Tunneling of Charged Dirac Particles from Rotating and Accelerating Black Holes

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A Dissertation
Submitted in the Partial Fulfillment of the
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CERTIFICATE

A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF PHILOSOPHY

We accept this dissertation as conforming to the required standard

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2011

Dedicated to

My parents


بسم الله الرحمن الرحيم

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Mudassar Rehman

Abstract

In the first chapter of this thesis we have given some definitions and equations which are being used in our work. In the second chapter we have explained quantum tunneling and also found the tunneling probability of fermion particles from Riessner-Nordstrom black hole and Kerr-Newman black hole. For fermion particles we have used the Dirac equation. In the third chapter we have calculated the tunneling probability of fermion particles from rotating and accelerating charged black hole, following the work of Kerner and Mann at the outer horizon. From this Hawking temperature has been worked out. Also we have calculated the tunneling probability of fermion particles at acceleration horizon and found the explicit expression for the action as well. Then finally in the fourth chapter we have calculated the tunneling probability of scalar particles at the outer horizon and acceleration horizon. For scalar particles we have used Klein-Gordon equation. The final results of scalar particles and fermion particles are the same. We have given a conclusion at the end.

Contents

Chapter 1

Introduction

1.1 The Einstein Field Equations

The Einstein field equations (EFE) are a set of 10 coupled hyperbolic elliptic nonlinear partial differential equations which describe the gravitaional effects produced by a given mass. These are 16 equations but due to symmetry of G_{ab} and T_{ab}, the actual number of equations reduces to 10. The EFE are given by

$$G_{ab} = \kappa T_{ab}, \tag{1.1}$$

where $\kappa = 8\pi G/c^4$, G is Newton's gravitational constant and c is the speed of light. In the geometrized units G and c are taken to be 1. T_{ab} is the stress energy momentum tensor, and

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R, \tag{1.2}$$

is the Einstein tensor, R_{ab} the Ricci curvature tensor , g_{ab} the metric tensor, and R the scalar curvature. With the use of Eq. (1.2), Eq. (1.1) becomes [1, 2]

$$R_{ab} - \frac{1}{2}g_{ab}R = \kappa T_{ab}. \tag{1.3}$$

The tensor on the right hand side describes the distribution and motion of matter in space and the left hand side describes the geometry of space. A manifold on which Einstein tensor is zero is called the Einstein space. The EFE relate the spacetime curvature which is expressed by the Einstein tensor G_{ab}, with the energy and momentum within that spacetime which is expressed by the stress energy momentum tensor T_{ab}. Eq. (1.1) tells that mass curves the geometry of spacetime, and the geometry of spacetime in turn tells masses how to move. EFE are very complicated despite their simple appearance. For a given energy momentum tensor, T_{ab}, EFE are the equations for the metric g_{ab} because both Ricci curvature tensor R_{ab} and scalar curvature R depend on the metric components as

$$R_{ab} = R^c_{acb} = \Gamma^c_{ab,c} - \Gamma^c_{ac,b} + \Gamma^c_{ec}\Gamma^e_{ab} - \Gamma^c_{eb}\Gamma^e_{ca}, \tag{1.4}$$

where Γ^e_{ab} stands for the Christofell symbol which is defined by

$$\Gamma^e_{ab} = \frac{1}{2} g^{ec} \left(g_{ac,b} + g_{bc,a} - g_{ab,c} \right), \tag{1.5}$$

and

$$R = g^{ab} R_{ab}. \tag{1.6}$$

Einstein modified the field equations and introduced the cosmological constant Λ, thus modified EFE take the form

$$R_{ab} - \frac{1}{2} g_{ab} R + g_{ab}\Lambda = \kappa T_{ab}, \tag{1.7}$$

Einstein introduced the cosmological constant Λ for the static universe but later this effort became unsuccessfull because firstly the universe was not stable on behalf of this theory and our universe is not static but expanding. So Einstein called it "the biggest blunder he ever made". Contracting Eq. (1.3)

$$g^{ab} \left(R_{ab} - \frac{1}{2} g_{ab} R \right) = g^{ab} \left(\kappa T_{ab} \right) \tag{1.8}$$

shows that $R - \frac{1}{2} 4R = \kappa T$ which implies that $R = -\kappa T$ so Eq. (1.3) can also be written in the form

$$R_{ab} = \kappa \left(T_{ab} - \frac{1}{2} T g_{ab} \right), \tag{1.9}$$

which is sometimes more convenient in practical calculations. In particular this form easily gives the vacuum field equations in an empty spacetime (If energy momentum tensor is taken to be zero in the region under consideration then the EFE are called vacuum field equations). Thus

$$T_{ab} = 0 \Longrightarrow T = 0 \Longrightarrow R_{ab} = 0, \tag{1.10}$$

which is the vacuum field equation for $\Lambda = 0$. So vacuum solutions are characterised by the vanishing of the Ricci tensor. If the cosmological constant is taken to be non zero then vacuum field equations will be

$$R_{ab} = \Lambda g_{ab}. \tag{1.11}$$

For weak gravitational field and low velocities EFE reduce to Newton's laws of gravitation.

1.2 Black Holes

A black hole is a region of space from which nothing, not even light, can escape. It is an object whose gravity is so strong, the escape velocity will also be so strong, that speed of any thing, not even light, can reach the value of escape velocity. Hence when an object falls into a black hole then it can never emit that black hole. The boundary of a black hole is called an event horizon. An observor outside the event horizon cannot gain any information from the region inside the event horizon. It is called "black" because it absorbs all the light that hits it, reflecting nothing, just like a perfect black body in thermodynamics [3].

Black holes are the evolutionary endpoints of stars whose mass is 10 to 15 times the mass of the Sun. If a star undergoes a supernova explosion, it may leave behind a fairly massive burned out stellar remnant. With no outward forces to oppose gravitational forces, the remnant will collapse in itself. The star eventually collapses to the point of zero volume and infinite density, creating a "singularity". Around the singularity there is a region where the force of gravity is so strong that not even light can escape. Thus, no information can reach us from this region.

Consider a very compact and massive star. Its gravity can be increased if the star shrinks or if mass is added to it. According to the general theory of relativity gravity does influence the properties of light. When a light travels radially outward leaving the surface of a star it has to do work to over come the surface gravity of star so its energy and frequency will be somewhat diminished. This famous red shift has been measured in light in the (weak) gravitational fields of Sun and Earth. But for compact and massive objects red shift may be enormous. When escape velocity exceeds the speed of light then some thing odd must happen. This event comes about when the radius of a spherical uncharged star becomes less than $2GM/c^2$, where M is the mass of the star, c is the speed of light and G is Newton's gravitational constant. This radius is very small, for the Sun it is $1km$ and for Earth it is $1cm$. The radius of an ordinary star is very large but the radius of a neutron star approches this radius. Once the star reaches this radius then nothing can be viewed in the star thereafter. Therefore this spherical surface $r = 2GM/c^2$ acts like an event horizon [3].

As an observer cannot receive any information from inside the event horizon, black hole can be observed through its interaction with other matter like stars. A black hole can be inferred by tracking the movement of a group of stars that orbit a region in space. Alternatively, when gas falls into a stellar black hole from a companion star, the gas spirals inward, heating to very high temperatures and emitting large amount of radiation.

1.2.1 Event Horizon

Black holes are defined by the appearance of an event horizon. It is the boundary of spacetime such that light and matter can pass inward from outside the boundary but nothing can pass outward from inside that boundary. Thus an observor outside the event horizon cannot receive any information from the region inside the event horizon. Some times it is defined as the boundary within which the escape velocity of a black hole is greater than the speed of light.

1.2.2 Singularity

At the centre of a black hole gravitational singularity lies, it is the region where the spacetime curvature becomes infinite. For a non-rotating black hole this region is like a point but for a rotating black hole it forms a ring singularity. In both the cases the singularity region has zero volume, all the mass of a black hole is concentrated in this region so its density is also infinite.

1.2.3 Gravitational Collapse

When we throw any object upward on the earth it goes up to some distance then it comes back because gravity of Earth pulls it back. If we throw the same object with the same force on Mars or the Moon then it will cover large distance than it covers on the earth because both Mars and the Moon are less massive than the Earth. But if we give sufficient velocity to the body which is equal to the escape velocity of the Earth then it will never come back. As nothing can travel faster than the light so if a planet or star has escape velocity greater then the speed of light then general theory of relativity predicts black hole.

Black holes can be created by the gravitaional collapse in the lage stars. Normally there are two forces acting in the star one is the gravity which acts towards the centre of the star and second is the pressure produced by the nuclear fusion reaction inside the star.

A star goes to a gravitational collapse at the end of its life time. When all the energy sources are exhausted then a star goes to a gravitaional collapse. The three states of compact stars are white dwarfs, neutron stars and black holes.

White dwarf is that stage in which gravity is opposed by the electron degeneracy pressure; neutron stars is that stage in which gravity is opposed by the neutron degeneracy pressure and black hole is that stage in which physics at the centre is unknown. It takes tens of thousands of years to collapse to white dwarf, the collapse continues and white dwarf goes to the next stage called neutron star when it crosses the Chendrasekhar limit (The Chandrasekhar limit is the mass above which electron degeneracy pressure in the star's core is insufficient to balance the star's own gravitational self-attraction). And when it collapse to the Schwarzschild radius then it becomes a black hole. Black hole is the final stage of a star. Black holes created by the gravitaional collapse of individual stars are 2 to 100 times massive than our sun.

1.3 Solutions of Einstein Field Equations

In this section we discuss some important and well-known solutions of EFE that represent black holes.

1.3.1 The Schwarzschild Metric

In Einstein's theory of general relativity, the Schwarzschild solution describes the gravitational field outside a spherical, non-rotating mass such as a (non-rotating) star, planet, or black hole. It is also a good approximation to the gravitational field of a slowly rotating body like the

Earth or Sun. The cosmological constant is assumed to be equal to zero.

According to Birkhoff's theorem, the Schwarzschild solution is the most general spherically symmetric, vacuum solution of the EFE. A Schwarzschild black hole or static black hole is a black hole that has no charge and angular momentum.

Karl Schwarzschild found the solution in 1915 after the publication of Einstein's theory of general relativity. It was the first exact solution of the EFE other than the trivial flat space solution.

There is surrounding spherical surface around a Schwarzschild black hole, called the event horizon, which is situated at the Schwarzschild radius, often called the radius of a black hole. A Schwarzschild black hole of any mass could exist if conditions became sufficiently favorable to allow for its formation.

Mathematically this metric can be written as

$$c^2 d\tau^2 = \left(1 - \frac{r_s}{r}\right) c^2 dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{1.12}$$

where τ is the proper time measured by a clock moving with the particle, t is the time measured by a stationary clock at infinity, r is the radial coordinate, θ is colatitude, ϕ is longitude and r_s is the Schwazschild radius which is related to the mass of a black hole M as

$$r_s = \frac{2GM}{c^2},$$

where G is the gravitaional constant and c is the speed of light [4].

The Schwarzschild metric has singularities at $r = 0$ and $r = r_s$ because some of the metric components blow up at these values. However the singularity $r = r_s$ can be removed by the appropriate choice of coordinates such as Lemaitre coordinates, Eddington-Finkelstein coordinates, Kruskal-Szekeres coordinates, Novikov coordinates, or Gullstrand–Painlevé coordinates showing that this is a coordinate singularity. The Schwarzschild metric is expected to be valid outside the gravitating body, there is no problem for $R > r_s$. At $r = 0$ the curvature becomes infinite indicating the presence of a singularity. At this point the metric, and spacetime itself, is no longer well-defined. For a long time it was thought that such a solution was non-physical. However according to the general theory of relativity such solutions are now believed to exist and are termed black holes. When $r < r_s$ then the radial component and time component changes the direction that is radial component of the metric becomes time-like and the time-like component becomes space-like.

1.3.2 The Reissner-Nordström metric

The Reissner–Nordström (RN) metric is a static solution to the EFE in empty space, which corresponds to the gravitational field of a charged, non-rotating, spherically symmetric body of mass M. The only difference between Schwarzschild metric and the Reissner-Nordstrom metric is that the later includes charge. If we remove charge from this RN metric then it will reduce to the Schwarzschild metric disscussed above.

Mathematically this metric can be written as

$$c^2 d\tau^2 = \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right) c^2 dt^2 - \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)^{-1} dr^2 - r^2 d\Omega^2, \tag{1.13}$$

where τ is the proper time measured by a clock moving with the particle, c is the speed of light, t is the time coordinate measured by a stationary clock, r is the radial coordinate, Ω is the solid angle defined by

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2. \tag{1.14}$$

Here r_s is the Schwarzschild radius of the massive body which is related to the mass M as $r_s = 2GM/c^2$, where G is the gravitaional constant and r_Q is the length scale corresponding to the electric charge Q of the massive body

$$r_Q = \frac{Q^2 G}{4\pi\epsilon_0 c^4}, \tag{1.15}$$

where $1/4\pi\epsilon_0$ is Coulumb's force constant [5]. If we remove charge Q then we shall recover the Schwarzschild metric and if the term $r_s/r \rightarrow 0$ then we shall recover the Minkowski metric for special relativity. In practice, the ratio r_s/r is almost always extremely small. For example, the Schwarzschild radius r_s of the Earth is roughly 9 mm , whereas a satellite in a geosynchronous orbit has a radius r that is roughly four billion times larger, at 42,164 km. Even at the surface of the Earth, the corrections to Newtonian gravity are only one part in a billion. The ratio only becomes large close to black holes and other ultra-dense objects such as neutron stars. Horizons of the RN metric can be found by putting $g^{rr} = 0$

$$r_{\pm} = \frac{1}{2}\left(r_s \pm \sqrt{r_s^2 - 4r_Q^2}\right). \tag{1.16}$$

These concentric event horizons become degenerate at $r_s = 2r_Q$. Black holes with $r_s < 2r_Q$ are believed not to exist in nature because they would contain naked singularity, and their appearance will contradict Roger Penrose's cosmic censorship hypothesis which is generally believed to be true. The electromagnetic potential for this metric is

$$A^{\alpha} = \left(\frac{Q}{r}, 0, 0, 0\right). \tag{1.17}$$

1.3.3 The Kerr-Newman Metric

The Kerr–Newman metric is a solution of the EFE in general relativity which describs the spacetime geometry in the region surrounding a charged, rotating mass. It is assumed that the cosmological constant equals zero. It is the generallization of the Kerr metric. Mathematically in spherical polar coordinates (t, r, θ, ϕ) this metric can be written as

$$c^2 d\tau^2 = -\left(\frac{dr^2}{\Delta} + d\theta^2\right)\rho^2 + \left(cdt - \alpha \sin^2\theta d\phi\right)^2 \frac{\Delta}{\rho^2} - \left[\left(r^2 + \alpha^2\right) d\phi - \alpha cdt\right]^2 \frac{\sin^2\theta}{\rho^2}, \tag{1.18}$$

where

$$\begin{aligned} \alpha &= \frac{J}{Mc}, \\ \rho^2 &= r^2 + \alpha^2 \cos^2\theta, \\ \Delta &= r^2 - r_s r + \alpha^2 + r_Q^2. \end{aligned}$$

Here r_s is the Schwarzschild radius of the massive body, which is related to its mass M by

$$r_s = \frac{2GM}{c^2},$$

where G is the gravitational constant, c is the speed of light and r_Q is a length-scale corresponding to the electric charge Q of the mass

$$r_Q^2 = \frac{Q^2 G}{4\pi\epsilon_0 c^4},$$

where $1/4\pi\epsilon_0$ is Coulomb's force constant.

If the charge Q is zero then this metric reduces to the Kerr metric.

1.4 Accelerating and Rotating Charged Black Holes

In this section we will discuss a large family of exact solutions of Einstein-Maxwell field equations, that was presented by Plebański and Demiański in 1976 [6]. This family includes the famous Kerr-Newman rotating black hole and hence the Kerr, the Reissner- Nordström, and the Schwarzschild black holes. This family, includes in particular, solutions for accelerating and rotating black holes also. The general form of the metric thus contains seven free parameters which characterize the mass M, electric and magnetic charges e and g respectively, Kerr-like rotation a which is equal to angular momentum per unit mass that is $a = J/m$, the NUT (Newman-Unti-Tamburino) parameter l which is a twisting property of the surrounding spacetime [7], acceleration of the source α and the cosmological constant Λ.

Unfortunately, many particular and well-known spacetimes are not included explicitly in the original form of the line element. They can only be obtained from it by using certain transformations.

We shall start with the Plebański–Demiański metric [6] which is given by

$$ds^2 = \frac{1}{(1-\hat{p}\hat{r})^2}\left[\frac{\mathcal{Q}\left(d\hat{\tau} - \hat{p}^2 d\hat{\sigma}\right)^2}{\hat{r}^2+\hat{p}^2} - \frac{\mathcal{P}\left(d\hat{\tau} + \hat{r}^2 d\hat{\sigma}\right)^2}{\hat{r}^2+\hat{p}^2} - \frac{\hat{r}^2+\hat{p}^2}{\mathcal{P}}d\hat{p}^2 - \frac{\hat{r}^2+\hat{p}^2}{\mathcal{Q}}d\hat{r}^2\right], \quad (1.19)$$

where

$$\mathcal{P} = \hat{k} + 2\hat{n}\hat{p} - \hat{\epsilon}\hat{p}^2 + 2\hat{M}\hat{p}^3 - \left(\hat{k} + \hat{e}^2 + \hat{g}^2 + \frac{\Lambda}{3}\right)\hat{p}^4, \tag{1.20}$$

$$\mathcal{Q} = \left(\hat{k} + \hat{e}^2 + \hat{g}^2\right) - 2\hat{M}\hat{r} + \hat{\epsilon}\hat{r}^2 - 2\hat{n}\hat{r}^3 - \left(\hat{k} + \frac{\Lambda}{3}\right)\hat{r}^4. \tag{1.21}$$

Here $\hat{M}$, $\hat{n}$, $\hat{e}$, $\hat{g}$, $\hat{\epsilon}$, $\hat{k}$ and Λ are arbitrary real parameters. $\hat{M}$ and $\hat{n}$ are the mass and NUT parameters, the parameters $\hat{e}$ and $\hat{g}$ represents the electric and magnetic charges. Eq. (1.19) is the original form of the Plebański and Demiański metric. The modified form of the metric is obtained after rescaling [8, 10].

$$\hat{p} = \sqrt{\alpha\omega}p,. \qquad \hat{r} = \sqrt{\frac{\alpha}{\omega}}r, \qquad \hat{\sigma} = \sqrt{\frac{\omega}{\alpha^3}}\sigma, \qquad \hat{\tau} = \sqrt{\frac{\omega}{\alpha}}\tau, \tag{1.22}$$

with the relabelling of parameters

$$\hat{M} + i\hat{n} = \left(\frac{\alpha}{\omega}\right)^{3/2}(M + in), \qquad \hat{e} + i\hat{g} = \frac{\alpha}{\omega}(e + ig), \qquad \hat{\epsilon} = \frac{\alpha}{\omega}\epsilon, \qquad \hat{k} = \alpha^2 k. \tag{1.23}$$

With these changes the original metric takes the form

$$ds^2 = \frac{1}{(1-\alpha pr)^2}\left[\frac{Q\left(d\tau - \omega p^2 d\sigma\right)^2}{r^2 + \omega^2 p^2} - \frac{P\left(\omega d\tau + r^2 d\sigma\right)^2}{r^2 + \omega^2 p^2} - \frac{r^2 + \omega^2 p^2}{P}dp^2 - \frac{r^2 + \omega^2 p^2}{Q}dr^2\right], \tag{1.24}$$

where

$$P = P(p) = k + 2\omega^{-1}np - \epsilon p^2 + 2\alpha M p^3 - \left[\alpha^2\left(\omega^2 k + e^2 + g^2\right) + \omega^2\Lambda/3\right]p^4,$$

$$Q = Q(r) = \left(\omega^2 k + e^2 + g^2\right) - 2Mr + \epsilon r^2 - 2\alpha\omega^{-1}nr^3 - \left(\alpha^2 k + \Lambda/3\right)r^4, \tag{1.25}$$

where M, n, e, g, Λ, ϵ, k, α and ω are arbitrary real parameters.

It is not immediately obvious that the metric represented by Eq. (1.24) includes the Schwarzschild–de Sitter solution, the Reissner– Nordström solution, the Kerr metric, the NUT solution. It is necessary to introduce a specific shift in the coordinate p. In fact, this procedure is essential to obtain the correct metric for accelerating and rotating black holes. We therefore start with Eq. (1.24) with (1.25), and perform the coordinate transformation

$$p = \frac{l}{\omega} + \frac{a}{\omega}\cos\theta, \qquad \tau = t - \frac{(l+a)^2}{a}\phi, \qquad \sigma = -\frac{\omega}{a}\phi, \tag{1.26}$$

where a and l are new arbitrary parameters. By this procedure we obtain the following metric

$$\begin{aligned} ds^2 &= \frac{1}{\Omega^2}\left\{\frac{Q}{\rho^2}\left[dt - \left(a\left(1-\tilde{p}^2\right) + 2l\left(1-\tilde{p}\right)\right)d\phi\right]^2 - \frac{\rho^2}{Q}dr^2 - \frac{\tilde{P}}{\rho^2}\left[adt - \left(r^2 + (l+a)^2\right)d\phi\right]^2 \right. \\ & \left. -\frac{\rho^2}{\tilde{P}}\sin^2\theta d\theta^2\right\}, \end{aligned} \tag{1.27}$$

where

$$\begin{aligned} \Omega &= 1 - \frac{\alpha}{\omega}\left(l + a\cos\theta\right)r, \\ \rho^2 &= r^2 + \left(l + a\cos\theta\right)^2, \\ \tilde{P} &= \sin^2\theta\left(1 - a_3\cos\theta - a_4\cos^2\theta\right), \\ Q &= \left(\omega^2 k + e^2 + g^2\right) - 2Mr + \epsilon r^2 - 2\alpha\frac{n}{\omega}r^3 - \left(\alpha^2 k + \frac{\Lambda}{3}\right)r^4. \end{aligned}$$

Here we have put

$$\begin{aligned} a_3 &= 2\alpha\frac{a}{\omega}M - 4\left[\frac{\alpha^2}{\omega^2}\left(\omega^2 k + e^2 + g^2\right) + \frac{\Lambda}{3}\right]al, \\ a_4 &= -\left[\frac{\alpha^2}{\omega^2}\left(\omega^2 k + e^2 + g^2\right) + \frac{\Lambda}{3}\right]a^2. \end{aligned}$$

Here M, n, e, g, Λ, ϵ, k, α and ω are arbitrary real parameters, n is the Plebański-Demiański parameter and ω is the twist [9, 10]. Depending on the specific choice of these parameters and thus the number and types of the roots of the polynomials P and Q, and on the admitted range of the coordinates the above class of solutions contains a very large number of various spacetimes with different geometrical and physical properties. Those solutions are physically most interesting which describe black holes. In such a case it is convenient to express the parameters ϵ, n and k in terms of new parameters a, l as [9]

$$\begin{aligned}
\epsilon &= \frac{\omega^2 k}{a^2 - l^2} + 4\alpha \frac{l}{\omega} M - \left(a^2 + 3l^2\right) \left[\frac{\alpha^2}{\omega^2}\left(\omega^2 k + e^2 + g^2\right) + \frac{\Lambda}{3}\right], \qquad (1.28)\\
n &= \frac{\omega^2 k l}{a^2 - l^2} - \alpha \frac{\left(a^2 - l^2\right)}{\omega} M + \left(a^2 - l^2\right) l \left[\frac{\alpha^2}{\omega^2}\left(\omega^2 k + e^2 + g^2\right) + \frac{\Lambda}{3}\right]. \qquad (1.29)
\end{aligned}$$

$$k = \left(1 + 2\alpha \frac{l}{\omega} M - 3\alpha^2 \frac{l^2}{\omega^2}\left(e^2 + g^2\right) - l^2 \Lambda\right)\left(\frac{\omega^2}{a^2 - l^2} + 3\alpha^2 l^2\right)^{-1}. \qquad (1.30)$$

It is also assumed [9] that $|a_3|$ and $|a_4|$ are sufficiently small so that P has no roots within the considered range $\theta = [0, \pi]$. When $\alpha = 0$ the general metric reduces to Kerr-Newman-NUT-de Sitter solution. Further if $l = 0$ then it reduces to familiar forms of Kerr-Newman-de Sitter black hole spacetimes. If $\alpha = 0$ and the Kerr-like rotation vanishes that is $a = 0$ then general metric reduces to the charged NUT-de Sitter spacetime. When $\alpha = l = g = \Lambda = 0$ then the Kerr-Newman metric is deduced. Further Schwarzschild metric is directly obtained if electric charge and rotation parameter vanish that is $e = 0 = a$.

