Intuitionistic Fuzzy Γ-ideals of Γ-LA-semigroups

QUAID-I-AZAM UNIVERSITY

By

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CERTIFICATE

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We accept this dissertation as conforming to the required standard.

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Dedicated

To

my

Mother, Father and Rana Kuaser

Whose affection is reason of every success in my life.

Who've always given me perpetual love, care, and cheers. Whose prayers have always been a source of great inspiration for me and whose sustained hope in me led me to where I stand today.

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Jan, 2011 *Saleem Abdullah*

Contents

Preface

Set theory is a basis of modern mathematics, and notions of set theory are used in all formal descriptions. At the end of 19th century, G. Cantor introduced the concept of a "Set" on paper of mathematical basis. Many branches of modern mathematics are based on this concept. It has played a vital role in the developed of theory of mathematics. In 1965, L. A Zadeh introduced a new concept of fuzzy set in his seminal paper [28]. This paper has opened up new insights and applications in wide range of scientific fields. Many researcher, after the semblance of fuzzy set theory, engaged to applied fuzzy concept to algebra.

A. Rosenfeld was the first mathematician who used the fuzzy concept to algebra and he introduced fuzzy subgroup of group in his pioneer paper [23]. A. Rosenfeld is the father of fuzzy algebra. There have been a number of generalization of this fundamental concept. The idea of intuitionistic fuzzy set is one among them which is a major role in the fuzzy set theory. This idea was first introduced by K.T. Atanassov in his definitive paper's [5, 6]. Many mathematicians used this idea to algebra.

This dissertation consists of three chapters. Chapter one is of an introductory nature which provides basic definitions and reviews some of the background material which is needed for reading the subsequent chapters. Chapters one consists of four sections. Section 1 is of basic definitions of LA-semigroups. Section 2 is of basic definitions and results of $\Gamma-$LA-semigroups. Section 3 is of basic definitions and results of fuzzy subset. Last section of this chapter consists of basic concepts and results of an intuitionistic fuzzy set.

In chapter two, we apply the concept of intuitionistic fuzzy set to $\Gamma-$LA-semigroups. We introduce intuitionistic fuzzy (left, right, two sided, bi) $\Gamma-$ideals and investigate some related properties of them. We also prove some characterization theorems of intuitionistic fuzzy (left, right, two sided, bi) $\Gamma-$ideals.

In Chapter three, we introduce intuitionistic fuzzy prime $\Gamma-$ideals, intuitionistic fuzzy semi-prime $\Gamma-$ideals, intuitionistic fuzzy interior $\Gamma-$ideals, intuitionistic fuzzy quasi-$\Gamma-$ideals. Also, we investigate some different properties of these $\Gamma-$ideals in regular and intra-regular $\Gamma-$LA-semigroups.

Chapter 1

Fundamental Concepts

The aim of this chapter is to present a brief summary of basic definitions and preliminary result of LA-semigroups and $\Gamma-$LA-semigroups that will be of value for our later pursuits.

1.1 LA-semigroup; Basic definitions and Examples

Definition 1 *[10]* A groupoid $(S,.)$ is called a Left Almost semigroup, abbreviated as LA-semigroup if it satisfies left invertive law that is

$$(ab)c = (cb)a \quad \text{for all } a, b, c \in S$$

Example 2 *[19] Let $(\mathbb{Z}, +)$ denote the commutative group of integers under addition. Define a binary operation " $*$ " in $\mathbb{Z}$ as follows:*

$$a * b = b - a \quad \text{for } a, b \in \mathbb{Z},$$

where " $-$ " denotes the ordinary subtraction of integers. Then $(\mathbb{Z}, *)$ is an LA-semigroup.

Example 3 *[19] $(\mathbb{R}, +)$ denote the group of real numbers under addition. Define a binary operation " $*$ " in $\mathbb{R}$ as follows:*

$$a * b = b - a \quad \text{for } a, b \in \mathbb{R}.$$

Then $(\mathbb{R}, *)$ *is an LA-semigroup.*

Definition 4 A non-empty subset B of an LA-semigroup S is called a subLA-semigroup of S if $BB \subseteq B$.

Definition 5 An element of an LA-semigroup S is called an idempotent if $a^2 = a$.

Example 6 *Let* $S = \{1, 2, 3, 4, 5\}$ *be an LA-semigroup with multiplication defined by the Cayley table*

$\cdot$	1	2	3	4	5
1	1	2	3	4	5
2	2	2	2	4	5
3	3	2	1	4	5
4	5	5	5	2	4
5	4	4	4	5	2

1 *and* 2 *are idempotents in* S.

Theorem 7 *[10] In an LA-semigroup S, the medial law hold*

$$(ab)(cd) = (ac)(bd) \quad \text{for all } a, b, c, d \in S.$$

If S *is an LA-semigroup with left identity* e, *then*

$$a(bc) = b(ac) \quad \text{for all } a, b, c \in S.$$

1.1.1 Ideals in LA-semigroups

In this section, we have defined ideals of LA-semigroups and discussed a brief summary of basic definitions and preliminary results of the ideals of LA-semigroups.

Definition 8 *[20]*Let S be an LA-semigroup. A non-empty subset I of S is called a left (right) ideal of S if $SI \subseteq I$ $(IS \subseteq I)$. A non-empty subset I of S is called an ideal of S if it is a left and a right ideal of S.

Intersection of any family of left (right) ideals of an LA-semigroup S is either empty or a left (right) ideal of S. If A is a non-empty subset of S then intersection of all left (right) ideals of S which contains A is left (right) ideal of S containing A. Of course this is the smallest left (right) ideal of S containing A and is called left (right) ideal of S generated by A. If $A = \{a\}$, a singleton subset of S, then the left (right) ideal of S generated by A is called a principle left (right) ideal of S generated by A.

Union of left (right) ideals of an LA-semigroup S is a left (right) ideal of S.

Example 9 *Let $S = \{1, 2, 3, 4\}$ be an LA-semigroup with left identity 1 by the following cayley table*

$\cdot$	1	2	3	4
1	1	2	3	4
2	4	3	3	3
3	3	3	3	3
4	3	3	3	3

Then sets $L = \{2, 3\}$ and $I = \{2, 3, 4\}$ is a left and an ideal of S.

Definition 10 *[20]* An ideal I of an LA-semigroup S is called minimal if it does not contain any ideal of S other than itself.

Lemma 11 *[20] If S is an LA-semigroup with left identity e, then $SS = S$ and $S = eS = Se$.*

Remark 12 *[20] If S is an LA-semigroup with left identity e, then $SS = S$, but the converse is not necessarily true.*

Proposition 13 *[20] If S is an LA-semigroup with left identity e, then every right ideal of S is a left ideal of S.*

1.1.2 Ideals in regular LA-semigroups

Definition 14 *[20]* An LA-semigroup S is said to be regular if for each a in S there exists x in S such that $a = (ax)a$. If S is a regular LA-semigroup, then it is easy to see that $S = S^2$.

Lemma 15 *[20] Every right ideal of a regular LA-semigroup S is a left ideal.*

Lemma 16 *[20] If P and Q are right ideal of a regular LA-semigroup S, then $PQ = P \cap Q$.*

Lemma 17 *[20] If I is a right ideal of a regular LA-semigroup S, then $I = I^2$.*

1.1.3 Generalized bi-ideals, Bi-ideals, Interior ideals and Quasi-ideals of LA-semigroups

Definition 18 *[29] Let S be an LA-semigroup. A non-empty subset Q of S is called a quasi ideal of S if $QS \cap SQ \subseteq Q$.*

Definition 19 *[29] A sub LA-semigroup B of an LA-semigroup S is called a bi-ideal of S if $(BS)B \subseteq B$.*

Definition 20 *[29] A non-empty subset A of an LA-semigroup S is called a generalized bi-ideal of S if $(AS)A \subseteq A$.*

Definition 21 *[29] A non-empty subset A of an LA-semigroup S is called an interior ideal of S if $(SA)S \subseteq A$.*

Example 22 *Let S be an LA-semigroup as given in Example 9 and let $B = \{2,3\}$, $A = \{2,3,4\}$, $G = \{2,3\}$ and $Q = \{3,4\}$ be a bi-ideal, an interior ideal, a generalized bi-ideal and a quasi-ideal of S, respectively.*

Note that every two sided ideal of an LA-semigroup S is an interior ideal of S. Also note that every right ideal and every left ideal of an LA-semigroup S is a quasi ideal of S and intersection of quasi ideals of an LA-semigroup S is a quasi ideal of S. Also it is important to note that the intersection of a left ideal and a right ideal of an LA-semigroup S is a quasi ideal of S.

1.2 $\Gamma-$LA-semigroup; Basic definitions and Examples

In this section we have defined $\Gamma-$LA-semigroup and Γ-ideals, bi-Γ-ideals, prime Γ-ideals and semiprime Γ-ideals in Γ-LA-semigroups and discussed some fundamental concept of $\Gamma-$LA-semigroups.

Definition 23 *[24] Let S and Γ be non-empty sets. We call S to be a $\Gamma-$LA-semigroup if there exists a mapping $S \times \Gamma \times S \longrightarrow S$, written (a, γ, b) by $a\gamma b$, such that S satisfies the identity $(a\gamma b)\,\beta c = (c\gamma b)\,\beta a$ for all $a, b, c \in S$ and $\gamma, \beta \in \Gamma$.*

Example 24 *[24] Let S be an arbitrary LA-semigroup and Γ any non-empty set. Define a mapping $S \times \Gamma \times S \longrightarrow S$, by $a\gamma b = ab$ for all $a, b \in S$ and $\gamma \in \Gamma$. It is easy to see that S is a $\Gamma-$LA-semigroup. Indeed,*

$$
\begin{aligned}
(a\gamma b)\,\beta c &= (ab)\,\beta c = (ab)\,c = (cb)\,a \\
\textit{Now take } (c\gamma b)\,\beta a &= (cb)\,\beta a = (cb)\,a. \\
\textit{Hence } (a\gamma b)\,\beta c &= (c\gamma b)\,\beta a
\end{aligned}
$$

Remark 25 *[24] Every LA-semigroup implies a $\Gamma-$LA-semigroup. But Converse is not true in general.*

Example 26 *[24]* Let $S = \{0, i, -i\}$ and $\Gamma = S$. Then by defining $S \times \Gamma \times S \to S$ as $a\gamma b = a.\gamma.b$ for all $a, b \in S$ and $\gamma \in \Gamma$. It can be easily verified that S is a Γ-LA-semigroup under complex number multiplication while S is not an LA-semigroup.

Example 27 *[24]* Let $\Gamma = \{1, 2, 3\}$. Define a mapping $\mathbb{Z} \times \Gamma \times \mathbb{Z} \to \mathbb{Z}$ by $a\gamma b = b - \gamma - a$ for all $a, b \in \mathbb{Z}$ and $\gamma \in \Gamma$, where "$-$" is the usual subtraction of integers. Then $\mathbb{Z}$ is a Γ-LA-semigroup. Indeed

$$
\begin{aligned}
(a\gamma b)\mu c &= (b - \gamma - a)\mu c \\
&= c - \mu - (b - \gamma - a) \\
&= c - \mu - b + \gamma + a.
\end{aligned}
$$

and

$$\begin{aligned}(c\gamma b)\mu a &= (b-\gamma-c)\mu a\\ &= a-\mu-(b-\gamma-c)\\ &= a-\mu-b+\gamma+c\\ &= c-\mu-b+\gamma+a.\end{aligned}$$

Which implies $(a\gamma b)\mu c = (c\gamma b)\mu a$ for all $a, b, c \in \mathbb{Z}$ and $\gamma, \mu \in \Gamma$.

Example 28 *[24]* Let S be a Γ-LA-semigroup and γ a fixed element in Γ. We define $a * b = a\gamma b$ for all $a, b \in S$. We can show that $(S, *)$ is an LA-semigroup and we denote this by S_γ.

Definition 29 *[24]* An element $e \in S$ is called a left identity of Γ-LA-semigroup if $e\gamma a = a$ for all $a \in S$ and $\gamma \in \Gamma$.

Lemma 30 *[24] If S is a $\Gamma-LA$-semigroup with left identity e, then $S\Gamma S = S$ and $S = e\Gamma S = S\Gamma e$.*

1.2.1 Γ-ideals in Γ-LA-semigroups

Definition 31 *[24]* Let S be a Γ-LA-semigroup. Then a non-empty subset M of S is called a subΓ-LA-semigroup of S if $a\gamma b \in M$ for all $a, b \in M$ and $\gamma \in \Gamma$.

Definition 32 *[24]* A subset I of Γ-LA-semigroup S is called a left(right) Γ-ideal of S if $S\Gamma I \subseteq I$ $(I\Gamma S \subseteq I)$ and is called Γ-ideal if it is a left as well as a right Γ-ideal of S.

Example 33 *Let $S = \{1, 2, 3, 4, 5\}$ and $\Gamma = \{\alpha, \beta, \gamma\}$ be two non-empty sets. Then*

S is a $\Gamma-LA$-semigroup by the following Cayley tables:

α	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	1	1	1
4	1	1	1	1	1
5	1	1	1	1	1

β	1	2	3	4	5
1	2	2	2	2	2
2	2	2	2	2	2
3	2	2	2	2	2
4	2	2	2	2	2
5	2	2	2	2	2

γ	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	1	1	1
4	1	1	1	1	1
5	1	1	1	3	3

Also S is non-associative becouse $(1\alpha 2)\beta 3 \neq 1\alpha(2\beta 3)$. Let $I = \{1,2,3\}$ be subset of S. Then clearly I is a left and a riht $\Gamma-$ideal of S. Let $A = \{1,2,3,4\}$ be subset of S. Then clearly A is a $\Gamma-$ideal of S

Proposition 34 *[24] If a Γ-LA-semigroup S has a left identity e, then every right Γ-ideal is a left Γ-ideal.*

Proof. *Let I be a right Γ-ideal of S. Then for $i \in I$, $s \in S$ and $\alpha \in \Gamma$, consider*

$$\begin{aligned} s\alpha i &= (e\gamma s)\alpha i, \text{ where } e \in S \text{ is a left identity and } \gamma \in \Gamma \\ &= (i\gamma s)\alpha e \in I. \end{aligned}$$

Hence I is a left Γ-ideal. ∎

Lemma 35 *[24] If I is a left Γ-ideal of a Γ-LA-semigroup S with left identity e, and if for any $a \in S$, there exists $\gamma \in \Gamma$, then $a\gamma I$ is a left Γ-ideal of S.*

Proof. Let I be a left Γ-ideal of S and consider

$$\begin{aligned} s\gamma(a\gamma i) &= (e\gamma s)\gamma(a\gamma i), \text{ where } e \text{ is left identity in } S \\ &= (e\gamma a)\gamma(s\gamma i), \text{ by } \Gamma\text{-medial.} \\ &= a\gamma(s\gamma i) \in a\gamma I. \end{aligned}$$

Hence $a\gamma I$ is a left Γ-ideal of S. ∎

Lemma 36 *[24] If I is a right Γ-ideal of a Γ-LA-semigroup S with left identity e, then $I\Gamma I$ is a Γ-ideal of S.*

Proof. Let $x \in I\Gamma I$, then $x = i\gamma j$ where $i, j \in I$ and $\gamma \in \Gamma$. Now consider

$$x\alpha s = (i\gamma j)\alpha s = (s\gamma j)\alpha i \in I\Gamma I.$$

This implies that $I\Gamma I$ is a right Γ-ideal and hence by proposition 34, $I\Gamma I$ is a Γ-ideal of S. ■

Corollary 37 *[24] If I is a left Γ-ideal of S then $I\Gamma I$ becomes a Γ-ideal of S.*

Definition 38 *[24]* A Γ-ideal I of S is called minimal Γ-ideal, if it does not properly contain any Γ-ideal of S.

Lemma 39 *[24] A proper Γ-ideal M of a Γ-LA-semigroup S with left identity e is minimal if and only if $M = a^2\Gamma M$, for all $a \in S$.*

Proof. Assume that M is a minimal Γ-ideal of S. Now as $M\Gamma M$ is a Γ-ideal of S so $M = M\Gamma M$. It is easy to see that $a^2\Gamma M$ is a Γ-ideal and is contained in M. But as M is minimal so $M = a^2\Gamma M$.

Conversely let $M = a^2\Gamma M$, for all $a \in S$. On contrary let K be a minimal Γ-ideal of S which is properly contained in M containing a, then $M = a^2\Gamma M \subseteq K$, which is a contradiction. ■

Definition 40 *[24]* A Γ-ideal I of a Γ-LA-semigroup S is called a Γ-idempotent if $I\Gamma I = I$ and if every Γ-ideal of S is Γ-idempotent, then S is called fully Γ-idempotent.

Definition 41 *[24]* The set of Γ-ideals of Γ-LA-semigroup S is said to be totally ordered under inclusion if for all Γ-ideals H, K, either $H \subseteq K$ or $K \subseteq H$ and we denote it by Γ-ideal(S).

Definition 42 *[24]* If S is a Γ-LA-semigroup S with left identity e, then the principal left Γ-ideal generated by x is defined as $\langle x \rangle = S\Gamma x = \{s\gamma x : s \in S\}$, for all $x \in S$ and $\gamma \in \Gamma$.

Definition 43 *[24]* Let G and Γ be non-empty sets If there exists a mapping $G \times \Gamma \times G \to G$,written (x, γ, y) by $x\gamma y$, G is called a Γ-medial if it satisfies the identity $(x\alpha y)\beta(l\gamma m) = (x\alpha l)\beta(y\gamma m)$ for all $x, y, l, m \in G$ and $\alpha, \beta, \gamma \in \Gamma$.

Theorem 44 *[24] If S is a Γ-LA-semigroup S with left identity e, then a left Γ-ideal P of S is qausi Γ-prime if and only if $a\alpha(S\beta b) \subseteq P$ implies $a \in P$ or $b \in P$, for all $a, b \in S$ and any $\alpha, \beta \in \Gamma$.*

Corollary 45 *[24] If S is a Γ-LA-semigroup with left identity e, then a left Γ-ideal P of S is qausi Γ-semiprime if and only if $a\alpha(S\beta a) \subseteq P$ implies $a \in P$, for all $a \in S$ and any $\alpha, \beta \in \Gamma$.*

Lemma 46 *[24] If I is a proper right(left) Γ-ideal of a Γ-LA-semigroup S with left identity e, then $e \notin I$.*

1.2.2 bi-Γ-ideals in Γ-LA-semigroups

Definition 47 *[24]* Let S be a Γ-LA-semigroup. A sub Γ-LA-semigroup B of S is said to be a bi-Γ-ideal of S if $(B\Gamma S)\Gamma B \subseteq B$.

Example 48 *[24] Let $S = \{1, 2, 3, 4, 5\}$ be non-empty set. Define a binary operation "$\cdot$" in S as follows:*

$\cdot$	1	2	3	4	5
1	x	x	x	x	x
2	x	x	x	x	x
3	x	x	x	x	x
4	x	x	x	x	x
5	x	x	3	x	x

Then $(S, \cdot)$ becomes a $\Gamma-LA$-semigroup, where $x \in \{1, 2, 4\}$. Now, let $\Gamma = \{1\}$ and define a mapping $S \times \Gamma \times S \to S$, by $a1b = ab$ for all $a, b \in S$. Then it is easy to see that S is a Γ-LA-semigroup. If we take $B = \{3, x\}$, then B becomes a bi-Γ-ideal of S.

