Numerical Solution of Williamson Fluid Flow Past a Stretching Cylinder and Heat Transfer

By

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IN

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CERTIFICATE

A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTER OF PHILOSOPHY

We accept this dissertation as conforming to the required standard

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2015

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Madiha Bibi

Contents

Chapter 1

Basic Definitions

1.1 Preliminaries

This chapter contains some basic definitions and the governing equations for the fluid problems.

1.2 Basic definitions

1.2.1 Fluid

A physical entity that changes its shape continuously at the expense of shear stress applied on it.

1.2.2 Flow

It is the process in which substance deformation enhances continuously in the result of effects of several forces.

1.2.3 Fluid mechanics

Branch of mechanics which is further classified into two sub branches i.e, fluid statics and fluid dynamics, study of stationary and moving fluids respectively.

1.3 Some useful physical properties of fluids

1.3.1 Density

Physical property of fluid can be defined as ratio of mass(m) per unit volume(V). It is denoted by the symbol ρ and mathematical expression is given as

$$\rho = \lim_{\partial V \to 0} \frac{\partial m}{\partial V} \tag{1.1}$$

Dimensional form of density is $[ML^{-3}]$ its unit is kg/m^3 .

1.3.2 Viscosity

The viscosity of the fluid is that physical property which offers resistance to flow. It is denoted by the symbol μ. Mathematical expression is given by

$$\mu = \frac{shear\ stress}{rate\ of\ shear\ strain} \tag{1.2}$$

μ is also known as absolute or dynamic viscosity. Dimension of viscosity is $[ML^{-1}T^{-1}]$.

1.3.3 Kinematic viscosity

It can be defined as fraction of dynamic viscosity verses density of fluid. It is denoted by the symbol ν. Mathematically

$$\nu = \frac{\mu}{\rho} \tag{1.3}$$

In this expression ρ is for density and dimensional form for the kinematic viscosity is $[L^2T^{-1}]$.

1.3.4 Thermal conductivity

It is the heat conducting ability measure of fluid. SI unit for the thermal conductivity is W/mK and its dimension is $[MLT^{-3}\theta^{-1}]$.

1.4 Classification of fluids

1.4.1 Ideal/Inviscid fluids

The fluids with zero viscosity are said to be ideal or inviscid fluids. In reality the existence of fluid with zero viscosity is almost impossible. But is some of engineering problems the fluids with very low viscosity are considered. Generally gases are considered as low viscosity fluids.

1.4.2 Real fluids

Contrast of ideal fluids are called real fluids or the fluids with non-zero viscosity are called real fluids. Classification of real fluids is different on the basis of different properties. If the density is constant the fluid is **incompressible** and for variable viscosity the fluid is **compressible**. Main classification of fluids based on the ratio of shear stress and rate of shear strain and the division is **Newtonian** and **non-Newtonian**.

1.4.3 Newtonian fluids

The real fluids for which shear stress is directly and linearly proportional to deformation rate are called Newtonian fluids. Water is very common example. Mathematical expression is as follows

$$\tau_{yx} = \mu \frac{du}{dy} \tag{1.4}$$

where τ_{yx} is the x-component of shear stress applied on the plane perpendicular to y-axis and u is x-component of the velocity. μ is he dynamic viscosity.

1.4.4 Non-Newtonian fluids

The real fluids for which shear stress is not linearly proportional to deformation rate are called non-Newtonian fluids. Infact these type of fluids obey power law fluid model. Mathematically

$$\tau_{yx} = k(\frac{du}{dy})^n, \quad n \neq 1 \tag{1.5}$$

where k and n are called consistency and flow behavior index respectively. Above expression shows that for the value $n = 1$ and by considering $k = \mu$ power-law reduces to the newtion law of viscosity. Most common examples are ketchup, blood and paints etc.

1.5 Types of flow

1.5.1 Laminar flow

When the particles of fluid follow a definite path then the flow will be laminar. In laminar flow fluid particles will not disturb the path of each other. Laminar flow is more commonly in small tubes or channels and also for the relatively high viscosity fluids. For example oil flow in thin tubes and blood flow in capillaries.

1.5.2 Turbulent flow

When the particles of fluid don't follow a definite path then the flow will be turbulent. In turbulent flow particles of fluid perform irregular fluctuations or disturb path of each other. For example smoke rising from cigarette shows laminar flow and changes into turbulent flow because it eddies and swirls from its definite path.

1.5.3 Steady flow

If the properties of the fluid under study, at any point of flow field do not change with the passage of time then the flow of that fluid will be steady flow. Mathematically it can be written as

$$\frac{\partial \Upsilon}{\partial t} = 0, \tag{1.6}$$

where Υ is any fluid property.

1.5.4 Unsteady flow

In unsteady flow properties of the fluid at any point of flow field change with the passage of time. Mathematically,

$$\frac{\partial \Upsilon}{\partial t} \neq 0 \tag{1.7}$$

1.5.5 Incompressible flow

Fluid flow in which flow field density remains constant is called incompressible flow. Mathematically it can be written as

$$\rho \neq \rho(x, y, z, t) \;\; or \;\; \rho = constant \tag{1.8}$$

or can be written as

$$\nabla.\mathbf{V} = 0 \tag{1.9}$$

1.5.6 Compressible flow

In contrast to incompressible flow if the density is variable during the flow field then it is called compressible flow. Mathematically,

$$\rho = \rho(x, y, z, t) \tag{1.10}$$

or can be written as

$$\nabla.\mathbf{V} \neq 0 \tag{1.11}$$

1.6 Relationship in cylindrical coordinates

In cylindrical coordinates(r, θ, z) the gradient (∇) operator is defined as

$$\nabla = \frac{\partial(.)}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial(.)}{\partial \theta}\hat{\theta} + \frac{\partial(.)}{\partial z}\hat{z} \tag{1.12}$$

where $\hat{r}, \hat{\theta}, \hat{z}$ are the three unit vectors. The divergence of velocity vector $\mathbf{V} = (v_r, v_\theta, v_z)$ is defined as

$$\nabla.\mathbf{V} = \frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \tag{1.13}$$

The divergence of the stress tensor $\mathbf{T}$ can be defined as

$$\begin{aligned} \nabla.\mathbf{T} &= [\frac{1}{r}\frac{\partial(r\tau_{rr})}{\partial r} + \frac{1}{r}\frac{(\tau_{r\theta})}{\partial\theta} + \frac{\partial(\tau_{rz})}{\partial z} - \frac{\tau_{\theta\theta}}{r}]\hat{r} \\ &+ [\frac{1}{r}\frac{\partial(r\tau_{\theta r})}{\partial r} + \frac{1}{r}\frac{\partial(\tau_{\theta\theta})}{\partial\theta} + \frac{\partial(r\tau_{\theta z})}{\partial z} + \frac{\tau_{r\theta}}{r}]\hat{\theta} \\ &+ [\frac{1}{r}\frac{\partial(r\tau_{zr})}{\partial r} + \frac{1}{r}\frac{\partial(\tau_{z\theta})}{\partial\theta} + \frac{\partial(\tau_{zz})}{\partial z}]\hat{z} \end{aligned} \tag{1.14}$$

The gradient of a vector quantity gives a second rank tensor. In cylindrical coordinates such tensor is defined as

$$\nabla\mathbf{V} = \begin{pmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r} & \frac{\partial v_r}{\partial z} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r} & \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_z}{\partial r} & \frac{1}{r}\frac{\partial v_z}{\partial\theta} & \frac{\partial v_z}{\partial z} \end{pmatrix}$$

1.7 Governing laws

1.7.1 Conservation law of mass

The law states that mass in an isolated system is neither created nor destroyed. Mass can be changed from one form to another but always remains conserved. Mathematical expression is defined as

$$\frac{\partial\rho}{\partial t} + \nabla.(\rho\mathbf{V}) = 0 \tag{1.15}$$

where ∇ is three dimensional differential operator. If the incompressible fluid is under consideration then density is constant and $Eq.$(1.15) reduces to

$$\nabla.\mathbf{V} = 0 \tag{1.16}$$

1.7.2 Conservation law of momentum

Law of conservation of momentum is another form of Newton's second law. This law depicts that sum of all the forces applied on an isolated system is equal to the rate of change of momentum of that system with respect to time. Fluid particles should obey this law in any state either moving or at rest. Mathematical expression of this law is defined as

$$\rho\frac{d\mathbf{V}}{dt} = div\mathbf{T} + \rho\mathbf{B} \tag{1.17}$$

For Navier-Stokes equation tensor is defined as

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 \tag{1.18}$$

$\mathbf{A}_1$ is defined as,

$$\mathbf{A}_1 = grad\mathbf{V} + (grad\mathbf{V})^t \tag{1.19}$$

In above expressions

$\rho = density$

$\frac{d}{dt} = Material\ time\ derivative$

$\mathbf{V} = Velocity\ vector\ field$

$\mathbf{T} = Cauchy's\ stress\ tensor$

$\mathbf{B} = Body\ force$

$p = Pressure$

$\mu = Dynamic\ viscosity$

The Cauchy's stress tensor in the form of matrix can be expressed as

$$\mathbf{T} = \begin{pmatrix} \sigma_{rr} & T_{r\theta} & T_{rz} \\ T_{\theta r} & \sigma_{\theta\theta} & T_{\theta z} \\ T_{zr} & T_{z\theta} & \sigma_{zz} \end{pmatrix}$$

In the tensor $\sigma_{rr}, \sigma_{\theta\theta}$ and σ_{zz} are the normal stesses and rest are called shear stresses. Component form of $Eq(1.17)$ is

$$\rho\frac{dv_r}{dt} = \frac{1}{r}\frac{\partial(r(\sigma_{rr}))}{\partial r} + \frac{1}{r}\frac{\partial(T_{r\theta})}{\partial\theta} + \frac{\partial(T_{rz})}{\partial z} - \frac{\sigma_{\theta\theta}}{r} + \rho b_r, \tag{1.20}$$

$$\rho\frac{dv_\theta}{dt} = \frac{1}{r}\frac{\partial(rT_{\theta r})}{\partial r} + \frac{\partial(\sigma_{\theta\theta})}{\partial\theta} + \frac{\partial(T_{\theta z})}{\partial z} + \frac{T_{r\theta}}{r} + \rho b_\theta, \tag{1.21}$$

$$\rho\frac{dv_z}{dt} = \frac{1}{r}\frac{\partial(rT_zr)}{\partial r} + \frac{1}{r}\frac{\partial(T_{\theta z})}{\partial\theta} + \frac{\partial(\sigma_{zz})}{\partial z} + \rho b_z, \tag{1.22}$$

where b_r, b_θ and b_z are the components of body force in the direction of r, θ and z respectively.

1.7.3 Conservation law of energy

The law states that energy of an isolated system neither be created nor destroyed but can be converted from one form to another and energy of the whole system remains conserved. Mathematical expression is defined as

$$\rho c_p \frac{d\phi}{dt} = k\nabla^2\phi + \mathbf{T}.\mathbf{L} \tag{1.23}$$

In the above expression

ϕ =Temperature c_p =Specific heat at constant pressure.

$\mathbf{T.L}$ =Viscous dissipation

k =Thermal conductivity

1.7.4 Fourier law of heat

Fourier law is defined as "Heat flux is proportional to temperature gradient". In vector form equation of law can be written as

$$\overrightarrow{q} = -k\nabla T \tag{1.24}$$

Minus sign is for the heat transfer in the decreasing temperature direction. k and ∇T represents thermal conductivity and temperature gradient respectively.