Therefore, the line element (1.27) is a very convenient metric representation of the complete class of accelerating, rotating and charged black holes of the Plebański-Demiański class. The metric is singularity free if $|a| < |l|$ and it has a Kerr-like ring singularity at $\rho = 0$ when $|a| \geq |l|$.

When $\alpha = 0$, Eq. (1.30) becomes $\omega^2 k = (1 - l^2\Lambda)(a^2 - l^2)$ and hence Eqs. (1.28) and (1.29) become

$$\begin{aligned}
\epsilon &= 1 - \left(\frac{1}{3}a^2 + 2l^2\right)\Lambda,\\
n &= l + \frac{1}{3}\left(a^2 - 4l^2\right) l\Lambda.
\end{aligned}$$

Then the metric is given by

$$\begin{aligned}
ds^2 &= \frac{1}{\Omega^2}\left\{\frac{Q}{\rho^2}\left[dt - \left(a\sin^2\theta + 4l\sin^2\left(\frac{\theta}{2}\right)\right)d\phi\right]^2 - \frac{\rho^2}{Q}dr^2\right.\\
&\left. -\frac{\tilde{P}}{\rho^2}\left[adt - \left(r^2 + (l+a)^2\right)d\phi\right]^2 - \frac{\rho^2}{\tilde{P}}\sin^2\theta d\theta^2\right\}, \qquad (1.31)
\end{aligned}$$

where

$$\begin{aligned}
\Omega &= 1, \\
\rho^2 &= r^2 + (l + a\cos\theta)^2, \\
\tilde{P} &= \sin^2\theta\left(1 + \frac{4}{3}\Lambda al\cos\theta + \frac{1}{3}\Lambda a^2\cos^2\theta\right), \\
Q &= \left(a^2 - l^2 + e^2 + g^2\right) - 2Mr + r^2 - \Lambda\left[\left(a^2 - l^2\right)l^2 + \left(\frac{1}{3}a^2 + 2l^2\right)r^2 + \frac{1}{3}r^4\right].
\end{aligned}$$

This is exactly the Kerr-Newman-NUT-de Sitter solution. It represents a non-accelerating black hole with mass M, electric and magnetic charges e and g, a rotation parameter a and a NUT parameter l.

If we take arbitrary α but $l = 0$ then Eq. (1.30) implies that $\omega^2 k = a^2$. It is then convenient to put $\omega = a$ and hence

$$\epsilon = 1 - \alpha^2\left(a^2 + e^2 + g^2\right) - \frac{1}{3}\Lambda a^2, \qquad k = 1, \qquad n = -\alpha a M.$$

So the metric takes the following form

$$ds^2 = \frac{1}{\Omega^2}\left\{\frac{Q}{\rho^2}\left[dt - a\sin^2\theta d\phi\right]^2 - \frac{\rho^2}{Q}dr^2 - \frac{\tilde{P}}{\rho^2}\left[adt - \left(r^2 + a^2\right)d\phi\right]^2 - \frac{\rho^2}{\tilde{P}}\sin^2\theta d\theta^2\right\}, \quad (1.32)$$

where

$$\begin{aligned}
\Omega &= 1 - \alpha r\cos\theta, \\
\rho^2 &= r^2 + a^2\cos^2\theta, \\
\tilde{P} &= \sin^2\theta\left(1 - 2\alpha M\cos\theta + \left[\alpha^2\left(a^2 + e^2 + g^2\right) + \frac{1}{3}\Lambda a^2\right]\cos^2\theta\right), \\
Q &= \left(\left(a^2 + e^2 + g^2\right) - 2Mr + r^2\right)\left(1 - \alpha^2 r^2\right) - \frac{1}{3}\Lambda\left(a^2 + r^2\right)r^2.
\end{aligned}$$

The metric represented by Eq. (1.32) nicely represents the singularity and horizon structure of an accelerating and rotating charged black hole in a de Sitter or anti-de Sitter background. When $\Lambda = 0$ the above metric corresponds precisely to that of Hong and Teo [11] which represents an accelerating and rotating black hole without any NUT-like behaviour and in which the acceleration is characterized by α. In this case, if $M^2 > a^2 + e^2 + g^2$, the expression for Q factorises as

$$Q = (r - r_-)(r - r_+)\left(1 - \alpha^2 r^2\right),$$

where

$$r_\pm = M \pm \sqrt{M^2 - a^2 - e^2 - g^2}.$$

This expression for $r_\pm$ are identicle to those for the locations of the outer and inner horizons of the non-accelerating Kerr-Newman black hole. However in this case, there is another horizon at $r = \alpha^{-1}$ called an acceleration horizon.

1.5 Electromagnetic Potential

The electromagnetic four-potential is a covariant four vector which consists of a scalar potential and the vector potential. By definition

$$A^\mu = \left(\frac{\Phi}{c}, \mathbf{A}\right), \tag{1.33}$$

where Φ is the electric potential, c is the speed of light and $\mathbf{A}$ is the magnetic potential also called vector potential and (A_x, A_y, A_z) are its components. The electric and magnetic field associated with this potential are given by

$$E = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{1.34}$$

$$B = \nabla \times \mathbf{A}. \tag{1.35}$$

The covariant form of electromagnetic four potential A^α is found by multiplying it by the Minkowski metric η with sign $(+ - --)$

$$A_\nu = \eta_{\mu\nu} A^\mu = \left(\frac{\Phi}{c}, -\mathbf{A}\right), \tag{1.36}$$

where $\eta_{\mu\nu}$ is the Minkowski metric given by

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.37}$$

The Maxwel field tensor is defined by

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}, \tag{1.38}$$

which implies that

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}, \tag{1.39}$$

and

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}. \tag{1.40}$$

The inner product of Eqs. (1.39) and (1.40) gives the Lorentz invariant $F^{\mu\nu}F_{\mu\nu} = 2\left(B^2 - E^2/c^2\right)$.

1.6 Dirac Equation

The Dirac equation is a relativistic quantum mechanical wave equation formulated by the British physicist Paul Dirac in 1928. It provides a description of elementary spin-1/2 particles, such as electrons, consistent with both the principles of quantum mechanics and the theory of special relativity. The equation demands the existence of antiparticles and actually predated their experimental discovery. This made the discovery of the positron, the antiparticle of the electron.

The Dirac equation originally proposed by Dirac is

$$\left(\beta mc^2 + \sum \alpha_K P_K c\right) \Psi(x,t) = i\hbar \frac{\partial \Psi(x,t)}{\partial t}, \tag{1.41}$$

where m is the rest mass of electron, c is the speed of light, P is the momentum operator x and t are the space and time coordinates, $\hbar = h/2\pi$ is the reduced Planck constant, also known as Dirac's constant. β and α_K are the 4×4 matrices and $\Psi(x,t)$ is the wave function. The matrices are all Hermitian and have squares equal to the identity matrix that is

$$\alpha_K^2 = \beta^2 = I_4, \tag{1.42}$$

and they all mutually anticommute so $\{\alpha_i, \alpha_j\} = 0$ and $\{\alpha_i, \beta\} = 0$, where $i, j = 1, 2, 3$. These are aslo called Dirac matrices. The Pauli matrices share the same properties as the Dirac matrices they are all Hermitian, square to identity matrix I, and anticommute. This allows one to immediately find a representation of the Dirac matrices in terms of the Pauli matrices as

$$\alpha_K = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}. \tag{1.43}$$

1.6.1 Covariant Form

The explicitly covariant form of the Dirac equation is

$$-i\hbar\gamma^\mu \partial_\mu \Psi + mc\Psi = 0, \tag{1.44}$$

where γ^μ are Dirac matrices, γ^0 is Hermitian, and the γ^k are anti-Hermitian, with the definition $\gamma^0 = \beta$ and $\gamma^k = \gamma^0 \alpha^k$. Thus

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{1.45}$$

These matrices satisfy the relation for the Minkowski metric

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I. \tag{1.46}$$

1.6.2 Dirac Equation in Electromagnetic Field

To consider problems in which an applied electromagnetic field interacts with the particles described by the Dirac equation, one uses the correspondence principle, and takes over into the theory the corresponding expression from classical mechanics, whereby the total momentum of a charged particle in an external field is modified as so $P \to P - \frac{q}{c}A$, where q is the charge of the particle. Thus in natural units the Dirac equation takes the form

$$\left[-i\gamma^\mu \left(\partial_\mu + iqA_\mu\right) + m\right] \Psi = 0. \tag{1.47}$$

Another useful form of the Dirac equation is

$$\gamma^\mu \left(D_\mu - \frac{iq_h A_\mu}{\hbar}\right) \Psi - \frac{im}{\hbar} \Psi = 0,$$

where $\mu = 0, 1, 2, 3$, q_h and m are the equivalent charge and mass of fermion particles and

$$D_\mu = \partial_\mu + \Omega_\mu, \qquad \Omega_\mu = \frac{1}{2} i \Gamma_\mu^{\alpha\beta} \Sigma_{\alpha\beta}, \qquad \Sigma_{\alpha\beta} = \frac{1}{4} i \left[\gamma^\alpha, \gamma^\beta\right].$$

1.7 Klein-Gordon Equation

The Klein–Gordon equation is a relativistic version of the Schrödinger equation. It is the equation of motion of a quantum scalar field, a field whose quanta are spinless particles. It cannot be straightforwardly interpreted as a Schrödinger equation for a quantum state, because it is second order in time and because it does not admit a positive definite conserved probability density. Still, with the appropriate interpretation, it does describe the quantum amplitude for finding a point particle in various places, the relativistic wavefunction, but the particle propagates both forwards and backwards in time. Any solution to the Dirac equation is automatically a solution to the Klein–Gordon equation, but the converse is not true.

Mathematically the Klein-Gordon equation can be written as

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi - \nabla^2\Psi + \frac{m^2c^2}{\hbar^2}\Psi = 0. \tag{1.48}$$

In natural units this becomes

$$-\partial_t^2\Psi + \nabla^2\Psi = m^2\Psi. \tag{1.49}$$

For the time independent case, the Klein-Gordon equation becomes

$$\left[\nabla^2 - \frac{m^2c^2}{\hbar^2}\right]\Psi(r) = 0. \tag{1.50}$$

In order to include the electromagnetic field, using gauge invariant we shall replace the derivative operators with the gauge covariant derivative operators. Thus the Klein-Gordon equation becomes

$$-\left(\partial_t - ieA_0\right)^2\Psi + \left(\partial_i - ieA_i\right)^2\Psi = m^2\Psi, \tag{1.51}$$

where A is the vector potential. In general relativity, we include the effect of gravity and the Klein-Gordon equation takes the form

$$-g^{\mu\nu}\partial_\mu\partial_\nu\Psi + g^{\mu\nu}\Gamma^\sigma_{\mu\nu}\partial_\sigma\Psi + \frac{m^2c^2}{\hbar^2}\Psi = 0. \tag{1.52}$$

Here $g^{\mu\nu}$ is the inverse of the metric tensor $g_{\mu\nu}$.

1.8 Fermions and Bosons

All fundamental particles in nature can be divided into one of two categories, Fermions or Bosons. Fermions are particles with half-integer spin such as $1/2, 3/2, 5/2$ and so on. They obey the Pauli-exclusion principle, that is, only one fermion can exist in a given quantum state. Examples of fermions are electrons, protons, neutrons, quarks, etc.

Bosons are the particles that have integer spin such as $0, 1, 2$ and so on. They do not obey the Pauli-exclusion principle, that is, in a given state there may be more than one bosons. Photon is the example of boson particles.

Any object which is comprised of an even number of fermions is a boson, while any particle which is comprised of an odd number of fermions is a fermion. For example, a proton is made of three quarks, hence it is a fermion. A helium atom is made of two protons, two neutrons and two electrons, hence it is a boson. A particle with 0 spin is called a sclar particle.

Chapter 2

Quantum Tunneling from Black Holes

2.1 Quantum Tunneling

Classically a black hole is the ultimate prison. Any thing that enters it can never come back. Thus a black hole can grow bigger with time. But quantum mechanically Stephen Hawking demonstrated that a black hole could radiate particles in the form of Hawking radiation. Thus it will lose energy and mass and will shrink and finally evapurate completely [13].

How does this happen? When an object which is classically stable becomes quantum mechanically unstable it is natural to suspect tunneling. Indeed, when Hawking first proved the existence of black hole radiation [14], he described it as tunneling triggered by vacuum fluctuations near the horizon. The idea is that when a virtual particle is created just inside the horizon the positive energy virtual particle can tunnel out. Alternatively, when a virtual particle is created just outside the horizon the negative energy virtual particle can tunnel inwards. In either case the negative energy particle is absorbed by the black hole which results in the decrease in the mass of a black hole and positive energy particle moves away from a black hole to infinity appearing as Hawking radiation.

This heuristic picture has obvious visual and intuitive appeal. But, oddly, actual derivations of Hawking radiation did not proceed in this way at all [14, 15]. There were two apparent hurdles, the first one was technical that there should be a coordinate system that should be well behaved at the horizon, none of the well known coordinate system was suitable, and the second was conceptual that there did not seems to be any barrier. Typically, whenever a tunneling event takes place there are two seperated classical turning points which are joined by a trajectory in complex time. In the WKB (Wentzel, Kramers, and Brillouin) or geometrical optics limit, the probability of tunneling is related to the imaginary part of the action for a classically forbidden trajectory via

$$\Gamma \sim \exp\left(-2\,\mathrm{Im}\, I\right),$$

where I is the action for the trajectory. Now the problem with the black hole is that if a particle is created even infinitesimally outside the horizon it can escape to infinity classically. Thus the turning point seems to have zero seperation and so it is not clear what joining trajectory is to be considered and where is the barrier?

Actually particles do tunnel out of a black hole, much as Hawking had first imagined. But they do this in a rather subtle way. There is no existing barrier which sets the scale of tunneling. Barrier is created by the outgoing particle itself. The crucial point is that energy must be conserved [16]. When a particle tunnels out the black hole will lose energy and since energy and radius of a black hole are related to each other so with the decrease of energy of the black hole its radius will also decrease as a function of energy of the outgoing particles. And black hole will shrink, and it is this contraction that sets the scale, the horizon receds from its original position to a new smaller radius. Moreover, the amount of contraction depends on the energy of the outgoing particles thus it is the tunneling particle itself that secretly defines the barrier [13].

In the WKB limit, the probability of tunneling would take the form [13]

$$\Gamma \sim \exp\left(-2\,\mathrm{Im}\, I\right) \approx \exp\left(-\beta E\right),$$

where $\exp\left(-\beta E\right)$ is the Boltzmann factor appropriate for an object with inverse temperature β. From this relation Hawking temperature can be obtained by comparing the coefficients of E of $-\beta E$ with $-2\,\mathrm{Im}\, I$.

There are two methods to find the action of the emitted particles. First one is the null geodesic method used by Parikh and Wilczek [17], which followed from the work of Kraus and Wilczek [18, 19]. And the other method is the Hamilton-Jacobi ansatz used by Angheben et al, which is an extension of the complex path analysis of Padmanabhan et al [20, 21].

The null geodesic method considers the null S-wave emitted from the black hole. Based on the previous work analyzing the full action in detail [18, 19], the only part of the action that contributes to an imaginary term is

$$\operatorname{Im} I = \operatorname{Im} \int_{r_{in}}^{r_{out}} P_r dr,$$

where P_r is the momentum of the emitted null S-wave. Then by using the Hamilton's equation and the knowledge of the null geodesics it is possible to calculate the imaginary part of the action.

The Hamilton-Jacobi ansatz involves consideration of an emitted particle, ignoring its self-gravitation, and assumes that its action satisfies the relativistic Hamilton-Jacobi equation. From the symmetries of the metric, one picks an appropriate ansatz for the form of the action. This method is motivated by applying the WKB (WKB method is a method for finding approximate solutions to linear partial differential equations with spatially varying coefficients) approximation to the Klein-Gordon equation. This method can be extended to other types of particles (i.e. other than scalar particles) by applying the WKB approximation to other wave equations such as the Dirac equation to model spin-1/2 fermions.

The tunnelling method can even be applied to horizons that are not black hole horizons, such as Rindler Spacetimes [20]. The tunneling method can be extended beyond the emission of scalar particles that have spin. Kerner and Mann was first to model Dirac particles [22].

2.2 Tunneling from the Reissner-Nordström Black Hole with Magnetic Charges

In this section we shall find the tunneling probability of charged and magnetized fermions from the Reissner-Nordstrom black hole. The metric of Reissner-Nordstrom black hole with magnetic charges is [23]

$$\begin{aligned} ds^2 &= -\left(1-\frac{2M}{r}+\frac{Q_e^2+Q_m^2}{r^2}\right)dt^2+\left(1-\frac{2M}{r}+\frac{Q_e^2+Q_m^2}{r^2}\right)^{-1}dr^2 \\ &\quad +r^2\left(d\theta^2+\sin^2\theta d\phi^2\right), \end{aligned} \tag{2.1}$$

where Q_e and Q_m are the electric and magnetic charges and M is the mass of the black hole. The above metric can also be written as

$$ds^2 = -f(r)\,dt^2 + f^{-1}(r)\,dr^2 + r^2\left(d\theta^2+\sin^2\theta d\phi^2\right), \tag{2.2}$$

where

$$f(r) = \left(1-\frac{2M}{r}+\frac{Q_h^2}{r^2}\right), \tag{2.3}$$

where we have used $Q_h^2 = Q_e^2 + Q_m^2$. The horizons of the metric can be calcutated by putting

$$\left(1-\frac{2M}{r}+\frac{Q_h^2}{r^2}\right) = 0,$$

or

$$r^2 - 2Mr + Q_h^2 = 0,$$

which gives

$$r_{\pm} = M \pm \sqrt{M^2 - Q_h^2}, \tag{2.4}$$

where $r_{\pm}$ represent the outer and inner horizons respectively. The electromagnetic vector potential for this metric is

$$A_\mu = \left(-\frac{Q_h}{r}, 0, 0, 0\right). \tag{2.5}$$

The Dirac equation

$$\gamma^\mu\left(D_\mu - \frac{iq_hA_\mu}{\hbar}\right)\Psi - \frac{im}{\hbar}\Psi = 0, \tag{2.6}$$

will be solved to calculate the tunneling probability, where the Greek indices $\mu, \nu = 0, 1, 2, 3$, and q_h and m are the equivalent charge and mass of the fermions and

$$D_\mu = \partial_\mu + \Omega_\mu, \qquad \Omega_\mu = \frac{1}{2}i\Gamma_\mu^{\alpha\beta}\Sigma_{\alpha\beta}, \qquad \Sigma_{\alpha\beta} = \frac{1}{4}i\left[\gamma^\alpha, \gamma^\beta\right]. \tag{2.7}$$

Here $[\gamma^\alpha, \gamma^\beta]$ is antisymmetric that is $[\gamma^\alpha, \gamma^\beta] = 0$, if $\alpha = \beta$ and $[\gamma^\alpha, \gamma^\beta] = -[\gamma^\beta, \gamma^\alpha]$, if $\alpha \neq \beta$.The Gamma matrices can be choosen as

$$\begin{aligned}\gamma^t &= \frac{1}{\sqrt{f}}\gamma^0, \qquad \gamma^r = \sqrt{f}\gamma^3, \\ \gamma^\theta &= \frac{1}{r}\gamma^1, \qquad \gamma^\phi = \frac{1}{r\sin\theta}\gamma^2,\end{aligned} \tag{2.8}$$

while

$$\gamma^0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}. \tag{2.9}$$

Here σ^i $(i = 1, 2, 3)$ are the pauli sigma matrices. To solve Eq. (2.6) we employ the following ansatz

$$\Psi_\uparrow(t, r, \theta, \phi) = \begin{pmatrix} A(t, r, \theta, \phi)\xi_\uparrow \\ B(t, r, \theta, \phi)\xi_\uparrow \end{pmatrix} \exp\left(\frac{i}{\hbar} I_\uparrow(t, r, \theta, \phi)\right), \tag{2.10}$$

$$\Psi_\downarrow(t, r, \theta, \phi) = \begin{pmatrix} C(t, r, \theta, \phi)\xi_\downarrow \\ D(t, r, \theta, \phi)\xi_\downarrow \end{pmatrix} \exp\left(\frac{i}{\hbar} I_\downarrow(t, r, \theta, \phi)\right), \tag{2.11}$$

where $\xi_{\uparrow/\downarrow}$ are the eigen vectors of σ^3, $I_{\uparrow/\downarrow}$ is the action of the radiant spin particles, $A(t, r, \theta, \phi)$ and $B(t, r, \theta, \phi)$ correspond to the ingoing and outgoing modes with spin up while $C(t, r, \theta, \phi)$ and $D(t, r, \theta, \phi)$ correspond to the outgoing and ingoing modes with spin down. We are only interested in the spin up case because the spin down case is fully similar to the spin up case other than some changes in the sign. Using the antisymmetric prorperty of $[\gamma^\alpha, \gamma^\beta]$ in Eq. (2.7) we get $D_\mu = \partial_\mu$. Thus reduced form of Dirac equation becomes

$$\gamma^\mu\left(\partial_\mu - \frac{iq_h A_\mu}{\hbar}\right)\Psi - \frac{im}{\hbar}\Psi = 0, \tag{2.12}$$

which in expanded form becomes

$$\gamma^t\partial_t\Psi + \gamma^r\partial_r\Psi + \gamma^\theta\partial_\theta\Psi + \gamma^\phi\partial_\phi\Psi - \gamma^t\frac{iq_h A_t}{\hbar}\Psi - \frac{im}{\hbar}\Psi = 0 \tag{2.13}$$

Now we solve the first term of Eq. (2.13)

$$\gamma^t \partial_t \Psi = \frac{1}{\sqrt{f(r)}} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \partial_t \begin{pmatrix} A(t,r,\theta,\phi)\,\xi_\uparrow \\ B(t,r,\theta,\phi)\,\xi_\uparrow \end{pmatrix} \exp\left(\frac{i}{\hbar} I_\uparrow (t,r,\theta,\phi)\right).$$

Evaluating the derivative with respect to t we get

$$\gamma^t \partial_t \Psi = \frac{1}{\sqrt{f(r)}} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \left(\partial_t A + \frac{iA}{\hbar}\partial_t I_\uparrow\right)\xi_\uparrow \\ \left(\partial_t B + \frac{iB}{\hbar}\partial_t I_\uparrow\right)\xi_\uparrow \end{pmatrix} \exp\left(\frac{i}{\hbar} I_\uparrow\right),$$

or

$$\gamma^t \partial_t \Psi = \begin{pmatrix} \frac{i}{\sqrt{f(r)}}\left(\partial_t A + \frac{iA}{\hbar}\partial_t I_\uparrow\right) \\ 0 \\ -\frac{i}{\sqrt{f(r)}}\left(\partial_t B + \frac{iB}{\hbar}\partial_t I_\uparrow\right) \\ 0 \end{pmatrix} \exp\left(\frac{i}{\hbar} I_\uparrow\right). \tag{2.14}$$

Similarly remaining terms of Eq. (2.13) becomes

$$\gamma^r \partial_r \Psi = \begin{pmatrix} \sqrt{f(r)}\left(\partial_r B + \frac{iB}{\hbar}\partial_r I_\uparrow\right) \\ 0 \\ \sqrt{f(r)}\left(\partial_r A + \frac{iA}{\hbar}\partial_r I_\uparrow\right) \\ 0 \end{pmatrix} \exp\left(\frac{i}{\hbar} I_\uparrow\right), \tag{2.15}$$

$$\gamma^\theta \partial_\theta \Psi = \begin{pmatrix} 0 \\ \frac{1}{r}\left(\partial_\theta B + \frac{iB}{\hbar}\partial_\theta I_\uparrow\right) \\ 0 \\ \frac{1}{r}\left(\partial_\theta A + \frac{iA}{\hbar}\partial_\theta I_\uparrow\right) \end{pmatrix} \exp\left(\frac{i}{\hbar} I_\uparrow\right), \tag{2.16}$$

$$\gamma^\phi \partial_\phi \Psi = \begin{pmatrix} 0 \\ \frac{i}{r\sin\theta}\left(\partial_\phi B + \frac{iB}{\hbar}\partial_\phi I_\uparrow\right) \\ 0 \\ \frac{i}{r\sin\theta}\left(\partial_\phi A + \frac{iA}{\hbar}\partial_\phi I_\uparrow\right) \end{pmatrix} \exp\left(\frac{i}{\hbar} I_\uparrow\right), \tag{2.17}$$

$$\gamma^t \frac{iq_h A_t}{\hbar}\Psi = \begin{pmatrix} -\frac{q_h A A_t}{f(r)\hbar} \\ 0 \\ \frac{q_h B A_t}{\hbar} \\ 0 \end{pmatrix} \exp\left(\frac{i}{\hbar} I_\uparrow\right), \tag{2.18}$$

$$\frac{im}{\hbar}\Psi = \begin{pmatrix} \frac{imA}{\hbar} \\ 0 \\ \frac{imB}{\hbar} \\ 0 \end{pmatrix} \exp\left(\frac{i}{\hbar} I_\uparrow\right). \tag{2.19}$$