Remark 49 *[24] Example 48 shows that bi-Γ-ideals in Γ-LA-semigroups are infact a generalization of bi-ideals in LA-semigroups (for a suitable choice of Γ).*

Example 50 *Let S be Γ-LA-semigroups as given in Example 33 and let $B = \{1, 2, 4\}$ be subset of S. Then clearly B is bi-Γ-ideal of S.*

Proposition 51 *[24] Let A be a left Γ-ideal and B be a bi-Γ-ideal of a $\Gamma-LA$-semigroup S with left identity e. Then $B\Gamma A$ and $(A\Gamma A)\Gamma B$ are bi-Γ-ideals of S.*

Proof. To show that $B\Gamma A$ is a bi-Γ-ideal of S, let us consider

$$\begin{aligned}((B\Gamma A)\Gamma S)\Gamma(B\Gamma A) &= ((S\Gamma A)\Gamma B)\Gamma(B\Gamma A)\\ &= ((B\Gamma A)\Gamma B)\Gamma(S\Gamma A) \subseteq ((B\Gamma S)\Gamma B)\Gamma A \subseteq B\Gamma A.\end{aligned}$$

Also by Γ-medial law, it can be verified that $(B\Gamma A)\Gamma(B\Gamma A) = (B\Gamma B)\Gamma(A\Gamma A) \subseteq B\Gamma A$. Hence $B\Gamma A$ is a bi-Γ-ideal of S. Now by Corollary 37, Γ-medial law and the fact that $S\Gamma S = S$, we have

$$\begin{aligned}(((A\Gamma A)\Gamma B)\Gamma S)\Gamma((A\Gamma A)\Gamma B) &= (((A\Gamma A)\Gamma S)\Gamma(B\Gamma S))\Gamma((A\Gamma A)\Gamma B)\\ &\subseteq ((A\Gamma A)\Gamma(B\Gamma S))\Gamma((A\Gamma A)\Gamma B)\\ &= ((A\Gamma A)\Gamma(A\Gamma A))\Gamma((B\Gamma S)\Gamma B) \subseteq (A\Gamma A)\Gamma B.\end{aligned}$$

Hence $(A\Gamma A)\Gamma B$ is a bi-Γ-ideal of S. ■

Proposition 52 *[24] The product of two bi-Γ-ideals of a $\Gamma-LA$-semigroup S with left identity e is again a bi-Γ-ideal of S.*

Proof. Let H and K be two bi-Γ-ideals of S. Then using Γ-medial law and $S\Gamma S = S$, we get

$$\begin{aligned}((H\Gamma K)\Gamma S)\Gamma(H\Gamma K) &= ((H\Gamma K)\Gamma(S\Gamma S))\Gamma(H\Gamma K)\\ &= ((H\Gamma S)\Gamma(K\Gamma S))\Gamma(H\Gamma K)\\ &= ((H\Gamma S)\Gamma H)\Gamma((K\Gamma S)\Gamma K) \subseteq H\Gamma K.\end{aligned}$$

Hence $H\Gamma K$ is a bi-Γ-ideal of S. ■

Theorem 53 *[24] Let S be a Γ-LA-semigroup and H_i be bi-Γ-ideals of S for all $i \in I$. If $\cap_{i\in I} H_i \neq \varnothing$, then $\cap_{i\in I} H_i$ is a bi-Γ-ideal of S.*

Proof. Let S be a Γ-LA-semigroup and H_i be bi-Γ-ideals of S for all $i \in I$. Assume that $\cap_{i\in I} H_i \neq \varnothing$. Let $x, y \in \cap_{i\in I} H_i$, $s \in S$ and $\alpha, \beta \in \Gamma$. Now $x, y \in H_i$ for all $i \in I$ and since for each $i \in I$, H_i is a bi-Γ-ideal of S, so $x\alpha y \in H_i$ and $(x\alpha s)\beta y \in (H_i\Gamma S)\Gamma H_i \subseteq H_i$ for all $i \in I$. Therefore $x\alpha y \in \cap_{i\in I} H_i$ and $(x\alpha s)\beta y \in \cap_{i\in I} H_i$. Hence $\cap_{i\in I} H_i$ is a bi-Γ-ideal of S for all $i \in I$. ■

Theorem 54 *[24] If B is Γ-idempotent bi-Γ-ideal of a Γ-LA-semigroup S with left identity e, then B is a Γ-ideal of S.*

Proof. Consider

$$\begin{aligned} B\Gamma S &= (B\Gamma B)\Gamma S = (S\Gamma B)\Gamma B = (S\Gamma(B\Gamma B))\Gamma B \\ &= ((B\Gamma B)\Gamma S)\Gamma B = (B\Gamma S)\Gamma B \subseteq B. \end{aligned}$$

Which implies that B is a right Γ-ideal and so is left Γ-ideal of S. Hence B is a Γ-ideal of S. ■

Lemma 55 *[24] If B is a proper bi-Γ-ideal of a Γ-LA-semigroup S with left identity e, then $e \notin B$.*

Proof. On contrary let $e \in B$. Now consider $s\alpha b = (e\gamma s)\alpha b \in B$. Also for any $s \in S$ and any $\gamma \in \Gamma$,we have $s = (e\gamma e)\gamma s = (s\gamma e)\gamma e \in (S\Gamma B)\Gamma B \subseteq B$ which implies that $S \subseteq B$. A contradiction to the hypothesis. Hence $e \notin B$. ■

Proposition 56 *[24] If H and K are bi-Γ-ideals of a Γ-LA-semigroup S with left identity e, then the following assertions are equivalent:*

(1) every bi-Γ-ideal of S is Γ-idempotent,

(2) $H \cap K = H\Gamma K$,

(3) the Γ-ideals of S form a semilattice $(L_S, \wedge)$, where $H \wedge K = H\Gamma K$.

Proof. $(1) \Rightarrow (2)$: By lemma 54, it is obvious that $H\Gamma K \subseteq H \cap K$. For reverse inclusion, as $H \cap K \subseteq H$ and also $H \cap K \subseteq K$, so $(H \cap K)\Gamma(H \cap K) \subseteq H\Gamma K$ which implies that $H \cap K \subseteq H\Gamma K$. Hence $H \cap K = H\Gamma K$.

$(2) \Rightarrow (3)$: $H \wedge K = H\Gamma K = H \cap K = K \cap H = K \wedge H$. Also $H \wedge H = H\Gamma H = H \cap H = H$. Similarly associativity follows. Hence $(L_S, \wedge)$ is a semilattice.

$(3) \Rightarrow (1)$: $H = H \wedge H = H\Gamma H$. ■

Definition 57 *[24]* A bi-Γ-ideal P of a Γ-LA-semigroup S is said to be prime bi-Γ-ideal if for all bi-Γ-ideals A and B of S, $A\Gamma B \subseteq P$ implies either $A \subseteq P$ or $B \subseteq P$.

Definition 58 *[24]* The set of bi-Γ-ideals of S is totally ordered under inclusion if for all bi-Γ-ideals I, J either $I \subseteq J$ or $J \subseteq I$.

The following theorem gives necessary and sufficient conditions for a bi-Γ-ideal to be a prime bi-Γ-ideal.

Theorem 59 *[24] Every bi-Γ-ideal of a $\Gamma-$LA-semigroup S with left identity e is a prime if and only if it is Γ-idempotent and the set of bi-Γ-ideals of S is totally ordered under inclusion.*

1.2.3 Γ-ideals in regular $\Gamma-$LA-semigroups

Definition 60 *[24]* A Γ-LA-semigroup S is said to be a regular Γ-LA-semigroup if for each a in S there exists $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$.

Definition 61 A Γ-LA-semigroup S is called an Γ-LA-band if its all the elements are idempotents.

Definition 62 A Γ-LA-semigroup S is said to be a intra-regular Γ-LA-semigroup if for each a in S there exists $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = (x\alpha a)\beta(a\gamma b)$.

Lemma 63 *[24] Every right Γ-ideal of a regular $\Gamma-$LA-semigroup is a Γ-ideal.*

Lemma 64 *[24] Every regular $\Gamma-$LA-semigroup is fully Γ-idempotent.*

Proof. Let S be a regular $\Gamma-$LA-semigroup and I be a Γ-ideal of S. It is always true that $I\Gamma I \subseteq I$. Now if $a \in I$, then as S is regular $\Gamma-$LA-semigroup, so there exists $b \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha b)\beta a \in I\Gamma I$. Thus $I \subseteq I\Gamma I$, and hence S is fully Γ-idempotent. ■

Lemma 65 *[24] If S is a regular $\Gamma-$LA-semigroup, then $H\Gamma K = H \cap K$, where H is right Γ-ideal and K is left Γ-ideal.*

Proof. Let H and K be right and left Γ-ideals of S with $H\Gamma K \subseteq H \cap K$. Now, let $x \in H \cap K$, then there exist $y \in S$ and $\alpha, \beta \in \Gamma$ such that $x = (x\alpha y)\beta x \in H\Gamma K$. Hence $H\Gamma K = H \cap K$. ■

1.2.4 Prime Γ-ideals and Semi-prime Γ-ideals in Γ−LA-semigroups

Definition 66 *[24]* A Γ-ideal P of Γ-LA-semigroup S is said to be prime if $A\Gamma B \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$, for all Γ-ideals A and B in S.

Example 67 *Let S be Γ-LA-semigroups as given in Example 33 and let $P = \{1, 2, 3, 4\}$ be subset of S. Then clearly P is a prime Γ-ideal of S.*

Definition 68 *[24]* A Γ-ideal P is called semiprime if $I\Gamma I \subseteq P$ implies that $I \subseteq P$, for any Γ-ideals I of S. If every Γ-ideal of Γ-LA-semigroup S is semiprime, then S is said to be fully semiprime and if every Γ-ideal is prime, then S is called fully prime.

Theorem 69 *[24] A Γ−LA-semigroup S with left identity e is fully prime if and only if every Γ-ideal in S is Γ-idempotent and Γ-ideal(S) is totally ordered under inclusion.*

Proof. Let S be fully Γ-prime. Let I be a Γ-ideal in S. Then by Lemma 36, $I\Gamma I$ will also be a Γ-ideal in S and hence $I\Gamma I \subseteq I$. Also $I\Gamma I \subseteq I\Gamma I$. But as S is fully Γ-prime, so it implies that $I \subseteq I\Gamma I$. Thus $I\Gamma I = I$ and hence I is Γ-idempotent. Now let H, K be Γ-ideals of S and $H\Gamma K \subseteq H$, $H\Gamma K \subseteq K$ which imply that $H\Gamma K \subseteq H \cap K$. Now as $H \cap K$ is prime, so $H \subseteq H \cap K$ or $K \subseteq H \cap K$ which further imply that $H \subseteq K$ or $K \subseteq H$. Hence Γ-ideal(S) is totally ordered under inclusion. Conversely, let every Γ-ideal is Γ-idempotent and Γ-ideal(S) is totally ordered under inclusion. Let I, J and P be Γ-ideals in S with $I\Gamma J \subseteq P$ such that $I \subseteq J$. As I is Γ-idempotent, so $I = I\Gamma I \subseteq I\Gamma J \subseteq P$ which imply that S is fully prime. ∎

Theorem 70 *[24] A regular Γ−LA-semigroup S is fully prime if and only if Γ-ideal(S) is totally ordered under inclusion.*

Proof. Proof follows from Theorem 69 and Lemma 65. ∎

Definition 71 *[24]* A Γ-ideal I of a regular Γ-LA-semigroup S is said to be strongly irreducible if for Γ-ideals P and Q of S, $P \cap Q \subseteq I$ implies that either $P \subseteq I$ or $K \subseteq I$.

Theorem 72 *[24] Every Γ-ideal in a regular Γ−LA-semigroup S is prime if and only if it is strongly irreducible.*

Proof. Assume that P is a prime Γ-ideal of S. Then there exist Γ-ideals A and B in S such that $A\Gamma B \subseteq P$. Now by Lemma 65 $A\Gamma B = A \cap B$ implies that either $A \subseteq P$ or $B \subseteq P$. Hence P is strongly irreducible.

Conversely, let every Γ-ideal of a regular $\Gamma-$LA-semigroup S is strongly irreducible. Then for any Γ-ideals A and B of S, $A \cap B \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$. But by Lemma 65, $A\Gamma B = A \cap B$. Hence P is a prime Γ-ideal of S. ∎

1.3 Fuzzy Sets

The fundamental concept of a fuzzy set, introduced by L. A. Zadeh in his paper [28] of 1965, provides a natural frame-work for generalizing several basic notions of algebra. Kuroki initiated the theory of fuzzy semigroups in his papers [15, 16]. a Systematic exposition of fuzzy semigroups by Mordeson et al. appeared in [17], where one can find theoretical results on fuzzy semigroups and their use in fuzzy coding, fuzzy finite state machines and fuzzy languages. Fuzziness has a natural place in the field of formal languages. The monograph by Mordeson and Malik [18] deals with the application of fuzzy approach to the concept of automata and formal languages.

Definition 73 *[28]* A fuzzy set f of a universe X is a function from X into the unit closed interval $[0,1]$, i.e. $f : X \rightarrow [0,1]$. For any fuzzy sets f and g of X, $f \leq g$ means that $f(x) \leq g(x)$ for all $x \in X$. The symbol $f \wedge g$ and $f \vee g$ means the following fuzzy set of X :

$$\begin{aligned}(f \wedge g)(x) &= f(x) \wedge g(x) \\ (f \vee g)(x) &= f(x) \vee g(x)\end{aligned}$$

for all $x \in X$. This definition is clearly depicted in the following figures.

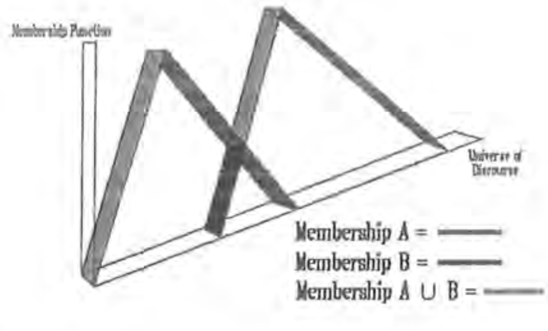

Figure 1

Figure 2

Definition 74 *[28]* The complement of fuzzy set f is denoted by f^c and is defined as $f^c(x) = 1 - f(x)$ for all $x \in X$.

Figure 3

Let f and g be two fuzzy sets of S. Then the product $f \circ g$ is defined by

$$(f \circ g)(x) = \begin{cases} \vee_{x=yz} \{f(y) \wedge g(z)\}, & \textit{if} \; \exists \; y, z \in S, \text{ such that } x = yz \\ 0 & \text{otherwise} \end{cases}$$

Definition 75 *[28]* Let f be a fuzzy set of universe X. Then set is denoted by $f_t = \{x : f(x) \geq t \text{ for all } t \in [0,1]\}$ is called level set of f.

Definition 76 *[28]*Let $f, g.h$ be any fuzzy set of universe X. then the following properties hold,

(*i*) $(f \cup g)' = f' \cap g'$ and $(f \cap g)' = f' \cup g'$ (De Morgan's laws).

(*ii*) $h \cap (f \cup g) = (h \cap f) \cup (h \cap g)$ Distributive laws.

(*iii*) $h \cup (f \cap g) = (h \cup f) \cap (h \cup g)$.

Definition 77 *[28]* A fuzzy set f of universe X is convex if and only if the sets Γ_t defined by

$$\Gamma_t = \{x : f(x) \geq t\}$$

are convex for all t in the interval $(0, 1]$.

Theorem 78 *[28]If f and g are convex, then $f \cap g$ is also convex.*

Definition 79 *[28]A fuzzy set f of universe X is bounded if and only if the sets $\Gamma_t = \{x : f(x) \geq t\}$ are bounded for all $t > 0$; that is, for every $t > 0$ there exists a finite $R(t)$ such that $\|x\| \leq R(t)$ for all x in Γ_t.*

Definition 80 Let X be a non empty set and A be a subset of X. Then the characteristic functions of A is the function C_A of X into $\{0, 1\}$ defined by

$$C_A = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Lemma 81 *Let A and B be non-empty subsets of an LA-semigroup S. Then the following hold.*

(1) $C_A \wedge C_B = C_{A \cap B}$.

(2) $C_A \circ C_B = C_{A \circ B}$.

We note that the LA-semigroup S can be considered a fuzzy subset of itself and we write $S = C_S$, i.e. $S(x) = 1$ for all $x \in S$.

Lemma 82 *Let A be non-empty subset of a Γ-LA-semigroup S. Then A is a left (resp, right) if and only if C_A is a fuzzy left 9resp, right) ideal of S.*

1.4 Intuitionistic Fuzzy Sets

In this section, we shall define intuitionistic fuzzy set which was given by K. T. Atanassov in his pioneer paper [2] and we shall discuss some basic results of intuitionistic fuzzy sets.

Definition 83 *[2]* Let E be a nonempty fixed set. An intuitionistic fuzzy set (briefly, IFS) A is an object having the form

$$A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in E\}$$

where the functions $\mu_A : E \longrightarrow I = [0,1]$ and $\gamma_A : E \longrightarrow I = [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in E$. An intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in E\}$ in E can be identified to an ordered pair $\langle \mu_A, \gamma_A\rangle$ in $I^E \times I^E$. For the sake of simplicity, we use the symbol $A = \langle \mu_A, \gamma_A\rangle$ for the IFS $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in E\}$.

Definition 84 [7, 8] If $A = \langle \mu_A, \gamma_A\rangle$ and $B = \langle \mu_B, \gamma_B\rangle$ are two IFSs of the set E, then

$A \subseteq B$ iff $\forall x \in E$, $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$,
$A = B$ iff $\forall x \in E$, $\mu_A(x) = \mu_B(x)$ and $\gamma_A(x) = \gamma_B(x)$,
$\Box A = \{x,\ \mu_A(x),\ 1 - \mu_A(x) | x \in E\}$,
$\Diamond\ A = \{x,\ 1 - \gamma_A(x),\ \gamma_A(x) | x \in E\}$.
$\bar{A} = \{x, \gamma_A(x),\ \mu_A(x) | x \in E\}$
$A \cup B = \{x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\gamma_A(x), \gamma_B(x)\} | x \in E\}$
$A \cap B = \{x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\gamma_A(x), \gamma_B(x)\} | x \in E\}$
$A + B = \{x, \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x),\ \gamma_A(x) \cdot \gamma_B(x) | x \in E\}$
$A \cdot B = \{x, \mu_A(x) \cdot \mu_B(x),\ \gamma_A(x) + \gamma_B(x) - \gamma_A(x) \cdot \gamma_B(x) | x \in E\}$

Definition 85 *[30]* For an IFS A of the set E and for any positive integer n,

(*i*) $A^n = \{x,\ [\mu_A(x)]^n,\ 1 - [1 - \gamma_A(x)]^n | x \in E\}$, where $0 \leq [\mu_A(x)]^n + 1 - [1 - \gamma_A(x)]^n \leq 1$,

(*ii*) $nA = \{x,\ 1 - [1 - \mu_A(x)]^n,\ [\gamma_A(x)]^n | x \in E\}$, where $0 \leq 1 - [1 - \mu_A(x)]^n + [\gamma_A(x)]^n \leq 1$.

Proposition 86 *[30] For an IFS A of the set E and for any integer n,*

(*i*) $\Box A^n = (\Box A)^n$,

(*ii*) $\Diamond A^n = (\Diamond A)^n$,

(*iii*) *if* $\pi_A(x) = 0$, *then* $\pi_{A^n}(x) = 0$,

(*iv*) $A^m \subseteq A^n$, *where* m *and* n *are both positive integers and* $m \geq n$,

(*v*) *if* A *is totally intuitionistic, then* A^n *is also so.*

Proposition 87 *[30] For an IFS A of the set E and for any integer n,*

(*i*) $\Box nA = n\,(\Box A)$,

(*ii*) $\Diamond nA = n(\Diamond A)$,

(*iii*) *if* $\pi_A(x) = 0$, *then* $\pi_{nA}(x) = 0$,

(*iv*) $mA \subseteq nA$, *where* m *and* n *are both positive integers and* $m \geq n$,

(*v*) *if* A *is totally intuitionistic, then* nA *is also so.*

Proposition 88 *[30] Consider two IFSs A and B of the set E, for any positive integer n,*

(*i*) *if* $A \subseteq B$, *then* $A^n \subseteq B^n$,

(*ii*) *if* $A \subseteq B$, *then* $nA \subseteq nB$,

(*iii*) $(A \cap B)^n = A^n \cap B^n$,

(*iv*) $(A \cup B)^n = A^n \cup B^n$,

(*v*) $n\,(A \cap B) = nA \cap nB$,

(*vi*) $n\,(A \cup B) = nA \cup nB$.

Chapter 2

Intuitionistic Fuzzy Set in $\Gamma-$LA-semigroups

2.1 Introduction

In this chapter, we have defined intuitionistic fuzzy left (right) $\Gamma-$ideals, intuitionistic fuzzy $\Gamma-$ideals and intuitionistic fuzzy bi-$\Gamma-$ideals of $\Gamma-$LA-semigroups S and some related properties are investigated. We have also defined intuitionistic fuzzy $\Gamma-$due. Characterizations of intuitionistic fuzzy left (right) $\Gamma-$ideals are given. Lastly, we have showed that every $\Gamma-$LA band is intuitionistic fuzzy $\Gamma-$due.

In what follows, S denote as $\Gamma-$LA-semigroup, unless otherwise specified.

2.2 Intuitionistic Fuzzy $\Gamma-$ideals and Intuitionistic Fuzzy bi-$\Gamma-$ideals in $\Gamma-$LA-semigroups

Definition 89 An IFS $A = \langle \mu_A, \gamma_A \rangle$ in S is called an intuitionistic fuzzy $\Gamma-$subLA-semigroup of S if

$(IF1)$ $\mu_A(x\gamma y) \geq \mu_A(x) \wedge \mu_A(y)$,

$(IF2)$ $\gamma_A(x\gamma y) \leq \gamma_A(x) \vee \gamma_A(y)$, for all $x, y \in S$ and $\gamma \in \Gamma$.

Definition 90 An IFS $A = \langle \mu_A, \gamma_A \rangle$ in S is called an intuitionistic fuzzy right $\Gamma-$ideal of S if

$(IF3)$ $\mu_A(x\gamma y) \geq \mu_A(x)$,

$(IF4)$ $\gamma_A(x\gamma y) \leq \gamma_A(x)$, for all $x, y \in S$ and $\gamma \in \Gamma$.

Definition 91 An IFS $A = \langle \mu_A, \gamma_A \rangle$ in S is called an intuitionistic fuzzy left $\Gamma-$ideal of S if

$(IF5)$ $\mu_A(x\gamma y) \geq \mu_A(y)$,

$(IF6)$ $\gamma_A(x\gamma y) \leq \gamma_A(y)$, for all $x, y \in S$ and $\gamma \in \Gamma$.

An IFS $A = \langle \mu_A, \gamma_A \rangle$ in S is called an intuitionistic fuzzy $\Gamma-$ideal of S if $A = \langle \mu_A, \gamma_A \rangle$ is both intuitionistic fuzzy left and right $\Gamma-$ideal of S.

Example 92 *Let $S = \{-i, 0, i\}$ and $\Gamma = S$. Then by defining $S \times \Gamma \times S \rightarrow S$ as $a\gamma b = a.\gamma.b$ for all $a, b \in S$ and $\gamma \in \Gamma$. It can be easily verified that S is a $\Gamma-LA$-semigroup under complex number multiplication while S is not an LA-semigroup. Let $A = \langle \mu_A, \gamma_A \rangle$ be IFS on S. Define $\mu_A : S \longrightarrow [0,1]$ by $\mu_A(0) = 0.7, \mu_A(i) = \mu_A(-i) = 0.5$ and $\gamma_A : S \longrightarrow [0,1]$ by $\gamma_A(0) = 0.2, \gamma_A(i) = \gamma_A(-i) = 0.4$. Then clearly $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy $\Gamma-$ ideal of $\Gamma-LA$-semigroup S.*

Example 93 *Let $S = \{1, 2, 3, 4, 5\}$ be a $\Gamma-LA$-semigroup as given in Example 33. Let $A = \langle \mu_A, \gamma_A \rangle$ be an IFS in S and defined as:*

$$\begin{aligned} \mu_A(1) &= 0.9 = \mu_A(2),\ \mu_A(3) = 0.7,\ \mu_A(4) = 0.5,\ \mu_A(5) = 0.3 \\ \gamma_A(1) &= 0.1 = \gamma_A(2),\ \gamma_A(3) = 0.3,\ \gamma_A(4) = 0.5,\ \gamma_A(5) = 0.7 \end{aligned}$$

Then by routine calculation $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy $\Gamma-$ideal of S.