1.8 Types of heat transfer

1.8.1 Conduction

It is the process in which heat transfer occurs between the physical objects.

1.8.2 Convection

It is the process in which heat or energy transfer between the objects and its environment due to a fluid flow.

1.9 Heat generation/absorption

Production of heat in any body by any method is called heat generation. But the gaining of heat by any source is named as absorption of heat. It is represented by β and mathematically it can be written as

$$\beta = \frac{Ql}{\rho U_\circ c_p} \tag{1.25}$$

where Q is volumetric rate of heat generation or absorption. $Q^+ > 0$ represents the heat source and $Q^- < 0$ represents the heat sink. l is characteristic length and $U_\circ$ is positive constant and c_p is the specific heat capacity.

1.10 Boundary layer

In the early 19^{th} century the idea of boundary layer is given by Ludwig Prandtl. Boundary layer is basically the region or layer near the surface at which the fluid flows, where the viscosity effects are maximum and these effects vanishes at the finite distance from the surface where the velocity field becomes free-stream velocity. By introducing this concept the solution of Navier-Stokes becomes more easier.

1.11 Dimensionless numbers

1.11.1 Reynolds number

It is a dimensional quantity and defined as the ratio of inertial and viscous forces. Reynolds number Re decides the flow type either laminar or turbulent. For small value of Re, flow is laminar and for large value flow is turbulent. The value of Re, can be decided by inverse relation with viscous forces or by direct relation with inertial forces. Mathematically it can be written as

$$Re = \frac{UL}{\nu} \tag{1.26}$$

where U, L and ν is free-stream velocity, characteristics length and kinematic viscosity respectively.

1.11.2 Nusselt number

It is a dimensionless quantity that stands behind the name of German Mathematician Nusselt. Nusselt number is the ratio of convective verses conductive heat transfer. Mathematically it can be written as

$$Nu = \frac{hL}{k} \tag{1.27}$$

In above expression h represents the convective heat transfer and k represents the conductive heat transfer coefficients.

1.11.3 Prandtl number

It is ratio of momentum diffusivity(ν) to thermal diffusivity(α). Mathematically,

$$Pr = \frac{\nu}{\alpha} \tag{1.28}$$

where

α =Thermal diffusivity

ν =Kinematic viscosity.

1.11.4 Weissenberg number

A dimensionless number introduce by Karl Weissenberg. The ratio of the stress relaxation time of the fluid to a specific process time. Mathematically can be written as

$$\lambda = \frac{\tau_r}{\tau_s} \tag{1.29}$$

In above expression λ represents the weissenberg number and τ_r is relaxation time (time taken by a system to come its original position after a disturbance applied on it by any source) τ_s is specific process time.

1.11.5 Skin friction

A resistance or friction between the fluid and the surface at which the fluid flow or it can be defined as the friction between the moving fluid and the surface enclosing it.

1.11.6 Curvature parameter

Curvature is a measure of turning of a curve. By definition it is the magnitude of rate of change to angle φ with respect to the measure of arc length s of curve. Curvature is denoted by γ. Mathematical expression can be written as

$$\gamma = |\frac{d\varphi}{ds}| \tag{1.30}$$

Chapter 2

Introduction

2.1 Introduction

For the study of non-Newtonian behavior of fluids, the most under consideration fluids for research are pseudoplastic fluids. As we know Williamson fluid is pseudoplastic and have great importance in industry such as in chemical industry used for extrusion of polymer sheets, preperation of emulsions and adhesives etc. Also have wide range applications in petroleum industry and power engineering. Most of the non-Newtonian models have been proposed in corporated with navier stokes equation for the explanation of rheological properties of fluids. Several models have been introduced to explain the pseudoplastic fluids for example Power law model, Carreaus model, Cross model, Ellis model and Williamson fluid model. In Williamson fluid model, minimum viscosity ($\mu_\circ$) as well as maximum viscosity (μ_∞) both are considered. Williamson [1] explained the pseudoplastic materials and introduced a model equation to describe the pseudoplastic fluid flow and proved the results experimentally. Lyubimov and Perminov [2] discussed the thin layer of a Williamson fluid over an inclined surface with effects of gravitational force. Dapra and Scarpi [3] proposed the perturbation solution for a Williamson fluid which is injected into fractured rock. Peristaltic flow of a Williamson fluid has been discussed by Nadeem et al. [4]. The analytical solution for Williamson fluid with chemical reactive species by Scalling transformation and Homotopy Analysis Method (HAM) is found by Najeeb Alam Khan et al. [5]

Williamson fluid past a stretching cylinder is under consideration and flow due to stretching boundary has importance in the extrusion of plastic and metal industries. Sakiadis [6] started the study of two-dimensional fluid flow over a stretching surface moving with constant velocity. Crane [7] proposed the exact solution of two-dimensional Navier Stokes equation for streching surface. Wang [8] extend the concept of Crane and presented the solution for three-dimensional stretching surface. Fang et al. [9] have observed the flow between two stretching disks. Numerical solution of boundary layer flow with effects of heat transfer over the stretched porous cylinder discussed by Xinhui Si et al. [10].

Incompressible viscous fluid flow and heat transfer through a stretching surface gained much attention in most of the manufacturing processes, such as drawing of copper wires and glass blowing. In study of heat transfer over a stretching sheet the most valued application is sheet extrusion either plastic sheets or metal sheets. During extrusion process observation of cooling and heat transfer is very important because it effects the final products. Conventionally water and air are used as cooling mediums but recently it has been proposed that water should be replaced by a medium with slower rate of solidfication. Carraagher et al. [11] investigate the boundary layer flow and heat transfer over a stretching surface with the condition that temperature difference between the surface and an ambient fluid is proportional to the power of distance from a fixed point. Tsou et al. [12] extended the study of Sakiadis' and considered the heat transfer over the stretching continuous surface and proved the Sakiadis' results experimentally. Afterwards many researchers extended the work of Crane' such as Gupta and Gupta [13], Dutta [14] considering the heat transfer phenomena under different physical conditions. Ishak [15] and Nadeem [16] also discussed the results of heat transfer. Recently experimental and numerical study of heat transfer and flow friction is discussed by Yu Rao et al. [17].

In recent few years the study of heat generation became more popular due to uncountable applications in nuclear reactor engineering and scientific instrumentation. And heat absorption chillers and heat pumps are important in industry due to advantages in renewable utilization and waste heat recovery. The energy conservation is very important and most under discussion areas all over the world. That efforts in absorption technologies perform a distinct role in global energy and environmental issues. Some of the absorption

technologies are absorption heat pump (AHP), generator absorber heat exchange (GAX), compression-absorption heat pump (CAHP), Open-cycle absrption heat pump (OAHP), etc. There are many residential and commercial applications of absorption heating systems because of huge amount of energy consumption. For example Direct-fired absorption chiller/heater works on absorption pump theory, Latent heat recovery of vapor works on OAPH, Hybrid CAHP heating systems, district heating systems, thermal energy storage and transportation, etc are civil applications. Absorption-assisted drying, Absorption-assisted evaporation, Absorption-assisted distillation, etc are industrial applications. Internal heat generation with multi-boiling effects on cylindrical bodies are disscused by Rybchinskaya et al [18]. Uniform and non-uniform heat generation results for cylindrical, rectangular and longitudinal surfaces examined by H.C.Unal [19, 20]. The effects on unsteady flow in the presence of heat generation with mixed convection and magnetic force examined numerically by Tapas Ray Mahapatra et al [21]. Recently boundary layer flow problem with effect of thermal conductivity and heat generation solved numerically and to remove the highly non-linearity of momentum and heat equation, differential transformation method (DTM) is used by Mohsen Torabi et al [22].
As we know thermal conductivity is a material property and changes with the variation in temperature. It depends on material, if fluid is electrically conducting then temperature flow is increased. Study of variable thermal conductivity is important in electrolytes which have important part in perparation of D.C. batteries. Chaim [23] studied that variable thermal conductivity vary linearly with temperature. This problem is solved by Shooting method. Heat transfer with heat generation or absorption is considered. C_f skin friction and Nu_x is Nusselt number and the effects of pertinent parameters e.g; curvature parameter γ, Prandtl number Pr, Thermal conductivity variable ε and heat generation coefficient β have been discussed.

2.2 Dissertation objectives

The main objectives of this dissertation are:

- To investigate the heat transfer, variable thermal conductivity for the boundary layer flow past a stretching cylinder.

- To find out solution for the heat transfer,variable thermal conductivity in the presence of heat generation/absorption for the boundary layer Williamson fluid flow past a stretching cylinder.

2.3 Method of finding solution

Practical problems mostly involve nonlinearity and in this dissertation the considered fluid problem having non-linear differential equations. To solve the system of non-linear differential equation there are many analytical and numerical methods. Analytical methods for example ADM, HAM and Perturbation method etc, are mostly used. There are also several numerical techniques to solve for non-linear equations for example keller box technique, Finite difference scheme and Shooting method etc. To solve the problem of this dissertation the Shooting method is used because of its advantages, for example it convert boundary value problem to initial value problem etc.

Chapter 3

Boundary Layer Flow past a Stretching Cylinder and Heat Transfer with Variable Thermal Conductivity

3.1 Introduction

In this chapter we have discussed the solution for heat transfer along variable thermal conductivity for the incompressible, viscous fluid pass over a stretching cylinder. After applying assumptions, Navier-Stokes equation give birth to non-linear partial differential equation and further converted into ordinary differential equation with the help of similarity transformation. Effects of different parameters on velocity and temperature profile studied through graphs by using shooting method, whereas influence of parameters on the Nusselt number and skin friction are analyzed through tables.

