Application of Wiener-Hopf Technique on Diffraction Problems

By

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QUAID-I-AZAM UNIVERSITY
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717

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A THESIS

SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN

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Supervised By

Prof. Dr. Muhammad Ayub

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Dedicated to

- **My Father (late),**
  Who sacrificed his today for our tomorrow

- **My Mother (late),**
  Who is and will always be with me with her prayers

- **My Elder Brother**
  Who is and has always been a driving force to me and I never feel alone especially after the sad demise of our parents

- **My Bhabi**
  Who cares and pray for me like her real brother

- **My Wife**
  Without her patience, understanding, support and all love, the completion of this thesis would have not been possible.

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Muhammad Ramzan

CERTIFICATE

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We accept this dissertation as conforming to the required standard

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Nomenclature

Symbol	Narration
$\widetilde{A}$, A, A_1, A_2	undetermined coefficients
a	step height
a_0, a_1, a_2	Taylor series coefficients
$\mathbf{B}$	magnetic induction
B_1, B_2	constants
b_0, b_1, b_2,	Taylor series coefficients
c	speed of sound
C_n	Fourier coefficients
C_1, C_2, C, $C_1^{/}$, $C_2^{/}$	constants
$D_+^*(\alpha)$	constant
$\mathbf{D}$	electric displacement
D_1, D_2, D_3	regular known functions
$\mathbf{E}$	electric field strength
E_x, E_y	tagential components of electric field
$f_1(t)$	periodic function
$F(\alpha)$	Fourier transform of $f(x)$
$f(y)$, $f_m(\alpha)$, $F_+^*(\alpha)$, $F_+(\alpha, 0, \omega)$	substitutional constants
$F_\pm(\alpha)$	split analytic function
$f(z)$, $f(\alpha)$	analytic function
$\mathcal{F}(z)$	fresenel function
$G(\alpha)$	Green function
$\hat{G}(\alpha)$	kernel

$\hat{G}_+(\alpha)$, $\hat{G}_-(\alpha)$	split kernel functions
$G_1(\alpha)$, $G_2(\alpha)$	unknown strip parameters
$\mathbf{H}$	magnetic field strength
$\hat{H}(\alpha)$, $\hat{H}_+(\alpha)$, $\hat{H}_-(\alpha)$	analytical function
I, I_1, I_2, I_3, I_4, I_5	unknown integrals
J_1, J_2	substitutional constants
$\mathbf{J}$	electric source density
k_1, k, k_{1xz}	wave number
$\mathbf{k}$	magnetoelectric material parameters
$K(\alpha)$	branch cut
$K_m(\alpha)$	m branch cuts
$\mathbf{K}$	magnetic field density
l	strip length
$L(\alpha)$	kernel
$L_+(\alpha)$, $L_-(\alpha)$	split kernel function
$M(\alpha)$	substitutional function
$M_\pi(z)$	Maliuzhinetz function
$N(\alpha)$	kernel
$N_+(\alpha)$, $N_-(\alpha)$	split kernel function
$N_1(\alpha)$, $N_2(\alpha)$	unknown strip parameters
$P_1(\alpha)$, $P_2(\alpha)$	unknown strip parameters
$\mathbf{Q}_{1t}$, $\mathbf{Q}_{2t}$	fields in xz-plane
$\mathbf{Q}_{1x}$, $\mathbf{Q}_{1y}$, $\mathbf{Q}_{1z}$,	scalar fields
$\mathbf{Q}_{2x}$, $\mathbf{Q}_{2y}$, $\mathbf{Q}_{2z}$	scalar fields
Q_1^2	substitutional constant

r	distance between the origin and the obsever
r_0	distance between the origin and the source
$R_+(\alpha),\ R_-(\alpha)$	split functions
$R_1,\ R_2$	constants
$S_+(\alpha),\ S_-(\alpha),\ S_+^*(\alpha)$	unknown strip parameters
S_-	substitutional constant
$\mathbf{S}_1,\ \mathbf{S}_2$	source densities
$T(\alpha),\ T_+(\alpha),\ T_-(\alpha)$	unknown strip parameter
t	time
$\check{T}$	substitutional constant
T_0	corresponding period
$u_0 f_1(t)$	velocity of the oscillating strip
$U_+(\alpha),\ U_-(\alpha)$	unknown strip parameters
$V_+(\alpha),\ V_-(\alpha)$	unknown strip parameters
$W_+(\alpha),\ W_-(\alpha)$	unknown strip parameters
$W_{m,n}$	Whittaker function
α	Fourier transform parameter
α_{BF}	bi-isotropic scalar
β_{BF}	bi-isotropic scalar
δ	Dirac delta
∇	del operator
ε	permitivity scalar
ζ	integral parameter