Theorem 94 *Let S be a $\Gamma-LA$-semigroup with left identity. Then, every an intuitionistic fuzzy right $\Gamma-$ ideal of S is an intuitionistic fuzzy left $\Gamma-$ ideal of S.*

Proof. Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy right $\Gamma-$ ideal of S and let $x, y \in S$ and $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned}
\mu_A(x\alpha y) &= \mu_A((e\beta x)\alpha y) = \mu_A((y\beta x)\alpha e) \\
&\geq \mu_A(y\beta x) \geq \mu_A(y) \\
\mu_A(x\alpha y) &\geq \mu_A(y)
\end{aligned}$$

and

$$\begin{aligned}
\gamma_A(x\alpha y) &= \gamma_A((e\beta x)\alpha y) = \gamma_A((y\beta x)\alpha e) \\
&\leq \gamma_A(y\beta x) \leq \gamma_A(y) \\
\gamma_A(x\alpha y) &\leq \gamma_A(y)
\end{aligned}$$

Hence $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy left $\Gamma-$ ideal of S. ■

Corollary 95 *In $\Gamma-$LA-semigroup S with left identity, every intuitionistic fuzzy right $\Gamma-$ ideal is an intuitionistic fuzzy $\Gamma-$ ideal.*

Theorem 96 *Let $\{A_i\}_{i\in\Lambda}$ be a family of intuitionistic fuzzy $\Gamma-$ideals of $\Gamma-$LA-semigroup S. Then $\bigcap\limits_{i\in\Lambda} A_i$ is an intuitionistic fuzzy $\Gamma-$ideal of S and defined as*

$$\begin{aligned}
\bigcap_{i\in\Lambda} A_i &= \langle \bigwedge_{i\in\Lambda} \mu_{A_i}, \bigvee_{i\in\Lambda} \gamma_{A_i} \rangle \text{ and} \\
\bigwedge_{i\in\Lambda} \mu_{A_i}(x) &= \inf\{\mu_{A_i}(x) \,:\, i \in \Lambda,\ x \in S\} \\
\bigvee_{i\in\Lambda} \gamma_{A_i}(x) &= \sup\{\gamma_{A_i}(x) \,:\, i \in \Lambda,\ x \in S\}
\end{aligned}$$

Proof. Let $\{A_i\}_{i\in\Lambda}$ be intuitionistic fuzzy $\Gamma-$ideals of $\Gamma-$LA-semigroup S and let for any $x, y \in S$ and $\gamma \in \Gamma$.

$$\begin{aligned}
\bigwedge_{i\in\Lambda} \mu_{A_i}(x\gamma y) &\geq \bigwedge_{i\in\Lambda} \mu_{A_i}(x) \\
\bigvee_{i\in\Lambda} \gamma_{A_i}(x\gamma y) &\leq \bigvee_{i\in\Lambda} \gamma_{A_i}(x)
\end{aligned}$$

and

$$\begin{aligned}
\bigwedge_{i\in\Lambda} \mu_{A_i}(x\gamma y) &\geq \bigwedge_{i\in\Lambda} \mu_{A_i}(y) \\
\bigwedge_{i\in\Lambda} \gamma_{A_i}(x\gamma y) &\leq \bigwedge_{i\in\Lambda} \gamma_{A_i}(y)
\end{aligned}$$

Hence $\cap A_i = \langle \wedge\mu_{A_i}, \vee\gamma_{A_i} \rangle$ is an intuitionistic fuzzy $\Gamma-$ideal of $\Gamma-$LA-semigroup S.
■

Theorem 97 *Let $IF(S)$ be denote the set of all intuitionistic fuzzy left(right) $\Gamma-$ideal of $\Gamma-$LA-semigroup S. Then $(IF(S), \subseteq, U, \cap)$ is a lattice.*

Proof. For all $A, B, C \in IF(S)$

1) Reflexive: Since

$$\mu_A(x) \leq \mu_A(x) \text{ and } \gamma_A(x) \geq \gamma_A(x)$$

Thus $A \subseteq B$

2) Antisymmetric: For all $A, B \in IF(S)$ such that $A \subseteq B$ and $B \subseteq A$. Then

$$\mu_A(x) \leq \mu_B(x), \gamma_A(x) \geq \gamma_B(x)$$

and

$$\mu_B(x) \leq \mu_A(x), \gamma_B(x) \geq \gamma A(x)$$

for all $x \in S$. Thus $A = B$

3) Transitive: For all $A, B, C \in IF(S)$ such that

$$A \subseteq B \text{ and } B \subseteq C.$$

Then

$$\begin{aligned} \mu_A(x) &\leq \mu_B(x), \gamma_A(x) \geq \gamma_B(x) \\ \mu_B(x) &\leq \mu_C(x), \gamma_B(x) \geq \gamma_C(x) \end{aligned}$$

it is follows that

$$\mu_A(x) \leq \mu_C(x), \gamma_A(x) \geq \gamma_C(x).$$

Thus $A \subseteq C$. Hence $(IF(S), \subseteq)$ is poset. Now for lattice we have see that sup and inf of any two intuitionistic fuzzy set $A, B \in (IF(S)$

inf: For any two $A, B \in (IF(S)$, $\inf\{A, B\} = A \cap B$

$$A \cap B = \{\mu_A \wedge \mu_B, \gamma_A \vee \gamma_B\}.$$

Now, we show that $A \cap B$ is an intuitionistic fuzzy right $\Gamma-$ideal of $\Gamma-$LA-semigroup S. For any $x, y \in S$ and $\alpha \in \Gamma$

$$\begin{aligned} (\mu_A \wedge \mu_B)(x\alpha y) &= \mu_A(x\alpha y) \wedge \mu_A(x\alpha y) \\ &\geq \mu_A(x) \wedge \mu_A(x) = (\mu_A \wedge \mu_B)(x) \\ (\mu_A \wedge \mu_B)(x\alpha y) &\geq (\mu_A \wedge \mu_B)(x) \end{aligned}$$

and

$$\begin{aligned}(\gamma_A \vee \gamma_B)(x\alpha y) &= \gamma_A(x\alpha y) \vee \gamma_A(x\alpha y) \\ &\leq \gamma_A(x) \vee \gamma_A(x) = (\gamma_A \vee \gamma_B)(x) \\ (\gamma_A \vee \gamma_B)(x\alpha y) &\leq (\gamma_A \vee \gamma_B)(x)\end{aligned}$$

$A \cap B$ is an intuitionistic fuzzy right $\Gamma-$ideal of $\Gamma-$LA-semigroup S. This mean $A \cap B \in IF(S)$, $\inf\{A, B\}$ exist in $IF(S)$.

For any two $A, B \in (IF(S)$, $\sup\{A, B\} = A \cup B$

$$\begin{aligned}A \cup B &= \{\mu_A \vee \mu_B, \gamma_A \wedge \gamma_B\} \\ (\mu_A \vee \mu_B)(x\alpha y) &= \mu_A(x\alpha y) \vee \mu_A(x\alpha y) \\ &\geq \mu_A(x) \vee \mu_A(x) = (\mu_A \vee \mu_B)(x) \\ (\mu_A \vee \mu_B)(x\alpha y) &\geq (\mu_A \vee \mu_B)(x)\end{aligned}$$

and

$$\begin{aligned}(\gamma_A \wedge \gamma_B)(x\alpha y) &= \gamma_A(x\alpha y) \wedge \gamma_A(x\alpha y) \\ &\leq \gamma_A(x) \wedge \gamma_A(x) = (\gamma_A \wedge \gamma_B)(x) \\ (\gamma_A \wedge \gamma_B)(x\alpha y) &\leq (\gamma_A \wedge \gamma_B)(x)\end{aligned}$$

$A \cup B$ is an intuitionistic fuzzy right $\Gamma-$ideal of $\Gamma-$LA-semigroup S. This mean $A \cup B \in IF(S)$, $\sup\{A, B\}$ exist in $IF(S)$. Hence $(IF(S), \subseteq, U, \cap)$ is a lattice. ■

Definition 98 Let $A = \langle \mu_A, \gamma_A \rangle$ and $B = (\mu_{B,} \gamma_B)$ be two intuitionistic fuzzy sets in $\Gamma-$LA-semigroup S. Then product of $A = \langle \mu_A, \gamma_A \rangle$ and $B = (\mu_{B,} \gamma_B)$ is denoted by $A \circ_\Gamma B$ defined as

$$\mu_{A\circ_\Gamma B}(x) = \begin{cases} \bigvee\limits_{x=y\alpha z} \{\mu_A(y) \wedge \mu_B(z)\}, \textit{ If } x = y\alpha z \textit{ for some } x, y \in S \textit{ and } \alpha \in \Gamma. \\ 0 \qquad \textit{otherwise} \end{cases}$$

$$\gamma_{A\circ_\Gamma B}(x) = \begin{cases} \bigwedge\limits_{x=y\alpha z} \{\gamma_A(y) \vee \gamma_A(z)\}, \textit{ If } x = y\alpha z \textit{ for some } x, y \in S \textit{ and } \alpha \in \Gamma. \\ 1 \qquad \textit{otherwise} \end{cases}$$

Thus $(IF(S), \circ_\Gamma)$ is a $\Gamma-$LA-semigroup. ∎

Proposition 100 *Let S be a $\Gamma-LA$-semigroup with left identity. If $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy right $\Gamma-$ideal of $\Gamma-LA$-semigroup S, then $A \circ_\Gamma A$ is an intuitionistic fuzzy $\Gamma-$ideal of S.*

Proof. Since $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy right $\Gamma-$ideal of S, so $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy left $\Gamma-$ideal of S. Let $a, b \in S$ and $\alpha, \gamma \in \Gamma$. If $a \neq x\gamma y$, then

$$\mu_{A\circ_\Gamma A}(a) = 0 \text{ and } \mu_{A\circ_\Gamma A}(a\alpha b) \geq \mu_{A\circ_\Gamma A}(a)$$

and

$$\gamma_{A\circ_\Gamma A}(a) = 1 \text{ and } \gamma_{A\circ_\Gamma A}(a\alpha b) \leq \gamma_{A\circ_\Gamma A}(a).$$

Otherwise

$$\begin{aligned}
\mu_{A\circ_\Gamma A}(a) &= \bigvee_{a=x\gamma y} \{\mu_A(x) \wedge \mu_A(y)\} \\
\text{If } a &= x\gamma y, \text{ then } a\alpha b = (x\gamma y)\alpha b = (b\gamma y)\alpha x\ . \\
\mu_{A\circ_\Gamma A}(a) &= \bigvee_{a=x\gamma y} \{\mu_A(y) \wedge \mu_A(x)\} \\
\mu_{A\circ_\Gamma A}(a) &\leq \bigvee_{a=x\gamma y} \{\mu_A(b\gamma y) \wedge \mu_A(x)\} \text{ since } A \text{ is IF left } \Gamma-\text{ideal} \\
&\leq \bigvee_{a\alpha b=(b\gamma y)\alpha x} \{\mu_A(b\gamma y) \wedge \mu_A(x)\} = \mu_{A\circ_\Gamma A}(a\alpha b) \\
\mu_{A\circ_\Gamma A}(a\alpha b) &\geq \mu_{A\circ_\Gamma A}(a) \\
\text{and } \gamma_{A\circ_\Gamma A}(a) &= \bigwedge_{a=x\gamma y} \{\gamma_A(x) \vee \gamma_A(y)\} \\
\gamma_{A\circ_\Gamma A}(a) &= \bigwedge_{a=x\gamma y} \{\gamma_A(y) \vee \gamma_A(x)\} \\
\gamma_{A\circ_\Gamma A}(a) &\geq \bigwedge_{a=x\gamma y} \{\gamma_A(b\gamma y) \vee \gamma_A(x)\} \text{ since } A \text{ is IF left } \Gamma-\text{ideal} \\
&\geq \bigwedge_{a\alpha b=(b\gamma y)\alpha x} \{\gamma_A(b\gamma y) \vee \gamma_A(x)\} = \gamma_{A\circ_\Gamma A}(a\alpha b) \\
\gamma_{A\circ_\Gamma A}(a\alpha b) &\leq \gamma_{A\circ_\Gamma A}(a)
\end{aligned}$$

Hence $A \circ_\Gamma A = \langle \mu_{A\circ_\Gamma A}, \gamma_{A\circ_\Gamma A} \rangle$ is an intuitionistic fuzzy right $\Gamma-$ideal of S, and by Theorem 93, $A \circ_\Gamma A = \langle \mu_{A\circ_\Gamma A}, \gamma_{A\circ_\Gamma A} \rangle$ is an intuitionistic fuzzy left $\Gamma-$ideal of S. Therefore $A \circ_\Gamma A = \langle \mu_{A\circ_\Gamma A}, \gamma_{A\circ_\Gamma A} \rangle$ is an intuitionistic fuzzy $\Gamma-$ideal of S. ∎

Theorem 101 *Let S be a $\Gamma-LA$-semigroup with left identity. Then for any A, B, C in $IF(S)$. We have $A \circ_\Gamma (B \circ_\Gamma C) = B \circ_\Gamma (A \circ_\Gamma C)$*

Lemma 99 *Let $A = \langle \mu_A, \gamma_A \rangle$ and $B = (\mu_{B,} \gamma_B)$ be any two intuitionistic fuzzy right(left) $\Gamma-$ideals of $\Gamma-$LA-semigroup S with left identity. Then $A \circ_\Gamma B$ is also intuitionistic fuzzy right(left) $\Gamma-$ideal of S.*

Theorem 100 *Let $IF(S)$ denote the set of all intuitionistic fuzzy left(right) $\Gamma-$ideals of $\Gamma-$LA-semigroup S with left identity. Then $(IF(S), \circ_\Gamma)$ is $\Gamma-$LA-semigroup.*

Proof. Clearly $(IF(S), \circ_\Gamma)$ is closed by Lemma 99. Now for any $A = \langle \mu_A, \gamma_A \rangle$, $B = (\mu_{B,} \gamma_B)$ and $C = (\mu_{C,} \gamma_C) \in IF(S)$,

$$\begin{aligned}
\mu_{(A\circ_\Gamma B)\circ_\Gamma C}(x) &= \bigvee_{x=y\alpha z} \{\mu_{A\circ_\Gamma B}(y) \wedge \mu_C(z)\} \\
&= \bigvee_{x=y\alpha z} \{\vee_{y=p\beta q}\{\mu_A(p) \wedge \mu_B(q)\} \wedge \mu_C(z)\} \\
&= \bigvee_{x=(p\beta q)\alpha z} \{\mu_A(p) \wedge \mu_B(q) \wedge \mu_C(z)\} \\
&= \bigvee_{x=(z\beta q)\alpha p} \{\mu_C(z) \wedge \mu_B(q) \wedge \mu_A(p)\} \\
&\leq \bigvee_{x=w\alpha p} \{\vee_{w=z\beta q}\{\mu_C(z) \wedge \mu_B(q)\} \wedge \mu_C(p)\} \\
&= \bigvee_{x=w\alpha p} \{\mu_{C\circ_\Gamma B}(w) \wedge \mu_A(p)\} = \mu_{(C\circ_\Gamma B)\circ_\Gamma A}(x)
\end{aligned}$$

This implies $\mu_{(A\circ_\Gamma B)\circ_\Gamma C}(x) \leq \mu_{(C\circ_\Gamma B)\circ_\Gamma A}(x)$.

Similarly $\mu_{(C\circ_\Gamma B)\circ_\Gamma A}(x) \leq \mu_{(A\circ_\Gamma B)\circ_\Gamma C}(x)$ and thus $\mu_{(A\circ_\Gamma B)\Gamma C}(x) = \mu_{(C\circ_\Gamma B)\circ_\Gamma A}(x)$

Now,

$$\begin{aligned}
\gamma_{(A\circ_\Gamma B)\circ_\Gamma C}(x) &= \bigwedge_{x=y\alpha z} \{\gamma_{A\circ_\Gamma B}(y) \vee \gamma_C(z)\} \\
&= \bigwedge_{x=y\alpha z} \{\wedge_{y=m\beta n}\{\gamma_A(m) \vee \gamma_B(n)\} \vee \gamma_C(z)\} \\
&= \bigwedge_{x=(m\beta n)\alpha z} \{\gamma_A(m) \vee \gamma_B(n) \vee \gamma_C(z)\} \\
&= \bigwedge_{x=(z\beta n)\alpha m} \{\gamma_C(z) \vee \gamma_B(n) \vee \gamma_A(m)\} \\
&\geq \bigwedge_{x=l\alpha m} \{\wedge_{x=z\beta n}\{\gamma_C(z) \vee \gamma_B(n)\} \vee \gamma_A(m)\} \\
&= \bigwedge_{x=l\alpha m} \{\gamma_{A\circ_\Gamma B}(l) \vee \gamma_C(m)\} = \gamma_{(C\circ_\Gamma B)\circ_\Gamma A}(x) \\
\gamma_{(A\circ_\Gamma B)\circ_\Gamma C}(x) &\geq \gamma_{(C\circ_\Gamma B)\circ_\Gamma A}(x)
\end{aligned}$$

Similarly $\gamma_{(C\circ_\Gamma B)\circ_\Gamma A}(x) \geq \gamma_{(A\circ_\Gamma B)\circ_\Gamma C}(x)$ and thus $\gamma_{(A\circ_\Gamma B)\circ_\Gamma C}(x) = \gamma_{(C\circ_\Gamma B)\circ_\Gamma A}(x)$

Hence

$$(A \circ_\Gamma B) \circ_\Gamma C = (C \circ_\Gamma B) \circ_\Gamma A$$

Proof. Let $x \in S$ and $A = \langle \mu_A, \gamma_A \rangle, B = \langle \mu_B, \gamma_B \rangle, C = \langle \mu_C, \gamma_C \rangle$ be any IFS of S. Then

$$\begin{aligned}
\mu_{A \circ_\Gamma (B \circ_\Gamma C)}(x) &= \bigvee_{x = y\alpha z} \{\mu_A(y) \wedge \mu_{B \circ_\Gamma C}(z)\} \\
&= \bigvee_{x = y\alpha z} \{\mu_A(y) \wedge [\bigvee_{z = s\beta t} \{\mu_B(s) \wedge \mu_C(t)\}]\} \\
&= \bigvee_{x = y\alpha(s\beta t)} \{\mu_A(y) \wedge \mu_B(s) \wedge \mu_C(t)\} \\
&= \bigvee_{x = s\alpha(y\beta t)} \{\mu_B(s) \wedge \mu_A(y) \wedge \mu_C(t)\} \\
\text{since } \mu_A(y) \wedge \mu_C(t) &\leq \bigvee_{y\alpha t = a\gamma b} \{\mu_A(a) \wedge \mu_C(b)\} \\
\text{so} \qquad &\leq \bigvee_{x = s\alpha(y\beta t)} \{\mu_B(s) \wedge [\bigvee_{y\beta t = a\gamma b} \{\mu_A(a) \wedge \mu_C(b)\}]\} \\
&= \bigvee_{x = s\alpha(y\beta t)} \{\mu_B(s) \wedge \mu_{A \circ_\Gamma C}(y\beta t)\} \\
&\leq \bigvee_{x = p\alpha q} \{\mu_B(p) \wedge \mu_{A \circ_\Gamma C}(q)\} = \mu_{B \circ_\Gamma (A \circ_\Gamma C)}(x) \\
\mu_{A\Gamma(B\Gamma C)}(x) &\leq \mu_{B\Gamma(A\Gamma C)}(x) \Longrightarrow \mu_{A\Gamma(B\Gamma C)} \leq \mu_{B\Gamma(A\Gamma C)}. \\
\text{Similarly } \gamma_{A\Gamma(B\Gamma C)}(x) &\geq \gamma_{B\Gamma(A\Gamma C)}(x) \Longrightarrow \gamma_{A\Gamma(B\Gamma C)} \geq \gamma_{B\Gamma(A\Gamma C)}
\end{aligned}$$

Now,

$$\begin{aligned}
\gamma_{A \circ_\Gamma (B \circ_\Gamma C)}(x) &= \bigwedge_{x = y\alpha z} \{\gamma_A(y) \vee \gamma_{B \circ_\Gamma C}(z)\} \\
&= \bigwedge_{x = y\alpha z} \{\gamma_A(y) \vee [\bigwedge_{z = s\beta t} \{\gamma_B(s) \vee \gamma_C(t)\}]\} \\
&= \bigwedge_{x = y\alpha(s\beta t)} \{\gamma_A(y) \vee \gamma_B(s) \vee \gamma_C(t)\} \\
&= \bigwedge_{x = s\alpha(y\beta t)} \{\gamma_B(s) \vee \gamma_A(y) \vee \gamma_C(t)\} \\
\text{since } \gamma_A(y) \wedge \gamma_C(t) &\geq \bigwedge_{y\alpha t = a\gamma b} \{\gamma_A(a) \wedge \gamma_C(b)\} \\
&\geq \bigwedge_{x = s\alpha(y\beta t)} \{\gamma_B(s) \vee [\bigwedge_{y\beta t = a\gamma b} \{\gamma_A(a) \vee \gamma_C(b)\}]\}, \\
\text{so} \qquad &= \bigwedge_{x = s\alpha(y\beta t)} \{\gamma_B(s) \vee \gamma_{A \circ_\Gamma C}(y\beta t)\} \\
&\geq \bigwedge_{x = p\alpha q} \{\gamma_B(p) \vee \gamma_{A \circ_\Gamma C}(q)\} = \gamma_{B \circ_\Gamma (A \circ_\Gamma C)}(x)
\end{aligned}$$

Thus $A \circ_\Gamma (B \circ_\Gamma C) \leq B \circ_\Gamma (A \circ_\Gamma C)$ and similarly $A \circ_\Gamma (B \circ_\Gamma C) \geq B \circ_\Gamma (A \circ_\Gamma C)$. Hence $A \circ_\Gamma (B \circ_\Gamma C) = B \circ_\Gamma (A \circ_\Gamma C)$. ■

