Solution of differential equation by variational iteration method

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A Dissertation Submitted in the partial fulfillment of the requirements for the degree of

MASTER OF PHILOSOPHY

IN

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CERTIFICATE

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We accept this dissertation as conforming to the required standard

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Dedicated

To

Ammi & Abbu

Who are the most precious gems
of my life.
Who've always given me perpetual love,
care, and cheers. Whose prayers have
always been a source of great
inspiration for me and whose
sustained hope in me led me
to where I stand today.

&

Then to my lovely Sisters, and very sweet brother.

Acknowledgement

All praises to almighty Allah, the most beneficent and the most merciful, who created this universe and gave us the idea to discover. I am highly grateful to Almighty Allah for His blessing, guidance and help in each and every step of my life. He blessed us with the **Holy Prophet Muhammad** (Sal-Allah-Hu-Alai-Hi Wa-aali-hi Wa-sallam), who is forever source of guidance and knowledge for humanity.

I express my deepest and heart-felt gratitude to my supervisor, **Dr. Sohail Nadeem** for his knowledgeable discussions, valuable guidance and inexhaustible inspiration throughout my research. His sympathetic attitude and encouragement prop up me to work harder with keen interest. What ever I had learned and achieved is all because of his superb support.

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My love and gratitude from the core of heart to my loving parents, caring brothers and sisters for their prayers, support and encouragement, who have always given me love, care and cheer and whose sustained hope in me led me to where I stand today.

I have no words to express my sweet sensation of thanks for my friends **Muhammad Ali, Majid, Zayafat, Zaheer, Qasim, salman.**

May Almighty Allah shower His choicest blessing and prosperity on all those who assisted me in any way during completion of my thesis.

Preface

The exact solution of linear and non-linear differential equation are very rare.Because in nature every thing is nonlinear and most of the practical applications have been model and developed linear and non-linear ordinary as well as partial differentials equations. There are many perturbation methods to solve non-linear differential equations.However, in perturbation method there should be small or large parameter and solution is only valid for small values of those parameters. Recently, various analytical methods have been developed to find the solution of linear and non-linear differential equations. Prof.He in 1999 has established a new method known as variational iteration method. The beauty of this method is, it is relatively simpler analytical method. After the initiation of this method various researcher have discuss this method.Mention may be made to the works of [1-20]. The reliability of the method and the reduction in the size of computational domain give this method as wider applicability. The method has wider applicability. This method gives rapidly convergent successive approximations of the exact solution, if such a solution exists . For concrete problems, a few numbers of approximations can be used for numerical purposes with high degree of accuracy. The VIM does not require specific transformations for non-linear terms as required by some existing techniques.

Keeping in mind the importance of VIM, the aim of the present dissertation is to solve few problems with the help of VIM. The dissertation is arranged as follow:

In chapter one, two problems of standard Blasius quations have been solved with VIM.

Chapter two is devoted to the study of MHD boundary layer problem of viscous fluid with the help of VIM.

Contents

Chapter 1

The variational iteration method for solving two forms of Blasius Equations on half infinite domain

1.1 Introduction

Blasius equation is one of the basic equations of fluid dynamics[1]. Blasius equation describes the velocity profile of the fluid[2] in the boundary layer theory on a half-infinite interval. The variational iteration method is applied for a reliable treatment of two forms of the third order nonlinear Blasius equation which comes from boundary layer[3] equations. The series solution is obtained without restrictions on the nonlinearity behavior. The obtained series solution is combined with the diagonal Pade approximants to handle the boundary condition at infinity for only one of these forms. This work is due to Abdul-Majid Wazwaz [5]. The necessary details

[1] Fluid dynamics is the sub-discipline of fluid mechanics dealing with fluids (liquids and gases) in motion. It has several subdisciplines itself, including aerodynamics (the study of gases in motion) and hydrodynamics (the study of liquids in motion).

[2] A fluid is defined as a substance that continuously deforms (flows) under an applied shear stress (stress along the tangent) regardless of the magnitude of the applied stress.

[3] A boundary layer is that layer of fluid in the immediate vicinity of a bounding surface. In the Earth's atmosphere, the planetary boundary layer is the air layer near the ground affected by diurnal heat, moisture or momentum transfer to or from the surface. On an aircraft wing the boundary layer is the part of the flow close to the wing. The boundary layer effect occurs at the field region in which all changes occur in the flow pattern. The boundary layer distorts surrounding nonviscous flow. It is a phenomenon of viscous forces. This effect is related to the Reynolds number.

in the paper are incorporated in this chapter.

1.2 Variational Iteration Method

To illustrate the basic idea of varational iteration method, we consider the general nonlinear differential equation as follows

$$L[u(x)] + N[u(x)] = g(x), \tag{1.1}$$

where L and N are linear and nonlinear operators respectively and $g(x)$ is the given continous function. The basic idea of the method is to construct a correction function as

$$u_{n+1}(x) = u_n(x) + \int_{x_0}^{x} \lambda[Lu_n(t) + N\widetilde{u_n}(t) - g(t)]dt, \quad n \geq 0, \tag{1.2}$$

in which λ is a general lagrange multiplier which can be identified optimally via variational theory, u_n is the nth approximate solution and $\widetilde{u}_n$denotes a restricted variation i.e. $\delta\widetilde{u}_n = 0$.

For linear problems, its exact solution can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified. Here an analytic treatment will be approached to find the numerical values of $u''(0)$ for the both Blasius equations. We will apply Padé approximants for numerical approximation on series, which is obtained after applying variational iteration method.

1.3 The first form of Blasius equation

The first form of Blasius equation along with its boundary conditions are defined as

$$u'''(x) + \frac{1}{2}u(x)u''(x) = 0, \tag{1.3}$$

$$u(0) = 0, \quad u'(0) = 1, \quad u'(\infty) = 0.$$

To solve the above boundary value problem with the help of variational iteration method, we first define the initial conditions. Since the differential equation is of third order we need three initial conditions, two initial conditions are already defined and one extra initial condition can be written as

$$u''(0) = A.$$

In the above condition A is constant which is that appropiate value at which the first derivative at infinity is zero. The correction functional for (1.3) takes the following form

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)(\frac{\partial^3 u_n(\xi)}{\partial \xi^3} + \frac{1}{2}\ddot{u}_n(\xi)\frac{\partial^2 \ddot{u}_n(\xi)}{\partial \xi^2})d\xi. \tag{1.4}$$

Using the concept of lagrange multiplier, the above equation can be written as

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\frac{\partial^2 u_n(\xi)}{\partial \xi^2} \mid_0^x - \int_0^x \lambda' \frac{\partial^2 u_n}{\partial \xi^2} d\xi. \tag{1.5}$$

With the help of integration by parts, we arrive at

$$u_{n+1}(x) = u_n(x) + \lambda(x)\frac{\partial^2 u_n(x)}{\partial \xi^2} - \lambda'(x)\frac{\partial u_n(x)}{\partial \xi} + \lambda''(x)u_n(x) - \int_0^x \lambda''' u_n(\xi)d\xi. \tag{1.6}$$

From Eq.(1.6), we can calculate the value of lagrange multiplier λ as

$$\lambda'''_{\xi=x} = 0. \tag{1.7}$$

The corresponding boundary conditions from Eq.(1.6) can be written as

$$1 + \lambda'' \mid_{\xi=x} = 0,$$

$$\lambda' \mid_{\xi=x} = 0,$$

$$\lambda \mid_{\xi=x} = 0. \tag{1.8}$$

The solution of Eq.(1.7) satisfying the boundary conditions (1.8) can be directly written as

$$\lambda = -\frac{(\xi - x)^2}{2}. \tag{1.9}$$

Substituting the value of λ in Eq.(1.4) , we obtain

$$u_{n+1}(x) = u_n(x) - \frac{1}{2}\int_0^x (\xi - x)^2 (\frac{\partial^3 u_n(\xi)}{\partial \xi^3} + \frac{1}{2}\tilde{u}_n(\xi)\frac{\partial^2 \tilde{u}_n(\xi)}{\partial \xi^2}) d\xi, \qquad n \geq 0. \tag{1.10}$$