Using Eqs. (2.14) to (2.19) in (2.13) and simplifying we get a matrix of order 4×1. Thus we get a set of four equations. After dividing these four equations by the exponential term and multiplying by $\hbar$ and taking leading order terms to $\hbar$ we obtain

$$\frac{iA\left(\partial_t I_\uparrow - q_h A_t\right)}{\sqrt{f\left(r\right)}} + B\sqrt{f\left(r\right)}\partial_r I_\uparrow - Am = 0, \tag{2.20}$$

$$-\frac{B}{r}\left(\partial_\theta I_\uparrow + \frac{i}{\sin\theta}\partial_\phi I_\uparrow\right) = 0, \tag{2.21}$$

$$-\frac{iB\left(\partial_t I_\uparrow - q_h A_t\right)}{\sqrt{f\left(r\right)}} + A\sqrt{f\left(r\right)}\partial_r I_\uparrow - Bm = 0, \tag{2.22}$$

$$-\frac{A}{r}\left(\partial_\theta I_\uparrow + \frac{i}{\sin\theta}\partial_\phi I_\uparrow\right) = 0. \tag{2.23}$$

Solving these equations is not an easy task but taking into account the existence of time-like Killing vector $\left(\frac{\partial}{\partial_t}\right)^\alpha$, we carry out the following seperation of variables

$$I_\uparrow = -\omega t + W\left(r\right) + J\left(\theta, \phi\right). \tag{2.24}$$

Using Eq. (2.24) into Eqs. (2.20) to (2.23) the resulting four equations take the form

$$
\begin{aligned}
-\frac{iA\left(\omega+q_hA_t\right)}{\sqrt{f\left(r\right)}}+B\sqrt{f\left(r\right)}W'(r)-Am &= 0, &(2.25)\\
-\frac{B}{r}\left(J_\theta+\frac{i}{\sin\theta}J_\phi\right) &= 0, &(2.26)\\
\frac{iB\left(\omega+q_hA_t\right)}{\sqrt{f\left(r\right)}}+A\sqrt{f\left(r\right)}W'(r)-Bm &= 0, &(2.27)\\
-\frac{A}{r}\left(J_\theta+\frac{i}{\sin\theta}J_\phi\right) &= 0. &(2.28)
\end{aligned}
$$

Eqs. (2.26) and (2.28) give the same solution regardless of the values of A or B showing that $J\left(\theta,\phi\right)$ for the ingoing and outgoing particles is same. On the other hand from Eqs. (2.25) and (2.27) , for $m=0$, we get two possible solutions

$$
\begin{aligned}
A &= -iB, \qquad W'(r)=W_+'\left(r\right)=\frac{\omega+q_hA_t}{f\left(r\right)}, &(2.29)\\
A &= iB, \qquad W'(r)=W_-'\left(r\right)=-\frac{\omega+q_hA_t}{f\left(r\right)}, &(2.30)
\end{aligned}
$$

where $W_\pm\left(r\right)$ respectively represent the outgoing and ingoing solutions. And when $m\neq0$ then one gets

$$\left(\frac{A}{B}\right)^2=\frac{-i\left(\omega+q_hA_t\right)+f\left(r\right)m}{i\left(\omega+q_hA_t\right)+f\left(r\right)m}. \qquad (2.31)$$

At the horizon $r=r_+$ we get from Eq. (2.31) $\left(\frac{A}{B}\right)^2=-1$ which gives $A^2=-B^2$, and we get the same result as the massless case. Therefore for the massive and massless case we have

$$W_+=\int\frac{\omega+q_hA_t}{f\left(r\right)}dr.$$

Here $r=r_+$ is the singularity so using residue theory this integral gives

$$W_+=\frac{i\pi\left(\omega-\omega_0\right)\left(M^2+M\sqrt{M^2-Q_h^2}-\frac{1}{2}Q_h^2\right)}{\sqrt{M^2-Q_h^2}}, \qquad (2.32)$$

where $\omega_0 = q_h V_0 = q_h Q_h / r_+$. The total tunneling probability can be written as [29, 30]

$$\Gamma \sim \frac{P(out)}{P(in)} = \frac{\exp\left[-2\left(\operatorname{Im} W_+ + \operatorname{Im} J\right)\right]}{\exp\left[-2\left(\operatorname{Im} W_- + \operatorname{Im} J\right)\right]} = \exp\left[-4 \operatorname{Im} W_+\right]. \tag{2.33}$$

With the use of Eq. (2.32) the above equation takes the form

$$\Gamma = \exp\left(\frac{-4\pi\left(\omega-\omega_0\right)\left(M^2 + M\sqrt{M^2 - Q_h^2} - \frac{1}{2}Q_h^2\right)}{\sqrt{M^2 - Q_h^2}}\right), \tag{2.34}$$

comparing this with $\Gamma = \exp\left(-\beta E\right)$ where $\beta = 1/T$, this gives the expected Hawking temperature as

$$T = \frac{\sqrt{M^2 - Q_h^2}}{4\pi\left(M^2 + M\sqrt{M^2 - Q_h^2} - \frac{1}{2}Q_h^2\right)}.$$

2.3 Tunneling From Kerr-Newman Black Holes

In this section we shall study the charged fermion tunneling from the Kerr-Newman black holes. We shall consider charged particles emission from Kerr-Newman specetime. The Kerr-Newman metric is given by [22]

$$ds^2 = -f\left(r,\theta\right)dt^2 + \frac{dr^2}{g\left(r,\theta\right)} - 2H\left(r,\theta\right)dtd\phi + K\left(r,\theta\right)d\phi^2 + \Sigma\left(r,\theta\right)d\theta^2, \tag{2.35}$$

with electromagnetic vector potential

$$A_a = -\frac{er}{\Sigma\left(r,\theta\right)}\left[\left(dt\right)_a - a\sin^2\theta\left(d\phi\right)_a\right], \tag{2.36}$$

and where $f(r,\theta)$, $g(r,\theta)$, $H(r,\theta)$, $K(r,\theta)$ and $\Sigma(r,\theta)$ are given below

$$
\begin{aligned}
f(r,\theta) &= \frac{\Delta(r) - a^2 \sin^2\theta}{\Sigma(r,\theta)}, &(2.37)\\
g(r,\theta) &= \frac{\Delta(r)}{\Sigma(r,\theta)}, &(2.38)\\
H(r,\theta) &= \frac{a\sin^2\theta\left(r^2+a^2-\Delta(r)\right)}{\Sigma(r,\theta)}, &(2.39)\\
K(r,\theta) &= \frac{\left(r^2+a^2\right)^2 - \Delta(r)\, a^2 \sin^2\theta}{\Sigma(r,\theta)}\sin^2\theta, &(2.40)
\end{aligned}
$$

and

$$
\begin{aligned}
\Sigma(r,\theta) &= r^2 + a^2\cos^2\theta, &(2.41)\\
\Delta(r) &= r^2 + a^2 + e^2 - 2Mr. &(2.42)
\end{aligned}
$$

We shall consider a non-extremal black hole so that $M^2 > a^2 + e^2$. To calculate the horizons we put $1/g_{11} = 0$ which gives

$$g(r,\theta) = 0.$$

Using Eq. (2.38) and (2.42) it becomes

$$r^2 + a^2 + e^2 - 2Mr = 0,$$

which is a quadratic equation in r, solving this using quadratic farmula we get

$$r_{\pm} = M \pm \sqrt{M^2 - a^2 - e^2}. \tag{2.43}$$

We introduce a new function which will be needed later as $F(r,\theta) = -\left(g^{tt}\right)^{-1}$ which gives

$$F(r,\theta) = f(r,\theta) + \frac{H^2(r,\theta)}{K(r,\theta)} = \frac{\Delta(r)\,\Sigma(r,\theta)}{\left(r^2+a^2\right)^2 - \Delta(r)\, a^2\sin^2\theta}. \tag{2.44}$$

The angular velocity at the black hole horizon becomes [24]

$$\Omega_H = \frac{H(r_+,\theta)}{K(r_+,\theta)} = \frac{a}{r_+^2 + a^2}. \tag{2.45}$$

We shall only show the calculations for the spin up case because the final result is same in both the cases other than some change in the sign. Dirac equation will be solved for this purpose which is given below

$$i\gamma^{\mu}\left(D_{\mu}-\frac{iq}{\hbar}A_{\mu}\right)\Psi+\frac{m}{\hbar}\Psi=0, \tag{2.46}$$

where

$$\begin{aligned} D_{\mu} &= \partial_{\mu}+\Omega_{\mu}, && (2.47)\\ \Omega_{\mu} &= \frac{1}{2}i\Gamma_{\mu}^{\alpha\beta}\Sigma_{\alpha\beta}, && (2.48)\\ \Sigma_{\alpha\beta} &= \frac{1}{4}i\left[\gamma^{\alpha},\gamma^{\beta}\right]. && (2.49)\end{aligned}$$

Here $\left[\gamma^{\alpha},\gamma^{\beta}\right]$ is antisymmetric that is $\left[\gamma^{\alpha},\gamma^{\beta}\right]=0$ if $\alpha=\beta$ and $\left[\gamma^{\alpha},\gamma^{\beta}\right]=-\left[\gamma^{\beta},\gamma^{\alpha}\right]$ if $\alpha\neq\beta$. We shall choose a representation for the γ^{μ} matrices in the form

$$\begin{aligned}\gamma^{t} &= \frac{1}{\sqrt{F(r,\theta)}}\gamma^{0},\gamma^{r}=\sqrt{g(r,\theta)}\gamma^{3},\gamma^{\theta}=\frac{1}{\sqrt{\Sigma(r,\theta)}}\gamma^{1},\\ \gamma^{\phi} &= \frac{1}{\sqrt{K(r,\theta)}}\left(\gamma^{2}+\frac{H(r,\theta)}{\sqrt{F(r,\theta)K(r,\theta)}}\gamma^{0}\right),\end{aligned} \tag{2.50}$$

where γ^{a}'s are the following matices for Minkowski space

$$\begin{aligned}\gamma^{0} &= \begin{pmatrix}0 & I\\ -I & 0\end{pmatrix},\gamma^{1}=\begin{pmatrix}0 & \sigma^{1}\\ \sigma^{1} & 0\end{pmatrix},\\ \gamma^{2} &= \begin{pmatrix}0 & \sigma^{2}\\ \sigma^{2} & 0\end{pmatrix},\gamma^{3}=\begin{pmatrix}0 & \sigma^{3}\\ \sigma^{3} & 0\end{pmatrix}.\end{aligned} \tag{2.51}$$

The Pauli matrices are

$$\sigma^{1}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\sigma^{2}=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\sigma^{3}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{2.52}$$

The resulting γ^5 matix is

$$\gamma^5 = i\gamma^t\gamma^r\gamma^\theta\gamma^\phi = \sqrt{\frac{g}{FK\Sigma}} \begin{pmatrix} -I + \frac{H}{\sqrt{FK}}\sigma^2 & 0 \\ 0 & I + \frac{H}{\sqrt{FK}}\sigma^2 \end{pmatrix}. \quad (2.53)$$

For the spin up case ansatz for the Dirac field is

$$\begin{aligned} \Psi_\uparrow(t,r,\theta,\phi) &= \begin{pmatrix} A(t,r,\theta,\phi)\,\xi_\uparrow \\ B(t,r,\theta,\phi)\,\xi_\uparrow \end{pmatrix} \exp\left(\frac{i}{\hbar} I_\uparrow(t,r,\theta,\phi)\right), \\ \Psi_\uparrow(t,r,\theta,\phi) &= \begin{pmatrix} A(t,r,\theta,\phi) \\ 0 \\ B(t,r,\theta,\phi) \\ 0 \end{pmatrix} \exp\left(\frac{i}{\hbar} I_\uparrow(t,r,\theta,\phi)\right). \end{aligned} \quad (2.54)$$

Using the antisymmetric property of $[\gamma^\alpha, \gamma^\beta]$ value of Ω_μ becomes zero. Thus Dirac equation reduces to

$$i\gamma^\mu \left(\partial_\mu - \frac{iq}{\hbar} A_\mu\right)\Psi + \frac{m}{\hbar}\Psi = 0, \quad (2.55)$$

which in expanded form becomes

$$i\gamma^t\partial_t\Psi + i\gamma^r\partial_r\Psi + i\gamma^\theta\partial_\theta\Psi + i\gamma^\phi\partial_\phi\Psi + \frac{q}{\hbar}\gamma^t A_t\Psi + \frac{q}{\hbar}\gamma^\phi A_\phi\Psi + \frac{m}{\hbar}\Psi = 0. \quad (2.56)$$

To solve Eq. (2.56) consider the first term of Eq. (2.56)

$$i\gamma^t\partial_t\Psi = i\frac{1}{\sqrt{F(r,\theta)}}\gamma^0\partial_t \begin{pmatrix} A(t,r,\theta,\phi)\,\xi_\uparrow \\ B(t,r,\theta,\phi)\,\xi_\uparrow \end{pmatrix} \exp\left(\frac{i}{\hbar} I_\uparrow(t,r,\theta,\phi)\right).$$

Evaluating the derivative with respect to t we get

$$i\gamma^t\partial_t\Psi = \frac{i}{\sqrt{F(r,\theta)}} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \left(\partial_t A + \frac{iA}{\hbar}\partial_t I_\uparrow\right)\xi_\uparrow \\ \left(\partial_t B + \frac{iB}{\hbar}\partial_t I_\uparrow\right)\xi_\uparrow \end{pmatrix} \exp\left(\frac{i}{\hbar} I_\uparrow\right),$$

or

$$i\gamma^t\partial_t\Psi = \frac{i}{\sqrt{F(r,\theta)}}\begin{pmatrix} \left(\partial_t B + \frac{iB}{\hbar}\partial_t I_\uparrow\right) I\xi_\uparrow \\ -\left(\partial_t A + \frac{iA}{\hbar}\partial_t I_\uparrow\right) I\xi_\uparrow \end{pmatrix}\exp\left(\frac{i}{\hbar}I_\uparrow\right),$$

or

$$i\gamma^t\partial_t\Psi = \begin{pmatrix} \frac{i}{\sqrt{F(r,\theta)}}\left(\partial_t B + \frac{iB}{\hbar}\partial_t I_\uparrow\right) \\ 0 \\ -\frac{i}{\sqrt{F(r,\theta)}}\left(\partial_t A + \frac{iA}{\hbar}\partial_t I_\uparrow\right) \\ 0 \end{pmatrix}\exp\left(\frac{i}{\hbar}I_\uparrow\right). \quad (2.57)$$

Similarly remaining terms of Dirac equation become

$$i\gamma^r\partial_r\Psi = \begin{pmatrix} i\sqrt{g(r,\theta)}\left(\partial_r B + \frac{iB}{\hbar}\partial_r I_\uparrow\right) \\ 0 \\ i\sqrt{g(r,\theta)}\left(\partial_r A + \frac{iA}{\hbar}\partial_r I_\uparrow\right) \\ 0 \end{pmatrix}\exp\left(\frac{i}{\hbar}I_\uparrow\right), \quad (2.58)$$

$$i\gamma^\theta\partial_\theta\Psi = \begin{pmatrix} 0 \\ \frac{i}{\sqrt{\Sigma(r,\theta)}}\left(\partial_\theta B + \frac{iB}{\hbar}\partial_\theta I_\uparrow\right) \\ 0 \\ \frac{i}{\sqrt{\Sigma(r,\theta)}}\left(\partial_\theta A + \frac{iA}{\hbar}\partial_\theta I_\uparrow\right) \end{pmatrix}\exp\left(\frac{i}{\hbar}I_\uparrow\right), \quad (2.59)$$

$$i\gamma^\phi\partial_\phi\Psi = \begin{pmatrix} \frac{iH(r,\theta)}{K(r,\theta)\sqrt{F(r,\theta)}}\left(\partial_\phi B + \frac{iB}{\hbar}\partial_\phi I_\uparrow\right) \\ -\frac{1}{\sqrt{K(r,\theta)}}\left(\partial_\phi B + \frac{iB}{\hbar}\partial_\phi I_\uparrow\right) \\ -\frac{iH(r,\theta)}{K(r,\theta)\sqrt{F(r,\theta)}}\left(\partial_\phi A + \frac{iA}{\hbar}\partial_\phi I_\uparrow\right) \\ -\frac{1}{\sqrt{K(r,\theta)}}\left(\partial_\phi A + \frac{iA}{\hbar}\partial_\phi I_\uparrow\right) \end{pmatrix}\exp\left(\frac{i}{\hbar}I_\uparrow\right), \quad (2.60)$$

$$\frac{q}{\hbar}\gamma^t A_t\Psi = \begin{pmatrix} \frac{qBA_t}{\hbar\sqrt{F(r,\theta)}} \\ 0 \\ -\frac{qAA_t}{\hbar\sqrt{F(r,\theta)}} \\ 0 \end{pmatrix}\exp\left(\frac{i}{\hbar}I_\uparrow\right), \quad (2.61)$$

$$\frac{q}{\hbar}\gamma^{\phi}A_{\phi}\Psi=\begin{pmatrix}\frac{qBA_{\phi}H(r,\theta)}{\hbar K(r,\theta)\sqrt{F(r,\theta)}}\\ \frac{iqBA_{\phi}}{\hbar\sqrt{K(r,\theta)}}\\ -\frac{qAA_{\phi}H(r,\theta)}{\hbar K(r,\theta)\sqrt{F(r,\theta)}}\\ \frac{iqAA_{\phi}}{\hbar\sqrt{K(r,\theta)}}\end{pmatrix}\exp\left(\frac{i}{\hbar}I_{\uparrow}\right), \tag{2.62}$$

$$\frac{m}{\hbar}\Psi=\begin{pmatrix}\frac{mA}{\hbar}\\ 0\\ \frac{mB}{\hbar}\\ 0\end{pmatrix}\exp\left(\frac{i}{\hbar}I_{\uparrow}\right). \tag{2.63}$$

Using Eqs. (2.57) to (2.63) in (2.56) and simplifying we get a matrix of order 4×1. Thus we get a set of four equations. After dividing these four equations by the exponential term and multiplying by $\hbar$ and taking leading order terms to $\hbar$ these four equations take the following form

$$\begin{aligned}0 &= -B\left[\frac{1}{\sqrt{F(r,\theta)}}\partial_t I_{\uparrow}+\sqrt{g(r,\theta)}\partial_r I_{\uparrow}+\frac{H(r,\theta)}{K(r,\theta)\sqrt{F(r,\theta)}}\partial_{\phi}I_{\uparrow}\right.\\ &\quad\left.+\frac{qer}{\Sigma(r,\theta)\sqrt{F(r,\theta)}}\left(1-\frac{H(r,\theta)}{K(r,\theta)}a\sin^2\theta\right)\right]+Am, \qquad (2.64)\\ 0 &= -B\left[\frac{i}{\sqrt{K(r,\theta)}}\left(\partial_{\phi}I_{\uparrow}-\frac{qer}{\Sigma(r,\theta)}a\sin^2\theta\right)+\frac{1}{\sqrt{\Sigma(r,\theta)}}\partial_{\theta}I_{\uparrow}\right], \qquad (2.65)\\ 0 &= A\left[\frac{1}{\sqrt{F(r,\theta)}}\partial_t I_{\uparrow}-\sqrt{g(r,\theta)}\partial_r I_{\uparrow}+\frac{H(r,\theta)}{K(r,\theta)\sqrt{F(r,\theta)}}\partial_{\phi}I_{\uparrow}\right.\\ &\quad\left.+\frac{qer}{\Sigma(r,\theta)\sqrt{F(r,\theta)}}\left(1-\frac{H(r,\theta)}{K(r,\theta)}a\sin^2\theta\right)\right]+Bm, \qquad (2.66)\\ 0 &= -A\left[\frac{i}{\sqrt{K(r,\theta)}}\left(\partial_{\phi}I_{\uparrow}-\frac{qer}{\Sigma(r,\theta)}a\sin^2\theta\right)+\frac{1}{\sqrt{\Sigma(r,\theta)}}\partial_{\theta}I_{\uparrow}\right]. \qquad (2.67)\end{aligned}$$

When $m\neq 0$ Eqs. (2.64) and (2.66) couple whereas when $m=0$ they decouple. we shall apply the standard ansatz

$$I_{\uparrow}=-Et+J\phi+W(r,\theta), \tag{2.68}$$

and insert it into Eqs. (2.64) to (2.67). We expand the equations near the horizon and get

$$0 = -B\left(\frac{-E+\Omega_H J+\frac{qer_+}{r_+^2+a^2}}{\sqrt{F_r(r_+,\theta)(r-r_+)}}+\sqrt{g_r(r_+,\theta)(r-r_+)}W_r(r,\theta)\right)+Am, \tag{2.69}$$

$$0 = -B\left(\frac{i}{\sqrt{K(r_+,\theta)}}\left(J-\frac{qer_+}{\Sigma(r_+,\theta)}a\sin^2\theta\right)+\frac{1}{\sqrt{\Sigma(r_+,\theta)}}W_\theta(r,\theta)\right), \tag{2.70}$$

$$0 = A\left(\frac{-E+\Omega_H J+\frac{qer_+}{r_+^2+a^2}}{\sqrt{F_r(r_+,\theta)(r-r_+)}}-\sqrt{g_r(r_+,\theta)(r-r_+)}W_r(r,\theta)\right)+Bm, \tag{2.71}$$

$$0 = -A\left(\frac{i}{\sqrt{K(r_+,\theta)}}\left(J-\frac{qer_+}{\Sigma(r_+,\theta)}a\sin^2\theta\right)+\frac{1}{\sqrt{\Sigma(r_+,\theta)}}W_\theta(r,\theta)\right), \tag{2.72}$$

where

$$g_r(r_+,\theta) = \frac{\Delta_r(r_+)}{\Sigma(r_+,\theta)}=\frac{2(r_+-M)}{r_+^2+a^2\cos^2\theta},$$
$$F_r(r_+,\theta) = \frac{\Delta_r(r_+)\Sigma(r_+,\theta)}{\left(r_+^2+a^2\right)^2}=\frac{2(r_+-M)\left(r_+^2+a^2\cos^2\theta\right)}{\left(r_+^2+a^2\right)^2}.$$

In the massless case it is possible to pull $1/\sqrt{\Sigma(r_+,\theta)}$ out of Eqs. (2.69) and (2.71) , making these equations independent of θ. Also Eqs. (2.70) and (2.72) have no explicit r dependence. From this it is possible to conclude that near the black hole horizon it is possible to further seperate the fuction W

$$W(r,\theta)=R(r)+\Theta(\theta).$$

Then Eqs. (2.70) and (2.72) give the same equation for Θ regardless of the values of A or B. Eqs. (2.69) and (2.71) have two possible solutions

$$B = 0 \quad \text{and} \quad R'(r)=R'_+(r)=\frac{\left(E-\Omega_H J-\frac{qer_+}{r_+^2+a^2}\right)\left(r_+^2+a^2\right)}{\Delta_r(r_+)(r-r_+)},$$
$$A = 0 \quad \text{and} \quad R'(r)=R'_-(r)=\frac{-\left(E-\Omega_H J-\frac{qer_+}{r_+^2+a^2}\right)\left(r_+^2+a^2\right)}{\Delta_r(r_+)(r-r_+)},$$

where prime denotes a derivative with respect to r and $R_{+/-}$ correspond to outgoing/ingoing solutions.