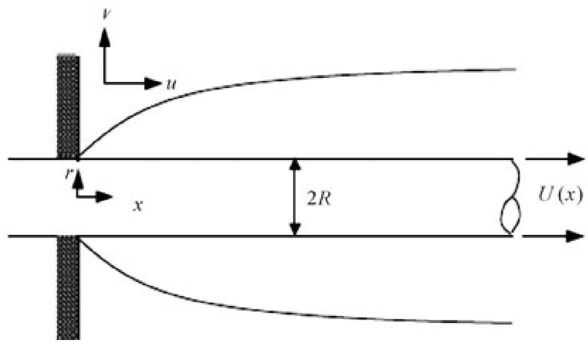

Figure 3.1: Physical model with coordinate system

3.2 Mathematical formulation

The assumptions are steady, incompressible, axisymmetric and two dimensional viscous boundary layer flow past the continuously stretching cylinder. For temperature profile heat equation is considered with thermal conductivity. In he absence of body force the governing equations i.e, continuity equation, momentum and heat equation are given below

$$\nabla.\mathbf{V} = 0 \tag{3.1}$$

$$\rho\frac{D\mathbf{V}}{Dt} = \nabla.\mathbf{T} \tag{3.2}$$

$$\rho c_p\frac{DT}{Dt} = \nabla.(k\nabla T) \tag{3.3}$$

In the above expressions ∇ is differential operator, $\mathbf{V}$, ρ, $\frac{D}{Dt}$ and $\mathbf{T}$ are the velocity of fluid, density of fluid, material derivative and cauchy stress tensor respectively. In the heat equation c_p, k and T are the specific heat capacity at constant pressure, thermal conductivity and temperature of flow field respectively. Mathematical expressions of $\mathbf{V}$, ∇, $\mathbf{T}$ and $\frac{D}{Dt}$ are given as

$$\mathbf{V} = [(v(x,r), 0, u(x,r))], \tag{3.4}$$

$$\nabla = \frac{\partial(.)}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial(.)}{\partial\theta}\hat{\theta} + \frac{\partial(.)}{\partial x}\hat{x}, \tag{3.5}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V}.\nabla \tag{3.6}$$

Here flow is independent of time that is steady flow so $Eq(3.6)$ reduces to

$$\frac{D}{Dt} = \mathbf{V}.\nabla \tag{3.7}$$

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A_1} \tag{3.8}$$

In $Eq(3.4)$ u and v are the axial and radial component of velocity. In $Eq(3.8)$ p, I and μ are the pressure, identity tensor and dynamic viscosity respectively. $\mathbf{A_1}$ is the first Rivilin-Ericksen tensor which has mathematical expression as follows

$$\mathbf{A_1} = \nabla\mathbf{V} + (\nabla\mathbf{V})^t \tag{3.9}$$

In $Eq(3.9)$ $\nabla(\mathbf{V})$ is the gradient of velocity vector given as

$$\nabla\mathbf{V} = \begin{bmatrix} \frac{\partial v}{\partial r} & \frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{w}{r} & \frac{\partial v}{\partial x} \\ \frac{\partial w}{\partial r} & \frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{v}{r} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial r} & \frac{1}{r}\frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial x} \end{bmatrix} \tag{3.10}$$

After putting the above expression in $Eq(3.9)$, final form will be

$$\mathbf{A_1} = \begin{bmatrix} 2\frac{\partial v}{\partial r} & \frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{w}{r} & \frac{\partial v}{\partial x} + \frac{\partial u}{\partial r} \\ \frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{w}{r} & \frac{2}{r}\frac{\partial w}{\partial r} + 2\frac{v}{r} & \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial r} & \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial x} & 2\frac{\partial u}{\partial x} \end{bmatrix} \tag{3.11}$$

Apply velocity profile $\mathbf{V} = [v(x, r), 0, u(x, r)]$ in above expression then first Rivilin-Ericksen tensor reduced to

$$\mathbf{A_1} = \begin{bmatrix} 2\frac{\partial v}{\partial r} & 0 & \frac{\partial v}{\partial x} + \frac{\partial u}{\partial r} \\ 0 & 2\frac{v}{r} & 0 \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial r} & 0 & 2\frac{\partial u}{\partial x} \end{bmatrix} \tag{3.12}$$

After applying $Eq(3.4)$ and $Eq(3.5)$ to continuity equation gives the expression

$$\frac{\partial(rv)}{\partial r} + \frac{\partial(ru)}{\partial x} = 0 \tag{3.13}$$

By putting the $Eq(3.8)$ in $Eq(3.2)$ gives the expression

$$\rho\frac{D\mathbf{V}}{Dt} = -\nabla p + \mu\nabla.\mathbf{A_1} \tag{3.14}$$

In the absence of pressure gradient above expression reduces to

$$\rho\frac{D\mathbf{V}}{Dt} = \mu\nabla.\mathbf{A_1} \tag{3.15}$$

or

$$\frac{D\mathbf{V}}{Dt} = \nu\nabla.\mathbf{A_1} \tag{3.16}$$

where $\nu = \frac{\mu}{\rho}$ known as kinematic viscosity. Application of material time derivative upon the velocity vector have the component form

$$(\frac{D\mathbf{V}}{Dt})_x = \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} \tag{3.17}$$

$$(\frac{D\mathbf{V}}{Dt})_r = \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial r} \tag{3.18}$$

$$(\frac{D\mathbf{V}}{Dt})_\theta = \frac{dw}{dt} = 0 \tag{3.19}$$

Also component form of $\nabla.\mathbf{A_1}$ have to be determined

$$(\nabla.\mathbf{A_1})_x = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{\partial^2 u}{\partial x^2} \tag{3.20}$$

$$(\nabla.\mathbf{A_1})_r = \frac{2}{r}\frac{\partial}{\partial r}(r\frac{\partial v}{\partial r}) + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial r} - \frac{2v}{r^2} \tag{3.21}$$

$$(\nabla.\mathbf{A_1})_\theta = 0 \tag{3.22}$$

Inserting $Eq(3.17)$ to $Eq(3.22)$ in $Eq(3.14)$ gives the expressions as follows

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = \nu(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{\partial^2 u}{\partial x^2}) \tag{3.23}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial r} = \nu(\frac{2}{r}\frac{\partial}{\partial r}(r\frac{\partial v}{\partial r}) + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial r} - \frac{2v}{r^2}) \tag{3.24}$$

θ-component is zero shown by $Eq(3.19)$ and $Eq(3.22)$. Now $Eq(3.3)$ which is heat equation, have to be determined. For calculation material time derivative at left hand side of $Eq(3.3)$ have to be calculated as

$$\frac{DT}{Dt} = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r} \tag{3.25}$$

And right hand side of $Eq(3.3)$ is calculated as

$$\nabla.(k\nabla T) = \frac{1}{r}\frac{\partial}{\partial r}(kr\frac{\partial T}{\partial r}) + \frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}) \tag{3.26}$$

Insertion of $Eq(3.25)$ and $Eq(3.26)$ in $Eq(3.3)$ gives the expression for temperature equation as

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}(\alpha r\frac{\partial T}{\partial r}) + \frac{\partial}{\partial x}(\alpha\frac{\partial T}{\partial x}) \tag{3.27}$$

where $\alpha = \frac{k}{\rho c_p}$ and known as thermal diffusivity. Now apply the boundary layer approximation and according to its rules taking assumptions as

$$u = O(1),\ x = O(1),\ v = O(\delta),\ T = O(1),\ \nu = O(\delta^2),\ \alpha = O(\delta^2) \qquad (3.28)$$

Apply above orders in $Eq(3.23) - Eq(3.24)$ and $Eq(3.27)$ and after certain calculations get the following reduced equations

$$\frac{\partial(rv)}{\partial r} + \frac{\partial(ru)}{\partial x} = 0 \qquad (3.29)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = \frac{v}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) \qquad (3.30)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}(ar\frac{\partial T}{\partial r}) \qquad (3.31)$$

Including boundary conditions

$$u(x,r) = U(x),\ v(x,r) = 0,\ T(x,r) = T_w\ at\ r = R$$

$$u(x,r) \to 0,\ T(x,r) \to T_\infty \quad r \to \infty \qquad (3.32)$$

where $U(x) = \frac{Ux}{l}$ denotes the stretching velocity in which U is reference velocity.l,T_w and T_∞ denotes characteristic length,surface temperature and extreme temperature respectively. We can introduce a stream function which satisfy the continuity equation, such that

$$u = \frac{1}{r}\frac{\partial \psi}{\partial r}, \quad v = -\frac{1}{r}\frac{\partial \psi}{\partial x} \qquad (3.33)$$

Similarity transformations for the governing equations can be defined as

$$\eta = \frac{r^2 - R^2}{2R}\sqrt{\frac{U}{\nu x}}, \quad \psi = \sqrt{U\nu x}Rf(\eta) \qquad (3.34)$$

$$\phi = \frac{T - T_\infty}{T_w - T_\infty}, \quad \alpha = \alpha_\infty(1 + \varepsilon\phi) \qquad (3.35)$$

In the above expression a_∞ is the thermal conductivity at a large distance away from the cylinder and ε is small number. $Eq(3.29)$ identically satisfied for these transformations but after applying above transformations to the governing equations that are $Eq(3.30 - 3.32)$ get following expressions

$$(1 + 2\gamma\eta)f''' + 2\gamma f'' - f'^2 + ff'' = 0 \qquad (3.36)$$

$$(1 + 2\gamma\eta)\phi'' + 2\gamma\phi' + P_r f\phi' + \varepsilon((1 + 2\gamma\eta)(\phi\phi'' + \phi'^2)) + 2\gamma\phi\phi') = 0 \qquad (3.37)$$

with the boundary conditions

$$f(0) = 0,\ f'(0) = 1,\ \phi(0) = 1,$$
$$f' \to 0,\ \phi \to 0\ at\ \eta \to \infty \tag{3.38}$$

In above expressions the prime represents the derivative with respect to η. Pr and γ Prandtl number and curvature parameter are defined below

$$P_r = \frac{\nu}{\alpha_\infty},\ \ \gamma = \frac{1}{R}\sqrt{\frac{x\nu}{U}} \tag{3.39}$$

3.3 Skin friction coefficient and local Nusselt number

The skin friction coefficient of the considered surface is

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} \tag{3.40}$$

In the above expression τ_w represents the shear stress at the surface of cylinder is defined as

$$\tau_w = \mu(\frac{\partial u}{\partial r})_{r=R} \tag{3.41}$$

The transformations defined for this problem also applied on shear stress and get the expression

$$\tau_w = \mu U\sqrt{\frac{U}{\nu x}} f''(0) \tag{3.42}$$

Put value of τ_w which is obtained after applying transformations into the $Eq(3.40)$ and get the expression

$$\frac{1}{2}C_f Re_x^{\frac{1}{2}} = f''(0) \tag{3.43}$$

Now the local Nusselt number of temperature distribution is

$$Nu_x = \frac{xq_w}{\alpha_\infty(T_w - T_\infty)} \tag{3.44}$$

where q_w is the degree of heat transfer at the surface of cylinder and defined as

$$q_w = -\alpha_\infty(\frac{\partial T}{\partial r})_{r=R} \tag{3.45}$$

After applying the defined transformation to q_w get the following expression

$$q_w = -\alpha_\infty(T_w - T_\infty)\sqrt{\frac{U}{\nu x}}\phi'(0) \tag{3.46}$$

Using $Eq(3.46)$ in $Eq(3.44)$ in order to get the expression for local Nusselt number that is

$$Nu_x Re_x^{-\frac{1}{2}} = -\phi'(0) \tag{3.47}$$

Here $Re_x = \frac{Ux}{\nu}$ is the Reynolds number.