η_{1i}, η_{2i}	surface impedances
$\eta_{1_{BF}}$, $\eta_{2_{BF}}$	surface impedances in Beltrami fields
η_{1r}, η_{3r}	surface reactances
ϕ	diffracted field
$\phi^{*}(\alpha)$	subsitutional constant
$\phi(z)$	analytic function
ϕ_i	incident field
ϕ_t	total field
ϕ^{int}, Φ^{int}	interacted field
ϕ^{sep}, Φ^{sep}	separated field
$\overline{\Phi}$	Fourier transform in Beltrami field
$\gamma(\alpha)$, $\gamma_1(\alpha)$	branch cuts
$\kappa(\alpha)$	branch cut
$\kappa_{BF}(\alpha)$	branch cut in Beltrami field
χ, k	magnetoelectric material parameters
μ	permeability scalar
ω	angular frequency
ω_0	non zero fundamental frequency
w	temporal Fourier transform parameter
Ω, Ω_1	dimensionless parameter
ρ	distance from the obsever to the origin
ρ_0	distance from the source to the origin
Ψ_1^r	reflected field
Ψ_t	total field
Ψ^i	incident field

$\Psi_1,\ \Psi_{11}^d,\ \Psi_{12}^d$	scattered fields
ψ	temporal Fourier transform of ϕ
$\overline{\psi}$	spatial Fourier transform of ψ
$\overline{\Psi}_t(in\ 2D),\ \widetilde{\Psi}_t(in\ \ 3D)$	Fourier transform of Ψ_t
$\overline{\Psi}_1(in\ 2D),\ \widetilde{\Psi}_1(in\ 3D)$	Fourier transform of Ψ_1
$\overline{\Psi}_2(in\ 2D),\ \widetilde{\Psi}_2(in\ 3D)$	Fourier transform of Ψ_2
$\overline{\Psi}^i(in\ 2D),\ \widetilde{\Psi}^i(in\ 3D)$	Fourier transform of Ψ^i
$\overline{\Psi}_1^r$	Fourier transform of Ψ_1^r
$\overline{\Psi}_0$	Fourier transform of $(\Psi^i + \Psi_1^r)$
$\overline{\Psi}_t(in\ 2D),\ \widetilde{\Psi}_t(in\ 3D)$	Fourier transform of Ψ_t
$\overline{\Psi}_1(in\ 2D),\ \widetilde{\Psi}_1(in\ 3D)$	Fourier transform of Ψ_1
$\overline{\Psi}_2(in\ 2D), \widetilde{\Psi}_2(in\ 3D)$	Fourier transform of Ψ_2
$\overline{\Psi}^i(in\ 2D),\ \widetilde{\Psi}^i(in\ 3D)$	Fourier transform of Ψ^i
$\overline{\Psi}_1^r$	Fourier transform of Ψ_1^r
$\psi_+(\alpha),\ \psi_-(\alpha)$	unknown functions
$\Psi^*(\alpha)$	subtitutional constant
σ	real part of complex number α
τ	imaginary part of the complex number α

List of Figures

Fig No.	Caption	PageNo.

Abstract

The Wiener-Hopf technique (WH) provides a significant extension of the large variety of problems that can be solved by Fourier, Laplace and Mellin integral transforms. The WH technique leads to an approach for considering the diffraction of waves by half planes, step discontinuities and strips of numerous nature in a variety of mediums.

The aim of the present thesis is to address the new horizons in the field of diffraction. This includes the diffraction of a magnetic line source by an impedance step is a problem of unique nature where the half planes and step are characterized by different surface impedances. The problem is solved using Wiener Hopf technique and Fourier transform. The scattered field in the far zone is determined by the method of steepest descent. Graphical results for the solution has also been presented.

A delicate extension to the above mentioned problem discusses diffraction of a line source and a point source by a reactive step with two half planes joined by a step are of different reactances. Using the Fourier transform, the diffraction problem is first reduced to modified Wiener-Hopf equation of second kind whose solution contains infinitely many constants satisfying an infinite system of linear equations. Numerical solution of this system is obtained for various values of surface impedances and the height of the step, from which the effects of these parameters on the diffraction phenomenon are studied.