The probabilities of crossing the horizon in each direction are proportional to

$$prob\,[out] \;\propto\; \exp\left[-2\,\mathrm{Im}\, I\right] = \exp\left[-2\left(\mathrm{Im}\, R_{+} + \mathrm{Im}\,\Theta\right)\right], \tag{2.73}$$

$$prob\,[in] \;\propto\; \exp\left[-2\,\mathrm{Im}\, I\right] = \exp\left[-2\left(\mathrm{Im}\, R_{-} + \mathrm{Im}\,\Theta\right)\right]. \tag{2.74}$$

For probabilities to be correctly normalized that is a particle coming from outside to inside the horizon have a 100% chance to enter the black hole we shall divide each equation by Eq. (2.74) so that probability of entering the black hole will be equal to 1 and this implies that the probability of a particle tunneling from inside to outside the horizon is

$$\Gamma \propto \frac{prob\,[out]}{prob\,[in]} = \exp\left[-4\,\mathrm{Im}\, R_{+}\right]. \tag{2.75}$$

Solving for R_{+} yields

$$R_{+}\left(r\right) = \int \frac{\left(E - \Omega_{H} J - \frac{qer_{+}}{r_{+}^{2}+a^{2}}\right)\left(r_{+}^{2} + a^{2}\right)}{\Delta_{r}\left(r_{+}\right)\left(r - r_{+}\right)} dr.$$

Here $r = r_{+}$ is a simple pole so using residue theory we get

$$R_{+}\left(r\right) = \frac{\pi i\left(E - \Omega_{H} J - \frac{qer_{+}}{r_{+}^{2}+a^{2}}\right)\left(r_{+}^{2} + a^{2}\right)}{2\left(r_{+} - M\right)},$$

or

$$\mathrm{Im}\, R_{+}\left(r\right) = \frac{\pi\left(E - \Omega_{H} J - \frac{qer_{+}}{r_{+}^{2}+a^{2}}\right)\left(r_{+}^{2} + a^{2}\right)}{2\left(r_{+} - M\right)}. \tag{2.76}$$

Thus resulting tunneling probility is

$$\Gamma = \exp\left[-2\pi \frac{r_{+}^{2} + a^{2}}{\left(r_{+} - M\right)}\left(E - \Omega_{H} J - \frac{qer_{+}}{r_{+}^{2} + a^{2}}\right)\right]. \tag{2.77}$$

This gives the expected Hawking temperature for a charged rotating black hole [25, 26].

$$T_{H} = \frac{r_{+} - M}{2\pi\left(r_{+}^{2} + a^{2}\right)} = \frac{\sqrt{M^{2} - a^{2} - e^{2}}}{2\pi\left[2M\left(M + \sqrt{M^{2} - a^{2} - e^{2}}\right) - e^{2}\right]}. \tag{2.78}$$

For the massive case Eqs. (2.69) and (2.71) couple. We shall eleminate the function $W(r,\theta)$ from these two equations. For this purpose multiplying Eq. (2.71) by B and Eq. (2.69) by A and subtracting

$$0 = \frac{2AB\left(E - \Omega_H J - \frac{qer_+}{r_+^2+a^2}\right)}{\sqrt{F_r(r_+,\theta)(r-r_+)}} + mA^2 - mB^2,$$

or

$$\begin{aligned} 0 &= m\sqrt{F_r(r_+,\theta)(r-r_+)}\left(\frac{A}{B}\right)^2 + 2\left(E - \Omega_H J - \frac{qer_+}{r_+^2+a^2}\right)\left(\frac{A}{B}\right) \\ &\quad -m\sqrt{F_r(r_+,\theta)(r-r_+)}, \end{aligned}$$

which is quadratic equation in A/B. Solving for A/B yields

$$\frac{A}{B} = \frac{-\left(E - \Omega_H J - \frac{qer_+}{r_+^2+a^2}\right) \pm \sqrt{\left(E - \Omega_H J - \frac{qer_+}{r_+^2+a^2}\right)^2 + m^2 F_r(r_+,\theta)(r-r_+)}}{m\sqrt{F_r(r_+,\theta)(r-r_+)}}. \tag{2.79}$$

Taking limite $r \to r_+$ we get

$$\lim_{r\to r_+}\frac{A}{B} = \begin{cases} 0 & \text{for upper sign} \\ -\infty & \text{for lower sign} \end{cases} \tag{2.80}$$

Consequently at the horizon either $\frac{A}{B} \to 0$ or $\frac{A}{B} \to -\infty$, that is either $A \to 0$ or $B \to 0$. For $A \to 0$ at the horizon. We shall solve Eq. (2.71) in terms of m and insert into Eq. (2.69), obtaining

$$W_r(r,\theta) = \frac{\left(E - \Omega_H J - \frac{qer_+}{r_+^2+a^2}\right)}{\sqrt{F_r(r_+,\theta)\, g_r(r_+,\theta)}\,(r-r_+)}\frac{\left(1+\frac{A^2}{B^2}\right)}{\left(1-\frac{A^2}{B^2}\right)}. \tag{2.81}$$

Since $\frac{A}{B}$ is zero at the horizon so integrating the above expression around the simple pole $r = r_+$ we get the same result as was in the massless case, that is

$$W(r,\theta) \equiv R_+(r) = \frac{\pi i\left(E - \Omega_H J - \frac{qer_+}{r_+^2+a^2}\right)\left(r_+^2+a^2\right)}{2\left(r_+ - M\right)}. \tag{2.82}$$

When $B \to 0$ at the horizon we can simply rewrite the Eq. (2.81) in terms of $\frac{B}{A}$ as

$$W_r(r,\theta) = \frac{-\left(E - \Omega_H J - \frac{qer_+}{r_+^2 + a^2}\right)}{\sqrt{F_r(r_+,\theta)\, g_r(r_+,\theta)}\,(r - r_+)} \frac{\left(1 + \frac{B^2}{A^2}\right)}{\left(1 - \frac{B^2}{A^2}\right)}. \tag{2.83}$$

Integrating around the pole we get

$$W(r,\theta) \equiv R_-(r) = \frac{-\pi i \left(E - \Omega_H J - \frac{qer_+}{r_+^2 + a^2}\right)\left(r_+^2 + a^2\right)}{2\,(r_+ - M)}. \tag{2.84}$$

As the result of integrating $W(r,\theta)$ in the massive case is same as in the massless case so we recover the tunneling probability (2.77) and Hawking temperature (2.78) for the Kerr-Newman black hole.

For the spin down case calculations proceed in the same way as in the spin up case. Equations for spin down case are similar to the equations for spin up case and we get the same temperature and tunneling probability showing that both spin up and spin down particles are emitted at the same rate.

Chapter 3

Tunneling from Rotating and Accelerating Black Hole

3.1 Dirac Equation in the Background of Rotating and Accelerating Black Holes

In this chapter we shall find the quantum tunneling for spin$-1/2$ fermion particles using the Hamilton Jacobi equation as an ansatz. Dirac equation will be used for this purpose.

The Plebanski-Demianski metric [6] covers a large family of solutions which also include accelerating and rotating charged black holes with cosmological constant $\Lambda = 0$. In spherical polar coordinates (t, r, θ, ϕ) this metric can be written as [27].

$$ds^2 = -\frac{1}{\Omega^2}\left[\frac{Q}{\rho^2}\left[dt - a\sin^2\theta d\phi\right]^2 - \frac{\rho^2}{Q}dr^2 - \frac{\rho^2}{P}d\theta^2 - \frac{P}{\rho^2}\sin^2\theta\left[adt - \left(r^2+a^2\right)d\phi\right]^2\right], \quad (3.1)$$

which in expanded form becomes

$$\begin{aligned} ds^2 &= -\frac{1}{\Omega^2}\left[\frac{Q}{\rho^2}dt^2 + \frac{Qa^2\sin^4\theta}{\rho^2}d\phi^2 - \frac{Q}{\rho^2}2a\sin^2\theta dtd\phi - \frac{\rho^2}{Q}dr^2 - \frac{\rho^2}{P}d\theta^2\right. \\ &\left. -\frac{P\sin^2\theta}{\rho^2}a^2dt^2 - \frac{P\sin^2\theta}{\rho^2}\left(r^2+a^2\right)^2d\phi^2 + \frac{P\sin^2\theta}{\rho^2}2a\left(r^2+a^2\right)dtd\phi\right], \quad (3.2)\end{aligned}$$

where

$$\Omega = 1 - \alpha r \cos\theta, \tag{3.3}$$

$$\rho^2 = r^2 + a^2 \cos^2\theta, \tag{3.4}$$

$$P = 1 - 2\alpha M \cos\theta + \left[\alpha^2 \left(a^2 + e^2 + g^2\right)\right] \cos^2\theta, \tag{3.5}$$

$$Q = \left[\left(a^2 + e^2 + g^2\right) - 2Mr + r^2\right] \left(1 - \alpha^2 r^2\right). \tag{3.6}$$

Here M, e, g, a, and α are the arbitrary parameters. M is the mass of the black hole, e and g are its electric and magnetic charges, a is rotation, and α is the acceleration of the black hole. In Eq. (3.2) rearranging the terms, we get

$$\begin{aligned} ds^2 = & -\left(\frac{Q - a^2 P \sin^2\theta}{\rho^2\Omega^2}\right) dt^2 + \left(\frac{\rho^2}{Q\Omega^2}\right) dr^2 + \left(\frac{\rho^2}{P\Omega^2}\right) d\theta^2 \\ & + \left(\frac{\sin^2\theta \left[P\left(r^2 + a^2\right)^2 - a^2 Q \sin^2\theta\right]}{\rho^2\Omega^2}\right) d\phi^2 \\ & - \left(\frac{2a\sin^2\theta \left[P\left(r^2 + a^2\right) - Q\right]}{\rho^2\Omega^2}\right) dt d\phi. \end{aligned} \tag{3.7}$$

We can also write the above metric in the form [22]

$$ds^2 = -f(r,\theta)\, dt^2 + \frac{dr^2}{g(r,\theta)} + \Sigma(r,\theta)\, d\theta^2 + K(r,\theta)\, d\phi^2 - 2H(r,\theta)\, dt d\phi, \tag{3.8}$$

where $f(r,\theta)$, $g(r,\theta)$, $\Sigma(r,\theta)$, $K(r,\theta)$, and $H(r,\theta)$ are defined below

$$f(r,\theta) = \left(\frac{Q - a^2 P \sin^2\theta}{\rho^2\Omega^2}\right), \tag{3.9}$$

$$g(r,\theta) = \frac{Q\Omega^2}{\rho^2}, \tag{3.10}$$

$$\Sigma(r,\theta) = \left(\frac{\rho^2}{P\Omega^2}\right), \tag{3.11}$$

$$K(r,\theta) = \left(\frac{\sin^2\theta\left[P\left(r^2+a^2\right)^2 - a^2 Q\sin^2\theta\right]}{\rho^2\Omega^2}\right), \tag{3.12}$$

$$H(r,\theta) = \left(\frac{a\sin^2\theta\left[P\left(r^2+a^2\right) - Q\right]}{\rho^2\Omega^2}\right). \tag{3.13}$$

The electromagnetic vector potential of this metric is given by [27]

$$A = \frac{-er\left[dt - a\sin^2\theta d\phi\right] - g\cos\theta\left[adt - \left(r^2+a^2\right)d\phi\right]}{r^2 + a^2\cos^2\theta}. \tag{3.14}$$

Opening the brackets and rearranging the terms we get

$$A = \left(\frac{-er - ag\cos\theta}{r^2 + a^2\cos^2\theta}\right)dt + \left(\frac{aer\sin^2\theta + g\left(r^2+a^2\right)\cos\theta}{r^2 + a^2\cos^2\theta}\right)d\phi. \tag{3.15}$$

The event horizons can be calculated by putting

$$\left(g_{rr}\right)^{-1} = 0,$$

which implies that

$$\left(\frac{1}{g(r,\theta)}\right)^{-1} = 0.$$

Using the value of $g(r,\theta)$ from Eq. (3.10) this becomes

$$\left(\frac{Q\Omega^2}{\rho^2}\right) = 0,$$

or

$$Q\Omega^2 = 0.$$

Using the values of Ω and Q from Eqs. (3.3) and (3.6) we get

$$\left[\left(a^2+e^2+g^2\right)-2Mr+r^2\right]\left(1-\alpha^2r^2\right)\left(1-\alpha r\cos\theta\right)^2=0.$$

Solving these equations, we obtain

$$r=\frac{1}{\alpha\cos\theta},\; r=\pm\frac{1}{\alpha},\; \text{and } r_\pm=M\pm\sqrt{M^2-a^2-e^2-g^2}, \tag{3.16}$$

where $r_\pm$ represent the outer and inner horizons corresponding to the Kerr-Newman black holes. Here the other horizons are acceleration horizons. The angular velocity at the black hole horizon is defined as [22]

$$\Omega_H=\frac{H\left(r_+,\theta\right)}{K\left(r_+,\theta\right)}, \tag{3.17}$$

which after using the values of $H\left(r_+,\theta\right)$ and $K\left(r_+,\theta\right)$ from Eqs. (3.12) and (3.13) becomes

$$\Omega_H=\frac{a}{\left(r_+^2+a^2\right)}, \tag{3.18}$$

where we have used $Q\left(r_+\right)=0$. Now we define the function

$$F\left(r,\theta\right)=-\left(g^{tt}\right)^{-1},$$

which implies

$$F\left(r,\theta\right)=f\left(r,\theta\right)+\frac{H^2\left(r,\theta\right)}{K\left(r,\theta\right)}.$$

Using the values of $f\left(r,\theta\right)$, $K\left(r,\theta\right)$ and $H\left(r,\theta\right)$ from Eqs. (3.9), (3.12) and (3.13) and after simplification we get

$$F\left(r,\theta\right)=\frac{PQ\rho^2}{\Omega^2\left[P\left(r^2+a^2\right)^2-a^2Q\sin^2\theta\right]}. \tag{3.19}$$

We will only show the calculations for the spin up case. The calculations for the spin down case are similar, apart from some changes in the sign. The Dirac equation with electric charge is [28]

$$i\gamma^\mu\left(D_\mu-\frac{iq}{\hbar}A_\mu\right)\Psi+\frac{m}{\hbar}\Psi=0, \tag{3.20}$$

where the Greek indices $\mu = (0, 1, 2, 3)$ and m is the mass of the fermion particles and q is its charge and

$$D_\mu = \partial_\mu + \Omega_\mu, \tag{3.21}$$

where ∂_μ is the partial derivative and Ω_μ is defined below

$$\Omega_\mu = \frac{1}{2} i \Gamma_\mu^{\alpha\beta} \Sigma_{\alpha\beta}, \tag{3.22}$$

with

$$\Sigma_{\alpha\beta} = \frac{1}{4} i \left[\gamma^\alpha, \gamma^\beta\right]. \tag{3.23}$$

Here $\left[\gamma^\alpha, \gamma^\beta\right]$ is antisymmetric i.e.

$$\left[\gamma^\alpha, \gamma^\beta\right] = \begin{cases} -\left[\gamma^\beta, \gamma^\alpha\right] & \text{when} \quad \alpha \neq \beta, \\ 0 & \text{when} \quad \alpha = \beta \;. \end{cases} \tag{3.24}$$

The quantities γ's are defined as

$$\begin{aligned} \gamma^t &= \frac{1}{\sqrt{F(r,\theta)}} \gamma^0, \qquad \gamma^r = \sqrt{g(r,\theta)} \gamma^3, \qquad \gamma^\theta = \frac{1}{\sqrt{\Sigma(r,\theta)}} \gamma^1, \\ \gamma^\phi &= \frac{1}{\sqrt{K(r,\theta)}} \left(\gamma^2 + \frac{H(r,\theta)}{\sqrt{F(r,\theta) K(r,\theta)}} \gamma^0 \right), \end{aligned} \tag{3.25}$$

where the matrices γ^μ are

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \\ \gamma^2 &= \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}. \end{aligned} \tag{3.26}$$

The Pauli matrices σ^i are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.27}$$

The eigenvector of σ^3 for the spin up and spin down cases are

$$\xi_{\uparrow} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_{\downarrow} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.28}$$

Note that

$$\gamma^5 = i\gamma^t\gamma^r\gamma^\theta\gamma^\phi = \sqrt{\frac{g}{FK\Sigma}} \begin{pmatrix} -I + \frac{H}{\sqrt{FK}}\sigma^2 & 0 \\ 0 & I + \frac{H}{\sqrt{FK}}\sigma^2 \end{pmatrix}, \tag{3.29}$$

is the resulting γ^5 matrix. The solution of the spin up and spin down particles respectively can be assumed to be [28]

$$\Psi_{\uparrow}(t, r, \theta, \phi) = \begin{pmatrix} A(t, r, \theta, \phi)\,\xi_{\uparrow} \\ B(t, r, \theta, \phi)\,\xi_{\uparrow} \end{pmatrix} \exp\left[\frac{i}{\hbar} I_{\uparrow}(t, r, \theta, \phi)\right], \tag{3.30}$$

and

$$\Psi_{\downarrow}(t, r, \theta, \phi) = \begin{pmatrix} A(t, r, \theta, \phi)\,\xi_{\downarrow} \\ B(t, r, \theta, \phi)\,\xi_{\downarrow} \end{pmatrix} \exp\left[\frac{i}{\hbar} I_{\downarrow}(t, r, \theta, \phi)\right], \tag{3.31}$$

where $I_{\uparrow/\downarrow}$ denote the action of the emitted spin up and spin down particles, respectively. We shall only show the spin up case since the spin down case is similar except for some changes in the sign. Now using Eqs. (3.22) and (3.23) in (3.21) we get

$$D_\mu = \partial_\mu + \frac{1}{8} i^2 \Gamma_\mu^{\alpha\beta} \left[\gamma^\alpha, \gamma^\beta\right].$$

Giving variation to α and β this becomes

$$\begin{aligned} D_\mu \;=\; & \partial_\mu + \frac{1}{8} i^2 \left[\Gamma_\mu^{00}\left[\gamma^0, \gamma^0\right] + \Gamma_\mu^{01}\left[\gamma^0, \gamma^1\right] + \Gamma_\mu^{02}\left[\gamma^0, \gamma^2\right] + \Gamma_\mu^{03}\left[\gamma^0, \gamma^3\right]\right. \\ & + \Gamma_\mu^{10}\left[\gamma^1, \gamma^0\right] + \Gamma_\mu^{11}\left[\gamma^1, \gamma^1\right] + \Gamma_\mu^{12}\left[\gamma^1, \gamma^2\right] + \Gamma_\mu^{13}\left[\gamma^1, \gamma^3\right] \\ & + \Gamma_\mu^{20}\left[\gamma^2, \gamma^0\right] + \Gamma_\mu^{21}\left[\gamma^2, \gamma^1\right] + \Gamma_\mu^{22}\left[\gamma^2, \gamma^2\right] + \Gamma_\mu^{23}\left[\gamma^2, \gamma^3\right] \\ & \left. + \Gamma_\mu^{30}\left[\gamma^3, \gamma^0\right] + \Gamma_\mu^{31}\left[\gamma^3, \gamma^1\right] + \Gamma_\mu^{32}\left[\gamma^3, \gamma^2\right] + \Gamma_\mu^{33}\left[\gamma^3, \gamma^3\right]\right]. \end{aligned}$$

Using the antisymmetric property i.e. $[\gamma^\alpha, \gamma^\beta] = 0$, if $\alpha = \beta$ and $[\gamma^\alpha, \gamma^\beta] = -[\gamma^\beta, \gamma^\alpha]$, if $\alpha \neq \beta$ this becomes

$$\begin{aligned} D_\mu &= \partial_\mu - \frac{1}{8}\left[g^{11}\Gamma^0_{\mu 1}\left[\gamma^0, \gamma^1\right] + g^{22}\Gamma^0_{\mu 2}\left[\gamma^0, \gamma^2\right] + g^{33}\Gamma^0_{\mu 3}\left[\gamma^0, \gamma^3\right]\right. \\ &\quad -g^{11}\Gamma^0_{\mu 1}\left[\gamma^0, \gamma^1\right] + g^{22}\Gamma^1_{\mu 2}\left[\gamma^1, \gamma^2\right] + g^{33}\Gamma^1_{\mu 3}\left[\gamma^1, \gamma^3\right] \\ &\quad -g^{22}\Gamma^0_{\mu 2}\left[\gamma^0, \gamma^2\right] - g^{22}\Gamma^1_{\mu 2}\left[\gamma^1, \gamma^2\right] + g^{33}\Gamma^2_{\mu 3}\left[\gamma^2, \gamma^3\right] \\ &\quad \left.-g^{33}\Gamma^0_{\mu 3}\left[\gamma^0, \gamma^3\right] - g^{33}\Gamma^1_{\mu 3}\left[\gamma^1, \gamma^3\right] - g^{33}\Gamma^2_{\mu 3}\left[\gamma^2, \gamma^3\right]\right], \end{aligned}$$

which after cancelling the identical terms reduces to

$$D_\mu = \partial_\mu.$$

Thus the Dirac Eq. (3.20) becomes

$$i\gamma^\mu\left(\partial_\mu - \frac{iq}{\hbar}A_\mu\right)\Psi + \frac{m}{\hbar}\Psi = 0. \tag{3.32}$$

Giving variation on μ we get

$$i\gamma^t\partial_t\Psi + i\gamma^r\partial_r\Psi + i\gamma^\theta\partial_\theta\Psi + i\gamma^\phi\partial_\phi\Psi + \gamma^t\frac{q}{\hbar}A_t\Psi + \gamma^\phi\frac{q}{\hbar}A_\phi\Psi + \frac{m}{\hbar}\Psi = 0. \tag{3.33}$$

Now consider the first term of Eq. (3.33)

$$i\gamma^t\partial_t\Psi = i\frac{1}{\sqrt{F(r,\theta)}}\gamma^0\partial_t\begin{pmatrix} A(t,r,\theta,\phi)\,\xi_\uparrow \\ B(t,r,\theta,\phi)\,\xi_\uparrow \end{pmatrix}\exp\left[\frac{i}{\hbar}I_\uparrow(t,r,\theta,\phi)\right].$$

Putting the value of γ^0 from Eq. (3.26) this becomes

$$i\gamma^t\partial_t\Psi = \frac{i}{\sqrt{F(r,\theta)}}\partial_t\begin{pmatrix} B\exp\left[\frac{i}{\hbar}I_\uparrow\right]I\xi_\uparrow \\ -A\exp\left[\frac{i}{\hbar}I_\uparrow\right]I\xi_\uparrow \end{pmatrix}.$$

Taking derivative of the matrix with respect to t we get

$$i\gamma^t \partial_t \Psi = \frac{i}{\sqrt{F(r,\theta)}} \begin{pmatrix} \left[\partial_t B + \frac{i}{\hbar} B \partial_t I_\uparrow\right] I\xi_\uparrow \\ \left[-\partial_t A - \frac{i}{\hbar} A \partial_t I_\uparrow\right] I\xi_\uparrow \end{pmatrix} \exp\left[\frac{i}{\hbar} I_\uparrow\right].$$

Now

$$I\xi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

therefore the above equation becomes

$$i\gamma^t \partial_t \Psi = \begin{pmatrix} \frac{i}{\sqrt{F(r,\theta)}} \left[\partial_t B + \frac{i}{\hbar} B \partial_t I_\uparrow\right] \\ 0 \\ \frac{i}{\sqrt{F(r,\theta)}} \left[-\partial_t A - \frac{i}{\hbar} A \partial_t I_\uparrow\right] \\ 0 \end{pmatrix} \exp\left[\frac{i}{\hbar} I_\uparrow\right]. \quad (3.34)$$

Now consider the second term of Eq. (3.33)

$$i\gamma^r \partial_r \Psi = i\sqrt{g(r,\theta)}\gamma^3 \partial_r \begin{pmatrix} A\xi_\uparrow \\ B\xi_\uparrow \end{pmatrix} \exp\left[\frac{i}{\hbar} I_\uparrow\right].$$

Putting the value of γ^3 from Eq. (3.26) and taking derivative with respect to r this becomes

$$i\gamma^r \partial_r \Psi = i\sqrt{g(r,\theta)} \begin{pmatrix} \left[\partial_r B + \frac{i}{\hbar} B.\left(\partial_r I_\uparrow\right)\right] \sigma^3 \xi_\uparrow \\ \left[\partial_r A + \frac{i}{\hbar} A.\left(\partial_r I_\uparrow\right)\right] \sigma^3 \xi_\uparrow \end{pmatrix} \exp\left[\frac{i}{\hbar} I_\uparrow\right].$$

