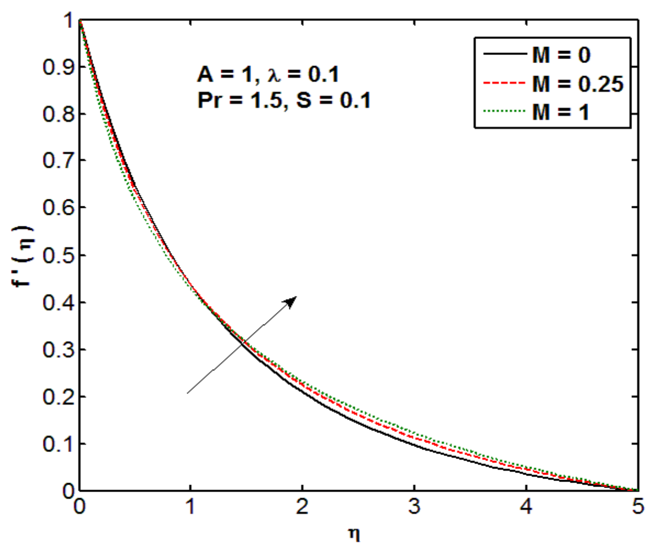

Figure 4.4: Effect of curvature parameter M on velocity profile $f^{'}(\eta)$ for $n = 0.2$.

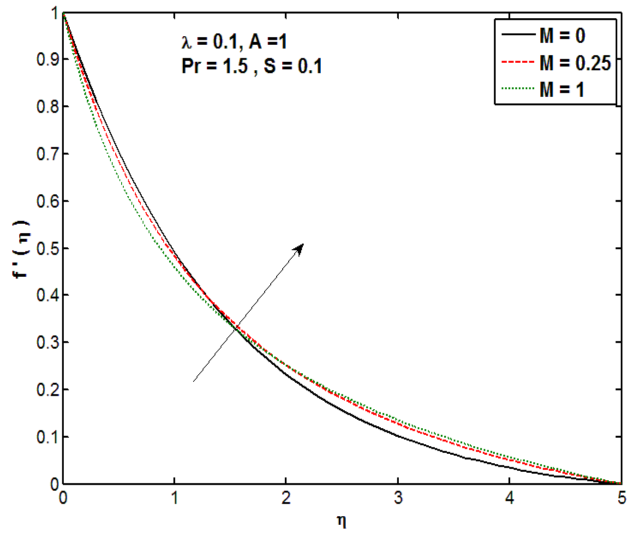

Figure 4.5: Impact of curvature parameter M on velocity profile $f^{'}(\eta)$ for $n = 1$.

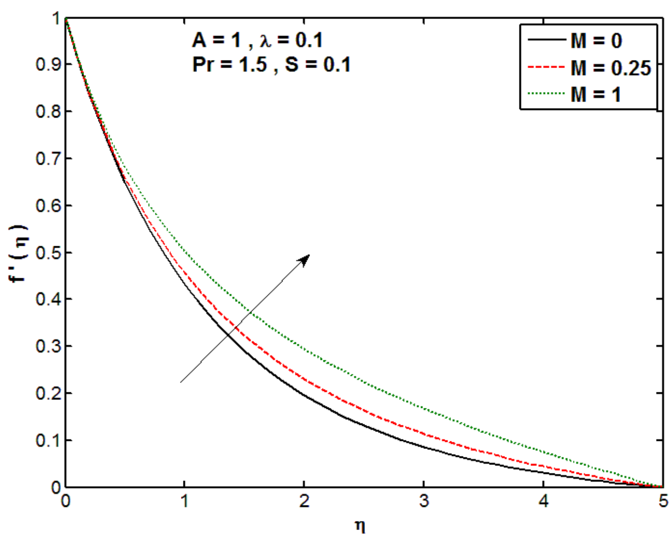

Figure 4.6: Impact of curvature parameter M on velocity profile $f^{'}(\eta)$ for $n = 2$.