Lemma 103 *Let S be a $\Gamma-$LA-semigroup and $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy right $\Gamma-$ideal of S and $B = \langle \mu_B, \gamma_B \rangle$ be an intuitionistic fuzzy left $\Gamma-$ideal of S. Then $A \circ_\Gamma B \subseteq A \cap B$*

Proof. Let for any $x \in S$ and $\alpha \in \Gamma$. If $x \neq y\alpha z$ for any $y, z \in S$, then

$$\mu_{A \circ_\Gamma B}(x) = 0 \leq \mu_{A \cap B}(x) = \mu_A \wedge \mu_B(x).$$

Otherwise

$$\begin{aligned}
\mu_{A\circ_\Gamma B}(x) &= \bigvee_{x=y\alpha z} \{\mu_A(y) \wedge \mu_B(z)\} \\
&\leq \bigvee_{x=y\alpha z} \{\mu_A(y\alpha z) \wedge \mu_B(y\alpha z)\} \\
&= \bigvee_{x=y\alpha z} \{\mu_A(x) \wedge \mu_B(x)\} \\
\mu_{A\circ_\Gamma B}(x) &\leq (\mu_A \wedge \mu_B)(x) \Longrightarrow \mu_{A\circ_\Gamma B} \leq (\mu_A \wedge \mu_B).
\end{aligned}$$

If $x \neq y\alpha z$ for any $y, z \in S$ and $\alpha \in \Gamma$, then

$$\gamma_{A\circ_\Gamma B}(x) = 1 \geq \gamma_{A\cap B}(x) = \gamma_A \wedge \gamma_B(x).$$

Otherwise

$$\begin{aligned}
\mu_{A\circ_\Gamma B}(x) &= \bigwedge_{x=y\alpha z} \{\gamma_A(y) \vee \gamma_B(z)\} \\
&\leq \bigwedge_{x=y\alpha z} \{\gamma_A(y\alpha z) \vee \gamma_B(y\alpha z)\} \\
&= \bigwedge_{x=y\alpha z} \{\gamma_A(x) \vee \gamma_B(x)\} \\
\gamma_{A\circ_\Gamma B}(x) &\leq (\gamma_A \vee \gamma_B)(x) \Longrightarrow \gamma_{A\circ_\Gamma B} \leq (\gamma_A \vee \gamma_B)
\end{aligned}$$

Hence $A \circ_\Gamma B = \langle \mu_{A\circ_\Gamma B}, \gamma_{A\circ_\Gamma B} \rangle \subseteq \langle \mu_A \wedge \mu_B, \gamma_A \vee \gamma_B \rangle = A \cap B$. ∎

Corollary 104 *Let S be a $\Gamma-$LA-semigroup. Then $A \circ_\Gamma B \subseteq A \cap B$, for every intuitionistic fuzzy $\Gamma-$ideals $A = \langle \mu_A, \gamma_A \rangle$ and $B = \langle \mu_B, \gamma_B \rangle$ of S.*

Remark 105 *Let S be a $\Gamma-$LA-semigroup with left identity e. Then $A \circ_\Gamma B \subseteq A \cap B$ for every intuitionistic fuzzy right $\Gamma-$ideals $A = \langle \mu_A, \gamma_A \rangle$ and $B = \langle \mu_B, \gamma_B \rangle$ of S.*

Theorem 106 *Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy left(resp, right) $\Gamma-$ ideal of $\Gamma-$LA-semigroup S. Then $\Box A = \langle \mu_A, \overline{\mu_A} \rangle$ is an intuitionistic fuzzy left(resp, right) $\Gamma-$ ideal of S, where $\overline{\mu_A} = 1 - \mu_A$*

Proof. Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy left $\Gamma-$ ideal of $\Gamma-$LA-semigroup S and let for any $x, y \in S$ and $\gamma \in \Gamma$. Then

$$\begin{aligned} \mu_A(x\gamma y) &\geq \mu_A(y) \\ -\mu_A(x\gamma y) &\leq -\mu_A(y) \\ 1 - \mu_A(x\gamma y) &\leq 1 - \mu_A(y) \\ \bar{\mu_A}(x\gamma y) &\leq \bar{\mu_A}(y) \end{aligned}$$

Hence $\Box A = \langle \mu_A, \overline{\mu_A} \rangle$ is an intuitionistic fuzzy left $\Gamma-$ideal of $\Gamma-$LA-semigroup S. Similarly for intuitionistic fuzzy right $\Gamma-$ideal of $\Gamma-$LA-semigroup S ■

Definition 106 Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy sub$\Gamma-$LA-semigroup of $\Gamma-$LA-semigroup S. Then $A = \langle \mu_A, \gamma_A \rangle$ is called an intuitionistic fuzzy bi-$\Gamma-$ideal of S, if

$(IF7))$ $\mu_A((x\alpha y)\beta z) \geq \min\{\mu_A(x), \mu_A(z)\}$,

$(IF8)$ $\gamma_A((x\alpha y)\beta z) \leq \max\{\gamma_A(x), \gamma_A(z)\}$, for all $x, y, z \in S$ and $\alpha, \beta \in S$.

Example 107 *Let $S = \{1, 2, 3, 4, 5\}$ be non-empty set. Then $(S, *)$ is LA-semigroup by the following table.*

*	1	2	3	4	5
1	3	3	3	3	3
2	3	3	3	3	3
3	3	3	3	3	3
4	3	3	3	3	3
5	3	4	4	3	3

Now let $S = \{1, 2, 3, 4, 5\}$ and $\Gamma = \{2\}$ be two non-empty sets and define a mapping $S \times \Gamma \times S \longrightarrow S$, by $a2b = a * b$ for all $a, b \in S$. Then it is easy to see that S is a $\Gamma-$LA-semigroup. Let $A = \langle \mu_A, \gamma_A \rangle$ be IFS, define by $\mu_A(1) = \mu_A(2) = \mu_A(3) = 0.7$, $\mu_A(4) = 0.5$, $\mu_A(5) = 0.2$. and $\gamma_A(1) = \gamma_A(2) = \gamma_A(3) = 0.2$, $\gamma_A(4) = 0.4$, $\gamma_A(1) = 0.7$. Then, clearly $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy bi-$\Gamma-$ideal of S.

Example 108 *Let $S = \{1, 2, 3, 4, 5\}$ be a $\Gamma-$LA-semigroup as given in Example 33. Let $A = \langle \mu_A, \gamma_A \rangle$ be an IFS in S and defined as:*

$$\begin{aligned} \mu_A(1) &= 0.85 = \mu_A(2), \ \mu_A(3) = 0.7, \ \mu_A(4) = 0.65, \ \mu_A(5) = 0.3 \\ \gamma_A(1) &= 0.15 = \gamma_A(2), \ \gamma_A(3) = 0.3, \ \gamma_A(4) = 0.35, \ \gamma_A(5) = 0.7 \end{aligned}$$

Then by routine calculation $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy bi-$\Gamma-$ideal of S.

Definition 109 $A = \langle \mu_A, \gamma_A \rangle$ be an IFS in S and $\alpha \in [0,1]$. Then sets

$$\mu_{A,\alpha}^{\geq} := \{x \in S : \mu_A(x) \geq \alpha\} \text{ and } \gamma_{A,\alpha}^{\leq} := \{x \in S : \gamma_A(x) \leq \alpha\}$$

are called $\mu-$level $\alpha-$cut and $\gamma-$level $\alpha-$cut of A, respectively.

Theorem 110 *Let $A = \langle \mu_A, \gamma_A \rangle$ be an IFS in $\Gamma-$LA-semigroup S. Then $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy left(resp, right) $\Gamma-$ ideal of $\Gamma-$LA-semigroup S if and only if $\mu-$level $\alpha-$cut, $\mu_{A,\alpha}^{\geq}$ and $\gamma-$level $\alpha-$cut ,$\gamma_{A,\alpha}^{\leq}$ of A respectively are left(resp, right) $\Gamma-$ ideal of $\Gamma-$LA-semigroup S.*

Proof. Let $\alpha \in [0,1]$. Suppose $\mu_{A,\alpha}^{\geq} (\neq \Phi)$ and $\gamma_{A,\alpha}^{\leq} (\neq \Phi)$, are left $\Gamma-$ ideal of $\Gamma-$LA-semigroup S. We must show that $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy left $\Gamma-$ ideal of S. Suppose $A = \langle \mu_A, \gamma_A \rangle$ is not an intuitionistic fuzzy left $\Gamma-$ ideal of S, then there exits $x_\circ, y_\circ$ in S and $\gamma \in \Gamma$ such that

$$\mu_A(x_\circ \gamma y_\circ) < \mu_A(y_\circ).$$

Taking

$$\alpha_\circ = \frac{1}{2}\{\mu_A(x_\circ \gamma y_\circ) + \mu_A(y_\circ)\}.$$

We have $\mu_A(x_\circ \gamma y_\circ) < \alpha_\circ < \mu_A(y_\circ)$. It follows that $y_\circ \in \mu_{A,\alpha}^{\geq}$ and $x_\circ \in S$ and $\gamma \in \Gamma$ but $x_\circ \gamma y_\circ \notin \mu_{A,\alpha}^{\geq}$. Which is a contradiction. Thus

$$\mu_A(x \gamma y) \geq \mu_A(y)$$

for all $x, y \in S$ and $\gamma \in \Gamma$, and now

$$\gamma_A(x_\circ \gamma y_\circ) > \gamma_A(y_\circ).$$

Taking

$$\alpha_\circ = \frac{1}{2}\{\gamma_A(x_\circ\gamma y_\circ) + \gamma_A(y_\circ)\}.$$

We have $\gamma_A(x_\circ\gamma y_\circ) < \alpha_\circ < \gamma_A(y_\circ)$. It follows that $y_\circ \in \gamma^{\geq}_{A,\alpha}$ and $x_\circ \in S$, $\gamma \in \Gamma$, this implies $x_\circ\gamma y_\circ \notin \gamma^{\geq}_{A,\alpha}$. Which is again a contradiction, thus

$$\gamma_A(x_\circ\gamma y_\circ) \leq \gamma_A(y_\circ)$$

Hence $A = \langle\mu_A, \gamma_A\rangle$ is an intuitionistic fuzzy $\Gamma-$ ideal of $\Gamma-$LA-semigroup S.

Conversely, suppose $A = \langle\mu_A, \gamma_A\rangle$ is an intuitionistic fuzzy left $\Gamma-$ ideal of $\Gamma-$LA-semigroup S. Let $\alpha \in [0,1]$ and for any $x \in S$, $\gamma \in \Gamma$ and $y \in \mu^{\geq}_{A,\alpha}$. Then

$$\begin{aligned}\mu_A(x\gamma y) &\geq \mu_A(y) \geq \alpha \\ \mu_A(x\gamma y) &\geq \alpha\end{aligned}$$

$x\gamma y \in \mu^{\geq}_{A,\alpha}$ for all $x \in S$, $\gamma \in \Gamma$ and $y \in S$. Hence $\mu^{\geq}_{A,\alpha}$ is left $\Gamma-$ ideal of $\Gamma-$LA-semigroup S. Now let $x \in S$, $\gamma \in \Gamma$ and $y \in \gamma^{\geq}_{A,\alpha}$. Then

$$\gamma_A(x\gamma y) \leq \gamma_A(y) \leq \alpha$$

$x\gamma y \in \gamma^{\geq}_{A,\alpha}$ for all $x \in S$, $\gamma \in \Gamma$ and $y \in S$. Hence $\gamma^{\geq}_{A,\alpha}$ is a left $\Gamma-$ ideal of $\Gamma-$LA-semigroup S. ■

Theorem 112 *Let $A = \langle\mu_A, \gamma_A\rangle$ be an intuitionistic fuzzy set of $\Gamma-$LA-semigroup S. Then $A = \langle\mu_A, \gamma_A\rangle$ is an intuitionistic fuzzy bi-$\Gamma-$ideal of $\Gamma-$LA-semigroup S if and only if μ-level α-cut and γ-level α-cut of A are bi-$\Gamma-$ideals of S.*

Proof. Let $\alpha \in [0,1]$ and let $x, y \in \mu^{\geq}_{A,\alpha}$. Then $\mu_A(x) \geq \alpha$ and $\mu_A(y) \geq \alpha$. By $(IF1)$,

$$\begin{aligned}\mu_A(x\alpha y) &\geq \min\{\mu_A(x), \mu_A(y)\} \geq \alpha \\ \mu_A(x\alpha y) &\geq \alpha \text{ so that } x\alpha y \in \mu^{\geq}_{A,\alpha}.\end{aligned}$$

If $x, y \in \gamma^{\leq}_{A,\alpha}$, then $\gamma_A(x) \leq \alpha$ and $\gamma_A(y) \leq \alpha$. By $(IF2)$,

$$\begin{aligned}\gamma_A(x\alpha y) &\leq \max\{\gamma_A(x), \gamma_A(y)\} \leq \alpha \\ \gamma_A(x\alpha y) &\leq \alpha \text{ so that } x\alpha y \in \gamma^{\leq}_{A,\alpha}\end{aligned}$$

Hence $\mu^{\geq}_{A,\alpha}$ and $\gamma^{\leq}_{A,\alpha}$ are sub$\Gamma-$LA-semigroups of S. Now let $y \in S$ and $x, z \in \mu^{\geq}_{A,\alpha}$, then $\mu_A(x) \geq \alpha$ and $\mu_A(z) \geq \alpha$. From $(IF3)$ that

$$\begin{aligned} \mu_A((x\alpha y)\beta z) &\geq \min\{\mu_A(x), \mu_A(z)\} \geq \alpha \\ \mu_A((x\alpha y)\beta z) &\geq \alpha \text{ so that } (x\alpha y)\beta z \in \mu^{\geq}_{A,\alpha} \end{aligned}$$

If $y \in S$ and $x, z \in \gamma^{\leq}_{A,\alpha}$, then $\gamma_A(x) \leq \alpha$ and $\gamma_A(z) \leq \alpha$. From $(IF4)$ that

$$\begin{aligned} \gamma_A((x\alpha y)\beta z) &\leq \max\{\gamma_A(x), \gamma_A(z)\} \leq \alpha \\ \gamma_A((x\alpha y)\beta z) &\leq \alpha \text{ so that } (x\alpha y)\beta z \in \gamma^{\leq}_{A,\alpha} \end{aligned}$$

Hence $\mu^{\geq}_{A,\alpha}$ and $\gamma^{\leq}_{A,\alpha}$ are bi-$\Gamma-$ideals of S.

Conversely, let $\alpha \in [0,1]$ and suppose that $\mu^{\geq}_{A,\alpha}(\neq \varnothing)$ and $\gamma^{\leq}_{A,\alpha}(\neq \varnothing)$ are bi-$\Gamma-$ideals of $\Gamma-$LA-semigroup S. Let $A = \langle \mu_A, \gamma_A \rangle$ be an IFS in S. We must show that $A = \langle \mu_A, \gamma_A \rangle$ is satisfying conditions $(IF1)$ to $(IF4)$. If condition $(IF1)$ is false, then there exist $x_\circ, y_\circ$ in S and $\alpha \in \Gamma$ such that

$$\begin{aligned} \mu_A(x_\circ \alpha y_\circ) &< \mu_A(x_\circ) \wedge \mu_A(y_\circ). \\ \text{Let } \alpha_\circ &= \frac{1}{2}[\mu_A(x_\circ \alpha y_\circ) + \mu_A(x_\circ) \wedge \mu_A(y_\circ)]. \text{ Then} \\ \mu_A(x_\circ \alpha y_\circ) &< \alpha_\circ < \mu_A(x_\circ) \wedge \mu_A(y_\circ) \\ x_\circ &\in \mu^{\geq}_{A,\alpha_\circ} \text{ and } y_\circ \in \mu^{\geq}_{A,\alpha_\circ} \text{ but } x_\circ \alpha y_\circ \notin \mu^{\geq}_{A,\alpha_\circ} \end{aligned}$$

which is a contradiction. Hence condition $(IF1)$ is true. The proof of other conditions are similar to the case $(IF1)$, we omit the proof. ■

Example 113 *Let $S = \{-i, 0, i\}$ and $\Gamma = S$. Then by defining $S \times \Gamma \times S \to S$ as $a\gamma b = a.\gamma.b$ for all $a, b \in S$ and $\gamma \in \Gamma$. It can be easily verified that S is a $\Gamma-$LA-semigroup under complex number multiplication while S is not an LA-semigroup. $A = \langle \mu_A, \gamma_A \rangle$ be IFS on S. $\mu_A : S \longrightarrow [0,1]$ by $\mu_A(0) = 0.7, \mu_A(i) = \mu_A(-i) = 0.5$ and $\gamma_A(0) = 0.2, \gamma_A(i) = \gamma_A(-i) = 0.4$*

Now let $\alpha \in [0,1]$,

$$\mu_{A,\alpha}^{\geq}(x) = \begin{cases} S & \text{If } \alpha \in (0,0.5] \\ \{0\} & \text{If } \alpha \in (0.5,0.7] \\ \Phi & \text{If } \alpha \in (0.7,1] \end{cases}$$

$$\gamma_{A}^{\leq}(x) = \begin{cases} \Phi & \text{If } \alpha \in [0,0.2) \\ \{0\} & \text{If } \alpha \in [0.2,0.4) \\ S & \text{If } \alpha \in [0.4,1) \end{cases}$$

Then by Theorem 111, $A = \langle \mu_A, \gamma_A \rangle$ *is an intuitionistic fuzzy* $\Gamma-$ *ideal of* $\Gamma-LA$*-semigroup* S.

Example 114 *Let* $S = \{1,2,3,4,5\}$ *with binary operation* " $*$ ". *Then* $(S,*)$ *is an LA-semigroup by the following table:*

$*$	1	2	3	4	5
1	2	2	2	2	2
2	2	2	2	2	2
3	2	2	2	2	2
4	2	2	2	2	2
5	2	3	3	2	2

Now let $S = \{1,2,3,4,5\}$ *and* $\Gamma = \{1\}$ *and define a mapping* $S \times \Gamma \times S \longrightarrow S$, *by* $a1b = a * b$ *for all* $a, b \in S$. *Then it is easy to see that* S *is a* $\Gamma-LA$*-semigroup. Let* $A = \langle \mu_A, \gamma_A \rangle$ *be an intuitionistic fuzzy set defined by* $\mu_A(1) = \mu_A(2) = \mu_A(3) = 0.7$, $\mu_A(4) = 0.5$, $\mu_A(5) = 0.2$. *and* $\gamma_A(1) = \gamma_A(2) = \gamma_A(3) = 0.2$, $\gamma_A(4) = 0.4$, $\gamma_A(1) =$

0.7. *Now we find its level sets* $\mu^{\geq}_{\bar{A},\alpha}$ *and* $\gamma^{\leq}_{\bar{A},\alpha}$ *of* A.

$$\mu^{\geq}_{\bar{A},\alpha}(x) = \begin{cases} S & \text{If } \alpha \in (0, 0.2] \\ \{1,2,3,4\} & \text{If } \alpha \in (0.2, 0.5] \\ \{1,2,3\} & \text{If } \alpha \in (0.5, 0.7] \\ \Phi & \text{If } \alpha \in (0.7, 1] \end{cases}$$

$$\gamma^{\leq}_{\bar{A}}(x) = \begin{cases} \Phi & \text{If } \alpha \in [0, 0.2) \\ \{1,2,3,\} & \text{If } \alpha \in [0.2, 0.5) \\ \{1,2,3,4\} & \text{If } \alpha \in [0.4, 0.7) \\ S & \text{If } \alpha \in [0.7, 1) \end{cases}$$

By using Theorem 111, $A = \langle \mu_A, \gamma_A \rangle$ *is an intuitionistic fuzzy* $\Gamma-$*ideal of* $\Gamma-LA$-*semigroup* S. *By routine calculation* $A = \langle \mu_A, \gamma_A \rangle$ *is an intuitionistic fuzzy bi-*$\Gamma-$*ideal of* $\Gamma-LA$-*semigroup* S.

Example 115 *Let* $S = \{1, 2, 3, 4, 5\}$ *be a* $\Gamma-LA$-*semigroup as given in Example 33. Let* $A = \langle \mu_A, \gamma_A \rangle$ *be an IFS in* S *and defined as:*

$$\begin{aligned} \mu_A(1) &= 0.8 = \mu_A(2),\ \mu_A(3) = 0.7,\ \mu_A(4) = 0.6,\ \mu_A(5) = 0.3 \\ \gamma_A(1) &= 0.1 = \gamma_A(2),\ \gamma_A(3) = 0.3,\ \gamma_A(4) = 0.4,\ \gamma_A(5) = 0.7 \end{aligned}$$

Then

$$\mu_{\overline{A},\alpha}^{\geq}(x) = \begin{cases} S & \text{If } \alpha \in (0, 0.3] \\ \{1,2,3,4\} & \text{If } \alpha \in (0.3, 0.6] \\ \{1,2,3\} & \text{If } \alpha \in (0.6, 0.7] \\ \{1,2\} & \text{If } \alpha \in (0.7, 0.8] \\ \Phi & \text{If } \alpha \in (0.7, 1] \end{cases}$$

$$\gamma_{\overline{A}}^{\leq}(x) = \begin{cases} \Phi & \text{If } \alpha \in [0, 0.1) \\ \{1,2\} & \text{If } \alpha \in [0.2, 0.3) \\ \{1,2,3\} & \text{If } \alpha \in [0.3, 0.4) \\ \{1,2,3,4\} & \text{If } \alpha \in [0.4, 0.7) \\ S & \text{If } \alpha \in [0.7, 1) \end{cases}$$

Thus by Theorem 111, $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy $\Gamma-$ideal and intuitionistic fuzzy bi-$\Gamma-$ideal of S.

For the generalization, see the following theorem.