To solve Eq. (1.10), we select the following initial guess which satisfy the imposed initial conditions

$$u_0 = x + \frac{1}{2}Ax^2. \tag{1.11}$$

Using (1.10), we obtain the successive approximations

$$\begin{aligned}
u_1(x) &= -\frac{1}{240}A^2x^5 - \frac{1}{48}Ax^4 + \frac{1}{2}Ax^2 + x, && (1.13)\\
u_2(x) &= -\frac{1}{5702400}A^4x^{11} - \frac{1}{518400}A^3x^{10} - \frac{1}{193536}A^2x^9 + \frac{11}{161280}A^3x^8 \\
&\quad + \frac{11}{20160}A^2x^7 + \frac{1}{960}Ax^6 - \frac{1}{240}A^2x^5 - \frac{1}{48}Ax^4 + \frac{1}{2}Ax^2 + x, && (1.14)\\
u_3(x) &= -\frac{x^{23}A^8}{6282355064832000} - \frac{x^{22}A^7}{273145872384000} - \frac{197x^{21}A^6}{6290632212480000} + \\
&\quad (-\frac{1}{8472231936000}A^5 + \frac{83}{571875655680000}A^7)x^{20} + (\frac{83}{28593782784000}A^6 \\
&\quad -\frac{1}{6049173602304}A^4)x^{19} + \frac{14057x^{18}A^5}{65665713408000} + (\frac{1829}{26531463168000}A^4 \\
&\quad -\frac{5449}{125076897792000}A^6)x^{17} + (-\frac{5449}{7357464576000}A^5 + \frac{17}{208089907200}A^3)x^{16} \\
&\quad -\frac{1147x^{15}A^4}{253609574400} + (\frac{10033}{1394852659200}A^5 - \frac{967}{84536524800}A^3)x^{14} \\
&\quad +(\frac{10033}{99632332800}A^4 - \frac{1}{105431040}A^2)x^{13} + \frac{1157x^{12}A^3}{2554675200} + (-\frac{5}{4257792}A^4 \\
&\quad +\frac{23}{35481600}A^2)x^{11} - \frac{5x^{10}A^3}{387072} - \frac{43x^9A^2}{967680} + (\frac{11}{161280}A^3 - \frac{1}{21504}A)x^8 \\
&\quad +\frac{11x^7A^2}{20160} + \frac{x^6A}{960} - \frac{x^5A^2}{240} - \frac{x^4A}{48} + \frac{x^2A}{2} + x + \ldots && (1.15)
\end{aligned}$$

It is obvious that only three iterations are used to obtain the approximation $u_3(x)$. Other methods require many iterations to get this result.

1.4 The Padé approximants

To study the mathematical behavior of $u(x)$, it is normal to derive approximation for $u''(0) = A > 0$. It was formally shown by $[15-18]$ that this goal can be achieved by forming Pade' approximants [19] which have the advantage of manipulating the polynomial approximation into a rational function to gain more information about $u(x)$. It is well- known that Pade' approximants will converge on the entire real axis if $u(x)$ is free of singularities on the real axis . Moreover, it is to be noted that Pade'-finding algorithms are built-in utilities in most manipulation languages such as Maple and Mathematica.

More importantly, the diagonal approximant is the most accurate approximant, therefore we will construct only diagonal approximants in the following discussions. Using the boundary condition $u'(\infty) = 0$, the diagonal approximant $[M/M]$ vanishes if the coefficient of x with the highest power in the numerator vanishes. The diagonal approximants will be determined for $u'(x)$ where

Table 1

Pade approximants	$A = u''(0)$
[2/2]	0.5773502693
[3/3]	0.516397793
[4/4]	0.5227030796
[5/5]	Complex numbers
[6/6]	0.5217102130
[7/7]	0.5026354150
[8/8]	Complex umbers
[9/9]	Complex numbers
[10/10]	0.4672639966
[11/11]	0.5176098151

$$\begin{aligned} u'(x) &= -\frac{A^8x^{22}}{273145872384000} - \frac{A^7x^{21}}{12415721472000} - \frac{197A^6x^{20}}{299553914880000} + (-\frac{1}{423611596800}A^5 \\ &+ \frac{83}{28593782784000}A^7)x^{19} + (-\frac{1}{318377558016}A^4 + \frac{83}{1504935936000}A^6)x^{18} \\ &+ \frac{14057A^5x^{17}}{3648061856000} + (\frac{1829}{1560674304000}A^4 - \frac{5449}{7357464576000}A^6)x^{16} \\ &+ \frac{17}{13005619200}A^3 - \frac{5449}{459841536000}A^5)x^{15} - \frac{1147A^4x^{14}}{16907304960} + (\frac{10033}{99632332800}A^5 \\ &- \frac{967}{6038323200}A^3)x^{13} + (\frac{10033}{7664025600}A^4 - \frac{1}{8110080}A^2)x^{12} + \frac{1157A^3x^{11}}{212889600} \\ &+ (\frac{23}{3225600}A^2 - \frac{5}{387072}A^4)x^{10} - \frac{25A^3x^9}{193536} - \frac{43A^2x^8}{107520} \\ &+ (-\frac{1}{2688}A + \frac{11}{20160}A^3)x^7 + \frac{11A^2x^6}{2880} + \frac{Ax^5}{160} - \frac{A^2x^4}{48} - \frac{Ax^3}{12} + Ax + 1. \end{aligned}$$

Using the Mathematica built -in utilities to solve the resulting polynomials gives the values of the initial slope $u''(0) = A$ listed in above table. The best approximation is $A = u''(0) = 0.5227030796$.

1.5 The second form of Blasius equation

The second form of Blasius equation along with the boundary conditions take the form

$$u'''(x) + \frac{1}{2}u(x)u''(x) = 0, \tag{1.15}$$

$$u(0) = 0, \quad u'(0) = 0, \quad u'(\infty) = 1.$$

Let us introduce new variables

$$\begin{aligned} y(x) &= Bu(Bx), \\ u(x) &= \frac{1}{B}y(\frac{1}{B}x). \end{aligned} \tag{1.16}$$

In which B is an unknown which we will determine. with the help of Eq.(1.16) , Eq (1.15) take the following form

$$y'''(x) + \frac{1}{2}y(x)y''(x) = 0, \tag{1.17}$$

$$y(0) = 0, \quad y'(0) = 0, \quad y'(x) = B^2u'(Bx) \to B^2 \ as\ x \to \infty.$$

Incorporating a new condition $y''(0) = 1$, $u''(0) = \frac{1}{B^3}$. From this conditions it is obvious that

$$B = \sqrt{y'(\infty)}. \tag{1.18}$$

The correction functional for (1.17) is defined as

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi)(\frac{\partial^3 \tilde{y}_n(\xi)}{\partial \xi^3} + \frac{1}{2} y_n(\xi) \frac{\partial^2 \tilde{y}_n(\xi)}{\partial \xi^2}) d\xi. \tag{1.19}$$

Using the same procedure as discussed in previous section, the lagrange multiplier λ can be calculated from the following problem

$$\lambda'''_{\xi=x} = 0,$$

$$1 + \lambda'' \mid_{\xi=x} = 0,$$

$$\lambda' \mid_{\xi=x} = 0,$$

$$\lambda \mid_{\xi=x} = 0.$$

The above problem gives

$$\lambda = -\frac{(\xi - x)^2}{2}. \tag{1.20}$$

Substituting this value of the Lagrangian multiplier in to the functional (1.19), we obtain the following formula

$$y_{n+1}(x) = y_n(x) - \frac{1}{2} \int_0^x (\xi - x)^2 (\frac{\partial^3 y_n(\xi)}{\partial \xi^3} + \frac{1}{2} y_n(\xi) \frac{\partial^2 y_n(\xi)}{\partial \xi^2}) d\xi, \quad n \geq 0. \tag{1.21}$$