Now

$$\sigma^3 \xi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

thus the above equation becomes

$$i\gamma^r \partial_r \Psi = \begin{pmatrix} i\sqrt{g(r,\theta)}\left[\partial_r B + \frac{i}{\hbar} B.\partial_r I_\uparrow\right] \\ 0 \\ i\sqrt{g(r,\theta)}\left[\partial_r A + \frac{i}{\hbar} A.\partial_r I_\uparrow\right] \\ 0 \end{pmatrix} \exp\left[\frac{i}{\hbar} I_\uparrow\right]. \tag{3.35}$$

The third term of Eq. (3.33) is

$$i\gamma^\theta \partial_\theta \Psi = i\frac{1}{\sqrt{\Sigma(r,\theta)}}\gamma^1 \partial_\theta \begin{pmatrix} A(t,r,\theta,\phi)\,\xi_\uparrow \\ B(t,r,\theta,\phi)\,\xi_\uparrow \end{pmatrix} \exp\left[\frac{i}{\hbar} I_\uparrow(t,r,\theta,\phi)\right].$$

Putting the value of γ^1 from Eq. (3.26) this takes the form

$$i\gamma^\theta \partial_\theta \Psi = \frac{i}{\sqrt{\Sigma(r,\theta)}} \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \partial_\theta \begin{pmatrix} A\exp\left[\frac{i}{\hbar} I_\uparrow\right]\xi_\uparrow \\ B\exp\left[\frac{i}{\hbar} I_\uparrow\right]\xi_\uparrow \end{pmatrix},$$

Taking derivative of the matrix with respect to θ we get

$$i\gamma^\theta \partial_\theta \Psi = \frac{i}{\sqrt{\Sigma(r,\theta)}} \begin{pmatrix} \left[\partial_\theta B + \frac{i}{\hbar} B.\partial_\theta I_\uparrow\right]\sigma^1\xi_\uparrow \\ \left[\partial_\theta A + \frac{i}{\hbar} A.\partial_\theta I_\uparrow\right]\sigma^1\xi_\uparrow \end{pmatrix} \exp\left[\frac{i}{\hbar} I_\uparrow\right].$$

Since

$$\sigma^1\xi_\uparrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

thus the above equation takes the form

$$i\gamma^\theta \partial_\theta \Psi = \begin{pmatrix} 0 \\ \frac{i}{\sqrt{\Sigma(r,\theta)}}\left[\partial_\theta B + \frac{i}{\hbar} B.\partial_\theta I_\uparrow\right] \\ 0 \\ \frac{i}{\sqrt{\Sigma(r,\theta)}}\left[\partial_\theta A + \frac{i}{\hbar} A.\partial_\theta I_\uparrow\right] \end{pmatrix} \exp\left[\frac{i}{\hbar} I_\uparrow\right]. \tag{3.36}$$

Similarly the fourth term of Eq. (3.33) becomes

$$\begin{aligned} i\gamma^{\phi}\partial_{\phi}\Psi &= \frac{i}{\sqrt{K(r,\theta)}}\gamma^{2}\partial_{\phi}\begin{pmatrix} A\xi_{\uparrow} \\ B\xi_{\uparrow} \end{pmatrix}\exp\left[\frac{i}{\hbar}I_{\uparrow}\right] \\ &+\frac{i}{\sqrt{K(r,\theta)}}\frac{H(r,\theta)}{\sqrt{F(r,\theta)K(r,\theta)}}\gamma^{0}\partial_{\phi}\begin{pmatrix} A\xi_{\uparrow} \\ B\xi_{\uparrow} \end{pmatrix}\exp\left[\frac{i}{\hbar}I_{\uparrow}\right]. \end{aligned}$$

Putting the values of γ^2 and γ^0 from Eq. (3.26) and taking derivative of matrices with respect to ϕ we get

$$\begin{aligned} i\gamma^{\phi}\partial_{\phi}\Psi &= \frac{i}{\sqrt{K(r,\theta)}}\begin{pmatrix} \left[\partial_{\phi}B+\frac{i}{\hbar}B(\partial_{\phi}I_{\uparrow})\right]\sigma^{2}\xi_{\uparrow} \\ \left[\partial_{\phi}A+\frac{i}{\hbar}A(\partial_{\phi}I_{\uparrow})\right]\sigma^{2}\xi_{\uparrow} \end{pmatrix}\exp\left[\frac{i}{\hbar}I_{\uparrow}\right] \\ &+\frac{iH(r,\theta)}{K(r,\theta)\sqrt{F(r,\theta)}}\begin{pmatrix} \left[\partial_{\phi}B+\frac{i}{\hbar}B\partial_{\phi}(I_{\uparrow})\right]I\xi_{\uparrow} \\ -\left[\partial_{\phi}A+\frac{i}{\hbar}A\partial_{\phi}(I_{\uparrow})\right]I\xi_{\uparrow} \end{pmatrix}\exp\left[\frac{i}{\hbar}I_{\uparrow}\right]. \end{aligned}$$

Now

$$\sigma^{2}\xi_{\uparrow}=\begin{pmatrix} 0 \\ i \end{pmatrix}, \quad \text{and} \quad I\xi_{\uparrow}=\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

therefore the above equation becomes

$$i\gamma^{\phi}\partial_{\phi}\Psi=\begin{pmatrix} \frac{iH(r,\theta)}{K(r,\theta)\sqrt{F(r,\theta)}}\left[\partial_{\phi}B+\frac{i}{\hbar}B\partial_{\phi}(I_{\uparrow})\right] \\ -\frac{1}{\sqrt{K(r,\theta)}}\left[\partial_{\phi}B+\frac{i}{\hbar}B(\partial_{\phi}I_{\uparrow})\right] \\ -\frac{iH(r,\theta)}{K(r,\theta)\sqrt{F(r,\theta)}}\left[\partial_{\phi}A+\frac{i}{\hbar}A\partial_{\phi}(I_{\uparrow})\right] \\ -\frac{1}{\sqrt{K(r,\theta)}}\left[\partial_{\phi}A+\frac{i}{\hbar}A(\partial_{\phi}I_{\uparrow})\right] \end{pmatrix}\exp\left[\frac{i}{\hbar}I_{\uparrow}\right]. \tag{3.37}$$

Now consider the fifth term of Eq. (3.33)

$$\gamma^{t}\frac{q}{\hbar}A_{t}\Psi=\frac{1}{\sqrt{F(r,\theta)}}\gamma^{0}\frac{q}{\hbar}A_{t}\begin{pmatrix} A\xi_{\uparrow} \\ B\xi_{\uparrow} \end{pmatrix}\exp\left[\frac{i}{\hbar}I_{\uparrow}\right].$$

Putting the value of γ^0 from Eq. (3.26) we obtain

$$\gamma^t \frac{q}{\hbar} A_t \Psi = \frac{1}{\sqrt{F(r,\theta)}} \frac{q}{\hbar} A_t \begin{pmatrix} BI\xi_\uparrow \\ -AI\xi_\uparrow \end{pmatrix} \exp\left[\frac{i}{\hbar} I_\uparrow\right]$$

Now

$$I\xi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

therefore the above equation becomes

$$\gamma^t \frac{q}{\hbar} A_t \Psi = \begin{pmatrix} \frac{1}{\sqrt{F(r,\theta)}} \frac{q}{\hbar} A_t . B \\ 0 \\ -\frac{1}{\sqrt{F(r,\theta)}} \frac{q}{\hbar} A_t . A \\ 0 \end{pmatrix} \exp\left[\frac{i}{\hbar} I_\uparrow\right]. \tag{3.38}$$

Similarly the sixth term of Eq. (3.33) takes the form

$$\begin{aligned} \gamma^\phi \frac{q}{\hbar} A_\phi \Psi &= \frac{1}{\sqrt{K(r,\theta)}} \gamma^2 \frac{q}{\hbar} A_\phi \begin{pmatrix} A\xi_\uparrow \\ B\xi_\uparrow \end{pmatrix} \exp\left[\frac{i}{\hbar} I_\uparrow\right] \\ &+ \frac{1}{\sqrt{K(r,\theta)}} \frac{H(r,\theta)}{\sqrt{F(r,\theta) K(r,\theta)}} \gamma^0 \frac{q}{\hbar} A_\phi \begin{pmatrix} A\xi_\uparrow \\ B\xi_\uparrow \end{pmatrix} \exp\left[\frac{i}{\hbar} I_\uparrow\right]. \end{aligned}$$

Putting the values of γ^2 and γ^0 from Eq. (3.26) this becomes

$$\begin{aligned} \gamma^\phi \frac{q}{\hbar} A_\phi \Psi &= \frac{1}{\sqrt{K(r,\theta)}} \frac{q}{\hbar} A_\phi \begin{pmatrix} B\sigma^2\xi_\uparrow \\ A\sigma^2\xi_\uparrow \end{pmatrix} \exp\left[\frac{i}{\hbar} I_\uparrow\right] \\ &+ \frac{H(r,\theta)}{K(r,\theta)\sqrt{F(r,\theta)}} \frac{q}{\hbar} A_\phi \begin{pmatrix} BI\xi_\uparrow \\ -AI\xi_\uparrow \end{pmatrix} \exp\left[\frac{i}{\hbar} I_\uparrow\right]. \end{aligned}$$

As

$$\sigma^2\xi_\uparrow = \begin{pmatrix} 0 \\ i \end{pmatrix}, \quad \text{and} \quad I\xi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

the above equation becomes

$$\gamma^{\phi}\frac{q}{\hbar}A_{\phi}\Psi = \begin{pmatrix} \frac{H(r,\theta)}{K(r,\theta)\sqrt{F(r,\theta)}}\frac{q}{\hbar}A_{\phi}.B \\ \frac{i}{\sqrt{K(r,\theta)}}\frac{q}{\hbar}A_{\phi}.B \\ -\frac{H(r,\theta)}{K(r,\theta)\sqrt{F(r,\theta)}}\frac{q}{\hbar}A_{\phi}.A \\ \frac{i}{\sqrt{K(r,\theta)}}\frac{q}{\hbar}A_{\phi}.A \end{pmatrix}\exp\left[\frac{i}{\hbar}I_{\uparrow}\right]. \tag{3.39}$$

The last term of Eq. (3.33) is

$$\frac{m}{\hbar}\Psi = \begin{pmatrix} \frac{m}{\hbar}A\xi_{\uparrow} \\ \frac{m}{\hbar}B\xi_{\uparrow} \end{pmatrix}\exp\left[\frac{i}{\hbar}I_{\uparrow}\right],$$

As

$$\xi_{\uparrow} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

the above equation becomes

$$\frac{m}{\hbar}\Psi = \begin{pmatrix} \frac{m}{\hbar}A \\ 0 \\ \frac{m}{\hbar}B \\ 0 \end{pmatrix}\exp\left[\frac{i}{\hbar}I_{\uparrow}\right]. \tag{3.40}$$

Now Eq. (3.33) on using Eqs. (3.34) - (3.40) and dividing by the exponential term becomes

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \left[\frac{i}{\sqrt{F(r,\theta)}}\left[[\partial_t B] + \frac{i}{\hbar}B\left(\partial_t I_\uparrow\right)\right] + i\sqrt{g\left(r,\theta\right)}\left[\partial_r B + \frac{i}{\hbar}B.\partial_r I_\uparrow\right] \right. \\ + \frac{iH(r,\theta)}{K(r,\theta)\sqrt{F(r,\theta)}}\left[\partial_\phi B + \frac{i}{\hbar}B\partial_\phi I_\uparrow\right] + \frac{1}{\sqrt{F(r,\theta)}}\frac{q}{\hbar}A_t.B \\ \left. + \frac{H(r,\theta)}{K(r,\theta)\sqrt{F(r,\theta)}}\frac{q}{\hbar}A_\phi.B + \frac{m}{\hbar}A\right] \\ \left[\frac{i}{\sqrt{\Sigma(r,\theta)}}\left[\partial_\theta B + \frac{i}{\hbar}B.\partial_\theta I_\uparrow\right] - \frac{1}{\sqrt{K(r,\theta)}}\left[\partial_\phi B + \frac{i}{\hbar}B\left(\partial_\phi I_\uparrow\right)\right]\right. \\ \left. + \frac{i}{\sqrt{K(r,\theta)}}\frac{q}{\hbar}A_\phi.B\right] \\ \left[\frac{i}{\sqrt{F(r,\theta)}}\left[-[\partial_t A] - \frac{i}{\hbar}A\partial_t I_\uparrow\right] + i\sqrt{g\left(r,\theta\right)}\left[\partial_r A + \frac{i}{\hbar}A.\partial_r I_\uparrow\right]\right. \\ - \frac{iH(r,\theta)}{K(r,\theta)\sqrt{F(r,\theta)}}\left[\partial_\phi A + \frac{i}{\hbar}A\partial_\phi I_\uparrow\right] - \frac{1}{\sqrt{F(r,\theta)}}\frac{q}{\hbar}A_t.A \\ \left. - \frac{H(r,\theta)}{K(r,\theta)\sqrt{F(r,\theta)}}\frac{q}{\hbar}A_\phi.A + \frac{m}{\hbar}B\right] \\ \left[\frac{i}{\sqrt{\Sigma(r,\theta)}}\left[\partial_\theta A + \frac{i}{\hbar}A.\partial_\theta I_\uparrow\right] - \frac{1}{\sqrt{K(r,\theta)}}\left[\partial_\phi A + \frac{i}{\hbar}A\partial_\phi I_\uparrow\right]\right. \\ \left. + \frac{i}{\sqrt{K(r,\theta)}}\frac{q}{\hbar}A_\phi.A\right] \end{pmatrix}.$$

Multiplying the above matrix by $\hbar$ and neglecting the terms containg $\hbar$ we get

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -B\left[\frac{1}{\sqrt{F(r,\theta)}}\partial_t I_\uparrow + \sqrt{g\left(r,\theta\right)}\partial_r I_\uparrow + \frac{H(r,\theta)}{K(r,\theta)\sqrt{F(r,\theta)}}\partial_\phi I_\uparrow \right. \\ \left. - \frac{1}{\sqrt{F(r,\theta)}}qA_t - \frac{H(r,\theta)}{K(r,\theta)\sqrt{F(r,\theta)}}qA_\phi\right] + mA \\ -B\left[\frac{1}{\sqrt{\Sigma(r,\theta)}}\partial_\theta I_\uparrow + \frac{i}{\sqrt{K(r,\theta)}}\partial_\phi I_\uparrow - \frac{i}{\sqrt{K(r,\theta)}}qA_\phi\right] \\ A\left[\frac{1}{\sqrt{F(r,\theta)}}\partial_t I_\uparrow - \sqrt{g\left(r,\theta\right)}\partial_r I_\uparrow + \frac{H(r,\theta)}{K(r,\theta)\sqrt{F(r,\theta)}}\partial_\phi I_\uparrow \right. \\ \left. - \frac{1}{\sqrt{F(r,\theta)}}qA_t - \frac{H(r,\theta)}{K(r,\theta)\sqrt{F(r,\theta)}}qA_\phi\right] + mB \\ -A\left[\frac{1}{\sqrt{\Sigma(r,\theta)}}\partial_\theta I_\uparrow + \frac{i}{\sqrt{K(r,\theta)}}\partial_\phi I_\uparrow - \frac{i}{\sqrt{K(r,\theta)}}qA_\phi\right] \end{pmatrix}. \tag{3.41}$$

From this matrix of order (4×1) we get a set of four equations

$$0 = -B\left[\frac{1}{\sqrt{F(r,\theta)}}\partial_t I_\uparrow + \sqrt{g(r,\theta)}\partial_r I_\uparrow + \frac{H(r,\theta)}{K(r,\theta)\sqrt{F(r,\theta)}}\partial_\phi I_\uparrow - \frac{1}{\sqrt{F(r,\theta)}}qA_t - \frac{H(r,\theta)}{K(r,\theta)\sqrt{F(r,\theta)}}qA_\phi\right] + mA, \tag{3.42}$$

$$0 = -B\left[\frac{1}{\sqrt{\Sigma(r,\theta)}}\partial_\theta I_\uparrow + \frac{i}{\sqrt{K(r,\theta)}}\partial_\phi I_\uparrow - \frac{i}{\sqrt{K(r,\theta)}}qA_\phi\right], \tag{3.43}$$

$$0 = A\left[\frac{1}{\sqrt{F(r,\theta)}}\partial_t I_\uparrow - \sqrt{g(r,\theta)}\partial_r I_\uparrow + \frac{H(r,\theta)}{K(r,\theta)\sqrt{F(r,\theta)}}\partial_\phi I_\uparrow - \frac{1}{\sqrt{F(r,\theta)}}qA_t - \frac{H(r,\theta)}{K(r,\theta)\sqrt{F(r,\theta)}}qA_\phi\right] + mB, \tag{3.44}$$

$$0 = -A\left[\frac{1}{\sqrt{\Sigma(r,\theta)}}\partial_\theta I_\uparrow + \frac{i}{\sqrt{K(r,\theta)}}\partial_\phi I_\uparrow - \frac{i}{\sqrt{K(r,\theta)}}qA_\phi\right]. \tag{3.45}$$

As the metric coefficients do not depend on t and ϕ coordinates therefore we can apply the following ansatz for solving the above system of equations [22]

$$I_\uparrow = -Et + J\phi + W(r,\theta), \tag{3.46}$$

where E and J denote the energy and angular momentum of the emitted particles. We find the derivatives of $g(r,\theta)$ and $F(r,\theta)$ at the outer horizon one by one. Taking derivative of Eqs. (3.10) and (3.19) with respect to r we get

$$\partial_r g(r_+,\theta) = \frac{(1-\alpha r_+\cos\theta)^2(2r_+ - 2M)\left(1-\alpha^2 r_+^2\right)}{\left(r_+^2 + a^2\cos^2\theta\right)}, \tag{3.47}$$

and

$$\partial_r F(r_+,\theta) = \frac{\left(r_+^2 + a^2\cos^2\theta\right)(2r_+ - 2M)\left(1-\alpha^2 r_+^2\right)}{(1-\alpha r_+\cos\theta)^2\left(r_+^2+a^2\right)^2}, \tag{3.48}$$

where we have used the result $Q(r_+) = 0$. Also we have

$$g(r_+,\theta) = 0, \qquad F(r_+,\theta) = 0. \tag{3.49}$$

Expanding $g(r,\theta)$ and $F(r,\theta)$ using Taylor's theorem near the outer horizon and neglecting squares and higher powers and using Eq. (3.49) we get

$$g(r,\theta) = (r - r_+)\,\partial_r g(r_+,\theta), \qquad F(r,\theta) = (r - r_+)\,\partial_r F(r_+,\theta). \tag{3.50}$$

We expand Eqs. (3.42) to (3.45) near the black hole horizon one by one. Using Eq. (3.46) in (3.42) we get

$$\begin{aligned} 0 &= -B\Bigg[\frac{1}{\sqrt{(r-r_+)\,\partial_r F(r_+,\theta)}}(-E) + \sqrt{(r-r_+)\,\partial_r g(r_+,\theta)}\,(\partial_r W) \\ &\quad + \frac{H(r_+,\theta)}{K(r_+,\theta)\sqrt{(r-r_+)\,\partial_r F(r_+,\theta)}}J \\ &\quad - \frac{1}{\sqrt{(r-r_+)\,\partial_r F(r_+,\theta)}}qA_t(r_+,\theta) - \frac{H(r_+,\theta)}{K(r_+,\theta)\sqrt{(r-r_+)\,\partial_r F(r_+,\theta)}}qA_\phi(r_+,\theta)\Bigg] + mA. \end{aligned}$$

Using Eq. (3.17) and putting the values of $A_t(r_+,\theta)$ and $A_\phi(r_+,\theta)$ from Eq. (3.15) this becomes

$$-B\left[\frac{\left(-E + \Omega_H J + \frac{qer_+}{(r_+^2+a^2)}\right)}{\sqrt{(r-r_+)\,\partial_r F(r_+,\theta)}} + \sqrt{(r-r_+)\,\partial_r g(r_+,\theta)}\,(\partial_r W)\right] + mA = 0. \tag{3.51}$$

Using Eq. (3.46) in (3.43) and putting the values of $\Sigma(r_+,\theta)$ and $K(r_+,\theta)$ from Eqs. (3.11) and (3.12) we get

$$0 = -B\left[\frac{1}{\sqrt{\left(\frac{\rho^2(r_+,\theta)}{P\Omega^2(r_+,\theta)}\right)}}\partial_\theta W + \frac{i\rho(r_+,\theta)\,\Omega(r_+,\theta)}{\sqrt{\sin^2\theta P\left(r_+^2+a^2\right)^2}}J - \frac{i\rho(r_+,\theta)\,\Omega(r_+,\theta)}{\sqrt{\sin^2\theta P\left(r_+^2+a^2\right)^2}}qA_\phi(r_+,\theta)\right].$$

Putting the value of $A_\phi(r_+,\theta)$ from Eq. (3.15) this becomes

$$-B\left[\sqrt{\frac{P\Omega^2(r_+,\theta)}{\rho^2(r_+,\theta)}}\partial_\theta W + \frac{i\rho(r_+,\theta)\,\Omega(r_+,\theta)}{\sqrt{\sin^2\theta P\left(r_+^2+a^2\right)^2}}\left(J - q\left(\frac{aer_+\sin^2\theta + g\left(r_+^2+a^2\right)\cos\theta}{r_+^2+a^2\cos^2\theta}\right)\right)\right] = 0. \tag{3.52}$$

Using Eqs. (3.46) and (3.17) in (3.44) and simplifiying we obtain

$$\begin{aligned} 0 &= A\left[\frac{1}{\sqrt{(r-r_+)\,\partial_r F(r_+,\theta)}}(-E) - \sqrt{(r-r_+)\,\partial_r g(r_+,\theta)}\partial_r W + \frac{\Omega_H J}{\sqrt{(r-r_+)\,\partial_r F(r_+,\theta)}}\right. \\ &\left. - \frac{1}{\sqrt{(r-r_+)\,\partial_r F(r_+,\theta)}}\left(qA_t(r_+,\theta) + q\Omega_H A_\phi(r_+,\theta)\right)\right] + Bm. \end{aligned}$$

Putting the values of $A_t(r_+,\theta)$ and $A_\phi(r_+,\theta)$ from Eq. (3.15) and after simplification this becomes

$$A\left[\frac{\left(-E+\Omega_H J + \frac{qer_+}{(r_+^2+a^2)}\right)}{\sqrt{(r-r_+)\,\partial_r F(r_+,\theta)}} - \sqrt{(r-r_+)\,\partial_r g(r_+,\theta)}\partial_r W\right] + Bm = 0. \tag{3.53}$$

Now using Eq. (3.46) in (3.45) we get

$$-A\left[\frac{1}{\sqrt{\Sigma(r_+,\theta)}}\partial_\theta W + \frac{i}{\sqrt{K(r_+,\theta)}}J - \frac{i}{\sqrt{K(r_+,\theta)}}qA_\phi(r_+,\theta)\right] = 0,$$

and putting the values of $\Sigma(r_+,\theta)$, $K(r_+,\theta)$ and $A_\phi(r_+,\theta)$ from Eqs. (3.11), (3.12) and (3.15) and simplifying we get

$$-A\left[\sqrt{\frac{P\Omega^2(r_+,\theta)}{\rho^2(r_+,\theta)}}\partial_\theta W + \frac{i\rho(r_+,\theta)\,\Omega(r_+,\theta)}{\sqrt{\sin^2\theta P\left(r_+^2+a^2\right)^2}}\left(J - q\left(\frac{aer_+\sin^2\theta + g\left(r_+^2+a^2\right)\cos\theta}{r_+^2+a^2\cos^2\theta}\right)\right)\right] = 0. \tag{3.54}$$

Near the black hole horizon it is possible to sperate $W(r,\theta)$ as

$$W(r,\theta) = R(r) + \Theta(\theta). \tag{3.55}$$

We use this relation and divide our solution into two parts, the massless and the massive case.