Theorem 116 *Every intuitionistic fuzzy left(right) $\Gamma-$ideal of $\Gamma-LA$-semigroup S is an intuitionistic fuzzy bi-$\Gamma-$ideal of $\Gamma-LA$-semigroup S. But the converse is not true in general.*

Proof. Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy left $\Gamma-$ideal of $\Gamma-$LA-semigroup S. if $w, x, y \in S$ and $\alpha, \gamma \in \Gamma$, then

$$\begin{aligned} \mu_A((x\alpha w)\gamma y) &\geq \mu_A(y) \\ \mu_A((x\alpha w)\gamma y) &= \mu_A((y\alpha w)\gamma x) \geq \mu_A(y) \\ \mu_A((x\alpha w)\gamma y) &\geq \min\{\mu_A(z), \mu_A(y)\} \end{aligned}$$

and

$$\begin{aligned}
\gamma_A((x\alpha w)\gamma y) &\leq \gamma_A(y) \\
\gamma_A((x\alpha w)\gamma y) &= \gamma_A((y\alpha w)\gamma x) \leq \gamma_A(x) \\
\gamma_A((x\alpha w)\gamma y) &\leq \max\{\gamma_A(x), \gamma_A(y)
\end{aligned}$$

for all $x, w, y \in S$. Hence $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy bi-$\Gamma-$ideal of $\Gamma-$LA-semigroup S. ■

Example 117 *Let $S = \{1, 2, 3, 4, 5\}$ be a $\Gamma-$LA-semigroup as given in Example 33. Let $A = \langle \mu_A, \gamma_A \rangle$ be an IFS in S and defined as:*

$$\begin{aligned}
\mu_A(1) &= 0.9 = \mu_A(2), \ \mu_A(3) = 0.4, \ \mu_A(4) = 0.6, \ \mu_A(5) = 0.3 \\
\gamma_A(1) &= 0.1 = \gamma_A(2), \ \gamma_A(3) = 0.6, \ \gamma_A(4) = 0.4, \ \gamma_A(5) = 0.7
\end{aligned}$$

Then clearly $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy bi-$\Gamma-$ideal of S and $A = \langle \mu_A, \gamma_A \rangle$ is not an ituitionistic fuzzy left $\Gamma-$ideal of S. As

$$\mu_A(5\gamma 4) \ngeq \mu_A(4) \text{ and } \gamma_A(5\gamma 4) \nleq \gamma_A(4)$$

Definition 118 Let $f : S \longrightarrow S_1$ be homomorphism from $\Gamma-$LA-semigroup S to $\Gamma-$LA-semigroup S_1. If $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy set in S_1, then the preimage of $A = \langle \mu_A, \gamma_A \rangle$ is denoted by $f^{-1}(A) = \langle f^{-1}(\mu_A), f^{-1}(\gamma_A) \rangle$ and defined as $f^{-1}(\mu_A(x)) = (\mu_A(f(x)))$ and $f^{-1}(\gamma_A(x)) = (\gamma_A(f(x)))$

Theorem 119 *Let the pair of mappings $f : S \longrightarrow S_1$ and $h : \Gamma \longrightarrow \Gamma_1$ be homomorphisms of $\Gamma-$LA-semigroup. If $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy left(resp, right) $\Gamma-$ideal of $\Gamma-$LA-semigroup S_1, then $f^{-1}(A) = \langle f^{-1}(\mu_A), f^{-1}(\gamma_A) \rangle$ is an intuitionistic fuzzy left(resp, right) $\Gamma-$ideal of $\Gamma-$LA-semigroup S.*

Proof. Let $x, y \in S$, $\alpha \in \Gamma$ and $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy left $\Gamma-$ideal of $\Gamma-$LA-semigroup S_1. Then

$$\begin{aligned}
f^{-1}(\mu_A(x\alpha y) &= (\mu_A(f(x\alpha y))) = (\mu_A(f(x)h(\alpha)f(y))) \\
&\geq \mu_A(f(y)) = f^{-1}\mu_A(y) \\
f^{-1}(\mu_A(x\alpha y) &\geq f^{-1}\mu_A(y)
\end{aligned}$$

and

$$\begin{aligned} f^{-1}(\gamma_A(x\alpha y)) &= (\gamma_A(f(x\alpha y))) = (\gamma_A(f(x)h(\alpha)f(y))) \\ &\leq \gamma_A(f(y)) = f^{-1}(\gamma_A(y)) \\ f^{-1}(\gamma_A(x\alpha y)) &\leq f^{-1}(\gamma_A(y) \end{aligned}$$

for all $x, y \in S$ and $\alpha \in \Gamma$. Hence $f^{-1}(A) = \langle f^{-1}(\mu_A), f^{-1}(\gamma_A)\rangle$ is an intuitionistic fuzzy left $\Gamma-$ideal of $\Gamma-$LA-semigroup S. Similarly for an intuitionistic fuzzy right $\Gamma-$ideal of $\Gamma-$LA-semigroup S. ∎

Definition 120 Let $f : [0,1] \longrightarrow [0,1]$ is an increasing function and $A = \langle \mu_A, \gamma_A \rangle$ be an IFS of $\Gamma-$LA-semigroup S. Then $A_f = \langle \mu_{A_f}, \gamma_{A_f}\rangle$ be an IFS of $\Gamma-$LA-semigroup S and defined as $\mu_{A_f}(x) = f(\mu_A(x))$ and $\gamma_{A_f}(x) = f(\gamma_A(x))$ for all $x \in S$.

Proposition 121 *Let S be a $\Gamma-$LA-semigroup. If $A = \langle \mu_A, \gamma_A\rangle$ is an intuitionistic fuzzy left(resp, right) $\Gamma-$ideal of S, then $A_f = \langle \mu_{A_f}, \gamma_{A_f}\rangle$ is an intuitionistic fuzzy left(resp, right) $\Gamma-$ideal of S.*

Proof. Let $A = \langle \mu_A, \gamma_A\rangle$ be an intuitionistic fuzzy left $\Gamma-$ideal of S. Let for any $x, y \in S$, $\alpha \in \Gamma$ and $A_f = \langle \mu_{A_f}, \gamma_{A_f}\rangle$ be an IFS of S. Then

$$\mu_{A_f}(x\alpha y) = f(\mu_A(x\alpha y)) \geq f(\mu_A(y))$$

and

$$\begin{aligned} \gamma_{A_f}(x\alpha y) &= f(\gamma_A(x\alpha y)) \leq f(\gamma_A(y)) \\ \mu_{A_f}(x\alpha y) &\geq f(\mu_A(y)) \text{ and } \gamma_{A_f}(x\alpha y) \leq f(\gamma_A(y)) \end{aligned}$$

for all $x, y \in S$. Hence $A_f = \langle \mu_{A_f}, \gamma_{A_f}\rangle$ is an intuitionistic fuzzy left $\Gamma-$ideal of S. ∎

Proposition 122 *Let $A = \langle \mu_A, \gamma_A\rangle$ be an intuitionistic fuzzy left $\Gamma-$ideal of left zero $\Gamma-$LA-semigroup S. Then $A(x) = A(z)$ for all $x, z \in S$.*

Proof. Let $x, z \in S$ and $\alpha \in \Gamma$. Since S is left zero $\Gamma-$LA-semigroup S, so $x\alpha z = x$ and $z\alpha x = z$, we have

$$\begin{aligned} \mu_A(x) &= \mu_A(x\alpha z) \geq \mu_A(z) \Longrightarrow \mu_A(x) \geq \mu_A(z) \\ \mu_A(z) &= \mu_A(z\alpha x) \geq \mu_A(x) \Longrightarrow \mu_A(z) \geq \mu_A(x) \\ \mu_A(x) &= \mu_A(z) \end{aligned}$$

and

$$\begin{aligned}
\gamma_A(x) &= \gamma_A(x\alpha z) \leq \gamma_A(z) \Longrightarrow \gamma_A(x) \leq \gamma_A(z) \\
\gamma_A(z) &= \gamma_A(z\alpha x) \leq \gamma_A(x) \Longrightarrow \gamma_A(z) \leq \gamma_A(x) \\
\gamma_A(x) &= \gamma_A(z),
\end{aligned}$$

for all $x, z \in S$. Hence $A(x) = A(z)$ for all $x, z \in S$. ∎

Proposition 123 *Let I be a left $\Gamma-$ideal of $\Gamma-$LA-semigroup S. Then $A = (x_{I,}\bar{x_I})$ is an intuitionistic fuzzy left $\Gamma-$ideal of $\Gamma-$LA-semigroup S, where x_I is characteristic functions and $\bar{x_I} = 1 - x_I$*

Proof. Let $y, z \in S$ and $\alpha \in \Gamma$ and $A = (x_{I,}\bar{x_I})$ be IFS of S. Since I left $\Gamma-$ideal of $\Gamma-$LA-semigroup S, so we have two cases i) if $y \in I$ and ii) $y \notin I$

case i) if $y \in I$, then $y\alpha z \in I$

$$x_I(y) = 1 \text{ and } x_I(y\alpha z) = 1$$

and also

$$x_I(y\alpha z) = 1 = x_I(y)$$

ii) if $y \notin I$, then

$$\begin{aligned}
x_I(y) &= 0 \text{ and } x_I(y\alpha z) \geq 0 \\
x_I(y\alpha z) &\geq 0 = x_I(y) \Longrightarrow x_I(y\alpha z) \geq x_I(y)
\end{aligned}$$

if $y \in I$

$$\begin{aligned}
1 - x_I(y) &= 1 - 1 = 0 \text{ and } 1 - x_I(y\alpha z) = 1 - 1 = 0 \\
\bar{x_I}(y\alpha z) &= \bar{x_I}(y)
\end{aligned}$$

if $y \notin I$ then

$$\begin{aligned}
\bar{x_I}(x) &= 1 - x_I(y) = 1 - 0 = 1 \\
\bar{x_I}(y\alpha z) &\leq \bar{x_I}(x)
\end{aligned}$$

Hence $A = (x_{I,}\bar{x_I})$ is an intuitionistic fuzzy left $\Gamma-$ideal of a $\Gamma-$LA-semigroup S. ∎

Proposition 124 *Let I be a right $\Gamma-$ideal of $\Gamma-LA$-semigroup S. Then $A = (x_I, \bar{x_I})$ is an intuitionistic fuzzy right $\Gamma-$ideal of $\Gamma-LA$-semigroup S. Where x_I is characteristic functions and $\bar{x_I} = 1 - x_I$*

Theorem 125 *Let $A = \langle \mu_A, \gamma_A \rangle$ and $B = \langle \mu_B, \gamma_B \rangle$ be any two intuitionistic fuzzy bi- $\Gamma-$ideals of $\Gamma-LA$-semigroup S. Then $A \cap B$ is also an intuitionistic fuzzy bi-$\Gamma-$ideal of $\Gamma-LA$-semigroup S.*

Proof. *Let $A = \langle \mu_A, \gamma_A \rangle$ and $B = \langle \mu_B, \gamma_B \rangle$ be any two intuitionistic fuzzy bi-$\Gamma-$ideals of $\Gamma-LA$-semigroup S. Let $x, y \in S$ and $\alpha \in \Gamma$. Then*

$$\begin{aligned}
(\mu_A \wedge \mu_B)(x\alpha y) &= \mu_A(x\alpha y) \wedge \mu_B(x\alpha y) \\
&\geq \min\{\mu_A(x), \mu_A(y)\} \wedge \min\{\mu_B(x), \mu_B(y)\} \\
&= \min\{\mu_A(x) \wedge \mu_B(x), \mu_A(y) \wedge \mu_B(y)\} \\
&= \min\{(\mu_A \wedge \mu_B)(x), (\mu_A \wedge \mu_B)(y)\} \\
(\mu_A \wedge \mu_A)(x\alpha y) &\geq \min\{(\mu_A \wedge \mu_B)(x), (\mu_A \wedge \mu_B)(y)\}
\end{aligned}$$

and

$$\begin{aligned}
(\gamma_A \vee \gamma_B)(x\alpha y) &= \gamma_A(x\alpha y) \vee \gamma_B(x\alpha y) \\
&\leq \max\{\gamma_A(x), \gamma_B(y)\} \vee \max\{\gamma_B(x), \gamma_B(y)\} \\
&= \max\{\gamma_A(x) \vee \gamma_B(x), \gamma_A(y) \vee \gamma_B(y)\} \\
&= \max\{(\gamma_A \vee \gamma_B)(x), (\gamma_A \vee \gamma_B)(y)\} \\
(\gamma_A \vee \gamma_B)(x\alpha y) &\leq \max\{(\gamma_A \vee \gamma_B)(x), (\gamma_A \vee \gamma_B)(y)\}
\end{aligned}$$

Thus $A \cap B$ is an intuitionistic fuzzy sub$\Gamma-LA$-semigroup of S. Now, let $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned}
(\mu_A \wedge \mu_B)((x\alpha y)\beta z) &= \mu_A((x\alpha y)\beta z) \wedge \mu_A((x\alpha y)\beta z) \\
&\geq \min\{\mu_A(x), \mu_A(z)\} \wedge \min\{\mu_B(x), \mu_B(z)\} \\
&= \min\{\mu_A(x) \wedge \mu_B(x), \mu_A(z) \wedge \mu_B(z)\} \\
&= \min\{(\mu_A \wedge \mu_B)(x), (\mu_A \wedge \mu_B)(z)\} \\
(\mu_A \wedge \mu_B)((x\alpha y)\beta z) &\geq \min\{(\mu_A \wedge \mu_B)(x), (\mu_A \wedge \mu_B)(z)\}
\end{aligned}$$

and

$$
\begin{aligned}
(\gamma_A \vee \gamma_B)((x\alpha y)\beta z) &= \gamma_A((x\alpha y)\beta z) \vee \gamma_B((x\alpha y)\beta z) \\
&\leq \max\{\gamma_A(x), \gamma_A(z)\} \vee \max\{\gamma_B(x), \gamma_B(z)\} \\
&= \max\{\gamma_A(x) \vee \gamma_B(x), \gamma_A(z) \vee \gamma_B(z)\} \\
&= \max\{(\gamma_A \vee \gamma_B)(x), (\gamma_A \vee \gamma_B)(z)\} \\
(\gamma_A \vee \gamma_B)((x\alpha y)\beta z) &\leq \max\{(\gamma_A \vee \gamma_B)(x), (\gamma_A \vee \gamma_B)(z)\}
\end{aligned}
$$

for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. Hence $A \cap B$ is an intuitionistic fuzzy bi- $\Gamma-$ideal of $\Gamma-LA$-semigroup S. ■

Corollary 126 *Let $\{A_i\}_{i\in\Delta}$ be a family of intuitionistic fuzzy bi-$\Gamma-$ideals of $\Gamma-LA$-semigroup S. Then $\bigcap\limits_{i\in\Delta} A_i$ is also an intuitionistic fuzzy bi-$\Gamma-$ideal of $\Gamma-LA$-semigroup S.*

Theorem 127 *If $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy bi-$\Gamma-$ideal of $\Gamma-LA$-semigroup S, then $\Box A = \langle \mu_A, \bar{\mu_A} \rangle$ is also intuitionistic fuzzy bi-$\Gamma-$ideal of $\Gamma-LA$-semigroup S.*

Proof. Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy bi-$\Gamma-$ideal of $\Gamma-$LA-semigroup S. Let $x, y, \in S$ and $\alpha, \in \Gamma$. Then

$$
\begin{aligned}
\mu_A(x\alpha y) &\geq \min\{\mu_A(x), \mu_A(y)\} \text{ and} \\
\bar{\mu_A}(x\alpha y) &= 1 - \mu_A(x\alpha y) \\
&\leq 1 - \min\{\mu_A(x), \mu_A(y)\} \\
&= \max\{1 - 1\mu_A(x), 1 - \mu_A(y)\} \\
\bar{\mu_A}(x\alpha y) &\leq \max\{\bar{\mu_A}(x), \bar{\mu_A}(y)\}
\end{aligned}
$$

and thus $\Box A = \langle \mu_A, \bar{\mu_A} \rangle$ is an intuitionistic fuzzy $\Gamma-$subLA-semigroup of S. Now, let $x, y, \in S$ and $\alpha, \in \Gamma$. Then

$$
\begin{aligned}
\mu_A((x\alpha y)\beta z) &\geq \min\{\mu_A(x), \mu_A(z)\} \text{ and} \\
\bar{\mu_A}((x\alpha y)\beta z) &= 1 - \mu_A((x\alpha y)\beta z) \\
&\leq 1 - \min\{\mu_A(x), \mu_A(z)\} \\
&= \max\{1 - \mu_A(x), 1 - \mu_A(z)\} \\
\bar{\mu_A}((x\alpha y)\beta z) &\leq \max\{\bar{\mu_A}(x), \bar{\mu_A}(z)\}
\end{aligned}
$$

for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. Hence $\Box A = \langle \mu_A, \bar{\mu_A} \rangle$ is an intuitionistic fuzzy bi-$\Gamma-$ideal of S. ■

Theorem 128 *Let $A = \langle \mu_A, \gamma_A \rangle$ be IFS in S. Then $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy bi-$\Gamma-$ideal of S if and only if the fuzzy sets μ_A and $\bar{\gamma_A}$ are fuzzy bi-$\Gamma-$ideals of S.*

Proof. Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy bi-$\Gamma-$ideal of S. Then clearly μ_A is fuzzy bi-$\Gamma-$ideal. Let $x, y, z \in S$ and $\alpha, \beta \in \Gamma$.

$$\begin{aligned}
\bar{\gamma_A}(x\alpha y) &= 1 - \gamma_A(x\alpha y) \\
&\geq 1 - \max\{\gamma_A(x), \gamma_A(y)\} \\
&= \min\{1 - \gamma_A(x), 1 - \gamma_A(y)\} \\
\bar{\gamma_A}(x\alpha y) &\geq \min\{\bar{\gamma_A}(x), \bar{\gamma_A}(y)\}
\end{aligned}$$

$$\begin{aligned}
\bar{\gamma_A}((x\alpha y)\beta z) &= 1 - \gamma_A((x\alpha y)\beta z) \\
&\geq 1 - \max\{\gamma_A(x), \gamma_A(z)\} \\
&= \min\{1 - \gamma_A(x), 1 - \gamma_A(z)\} \\
\bar{\gamma_A}((x\alpha y)\beta z) &\geq \min\{\bar{\gamma_A}(x), \bar{\gamma_A}(z)
\end{aligned}$$

Hence μ_A and $\bar{\gamma_A}$ are fuzzy bi-$\Gamma-$ideals.

Conversely, let μ_A and $\bar{\gamma_A}$ are fuzzy bi-$\Gamma-$ideal of S. Let $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. then

$$\begin{aligned}
1 - \gamma_A(x\alpha y) &= \bar{\gamma_A}(x\alpha y) \\
&\geq \min\{\bar{\gamma_A}(x), \bar{\gamma_A}(y)\} \\
&= \min\{1 - \gamma_A(x), 1 - \gamma_A(y)\} \\
1 - \gamma_A(x\alpha y) &\geq 1 - \max\{\gamma_A(x), \gamma_A(y)\} \\
\gamma_A(x\alpha y) &\leq \max\{\gamma_A(x), \gamma_A(y)\}
\end{aligned}$$

and

$$\begin{aligned}
1-\gamma_A((x\alpha y)\beta z) &= \bar{\gamma_A}(x\alpha y)\beta z) \\
&\geq \min\{\bar{\gamma_A}(x), \bar{\gamma_A}(z)\} \\
&= \min\{1-\gamma_A(x), 1-\gamma_A(z)\} \\
1-\gamma_A((x\alpha y)\beta z) &\geq 1-\max\{\gamma_A(x), \gamma_A(z)\} \\
\gamma_A((x\alpha y)\beta z) &\leq \max\{\gamma_A(x), \gamma_A(z)\}
\end{aligned}$$

Hence $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy bi-$\Gamma-$ideal of S. ■

Proposition 129 *Let the pair of $f : S \longrightarrow S_1$ and $h : \Gamma \longrightarrow \Gamma_1$ be hommorphism from $\Gamma-LA$-semigroup S to Γ_1-LA-semigroup S_1. If $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy bi-$\Gamma-$ideal of S_1, then pre-image $f^{-1}(A) = \langle f^{-1}(\mu_A), f^{-1}(\gamma_A) \rangle$ of $A = \langle \mu_A, \gamma_A \rangle$ under f is an intuitionistic fuzzy bi-$\Gamma-$ideal of S.*

Proof. Let $x, y \in S$ and $\gamma \in \Gamma$. Then

$$\begin{aligned}
f^{-1}(\mu_A)(x\gamma y) &= (\mu_A)(f(x\gamma y)) = (\mu_A)(f(x)h(\gamma)f(y)) \\
&\geq \min\{\mu_A(f(x), \mu_A(f(y)) \text{ since } A \text{ is IF } bi-\Gamma-ideal \text{ of } S_1 \\
&= \min\{f^{-1}(\mu_A(x)), f^{-1}(\mu_A(y))\} \\
f^{-1}(\mu_A)(x\gamma y) &\geq \min\{f^{-1}(\mu_A(x)), f^{-1}(\mu_A(y))\}
\end{aligned}$$

and

$$\begin{aligned}
f^{-1}(\gamma_A)(x\gamma y) &= (\gamma_A)(f(x\gamma y)) = (\gamma_A)(f(x)h(\gamma)f(y)) \\
&\leq \max\{\gamma_A(f(x), \gamma_A(f(y)) \text{ since } A \text{ is IF bi-}\Gamma-\text{ideal of } S_1 \\
&= \max\{f^{-1}(\gamma_A(x)), f^{-1}(\gamma_A(y))\} \\
f^{-1}(\gamma_A)(x\gamma y) &\leq \max\{f^{-1}(\gamma_A(x)), f^{-1}(\gamma_A(y))\}.
\end{aligned}$$