To solve Eq.(1.21), we select the following initial guess for y_0

$$y_0(x) = \frac{1}{2} x^2.$$

With the help of this initial guess the next iterations can be calculated for n $=0, 1, 2, \ldots\ldots.$which take the following form

$$
\begin{aligned}
y_1(x) &= \frac{1}{2}x^2 - \frac{1}{240}x^5, \\
y_2(x) &= \frac{1}{2}x^2 - \frac{1}{240}x^5 + \frac{11}{161280}x^8 - \frac{1}{5702400}x^{11}, \\
y_3(x) &= \frac{1}{2}x^2 - \frac{1}{240}x^5 + \frac{11}{161280}x^8 - \frac{5}{4257792}x^{11} + \frac{10033}{1394852659200}x^{14} - \\
&\frac{5449}{125076897792000}x^{17} + \frac{83}{571875655680000}x^{20} - \frac{1}{6282355064832000}x^{23}. \quad (1.22)
\end{aligned}
$$

It is worth noting that only three iterations are used to obtain the approximation $y_3(x)$. Considering $y = y_3(x)$ as the best approximation, thus

$$
\begin{aligned}
y'(x) &= x - \frac{1}{48}x^4 - \frac{11}{20160}x^7 + \frac{5}{387072}x^{10} + \frac{10033}{99632332800}x^{13} \\
&-\frac{5449}{7357464576000}x^{16} + \frac{5449}{28593782784000}x^{19} - \frac{1}{273145872384000}x^{22}. \quad (1.23)
\end{aligned}
$$

With the help of Eqs.(1.23) , (1.16) and using the Pade -approximation the best values of B are defined in Table 2 as

Table 2

x	B	$u''(0)$
2.0	1.313034017	0.4417454320
2.2	1.347736192	0.4084936660
2.4	1.373000106	0.3863565574
2.6	1.387743095	0.3741732832
2.8	1.388836100	0.3732905625

The series solution for the second form of Blasius equation is given as

$$
\begin{aligned}
u(x) &= \frac{x^2}{2B^3} - \frac{x^5}{240B^6} + \frac{11}{161280B^9}x^8 - \frac{5}{4257792B^{12}}x^{11} + \frac{10033}{1394852659200B^{15}}x^{14} \\
&-\frac{5449}{125076897792000B^{18}}x^{17} + \frac{83}{571875655680000B^{21}}x^{20} \\
&-\frac{1}{6282355064832000B^{24}}x^{23}, \quad (1.24)
\end{aligned}
$$

in which B is approximated by 1.388836100 from Table 2, and $u''(0) = 0.3732905625$.

Fig.1

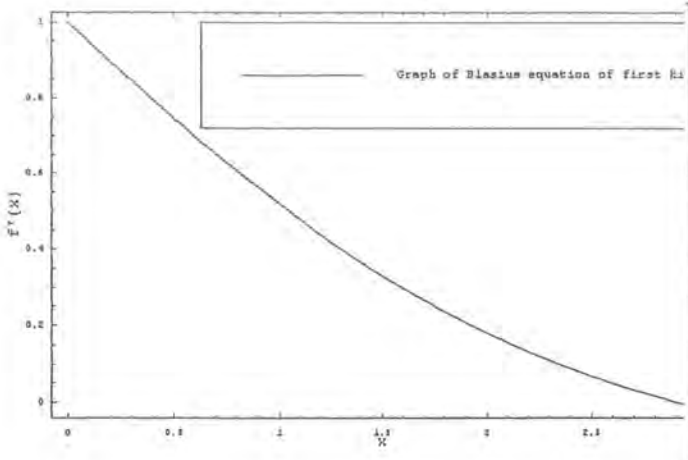
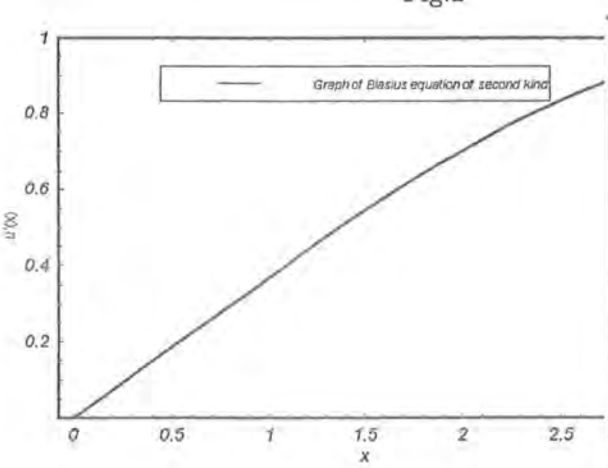


Fig.2



Chapter 2

Solution of boundary layer problem with the help of variational iterational method

2.1 Introduction

In this chapter,we have presented the solution of stretching problem with the help of variational iteration method. We have taken the general non-dimensional boundary layer equation of viscous fluid in the presence of magnetic field with the corresponding boundary conditions of linear stretching, nonlinear stretching and exponential stretching.

2.2 Mathematical formulations

We consider an incompressible and MHD[1] viscous fluid [2] bounded by a stretching sheet at $y = 0$. A non-uniform magnetic field $B(x)$ is applied normal to the non-linear stretching sheet. We further assume that an induced magnetic field is negligible. The boundary layer equations

[1] Magnetohydrodynamics (MHD) is the study of the interaction of conducting fluids with electromagnetic phenomena.

[2] A fluid whose viscosity is sufficiently large to make the viscous forces a significant part of the total force field in the fluid.

for viscous incompressible flow[3] are defined as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0, \tag{2.1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B^2(\eta)}{\rho}. \tag{2.2}$$

In above expressions, u and v indicate the velocity components in the x and y directions respectively , v is the kinematic viscosity, ρ is the fluid density , σ is the electrical conductivity of the fluid and the value of $B(x)$ is taken as

$$B(x) = B_o x^{\frac{(m-1)}{2}}, \tag{2.3}$$

in which $m = 1$, corresponds to the case of constant magnetic field.

The relevant boundary conditions for non linear sheet are defined as

$$u(x,0) = cx^m, v(x,0) = 0, u(x,y) \to 0 \text{ as } y \to \infty. \tag{2.4}$$

Introducing the following similarity transformations

$$\eta = \sqrt{\frac{c(m+1)}{2v}}x^{\frac{m-1}{2}}y, u = cx^m f'(\eta),$$

$$v = -\sqrt{\frac{cv(m+1)}{2}}x^{\frac{m-1}{2}}[f(\eta)+\frac{m-1}{m+1}\eta f'(\eta).$$

With the help of these transformations, the continuity equation (2.1) is identically satisfied and the momentum equation take the following form

$$f'''(\eta) - \beta f'^2(\eta) + f(\eta)f''(\eta) - Mf'(\eta) = 0 \tag{2.5}$$

[3]A flow, in which the volume and thus the density of the flowing fluid does not change during the flow.All the liquids are generally,considered to have incompressible flow.

The coressponding boundary condiions take the following form

$$f(0) = 0, f'(0) = 1, f' \to 0 \text{ as } \eta \to \infty, \tag{2.6}$$

where

$$\beta = \frac{2m}{1+m}, M = \frac{2\sigma B_o^2}{\rho c(1+m)}.$$

Note that in above equation $\beta = 1$ corresponds to linear stretching , $\beta = 2$ for exponential stretching and $\beta > 2$ corresponds to nonlinear stetching

2.3 Solution by variational iterational method

For the solution procedure we write Eq.(2.5) as

$$f_{n+1}(\eta) = f_n(\eta) + \int_0^\eta \lambda(\xi)(\frac{\partial^3 f_n(\xi)}{\partial\xi^3} - \beta\frac{\partial \bar{f}_n^2(\xi)}{\partial\xi} + \bar{f}_n(\xi)\frac{\partial^2 \bar{f}_n(\xi)}{\partial\xi^2} - M\frac{\partial \bar{f}_n(\xi)}{\partial\xi})d\xi. \tag{2.7}$$