3.2 The Massless Case

In the massless case we put $m = 0$ in Eqs. (3.51) - (3.54) and solve them. Taking derivative of Eq. (3.55) with respect to r we get

$$\partial_r W(r,\theta) = R'(r),$$

where prime denotes the derivative with respect to r. Using this Eq. (3.51) becomes

$$-B\left[\frac{\left(-E+\Omega_H J+\frac{qer_+}{(r_+^2+a^2)}\right)}{\sqrt{(r-r_+)\,\partial_r F(r_+,\theta)}}+\sqrt{(r-r_+)\,\partial_r g(r_+,\theta)}R'(r)\right]=0,$$

which implies that

$$B=0 \qquad \text{and} \qquad \left[\frac{\left(-E+\Omega_H J+\frac{qer_+}{(r_+^2+a^2)}\right)}{\sqrt{(r-r_+)\,\partial_r F(r_+,\theta)}}+\sqrt{(r-r_+)\,\partial_r g(r_+,\theta)}R'(r)\right]=0.$$

Consider

$$\left[\frac{\left(-E+\Omega_H J+\frac{qer_+}{(r_+^2+a^2)}\right)}{\sqrt{(r-r_+)\,\partial_r F(r_+,\theta)}}+\sqrt{(r-r_+)\,\partial_r g(r_+,\theta)}R'(r)\right]=0.$$

After rearranging the terms it takes the form

$$R'(r)=\frac{\left(E-\Omega_H J-\frac{qer_+}{(r_+^2+a^2)}\right)}{\sqrt{(r-r_+)\,\partial_r F(r_+,\theta)}}\frac{1}{\sqrt{(r-r_+)\,\partial_r g(r_+,\theta)}}.$$

Putting the values of $\partial_r F(r_+,\theta)$ and $\partial_r g(r_+,\theta)$ from Eqs. (3.47) and (3.48) and after simplification this becomes

$$R'(r)=\frac{\left(E-\Omega_H J-\frac{qer_+}{(r_+^2+a^2)}\right)(r_+^2+a^2)}{(r-r_+)(2r_+-2M)(1-\alpha^2 r_+^2)}.$$

So

$$B = 0 \qquad \text{and} \qquad R'(r) = R_+'(r) = \frac{\left(E - \Omega_H J - \frac{qer_+}{(r_+^2+a^2)}\right)(r_+^2 + a^2)}{(r - r_+)(2r_+ - 2M)(1 - \alpha^2 r_+^2)}. \tag{3.56}$$

where R_+ corresponds to the outgoing solutions. Eq. (3.53) becomes

$$A\left[\frac{\left(-E + \Omega_H J + \frac{qer_+}{(r_+^2+a^2)}\right)}{\sqrt{(r - r_+)\,\partial_r F(r_+, \theta)}} - \sqrt{(r - r_+)\,\partial_r g(r_+, \theta)}R'(r)\right] = 0,$$

which implies that

$$A = 0 \quad \text{and} \quad \left[\frac{\left(-E + \Omega_H J + \frac{qer_+}{(r_+^2+a^2)}\right)}{\sqrt{(r - r_+)\,\partial_r F(r_+, \theta)}} - \sqrt{(r - r_+)\,\partial_r g(r_+, \theta)}R'(r)\right] = 0.$$

The second equation can be written as

$$R'(r) = \frac{\left(-E + \Omega_H J + \frac{qer_+}{(r_+^2+a^2)}\right)}{(r - r_+)\sqrt{\partial_r F(r_+, \theta)\,\partial_r g(r_+, \theta)}}.$$

Putting the values of $\partial_r F(r_+, \theta)$ and $\partial_r g(r_+, \theta)$ from Eqs. (3.47) and (3.48) and after simplification this becomes

$$R'(r) = \frac{\left(-E + \Omega_H J + \frac{qer_+}{(r_+^2+a^2)}\right)(r_+^2 + a^2)}{(r - r_+)(2r_+ - 2M)(1 - \alpha^2 r_+^2)}.$$

So

$$A = 0 \qquad \text{and} \qquad R'(r) = R_-'(r) = -\frac{\left(E - \Omega_H J - \frac{qer_+}{(r_+^2+a^2)}\right)(r_+^2 + a^2)}{(r - r_+)(2r_+ - 2M)(1 - \alpha^2 r_+^2)}. \tag{3.57}$$

where R_- corresponds to the incoming solutions. From Eq. (3.56) solving for R_+ yields

$$R_+ = \int \frac{\left(E - \Omega_H J - \frac{qer_+}{(r_+^2+a^2)}\right)(r_+^2 + a^2)}{(r - r_+)(2r_+ - 2M)(1 - \alpha^2 r_+^2)} dr.$$

Taking constant terms outside the integral we get

$$R_+ = \frac{\left(E - \Omega_H J - \frac{qer_+}{(r_+^2+a^2)}\right)(r_+^2 + a^2)}{(2r_+ - 2M)(1 - \alpha^2 r_+^2)} \int \frac{1}{(r - r_+)} dr.$$

Here $r = r_+$ is a singularity which is a simple pole. Using the residue theory, we integrate it around the pole

$$R_+ = \frac{\pi i \left(E - \Omega_H J - \frac{qer_+}{(r_+^2+a^2)}\right)(r_+^2 + a^2)}{(2r_+ - 2M)(1 - \alpha^2 r_+^2)}. \tag{3.58}$$

Imaginary part of R_+ is

$$\operatorname{Im} R_+ = \frac{\pi}{2} \frac{\left(E - \Omega_H J - \frac{qer_+}{(r_+^2+a^2)}\right)(r_+^2 + a^2)}{(r_+ - M)(1 - \alpha^2 r_+^2)}. \tag{3.59}$$

Similarly from Eq. (3.57) we get

$$R_- = -\frac{\pi i \left(E - \Omega_H J - \frac{qer_+}{(r_+^2+a^2)}\right)(r_+^2 + a^2)}{(2r_+ - 2M)(1 - \alpha^2 r_+^2)}, \tag{3.60}$$

and

$$\operatorname{Im} R_- = -\frac{\pi}{2} \frac{\left(E - \Omega_H J - \frac{qer_+}{(r_+^2+a^2)}\right)(r_+^2 + a^2)}{(r_+ - M)(1 - \alpha^2 r_+^2)}. \tag{3.61}$$

From Eqs. (3.59) and (3.61) we have

$$\operatorname{Im} R_+ = -\operatorname{Im} R_-. \tag{3.62}$$

The probabilities of crossing the horizon in each direction are proportional to [22, 29, 30]

$$prob\,[out] \propto \exp\left[-2 \operatorname{Im} I\right] = \exp\left[-2 \left(\operatorname{Im} R_+ + \operatorname{Im} \Theta\right)\right], \tag{3.63}$$

and

$$prob\,[in] \propto \exp\left[-2 \operatorname{Im} I\right] = \exp\left[-2 \left(\operatorname{Im} R_- + \operatorname{Im} \Theta\right)\right]. \tag{3.64}$$

The probability of a particle tunneling from inside to outside the horizon is

$$\Gamma \propto \frac{prob\,[out]}{prob\,[in]} = \frac{\exp\left[-2\left(\operatorname{Im} R_{+} + \operatorname{Im} \Theta\right)\right]}{\exp\left[-2\left(\operatorname{Im} R_{-} + \operatorname{Im} \Theta\right)\right]},$$

which gives

$$\Gamma = \exp\left[-2\left(\operatorname{Im} R_{+} - \operatorname{Im} R_{-}\right)\right].$$

Using Eq. (3.62), we get

$$\Gamma = \exp\left[-4 \operatorname{Im} R_{+}\right].$$

Using Eq. (3.59) this takes the form

$$\Gamma = \exp\left[-4\frac{\frac{\pi}{2}\left(E - \Omega_H J - \frac{qer_+}{\left(r_+^2+a^2\right)}\right)\left(r_+^2 + a^2\right)}{\left(r_+ - M\right)\left(1 - \alpha^2 r_+^2\right)}\right],$$

which implies that

$$\Gamma = \exp\left[-2\pi\frac{\left(r_+^2 + a^2\right)}{\left(r_+ - M\right)\left(1 - \alpha^2 r_+^2\right)}\left(E - \Omega_H J - \frac{qer_+}{\left(r_+^2 + a^2\right)}\right)\right]. \tag{3.65}$$

Comparing this with $\Gamma = \exp\left(-\beta E\right)$ where $\beta = 1/T_H$ we get

$$T_H = \frac{\left(1 - \alpha^2 r_+^2\right)\left(r_+ - M\right)}{2\pi\left(r_+^2 + a^2\right)}. \tag{3.66}$$

3.3 The Massive Case

In the massive case we shall eliminate the function $\partial_r W$ from Eqs. (3.51) and (3.53). Multiplying Eq. (3.51) by A and Eq. (3.53) by B and subtracting yields

$$0 = -2AB\frac{\left(-E + \Omega_H J + \frac{qer_+}{\left(r_+^2+a^2\right)}\right)}{\sqrt{\left(r - r_+\right)\partial_r F\left(r_+, \theta\right)}} + mA^2 - mB^2.$$

Dividing by B^2 and rearranging the terms in the descending order we get

$$\begin{aligned} 0 &= m\sqrt{(r-r_+)\,\partial_r F(r_+,\theta)}\left(\frac{A}{B}\right)^2 + 2\left(E-\Omega_H J-\frac{qer_+}{(r_+^2+a^2)}\right)\left(\frac{A}{B}\right) \\ & -m\sqrt{(r-r_+)\,\partial_r F(r_+,\theta)}, \end{aligned}$$

which is a quadratic equation in A/B. Thus

$$\frac{A}{B} = \frac{-\left(E-\Omega_H J-\frac{qer_+}{(r_+^2+a^2)}\right) \pm \sqrt{\left(E-\Omega_H J-\frac{qer_+}{(r_+^2+a^2)}\right)^2 + m^2(r-r_+)\,\partial_r F(r_+,\theta)}}{m\sqrt{(r-r_+)\,\partial_r F(r_+,\theta)}}. \tag{3.67}$$

Now

$$\lim_{r\to r_+}\frac{A}{B} = \begin{cases} 0 & \text{for upper sign,} \\ -\infty & \text{for lower sign.} \end{cases} \tag{3.68}$$

Consequently at the horizon either $A/B \to 0$ or $A/B = -\infty$, i.e. either $A \to 0$ or $B \to 0$. When $A \to 0$ then Eq. (3.53) becomes

$$m = -\frac{A}{B}\left[\frac{\left(-E+\Omega_H J+\frac{qer_+}{(r_+^2+a^2)}\right)}{\sqrt{(r-r_+)\,\partial_r F(r_+,\theta)}} - \sqrt{(r-r_+)\,\partial_r g(r_+,\theta)}R'(r)\right].$$

Using in Eq. (3.51) and rearranging the terms we get

$$\left(A^2-B^2\right)\sqrt{(r-r_+)\,\partial_r g(r_+,\theta)}R'(r) = \left(A^2+B^2\right)\frac{\left(-E+\Omega_H J+\frac{qer_+}{(r_+^2+a^2)}\right)}{\sqrt{(r-r_+)\,\partial_r F(r_+,\theta)}},$$

or

$$\partial_r W(r,\theta) = R'_+(r) = \frac{\left(E-\Omega_H J-\frac{qer_+}{(r_+^2+a^2)}\right)\left(1+\frac{A^2}{B^2}\right)}{\sqrt{\partial_r g(r_+,\theta)\,\partial_r F(r_+,\theta)}\,(r-r_+)\left(1-\frac{A^2}{B^2}\right)}. \tag{3.69}$$

Smilarly for $B \to 0$, we get

$$\partial_r W(r,\theta) = R'_{-}(r) = -\frac{\left(E - \Omega_H J - \frac{qer_+}{(r_+^2+a^2)}\right)\left(1+\frac{B^2}{A^2}\right)}{\sqrt{\partial_r g(r_+,\theta)\,\partial_r F(r_+,\theta)}\,(r-r_+)\left(1-\frac{B^2}{A^2}\right)}. \tag{3.70}$$

From Eq. (3.69) solving for $R_+(r)$

$$R_+(r) = \int \frac{\left(E - \Omega_H J - \frac{qer_+}{(r_+^2+a^2)}\right)\left(1+\frac{A^2}{B^2}\right)}{\sqrt{\partial_r g(r_+,\theta)\,\partial_r F(r_+,\theta)}\,(r-r_+)\left(1-\frac{A^2}{B^2}\right)}dr.$$

Taking constant terms outside the integral we get

$$R_+(r) = \frac{\left(E - \Omega_H J - \frac{qer_+}{(r_+^2+a^2)}\right)}{\sqrt{\partial_r g(r_+,\theta)\,\partial_r F(r_+,\theta)}} \int \frac{\left(1+\frac{A^2}{B^2}\right)/\left(1-\frac{A^2}{B^2}\right)}{(r-r_+)}dr.$$

Here $r = r_+$ is the simple pole, thus using the residue theory and integrating around the pole this becomes

$$R_+ = \frac{\pi i\left(E - \Omega_H J - \frac{qer_+}{(r_+^2+a^2)}\right)}{\sqrt{\partial_r g(r_+,\theta)\,\partial_r F(r_+,\theta)}}.$$

Using the values of $\partial_r g(r_+,\theta)$ and $\partial_r F(r_+,\theta)$ from Eqs. (3.47) and (3.48) and simplifying this takes the form

$$R_+ = \frac{\pi i\left(E - \Omega_H J - \frac{qer_+}{(r_+^2+a^2)}\right)\left(r_+^2+a^2\right)}{(2r_+ - 2M)\left(1-\alpha^2 r_+^2\right)}. \tag{3.71}$$

Similarly from Eq. (3.70) we get

$$R_- = -\frac{\pi i\left(E - \Omega_H J - \frac{qer_+}{(r_+^2+a^2)}\right)\left(r_+^2+a^2\right)}{(2r_+ - 2M)\left(1-\alpha^2 r_+^2\right)}. \tag{3.72}$$

Now

$$\operatorname{Im} R_+ = \frac{\pi\left(E - \Omega_H J - \frac{qer_+}{(r_+^2+a^2)}\right)\left(r_+^2+a^2\right)}{(2r_+ - 2M)\left(1-\alpha^2 r_+^2\right)}, \tag{3.73}$$

and

$$\operatorname{Im} R_- = -\frac{\pi\left(E - \Omega_H J - \frac{qer_+}{(r_+^2+a^2)}\right)(r_+^2 + a^2)}{(2r_+ - 2M)\left(1 - \alpha^2 r_+^2\right)},$$

which implies that

$$\operatorname{Im} R_+ = -\operatorname{Im} R_-. \tag{3.74}$$

The probabilities of crossing the horizon in each direction are proportional to

$$prob\,[out] \propto \exp\left[-2\operatorname{Im} I\right] = \exp\left[-2\left(\operatorname{Im} R_+ + \operatorname{Im}\Theta\right)\right], \tag{3.75}$$

and

$$prob\,[in] \propto \exp\left[-2\operatorname{Im} I\right] = \exp\left[-2\left(\operatorname{Im} R_- + \operatorname{Im}\Theta\right)\right]. \tag{3.76}$$

The probability of a particle tunneling from inside to outside the horizon is

$$\Gamma \propto \frac{prob\,[out]}{prob\,[in]} = \frac{\exp\left[-2\left(\operatorname{Im} R_+ + \operatorname{Im}\Theta\right)\right]}{\exp\left[-2\left(\operatorname{Im} R_- + \operatorname{Im}\Theta\right)\right]}.$$

Using Eq. (3.74) this becomes

$$\Gamma = \exp\left(-4\operatorname{Im} R_+\right).$$

Using the value of $\operatorname{Im} R_+$ from Eq. (3.73) we get

$$\Gamma = \exp\left[-4\frac{\pi\left(E - \Omega_H J - \frac{qer_+}{(r_+^2+a^2)}\right)(r_+^2 + a^2)}{(2r_+ - 2M)\left(1 - \alpha^2 r_+^2\right)}\right],$$

or

$$\Gamma = \exp\left[-2\pi\frac{(r_+^2 + a^2)}{(r_+ - M)\left(1 - \alpha^2 r_+^2\right)}\left(E - \Omega_H J - \frac{qer_+}{(r_+^2 + a^2)}\right)\right], \tag{3.77}$$

From this we get

$$T_H = \frac{\left(1 - \alpha^2 r_+^2\right)(r_+ - M)}{2\pi\left(r_+^2 + a^2\right)} \tag{3.78}$$

which are same that we have obtained in the massless case. Thus the extra contributions at the horizon in the massive case vanish and we get the same result as in the massless case.

3.4 Calculation of the Action

Now we will solve Eqs. (3.51) to (3.54) near the black hole horizon to find the explicit expression for the action $I_\uparrow$. Now Eq. (3.51) using Eq. (3.55) becomes

$$-B\left[\frac{\left(-E+\Omega_H J+\frac{qer_+}{(r_+^2+a^2)}\right)}{\sqrt{(r-r_+)\,\partial_r F(r_+,\theta)}}+\sqrt{(r-r_+)\,\partial_r g(r_+,\theta)}R'(r)\right]+mA=0.$$

After simplification we have

$$R'(r)=\frac{mA}{B\sqrt{\partial_r g(r_+,\theta)\,(r-r_+)}}-\frac{\left(-E+\Omega_H J+\frac{qer_+}{(r_+^2+a^2)}\right)}{\sqrt{\partial_r F(r_+,\theta)\,\partial_r g(r_+,\theta)}\,(r-r_+)},$$

where A and B are functions of (t,r,θ,ϕ). Integrating with respect to r we get

$$R(r)=R_+(r)=\int\frac{mA}{B\sqrt{\partial_r g(r_+,\theta)\,(r-r_+)}}dr-\frac{\left(-E+\Omega_H J+\frac{qer_+}{(r_+^2+a^2)}\right)}{\sqrt{\partial_r F(r_+,\theta)\,\partial_r g(r_+,\theta)}}\ln(r-r_+)\,. \quad (3.79)$$

which corresponds to outgoing particles. Similarly from Eq. (3.53) we get

$$R(r)=R_-(r)=\int\frac{mB}{A\sqrt{\partial_r g(r_+,\theta)\,(r-r_+)}}dr+\frac{\left(-E+\Omega_H J+\frac{qer_+}{(r_+^2+a^2)}\right)}{\sqrt{\partial_r F(r_+,\theta)\,\partial_r g(r_+,\theta)}}\ln(r-r_+)\,. \quad (3.80)$$

which corresponds to incoming particles. Now from Eq. (3.52) or (3.54) we get the same equation regardless of the values of A or B

$$\left[\sqrt{\frac{P\Omega^2(r_+,\theta)}{\rho^2(r_+,\theta)}}\partial_\theta W+\frac{i\rho(r_+,\theta)\,\Omega(r_+,\theta)}{\sqrt{\sin^2\theta P\left(r_+^2+a^2\right)^2}}\left(J-q\left(\frac{aer_+\sin^2\theta+g\left(r_+^2+a^2\right)\cos\theta}{r_+^2+a^2\cos^2\theta}\right)\right)\right]=0.$$

Using Eq. (3.55) this becomes

$$\left[\sqrt{\frac{P\Omega^2\left(r_+,\theta\right)}{\rho^2\left(r_+,\theta\right)}}\left(\partial_\theta\Theta\right)+\frac{i\rho\left(r_+,\theta\right)\Omega\left(r_+,\theta\right)}{\sqrt{\sin^2\theta\left[P\left(r_+^2+a^2\right)^2\right]}}\left(J-q\left(\frac{aer_+\sin^2\theta+g\left(r_+^2+a^2\right)\cos\theta}{r_+^2+a^2\cos^2\theta}\right)\right)\right]=0. \tag{3.81}$$

Using the values of $\rho\left(r_+,\theta\right)$ and P from Eqs. (3.4) and (3.5) and after simplification this becomes

$$\begin{aligned}\partial_\theta\Theta &= \frac{iqaer_+ + iJa^2}{r_+^2+a^2}\frac{\sin\theta}{1-2\alpha M\cos\theta+\alpha^2\left(a^2+e^2+g^2\right)\cos^2\theta}\\ &+\frac{iqg\cos\theta-iJ}{\sin\theta\left[1-2\alpha M\cos\theta+\alpha^2\left(a^2+e^2+g^2\right)\cos^2\theta\right]}.\end{aligned}$$

Integrating with respect to θ we get

$$\Theta=\frac{iqaer_+ + iJa^2}{r_+^2+a^2}I_1+I_2, \tag{3.82}$$

where I_1 and I_2 are given below

$$I_1=\int\frac{\sin\theta}{1-2\alpha M\cos\theta+\alpha^2\left(a^2+e^2+g^2\right)\cos^2\theta}d\theta,$$

$$I_2=\int\frac{iqg\cos\theta-iJ}{\sin\theta\left[1-2\alpha M\cos\theta+\alpha^2\left(a^2+e^2+g^2\right)\cos^2\theta\right]}d\theta.$$

Now consider I_1

$$I_1=\int\frac{\sin\theta}{1-2\alpha M\cos\theta+\alpha^2\left(a^2+e^2+g^2\right)\cos^2\theta}d\theta.$$

Using substitution $z=\cos\theta$, we get

$$I_1=-\int\frac{1}{1-2\alpha Mz+\alpha^2\left(a^2+e^2+g^2\right)z^2}dz.$$

Integrating with respect to z and putting $z=\cos\theta$ we get

$$I_1=-\frac{1}{2\alpha\sqrt{M^2-a^2-e^2-g^2}}\ln\left(\frac{\alpha\cos\theta\left(a^2+e^2+g^2\right)-M-\sqrt{M^2-a^2-e^2-g^2}}{\alpha\cos\theta\left(a^2+e^2+g^2\right)-M+\sqrt{M^2-a^2-e^2-g^2}}\right). \tag{3.83}$$

Now consider I_2

$$I_2 = \int \frac{iqg\cos\theta - iJ}{\sin\theta\left[1 - 2\alpha M\cos\theta + \alpha^2\left(a^2 + e^2 + g^2\right)\cos^2\theta\right]}d\theta.$$

Using substitution $\cos\theta = z$, we have

$$I_2 = -\int \frac{iqgz - iJ}{\left(1 - z^2\right)\left[1 - 2\alpha Mz + \alpha^2\left(a^2 + e^2 + g^2\right)z^2\right]}dz.$$

Using partial fraction this becomes

$$I_2 = -\int \left[\frac{L_1}{1-z}dz + \frac{L_2}{1+z} + \frac{L_3 z + L_4}{1 - 2\alpha Mz + \alpha^2\left(a^2 + e^2 + g^2\right)z^2}\right], \tag{3.84}$$

where values of L_1, L_2, L_3, L_4 are given below

$$L_1 = \frac{iqg - iJ}{2\left[1 - 2\alpha M + \alpha^2\left(a^2 + e^2 + g^2\right)\right]}, \tag{3.85}$$

$$L_2 = -\frac{iqg + iJ}{2\left[1 + 2\alpha M + \alpha^2\left(a^2 + e^2 + g^2\right)\right]}, \tag{3.86}$$

$$L_3 = \frac{\alpha^2\left(a^2 + e^2 + g^2\right)\left[2iqg\left(1 + \alpha^2\left(a^2 + e^2 + g^2\right)\right) - 4\alpha MiJ\right]}{2\left[1 - 2\alpha M + \alpha^2\left(a^2 + e^2 + g^2\right)\right]\left[1 + 2\alpha M + \alpha^2\left(a^2 + e^2 + g^2\right)\right]}, \tag{3.87}$$

$$L_4 = -\frac{2iJ\alpha^2\left(a^2 + e^2 + g^2\right)\left[1 + \alpha^2\left(a^2 + e^2 + g^2\right)\right] - 8iJ\alpha^2 M^2 + 4iqg\alpha M}{2\left[1 - 2\alpha M + \alpha^2\left(a^2 + e^2 + g^2\right)\right]\left[1 + 2\alpha M + \alpha^2\left(a^2 + e^2 + g^2\right)\right]}. \tag{3.88}$$