3.4 Method of solution

Current problem have nonlinear equations in which velocity profile is of order three and temperature profile is of order two. Numerical technique that is Shooting method is used to solve this nonlinear problem. A basic rule for the shooting method is to convert the higher order equations to first order equations so here the nonlinear ordinary differential equations are converted into system of five first order ordinary differential equations. Then this first order system of differential equations is solved by Runge-Kutta-Fehlberg method. The $Eqs(3.36)-(3.37)$ can be written as

$$f''' = \frac{-2\gamma f'' + f'^2 - ff''}{1+2\gamma\eta} \tag{3.48}$$

$$\phi'' = \frac{-P_r f\phi' - \varepsilon(1+2\gamma\eta)\phi'^2 - 2\gamma(1+\varepsilon\phi)\phi'}{(1+2\gamma\eta)(1+\varepsilon\phi)} \tag{3.49}$$

Reduction of higher order equations to five first order differential equations is shown as

$$f = y_1,\ f' = y_2,\ f'' = y_3,\ f''' = y_3',\ \phi = y_4,\ \phi' = y_5,\ \phi'' = y_5' \tag{3.50}$$

Here the momentum and heat equations are coupled and these are converted into the system of five first order simultaneous equations as

$$y_1' = y_2 \tag{3.51}$$

$$y_2' = y_3 \tag{3.52}$$

$$y_3' = \frac{y_2^2 - 2\gamma y_3 - y_1 y_3}{1+2\gamma\eta} \tag{3.53}$$

$$y_4' = y_5 \tag{3.54}$$

$$y_5' = -\frac{P_r y_1 y_5 + \varepsilon(1+2\gamma\eta)y_5^2 + 2\gamma(1+\varepsilon y_4)y_5}{(1+2\gamma\eta)(1+\varepsilon y_4)} \tag{3.55}$$

where prime represents the derivative with respect to η and the transformed boundary conditions are

$$y_1(0) = 0,\ y_2(0) = 1,\ y_2(\infty) \to 0, y_4(0) = 1, y_4(\infty) \to 0 \tag{3.56}$$

For solving the system of five first order ordinary differential equation, five initial guesses should be required, as Runge-Kutta-Fehlberg method is used here. In $Eq(3.56)$ two initial guesses are given for f and one is given for ϕ but two initial guesses are unknown becaues these are given at $\eta \to \infty$. To find out these unknowns the most important step is to assume the approximate value of η at∞. After choosing the initial guesses solve the problem's governing equations to find the value of $f''(0)$ and $\phi'(0)$. The process of solution is continued by considering another value of η at ∞ repeat the process until two consecutive values of $f''(0)$ and $\phi'(0)$ are differ only after the significant digits. This final value of η at ∞ is considered for the calculation of velocity profile f and temperature profile ϕ in the boundary layer for the measurement of all physical parameters. When all the initial conditions become known then Runge-Kutta method applied to solve the system of ODEs.

3.5 Results and discussions

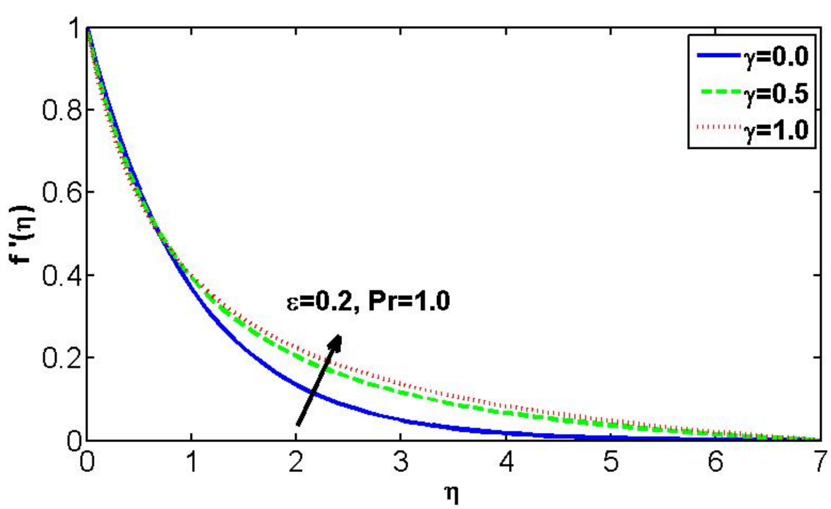

Figure 3.2: Velocity profile for different values of γ.

Figure. 3.2 shows the change in velocity profile $f'(\eta)$ with respect to curvature parameter. It is analyzed that there is a small change in the dynamic region $[0, 0.75]$ and after this region that is within $[0.75, \infty)$ the velocity profile going to zero, that means that velocity becomes the free stream velocity in the region $[0.75, \infty]$. The behavior of velocity profile $f'(\eta)$ is increasing with the increase in curvature because with the increment in curvature parameter γ cause reduction in the radius of curvature which leads to the decrease in resistance to flow because of less area of contact of fluid with cylinder. Boundary layer thickness increases with the increase in curvature parameter.

Table 3.1: Skin friction $f''(0)$ variation with respect to curvature parameter γ.

γ	0.0	0.25	0.5	0.75	1.0
$f''(0)$	-1.0002	-1.0960	-1.1922	-1.2874	-1.3815

Tabel. 3.1 describes how the skin friction coefficient is effected by curvature parameter γ. Now the expression of skin friction shows that $C_f \propto Re_x^{-\frac{1}{2}}$. As Re_x has negative fraction power which depicts that Re_x has an inverse relation with C_f that is increase in Re will decrease the C_f. *Table*. 3.1 shows that absolute value of skin friction increases with the increase in curvature parameter. Boundary layer thickness grows which causes an increase in skin friction coefficient.

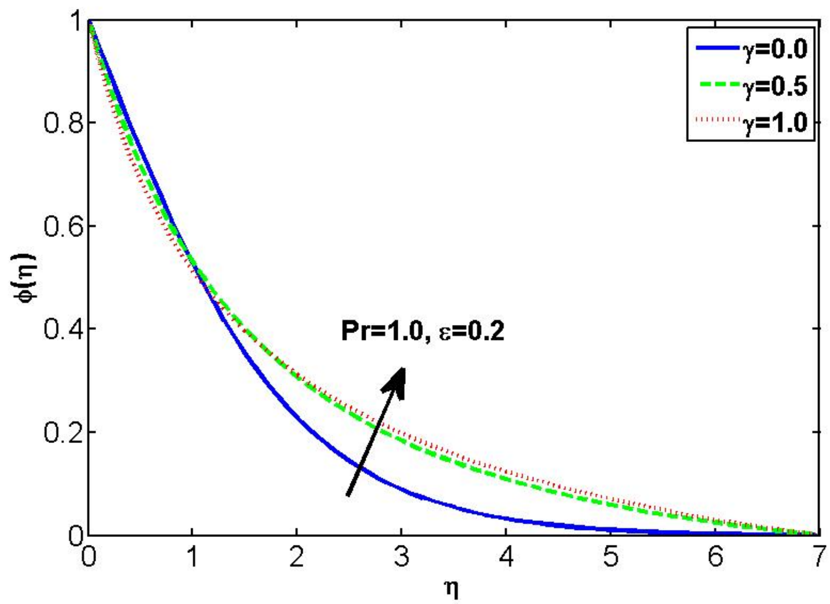

Figure 3.3: Temperature profile for different values of γ.

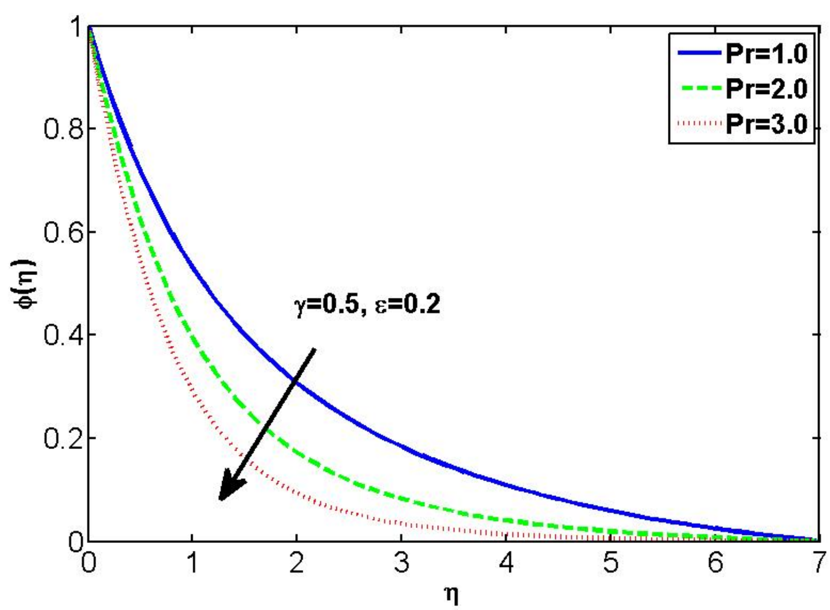

Figure 3.4: Temperature profile for different values of Pr.

Figure. 3.3 represents that how curvature parameter γ effects the temperature profile $\phi(\eta)$. Enhancement of curvature parameter causes the reduction of viscous forces which

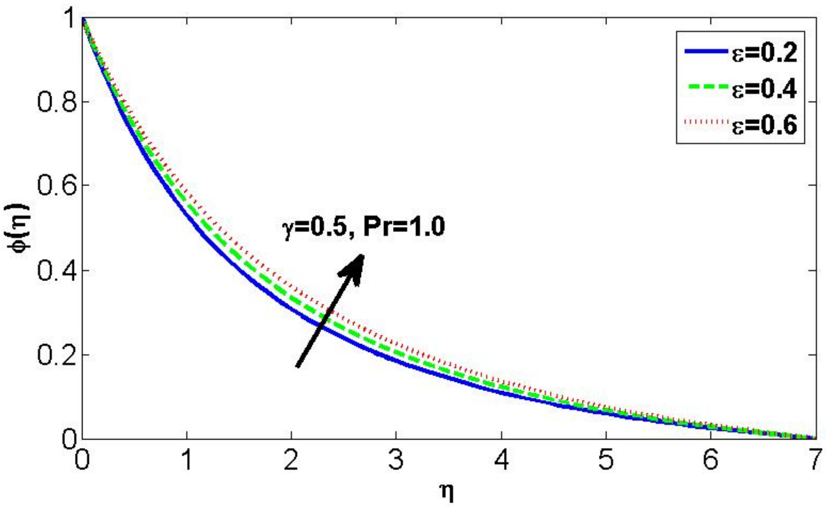

Figure 3.5: Temperature profile for different values of ε.

increases the heat transfer and ultimately increase the temperature $\phi(\eta)$ of fluid. Thermal boundary layer increases with the increase in curvature parameter.

Figure. 3.4 describes the effect of Prandtl number on temperature field. As Pr has an inverse relation with thermal diffusivity. So the increase in Pr causes decrease in thermal diffusion, which means the reduction of heat transfer. As a result due to increase in Pr causes fall in temperature $\phi(\eta)$. Thermal boundary layer reduces with the increase in Prandtl number.

Figure. 3.5 depicts the impact of thermal conductivity ε on temperature profile $\phi(\eta)$. With the increase in thermal conductivity ε there is an increase in conductance of fluid due to which heat transfers more rapidly. So temperature distribution $\phi(\eta)$ increases with the increase in thermal conductivity ε.

Table 3.2: Local Nusselt number($-\phi'(0)$) influenced by different parameters.

Pr	γ	$\varepsilon = 0.0$	$\varepsilon = 0.2$	$\varepsilon = 0.4$
1.0	0.0	0.5827	0.5071	0.4512
	0.25	0.6765	0.5954	0.5358
	0.5	0.7819	0.6943	0.6305
	0.75	0.8882	0.7935	0.7248
	1.0	0.9933	0.812	0.8174
ε	γ	$Pr = 1.0$	$Pr = 2.0$	$Pr = 3.0$
0.1	0	0.5418	0.8498	1.0878
	0.25	0.6325	0.9224	1.1609
	0.5	0.7343	1.0001	1.2330
	0.75	0.8368	1.0826	1.3068
	1.0	0.9378	1.1678	1.3827

Tabel. 3.2 shows the influence of parameters that is curvature parameter γ, thermal conductivity ε and Pr on the Nusselt number. It's expression $Eq(3.47)$ shows that $Nu_x \propto Re_x^{\frac{1}{2}}$, which means there is a direct relation that is increase in Re_x causes an increase in heat transfer. As increase in curvature parameter γ reduce the viscous forces, that leads to increase in Re_x ultimately causes the increase in rate of heat transfer. Also increase in Pr causes decrease in temperature of fluid, which will create a temperature gradient and heat tranfer rate increases. Table shows that Prandlt number Pr and curvature parameter γ increase the local Nusselt number. But increase in thermal conductivity ε reduces the local Nusselt number because conductivity increases the temperature of fluid due to which temperature difference reduces between wall and fluid and ultimately rate of heat transfer reduces.