Using integration by parts in above equation, we obtain the following problem to calculate the value of lagrange multiplier λ

$$\lambda'''_{\xi=\eta} = 0, \tag{2.7a}$$

with the conditions

$$1 + \lambda'' |_{\xi=\eta} = 0,$$

$$\lambda' |_{\xi=\eta} = 0,$$

$$\lambda |_{\xi=\eta} = 0.$$

The solution of λ can be directly written as

$$\lambda = -\frac{(\xi-\eta)^2}{2}. \tag{2.8}$$

Substituting this value of the Lagrangian multiplier into the Eq (2.7), we obtain

$$f_{n+1}(\eta) = f_n(\eta) - \frac{1}{2}\int_0^\eta (\xi-\eta)^2\left(\frac{\partial^3 f_n(\xi)}{\partial \xi^3} - \beta\frac{\partial \bar{f}_n^2(\xi)}{\partial \xi} + \bar{f}_n(\xi)\frac{\partial^2 \bar{f}_n(\xi)}{\partial \xi^2} - M\frac{\partial \bar{f}_n(\xi)}{\partial \xi}\right)d\xi, \qquad n \geq 0. \tag{2.9}$$

To find the solution of Eq.(2.9), we require initial guess which satisfy the conditions, thus we choose

$$f_0(\eta) = \eta + \frac{C\eta^2}{2}. \tag{2.10}$$

With the help of Eq.(2.10), for n =0, 1, 2, ..., Eq (2.9) give the following solutions

$$f_1(\eta) = \eta + \frac{C\eta^2}{2} + \frac{M\eta^3}{6} - \frac{C\eta^4}{48} + \frac{1}{24}CM\eta^4 - \frac{C^2\eta^5}{240} + \frac{\eta^3\beta}{6} + \frac{1}{12}C\eta^4\beta + \frac{1}{60}C^2\eta^5\beta,$$

$$\begin{aligned} f_2(\eta) \;=\; & \eta + \frac{C\eta^2}{2} + \frac{M\eta^3}{6} - \frac{C\eta^4}{48} + \frac{1}{24}CM\eta^4 - \frac{C^2\eta^5}{240} - \frac{M\eta^5}{120} + \frac{M^2\eta^5}{120} + \frac{C\eta^6}{960} - \frac{1}{180}CM\eta^6 \\ & + \frac{1}{720}CM^2\eta^6 + \frac{11}{20160}C^2\eta^7 - \frac{C^2M\eta^7}{1260} - \frac{M^2\eta^7}{2520} + \frac{11C^3\eta^8}{161280} + \frac{CM\eta^8}{10752} - \frac{CM^2\eta^8}{5376} \\ & - \frac{C^2\eta^9}{193536} + \frac{C^2M\eta^9}{25920} - \frac{C^2M^2\eta^9}{48384} - \frac{C^3\eta^{10}}{518400} + \frac{C^3M\eta^{10}}{259200} - \frac{C^4\eta^{11}}{5702400} + \frac{\eta^3\beta}{6} + \frac{1}{12}C\eta^4\beta \\ & - \frac{\eta^5\beta}{120} + \frac{1}{60}C^2\eta^5\beta + \frac{1}{40}M\eta^5\beta - \frac{1}{120}C\eta^6\beta + \frac{1}{72}CM\eta^6\beta - \frac{1}{315}C^2\eta^7\beta - \frac{M\eta^7\beta}{1260} \\ & + \frac{1}{504}C^2M\eta^7\beta + \frac{1}{840}M^2\eta^7\beta + \frac{C\eta^8\beta}{10752} - \frac{C^3\eta^8\beta}{2520} - \frac{13CM\eta^8\beta}{16128} + \frac{CM^2\eta^8\beta}{2016} + \frac{53C^2\eta^9\beta}{725760} \\ & - \frac{13C^2M\eta^9\beta}{51840} + \frac{C^2M^2\eta^9\beta}{18144} + \frac{7C^2\eta^{10}\beta}{345600} - \frac{13C^3M\eta^{10}\beta}{518400} + \frac{7C^4M\eta^{10}\beta}{3801600} + \frac{\eta^5\beta^2}{60} + \frac{C\eta^6\beta^2}{72} \\ & - \frac{\eta^7\beta^2}{2520} + \frac{1}{252}C^2\eta^7\beta^2 + \frac{1}{420}M\eta^7\beta^2 - \frac{5C\eta^8\beta^2}{8064} + \frac{C^3\eta^8\beta^2}{2016} + \frac{1}{672}CM\eta^8\beta^2 - \frac{37C^2\eta^9\beta^2}{120960} \\ & + \frac{C^2M\eta^9\beta^2}{2592} - \frac{C^3M\eta^{10}\beta^2}{14400} + \frac{C^3M\eta^{10}\beta^2}{25920} - \frac{C^4\eta^{11}\beta^2}{158400} + \frac{\eta^7\beta^3}{840} + \frac{C\eta^8\beta^3}{1008} + \frac{C^2\eta^9\beta^3}{2592} \\ & + \frac{C^3\eta^{10}\beta^3}{12960} + \frac{C^4\eta^{11}\beta^3}{142560}, \end{aligned}$$