Now for $x, y, z \in S$ and $\beta, \gamma \in \Gamma$,

$$\begin{aligned}
f^{-1}(\mu_A)((x\gamma y)\beta z) &= (\mu_A)(f((x\gamma y)\beta z) = (\mu_A)((f(x)h(\gamma)f(y))h(\beta)f(z)) \\
&\geq \min\{\mu_A((f(x), \mu_A f(z))\} \text{ since } A \text{ is IF } bi-\Gamma-ideal \text{ of } S_1 \\
&= \min\{f^{-1}(\mu_A(x)), f^{-1}(\mu_A(z))\} \\
f^{-1}(\mu_A)((x\gamma y)\beta z) &\geq \min\{f^{-1}(\mu_A(x)), f^{-1}(\mu_A(z))\}
\end{aligned}$$

and

$$\begin{aligned}
f^{-1}(\gamma_A)((x\gamma y)\beta z) &= (\gamma_A)(f((x\gamma y)\beta z)) = (\gamma_A)((f(x)h(\gamma)f(y))h(\beta)f(z)) \\
&\leq \max\{\gamma_A((f(x), \gamma_A f(z))\} \text{ since } A \text{ is IF } bi-\Gamma-ideal \text{ of } S_1 \\
&= \max\{f^{-1}(\gamma_A(x)), f^{-1}(\gamma_A(z))\} \\
f^{-1}(\gamma_A)((x\gamma y)\beta z) &\leq \max\{f^{-1}(\gamma_A(x)), f^{-1}(\gamma_A(z))\}
\end{aligned}$$

for all $x, y, z \in S$ and $\beta, \gamma \in \Gamma$. Thus $f^{-1}(A) = \langle f^{-1}(\mu_A), f^{-1}(\gamma_A)\rangle$ is an intuitionistic fuzzy bi-$\Gamma-$ideal of S. ■

Definition 130 An intuitionistic fuzzy bi-$\Gamma-$ideal $A = \langle \mu_A, \gamma_A\rangle$ of $\Gamma-$LA-semigroup of S is said to be a normal intuitionistic fuzzy bi-$\Gamma-$ideal if

$$\mu(0) = 1 \textit{ and } \gamma_A(0) = 0$$

Theorem 131 *Let $A^* = (\mu_{A^*}, \gamma_{A^*})$ be an intuitionistic fuzzy set in S defined by $\mu_{A^*}(x) = \mu_A(x)+1-\mu_A(0)$ and $\gamma_{A^*}(x) = \gamma_A(x)-\gamma_A(0)$ for all $x \in S$. If $A = \langle \mu_A, \gamma_A\rangle$ is an intuitionistic fuzzy bi-$\Gamma-$ideal of $\Gamma-$LA-semigroup S, then $A^* = (\mu_{A^*}, \gamma_{A^*})$ is a normal intuitionistic fuzzy bi-$\Gamma-$ideal of S which contains $A = \langle \mu_A, \gamma_A\rangle$.*

Proof. For all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$, we have

$$\mu_{A^*}(0) = \mu_A(0) + 1 - \mu_A(0) = 1 \text{ and } \gamma_{A^*}(0) = \gamma_A(0) - \gamma_A(0) = 0$$

Also, for all $x, y \in S$ and $\alpha \in \Gamma$, we have

$$\begin{aligned}
\mu_{A^*}(x\alpha y) &= \mu_A(x\alpha y) + 1 - \mu_A(0) \\
&\geq \min\{\mu_A(x), \mu_A(y)\} + 1 - \mu_A(0) \\
&= \min\{\mu_A(x) + 1 - \mu_A(0), \mu_A(y) + 1 - \mu_A(0)\} \\
\mu_{A^*}(x\alpha y) &\geq \min\{\mu_{A^*}(x), \mu_{A^*}(y)\} \\
\text{and } \gamma_{A^*}(x\alpha y) &= \gamma_A(x\alpha y) + 1 - \gamma_A(0) \\
&\leq \max\{\gamma_A(x), \gamma_A(y)\} + 1 - \gamma_A(0) \\
&= \max\{\gamma_A(x) + 1 - \gamma_A(0), \gamma_A(y) + 1 - \gamma_A(0)\} \\
\gamma_{A^*}(x\alpha y) &\leq \max\{\gamma_{A^*}(x), \gamma_{A^*}(y)\}
\end{aligned}$$

Now, let $x, y, z \in S$ and $\alpha, \beta \in \Gamma$, we have

$$\begin{aligned}
\mu_{A^*}((x\alpha y)\beta z) &= \mu_A((x\alpha y)\beta z) + 1 - \mu_A(0) \\
&\geq \min\{\mu_A(x), \mu_A(z)\} + 1 - \mu_A(0) \\
&= \min\{\mu_A(x) + 1 - \mu_A(0), \mu_A(z) + 1 - \mu_A(0)\} \\
\mu_{A^*}(x\alpha y) &\geq \min\{\mu_{A^*}(x), \mu_{A^*}(z)\} \\
\text{and } \gamma_{A^*}((x\alpha y)\beta z) &= \gamma_A((x\alpha y)\beta z) + 1 - \gamma_A(0) \\
&\leq \max\{\gamma_A(x), \gamma_A(z)\} + 1 - \gamma_A(0) \\
&= \max\{\gamma_A(x) + 1 - \gamma_A(0), \gamma_A(z) + 1 - \gamma_A(0)\} \\
\gamma_{A^*}(x\alpha y) &\leq \max\{\gamma_{A^*}(x), \gamma_{A^*}(z)\}
\end{aligned}$$

Therefore, $A^* = (\mu_{A^*}, \gamma_{A^*})$ is a normal intuitionistic fuzzy bi-$\Gamma-$ideal of S, and obviously $A \subseteq A^*$. ■

Corollary 132 *Let $A = \langle \mu_A, \gamma_A \rangle$ and $A^* = (\mu_{A^*}, \gamma_{A^*})$ be as in Theorem 131. If there exists $x \in S$ such that $A^*(x) = 0$, then $A(x) = 0$.*

Theorem 133 *Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy bi-$\Gamma-$ideal of $\Gamma-LA$-semigroup S and $f : [0,1] \longrightarrow [0,1]$ be an increasing function. Then the fuzzy set $A_f = \langle \mu_{A_f}, \gamma_{A_f} \rangle$ is defined by $\mu_{A_f}(x) = f[\mu_{A_f}(x)]$ and $\gamma_A(x) = f[\gamma_A(x)]$ is an intuitionistic fuzzy bi-$\Gamma-$ideal of S. In particular, if $f[A(0)] = \langle 1, 0 \rangle$, then $A_f = \langle \mu_{A_f}, \gamma_{A_f} \rangle$ is normal.*

Proof. *Let $x, y \in S$ and $\alpha \in \Gamma$. Then*

$$\begin{aligned}
\mu_{A_f}(x\alpha y) &= f[\mu_A(x\alpha y)] \\
&\geq f(\min\{\mu_A(x), \mu_A(y)\}) \\
&= \min\{f(\mu_A(x)), f(\mu_A(y))\} \\
\mu_{A_f}(x\alpha y) &\geq \min\{\mu_{A_f}(x), \mu_{A_f}(y)\} \\
\text{and } \gamma_{A_f}(x\alpha y) &= f[\gamma_A(x\alpha y)] \\
&\leq f(\max\{\gamma_A(x), \gamma_A(y)\}) \\
&= \min\{f(\gamma_A(x)), f(\gamma_A(y))\} \\
\gamma_{A_f}(x\alpha y) &\geq \min\{\gamma_{A_f}(x), \gamma_{A_f}(y)\}
\end{aligned}$$

Now, let $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. Then

$$
\begin{aligned}
\mu_{A_f}((x\alpha y)\beta z) &= f[\mu_A((x\alpha y)\beta z)] \\
&\geq f(\min\{\mu_A(x), \mu_A(z)\}) \\
&= \min\{f(\mu_A(x)), f(\mu_A(z))\} \\
\mu_{A_f}((x\alpha y)\beta z) &\geq \min\{\mu_{A_f}(x), \mu_{A_f}(z)\} \\
\text{and } \gamma_{A_f}((x\alpha y)\beta z) &= f[\gamma_A((x\alpha y)\beta z)] \\
&\leq f(\max\{\gamma_A(x), \gamma_A(z)\}) \\
&= \max\{f(\mu_A(x)), f(\mu_A(z))\} \\
\gamma_{A_f}((x\alpha y)\beta z) &\geq \min\{\gamma_{A_f}(x), \gamma_{A_f}(z)\}
\end{aligned}
$$

Therefore, $A_f = \langle \mu_{A_f}, \gamma_{A_f} \rangle$ is an intuitionistic fuzzy bi-$\Gamma-$ideal of S. If $f[A(0)] = \langle 1, 0 \rangle$, then $f[\mu_A(0)] = 1$ and $f[\gamma_A(x)] = 0$, Then $\mu_{A_f}(0) = 1$ and $\gamma_{A_f}(0) = 0$. Thus $A_f = \langle \mu_{A_f}, \gamma_{A_f} \rangle$ is normal. ∎

2.3 Intuitionistic Fuzzy $\Gamma-$ideals and Intuitionistic Fuzzy bi-$\Gamma-$ideals in Regular $\Gamma-$LA-semigroups and $\Gamma-$LA-bands

Definition 134 A $\Gamma-$LA-semigroup S is called an intuitionistic fuzzy left(resp, right) $\Gamma-$due, if every intuitionistic fuzzy left(resp, right) $\Gamma-$ideal of S is an intuitionistic fuzzy $\Gamma-$ideal of S. A $\Gamma-$LA-semigroup S is called intuitionistic fuzzy $\Gamma-$due, if S is both intuitionistic fuzzy left and right $\Gamma-$due.

Proposition 135 *Every intuitionistic fuzzy right $\Gamma-$ideal of regular $\Gamma-LA$-semigroup S is an intuitionistic fuzzy left $\Gamma-$ideal of S.*

Proof. Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy right $\Gamma-$ideal of S and $a, b \in S$ and $\gamma \in \Gamma$. Since S is regular, there exist $x \in S$, and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$.

Then

$$\begin{aligned}\mu_A(a\gamma b) &= \mu_A(((a\alpha x)\beta a)\gamma b)\\ &= \mu_A((b\beta a)\gamma(a\alpha x)) \geq \mu_A(b\beta a)\\ \mu_A(a\gamma b) &\geq \mu_A(b)\end{aligned}$$

and

$$\begin{aligned}\gamma_A(a\gamma b) &= \gamma_A(((a\alpha x)\beta a)\gamma b)\\ &= \gamma_A((b\beta a)\gamma(a\alpha x)) \geq \gamma_A(b\beta a)\\ \gamma_A(a\gamma b) &\geq \gamma_A(b)\end{aligned}$$

Hence $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy left $\Gamma-$ideal of S. ■

Corollary 136 *In a regular $\Gamma-LA$-semigroup S, every intuitionistic fuzzy right $\Gamma-$ideal of S is an intuitionistic fuzzy $\Gamma-$ideal of S.*

Proposition 137 *If $A = \langle \mu_A, \gamma_A \rangle$ and $B = \langle \mu_B, \gamma_B \rangle$ are any intuitionistic fuzzy right $\Gamma-$ideals of a regular $\Gamma-LA$-semigroup S, then $A \circ_\Gamma B = A \cap B$*

Proof. Since S is regular, by Proposition 135, every intuitionistic fuzzy right $\Gamma-$ideal of a regular $\Gamma-$LA-semigroup S is an intuitionistic fuzzy left $\Gamma-$ideal of S. By Lemma 103, $A \circ_\Gamma B \subseteq A \cap B$.

On other hand, let $a \in S$. Then there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$. Thus

$$\begin{aligned}(\mu_A \wedge \mu_B)(a) &= \mu_A(a) \wedge \mu_B(a)\\ &\leq \mu_A(a\alpha x) \wedge \mu_B(a)\\ &\leq \bigvee_{a=(a\alpha x)\beta a} \mu_A(a\alpha x) \wedge \mu_B(a)\\ (\mu_A \wedge \mu_B)(a) &\leq \mu_{A\circ_\Gamma B}(a) \Longrightarrow \mu_A \wedge \mu_B \leq \mu_{A\circ_\Gamma B}\end{aligned}$$

and

$$\begin{aligned}(\gamma_A \vee \gamma_B)(a) &= \gamma_A(a) \vee \gamma_B(a)\\ &\geq \gamma_A(a\alpha x) \vee \gamma_B(a)\\ &\geq \bigwedge_{a=(a\alpha x)\beta a} \gamma_A(a\alpha x) \vee \gamma_B(a)\\ (\gamma_A \vee \gamma_B)(a) &\geq \gamma_{A\circ_\Gamma B}(a) \Longrightarrow \gamma_A \vee \gamma_B \geq \gamma_{A\circ_\Gamma B}\end{aligned}$$

Thus $A \cap B \subseteq A\Gamma B$, therefore

$$A \circ_{\Gamma} B \subseteq A \cap B \text{ and } A \cap B \subseteq A \circ_{\Gamma} B \Longrightarrow A \cap B = A \circ_{\Gamma} B.$$

■

Theorem 138 *Let S be a regular $\Gamma-LA$-semigroup. Then every intuitionistic fuzzy bi-$\Gamma-$ideal of S is an intuitionistic fuzzy right(left) $\Gamma-$ideal of S.*

Proof. Since S is regular, so every bi-$\Gamma-$ideal of S is a right(left) $\Gamma-$ideal. Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy bi-$\Gamma-$ideal of S. Let $x, y \in S$ and $\alpha \in \Gamma$, $(x\Gamma S)\Gamma x$ is bi-$\Gamma-$ideal of S. Then $(x\Gamma S)\Gamma x$ is a right $\Gamma-$ideal of S. Since S is regular. We have $x\alpha y \in ((x\Gamma S)\Gamma x)\Gamma S \subseteq (x\Gamma S)\Gamma x$ which implies that $x\alpha y = (x\gamma y)\beta x$ for some $y \in S$ and $\beta, \gamma \in \Gamma$. Since $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy bi-$\Gamma-$ideal of S. It follows that

$$\begin{aligned} \mu_A(x\alpha y) &= \mu_A((x\gamma y)\beta x) \\ &\geq \min\{\mu_A(x), \mu_A(x)\} = \mu_A(x) \\ \mu_A(x\alpha y) &\geq \mu_A(x) \\ \text{and } \gamma_A(x\alpha y) &= \gamma_A((x\gamma y)\beta x) \\ &\leq \max\{\gamma_A(x), \gamma_A(x)\} = \gamma_A(x) \end{aligned}$$

Hence $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy right $\Gamma-$ideal. Similarly for left $\Gamma-$ideal.
■

Corollary 139 *Let S be a regular $\Gamma-LA$-semigroup. Then every intuitionistic fuzzy bi-$\Gamma-$ideal of S is an intuitionistic fuzzy $\Gamma-$ideal of S.*

Proof. Straightforward. ■

Proposition 140 *Let S be a regular $\Gamma-LA$-semigroup. Then S is an intuitionistic fuzzy right $\Gamma-$due.*

Theorem 141 *Let S be a regular $\Gamma-LA$-semigroup. If S is a left(resp, right) $\Gamma-$due. Then S is an intuitionistic fuzzy left(resp, right) $\Gamma-$due.*

Proof. Since S is left $\Gamma-$due, so every left $\Gamma-$ideal of S is $\Gamma-$ideal. Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy left $\Gamma-$ideal of S. Let $x, y \in S$ and $\alpha \in \Gamma$. Then $S\Gamma x$ is left $\Gamma-$ideal and $S\Gamma x$ is a two sided $\Gamma-$ideal of S. Since S is regular, we have $x\alpha y \in ((x\Gamma S)\Gamma x)\Gamma S \subseteq S\Gamma x$. It follows that there exist $z \in S$ and $\beta \in S$ such that $x\alpha y = z\beta x$.

$$\begin{aligned} \mu_A(x\alpha y) &= \mu_A(z\beta x) \geq \mu_A(x) \\ \text{And } \gamma_A(x\alpha y) &= \gamma_A(z\beta x) \leq \gamma_A(x) \end{aligned}$$

Hence $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy $\Gamma-$ideal of S. Thus S is an intuitionistic fuzzy left $\Gamma-$due. ■

Corollary 142 *A regular $\Gamma-$LA-semigroup S is $\Gamma-$due if and only if S is an intuitionistic fuzzy $\Gamma-$due.*

Proposition 143 *Let S be a $\Gamma-$LA-semigroup with left identity. Then S is an intuitionistic fuzzy right $\Gamma-$due.*

Proof. Straightforward. ■

Proposition 144 *Let S be a $\Gamma-$LA band. Then S is an intuitionistic fuzzy left $\Gamma-$due if and only if it is an intuitionistic fuzzy right $\Gamma-$due.*

Proof. Let S be an intuitionistic fuzzy left $\Gamma-$due and let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy right $\Gamma-$ideal of S. For any $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$. Then

$$\begin{aligned} \mu_A(x\alpha y) &= \mu_A((x\beta x)\alpha y) \\ &= \mu_A((y\beta x)\alpha x) \text{ by left invertible law} \\ &\geq \mu_A(y\beta x) \geq \mu_A(y) \\ \mu_A(x\alpha y) &\geq \mu_A(y) \text{ and} \\ \gamma_A(x\alpha y) &= \gamma_A((x\beta x)\alpha y) \\ &= \gamma_A((y\beta x)\alpha x) \text{ by left invertible law} \\ &\leq \gamma_A(y\beta x) \leq \gamma_A(y) \\ \mu_A(x\alpha y) &\leq \mu_A(y) \end{aligned}$$

Therefore $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy left $\Gamma-$ideal of S. Hence S is an intuitionistic fuzzy right $\Gamma-$due.

Conversely, suppose $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy left $\Gamma-$ideal S and $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$. Then

$$\begin{aligned}
\mu_A(x\alpha y) &= \mu_A((x\beta x)\alpha y) \\
&= \mu_A((y\beta x)\alpha x)) \geq \mu_A(x) \\
\mu_A(x\alpha y) &\geq \mu_A(x) \text{ and} \\
\gamma_A(x\alpha y) &= \gamma_A((x\beta x)\alpha y) \\
&= \gamma_A((y\beta x)\alpha x) \leq \gamma_A(x) \\
\gamma_A(x\alpha y) &\leq \gamma_A(x)
\end{aligned}$$

Therefore $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy right $\Gamma-$ideal of S. Hence S is an intuitionistic fuzzy left $\Gamma-$due. ■

Corollary 145 *Every $\Gamma-LA$ band is an intuitionistic fuzzy $\Gamma-$due.*

Theorem 146 *The concept of intuitionistic fuzzy right and left $\Gamma-$ideal in a $\Gamma-LA$ band are coincide.*

Proof. Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy right $\Gamma-$ideal in a $\Gamma-$LA band S. Now, for any $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$. Then

$$\begin{aligned}
\mu_A(x\alpha y) &= \mu_A((x\beta x)\alpha y) \\
&= \mu_A((y\beta x)\alpha x) \\
&\geq \mu_A(y\beta x) \geq \mu_A(y) \\
\mu_A(x\alpha y) &\geq \mu_A(y)
\end{aligned}$$

and

$$\begin{aligned}
\gamma_A(x\alpha y) &= \gamma_A((x\beta x)\alpha y) \\
&= \gamma_A((y\beta x)\alpha x) \\
&\leq \gamma_A(y\beta x) \leq \gamma_A(y) \\
\mu_A(x\alpha y) &\leq \mu_A(y)
\end{aligned}$$

Therefore $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy left $\Gamma-$ideal in a $\Gamma-$LA band S

Conversely, suppose that $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy left $\Gamma-$ideal in a $\Gamma-$LA band S and $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$. Then

$$\begin{aligned} \mu_A(x\alpha y) &= \mu_A((x\beta x)\alpha y) \\ &= \mu_A((y\beta x)\alpha y)) \geq \mu_A(y\beta x) \\ &\Longrightarrow \mu_A(x\alpha y) \geq \mu_A(x) \end{aligned}$$

and

$$\begin{aligned} \gamma_A(x\alpha y) &= \gamma_A((x\beta x)\alpha y) \\ &= \gamma_A((y\beta x)\alpha y) \geq \gamma_A(y\beta x) \\ &\Longrightarrow \gamma_A(x\alpha y) \geq \gamma_A(x) \end{aligned}$$

Therefore $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy right $\Gamma-$ideal in a $\Gamma-$LA band S. ■

Chapter 3

Intuitionistic Fuzzy Prime, Semi-prime, Interior and Quasi $\Gamma-$Ideals of $\Gamma-$LA-semigroups

3.1 Introduction

In this chapter, we have defined an intuitionistic fuzzy prime $\Gamma-$ideal, intuitionistic fuzzy semiprime $\Gamma-$ideal, intuitionistic fuzzy interior $\Gamma-$ideal and intuitionistic fuzzy quasi $\Gamma-$ideal of $\Gamma-$LA-semigroup S, then some related properties are investigated. Some characterizations of intuitionistic fuzzy prime, intuitionistic fuzzy semi-prime $\Gamma-$ideals, intuitionistic fuzzy interior $\Gamma-$ideals and intuitionistic fuzzy quasi $\Gamma-$ideals are given.

3.2 Intuitionistic fuzzy prime $\Gamma-$Ideals in $\Gamma-$LA-semigroup

Definition 147 Let $A = \langle \mu_A, \gamma_A \rangle$ be an IFS in $\Gamma-$LA-semigroup S. Then $A = \langle \mu_A, \gamma_A \rangle$ is called an intuitionistic fuzzy prime if

$(IFP1)$ $\inf_{\gamma \in \Gamma} \mu_A(x\gamma y) = \max\{\mu_A(x), \mu_A(y)\}$,

$(IFP2)$ $\sup_{\gamma \in \Gamma} \gamma_A(x\gamma y) = \min\{\gamma_A(x), \gamma_A(y)\}$, $\forall x, y \in S$ and $\gamma \in \Gamma$.

An intuitionistic fuzzy $\Gamma-$ideal $A = \langle \mu_A, \gamma_A \rangle$ of S is called an intuitionistic fuzzy prime $\Gamma-$ideal of S if it is an intuitionistic fuzzy prime.

Let $\mathcal{X}_P$ denote the characteristic function of a nonempty subset P of a $\Gamma-$LA-semigroup.