Using these values in Eq. (3.84), and after integrating the resulting expressions, and putting $z = cos\theta$ we get

$$\begin{aligned} I_2 = {} & \frac{iqg - iJ}{2\left[1 - 2\alpha M + \alpha^2\left(a^2 + e^2 + g^2\right)\right]}\ln\left(1 - \cos\theta\right) + \frac{iqg + iJ}{2\left[1 + 2\alpha M + \alpha^2\left(a^2 + e^2 + g^2\right)\right]}\ln\left(1 + \cos\theta\right) \\ & - \frac{iqg\left[1 + \alpha^2\left(a^2 + e^2 + g^2\right)\right] - 2\alpha MiJ}{2\left[1 - 2\alpha M + \alpha^2\left(a^2 + e^2 + g^2\right)\right]\left[1 + 2\alpha M + \alpha^2\left(a^2 + e^2 + g^2\right)\right]} \\ & \times \ln\left[1 - 2\alpha M\cos\theta + \alpha^2\left(a^2 + e^2 + g^2\right)\cos^2\theta\right] \\ & - \frac{\left(a^2 + e^2 + g^2\right)\left[-2iJ\alpha^2 - 2iJ\alpha^4\left(a^2 + e^2 + g^2\right) + 2iqg\alpha^3 M\right] + 4iJ\alpha^2 M^2 - 2iqg\alpha M}{4\alpha\sqrt{M^2 - a^2 - e^2 - g^2}\left[\left\{1 + \alpha^2\left(a^2 + e^2 + g^2\right)\right\}^2 - 4\alpha^2 M^2\right]} \\ & \times \ln\left[\frac{\alpha\left(a^2 + e^2 + g^2\right)\cos\theta - M - \sqrt{M^2 - a^2 - e^2 - g^2}}{\alpha\left(a^2 + e^2 + g^2\right)\cos\theta - M + \sqrt{M^2 - a^2 - e^2 - g^2}}\right]. \end{aligned} \tag{3.89}$$

Using Eqs. (3.83) and (3.89) in (3.82) we get

$$\begin{aligned}
\Theta &= -\frac{iqaer_{+}+iJa^{2}}{2\left(r_{+}^{2}+a^{2}\right)\alpha\sqrt{M^{2}-a^{2}-e^{2}-g^{2}}}\ln\left(\frac{\alpha\cos\theta\left(a^{2}+e^{2}+g^{2}\right)-M-\sqrt{M^{2}-a^{2}-e^{2}-g^{2}}}{\alpha\cos\theta\left(a^{2}+e^{2}+g^{2}\right)-M+\sqrt{M^{2}-a^{2}-e^{2}-g^{2}}}\right)\\
&+\frac{iqg-iJ}{2\left[1-2\alpha M+\alpha^{2}\left(a^{2}+e^{2}+g^{2}\right)\right]}\ln\left(1-\cos\theta\right)\\
&+\frac{iqg+iJ}{2\left[1+2\alpha M+\alpha^{2}\left(a^{2}+e^{2}+g^{2}\right)\right]}\ln\left(1+\cos\theta\right)\\
&-\frac{iqg\left[1+\alpha^{2}\left(a^{2}+e^{2}+g^{2}\right)\right]-2\alpha MiJ}{2\left[1-2\alpha M+\alpha^{2}\left(a^{2}+e^{2}+g^{2}\right)\right]\left[1+2\alpha M+\alpha^{2}\left(a^{2}+e^{2}+g^{2}\right)\right]}\\
&\times\ln\left[1-2\alpha M\cos\theta+\alpha^{2}\left(a^{2}+e^{2}+g^{2}\right)\cos^{2}\theta\right]\\
&-\frac{\left(a^{2}+e^{2}+g^{2}\right)\left[-2iJ\alpha^{2}-2iJ\alpha^{4}\left(a^{2}+e^{2}+g^{2}\right)+2iqg\alpha^{3}M\right]+4iJ\alpha^{2}M^{2}-2iqg\alpha M}{4\alpha\sqrt{M^{2}-a^{2}-e^{2}-g^{2}}\left[\left\{1+\alpha^{2}\left(a^{2}+e^{2}+g^{2}\right)\right\}^{2}-4\alpha^{2}M^{2}\right]}\\
&\times\ln\left[\frac{\alpha\left(a^{2}+e^{2}+g^{2}\right)\cos\theta-M-\sqrt{M^{2}-a^{2}-e^{2}-g^{2}}}{\alpha\left(a^{2}+e^{2}+g^{2}\right)\cos\theta-M+\sqrt{M^{2}-a^{2}-e^{2}-g^{2}}}\right]. \qquad (3.90)
\end{aligned}$$

Using Eqs. (3.79) and (3.90) in (3.55) we get

$$\begin{aligned}
W\left(r,\theta\right) &= \int\frac{mA}{B\sqrt{\partial_{r}g\left(r_{+},\theta\right)\left(r-r_{+}\right)}}dr-\frac{\left(-E+\Omega_{H}J+\frac{qer_{+}}{\left(r_{+}^{2}+a^{2}\right)}\right)}{\sqrt{\partial_{r}F\left(r_{+},\theta\right)\partial_{r}g\left(r_{+},\theta\right)}}\ln\left(r-r_{+}\right)\\
&-\frac{iqaer_{+}+iJa^{2}}{2\left(r_{+}^{2}+a^{2}\right)\alpha\sqrt{M^{2}-a^{2}-e^{2}-g^{2}}}\\
&\times\ln\left(\frac{\alpha\cos\theta\left(a^{2}+e^{2}+g^{2}\right)-M-\sqrt{M^{2}-a^{2}-e^{2}-g^{2}}}{\alpha\cos\theta\left(a^{2}+e^{2}+g^{2}\right)-M+\sqrt{M^{2}-a^{2}-e^{2}-g^{2}}}\right)\\
&+\frac{iqg-iJ}{2\left[1-2\alpha M+\alpha^{2}\left(a^{2}+e^{2}+g^{2}\right)\right]}\ln\left(1-\cos\theta\right)\\
&+\frac{iqg+iJ}{2\left[1+2\alpha M+\alpha^{2}\left(a^{2}+e^{2}+g^{2}\right)\right]}\ln\left(1+\cos\theta\right)\\
&-\frac{iqg\left[1+\alpha^{2}\left(a^{2}+e^{2}+g^{2}\right)\right]-2\alpha MiJ}{2\left[1-2\alpha M+\alpha^{2}\left(a^{2}+e^{2}+g^{2}\right)\right]\left[1+2\alpha M+\alpha^{2}\left(a^{2}+e^{2}+g^{2}\right)\right]}\\
&\times\ln\left[1-2\alpha M\cos\theta+\alpha^{2}\left(a^{2}+e^{2}+g^{2}\right)\cos^{2}\theta\right]\\
&-\frac{\left(a^{2}+e^{2}+g^{2}\right)\left[-2iJ\alpha^{2}-2iJ\alpha^{4}\left(a^{2}+e^{2}+g^{2}\right)+2iqg\alpha^{3}M\right]+4iJ\alpha^{2}M^{2}-2iqg\alpha M}{4\alpha\sqrt{M^{2}-a^{2}-e^{2}-g^{2}}\left[\left\{1+\alpha^{2}\left(a^{2}+e^{2}+g^{2}\right)\right\}^{2}-4\alpha^{2}M^{2}\right]}\\
&\times\ln\left[\frac{\alpha\left(a^{2}+e^{2}+g^{2}\right)\cos\theta-M-\sqrt{M^{2}-a^{2}-e^{2}-g^{2}}}{\alpha\left(a^{2}+e^{2}+g^{2}\right)\cos\theta-M+\sqrt{M^{2}-a^{2}-e^{2}-g^{2}}}\right]. \qquad (3.91)
\end{aligned}$$

which is the solution for the outgoing particles. Now if we consider the massless case ($m = 0$) then only third term of Eq. (3.92) will vanish and we will be left with

$$
\begin{aligned}
I_{\uparrow} &= -Et + J\phi - \frac{\left(-E + \Omega_H J + \frac{qer_+}{(r_+^2 + a^2)}\right)}{\sqrt{\partial_r F(r_+, \theta)\, \partial_r g(r_+, \theta)}} \ln(r - r_+) \\
&- \frac{iqaer_+ + iJa^2}{2(r_+^2 + a^2)\alpha\sqrt{M^2 - a^2 - e^2 - g^2}} \ln\left(\frac{\alpha\cos\theta(a^2 + e^2 + g^2) - M - \sqrt{M^2 - a^2 - e^2 - g^2}}{\alpha\cos\theta(a^2 + e^2 + g^2) - M + \sqrt{M^2 - a^2 - e^2 - g^2}}\right) \\
&+ \frac{iqg - iJ}{2[1 - 2\alpha M + \alpha^2(a^2 + e^2 + g^2)]} \ln(1 - \cos\theta) \\
&+ \frac{iqg + iJ}{2[1 + 2\alpha M + \alpha^2(a^2 + e^2 + g^2)]} \ln(1 + \cos\theta) \\
&- \frac{iqg\left[1 + \alpha^2\left(a^2 + e^2 + g^2\right)\right] - 2\alpha M iJ}{2[1 - 2\alpha M + \alpha^2(a^2 + e^2 + g^2)][1 + 2\alpha M + \alpha^2(a^2 + e^2 + g^2)]} \\
&\times \ln\left[1 - 2\alpha M\cos\theta + \alpha^2\left(a^2 + e^2 + g^2\right)\cos^2\theta\right] \\
&- \frac{\left(a^2 + e^2 + g^2\right)\left[-2iJ\alpha^2 - 2iJ\alpha^4\left(a^2 + e^2 + g^2\right) + 2iqg\alpha^3 M\right] + 4iJ\alpha^2 M^2 - 2iqg\alpha M}{4\alpha\sqrt{M^2 - a^2 - e^2 - g^2}\left[\{1 + \alpha^2(a^2 + e^2 + g^2)\}^2 - 4\alpha^2 M^2\right]} \\
&\times \ln\left[\frac{\alpha\left(a^2 + e^2 + g^2\right)\cos\theta - M - \sqrt{M^2 - a^2 - e^2 - g^2}}{\alpha(a^2 + e^2 + g^2)\cos\theta - M + \sqrt{M^2 - a^2 - e^2 - g^2}}\right].
\end{aligned}
\tag{3.93}
$$

which is the solution for the outgoing particles for the massless case ($m = 0$). Similarly Using Eqs. (3.80) and (3.90) in (3.55), and then using the resulting equation in (3.46) we get a solution for incoming particles as

$$
\begin{aligned}
I_{\uparrow} \;=\;& -Et + J\phi + \int \frac{mB}{A\sqrt{\partial_r g\left(r_+,\theta\right)\left(r-r_+\right)}}dr + \frac{\left(-E+\Omega_H J + \frac{qer_+}{\left(r_+^2+a^2\right)}\right)}{\sqrt{\partial_r F\left(r_+,\theta\right)\partial_r g\left(r_+,\theta\right)}}\ln\left(r-r_+\right) \\
& - \frac{iqaer_+ + iJa^2}{2\left(r_+^2+a^2\right)\alpha\sqrt{M^2-a^2-e^2-g^2}}\ln\left(\frac{\alpha\cos\theta\left(a^2+e^2+g^2\right)-M-\sqrt{M^2-a^2-e^2-g^2}}{\alpha\cos\theta\left(a^2+e^2+g^2\right)-M+\sqrt{M^2-a^2-e^2-g^2}}\right) \\
& + \frac{iqg-iJ}{2\left[1-2\alpha M+\alpha^2\left(a^2+e^2+g^2\right)\right]}\ln\left(1-\cos\theta\right) \\
& + \frac{iqg+iJ}{2\left[1+2\alpha M+\alpha^2\left(a^2+e^2+g^2\right)\right]}\ln\left(1+\cos\theta\right) \\
& - \frac{iqg\left[1+\alpha^2\left(a^2+e^2+g^2\right)\right]-2\alpha MiJ}{2\left[1-2\alpha M+\alpha^2\left(a^2+e^2+g^2\right)\right]\left[1+2\alpha M+\alpha^2\left(a^2+e^2+g^2\right)\right]} \\
& \times\ln\left[1-2\alpha M\cos\theta+\alpha^2\left(a^2+e^2+g^2\right)\cos^2\theta\right] \\
& - \frac{\left(a^2+e^2+g^2\right)\left[-2iJ\alpha^2-2iJ\alpha^4\left(a^2+e^2+g^2\right)+2iqg\alpha^3M\right]+4iJ\alpha^2M^2-2iqg\alpha M}{4\alpha\sqrt{M^2-a^2-e^2-g^2}\left[\left\{1+\alpha^2\left(a^2+e^2+g^2\right)\right\}^2-4\alpha^2M^2\right]} \\
& \times\ln\left[\frac{\alpha\left(a^2+e^2+g^2\right)\cos\theta-M-\sqrt{M^2-a^2-e^2-g^2}}{\alpha\left(a^2+e^2+g^2\right)\cos\theta-M+\sqrt{M^2-a^2-e^2-g^2}}\right].
\end{aligned}
\tag{3.94}
$$

Now if we consider the massless case ($m = 0$) then only the third term of Eq. (3.94) will vanish and we get

$$\begin{aligned}
I_{\uparrow} &= -Et + J\phi + \frac{\left(-E + \Omega_H J + \frac{qer_+}{(r_+^2+a^2)}\right)}{\sqrt{\partial_r F(r_+,\theta)\,\partial_r g(r_+,\theta)}} \ln(r - r_+) \\
&\quad - \frac{iqaer_+ + iJa^2}{2\left(r_+^2+a^2\right)\alpha\sqrt{M^2-a^2-e^2-g^2}} \ln\left(\frac{\alpha\cos\theta\left(a^2+e^2+g^2\right) - M - \sqrt{M^2-a^2-e^2-g^2}}{\alpha\cos\theta\left(a^2+e^2+g^2\right) - M + \sqrt{M^2-a^2-e^2-g^2}}\right) \\
&\quad + \frac{iqg - iJ}{2\left[1 - 2\alpha M + \alpha^2\left(a^2+e^2+g^2\right)\right]} \ln(1-\cos\theta) \\
&\quad + \frac{iqg + iJ}{2\left[1 + 2\alpha M + \alpha^2\left(a^2+e^2+g^2\right)\right]} \ln(1+\cos\theta) \\
&\quad - \frac{iqg\left[1+\alpha^2\left(a^2+e^2+g^2\right)\right] - 2\alpha MiJ}{2\left[1 - 2\alpha M + \alpha^2\left(a^2+e^2+g^2\right)\right]\left[1 + 2\alpha M + \alpha^2\left(a^2+e^2+g^2\right)\right]} \\
&\quad \times \ln\left[1 - 2\alpha M\cos\theta + \alpha^2\left(a^2+e^2+g^2\right)\cos^2\theta\right] \\
&\quad - \frac{\left(a^2+e^2+g^2\right)\left[-2iJ\alpha^2 - 2iJ\alpha^4\left(a^2+e^2+g^2\right) + 2iqg\alpha^3 M\right] + 4iJ\alpha^2M^2 - 2iqg\alpha M}{4\alpha\sqrt{M^2-a^2-e^2-g^2}\left[\left\{1+\alpha^2\left(a^2+e^2+g^2\right)\right\}^2 - 4\alpha^2M^2\right]} \\
&\quad \times \ln\left[\frac{\alpha\left(a^2+e^2+g^2\right)\cos\theta - M - \sqrt{M^2-a^2-e^2-g^2}}{\alpha\left(a^2+e^2+g^2\right)\cos\theta - M + \sqrt{M^2-a^2-e^2-g^2}}\right]
\end{aligned} \tag{3.95}$$

3.5 Tunneling Probability at the Acceleration Horizon

Other than the outer and inner horizons $r_{\pm}$ we have acceleration horizon given by

$$r_{\alpha} = \pm\frac{1}{\alpha}. \tag{3.96}$$

Now we find the probability of tunnelling a particle accross the acceleration horizon using the same procedure as that for the outer horizon and inner horizon. At acceleration horizon there occur some changes in the probability of tunnelling and the leading steps. We shall use + sign in the acceleration horizon in the calculation. There occur changes in the angular velocity given by Eq. (3.18) at the acceleration horizon as

$$\Omega_{H\alpha} = \frac{a}{(r_{\alpha}^2 + a^2)} \tag{3.97}$$

where r_α is given by Eq. (3.96). Similarly Eqs. (3.47) and (3.48) at the acceleration horizon take the form

$$\partial_r g\left(r_\alpha,\theta\right)=-\frac{2\alpha\left(1-\alpha r_\alpha\cos\theta\right)^2\left[r_\alpha^2-2Mr_\alpha+\left(a^2+e^2+g^2\right)\right]}{\left(r_\alpha^2+a^2\cos^2\theta\right)},\tag{3.98}$$

$$\partial_r F\left(r_\alpha,\theta\right)=-\frac{2\alpha\left(r_\alpha^2+a^2\cos^2\theta\right)\left[r_\alpha^2-2Mr_\alpha+\left(a^2+e^2+g^2\right)\right]}{\left(1-\alpha r_\alpha\cos\theta\right)^2\left(r_\alpha^2+a^2\right)^2}.\tag{3.99}$$

Eqs. (3.51) to (3.54) take the form

$$-B\left[\frac{\left(-E+\Omega_{H\alpha}J+\frac{qer_\alpha}{\left(r_\alpha^2+a^2\right)}\right)}{\sqrt{\left(r-r_\alpha\right)\partial_r F\left(r_\alpha,\theta\right)}}+\sqrt{\left(r-r_\alpha\right)\partial_r g\left(r_\alpha,\theta\right)}\left(\partial_r W\right)\right]+mA=0,\tag{3.100}$$

$$-B\left[\sqrt{\frac{P\Omega^2\left(r_\alpha,\theta\right)}{\rho^2\left(r_\alpha,\theta\right)}}\partial_\theta W+\frac{i\rho\left(r_\alpha,\theta\right)\Omega\left(r_\alpha,\theta\right)}{\sqrt{\sin^2\theta P\left(r_\alpha^2+a^2\right)^2}}\left(J-q\left(\frac{aer_\alpha\sin^2\theta+g\left(r_\alpha^2+a^2\right)\cos\theta}{r_\alpha^2+a^2\cos^2\theta}\right)\right)\right]=0,\tag{3.101}$$

$$A\left[\frac{\left(-E+\Omega_{H\alpha}J+\frac{qer_\alpha}{\left(r_\alpha^2+a^2\right)}\right)}{\sqrt{\left(r-r_\alpha\right)\partial_r F\left(r_\alpha,\theta\right)}}-\sqrt{\left(r-r_\alpha\right)\partial_r g\left(r_\alpha,\theta\right)}\partial_r W\right]+Bm=0,\tag{3.102}$$

$$-A\left[\sqrt{\frac{P\Omega^2\left(r_\alpha,\theta\right)}{\rho^2\left(r_\alpha,\theta\right)}}\partial_\theta W+\frac{i\rho\left(r_\alpha,\theta\right)\Omega\left(r_\alpha,\theta\right)}{\sqrt{\sin^2\theta P\left(r_\alpha^2+a^2\right)^2}}\left(J-q\left(\frac{aer_\alpha\sin^2\theta+g\left(r_\alpha^2+a^2\right)\cos\theta}{r_\alpha^2+a^2\cos^2\theta}\right)\right)\right]=0.\tag{3.103}$$

Now in the massless case ($m=0$) Eqs. (3.58) and (3.60) take the form

$$R_+=\frac{\pi i\left(E-\Omega_{H\alpha}J-\frac{qer_\alpha}{\left(r_\alpha^2+a^2\right)}\right)\left(r_\alpha^2+a^2\right)}{2\alpha\left[r_\alpha^2-2Mr_\alpha+\left(a^2+e^2+g^2\right)\right]},\tag{3.104}$$

$$R_-=\frac{\pi i\left(E-\Omega_{H\alpha}J-\frac{qer_\alpha}{\left(r_\alpha^2+a^2\right)}\right)\left(r_\alpha^2+a^2\right)}{2\alpha\left[r_\alpha^2-2Mr_\alpha+\left(a^2+e^2+g^2\right)\right]}.\tag{3.105}$$

The probability of tunneling of a particle accross the horizon becomes

$$\Gamma = \exp\left[-2\pi \frac{\left(r_\alpha^2 + a^2\right)\left(E - \Omega_{H\alpha} J - \frac{qer_\alpha}{\left(r_\alpha^2 + a^2\right)}\right)}{\alpha\left[r_\alpha^2 - 2Mr_\alpha + \left(a^2 + e^2 + g^2\right)\right]}\right]. \tag{3.106}$$

For the massive case Eqs. (3.67) and (3.68) become

$$\frac{A}{B} = \frac{-\left(E - \Omega_{H\alpha} J - \frac{qer_\alpha}{\left(r_\alpha^2 + a^2\right)}\right) \pm \sqrt{\left(E - \Omega_{H\alpha} J - \frac{qer_\alpha}{\left(r_\alpha^2 + a^2\right)}\right)^2 + m^2\left(r - r_\alpha\right)\partial_r F\left(r_\alpha, \theta\right)}}{m\sqrt{\left(r - r_\alpha\right)\partial_r F\left(r_\alpha, \theta\right)}}, \tag{3.107}$$

$$\lim_{r \to r_\alpha} \frac{A}{B} = \begin{cases} 0 & \text{for upper sign,} \\ -\infty & \text{for lower sign.} \end{cases} \tag{3.108}$$

Eqs. (3.71) and (3.72) change to

$$R_+ = \frac{\pi i\left(E - \Omega_{H\alpha} J - \frac{qer_\alpha}{\left(r_\alpha^2 + a^2\right)}\right)\left(r_\alpha^2 + a^2\right)}{2\alpha\left[r_\alpha^2 - 2Mr_\alpha + \left(a^2 + e^2 + g^2\right)\right]}, \tag{3.109}$$

$$R_- = -\frac{\pi i\left(E - \Omega_{H\alpha} J - \frac{qer_\alpha}{\left(r_\alpha^2 + a^2\right)}\right)\left(r_\alpha^2 + a^2\right)}{2\alpha\left[r_\alpha^2 - 2Mr_\alpha + \left(a^2 + e^2 + g^2\right)\right]}. \tag{3.110}$$

The probability of tunneling of a particle for the massive case accross the horizon takes the form

$$\Gamma = \exp\left[-2\pi \frac{\left(r_\alpha^2 + a^2\right)\left(E - \Omega_{H\alpha} J - \frac{qer_\alpha}{\left(r_\alpha^2 + a^2\right)}\right)}{\alpha\left[r_\alpha^2 - 2Mr_\alpha + \left(a^2 + e^2 + g^2\right)\right]}\right]. \tag{3.111}$$

In the massive case the extra contributions also vanish at the acceleration horizon as in the case of outer horizon. From Eq. (3.106) or (3.111) we get for the Hawking temperature

$$T_H = \frac{\alpha\left[r_\alpha^2 - 2Mr_\alpha + \left(a^2 + e^2 + g^2\right)\right]}{2\pi\left(r_\alpha^2 + a^2\right)}.$$

Chapter 4

Quantum Tunneling of Scalar Particles

In this chapter we shall find the quantum tunneling for scalar particles using the Hamilton Jacobi equation as an ansatz. Klein-Gordon equation will be used for this purpose.