$$
\begin{aligned}
f_3(\eta) \;=\; & \eta+\frac{C\eta^2}{2}+\frac{M\eta^3}{6}-\frac{C\eta^4}{48}+\frac{1}{24}CM\eta^4-\frac{C^2\eta^5}{240}-\frac{M\eta^5}{120}+\frac{M^2\eta^5}{120}+\frac{C\eta^6}{960}-\frac{1}{180}CM\eta^6\\
&+\frac{1}{720}CM^2\eta^6+\frac{11}{20160}C^2\eta^7+\frac{M\eta^7}{2520}-\frac{C^2M\eta^7}{1260}-\frac{M^2\eta^7}{1008}+\frac{M^3\eta^7}{5040}-\frac{C\eta^8}{21504}\\
&+\frac{11C\eta^8}{161280}+\frac{CM\eta^8}{2016}-\frac{13CM^2\eta^8}{26880}+\frac{CM^3\eta^8}{40320}-\frac{43C^2\eta^9}{967680}+\frac{3C^2M\eta^9}{17920}+\frac{19M^2\eta^9}{362880}\\
&-\frac{13}{241920}C^2M^2\eta^9-\frac{M^3\eta^9}{24192}-\frac{5}{387072}C^3\eta^{10}-\frac{49CM\eta^{10}}{4147200}+\frac{3C^3M\eta^{10}}{179200}+\\
&\frac{89CM^2\eta^{10}}{1814400}-\frac{5C^4\eta^{11}}{4257792}-\frac{11}{1209600}C^2M\eta^{11}-\frac{M^2\eta^{11}}{1425600}+\frac{2117C^2M^2\eta^{11}}{159667200}\\
&+\frac{M^3\eta^{11}}{332640}-\frac{113C^2M^3\eta^{11}}{79833600}-\frac{M^4\eta^{11}}{1425600}+\frac{1157C^2\eta^{12}}{2554675200}+\frac{CM\eta^{12}}{6082560}-\frac{8413C^3M\eta^{12}}{3832012800}\\
&-\frac{2011CM^2\eta^{12}}{1277337600}+\frac{2117C^3M^2\eta^{12}}{1916006400}+\frac{1381CM^3\eta^{12}}{638668800}-\frac{CM^4\eta^{12}}{4561920}-\frac{C^2\eta^{13}}{105431040}+\\
&\frac{10033}{99632332800}C^4\eta^{13}+\frac{251}{2348236800}C^3M^2\eta^{14}+\frac{2671C^2M\eta^{13}}{11070259200}-\frac{8413C^4M\eta^{13}}{49816166400}-\\
&-\frac{1871C^2M^2\eta^{13}}{2264371200}-\frac{31M^3\eta^{13}}{518918400}+\frac{367C^2M^3\eta^{13}}{754790400}+\frac{31M^4\eta^{13}}{518918400}-\frac{C^2M^4\eta^{13}}{59304960}\\
&-\frac{967\eta^{14}C^3}{84536524800}+\frac{10033C^5\eta^{14}}{1394852659200}+\frac{11CM^2\eta^{14}}{541900800}\frac{3421C^3M^2\eta^{14}}{211341131200}-\frac{541CM^3\eta^{14}}{7044710400}+\\
&\frac{367C^3M^3\eta^{14}}{10567065600}+\frac{127CM^4\eta^{14}}{3522355200}-\frac{1147}{253609574400}C^4\eta^{15}-\frac{571C^2M\eta^{15}}{253609574400}+\frac{109C^4M\eta^{15}}{5748019200}\\
&+\frac{1123C^2M^2\eta^{15}}{54344908800}-\frac{3421C^4M^2\eta^{15}}{317011968000}-\frac{5987C^2M^3\eta^{15}}{190207180800}-\frac{M^4\eta^{15}}{825552000}+\frac{17C^2M^4\eta^{15}}{2438553600}+\\
&\frac{17C^3\eta^{16}}{208089907200}-\frac{5449}{7357464576000}C^5\eta^{16}+\frac{9593C^3M\eta^{16}}{468202212000}+\frac{109C^5M\eta^{16}}{91968307200}+\\
&\frac{CM^3\eta^{16}}{1857945600}-\frac{383C^3M^3\eta^{16}}{73156608000}-\frac{1}{928972800}CM^4\eta^{16}+\frac{17C^3M^4\eta^{16}}{39016857600}+\frac{1829C^4\eta^{17}}{26531463168000}-\\
&\frac{5449C^6\eta^{17}}{125076897792000}-\frac{59399}{87553828454400}C^4M\eta^{17}-\frac{467C^2M^2\eta^{17}}{5306292633600}+\frac{15559C^4M^2\eta^{17}}{13680285696000}+\\
&\frac{1283C^2M^3\eta^{17}}{2842656768000}-\frac{383C^4M^3\eta^{17}}{1243662336000}-\frac{467}{1326573158400}C^2M^4\eta^{17}+\frac{14057C^5\eta^{18}}{656653713408000}+\\
&\frac{C^3M\eta^{18}}{159188779008}-\frac{13697C^5M\eta^{18}}{140711510016000}-\frac{2083C^3M^2\eta^{18}}{29847896064000}+\frac{15559}{246245142528000}C^5M^2\eta^{18}+\\
&\frac{2083C^3M^3\eta^{18}}{14923948032000}-\frac{C^3M^4\eta^{18}}{19898597376}-\frac{C^4\eta^{19}}{6049173602304}+\frac{83C^6\eta^{19}}{28593782784000}+\\
&\frac{1}{211805798400}C^4M\eta^{19}-\frac{13697C^6M\eta^{19}}{2673518690304000}-\frac{31907C^4M^2\eta^{19}}{1559552569344000}+\frac{C^4M^3\eta^{19}}{52951449600}+\\
&\frac{C^4M^4\eta^{19}}{378073350144}-\frac{1}{8472231936000}C^5\eta^{20}+\frac{83C^7\eta^{20}}{571875655680000}+\frac{197C^5M\eta^{20}}{149776957440000}\\
&-\frac{197C^5M^2\eta^{20}}{74888478720000}+\frac{C^5M^3\eta^{20}}{1059028992000}-\frac{197}{6290632212480000}C^6\eta^{21}+\frac{C^6M\eta^{21}}{6207860736000}\\
&-\frac{197C^6M^2\eta^{21}}{1572658053120000}-\frac{C^7\eta^{22}}{2731458723844000}+\frac{C^7M\eta^{22}}{136572936192000}-\frac{1}{6282355064832000}C^8\eta^{23}\\
&+\frac{\beta\eta^3}{6}+\frac{C\beta\eta^4}{12}-\frac{\eta^5\beta}{120}+\frac{1}{60}C^2\eta^5\beta+\frac{M\eta^5\beta}{40}-\frac{C\beta\eta^6}{120}+\frac{1}{72}CM\eta^6\beta+\frac{\eta^7\beta}{2520}-\frac{C^2\beta\eta^7}{315}-\\
&\frac{13M\beta\eta^7}{5040}+\frac{C^2M\eta^7\beta}{504}+\frac{11}{5040}\beta M^2\eta^7+\frac{103C\eta^8\beta}{161280}-\frac{1}{2520}C^3\eta^8\beta-\frac{7CM\eta^8\beta}{2880}+\frac{CM^2\eta^8\beta}{960}
\end{aligned}
$$

$$\frac{137}{362880}C^2\eta^9\beta+\frac{19M\eta^9\beta}{181440}-\frac{13C^2M\beta\eta^9}{17280}-\frac{M^2\eta^9\beta}{3360}+\frac{C^2M^2\eta^9\beta}{8640}+\frac{1}{10080}\beta M^3\eta^9-\frac{49C\eta^{10}\beta}{4147200}$$

$$+\frac{233}{2419200}C^3\eta^{10}\beta+\frac{127CM\eta^{10}\beta}{806400}-\frac{13C^3M\beta\eta^{10}}{172800}-\frac{1711}{7257600}CM^2\eta^{10}\beta+\frac{11CM^3\eta^{10}\beta}{302400}+$$

$$\frac{1447C^2\beta\eta^{11}}{10644800}+\frac{233C^4\beta\eta^{11}}{26611200}-\frac{M\eta^{11}\beta}{712800}+\frac{2599}{31933440}C^2\beta M\eta^{11}+\frac{3}{246400}M^2\eta^{11}\beta-$$

$$\frac{1207C^2M^2\eta^{11}\beta}{19958400}-\frac{17M^3\beta\eta^{11}}{1108800}+\frac{C^2M^3\eta^{11}\beta}{302400}+\frac{M^4\eta^{11}\beta}{570240}+\frac{C\eta^{12}\beta}{6082560}-\frac{4621}{766402560}C^3\eta^{12}\beta-$$

$$\frac{331CM\eta^{12}\beta}{79833600}+\frac{5603C^3M\beta\eta^{10}}{319334400}+\frac{2477}{159667200}CM^2\eta^{12}\beta-\frac{1207C^3M^2\eta^{12}\beta}{239500800}-\frac{149CM^3\beta\eta^{12}}{14515200}+$$

$$\frac{CM^4\beta\eta^{12}}{1900800}+\frac{129C^2\beta\eta^{13}}{410009600}-\frac{4201C^4\beta\eta^{13}}{3558297600}-\frac{3611C^2M\beta\eta^{13}}{1037836800}+\frac{431C^4M\eta^{13}\beta}{319334400}-\frac{31M^2\eta^{13}\beta}{172972800}+$$

$$\frac{33991C^2M^2\eta^{13}\beta}{4981616640}+\frac{53}{86486400}M^3\eta^{13}\beta-\frac{1271C^2M^3\eta^{13}\beta}{566092800}\frac{-163M^4\beta}{518918400}+\frac{C^2M^4\eta^{13}\beta}{24710400}+$$

$$\frac{27403C^3\eta^{14}\beta}{126804787200}-\frac{4201}{49816166400}C^5\beta\eta^{14}+\frac{11CM\eta^{14}\beta}{270950400}-\frac{13381C^3M\eta^{14}\beta}{10567065600}-$$

$$-\frac{2059}{5283532800}CM^3\eta^{14}\beta+\frac{167}{132088320}C^3M^2\eta^{14}\beta+\frac{139CM^3\eta^{14}\beta}{211341312}-\frac{1271C^3M^3\eta^{14}\beta}{792529920}$$

$$-\frac{CM^4\eta^{14}\beta}{5644800}-\frac{571C^2\eta^{15}\beta}{253609574400}+\frac{13063C^4\eta^{15}\beta}{186810624000}+\frac{2659C^2M\eta^{15}\beta}{42268262400}-\frac{16991C^4M\eta^{15}\beta}{80472268800}$$

$$-\frac{52103C^2M^2\eta^{15}\beta}{190207180800}+\frac{167C^4M^2\eta^{15}\beta}{1981324800}-\frac{M^3\eta^{15}\beta}{206388000}+\frac{5627C^2M^3\eta^{15}\beta}{22643712000}+\frac{M^4\eta^{15}\beta}{99066240}$$