3.6 Concluding remarks

Based on above discussion it is concluded that:

- Increase in curvature parameter γ of stretching cylinder raise or increase velocity $f'(\eta)$ and temperature profile $\phi(\eta)$ both.
- Increase in thermal conductivity parameter ε also enhances the temperature flow field $\phi(\eta)$.
- Increase in Prandtl number Pr reduces the temperature profile $\phi(\eta)$.

Chapter 4

Numerical Solution of Williamson Fluid Flow past a Stretching Cylinder and Heat Transfer with Variable Thermal Conductivity and Heat Generation/Absorption

4.1 Introduction

In this chapter the study of boundary layer steady flow for the Williamson fluid past a stretching cylinder has been done. Included the study of temperature profile having effects of heat transfer with variable thermal conductivity and heat generation/absorption. Modeling of problem and application of transformations over there shows non-linearity in problem. Numerical technique i.e. Shooting method is used to solve this non-linear problem and find out the velocity profile and temperature profile at different points. Effects and results of different flow parameters $\gamma, \lambda, \varepsilon, \beta$ and Pr are shown graphically. Nusselt number and skin friction dependence have analyzed in tabular form.

4.2 Mathematical formulation

Consider the steady, axisymmetric, incompressible and two dimensional boundary layer flow with Williamson fluid over a stretching cylinder. The flow is generated due to linear stretching. Also the study of heat transfer with thermal conductivity is assumed to be variable and heat generation/absorption. Following are the governing equations

$$\nabla.\mathbf{V} = 0 \tag{4.1}$$

$$\rho\frac{D\mathbf{V}}{Dt} = \nabla.\mathbf{T} \tag{4.2}$$

$$\rho c_p\frac{DT}{Dt} = \nabla.(k\nabla T) + Q(T - T_\infty) \tag{4.3}$$

In the above expressions ∇ is differential operator, $\mathbf{V}$, ρ and $\frac{D}{Dt}$ is velocity, density of fluid and material derivative respectively. $\mathbf{T}$ is the cauchy stress tensor. In the heat equation c_p is the specific heat capacity at constant pressure, k is the thermal conductivity and T is the temperature of flow field. Last term of Eq(4.3) is for the heat generation/absorption, in which Q is the average heat transfer coefficient. Mathematically the $\mathbf{V}$, ∇, $\mathbf{T}$ and $\frac{D}{Dt}$ are given as

$$\mathbf{V} = [(v(x,r), 0, u(x,r))], \tag{4.4}$$

$$\nabla = \frac{1}{r}\frac{\partial(r(.))}{\partial r} + \frac{1}{r}\frac{\partial(.)}{\partial\theta} + \frac{\partial(.)}{\partial x}, \tag{4.5}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V}.\nabla \tag{4.6}$$

Here flow is independent of time that is steady flow so Eq(4.6) reduces to

$$\frac{D}{Dt} = \mathbf{V}.\nabla \tag{4.7}$$

Constitutive equations of the Williamson fluid model are given as

$$\mathbf{T} = -p\mathbf{I} + \tau \tag{4.8}$$

$$\tau = [\mu_\infty + \frac{(\mu_\circ - \mu_\infty)}{1 - \Gamma\dot{\gamma}}]\mathbf{A_1} \tag{4.9}$$

In Eq(4.4) u and v are the axial and radial component of velocity. In Eq(4.8) p, I and τ are the pressure, identity vector and extra stress tensor. In Eq(4.9) Γ is time constant will be greater then zero i.e, $\Gamma > 0$. $\mu_\circ$ and μ_∞ are the limiting viscosities at zero and infinity shear rate and $\dot{\gamma}$ is defined as

$$\dot{\gamma} = \sqrt{\frac{1}{2}\pi} \tag{4.10}$$

where

$$\pi = \frac{1}{2}trace(\mathbf{A_1}^2) \tag{4.11}$$

Here consider only the case for $\mu_\infty = 0$ and $\Gamma\dot{\gamma} < 1$. Now extra stress tensor reduced to:

$$\tau = [\frac{\mu_\circ}{1-\Gamma\dot{\gamma}}]\mathbf{A_1} \tag{4.12}$$

Apply binomial expansion to $Eq(4.12)$) and get following expression

$$\tau = \mu_\circ[1+\Gamma\dot{\gamma}]A_1 \tag{4.13}$$

In above expression $\mathbf{A_1}$ is the first Rivilin-Ericksen tensor which has mathematical expression as follows

$$\mathbf{A_1} = \nabla\mathbf{V} + (\nabla\mathbf{V})^t \tag{4.14}$$

In $Eq(4.14)$ $\nabla\mathbf{V}$ is the gradient of velocity vector given as

$$\nabla\mathbf{V} = \begin{bmatrix} \frac{\partial v}{\partial r} & \frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{w}{r} & \frac{\partial v}{\partial x} \\ \frac{\partial w}{\partial r} & \frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{v}{r} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial r} & \frac{1}{r}\frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial x} \end{bmatrix} \tag{4.15}$$

After putting the above expression in $Eq(4.14)$, final form will be

$$\mathbf{A_1} = \begin{bmatrix} 2\frac{\partial v}{\partial r} & \frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{w}{r} & \frac{\partial v}{\partial x} + \frac{\partial u}{\partial r} \\ \frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{w}{r} & \frac{2}{r}\frac{\partial w}{\partial r} + 2\frac{v}{r} & \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial r} & \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial x} & 2\frac{\partial u}{\partial x} \end{bmatrix} \tag{4.16}$$

Now A_1^2 have to be calculated and its expression is as follows

$$\mathbf{A_1^2} = \begin{bmatrix} 4(\frac{\partial v}{\partial r})^2 + (\frac{\partial v}{\partial x} + \frac{\partial u}{\partial r})^2 & 0 & 2(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial r})(\frac{\partial v}{\partial r} + \frac{\partial u}{\partial x}) \\ 0 & 4\frac{v^2}{r^2} & 0 \\ 2(\frac{\partial v}{\partial r} + \frac{\partial u}{\partial x})(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial x}) & 0 & 4(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial x} + \frac{\partial u}{\partial r})^2 \end{bmatrix} \tag{4.17}$$

Put above expression into $Eq(4.11)$ and get following equation

$$\pi = 2[(\frac{\partial v}{\partial r})^2 + (\frac{\partial u}{\partial x})^2 + \frac{v^2}{r^2}] + (\frac{\partial v}{\partial x} + \frac{\partial u}{\partial r})^2 \tag{4.18}$$

Now by putting $Eq(4.18)$ into $Eq(4.10)$, get expression for $\dot{\gamma}$

$$\dot{\gamma} = [(\frac{\partial v}{\partial r})^2 + \frac{v^2}{r^2} + \frac{1}{2}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial r})^2]^{\frac{1}{2}} \tag{4.19}$$

The operator ∇ presented in $Eq(4.5)$ apply to the velocity profile (V) presented in $Eq(4.4)$ and get following expression

$$\frac{\partial(rv)}{\partial r} + \frac{\partial(ru)}{\partial x} = 0 \tag{4.20}$$

which is the continuity equation. Now apply $Eq(4.8)$ to the $Eq(4.2)$ and get following equation

$$\rho\frac{D\mathbf{V}}{Dt} = -\nabla p + \nabla.\tau \tag{4.21}$$

As assumption the stretching of cylinder is cause of flow so there is no pressure gradient which reduces the $Eq(4.21)$ to

$$\frac{D\mathbf{V}}{Dt} = \frac{1}{\rho}\nabla.\tau \tag{4.22}$$

The component form of material derivative of velocity vector (V) is

$$(\frac{D\mathbf{V}}{Dt})_x = \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} \tag{4.23}$$

$$(\frac{D\mathbf{V}}{Dt})_r = \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial r} \tag{4.24}$$

$$(\frac{D\mathbf{V}}{Dt})_\theta = \frac{dw}{dt} = 0 \tag{4.25}$$

Now the component form of extra stress tensor τ after applying divergence is

$$\begin{aligned}(\nabla.\tau)_x &= \mu_\circ[\frac{1}{r}[1+\Gamma\dot{\gamma}](\frac{\partial v}{\partial x} + \frac{\partial u}{\partial r}) + \Gamma\frac{\partial \dot{\gamma}}{\partial r}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial r}) + [1+\Gamma\dot{\gamma}](\frac{\partial^2 v}{\partial x \partial r} + \frac{\partial^2 u}{\partial r^2}) \\ &+ 2[1+\Gamma\dot{\gamma}]\frac{\partial^2 u}{\partial x^2} + 2\Gamma\frac{\partial u}{\partial x}\frac{\partial \dot{\gamma}}{\partial x}] \end{aligned} \tag{4.26}$$

$$\begin{aligned}(\nabla.\tau)_r &= \mu_\circ[\frac{2}{r}\frac{\partial v}{\partial r}[1+\Gamma\dot{\gamma}] + 2[1+\Gamma\dot{\gamma}]\frac{\partial^2 v}{\partial r^2} + 2\Gamma\frac{\partial v}{\partial r}\frac{\partial \dot{\gamma}}{\partial r} + \Gamma\frac{\partial \dot{\gamma}}{\partial x}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial r})] \\ &+ [1+\Gamma\dot{\gamma}](\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial r \partial x}) - \frac{2v}{r^2}[1+\Gamma\dot{\gamma}]] \end{aligned} \tag{4.27}$$

$$(\nabla.\tau)_\theta = 0 \tag{4.28}$$

By using $Eq(4.23) - Eq(4.28)$ in $Eq(4.22)$, get the following component form of governing equation

$$
\begin{aligned}
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} &= \nu[\frac{1}{r}[1+\Gamma\dot{\gamma}](\frac{\partial v}{\partial x} + \frac{\partial u}{\partial r}) + \Gamma\frac{\partial \dot{\gamma}}{\partial r}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial r}) + [1+\Gamma\dot{\gamma}](\frac{\partial^2 v}{\partial x \partial r} + \frac{\partial^2 u}{\partial r^2}) \\
&+ 2[1+\Gamma\dot{\gamma}]\frac{\partial^2 u}{\partial x^2} + 2\Gamma\frac{\partial u}{\partial x}\frac{\partial \dot{\gamma}}{\partial x}] \qquad (4.29) \\
u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial r} &= \nu[\frac{2}{r}\frac{\partial v}{\partial r}[1+\Gamma\dot{\gamma}] + 2[1+\Gamma\dot{\gamma}]\frac{\partial^2 v}{\partial r^2} + 2\Gamma\frac{\partial v}{\partial r}\frac{\partial \dot{\gamma}}{\partial r} + \Gamma\frac{\partial \dot{\gamma}}{\partial x}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial r})] \\
&+ [1+\Gamma\dot{\gamma}](\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial r \partial x}) - \frac{2v}{r^2}[1+\Gamma\dot{\gamma}]] \qquad (4.30) \\
0 &= 0 \qquad (4.31)
\end{aligned}
$$