4.1 Tunneling Probability at the Outer Horizon

In this section we shall find the quantum tunneling of charged scalar particles from the accelerating and rotating charged black hole at the outer horizon $r = r_+ = M + \sqrt{M^2 - a^2 - e^2 - g^2}$ with metric given by Eq. (3.8). Klein-Gordon equation will be solved for this purpose which is given for the scalar field Φ as

$$g^{\mu\nu}\left(\partial_\mu - \frac{iq}{\hbar}A_\mu\right)\left(\partial_\nu - \frac{iq}{\hbar}A_\nu\right)\Phi - \frac{m^2}{\hbar^2}\Phi = 0, \tag{4.1}$$

where $\mu, \nu = (0, 1, 2, 3)$ which correspond to (t, r, θ, ϕ) and m is the mass of the scalar particle, q is its charge, $g^{\mu\nu}$ is the inverse of metric tensor and A_μ is the vector potential which is given by Eq. (3.19). Giving variation to μ, ν in Eq. (4.1) we obtain

$$\begin{aligned} 0 &= g^{tt}\left(\partial_t - \frac{iq}{\hbar}A_t\right)^2\Phi + g^{rr}\left(\partial_r - \frac{iq}{\hbar}A_r\right)^2\Phi + g^{\theta\theta}\left(\partial_\theta - \frac{iq}{\hbar}A_\theta\right)^2\Phi + g^{\phi\phi}\left(\partial_\phi - \frac{iq}{\hbar}A_\phi\right)^2\Phi \\ &\quad + 2g^{t\phi}\left(\partial_t - \frac{iq}{\hbar}A_t\right)\left(\partial_\phi - \frac{iq}{\hbar}A_\phi\right)\Phi - \frac{m^2}{\hbar^2}\Phi. \end{aligned} \tag{4.2}$$

In order to solve the Klein-Gordon equation (4.1) in the background of rotating and accelerating black holes we shall assume the ansatz of the form for the metric (3.8) as

$$\Phi\left(t,r,\theta,\phi\right)=\exp\left(\frac{i}{\hbar}I\left(t,r,\theta,\phi\right)+I_1\left(t,r,\theta,\phi\right)\right), \tag{4.3}$$

where I is the action. For the simpification of Eq. (4.2), consider

$$\left(\partial_t-\frac{iq}{\hbar}A_t\right)^2\Phi=\left(\partial_t-\frac{iq}{\hbar}A_t\right)\left(\partial_t-\frac{iq}{\hbar}A_t\right)\Phi.$$

Using ansatz (4.3) this becomes

$$\left(\partial_t-\frac{iq}{\hbar}A_t\right)^2\Phi=\left(\partial_t-\frac{iq}{\hbar}A_t\right)\left(\partial_t-\frac{iq}{\hbar}A_t\right)\exp\left[\frac{i}{\hbar}I+I_1\right].$$

After simplifying we get

$$\begin{aligned}\left(\partial_t-\frac{iq}{\hbar}A_t\right)^2\Phi &= \exp\left(\frac{i}{\hbar}I+I_1\right)\left[\left(\frac{i}{\hbar}\partial_tI+\partial_tI_1\right)^2+\left(\frac{i}{\hbar}\partial_{tt}I+\partial_{tt}I_1\right)-\frac{iqA_t}{\hbar}\left(\frac{i}{\hbar}\partial_tI+\partial_tI_1\right)\right.\\ &\left.-\frac{iq\partial_tA_t}{\hbar}-\frac{iqA_t}{\hbar}\left(\frac{i}{\hbar}\partial_tI+\partial_tI_1\right)+\left(\frac{iqA_t}{\hbar}\right)^2\right].\end{aligned} \tag{4.4}$$

Similarly we obtain for other terms

$$\begin{aligned}\left(\partial_r-\frac{iq}{\hbar}A_r\right)^2\Phi &= \exp\left(\frac{i}{\hbar}I+I_1\right)\left[\left(\frac{i}{\hbar}\partial_rI+\partial_rI_1\right)^2+\left(\frac{i}{\hbar}\partial_{rr}I+\partial_{rr}I_1\right)-\frac{iqA_r}{\hbar}\left(\frac{i}{\hbar}\partial_rI+\partial_rI_1\right)\right.\\ &\left.-\frac{iq\partial_rA_r}{\hbar}-\frac{iqA_r}{\hbar}\left(\frac{i}{\hbar}\partial_rI+\partial_rI_1\right)+\left(\frac{iqA_r}{\hbar}\right)^2\right],\end{aligned} \tag{4.5}$$

$$\begin{aligned}\left(\partial_\theta-\frac{iq}{\hbar}A_\theta\right)^2\Phi &= \exp\left(\frac{i}{\hbar}I+I_1\right)\left[\left(\frac{i}{\hbar}\partial_\theta I+\partial_\theta I_1\right)^2+\left(\frac{i}{\hbar}\partial_{\theta\theta}I+\partial_{\theta\theta}I_1\right)-\frac{iqA_\theta}{\hbar}\left(\frac{i}{\hbar}\partial_\theta I+\partial_\theta I_1\right)\right.\\ &\left.-\frac{iq\partial_\theta A_\theta}{\hbar}-\frac{iqA_\theta}{\hbar}\left(\frac{i}{\hbar}\partial_\theta I+\partial_\theta I_1\right)+\left(\frac{iqA_\theta}{\hbar}\right)^2\right],\end{aligned} \tag{4.6}$$

$$\left(\partial_\phi - \frac{iq}{\hbar}A_\phi\right)^2 \Phi = \exp\left(\frac{i}{\hbar}I + I_1\right)\left[\left(\frac{i}{\hbar}\partial_\phi I + \partial_\phi I_1\right)^2 + \left(\frac{i}{\hbar}\partial_{\phi\phi} I + \partial_{\phi\phi} I_1\right) - \frac{iqA_\phi}{\hbar}\left(\frac{i}{\hbar}\partial_\phi I + \partial_\phi I_1\right)\right.$$
$$\left. -\frac{iq\partial_\phi A_\phi}{\hbar} - \frac{iqA_\phi}{\hbar}\left(\frac{i}{\hbar}\partial_\phi I + \partial_\phi I_1\right) + \left(\frac{iqA_\phi}{\hbar}\right)^2\right], \quad (4.7)$$

$$\left(\partial_t - \frac{iq}{\hbar}A_t\right)\left(\partial_\phi - \frac{iq}{\hbar}A_\phi\right)\Phi = \exp\left(\frac{i}{\hbar}I + I_1\right)\left[\left(\frac{i}{\hbar}\partial_t I + \partial_t I_1\right)\left(\frac{i}{\hbar}\partial_\phi I + \partial_\phi I_1\right)\right.$$
$$+\left(\frac{i}{\hbar}\partial_{\phi t} I + \partial_{\phi t} I_1\right) - \frac{iqA_\phi}{\hbar}\left(\frac{i}{\hbar}\partial_t I + \partial_t I_1\right) - \frac{iq\partial_t A_\phi}{\hbar}$$
$$\left. -\frac{iqA_t}{\hbar}\left(\frac{i}{\hbar}\partial_\phi I + \partial_\phi I_1\right) + \left(\frac{iq}{\hbar}\right)^2 A_t A_\phi\right]. \quad (4.8)$$

Using Eqs. (4.4) to (4.8) in (4.2), dividing by the exponential term and multiplying by $\hbar^2$ and taking leading order terms to $\hbar$ we obtain

$$0 = -g^{tt}\left(\partial_t I - qA_t\right)^2 - g^{rr}\left(\partial_r I - qA_r\right)^2 - g^{\theta\theta}\left(\partial_\theta I - qA_\theta\right)^2 - g^{\phi\phi}\left(\partial_\phi I - qA_\phi\right)^2$$
$$-2g^{t\phi}\left(\partial_t I - qA_t\right)\left(\partial_\phi I - qA_\phi\right) - m^2.$$

The above equation can also be written as

$$g^{\mu\nu}\left(\partial_\mu I - qA_\mu\right)\left(\partial_\nu I - qA_\nu\right) + m^2 = 0. \quad (4.9)$$

Expanding Eq. (4.9) and simplifying we get [31]

$$0 = -\frac{\left(\partial_t I - qA_t\right)^2}{F(r,\theta)} + g(r,\theta)\left(\partial_r I\right)^2 - \frac{2H(r,\theta)}{F(r,\theta)K(r,\theta)}\left(\partial_t I - qA_t\right)\left(\partial_\phi I - qA_\phi\right)$$
$$+\frac{f(r,\theta)}{F(r,\theta)K(r,\theta)}\left(\partial_\phi I - qA_\phi\right)^2 + \frac{\left(\partial_\theta I\right)^2}{\rho^2(r,\theta)} + m^2. \quad (4.10)$$

We shall chose the following ansatz for the calculation of tunneling probability [31]

$$I = -Et + W(r) + J\phi. \quad (4.11)$$

Using Eq. (4.11) in Eq. (4.10) we get

$$\begin{aligned} 0 &= -\frac{(E+qA_t)^2}{F(r,\theta)} + g(r,\theta)\,W'^2(r) + \frac{2H(r,\theta)}{F(r,\theta)\,K(r,\theta)}(E+qA_t)(J-qA_\phi) \\ &\quad + \frac{f(r,\theta)}{F(r,\theta)\,K(r,\theta)}(J-qA_\phi)^2 + m^2. \end{aligned} \tag{4.12}$$

After some algebra this takes the form

$$\begin{aligned} 0 &= -\frac{1}{F(r,\theta)}\left[(E+qA_t) - \frac{H(r,\theta)}{K(r,\theta)}(J-qA_\phi)\right]^2 \\ &\quad + \left[\frac{H^2(r,\theta)}{F(r,\theta)\,K(r,\theta)} + \frac{f(r,\theta)}{F(r,\theta)}\right]\frac{(J-qA_\phi)^2}{K(r,\theta)} + g(r,\theta)\,W'^2(r) + m^2. \end{aligned} \tag{4.13}$$

Here we have added and subtracted $\frac{H^2(r,\theta)}{F(r,\theta)K^2(r,\theta)}(J-qA_\phi)^2$ to make the first term a complete square. Simplifying $\left[\frac{H^2(r,\theta)}{F(r,\theta)K(r,\theta)} + \frac{f(r,\theta)}{F(r,\theta)}\right]$ we get 1 so Eq. (4.13) becomes

$$-\frac{1}{F(r,\theta)}\left[(E+qA_t) - \frac{H(r,\theta)}{K(r,\theta)}(J-qA_\phi)\right]^2 + \frac{(J-qA_\phi)^2}{K(r,\theta)} + g(r,\theta)\,W'^2(r) + m^2 = 0. \tag{4.14}$$

Near the horizon $r = r_+$

$$F(r,\theta) = \frac{\left(r_+^2 + a^2\cos^2\theta\right)(2r_+ - 2M)\left(1-\alpha^2 r_+^2\right)}{(1-\alpha r_+\cos\theta)^2\left(r_+^2+a^2\right)^2}(r-r_+), \tag{4.15}$$

$$g(r,\theta) = \frac{(1-\alpha r_+\cos\theta)^2(2r_+ - 2M)\left(1-\alpha^2 r_+^2\right)}{\left(r_+^2 + a^2\cos^2\theta\right)}(r-r_+), \tag{4.16}$$

$$\Omega_H = \frac{H(r_+,\theta)}{K(r_+,\theta)} = \frac{a}{r_+^2+a^2}, \tag{4.17}$$

$$K(r_+,\theta) = \left(\frac{P\sin^2\theta\left(r_+^2+a^2\right)^2}{\rho^2(r_+,\theta)\,\Omega^2(r_+,\theta)}\right), \tag{4.18}$$

$$H(r_+,\theta) = \left(\frac{aP\sin^2\theta\left(r_+^2+a^2\right)}{\rho^2(r_+,\theta)\,\Omega^2(r_+,\theta)}\right). \tag{4.19}$$

We shall expand Eq. (4.14) near the horizon $r = r_+$ using the above values similarly as was done in the case of Dirac particles. Thus we get

$$\begin{aligned} 0 &= -\frac{(1-\alpha r_+\cos\theta)^2\left(r_+^2+a^2\right)^2}{2\left(r_+^2+a^2\cos^2\theta\right)(r_+-M)\left(1-\alpha^2 r_+^2\right)(r-r_+)}\left(E-\Omega_H J-\frac{qer_+}{\left(r_+^2+a^2\right)}\right)^2+\frac{(J-qA_\phi)^2}{K(r_+,\theta)} \\ &+\frac{2(1-\alpha r_+\cos\theta)^2(r_+-M)\left(1-\alpha^2 r_+^2\right)(r-r_+)}{\left(r_+^2+a^2\cos^2\theta\right)}W'^2(r)+m^2. \end{aligned} \tag{4.20}$$

Here we have also used $qA_t(r_+,\theta)+\Omega_H qA_\phi(r_+,\theta) = -\frac{qer_+}{\left(r_+^2+a^2\right)}$. Solving this equation for $W(r)$ we get

$$\begin{aligned} W'^2(r) &= \frac{\left(r_+^2+a^2\right)^2}{4(r_+-M)^2\left(1-\alpha^2 r_+^2\right)^2(r-r_+)^2}\left(E-\Omega_H J-\frac{qer_+}{\left(r_+^2+a^2\right)}\right)^2-\left(\frac{(J-qA_\phi)^2}{K(r_+,\theta)}+m^2\right) \\ &\times\frac{\left(r_+^2+a^2\cos^2\theta\right)}{2(1-\alpha r_+\cos\theta)^2(r_+-M)\left(1-\alpha^2 r_+^2\right)(r-r_+)}, \end{aligned} \tag{4.21}$$

or

$$W_\pm(r) = \pm\int\frac{\left(r_+^2+a^2\right)}{2(r_+-M)\left(1-\alpha^2 r_+^2\right)(r-r_+)}dr$$

$$\times\sqrt{\left(E-\Omega_H J-\frac{qer_+}{\left(r_+^2+a^2\right)}\right)^2-\frac{2\left(r_+^2+a^2\cos^2\theta\right)\left(1-\alpha^2 r_+^2\right)(r_+-M)(r-r_+)}{(1-\alpha r_+\cos\theta)^2}\left(\frac{(J-qA_\phi)^2}{K(r_+,\theta)}+m^2\right)}.$$

Here $r = r_+$ is the singularity so using the residue theory for integrating the above equation we get

$$W_\pm(r) = \pm\frac{\pi i\left(E-\Omega_H J-\frac{qer_+}{\left(r_+^2+a^2\right)}\right)\left(r_+^2+a^2\right)}{2(r_+-M)\left(1-\alpha^2 r_+^2\right)}. \tag{4.22}$$

or

$$\operatorname{Im} W_\pm(r) = \pm\frac{\pi\left(E-\Omega_H J-\frac{qer_+}{\left(r_+^2+a^2\right)}\right)\left(r_+^2+a^2\right)}{2(r_+-M)\left(1-\alpha^2 r_+^2\right)}. \tag{4.23}$$

So the tunneling probability of outgoing scalar particles come out to be

$$\Gamma = \exp\left[-4\operatorname{Im} W_+\right].$$

Using the value of $\operatorname{Im} W_+$ in the above equation we get

$$\Gamma = \exp\left[-\frac{2\pi\left(r_+^2+a^2\right)}{\left(r_+-M\right)\left(1-\alpha^2 r_+^2\right)}\left(E-\Omega_H J-\frac{qer_+}{\left(r_+^2+a^2\right)}\right)\right]. \tag{4.24}$$

Note that the tunneling probability of scalar particles is same as in the case of Dirac particles. Thus we recover the Hawking temperature.

4.2 Tunneling Probability at the Acceleration Horizon

In this section we shall find the tunneling probability at the acceleration horizon $r_\alpha = \frac{1}{\alpha}$. The calculations for the probability at the acceleration horizon proceeds in the same way as in the case of outer horizon $r = r_+$. At acceleration horizon $r_\alpha = \frac{1}{\alpha}$ Eq. (4.20) takes the form

$$\begin{aligned} 0 &= \frac{(1-\cos\theta)^2\left(r_\alpha^2+a^2\right)^2}{2\alpha\left(r_\alpha^2+a^2\cos^2\theta\right)\left[r_\alpha^2-2Mr_\alpha+\left(a^2+e^2+g^2\right)\right](r-r_\alpha)}\left(E-\Omega_{H\alpha}J-\frac{qer_\alpha}{\left(r_\alpha^2+a^2\right)}\right)^2 \\ &\quad -\frac{2\alpha(1-\cos\theta)^2\left[r_\alpha^2-2Mr_\alpha+\left(a^2+e^2+g^2\right)\right](r-r_\alpha)}{\left(r_\alpha^2+a^2\cos^2\theta\right)}W'^2(r) \\ &\quad +\frac{\left(J-qA_\phi\left(r_\alpha,\theta\right)\right)^2}{K\left(r_\alpha,\theta\right)}+m^2. \end{aligned} \tag{4.25}$$

Solving this equation for $W(r)$ we get

$$\begin{aligned} W'^2(r) &= \frac{\left(r_\alpha^2+a^2\right)^2}{4\alpha^2\left[r_\alpha^2-2Mr_\alpha+\left(a^2+e^2+g^2\right)\right]^2(r-r_\alpha)^2}\left(E-\Omega_{H\alpha}J-\frac{qer_\alpha}{\left(r_\alpha^2+a^2\right)}\right)^2 \\ &\quad +\frac{\left(r_\alpha^2+a^2\cos^2\theta\right)}{2\alpha(1-\cos\theta)^2\left[r_\alpha^2-2Mr_\alpha+\left(a^2+e^2+g^2\right)\right](r-r_\alpha)}\left(\frac{\left(J-qA_\phi\left(r_\alpha,\theta\right)\right)^2}{K\left(r_\alpha,\theta\right)}+m^2\right). \end{aligned}$$

or

$$W_\pm(r) = \pm\int\frac{\left(r_\alpha^2+a^2\right)}{2\alpha\left[r_\alpha^2-2Mr_\alpha+\left(a^2+e^2+g^2\right)\right](r-r_\alpha)}dr$$

$$\times\sqrt{\left(E-\Omega_{H\alpha}J-\frac{qer_\alpha}{\left(r_\alpha^2+a^2\right)}\right)^2+\frac{2\alpha\left(r_\alpha^2+a^2\cos^2\theta\right)\left[r_\alpha^2-2Mr_\alpha+\left(a^2+e^2+g^2\right)\right](r-r_\alpha)}{\left(r_\alpha^2+a^2\right)^2(1-\cos\theta)^2}\left(\frac{\left(J-qA_\phi\left(r_\alpha,\theta\right)\right)^2}{K\left(r_\alpha,\theta\right)}+m^2\right)}.$$

Here $r = r_\alpha$ is the singularity. So integrating the above equation using the residue theory we get

$$W_{\pm}(r) = \pm \frac{\pi i \left(r_{\alpha}^{2} + a^{2}\right)}{2\alpha\left[r_{\alpha}^{2} - 2Mr_{\alpha} + (a^{2} + e^{2} + g^{2})\right]} \left(E - \Omega_{H\alpha} J - \frac{qer_{\alpha}}{(r_{\alpha}^{2} + a^{2})}\right), \tag{4.26}$$

or

$$\operatorname{Im} W_{\pm}(r) = \pm \frac{\pi \left(r_{\alpha}^{2} + a^{2}\right)}{2\alpha\left[r_{\alpha}^{2} - 2Mr_{\alpha} + (a^{2} + e^{2} + g^{2})\right]} \left(E - \Omega_{H\alpha} J - \frac{qer_{\alpha}}{(r_{\alpha}^{2} + a^{2})}\right). \tag{4.27}$$

Now the tunneling probability of outgoing scalar particles is found by the following farmula

$$\Gamma = \exp\left[-4 \operatorname{Im} W_{+}\right]. \tag{4.28}$$

Using the value of $\operatorname{Im} W_{+}$ in the above equation we get the resulting tunneling probability of scalar particles as

$$\Gamma = \exp\left[-\frac{2\pi \left(r_{\alpha}^{2} + a^{2}\right)}{\alpha\left[r_{\alpha}^{2} - 2Mr_{\alpha} + (a^{2} + e^{2} + g^{2})\right]} \left(E - \Omega_{H\alpha} J - \frac{qer_{\alpha}}{(r_{\alpha}^{2} + a^{2})}\right)\right]. \tag{4.29}$$

Note that the tunneling probility of scalar particles is the same as that of the Dirac particles at the acceleration horizon. Thus we recover the Hawking temperature for the acceleration horizon.

Chapter 5

Conclusion

In this thesis, in the first chapter, we have given some basic definitions and explained the Dirac and Klein-Gordon equations which have been used in our work. We have explained Einstein field equations (EFE) for general relativity and some of its well known solutions that represent black holes. Also we have explained the general Plebański-Demiański metric. This metric contains a large family of solutions of EFE which can be derived from this metric after applying some transformations. Putting $\alpha = 0$ the general metric reduces to the Kerr-Newman-NUT- de Sitter solution. Further if $l = 0$ then it reduces to familiar forms of Kerr-Newman-de Sitter black hole spacetimes. If $\alpha = 0$ and the Kerr-like rotation vanishes, that is $a = 0$, then general metric reduces to the charged NUT-de Sitter spacetime. When $\alpha = l = g = \Lambda = 0$ then the Kerr-Newman metric is deduced. Further Schwarzschild metric is directly obtained if electric charge and rotation parameter vanish that is $e = 0 = a$.

In the second chapter we have explained quantum tunneling through the black hole horizon. Thirty years ago, Stephen Hawking demonstrated that black hole is not fully black but can emit particles in the form of pure thermal spectrum. The idea is that when a virtual pair particle is created just inside or outside the event horizon, the negative energy particle is absorbed by the black hole and positive energy particle escapes to infinity. Thus energy and hence radius of a black hole decreases. The particles move in time so that its action becomes complex and only the imaginary part of the action contributes to the tunneling probability. The tunneling probability of particles from inside to outside the horizon is related to the imaginary part of the action. This action can be calculated by two methods, the null geodesic method or the Hamilton-Jacobi ansatz. We have used the Hamilton-Jacobi ansatz for this purpose. In the Hamilton-Jacobi ansatz one choose the ansatz for the action considering the symmetries and Killing vectors of the black hole. This method can be extended beyond scalar particles emission to charged scalar, uncharged fermion and charged fermion particles. In 2000 Parikh and Wilczek [13] proposed a simple paradigm to study the tunneling effect. According to their observation when a particle tunnels out, the energy of a black hole will decrease and its radius will shrink. Following their work massive, massless, charged and uncharged particles have been investigated from different spacetimes. But all of them involved scalar particles emission. Infact black hole can emit all sorts of particles (spin and spinless) like a black body at the Hawking temperature. Very recently Kerner and Mann [31] brought forward an approach to study the fermions tunneling. They used the general Dirac equation instead of Newman-Penrose formalism to determine the action of the radiant spin particles which is mainly based on the idea of Padmanabhan et al [21]. And the tunneling rate from inside to outside the horizon is calculated by dividing the probability of the outgoing modes with incoming modes. In this chapter we have reviewd two papers. In the first paper Qiang Li and Yi-Wen Han have calculated the tunneling probability of charged and magnetized fermions from the Riessner-Nordström (RN) black hole with magnetic charges which is a static, non rotating black hole characterised by mass M and charge Q_h only. For this purpose they have used the Dirac equation which involves gamma matrices γ^{μ}, charge q_h and mass m of fermions and wave function Ψ. There are subtle technical issues involved with choosing an appropriate ansatz for the Dirac field consistent with the choice of γ matrices, otherwise this will lead to a breakdown in the method. Here two cases arises: spin up and spin down. Both the cases are similar apart from some changes in the sign. Solving the Dirac equation for the spin up case they obtained a matrix of order 4×1, solving this matrix equal to zero matrix of the same order one gets a set of four complicated equations. To solve these equations they have used the WKB approximation and taken only leading order terms in $\hbar$. Then they have carried out an ansatz for the action $I_{\uparrow}$ which appears in the wave function $\Psi_{\uparrow}$ considering the symmetries and Killing vectors of the spacetime. So the four equations become simpler to solve. Then in the massless or massive case tunneling probability has been found by dividing the probabilities of outgoing modes to incoming modes. From the resulting tunneling probability it is clear that this does not depend on the mass of fermions. In the second paper Kerner and Mann have calculated the tunneling probability of spin-1/2 fermion particles from the Kerr-Newman black hole. Solving the Dirac equaion they have obtained four equations and taken leading order terms in $\hbar$ using the WKB approximation. They have expanded these equations near the horizon and found the tunneling probability and Hawking temperature, which is same for both the massless and the massive cases.

The third and fourth chapters contains our original work. We have followed the method of Kerner and Mann and calculated the tunneling probability of fermion particles from the rotating and accelerating black holes. After calculating the horizons and angular velocity of the black hole we have solved the Dirac equation and obtained four equations. Then using the WKB approximation we have taken leading order terms in $\hbar$ thus these equations become simple. Then we have expanded these equations near the horizon. The ansatz for the action $I_\uparrow$ has been choosen seeing the symmetries of the spacetime and near the horizon we have further separate the action. Then finding the imaginary part of the action we obtained the probabilities of particles for outgoing and incoming modes. The resulting tunneling probability of particles from inside to outside the horizon has been obtained by dividing the probabilities of outgoing modes to incoming modes. Then from this the Hawking temperature hase been worked out. The resulting tunneling probability is same for both the massless and the massive cases. When we put $\alpha = 0$ then we obtain the tunneling probability of Kerr-Newman black hole which is non accelerating. This shows the correctness of our results. We have also calculated the tunneling probability of fermions at the acceleration horizon using the same procedure as we have used in the case of outer horizon. We have also found the explicit expression for the action $I_\uparrow$. In the fourth chapter we have calculated the tunneling probability of scalar particles from the rotating and accelerating black hole using the Klein-Gordon equation. In this case we obtain only one equation. Then using the WKB approximation, taking leading order terms in $\hbar$, we got a simplified equation. While choosing the ansatz for the action $I_\uparrow$ we considered the symmetries of spacetime and also we have used the corotating frame because of this the term depending on θ disappears. Then finally finding the imaginary part of the action tunneling probability has been found. Same procedure has been adopted to find the tunneling probability at the acceleration horizon. We have found that for scalar particles and fermion particles final results are same indicating that both the particles emit at the same rate.

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