$$-\frac{61C^2M^4\eta^{15}\beta}{1828915200}-\frac{13973C^3\eta^{16}\beta}{4682022912000}+\frac{69271C^5\eta^{16}\beta}{6437781504000}+\frac{12421C^3M\eta^{16}\beta}{334430208000}-\frac{16991C^5M\eta^{16}\beta}{1287556300800}$$

$$+\frac{CM^2\eta^{16}\beta}{619315200}-\frac{25201C^3M^2\eta^{16}\beta}{292626432000}-\frac{377CM^3\eta^{16}\beta}{39016857600}+\frac{1957C^3M^3\eta^{16}\beta}{48771072000}+\frac{167CM^4\eta^{16}\beta}{19508428800}$$

$$-\frac{61C^3M^4\eta^{16}\beta}{292626643200}-\frac{86953C^4\eta^{17}\beta}{54721142784000}+\frac{69271C^6\eta^{17}\beta}{109442285568000}-\frac{467C^2M\eta^{17}\beta}{2653146316800}+$$

$$\frac{2299453C^4M\eta^{17}\beta}{218884571136000}+\frac{23719C^2M^2\eta^{17}\beta}{9949298688000}-\frac{459643C^4M^2\eta^{17}\beta}{36480761856000}-\frac{475C^2M^3\eta^{17}\beta}{79594389504}+$$

$$\frac{1957C^4M^3\eta^{17}\beta}{829108224000}+\frac{1349C^2M^4\eta^{17}\beta}{497464934400}-\frac{C^3\eta^{18}\beta}{159188779008}-\frac{3287051C^5\eta^{18}\beta}{7879844560896000}$$

$$-\frac{459643C^5M^2\eta^{18}\beta}{656653713408000}-\frac{295877C^3M^3\eta^{18}\beta}{179087376384000}+\frac{13C^3M^4\eta^{18}\beta}{34111881216}+\frac{41C^4M^3\eta^{19}\beta}{5618427494400}-$$

$$-\frac{131809C^4M\eta^{19}\beta}{1188230529024000}+\frac{23677C^6M\eta^{19}\beta}{314531610624000}+\frac{2449861C^4M^2\eta^{19}\beta}{7485852332851200}-\frac{41C^4M^3\eta^{19}\beta}{190625218560}$$

$$+\frac{13C^4M^4\eta^{19}\beta}{648125743104}+\frac{28247C^5\eta^{20}\beta}{8387509616640000}-\frac{1247C^7\eta^{20}\beta}{465972756480000}-\frac{167011C^5M\eta^{20}\beta}{6290632212480000}$$

$$+\frac{126277C^5M^2\eta^{20}\beta}{3145316106240000}-\frac{41C^5M^3\eta^{20}\beta}{3812504371200}+\frac{277C^6M^4\eta^{21}\beta}{362423255040000}-\frac{C^6M\eta^{21}\beta}{323326080000}+$$

$$\frac{126277C^6M^2\eta^{21}\beta}{66051638231040000}+\frac{29C^7\eta^{22}\beta}{341432340480000}-\frac{C^7M\eta^{22}\beta}{7113173760000}++\frac{29C^8\eta^{23}\beta}{7852943831040000}$$

$$+\frac{\eta^5\beta^2}{60}+\frac{C\eta^6\beta^2}{72}-\frac{\eta^7\beta^2}{630}+\frac{C^2\eta^7\beta^2}{252}+\frac{M\eta^7\beta^2}{252}-\frac{83C\eta^8\beta^2}{40320}+\frac{C^3\eta^8\beta^2}{2016}+\frac{CM\eta^8\beta^2}{336}$$

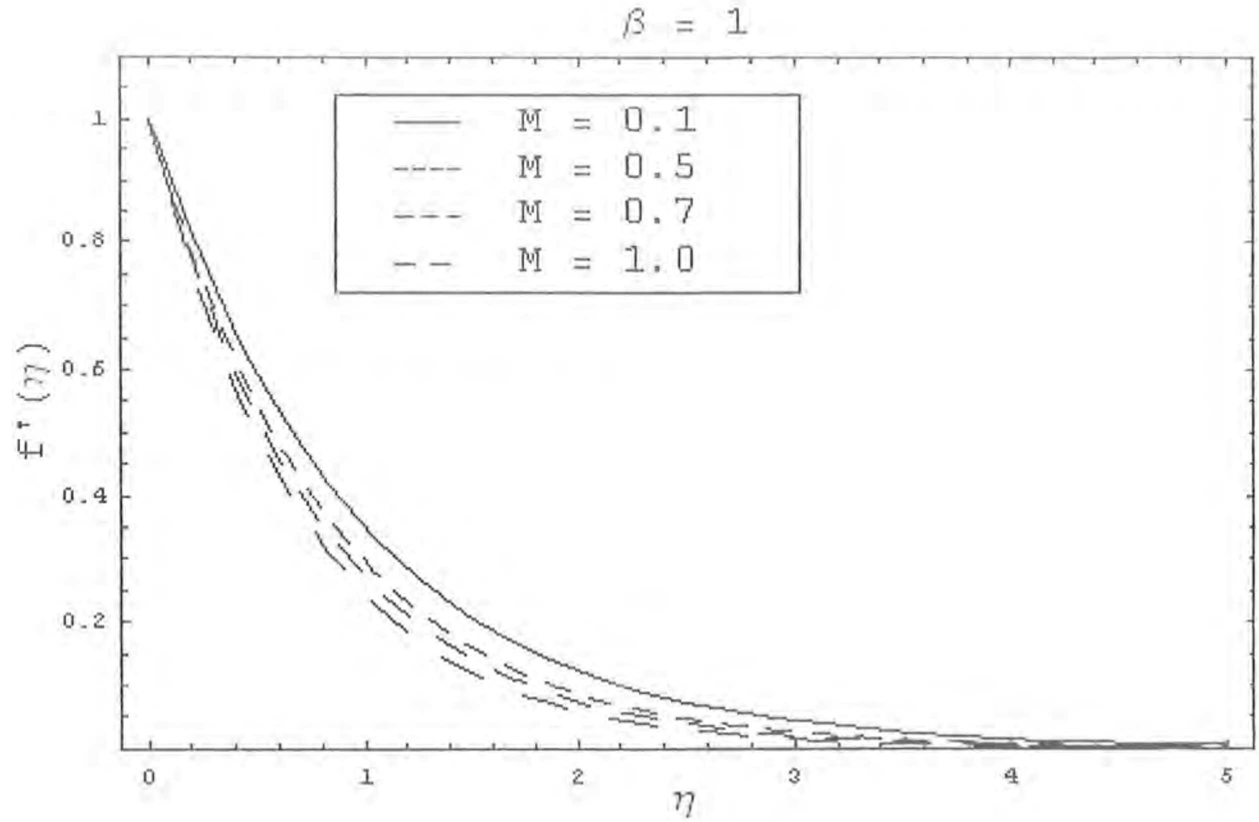
$$+\frac{19\eta^9\beta^2}{362880}-\frac{71C^2\eta^9\beta^2}{72576}-\frac{19M\eta^9\beta^2}{40320}-\frac{71C^2\eta^9\beta^2}{72576}-\frac{19M\eta^9\beta^2}{40320}+\frac{5C^2M\eta^9\beta^2}{6048}+$$

$$\frac{M^2\eta^9\beta^2}{2520}+\frac{787C\eta^{10}\beta^2}{7257600}-\frac{19C^3\eta^{10}\beta^2}{86400}-\frac{1867CM\eta^{10}\beta^2}{3628800}+\frac{C^3M\eta^{10}\beta^2}{12096}+\frac{83CM^2\eta^{10}\beta^2}{302400}$$