$Eq(4.31)$ shows that θ-component is identically satisfied. Here $\nu = \frac{\mu_\circ}{\rho}$ in which μ is dynamic viscosity and ν is kinematic viscosity. Now heat equation have to be calculated. By using $Eq(4.6)$ the material derivative of temperature T is

$$\frac{DT}{Dt} = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r} \qquad (4.32)$$

Apply the differential operator ∇ to the right side of $Eq(4.3)$ and get following expression

$$\nabla(k.\nabla T) = \frac{1}{r}\frac{\partial}{\partial r}(kr\frac{\partial T}{\partial r}) + \frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}) \qquad (4.33)$$

By putting $Eq(4.32) - (4.33)$ in $Eq(4.3)$ get following equation

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}(\alpha r\frac{\partial T}{\partial r}) + \frac{\partial}{\partial x}(\alpha\frac{\partial T}{\partial x}) + \frac{Q(T-T_\infty)}{\rho c_p} \qquad (4.34)$$

where $\alpha = \frac{k}{\rho c_p}$ and known as thermal diffusivity. Now apply the boundary layer approximation and according to its rules taking assumptions as

$$u = O(1),\ x = O(1),\ v = O(\delta),\ r = O(1),\ \Gamma = O(\delta),\ T = O(1),\ \nu = O(\delta^2),\ \alpha = O(\delta^2) \qquad (4.35)$$

Apply above orders in $Eq(4.20)$, $Eq(4.29)$ and $Eq(4.30)$ and after simplifying get the following reduced governing equations

$$
\begin{aligned}
\frac{\partial(rv)}{\partial r} + \frac{\partial(ru)}{\partial x} &= 0 \qquad (4.36) \\
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} &= \nu[\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{\Gamma}{\sqrt{2}r}(\frac{\partial u}{\partial r})^2 + \sqrt{2}\Gamma\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r^2}] \qquad (4.37) \\
u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r} &= \frac{1}{r}\frac{\partial}{\partial r}(\alpha r\frac{\partial T}{\partial r}) + \frac{Q(T-T_\infty)}{\rho c_p} \qquad (4.38)
\end{aligned}
$$

Respective boundary conditions are

$$u = U(x),\ v = 0,\ T = T_w\ at\ r = R$$

$$u \to 0,\ T \to T_\infty \quad r \to \infty \tag{4.39}$$

where $U(x) = \frac{U_\circ x}{l}$ denotes the stretching velocity in which $U_\circ$ is reference velocity and l, T_w and T_∞ are the characteristic length, surface temperature and the extreme temperature. We can introduce a stream function which satisfy the continuity equation, such that

$$u = \frac{1}{r}\frac{\partial \psi}{\partial r},\quad v = -\frac{1}{r}\frac{\partial \psi}{\partial x} \tag{4.40}$$

Similarity transformations for the governing equation can be defined as

$$\eta = \frac{r^2 - R^2}{2R}\sqrt{\frac{U}{\nu x}},\quad \psi = \sqrt{U\nu x}Rf(\eta) \tag{4.41}$$

$$\phi = \frac{T - T_w}{T_w - T_\infty},\quad \alpha = \alpha_\infty(1 + \varepsilon\phi) \tag{4.42}$$

In the above expression a_∞ is the thermal conductivity at a large distance away from the cylinder and ε is small number. $Eq(4.36)$ identically satisfied for these transformations but after applying above transformations to the governing equations that are $Eq(4.37 - 4.38)$ get following expressions

$$\begin{aligned} 2\gamma f'' + (1 + 2\eta\gamma)f''' + \frac{3}{\sqrt{2}}(1 + 2\eta\gamma)^{\frac{1}{2}}\gamma\lambda f''^2 & \\ +\sqrt{2}\lambda(1 + 2\eta\gamma)^{\frac{3}{2}}f''f''' + ff'' - f'^2 &= 0 \end{aligned} \tag{4.43}$$

$$\begin{aligned} \phi''(1 + 2\eta\gamma)(1 + \varepsilon\phi) + \phi'(2\gamma + P_r f + 2\epsilon\gamma\phi) & \\ +\phi'^2\varepsilon(1 + 2\eta\gamma) + P_r\phi\beta &= 0 \end{aligned} \tag{4.44}$$

Along with bundary conditions

$$f(0) = 0,\ f'(0) = 1,\ \phi(0) = 1,$$

$$f' \to 0,\ \phi \to 0\ at\ \eta \to \infty \tag{4.45}$$

In the above expressions the prime represents the derivative with respect to η. The dimensionless number Pr, γ, λ and β are the Prandtl number, curvature parameter, Weissenberg number and heat generation/absorption parameter, defined as

$$\gamma = \frac{1}{R}\sqrt{\frac{x\nu}{U}},\quad \lambda = \Gamma\sqrt{\frac{U^3}{\nu x}},\quad P_r = \frac{\nu}{\alpha_\infty},\ \beta = \frac{Qx}{\rho U c_p} \tag{4.46}$$

4.3 Skin friction coefficient and local Nusselt number

The skin friction coefficient of the considered surface is

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} \tag{4.47}$$

In the above expression τ_w represents the shear stress at the surface of cylinder is defined as for the Williamson fluid surface shear stress is defined as

$$\tau_w = \mu[\frac{\partial u}{\partial r} + \frac{\Gamma}{\sqrt{2}}(\frac{\partial u}{\partial r})^2]_{r=R} \tag{4.48}$$

The transformation defined for this problem also applied on shear stress and get the following expression

$$\tau_w = \mu[\frac{Ux}{l}\sqrt{\frac{U}{\nu l}}f''(0) + \frac{\Gamma}{\sqrt{2}}(\frac{Ux}{l}\sqrt{\frac{U}{\nu l}}f''(0))^2] \tag{4.49}$$

After putting the $Eq(4.49)$ in $Eq(4.47)$ get following expression

$$\frac{C_f Re_x^{\frac{1}{2}}}{2} = f''(0) + \frac{\lambda}{\sqrt{2}}f''^2(0) \tag{4.50}$$

Now the local Nusselt number of temperature distribution is

$$Nu_x = \frac{xq_w}{\alpha_\infty(T_w - T_\infty)} \tag{4.51}$$

where q_w is the measure of heat transfer at the surface of cylinder and defined as

$$q_w = -\alpha_\infty(\frac{\partial T}{\partial r})_{r=R} \tag{4.52}$$

After applying the defined transformation to q_w get the following expression

$$q_w = -\alpha_\infty(T_w - T_\infty)\sqrt{\frac{U}{\nu x}}\phi'(0) \tag{4.53}$$

Using $Eq(4.53)$ in $Eq(4.51)$ in order to get the expression for local Nusselt number that is

$$Nu_x Re_x^{-\frac{1}{2}} = -\phi'(0) \tag{4.54}$$

Here $Re_x = \frac{Ux}{\nu}$ is the Reynolds number.

4.4 Method of solution

Current problem have nonlinear equations in which velocity profile is of order three and temperature profile is of order two. Numerical technique that is shooting method is used to solve this nonlinear problem. A basic rule for the shooting method is to convert the higher order equations to first order equations so here the nonlinear ordinary differential equations are converted into system of five first order ordinary differential equations. Then this first order system of differential equations is solved by Runge-Kutta-Fehlberg method. The $Eqs(4.43) - (4.44)$ can be written as

$$f''' = \frac{f'^2 - f''[2\gamma + \frac{3\gamma\lambda}{\sqrt{2}}(1+2\eta\gamma)^{\frac{1}{2}} f'' + f]}{(1+2\eta\gamma)[1+\sqrt{2}\lambda(1+2\eta\gamma)^{\frac{1}{2}} f'']} \quad (4.55)$$

$$\phi'' = \frac{-\phi[2\gamma + P_r f + 2\varepsilon\gamma\phi] + \phi'^2\varepsilon[1+2\eta\gamma] + P_r\phi\beta}{[1+2\eta\gamma][1+\varepsilon\phi]} \quad (4.56)$$

Reduction of higher order equations to five first order differential equations is shown as

$$f = y_1,\ f' = y_2,\ f'' = y_3,\ f''' = y_3',\ \phi = y_4,\ \phi' = y_5,\ \phi'' = y_5' \quad (4.57)$$

System of first order simultaneous equations are

$$y_1' = y_2 \quad (4.58)$$

$$y_2' = y_3 \quad (4.59)$$

$$y_3' = \frac{y_2^2 - y_3[2\gamma + \frac{3\gamma\lambda}{\sqrt{2}}(1+2\eta\gamma)^{\frac{1}{2}} y_3 + y_1]}{(1+2\eta\gamma)[1+\sqrt{2}\lambda(1+2\eta\gamma)^{\frac{1}{2}} y_3]} \quad (4.60)$$

$$y_4' = y_5 \quad (4.61)$$

$$y_5' = \frac{-y_4[2\gamma + P_r y_1 + 2\varepsilon\gamma y_4] + y_5^2\varepsilon[1+2\eta\gamma] + P_r y_4\beta}{[1+2\eta\gamma][1+\varepsilon y_4]} \quad (4.62)$$

where prime represents the derivative with respect to η and the transformed boundary conditions are

$$y_1(0) = 0,\ y_2(0) = 1,\ y_2(\infty) \to 0, y_4(0) = 1, y_4(\infty) \to 0 \quad (4.63)$$

For solving the system of five first order ordinary differential equation, five initial guesses are required. In $Eq(4.63)$ two initial guesses are given for f and one is given for ϕ but two initial guesses are unknown becaues these are given at $\eta \to \infty$. To find out these unknowns the most important step is to assume the approximate value of η at∞. After choosing the initial guesses solve the problem's governing equations to find the value of $f''(0)$ and $\phi'(0)$. The process of solution is continued by considering another value of η at ∞ repeat the process until two consecutive values of $f''(0)$ and $\phi'(0)$ are differ only after the significant digits. This final value of η at ∞ is considered for the calculation of velocity profile f and temperature profile ϕ in the boundary layer for the measurement of all physical parameters. When all the initial conditions become known then Runge-Kutta method applied to solve the system of ODEs.

4.5 Results and discussions

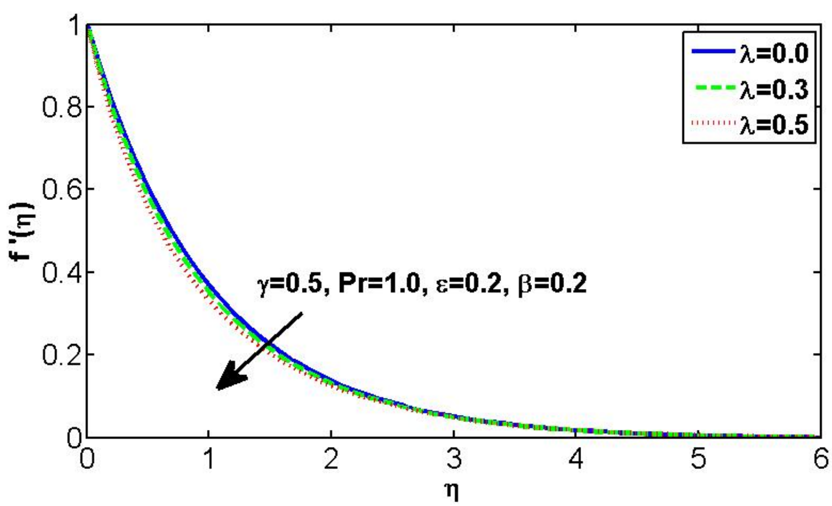

Figure 4.1: Velocity field f' for different values of λ.