Theorem 148 *Let S be a $\Gamma-$LA-semigroup and $\Phi \neq P \subseteq S$ is prime $\Gamma-$ideal of S. Then $A = \langle \mathcal{X}_P, \overline{\mathcal{X}_P} \rangle$ is an intuitionistic fuzzy prime $\Gamma-$ideal of S.*

Proof. Let $x, y \in S$ and $\gamma \in \Gamma$. If $x\Gamma y \in P$, then $x \in P$ or $y \in P$. Thus we have

$$\begin{aligned} \inf_{\gamma \in \Gamma} \mathcal{X}_P(x\gamma y) &= 1 \text{ and } \mathcal{X}_P(x) = 1 \text{ or } \mathcal{X}_P(y) = 1 \\ \inf_{\gamma \in \Gamma} \mathcal{X}_P(x\gamma y) &= 1 = \max\{\mathcal{X}_P(x), \mathcal{X}_P(y)\} \end{aligned}$$

and

$$\begin{aligned} 1 - \inf_{\gamma \in \Gamma} \mathcal{X}_P(x\gamma y) &= 0 \text{ and } 1 - \mathcal{X}_P(x) = 0 \text{ or } \mathcal{X}_P(y) = 0 \\ \sup_{\gamma \in \Gamma} \overline{\mathcal{X}_P}(x\gamma y) &= 0 \text{ and } \overline{\mathcal{X}_P}(x) = 0 \text{ or } \overline{\mathcal{X}_P}(y) = 0 \\ \sup_{\gamma \in \Gamma} \overline{\mathcal{X}_P}(x\gamma y) &= 0 = \min\left\{\overline{\mathcal{X}_P}(x), \overline{\mathcal{X}_P}(y)\right\} \end{aligned}$$

If $x\Gamma y \notin P$, then $x \notin P$ and $y \notin P$. Thus we have

$$\begin{aligned} \inf_{\gamma \in \Gamma} \mathcal{X}_P(x\gamma y) &= 0, \mathcal{X}_P(x) = 0 \text{ and } \mathcal{X}_P(y) = 0 \\ \inf_{\gamma \in \Gamma} \mathcal{X}_P(x\gamma y) &= 0 = \max\{\mathcal{X}_P(x), \mathcal{X}_P(y)\} \end{aligned}$$

and

$$\begin{aligned} 1 - \inf_{\gamma \in \Gamma} \mathcal{X}_P(x\gamma y) &= 1, 1 - \mathcal{X}_P(x) = 1 \text{ and } \mathcal{X}_P(y) = 1 \\ \sup_{\gamma \in \Gamma} \overline{\mathcal{X}_P}(x\gamma y) &= 1, \overline{\mathcal{X}_P}(x) = 1 \text{ and } \overline{\mathcal{X}_P}(y) = 1 \\ \sup_{\gamma \in \Gamma} \overline{\mathcal{X}_P}(x\gamma y) &= 1 = \min\left\{\overline{\mathcal{X}_P}(x), \overline{\mathcal{X}_P}(y)\right\} \end{aligned}$$

Hence $A = \langle \mathcal{X}_P, \overline{\mathcal{X}_P} \rangle$ is an intuitionistic fuzzy prime $\Gamma-$ideal of S. ∎

Theorem 149 *Let P be a non empty subset of S. If $A = \langle \mathcal{X}_P, \overline{\mathcal{X}_P} \rangle$ is an intuitionistic fuzzy prime $\Gamma-$ideal of S, then P is a prime $\Gamma-$ideal of S.*

Proof. Suppose that $A = \langle \mathcal{X}_P, \overline{\mathcal{X}_P} \rangle$ is an intuitionistic fuzzy prime $\Gamma-$ideal of S. Let $x, y \in S$ such that $x\Gamma y \in P$. Then $\mathcal{X}_P(x\gamma y) = 1$ for all $\gamma \in \Gamma$. So $\inf_{\gamma\in\Gamma} \mathcal{X}_P(x\gamma y) = 1$. Its follows from $(IFP1)$ that

$$1 = \inf_{\gamma\in\Gamma} \mathcal{X}_P(x\gamma y) = \max\{\mathcal{X}_P(x), \mathcal{X}_P(y)\}$$

Hence $\mathcal{X}_P(x) = 1$ or $\mathcal{X}_P(y) = 1$, so $x \in P$ or $y \in P$. Thus P is prime. Now from $(IFP1)$ that

$$\begin{aligned}
0 &= 1 - \inf_{\gamma\in\Gamma} \mathcal{X}_P(x\gamma y) = \sup_{\gamma\in\Gamma} \overline{\mathcal{X}_P}(x\gamma y) = \min\left\{\overline{\mathcal{X}_P}(x), \overline{\mathcal{X}_P}(y)\right\} \\
0 &= \min\{1 - \mathcal{X}_P(x), 1 - \mathcal{X}_P(y)\}
\end{aligned}$$

and so $1 - \mathcal{X}_P(x) = 0$ or $1 - \mathcal{X}_P(y) = 0 \Rightarrow \mathcal{X}_P(x) = 1$ or $\mathcal{X}_P(y) = 1$, so $x \in P$ or $y \in P$. Thus P is prime. ∎

Proposition 150 *If $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy prime $\Gamma-$ideal of S, then μ_A and $\overline{\gamma_A}$ are fuzzy prime $\Gamma-$ideals of S.*

Proof. Since $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy prime $\Gamma-$ideal of S. Then for any $x, y \in S$ and $\gamma \in \Gamma$, we have

$$\begin{aligned}
\inf_{\gamma\in\Gamma} \mu_A(x\gamma y) &= \max\{\mu_A(x), \mu_A(y)\} \text{ and} \\
\sup_{\gamma\in\Gamma} \gamma_A(x\gamma y) &= \min\{\gamma_A(x), \gamma_A(y)\} \\
1 - \sup_{\gamma\in\Gamma} \gamma_A(x\gamma y) &= 1 - \min\{\gamma_A(x), \gamma_A(y)\} \\
\inf_{\gamma\in\Gamma} \overline{\gamma_A}(x\gamma y) &= \max\{1 - \gamma_A(x), 1 - \gamma_A(y)\} \\
\inf_{\gamma\in\Gamma} \overline{\gamma_A}(x\gamma y) &= \max\{\overline{\gamma_A}(x), \overline{\gamma_A}(y)\}
\end{aligned}$$

Hence μ_A and $\overline{\gamma_A}$ are fuzzy prime $\Gamma-$ideals of S. ∎

Proposition 151 *If $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy prime $\Gamma-$ideal of S, then $\overline{\mu_A}$ and γ_A are anti fuzzy prime $\Gamma-$ideals of S.*

Theorem 152 *If $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy prime $\Gamma-$ideal of S, then $\Box A = \langle \mu_A, \overline{\mu_A} \rangle$ and $\Diamond A = \langle \overline{\gamma_A}, \gamma_A \rangle$ are intuitionistic fuzzy prime $\Gamma-$ideals of S.*

Proof. Since $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy prime $\Gamma-$ideal of S. Then for any $x, y \in S$ and $\gamma \in \Gamma$, we have

$$\begin{aligned} \inf_{\gamma\in\Gamma} \mu_A(x\gamma y) &= \max\{\mu_A(x), \mu_A(y)\} \\ 1 - \inf_{\gamma\in\Gamma} \mu_A(x\gamma y) &= 1 - \max\{\mu_A(x), \mu_A(y)\} \\ \sup_{\gamma\in\Gamma} (1 - \mu_A(x\gamma y)) &= \min\{1 - \mu_A(x), 1 - \mu_A(y)\} \\ \sup_{\gamma\in\Gamma} \overline{\mu_A}(x\gamma y) &= \min\{\overline{\mu_A}(x), \overline{\mu_A}(y)\} \end{aligned}$$

Hence $\Box A = \langle \mu_A, \overline{\mu_A} \rangle$ is an intuitionistic fuzzy prime $\Gamma-$ideal of S. Similarly, we have

$$\begin{aligned} \sup_{\gamma\in\Gamma} \gamma_A(x\gamma y) &= \min\{\gamma_A(x), \gamma_A(y)\} \\ 1 - \sup_{\gamma\in\Gamma} \gamma_A(x\gamma y) &= 1 - \min\{\gamma_A(x), \gamma_A(y)\} \\ \inf_{\gamma\in\Gamma} (1 - \gamma_A(x\gamma y)) &= \max\{1 - \gamma_A(x), 1 - \gamma_A(y)\} \\ \inf_{\gamma\in\Gamma} \overline{\gamma_A}(x\gamma y) &= \max\{\overline{\gamma_A}(x), \overline{\gamma_A}(y)\} \end{aligned}$$

Hence $\Diamond A = \langle \overline{\gamma_A}, \gamma_A \rangle$ is an intuitionistic fuzzy prime $\Gamma-$ideal of S. ■

Theorem 153 *Let $A = \langle \mu_A, \gamma_A \rangle$ be an IFS in $\Gamma-LA$-semigroup S. Then $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy prime $\Gamma-$ideal of S if and only if for any $s, t \in [0,1]$, the sets $U(\mu_A, s) = \{x \in S : \mu_A(x) \geq s\}$ and $L(\gamma_A, t) = \{x \in S : \gamma_A(x) \leq t\}$ are prime $\Gamma-$ideals of S.*

Proof. Suppose that $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy prime $\Gamma-$ideal of S. Let $s, t \in [0,1]$ such that $U(\mu_A, s)$ and $L(\mu_A, t)$ are non-empty. Now, let $x, y \in S$ such that $x\Gamma y \in U(\mu_A, s)$ then $\mu_A(x\gamma y) \geq s$ for all $\gamma \in \Gamma$. Then $\inf_{\gamma\in\Gamma} \mu_A(x\gamma y) \geq s$. Since

$$\begin{aligned} s &\leq \inf_{\gamma\in\Gamma} \mu_A(x\gamma y) = \max\{\mu_A(x), \mu_A(y)\} \\ s &\leq \max\{\mu_A(x), \mu_A(y)\} \\ \mu_A(x) &\geq s \text{ or } \mu_A(y) \geq s \end{aligned}$$

Hence $x \in U(\mu_A, s)$ or $y \in U(\mu_A, s)$. Thus $U(\mu_A, s)$ is a prime $\Gamma-$ideal of S. Now, let $x\Gamma y \in L(\mu_A, t)$. Then $\mu_A(x\gamma y) \geq t$ for all $\gamma \in \Gamma$. So, $\sup_{\gamma\in\Gamma} \gamma_A(x\gamma y) \geq t$. Since

$$\begin{aligned} s &\leq \sup_{\gamma\in\Gamma} \mu_A(x\gamma y) = \min\{\gamma_A(x), \gamma_A(y)\} \\ s &\leq \min\{\gamma_A(x), \gamma_A(y)\} \\ \gamma_A(x) &\geq t \text{ or } \gamma_A(y) \geq t \end{aligned}$$

Hence $x \in L(\gamma_A, t)$ or $y \in L(\gamma_A, t)$. Thus $L(\gamma_A, t)$ is a prime $\Gamma-$ideal of S.

Conversely, suppose that $U(\mu_A, s)$ and $L(\gamma_A, t)$ are prime $\Gamma-$ideals of S. Let $\inf_{\gamma\in\Gamma} \mu_A(x\gamma y) = s$ (Since $\mu_A(x\gamma y) \in [0,1]$ for all $\gamma \in \Gamma$, so $\inf_{\gamma\in\Gamma} \mu_A(x\gamma y)$ exists). Then $\mu_A(x\gamma y) \geq s$ for all $\gamma \in \Gamma$. So $x\gamma y \in U(\mu_A, s)$ for all $\gamma \in \Gamma$. Since $U(\mu_A, s)$ is prime. So $x \in U(\mu_A, s)$ or $y \in U(\mu_A, s) \Rightarrow \mu_A(x) \geq s$ or $\mu_A(y) \geq s$. Then

$$\max\{\mu_A(x), \mu_A(y)\} \geq s = \inf_{\gamma\in\Gamma} \mu_A(x\gamma y) \tag{3.1}$$

Since $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy $\Gamma-$ideal of S. So,

$$\mu_A(x\gamma y) \geq \max\{\mu_A(x), \mu_A(y)\} \ \forall \gamma \in \Gamma$$

$$\inf_{\gamma\in\Gamma} \mu_A(x\gamma y) \geq \max\{\mu_A(x), \mu_A(y)\} \tag{3.2}$$

From 3.1 and 3.2, we have

$$\inf_{\gamma\in\Gamma} \mu_A(x\gamma y) = \max\{\mu_A(x), \mu_A(y)\}$$

Now, let $\sup_{\gamma\in\Gamma} \gamma_A(x\gamma y) = t$ (Since $\mu_A(x\gamma y) \in [0,1]$ for all $\gamma \in \Gamma$, so $\sup_{\gamma\in\Gamma} \gamma_A(x\gamma y)$ exists). Then $\mu_A(x\gamma y) \leq t\ \forall \gamma \in \Gamma$. So $x\gamma y \in L(\gamma_A, t)$ for all $\gamma \in \Gamma$. Since $L(\gamma_A, t)$ is prime, so $x \in L(\gamma_A, t)$ or $y \in L(\gamma_A, t) \Rightarrow \gamma_A(x) \leq t$ or $\gamma_A(y) \leq t$. Then

$$\min\{\gamma_A(x), \gamma_A(y)\} \leq s = \sup_{\gamma\in\Gamma} \gamma_A(x\gamma y) \tag{3.3}$$

Since $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy $\Gamma-$ideal of S. So,

$$\gamma_A(x\gamma y) \leq \min\{\gamma_A(x), \gamma_A(y)\} \text{ for all } \gamma \in \Gamma$$

$$\sup_{\gamma\in\Gamma} \gamma_A(x\gamma y) \leq \min\{\gamma_A(x), \gamma_A(y)\} \tag{3.4}$$

From 3.3 and 3.4, we have

$$\sup_{\gamma\in\Gamma} \gamma_A(x\gamma y) = \min\{\gamma_A(x), \gamma_A(y)\}$$

Hence $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy prime $\Gamma-$ideal of S. ∎

3.3 Intuitionistic Fuzzy semi-prime $\Gamma-$Ideals in $\Gamma-$LA-semigroups

Definition 154 Let $A = \langle \mu_A, \gamma_A \rangle$ be an IFS in a $\Gamma-$LA-semigroup S. Then $A = \langle \mu_A, \gamma_A \rangle$ is called an intuitionistic fuzzy semi-prime if

$(IFP3)$ $\mu_A(x) \geq \mu_A(x\gamma x)$,

$(IFP4)$ $\gamma_A(x) \leq \mu\gamma_A(x\gamma x)$, $\forall x \in S$ and $\gamma \in \Gamma$. An intuitionistic fuzzy $\Gamma-$ideal is called an intuitionistic fuzzy semi-prime $\Gamma-$ideal of S if its intuitionistic fuzzy semi-prime.

Theorem 155 *Let S be a $\Gamma-LA$-semigroup and $\Phi \neq T \subseteq S$ is a semi-prime $\Gamma-$ideal of S. Then $A = \langle \mathcal{X}_T, \overline{\mathcal{X}_T} \rangle$ is an intuitionistic fuzzy semi-prime $\Gamma-$ideal of S.*

Proof. Let $x \in S$ and $\gamma \in \Gamma$. If $x\gamma x \in T$, then since T is semi-prime, we have $x \in T$. Thus

$$\begin{aligned} \mathcal{X}_T(x) &= 1 \geq \mathcal{X}_T(x\gamma x) \\ \text{and } \overline{\mathcal{X}_T}(x) &= 0 \leq \overline{\mathcal{X}_T}(x\gamma x) \end{aligned}$$

If $a\gamma a \notin T$, then

$$\begin{aligned} \mathcal{X}_T(x) &\geq 0 = \mathcal{X}_P(x\gamma x) \\ \text{and } \overline{\mathcal{X}_T}(x\gamma x) &= 1 - \mathcal{X}_T(x\gamma x) = 1 \geq \overline{\mathcal{X}_T}(x) \\ \overline{\mathcal{X}_T}(x) &\leq \overline{\mathcal{X}_T}(x\gamma x) \end{aligned}$$

Hence $A = \langle \mathcal{X}_T, \overline{\mathcal{X}_T} \rangle$ is an intuitionistic fuzzy semi-prime $\Gamma-$ideal of S. ∎

Theorem 156 *Let T be a non-empty subset of S. If $A = \langle \mathcal{X}_P, \overline{\mathcal{X}_P} \rangle$ is an intuitionistic fuzzy prime $\Gamma-$ideal of S, then T is a prime $\Gamma-$ideal of S.*

Proof. Let $A = \langle \mathcal{X}_T, \overline{\mathcal{X}_T} \rangle$ be an intuitionistic fuzzy prime $\Gamma-$ideal of S and $x\Gamma x \in T$. Then $\mathcal{X}_T(x\gamma x) = 1$ $\forall \gamma \in \Gamma$. Since from $(IFP3)$, so

$$\begin{aligned} \mu_A(x) &\geq \mathcal{X}_T(x\gamma x) = 1 \\ \mu_A(x) &\geq 1 \text{ and } \mu_A(x) \leq 1 \\ \mu_A(x) &= 1 \end{aligned}$$

Hence $x \in T$. Thus T is a semi-prime $\Gamma-$ideal of S. ∎

Theorem 157 *For any intuitionistic fuzzy sub$\Gamma-LA$-semigroup $A = \langle \mu_A, \gamma_A \rangle$ of S. If $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy semiprime, then $A(x) = A(x\gamma x)$ $\forall x \in S$ and $\gamma \in \Gamma$.*

Proof. Let $x \in S$. Then, since $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy sub$\Gamma-$LA-semigroup, so

$$\begin{aligned} \mu_A(x) &\geq \mu_A(x\gamma x) = \min\{\mu_A(x), \mu_A(x)\} = \mu_A(x) \\ \mu_A(x) &= \mu_A(x\gamma x). \end{aligned}$$

Also, we have

$$\begin{aligned} \gamma_A(x) &\leq \gamma_A(x\gamma x) = \max\{\gamma_A(x), \gamma_A(x)\} = \gamma_A(x) \\ \gamma_A(x) &= \gamma_A(x\gamma x) \end{aligned}$$

This completes the proof. ∎

Theorem 158 *If $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy semi-prime $\Gamma-$ideal of S, then $\Box A = \langle \mu_A, \overline{\mu_A} \rangle$ and $\Diamond A = \langle \overline{\gamma_A}, \gamma_A \rangle$ are intuitionistic fuzzy semi-prime $\Gamma-$ideals of S.*

Proof. Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy semi-prime $\Gamma-$ideal of S. Then we have

$$\begin{aligned} \mu_A(x) &\geq \mu_A(x\gamma x) \\ 1 - \mu_A(x) &\leq 1 - \mu_A(x\gamma x) \\ \overline{\mu_A}(x) &\leq \overline{\mu_A}(x\gamma x) \end{aligned}$$

Hence $\Box A = \langle \mu_A, \overline{\mu_A} \rangle$ is an intuitionistic fuzzy semi-prime $\Gamma-$ideal of S. Similarly we have

$$\begin{aligned} \gamma_A(x) &\leq \gamma_A(x\gamma x) \\ 1 - \gamma_A(x) &\geq 1 - \gamma_A(x\gamma x) \\ \overline{\gamma_A}(x) &\geq \overline{\gamma_A}(x\gamma x) \end{aligned}$$

Hence $\Diamond A = \langle \overline{\gamma_A}, \gamma_A \rangle$ is an intuitionistic fuzzy semi-prime $\Gamma-$ideal of S. ∎

Proposition 159 *If $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy semi-prime $\Gamma-$ideal of S, then μ_A and $\overline{\gamma_A}$ are fuzzy prime $\Gamma-$ideals of S.*

Proof. Straightforward. ∎

Proposition 160 *If $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy semi-prime $\Gamma-$ideal of S, then $\overline{\gamma_A}$ and γ_A are anti fuzzy prime $\Gamma-$ideals of S.*

Theorem 161 *Let $A = \langle \mu_A, \gamma_A \rangle$ be IFS in $\Gamma-$LA-semigroup S. Then $A = \langle \mu_A, \gamma_A \rangle$ intuitionistic fuzzy semi-prime $\Gamma-$ideal of S if and only if for any $s, t \in [0,1]$, the sets $U(\mu_A, s) = \{x \in S : \mu_A(x) \geq s\} \neq \Phi$ and $L(\gamma_A, t) = \{x \in S : \gamma_A(x) \leq t\} \neq \Phi$ are semiprime $\Gamma-$ideals of S.*

Proof. Suppose that $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy semi-prime $\Gamma-$ideal of S. Let $s, t \in [0,1]$ such that $U(\mu_A, s)$ and $L(\gamma_A, t)$ are non-empty. Now, let $x \in S$ such that $x\Gamma x \in U(\mu_A, s)$. Then $\mu_A(x\gamma x) \geq s$ for all $\gamma \in \Gamma$. Since

$$\begin{aligned}
\mu_A(x) &\geq \mu_A(x\gamma x) \geq s \\
\mu_A(x) &\geq s
\end{aligned}$$

Hence $x \in U(\mu_A, s)$. Thus $U(\mu_A, s)$ is a semi-prime $\Gamma-$ideal of S. Now, let $x\Gamma x \in L(\gamma_A, t)$. Then $\gamma_A(x\gamma x) \leq t$ for all $\gamma \in \Gamma$. Since

$$\begin{aligned}
\gamma_A(x) &\leq \gamma_A(x\gamma x) \leq t \\
\gamma_A(x) &\leq t
\end{aligned}$$

Hence $x \in L(\gamma_A, t)$. Thus $L(\gamma_A, t)$ is a semi-prime $\Gamma-$ideal of S.