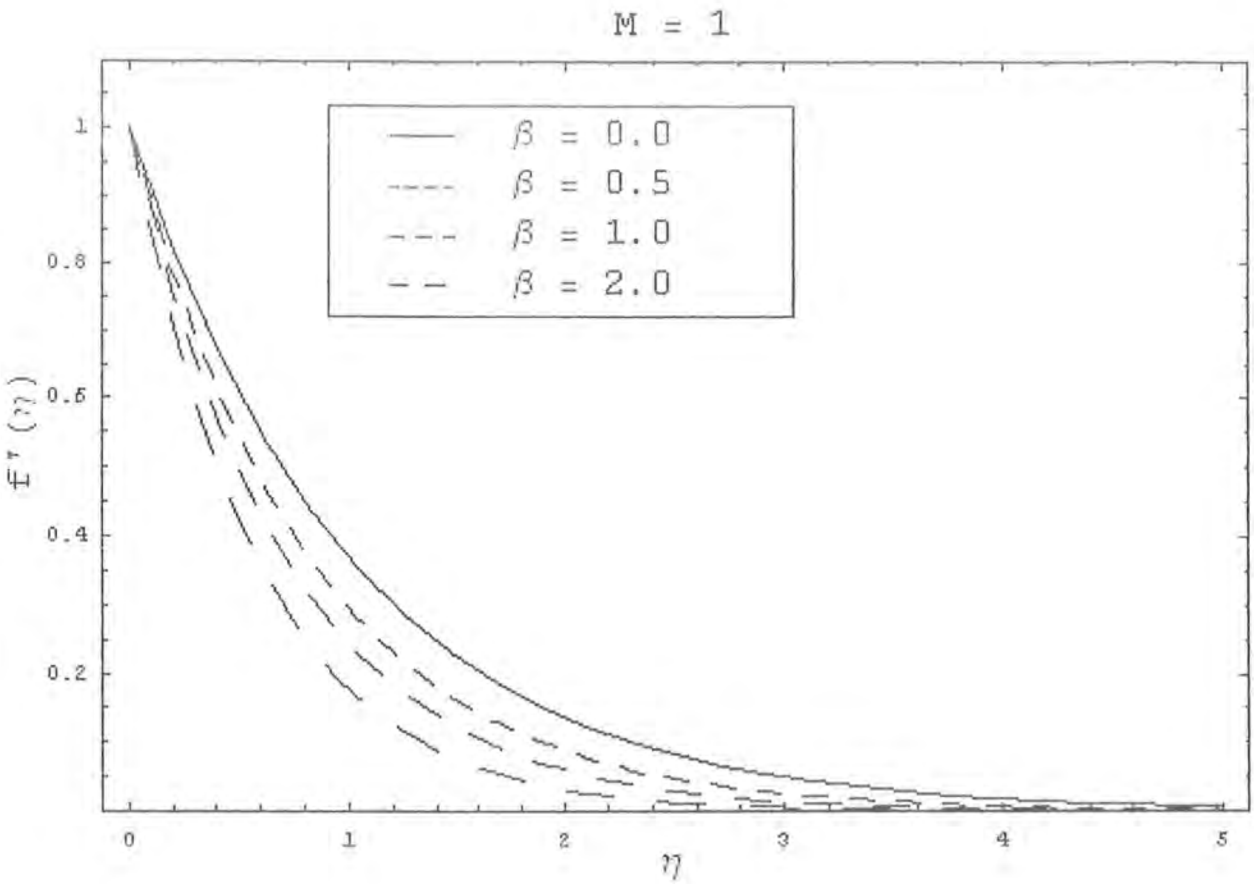
$$-\frac{\eta^{11}\beta^2}{1425600}+\frac{301C^2\eta^{11}\beta^2}{3801600}-\frac{19C^4\eta^{11}\beta^2}{950400}+\frac{M^2\eta^9\beta^2}{2520}+\frac{787C\eta^{10}\beta^2}{7257600}-\frac{19C^3\eta^{10}\beta^2}{86400}.$$

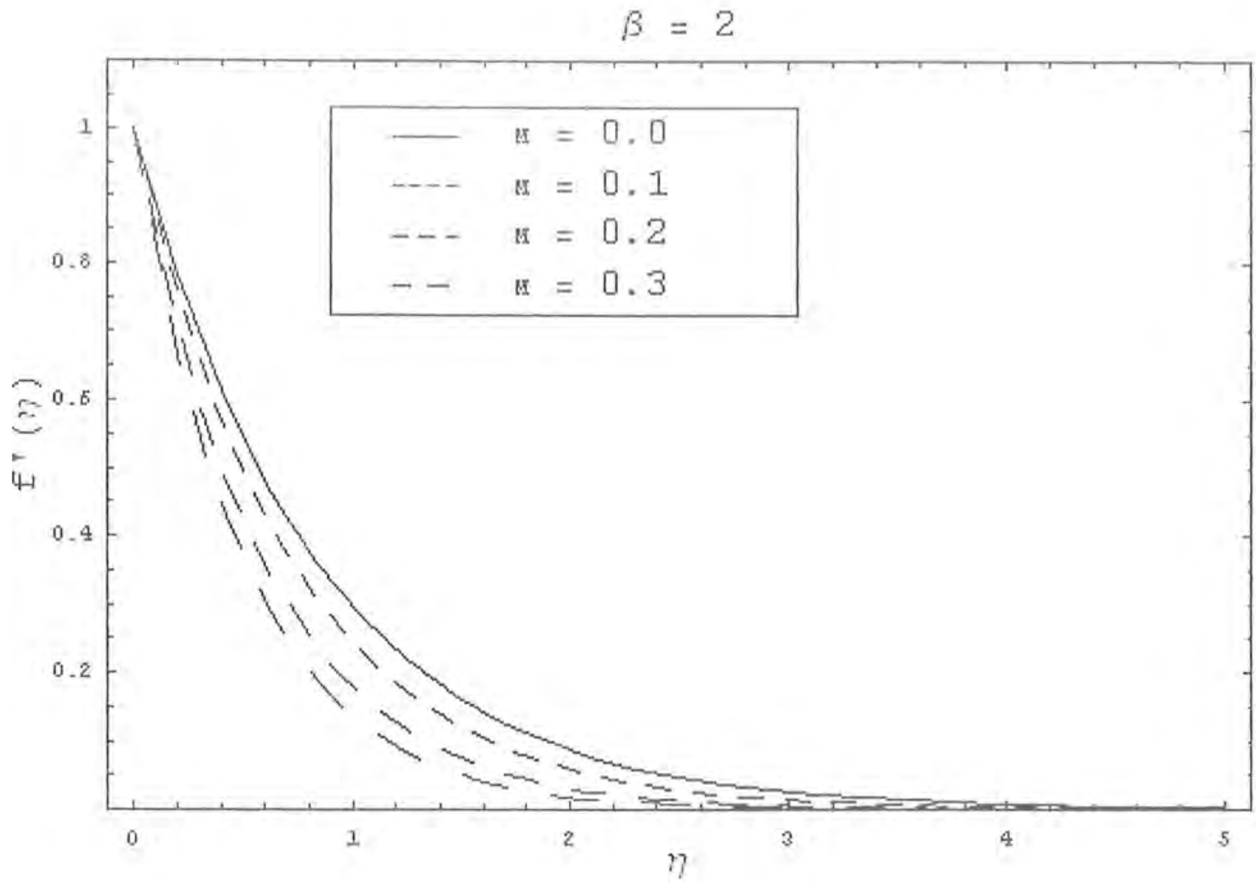
The value of C in above equations can be determined by using pade approximation method, which2 are shown as

β	M	[2/2]	[3/3]	[4/4]	[5/5]
0.1	1	-1.179033818	-1.104564697	Complex numbers	-1.194693313
0.2		-1.202717508	-1.134533595	-1.147845193	-1.142123790
0.3		-1.225948695	-1.163400118	-1.222358387	-1.228488722
0.4		-1.248858327	-1.191333856	-1.198831555	-1.197533244
0.5		-1.271527987	-1.218437609	-1.225567562	-1.224643090
1	0.1	-0.9698914631	Complex numbers	-0.9820379642	-.9544591990
	0.2	-1.023880173	Complex numbers	-1.031277264	-1.008337571
	0.3	-1.075227430	Complex numbers	-1.078368999	-1.058787633
	0.4	-1.124258739	-0.5709056973	-1.120222278	Complex numbers
	0.5	-1.171245043	-0.5607048323	-1.159426511	-1.151074249