Figure. 4.1 shows the effect of Weissenberg number λ on horizontal component of velocity. The velocity component approaches to zero asymptotically as $\eta \to \infty$, in this case velocity is free stream velocity. Also with the increases in λ the velocity component decreases. As we know that Weissenberg number λ is the ratio of relaxation time to specific process time therefore the decrease in specific process time will increase the Weissenberg number which depicts there is decrease in velocity component as well as decrease in boundary layer thickness and vice versa.

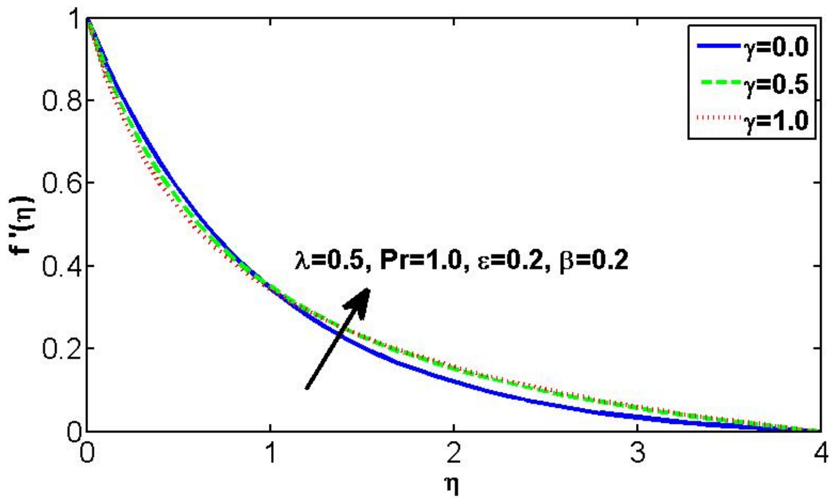

Figure 4.2: Velocity field f' for different values of γ.

Figure. 4.2 shows the change in velocity profile $f'(\eta)$ with respect to curvature parameter. It is analyzed that there is a small change in the dynamic region $[0, 0.75]$ and after this region that is within $[0.75, \infty)$ the velocity profile going to zero, that means that velocity becomes the free stream velocity in the region $[0.75, \infty]$. The behavior of velocity profile $f'(\eta)$ is increasing with the increase in curvature because with the increment in curvature parameter γ cause reduction in the radius of curvature which leads to the decrease in resistance to flow. Boundary layer thickness increases with the increase in curvature parameter.

Table 4.1: Skin friction for different values of γ and λ.

λ	γ	$f''(0)+\frac{\lambda}{\sqrt{2}}f''^2(0)$
0.1	0.1	-0.9776
	0.2	-1.0117
	0.3	-1.0456
0.2	0.1	-0.9118
	0.2	-0.9401
	0.3	-0.9680
0.3	0.1	0.8413
	0.2	-0.8634
	0.3	-0.8847

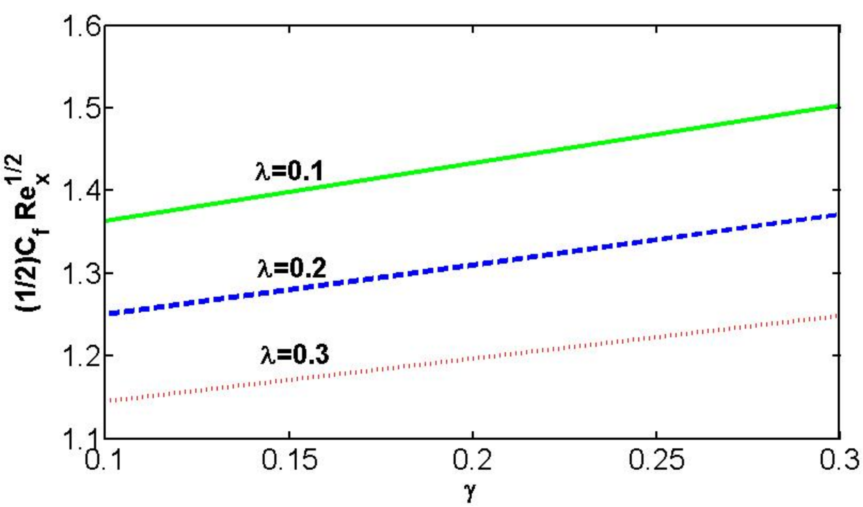

Figure 4.3: Skin friction for different values of γ and λ.

Analyzing the $Table$. 4.1, we see that $C_f \propto Re_x^{-\frac{1}{2}}$. Inverse relation shows that C_f decreases as Reynolds number increases. It is obvious that increase in Re_x causes the decrease in viscous forces, and due to this, there is reduction in $|C_f|$. In this case the dependence of C_f is on curvature parameter γ, we see that increase in γ causes increase of $|\frac{1}{2}C_f\sqrt{Re_x}|$ which means as γ increases, the viscous forces going to decrease. Graphical behavior is shown by $Figure$.4.3.

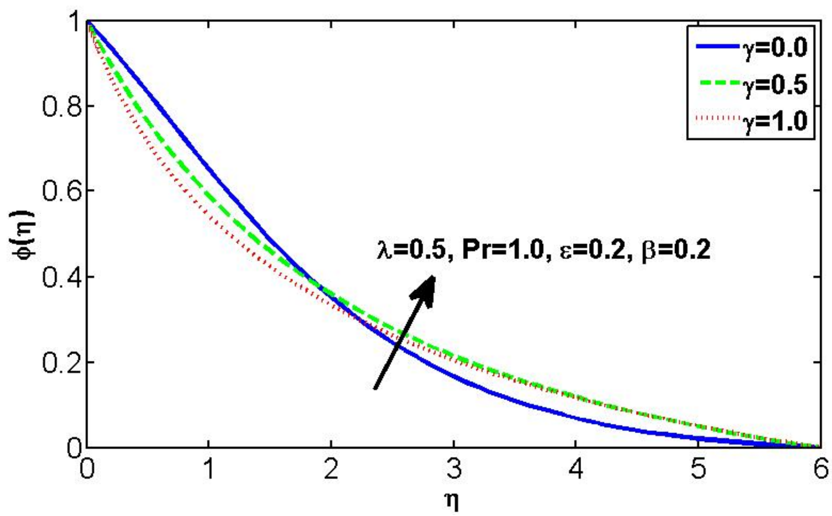

Figure 4.4: Influence of curvature parameter γ on temperature field.

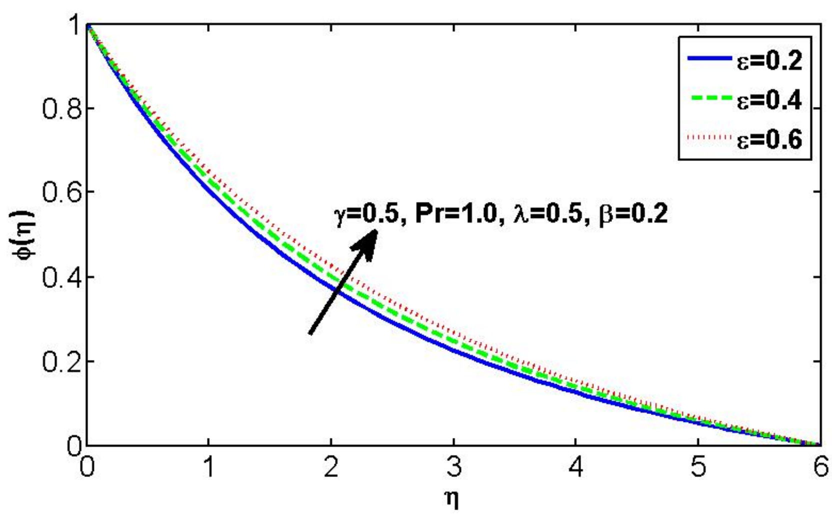

Figure 4.5: Influence of thermal conductivity ε on temperature profile

In $Figure$. 4.4, different values of curvature parameter γ shows that with the increase in curvature parameter there is increase in temperature profile and temperature boundary layer. So increase in γ accelerates the heat transfer. Thermal boundary layer increases with the increase in curvature parameter.

Figure. 4.5, shows that increase in thermal conductivity cause increase in temperature profile and temperature boundary layer which results spark in heat transfer.

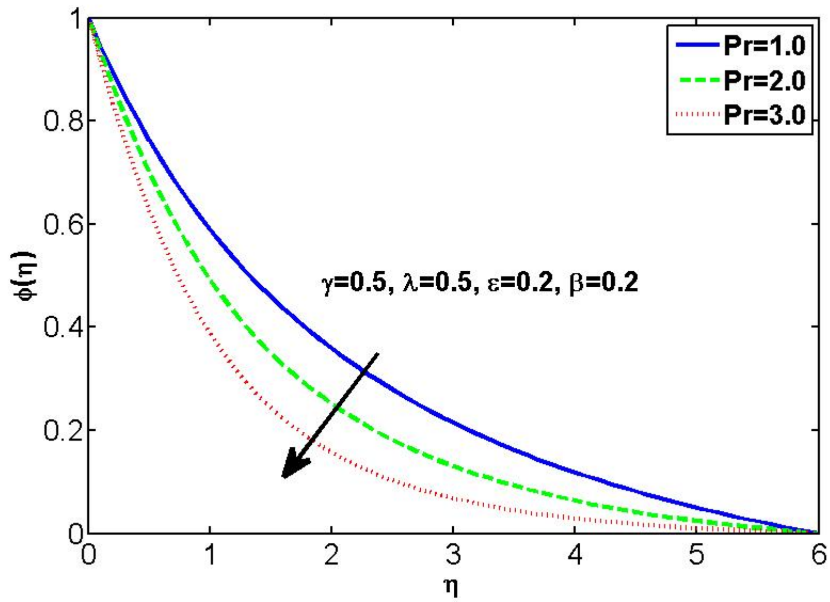

Figure 4.6: Influence of Prandtl number Pr on temperature profile.

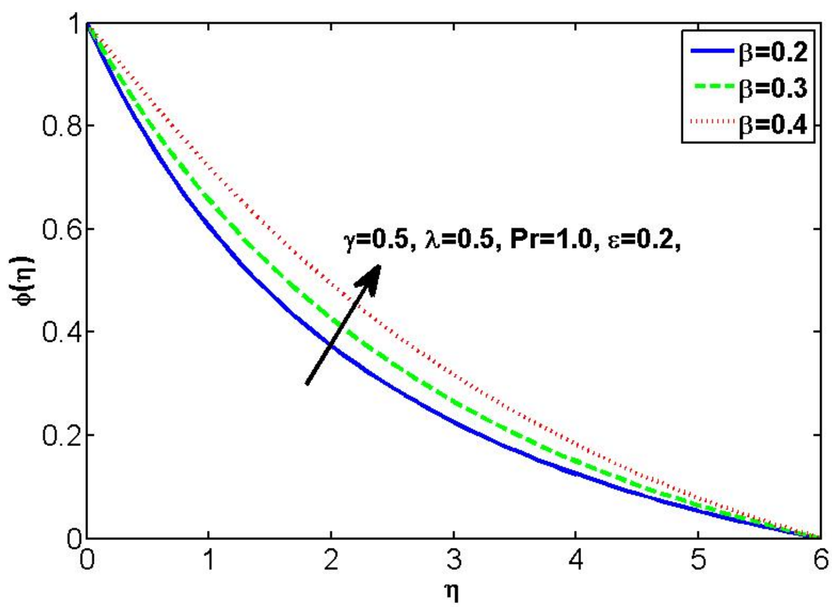

Figure 4.7: Influence of heat source $\beta > 0$ on temperature profile.