The next problem portraits diffraction of a plane acoustic wave from a rigid oscillating half plane to an oscillating rigid strip. The significance of the present analysis is that it recovered the results when a strip is widened to a half plane both mathematically and graphically.

Continuing this idea, a note on a plane wave diffraction by strip in Beltrami field highlights a major shortcoming and respective corrective measure. This problem is well supported by numerical discussion.

Preface

The Wiener-Hopf technique has been extensively used with a variety of applications in different research areas including diffraction problems. In available literature, one can find thousands and thousands problems that have been solved using Wiener-Hopf technique. Applications of Wiener-Hopf technique include noise reduction, mathematical finance, viscous problems and of course diffraction/scattering of waves.

The aim of the present thesis is to identify the new aspects in the field of diffraction. One of these is the introduction of a line source and point source diffraction from an impedance/reactive step, which have not been discussed in the literature prior to these considerations. Step discontinuities is an important topic in diffraction theory and is relevant to many engineering applications. Consideration of line source and point source is an important in the sense that the solution of line/point source problems are regarded as fundamental solutions of the problems. These are better substitutes for plane waves because a plane wave is considered to be coming from infinitely far away whereas for line/point sources we have a known position, *i.e.*, (x_0, y_0) and (x_0, y_0, z_0), respectively.

Scattering of waves by strip is an important and interesting topic both in acoustics and electromagnetics. It has attracted the attention of many researchers as a result of which abundant amount of literature is available on these topics. A variety of methods consisting of analytical, numerical and computational approaches have been adopted by various authors to study the scattering of waves by strips satisfying variety of boundary conditions.

With all the facts highlighted above this thesis runs as follows:

In chapter one, we discuss the history and literature survey. It also includes brief introduction to all the chapters. Chapter two includes some important preliminaries regarding Wiener-Hopf technique and other relevant methods which are used in the subsequent chapters.

Chapter three discusses the diffraction of a magnetic line source by an impedance step joined by two half planes in the case where the half planes and step are characterized by different surface impedances. The problem is solved using Wiener Hopf technique and Fourier transform. The scattered field in the far zone is determined by the method of steepest descent. Graphical results for the solution has also been presented. It is observed that if the source is shifted to a large distance these results differ from those of [55] by a multiplicative factor to the part of the scattered field containing the effects of incident and reflected waves. We have extended the problem of plane wave scattering [55] to the problem of scattering due to a magnetic line source situated at (x_0, y_0) because the line sources are considered as better substitute than the plane waves. It is perhaps the first attempt to look at the line source geometry with a step discontinuity. The introduction of line source changes the incident field and the method of solution requires a careful analysis in calculating the diffracted field. Using the Fourier transform, the diffraction problem is first reduced to modified Wiener-Hopf equation of second kind whose solution contains infinitely many constants satisfying an infinite system of linear equations. Numerical solution of this system is obtained for various values of surface impedances and the height of the step, from which the effects of these parameters on the diffraction phenomenon are studied. The possible excitation of the surface waves on the impedance surfaces can be neglected as the observation point is far from the surface while using the steepest descent method and the diffracted field dominates [72]. The contents of this chapter have been published in **IEEE Transactions on Antennas and Propagation. 57** (4), 1289 – 1293, 2009.

Chapter four presented the diffraction of a line and a point source by a reactive step joined by two half planes where the each half plane and step are characterized by different surface reactances have been studied. The problem is solved by using Wiener Hopf technique and the Fourier transform. The scattered field in the far zone is determined by the method of steepest descent. Graphical results for the line source are also presented. It is observed that if the source is shifted to a large distance the results of the line source differ from those of [57] by a multiplicative factor to the part of the scattered field containing the effects of incident and reflected waves. Subsequently, the point source diffraction is examined using the results obtained for a line source diffraction. These observations have been published in **Journal of Modern Optics. 56** (7), 893 – 902, 2009.