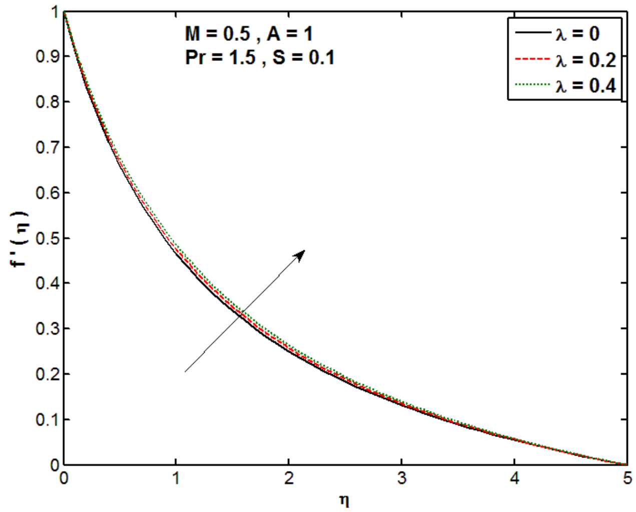

Figure 4.7: Effect of mixed convection parameter λ on velocity profile $f^{'}(\eta)$ for $n = 0.2$.

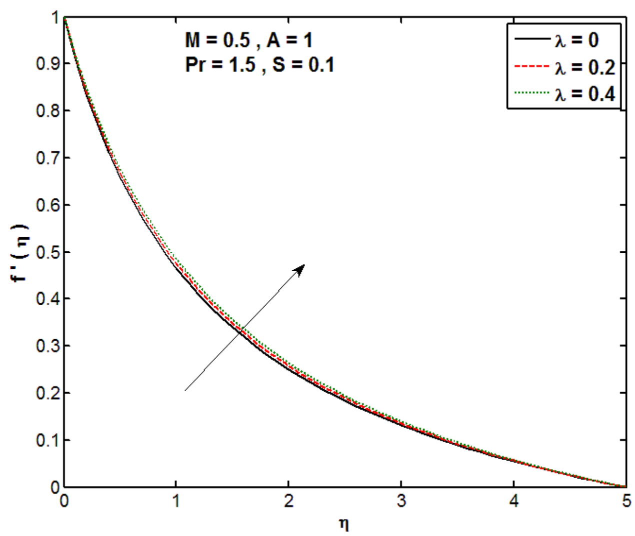

Figure 4.8: Effect of mixed convection parameter λ on velocity profile $f^{'}(\eta)$ for $n = 1$.

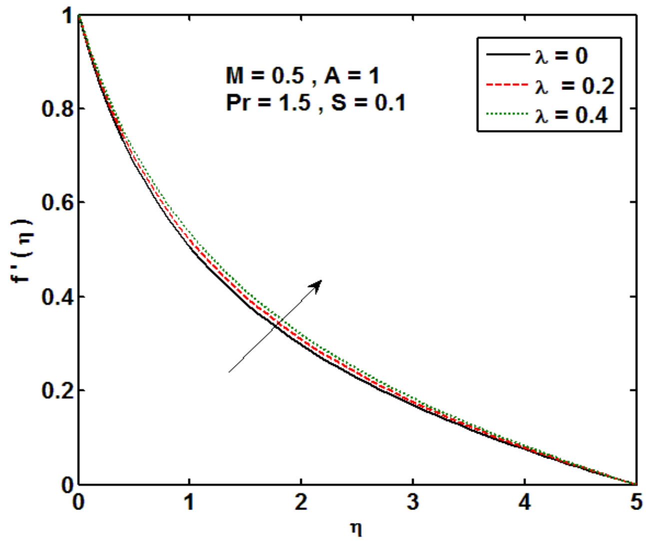

Figure 4.9: Effect of mixed convection parameter λ on velocity profile $f'(\eta)$ for $n = 2$.