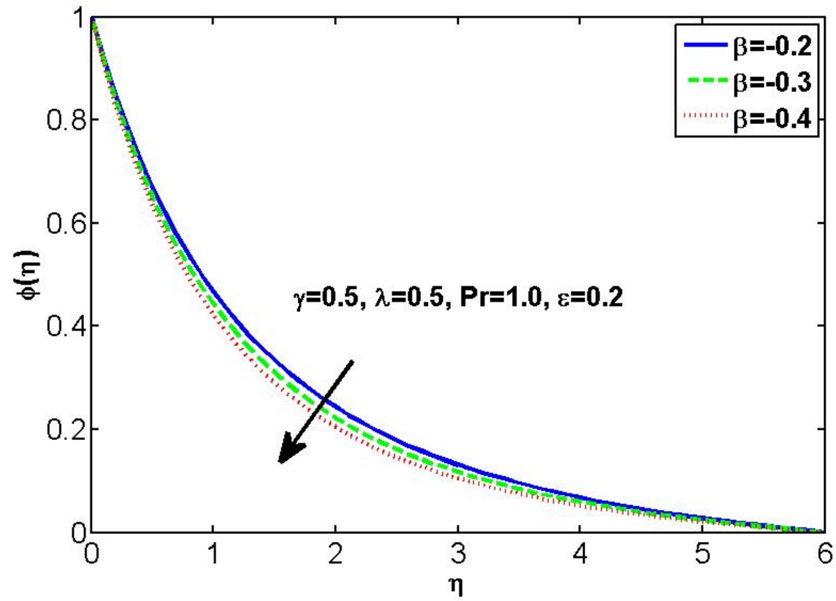

Figure 4.8: Influence of heat sink $\beta > 0$ on temperature profile.

Figure. 4.6, shows that increase in Prandtl number cause decrease in boundary layer thickness. In heat transfer problem Pr plays an important role to control the relative thickening of momentum and thermal boundary layer. Small value of Pr shows that heat diffuses quickly as compared to the velocity (momentum), which means that for liquid metals thermal boundary layer thickness is much greater than momentum boundary layer thickness. Therefore, Prandtl number can be used to increase or decrease the cooling rate in conducting flows.

Figure. 4.7, shows that the temperature field increases with the increase in heat source $\beta > 0$, because exothermic reactions occurred and heat released during these processes due to which heat of the system increases and thermal boundary layer increases.

Figure. 4.8, shows that temperature field decreases with the increase in heat sink $\beta < 0$, because endothermic reactions occurred and heat absorbed from the system due to which heat of system reduces and thermal boundary layer also reduces. This depicts that β has prime importance in heat transfer problems.

Table 4.2: Temperature gradient $-\phi(0)$ for different values of γ and β.

ε	Pr	β	γ	$-\phi'(0)$
0.1	1	0.1	0.1	0.5241
			0.2	0.5793
			0.3	0.6340
		0.2	0.1	0.4321
			0.2	0.4934
			0.3	0.5530
		0.3	0.1	0.3255
			0.2	0.3964
			0.3	0.4631

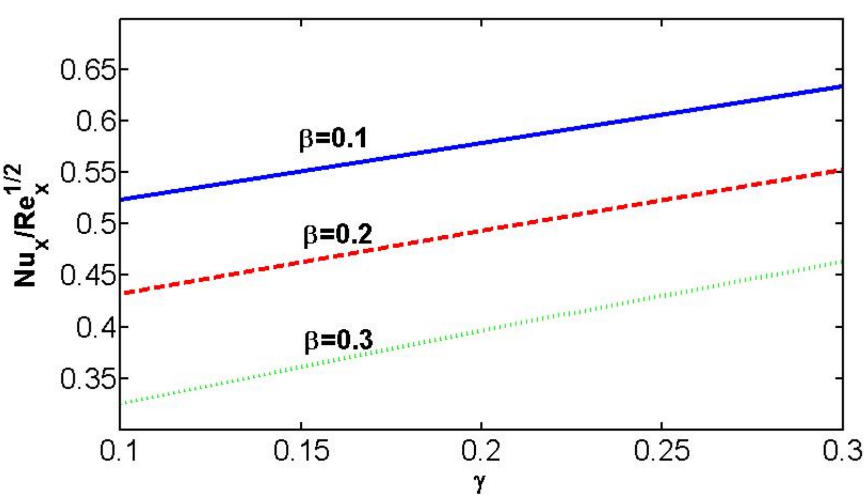

Figure 4.9: Heat transfer rate for different values of γ and β.

Table 4.3: Temperature gradient $-\phi(0)$ for different values of γ and Pr.

β	ε	Pr	γ	$-\phi'(0)$
0.1	0.1	1.0	0.1	0.5241
			0.2	0.5793
			0.3	0.6340
		2.0	0.1	0.7630
			0.2	0.7955
			0.3	0.8322
		3.0	0.1	0.9816
			0.2	1.0052
			0.3	1.0315

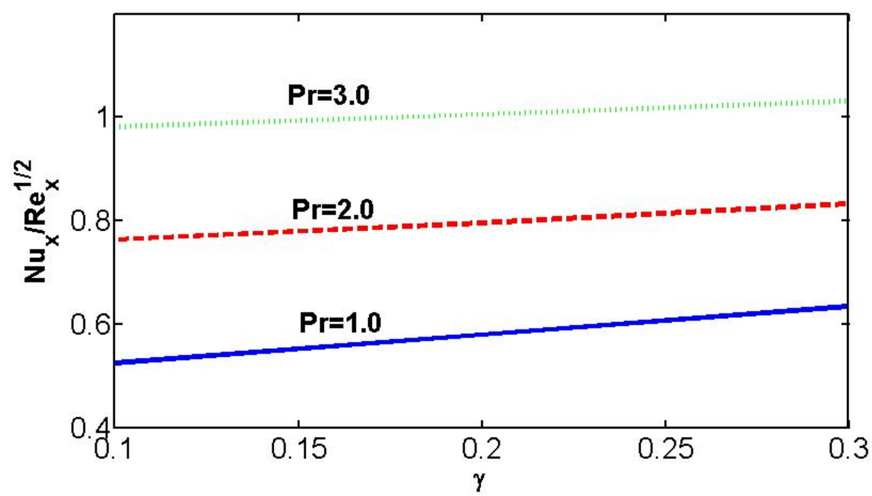

Figure 4.10: Heat transfer rate for different values of γ and Pr.

Table 4.4: Temperature gradient $-\phi(0)$ for different values of γ and ε.

β	Pr	ε	γ	$-\phi'(0)$
0.1	1	0.1	0.1	0.9816
			0.2	1.0052
			0.3	1.0315
		0.2	0.1	0.9212
			0.2	0.9440
			0.3	0.9695
		0.3	0.1	0.8692
			0.2	0.8913
			0.3	0.9162

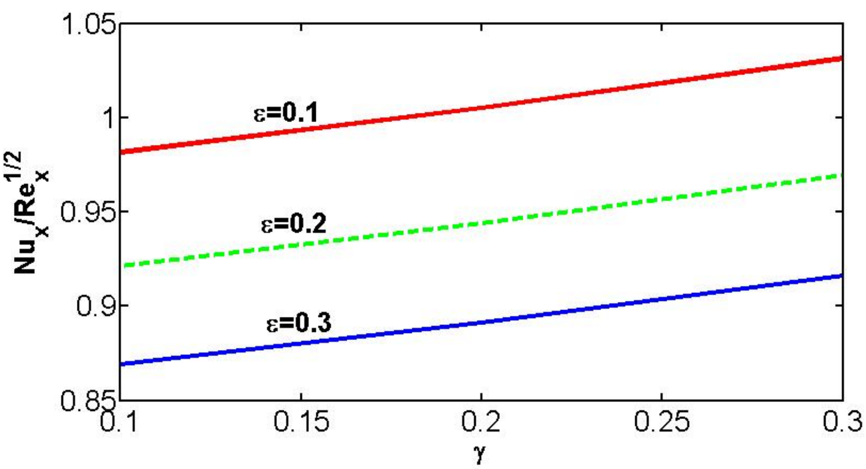

Figure 4.11: Heat transfer rate for different values of γ and ε.

$Eq(4.54)$ defines the relation between the coefficient of conventional heat transfer, i.e, Nusselt number Nu_x and Reynolds number Re_x, we note that $Nu_x \propto Re_x^{\frac{1}{2}}$ and $Re_x = \frac{Ux}{\nu}$, these relations shows that increase in viscosity decreases the Reynolds number and vice versa. So due to increase in curvature parameter γ viscosity decreases and Re_x increases which leads to increase in Nusselt number. And increase in Nu_x enhances the magnitude of rate of conventional heat transfer in case of stretching cylinder. Also, Nu_x depends on

β, Pr and ε, and their effects can be studied in $Table$ 4.2, 4.3 and 4.4. Graphical behavior shown by $Figure$. 4.9, 4.10 and 4.11.

$Table$. 4.2 shows the effect of heat source β on local Nusselt number. Increase in heat source β causes increase in temperature of fluid due to which heat transfer rate reduces from wall to fluid.

$Table$. 4.3 shows the effect of Prandtl number Pr on local Nusselt number. Increase in Pr causes decrease in temperature of fluid, which will create a temperature gradient between wall and fluid and heat transfer rate increases.

$Table$. 4.4 shows that increase in thermal conductivity ε reduces the local Nusselt number because conductivity increases the temperature of fluid due to which temperature difference reduces between wall and fluid and ultimately rate of heat transfer reduces.

4.6 Concluding remarks

There are different parameters involve in this problem. Especially curvature parameter plays an important role in affecting flow as well as temperature field. The Weissenberg number and Prandtl number controls the flow and temperature respectively, which is important for momentum and temperature study of different fluids.Variable thermal conductivity enhances the heat transfer. And reduction in viscosity of fluid also enhances the rate of conventional heat transfer.

- Increase in curvature parameter γ of stretching cylinder increase the velocity profile $f'(\eta)$ and temperature profile $\phi(\eta)$ both.
- Increase in Weissenberg number λ reduces the velocity profile $f'(\eta)$
- Increase in thermal conductivity parameter ε also enhances the temperature flow feild $\phi(\eta)$.
- Increase in Prandtl number Pr reduces the temperature profile $\phi(\eta)$.
- Heat generation/absorbtion parameter β shows two types of results, for $\beta > 0$ that is heat source enhances the temperature profile $\phi(\eta)$ and for $\beta < 0$ that is heat sink

reduces the temperature profile $\phi(\eta)$.

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