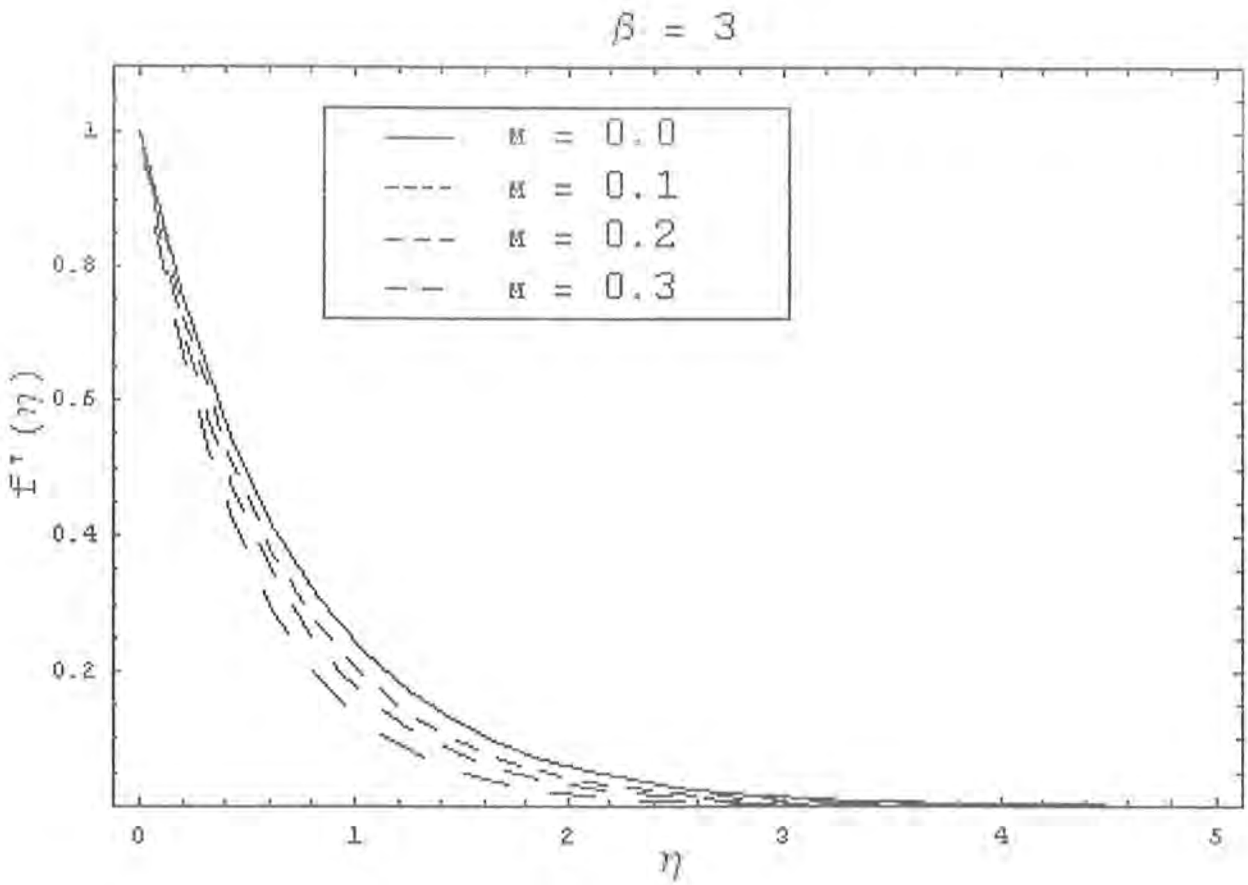


β = 1
M = 0.1
M = 0.5
M = 0.7
M = 1.0
f'(η)
η


M = 1
β = 0.0
β = 0.5
β = 1.0
β = 2.0
f'(η)
η
0
1
2
3
4
5
0.2
0.4
0.6
0.8
1



β = 2
m = 0.0
m = 0.1
m = 0.2
m = 0.3
f'(η)
η



2.4 Results and discussion

The boundary layer equations of viscous fluid in the presence of magnetic field has been solved analytically with the help of variational iteration method. The Pade approximation is used to find the bounded solution at infinity. Three types of stretching solutions cases named as linear, nonlinear stretching and exponential stretching have been discussed. To discuss the physical features we have plotted Figures 1 to 4.

In Fig.(2.1), the nondimensional velocity f' is plotted against η for various values of magnetic parameter M, the velocity field decreases and the boundary layer thickness also reduces for the case of linear stretching. The variation of β on thevelocity are shown in Fig.(2.2). It is depicted that the velocity field decreases with the increase in β. Moreover, the velocity field for linear stretching is less than for the nonlinear stretching and the boundary layer thickness is reduced rapidly in nonlinear stretching case. The variation of M for nonlinear stretching and exponential stretching are displayed in Figs (2.3) and (2.4). It is observed that the velocity decreases with the increase in M for both the cases. However, the boundary layer thickness is reduced rapidly as we increase the value of β.

Bibliography

[1] Z. Belhachmi, B. Bright, K. Taous, On the concave solutions of the Blasius equations, Acta Mat. Univ. Comenianae LXIX (2) (2000) 199–214.

[2] B.K. Datta, Analytic solution for the Blasius equation, Indian J. Pure Appl. Math. 34 (2) (2003) 237–240.

[3] H.K. Kuiken, On boundary layers in field mechanics that decay algebraically along stretches of wall that are not vanishing small, IMA J. Appl. Math. 27 (1981) 387–405.

[4] H.K. Kuiken, A backward free-convective boundary layer, Quart. J. Mech. Appl. Math. 34 (1981) 397–413.

[5] A. M Wazwaz, The variational iteration method for solving two forms of Blasius equation on a half-infinite domain, Appl. Math. Comput, 188 (2007) 485-491.

[6] J.H. He, Some asymptotic methods for strongly nonlinear equations, Int. J. Modern Phys. B 20 (10) (2006) 1141–1199.

[7] J.H. He, Non-perturbative methods for strongly nonlinear problems, Berlin: dissertation. de-Verlag im Internet GmbH, 2006.

[8] J.H. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media, Comput. Methods Appl. Mech. Eng. 167 (1998) 57–68.

[9] J.H. He, Variational iteration method for autonomous ordinary differential systems, Appl. Math. Comput. 114 (2/3) (2000) 115–123.

[10] J.H. He, Homotopy perturbation method: a new nonlinear technique, Appl. Math. Comput. 135 (2003) 73–79.

[11] S. Momani, Z. Odibat, Analytical approach to linear fractional partial differential equations arising in fluid mechanics, Phys. Lett. A 1 (53) (2006) 1–9.

[12] S. Momani, S. Abusaad, Application of he's variational-iteration method to Helmholtz equation, Chaos, Solitons & Fractals 27 (5) (2005) 1119–1123.

[13] A.M.Wazwaz, A comparison between the variational iteration method and Adomian decomposition method, J. Comput. Appl. Math., in press.

[14] A.M.Wazwaz, The variational iteration method for rational solutions for KdV, K(2,2), Burgers, and cubic Boussinesq equations, J. Comput. Appl. Math., in press.

[15] A.M.Wazwaz, Analytical approximations and Pade´ approximants for Volterra's population model, Appl. Math. Comput. 100 (1999) 13–25.

[16] A.M. Wazwaz, A study on aboundary-layer equation arising in an incompressible fluid, Appl. Math. Comput. 87 (1997) 199–204.

[17] A.M. Wazwaz, Partial Differential Equations: Methods and Applications, Balkema, The Netherlands, 2002.

[18] J. Boyd, Pade´ approximant algorithm for solving nonlinear ordinary differential equation boundary value problems on an unbounded domain, Comput. Phys. 11 (3) (1997) 299–303.

[19] G.A. Baker, Essentials of Pade´ approximants, Academic Press, London, 1975.

[20] H. Schlichting, Boundary Layer Theory, McGraw Hill, NY, 1968.