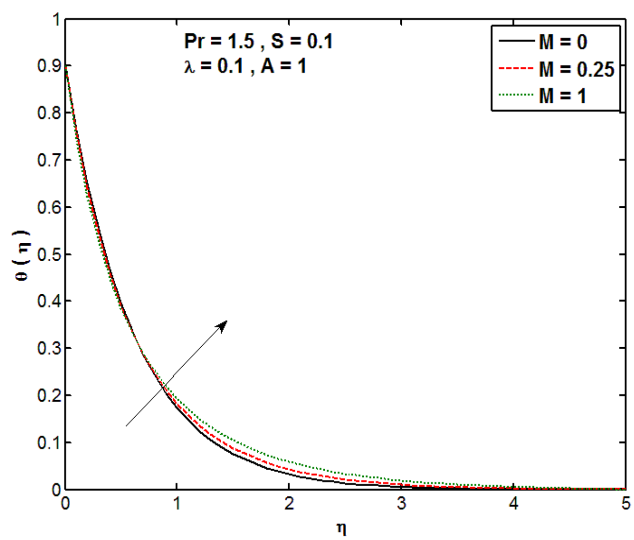

Figure 4.10: Effect of curvature parameter M on temperature profile $\theta(\eta)$ for $n = 0.2$.

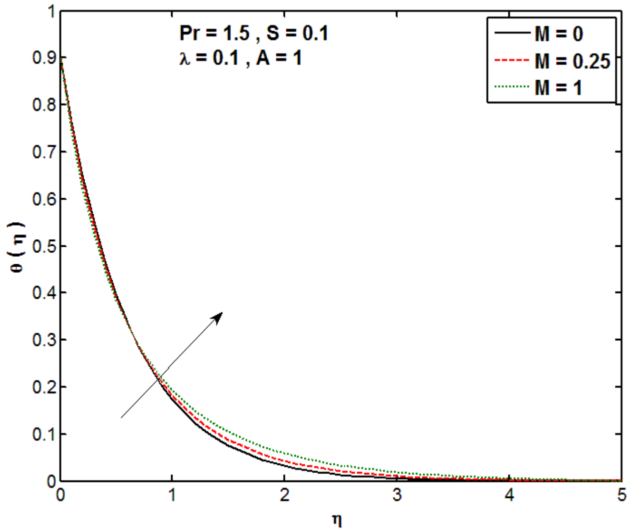

Figure 4.11: Impact of curvature parameter M on temperature profile $\theta(\eta)$ for $n = 1$.

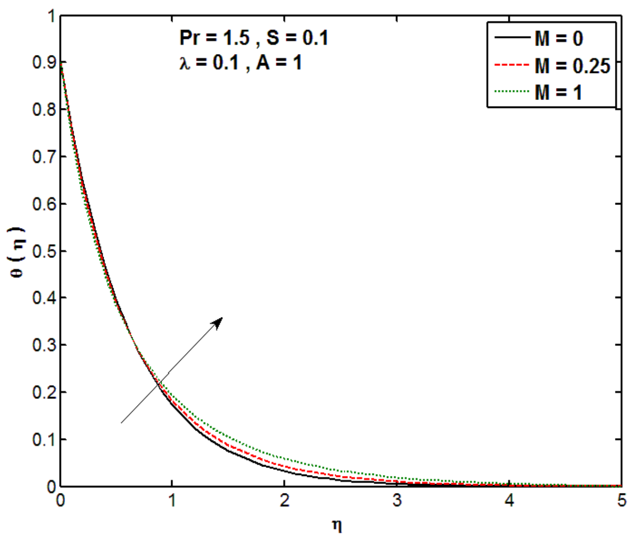

Figure 4.12: Impact of curvature parameter M on temperature profile $\theta(\eta)$ for $n = 2$.

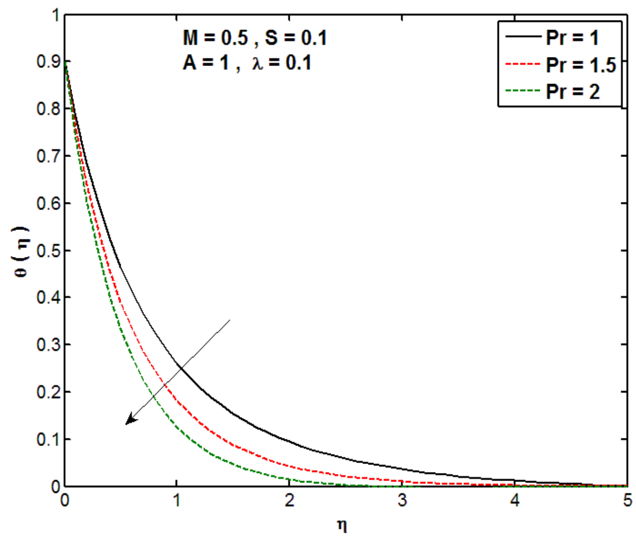

Figure 4.13: Effect of Prandlt number Pr on temperature profile $\theta(\eta)$ for $n = 0.2$.