Conversely, let $A = \langle \mu_A, \gamma_A \rangle$ be an IFS in S such that $U(\mu_A, s)$ and $L(\gamma_A, t)$ are semi-prime $\Gamma-$ideals of S and suppose that $A = \langle \mu_A, \gamma_A \rangle$ is not an intuitionistic fuzzy semi-prime $\Gamma-$ideal of S. Then there exist $x_\circ \in S$ such that $\mu_A(x_\circ) < \mu_A(x_\circ \gamma x_\circ)$. Let

$$\begin{aligned}
s_\circ &= \frac{1}{2}\left[\mu_A(x_\circ) + \mu_A(x_\circ \gamma x_\circ)\right]. \text{ Then} \\
\mu_A(x_\circ) &< s_\circ < \mu_A(x_\circ \gamma x_\circ)
\end{aligned}$$

Thus $x_\circ \gamma x_\circ \in U(\mu_A, s_\circ)$ but $x_\circ \notin U(\mu_A, s_\circ)$, a contradiction. Therefore $\mu_A(x) \geq \mu_A(x\gamma x)$ for all $x \in S$. Similarly $\gamma_A(x) \leq \gamma_A(x\gamma x)$ for all $x \in S$. Hence $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy semi-prime $\Gamma-$ideal of S. ∎

Theorem 162 *Let S be a left regular. Then, for every intuitionistic fuzzy right $\Gamma-$ideal $A = \langle \mu_A, \gamma_A \rangle$ of S, $A(x) = A(x\alpha x)$ $\forall x \in S$ and $\alpha \in \Gamma$.*

Proof. Let x be any element of S. Since S is left regular, so there exists $a \in S$ and $\alpha, \beta \in \Gamma$ such that $x = (a\alpha x)\beta x = (x\alpha x)\beta a$. Thus we have

$$\begin{aligned}
\mu_A(x) &= \mu_A((a\alpha x)\beta x) = \mu_A((x\alpha x)\beta a) \geq \mu_A(x\alpha x) \geq \mu_A(x) \\
\mu_A(x) &= \mu_A(x\alpha x) \text{ and} \\
\gamma_A(x) &= \gamma_A((a\alpha x)\beta x) = \gamma_A((x\alpha x)\beta a) \leq \gamma_A(x\alpha x) \leq \gamma_A(x) \\
\gamma_A(x) &= \gamma_A(x\alpha x)
\end{aligned}$$

Hence $A(x) = A(x\alpha x)$ $\forall x \in S$ and $\alpha \in \Gamma$. ■

Proposition 163 *If S is left(resp, right) regular, then for any intuitionistic fuzzy left(resp, right) $\Gamma-$ideal of S the following holds.*

$$A(a) = A(a\gamma a) \qquad \textit{for all } a \in S \textit{ and } \gamma \in \Gamma.$$

Proof. Let S be a left regular $\Gamma-$LA-semigroup. Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy left $\Gamma-$ideal of S. Let $a \in S$, since S is left regular, so there exist $x \in S$ and $\gamma, \beta \in \Gamma$ such that $a = x\beta(a\gamma a)$.

$$\begin{aligned}
\mu_A(a) &= \mu_A(x\beta(a\gamma a)) \geq \mu_A(a\gamma a) \geq \mu_A(a) \Rightarrow \mu_A(a) = \mu_A(a\gamma a) \\
\text{and } \lambda_A(a) &= \lambda_A(x\beta(a\gamma a)) \leq \lambda_A(a\gamma a) \leq \lambda_A(a) \Rightarrow \lambda_A(a) = \lambda_A(a\gamma a)
\end{aligned}$$

Hence $A(a) = A(a\gamma a)$ for all $a \in S$. ■

Proposition 164 *Every intuitionistic fuzzy right $\Gamma-$ideal of regular $\Gamma-LA$-semigroup S is $\Gamma-$idempotent.*

Proof. Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy right $\Gamma-$ideal of S. Since S regular, so by Proposition 137

$$\begin{aligned}
A\Gamma A &= A \cap A = A \\
A\Gamma A &= A
\end{aligned}$$

■

Definition 165 Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy $\Gamma-$subLA-semigroup of $\Gamma-$LA-semigroup. Then $A = \langle \mu_A, \gamma_A \rangle$ is called an intuitionistic fuzzy interior $\Gamma-$ideal (briefly, $IFI\Gamma I$) of S if

$(IFI1)$ $\mu_A((x\beta a)\gamma y)) \geq \mu_A(a)$,

$(IFI2)$ $\gamma_A((x\beta a)\gamma y)) \leq \gamma_A(a)$, *for all* $x, a, y \in S$.

Definition 166 Let $A = \langle \mu_A, \gamma_A \rangle$ be an IFS in $\Gamma-$LA-semigroup S. Then $A = \langle \mu_A, \gamma_A \rangle$ is called an intuitionistic fuzzy quasi $\Gamma-$ideal of S if

$(IFQ1)$ $\mu_A(x) \geq \min\{(\mu_A \circ_\Gamma \mathcal{S})(x), (\mathcal{S} \circ_\Gamma \mu_A)(x)\}$,

$(IFQ1)$ $\mu_A(x) \leq \max\{(\mu_A \circ_\Gamma \varphi)(x), (\varphi \circ_\Gamma \mu_A)(x)\}$, for all $x \in S$. Where $\mathcal{S}$ is fuzzy set of S which mapped on 1, and φ is fuzzy set of S which mapped on 0.

Theorem 167 *Let S be a $\Gamma-LA$-semigroup with left identity e such that $(x\alpha e)\Gamma S = x\Gamma S$ for all $x \in S$ and $\alpha \in \Gamma$. Then, every intuitionistic fuzzy quasi $\Gamma-$ideal of S is intuitionistic fuzzy bi-$\Gamma-$ideal.*

Proof. Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy quasi $\Gamma-$ideal of S and $x, y \in S$ and $\alpha \in \Gamma$. Then

$$\begin{aligned}
\mu_A(x\alpha y) &\geq \min\{(\mu_A \circ_\Gamma \mathcal{S})(x\alpha y), (\mathcal{S} \circ_\Gamma \mu_A)(x\alpha y)\} \\
&= \min\left\{ \bigvee_{x\alpha y = p\beta q} \{\mu_A(p) \wedge \mathcal{S}(q)\}, \bigvee_{x\alpha y = r\beta s} \{\mathcal{S}(r) \wedge \mu_A(s)\} \right\} \\
&= \min\left\{ \bigvee_{x\alpha y = p\beta q} \{\mu_A(p) \wedge 1\}, \bigvee_{x\alpha y = r\beta s} \{1 \wedge \mu_A(s)\} \right\} \\
\mu_A(x\alpha y) &\geq \min\{\mu_A(x), \mu_A(y)\} \text{ and}
\end{aligned}$$

$$\begin{aligned}
\gamma_A(x\alpha y) &\leq \max\{(\gamma_A \circ_\Gamma \varphi)(x\alpha y), (\varphi \circ_\Gamma \gamma_A)(x\alpha y)\} \\
\gamma_A(x\alpha y) &\leq \max\{(\gamma_A \circ_\Gamma \varphi)(x\alpha y), (\varphi \circ_\Gamma \gamma_A)(x\alpha y)\} \\
&= \max\left\{ \bigwedge_{x\alpha y = p\beta q} \{\gamma_A(p) \vee \varphi(q)\}, \bigwedge_{x\alpha y = r\beta s} \{\varphi(r) \wedge \gamma_A(s)\} \right\} \\
&= \max\left\{ \bigwedge_{x\alpha y = p\beta q} \{\gamma_A(p) \vee 0\}, \bigwedge_{x\alpha y = r\beta s} \{0 \vee \gamma_A(s)\} \right\} \\
\gamma_A(x\alpha y) &\leq \max\{\gamma_A(x), \gamma_A(y)\}
\end{aligned}$$

So, $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy sub$\Gamma-$LA-semigroup of S. Now for $x, y, z \in S$ and $\alpha, \beta \in \Gamma$.

$$\begin{aligned}
\mu_A((x\alpha y)\beta z) &\geq \min\{(\mu_A \circ_\Gamma \mathcal{S})((x\alpha y)\beta z), (\mathcal{S} \circ_\Gamma \mu_A)((x\alpha y)\beta z)\} \\
\text{Now, } (\mu_A \circ_\Gamma \mathcal{S})((x\alpha y)\beta z) &= \bigvee_{(x\alpha y)\beta z = a\gamma b} \{\mu_A(a) \wedge \mathcal{S}(b)\} \\
&= \bigvee_{(x\alpha y)\beta z = a\gamma b} \{\mu_A(a) \wedge 1\} \\
\text{Now, } (x\alpha y)\beta z &= (x\alpha y)(e\beta z) = (x\alpha e)(y\beta z) \in (x\alpha e)\Gamma S = x\Gamma S \\
\text{so } (x\alpha y)\beta z &= x\delta t \text{ for some } t \in S \text{ and } \delta \in \Gamma. \\
\text{So } (\mu_A \circ_\Gamma \mathcal{S})((x\alpha y)\beta z) &= \bigvee_{x\delta t = a\gamma b} \{\mu_A(a) \wedge 1\} \geq \mu_A(x) \text{ and} \\
(\mathcal{S} \circ_\Gamma \mu_A)((x\alpha y)\beta z) &= \bigvee_{(x\alpha y)\beta z = p\gamma q} \{\mathcal{S}(p) \wedge \mu_A(q)\} \\
&= \bigvee_{(x\alpha y)\beta z = p\gamma q} \{1 \wedge \mu_A(q)\} = \bigvee_{(x\alpha y)\beta z = p\gamma q} \{\mu_A(q)\} \\
(\mathcal{S} \circ_\Gamma \mu_A)((x\alpha y)\beta z) &\geq \mu_A(z) \\
\text{Thus } \mu_A((x\alpha y)\beta z) &\geq \min\{\mu_A(x), \mu_A(z)\} \text{ and} \\
\gamma_A((x\alpha y)\beta z) &\leq \max\{(\gamma_A \circ_\Gamma \varphi)((x\alpha y)\beta z), (\varphi \circ_\Gamma \gamma_A)((x\alpha y)\beta z)\} \\
\text{Now, } (\gamma_A \circ_\Gamma \varphi)((x\alpha y)\beta z) &= \bigwedge_{(x\alpha y)\beta z = a\gamma b} \{\gamma_A(a) \vee \varphi(b)\} \\
&= \bigwedge_{(x\alpha y)\beta z = a\gamma b} \{\gamma_A(a) \vee 0\} \\
\text{Now, } (x\alpha y)\beta z &= (x\alpha y)(e\beta z) = (x\alpha e)(y\beta z) \in (x\alpha e)\Gamma S = x\Gamma S \\
\text{so } (x\alpha y)\beta z &= x\delta t \text{ for some } t \in S \text{ and } \delta \in \Gamma. \\
\text{So } (\gamma_A \circ_\Gamma \varphi)((x\alpha y)\beta z) &= \bigwedge_{x\delta t = a\gamma b} \{\gamma_A(a) \vee 0\} \leq \gamma_A(x) \\
(\varphi \circ_\Gamma \gamma_A)((x\alpha y)\beta z) &= \bigwedge_{(x\alpha y)\beta z = p\gamma q} \{\varphi(p) \vee \gamma_A(q)\} \\
&= \bigwedge_{(x\alpha y)\beta z = p\gamma q} \{0 \vee \gamma_A(q)\} = \bigvee_{(x\alpha y)\beta z = p\gamma q} \{\gamma_A(q)\} \\
(\varphi \circ_\Gamma \gamma_A)((x\alpha y)\beta z) &\leq \gamma_A(z) \\
\gamma_A((x\alpha y)\beta z) &\leq \max\{\gamma_A(x), \gamma_A(z)\}
\end{aligned}$$

Hence $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy bi-$\Gamma-$ideal of S. ■

Theorem 168 *Let $A = \langle \mu_A, \gamma_A \rangle$ be an IFS in intra-regular $\Gamma-$LA-semigroup with left identity. Then $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy interior $\Gamma-$ideal of S if and only if $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy $\Gamma-$ideal of S.*

Proof. Direct part is obvious

Conversely, let $a, b \in S$ and $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy interior $\Gamma-$ideal of S. Since S is intra-regular, there exist $x, y, u, v \in S$ and $\beta, \gamma, \rho, \tau, \delta, \delta \in \Gamma$ such that $a = (x\rho a)\beta(a\gamma y)$ and $b = (u\tau b)\delta(b\delta v)$. Then we have

$$\begin{aligned}
\mu_A(a\alpha b) &= \mu_A(((x\rho a)\beta(a\gamma y))\alpha b) \\
&= \mu_A((a\beta((x\rho a)\gamma y))\alpha b) \quad \text{because } a\gamma(b\beta c) = b\gamma(a\beta c) \\
&\geq \mu_A((x\rho a)\gamma y) \geq \mu_A(a) \\
\mu_A(a\alpha b) &\geq \mu_A(a) \text{ and} \\
\gamma_A(a\alpha b) &= \gamma_A(((x\rho a)\beta(a\gamma y))\alpha b) \\
&= \gamma_A((a\beta((x\rho a)\gamma y))\alpha b) \quad \text{because } a\gamma(b\beta c) = b\gamma(a\beta c) \\
&\leq \gamma_A((x\rho a)\gamma y) \geq \mu_A(a) \\
\gamma_A(a\alpha b) &\leq \gamma_A(a)
\end{aligned}$$

Thus $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy right $\Gamma-$ideal of S. Also

$$\begin{aligned}
\mu_A(a\alpha b) &= \mu_A(a\alpha\{(u\tau b)\delta(b\delta v)\}) \\
&= \mu_A(a\alpha\{((b\delta v)\tau b)\delta u\}) \text{ by left invertive law} \\
&= \mu_A(((b\delta v)\tau b)\alpha(a\delta u)) \quad \text{because } a\gamma(b\beta c) = b\gamma(a\beta c) \\
\mu_A(a\alpha b) &\geq \mu_A(b) \text{ and} \\
\gamma_A(a\alpha b) &= \gamma_A(a\alpha\{(u\tau b)\delta(b\delta v)\}) \\
&= \gamma_A(a\alpha\{((b\delta v)\tau b)\delta u\}) \text{ by left invertive law} \\
&= \gamma_A(((b\delta v)\tau b)\alpha(a\delta u)) \quad \text{because } a\gamma(b\beta c) = b\gamma(a\beta c) \\
\gamma_A(a\alpha b) &\leq \gamma_A(b)
\end{aligned}$$

Thus $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy left $\Gamma-$ideal of S. Hence $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy $\Gamma-$ideal of S. ■

Theorem 169 *For $\Gamma-$LA-semigroup S, the following conditions are equivalent.*

(1) *S is regular*

(2) *$A \cap B = A \circ_\Gamma B$ for any intuitionistic fuzzy right $\Gamma-$ideal and intuitionistic fuzzy left $\Gamma-$ideal.*

Proof. (1) $\Rightarrow$ (2). Let $A = \langle \mu_A, \gamma_A \rangle$ be any intuitionistic fuzzy right $\Gamma-$ideal of S and $B = \langle \mu_B, \gamma_B \rangle$ be any intuitionistic fuzzy right $\Gamma-$ideal of S. Then by Lemma 103,

$$A \circ_\Gamma B \subseteq A \cap B. \tag{3.5}$$

Let a be any element of S. Since S is regular, then there exist an element $x \in S$ such that $a = (a\gamma x)\beta a$. Then we have

$$\begin{aligned}
\mu_{A\circ_\Gamma B}(a) &= \bigvee_{a=y\gamma z} \{\mu_A(y) \wedge \mu_B(z)\} \\
&\geq \{\mu_A(a\gamma x) \wedge \mu_B(a)\} \\
&\geq \mu_A(a) \wedge \mu_B(a) \\
\mu_{A\circ_\Gamma B}(a) &\geq (\mu_A \wedge \mu_B)(a) \\
\text{and } \lambda_{A\circ_\Gamma B}(a) &= \bigwedge_{a=y\gamma z} \{\lambda_A(y) \vee \lambda_B(z)\} \\
&\leq \lambda_A(a\gamma x) \vee \lambda_B(a) \\
&\leq \lambda_A(a) \vee \lambda_B(a) \\
\lambda_{A\circ_\Gamma B}(a) &\leq (\lambda_A \vee \lambda_B)(a)
\end{aligned}$$

So

$$A \cap B \subseteq A \circ_\Gamma B \tag{3.6}$$

From (3.5) and (3.6), we have $A \cap B = A \circ_\Gamma B$

(2) $\Rightarrow$ (1). Let R and L be left ant right $\Gamma-$ideals of $\Gamma-$LA-semigroup S respectively. In order to see that $R \cap L \subseteq R\Gamma L$ holds. Let $a \in R \cap L \Rightarrow (C_R \cap C_L)(a) = 1$ Then from Lemma 82, C_R and C_L are fuzzy right and left $\Gamma-$ideals of S respectively. Then we have

$$\begin{aligned}
C_{R\Gamma L}(a) &= (C_R \Gamma C_L)(a) \\
&= (C_R \cap C_L)(a) \\
C_{R\Gamma L}(a) &= 1
\end{aligned}$$

So $a \in R\Gamma L$. Then $R \cap L \subseteq R\Gamma L$ and $R\Gamma L \subseteq R \cap L$ is always hold. Therefore $R\Gamma L = R \cap L$. Hence S is regular.

Future Work

In our future research, we will concentrate on characterizations of different classes (regular, intra-regular and right weakly regular) of $\Gamma-$LA-semigroups in terms of intuitionistic fuzzy (left, right, bi, interior, quasi) $\Gamma-$ideals. We will also concentrate on defining fuzzy prime (semi-prime) bi-Γideals and intuitionistic fuzzy prime (semi-prime) bi-Γ-ideals. W will also concentrate on characterization of regular and intra-regular Γ-LA-semigroups by the properties of fuzzy prime(semi-prime) bi-Γideals and intuitionistic fuzzy prime (semi-prime) bi-Γ-ideals.

Hopefully, some new results in these topics can be obtained.

REFERENCES

[1] S. Abdullah, M. Aslam and T. Anwar, *A note on M-hypersystems and N-hypersystems in* $\Gamma-$*semihypergroups,* Quasigroups and Related Systems, 19 (2011).

[2] M. Aslam and S. Abdullah, *Direct product of intuitionistic fuzzy sets in LA-semigroups,* Ital J. of Pur. App. Maths, 33, In press.

[3] M. Aslam and S. Abdullah (Φ, Ψ)*-Intuitionistic fuzzy ideals of semigroups,* Ital J. of Pur. App. Maths, 32, In press.

[4] M. Aslam, S. Abdullah and N. Amin, *Characterization of* Γ*-LA-semigroups by generalized fuzzy* Γ*-ideals*, Accepted in International Journal of Mathematics and Statistics (IJMS).

[5] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy sets and Systems, 20 (1986), no,1,87-96.

[6] K. T. Atanassov K. *Review and new results on intuitionistic fuzzy sets.* Preprint IM-MFAIS-1-88, Sofia, 1988.

[7] K. T. Atanassov, *Intuitionistic fuzzy sets*, Theory and Applications, Springer-Verlag, Heidelberg, 1999

[8] K. T. Atanassov, *New operations defined over the intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 61 (1994) 137-142.

[9] Q. Iqbal; *Some studies in left almost-semigroups*, Ph.D. Thesis, Quaid-i-Azam University, Islamabad, 1991.

[10] M. A. Kazim and M. Naseerudin, *On almost semigroups*, Alig.Bull.Math, 2 (1972), 1-7.

[11] M. Khan and M.A. Khan *Characterizations of intra-regular LA-semigroup by their fuzzy ideals,* Journal of Mathematics Research, 2 (2010) 3 87-96.

[12] K. H. Kim and Y. B. Jun, *Intuitionistic fuzzy Ideal of semigroups,* Indian. J. Pure Appl. Math. 33 (2002), no. 4, 443-449.

[13] K. H. Kim and Y. B. Jun, *Intuitionistic fuzzy interior ideals of semigroups,* Int. J. Math. Sci. 27 (2001), no. 5, 261-142.

[14] Klaua and D. Zum , *Theory of Sets and Topology*, Berlin, 313–325 (1972).

[15] N. Kuroki; Fuzzy bi-ideals in Semigroups, Comment. Math. Univ. St. Paul, 28 (1979) 17-21.

[16] N. Kuroki; On fuzzy ideals and fuzzy bi-ideals in semigroups, Fuzzy Sets and Systems 5 (1981) 203-215.

[17] J. N. Mordeson, D. S. Malik and N. Kuroki; Fuzzy Semigroups, Studies in Fuzziness and Soft Computing Vol. 131, Springer-Verlag Berlin (2003).

[18] J. N. Mordeson and D. S. Malik; Fuzzy Automata and Languages, Theory and Applications, Computational Mathematical Series, Chap-man and Hall/CRC, Boca Raton (2002).

[19] Q. Mushtaq and S. M. Yousaf, *On LA-semigroups*, The Alig. Bull. Math, 8 (1978) 105-113.

[20] Q. Mushtaq and M. Khan, *Ideals in left almost semigroups,* Proceedings of 4th International Pure Mathematics Conference, 2003, 65-77.

[21] P. V. Protic and N. Stevanovic: *On Abel-Grassmann's groupoids (review)*, Proc. math. conf. in Pristina (1994), 31-38.

[22] P. V. Protic and N. Stevanovic; *The structural theorem for AG groupiods*, Ser. math. Inform, 10 (1995), 25-33.

[23] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl., 35(1971), 512–517.

[24] T. Shah and I. Rehman, *On* $\Gamma-$*ideals and* bi-$\Gamma-$ideals *in* $\Gamma-$LA-semigroup, Int. J. Algebra, 4 (2010), no. 267-276.

[25] T. Shah, I. Rehman and A.Khan, *Fuzzy* $\Gamma-$*ideals in* $\Gamma-$*AG groupoids*, (Submitted).

[26] T. Shah, I. Rehman and R. Chinram, *Some Characterization of Regular, Intra-regular* $\Gamma-$LA-semigroups, (Submitted).

[27] M. Uckun, M. A. Ozturk and Y. B. Jun, *Intuitionistic fuzzy sets in* $\Gamma-$*semigroups*, Bull. Korean. Math. Soc. 44 (2007), 359-367.

[28] L. A. Zadeh, *Fuzzy sets*, Information and Control, 8 (1965), 338-353.

[29] R. Zenab, *Some studies in fuzzy AG-Groupoids,* M.Phil. Thesis. Quaid-i-Azam university, Islamabad, 2009.

[30] W. Zeng and H. Li, *Note on some operations on intuitionistic fuzzy sets*, Fuzzy Sets and Systems 157 (2006) 990 – 991.