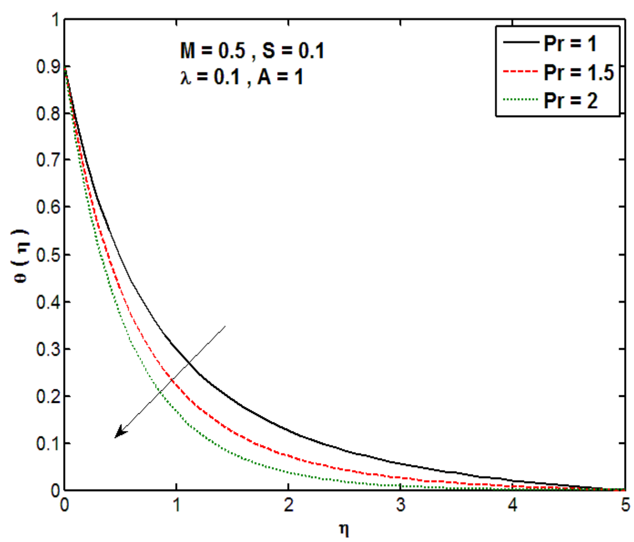

Figure 4.14: Effect of Prandlt number Pr on temperature profile $\theta(\eta)$ for $n = 1$.

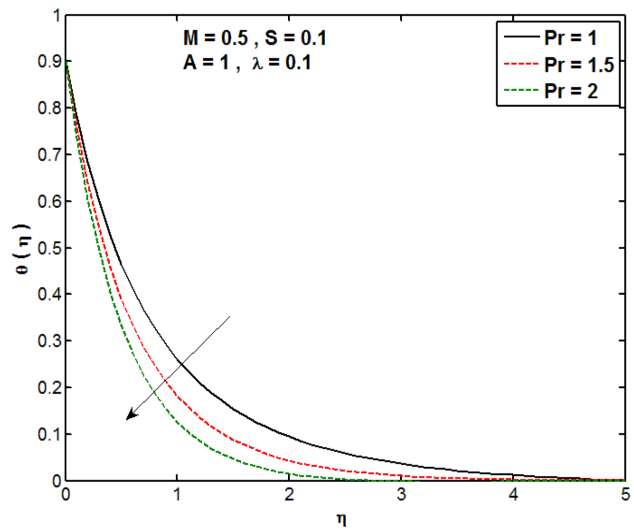

Figure 4.15: Effect of Prandtl number Pr on temperature profile $\theta(\eta)$ for $n = 2$.

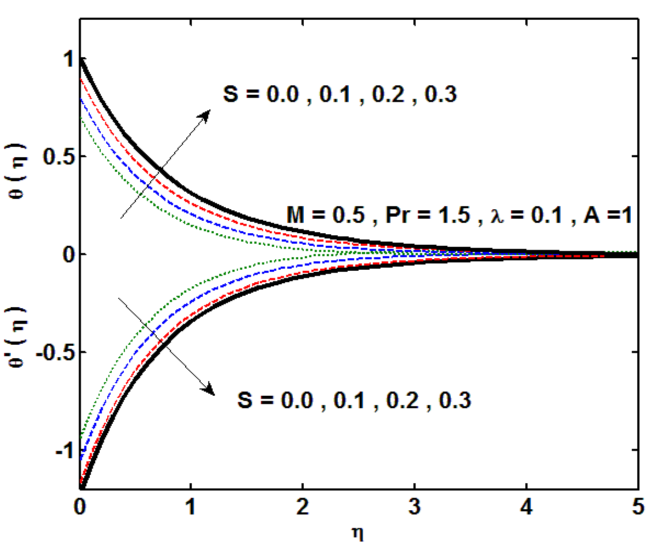

Figure 4.16: Effect of stratification parameter S on $\theta(\eta)$ and $\theta^{'}(\eta)$ for $n = 0.2$.

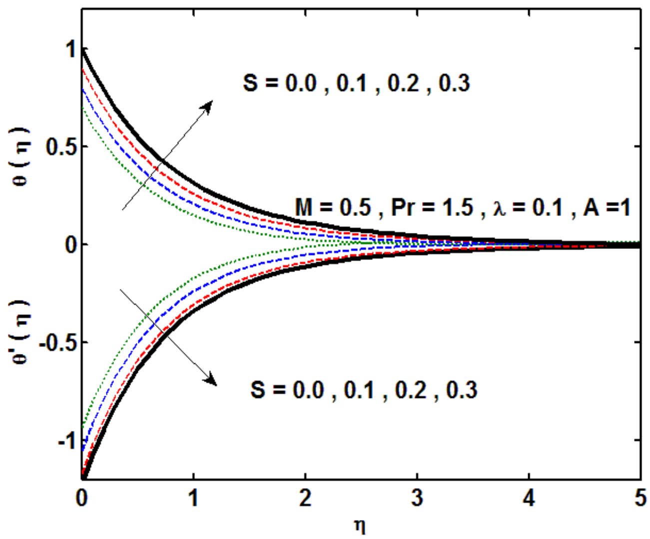

Figure 4.17: Effect of stratification parameter S on $\theta\,(\eta)$ and $\theta^{'}\,(\eta)$ for $n = 1$.

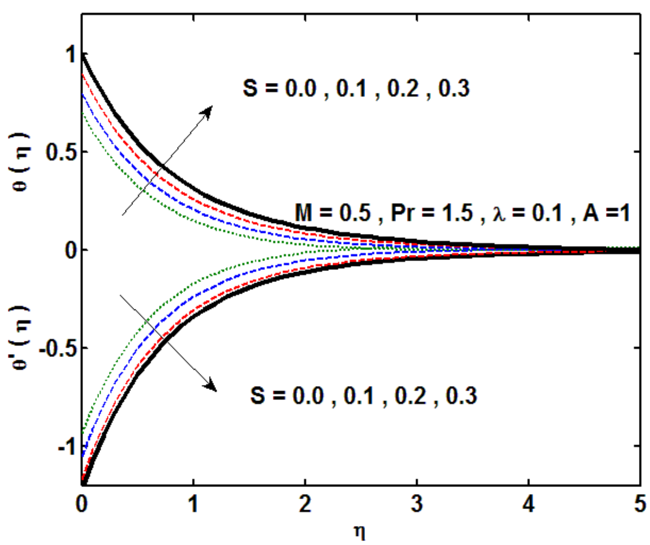

Figure 4.18: Effect of stratification parameter S on $\theta\,(\eta)$ and $\theta^{'}\,(\eta)$ for $n = 2$.

Table 4.1:The variation of skin-friction coefficient with respect to M, A and λ for $n = 0.2, 1$ and 2.

			$n = 0.2$	$n = 1$	$n = 2$
M	A	λ	$Af''(0) - [-f''(0)]^{0.2}$	$(1 + A)f''(0)$	$Af''(0) - (f''(0))^2$
0	1	0.1	-1.7743	-1.6793	-1.3896
0.5			-1.9032	-1.8424	-1.7097
1			-2.2842	-2.1342	-1.9451
0.5	1		-1.9032	-1.8424	-1.7097
	2		-2.6395	-2.4577	-2.1595
	3		-3.3041	-3.0506	-2.7143
	1	0.1	-1.9032	-1.8424	-1.7097
		0.2	-1.8569	-1.8036	-1.6696
		0.3	-1.7912	-1.7651	-1.6213

Table 4.2:The variation of $-\theta'(0)$ with respect to M, and, Pr for $n = 0.2, 1$ and 2.

			$n = 0.2$	$n = 1$	$n = 2$
Pr	M	S	-$\theta'(\eta)$	-$\theta'(\eta)$	-$\theta'(0)$
1	0.5	0.1	1.1786	1.2136	1.2832
2			1.6317	1.7179	1.8203
3			1.9754	2.1168	2.2406
1.5	0		1.3921	1.4113	1.4349
	0.5		1.4561	1.4832	1.5047
	1		1.6022	1.6314	1.6542
	0.5	0	1.5710	1.5444	1.5234
		0.1	1.5146	1.4832	1.4799
		0.2	1.4563	1.4319	1.4147

Figures. 4.1 , 4.2 and 4.3 describe the behavior of velocity profile for different values of A for $n = 0.2, 1$ and 2 respectively. It is observed that velocity increases as A increases. The effect of increasing values of the material parameter A was to enhance the velocity field and hence the boundary layer thickness.

Figures. 4.4 , 4.5and 4.6. velocity profiles are shown for different values of M. The velocity curves show that the rate of transport decreases with increasing distance (η) from the surface and vanishes asymptotically. It is examined that with increase of curvature parameter M velocity field increases. Reason behind is that by increasing curvature parameter M, radius of curvature decreases which implies that area of the cylinder with fluid decreases. Thus area of connectivity of fluid and cylinder decreases as a result less resistance is offered by surface of the cylinder. Therefore velocity of the fluid increases. Further more boundary layer is thicker for larger values of curvature parameter M for $n = 0.2, 1$ and 2. The velocity gradient increases for larger values of M, which produces larger skin friction coefficient.

Figures.4.7 , 4.8 and 4.9 demonstrate the behavior of mixed convection parameter λ for $n = 0.2, 1$ and 2 on velocity profile. It is perceived that velocity of the fluid increases with an increase in mixed convection parameter λ. Since λ is the ratio of buoyancy to inertial forces, by increasing mixed convection parameter λ buoyancy forces increases as a result velocity increase.

Figures. 4.10 , 4.11and 4.12 are plotted to see the influence of Prandlt's number for $n = 0.2, 1$ and 2 on temperature field. It is observed that temperature field decreases after an increase in the Prandtl number which reduces the thermal boundary layer thickness. The Prandtl number signifies the ratio of momentum diffusivity to thermal diffusivity. Fluids with small Prandtl number possess higher thermal conductivities and thicker thermal boundary layer structures. So, heat diffuses from the wall faster for large Pr with thinner boundary layers. Hence, the Prandtl number can control the rate of cooling in conducting flows.

Figures. 4.13 , 4.14 and 4.15 show the attribute of curvature parameter M on temperature field for $n = 0.2, 1$ and 2. It is observed that temperature increases with an increase in curvature parameter as surface area decreases. Due to this the transfer of energy increases.

Figures.4.16 , 4.17 and 4.18 gives the behavior of the stratification parameter S on the temperature field and the temperature gradient for $n = 0.2, 1$ and 2. The temperature in the boundary layer decreases, that results in a decreasing manner of the temperature gradient in absolute sense. The thermal boundary layer thickness also decreases with an increase in the stratification parameter S. With the increase in the stratification parameter, the buoyancy factor $T_w - T_\infty$ reduces within the boundary layer. Ambient thermal stratification causes a significant decrease in the local buoyancy level, which reduces the velocities in the boundary layer. All temperature profiles decay from the maximum value at the wall to zero in the free stream, that is, converge at the outer edge of the boundary layer.

Tables 4.1 present the values of Skin-friction coeffieint for different values of physical parameters. It is noted that the Skin-friction coefficient increases with increasing the physical parameters A, M.and decreasing by the increase of λ.

Tables 4.2 display the result of Local Nusselt's number for the different values of parameters M and Pr . It can be shown from the tables that as we increase the value of curvature parameter M and the Prandlt's number Pr the values of Nusselt's number increases.

4.6 Concluding remarks

The main findings of present analysis are listed below

- The behavior of curvature parameter M on velocity and temperature profile is same i.e. both velocity and temperature fields increases by increasing curvature parameter M.
- Velocity increases with an increase in material parameter A.
- By increase of mixed convection parameter λ velocity increases.
- With an increase in Prandtl number Pr temperature decreases.
- The Stratification parameter S decreases as temperature profile decreases.

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