Chapter five extends the problem of diffraction of a plane acoustic wave from a rigid oscillating half plane to an oscillating rigid strip. We have studied a new aspect in the diffraction theory, i.e., to consider an oscillating strip instead of a static strip and go a step further to understand the diffraction phenomenon from the oscillating strip. Oscillating nature of the strip is a unique kind of characteristic that has not been discussed in the literature formerly. Another feature of the work presented is to give a new idea that a strip reduces to a half plane as the strip length approaches to infinity. This is discussed in detail both mathematically and numerically. The problem is solved by using the temporal and spatial integral transform and the Wiener-Hopf technique. The scattered field in the far zone is determined by the method of steepest descent. The significance of the present analysis is that it recovered the results when a strip is widened to a half plane. Graphical results for the diffraction problem have also been presented. The findings of this chapter have been published in **Applied Mathematics and Computation. 214** , 201 – 209, 2009.

Continuing this idea that it reduces the results as the strip widened to a half plane, chapter six identifies a major short coming [104] which does not yield the right result as the strip reduces to a half plane. There was indeed a serious flaw and accordingly corrective measures were taken to solve it. Half plane [84] results are obtained both mathematically and graphically. The problem was solved by using the Wiener-Hopf technique and Fourier transform. The scattered field in the far zone was determined by the method of steepest descent. The significance of present analysis was that it recovered the results when a strip was widened into a half plane. These conclusions have been published in **Optics Express 16** (17), 13203 – 13217, 2008.

Contents

Chapter 1

Introduction

One of the most celebrated scientists of yore, Ibn-al-Haitham's claim to fame rests on his analysis of scattering of light by faceted objects during the 10th century A.D. Another renowned scientist Poincare followed suit through his pioneering work on scattering of sound. One of Poincare's [1] contemporaries, Sommerfeld [2] made his contributions towards scattering of electromagnetic waves. None of the above analytical works regarding scattering of light, sound and electromagnetic waves for some of the irregular geometric shapes was actually conclusive nor could this preliminary research lead to some mathematical solutions. Nevertheless, half plane, strip and wedge were among some of the regular geometric shapes for which considerable progress was achieved in finding definite solutions to the mathematical problems.

Sommerfeld's [2] preliminary research led to the solution of diffraction of plane waves from a half plane by exploiting image waves. These elucidations hold ground in the far field only and become unbounded at the incident or reflected shadow boundaries. Building on this ground breaking work, a number of other scientists came forward with their own exploratory effort on diffraction of electromagnetic and sound waves by a half plane. MacDonald [3] and Carslaw [4] are especially noteworthy for obtaining the diffraction of a line source and a point source field by a perfect half plane. Clemmow [5] and Senior

[6] were able to reach mathematical solutions vis-à-vis Fresnel functions bounded at the shadow boundaries. A uniform geometrical theory of diffraction was used later by Pathak and Kouyoumjian [7], thus venturing into the realm of improved solutions for the half plane with ideal boundaries. Bowman et al. [8] and Pierce [9] was successful in putting forward some very useful diffraction formulae in the next few years.

Williams [10] was the next in line with research on diffraction of waves by half planes with non-ideal boundary conditions and his plausible solution regarding surfaces of the half plane with identical point reacting impedance in the guise of an indefinite product has been widely eulogized. A closed form solution for the diffraction of a plane wave by a rigid-soft half plane was the achievement of Rawlins [11]. Subsequently, with the introduction of new physical applications, the scattering by half plane surfaces with more complicated boundaries was studied, e.g., impedance surfaces found application as absorbent liners in aero-engine exhausts. Wiener-Hopf (WH) technique was widely used for solution of half plane problems.

Two and three dimensional diffraction problems can easily be sorted out by employing the WH technique. Among the different feasible approaches towards the reduction of physical dimensions to a WH problem, the "Jones' method" proves the most uncomplicated and plausible option. Books compiled by Jones [12] and Noble [13] give an exhaustive description of application of Jones' method. Wickham [14], Kuiken [15], Shaw [16], Davis [17] and Soward's [18] ensuing work betrays different applications of WH technique to problems beyond the sphere of acoustics and electromagnetism. Their work quintessentially includes some innovative modifications of the old classical method [19] for handling problems with axial symmetry. The diffraction by arrays of parallel plates and diffraction by a plate with varying absorbing conditions on either face, as discussed in the scattering theory is worth reading for its physical and mathematical value. Notwithstanding some exceptional cases namely Heins [20, 21], Levine and Schwinger [22], Carlson and Heins [23] and Rawlins [24], these diffraction problems lead to matrix WH equations. However, no general method of solution of such equations has been evolved to date.

In order to solve a WH functional equation, disintegrating the kernel into a product of functions, one analytic and of algebraic nature in the upper half of the complex plane and the other analytical and of algebraic origin in the lower half plane, is mandatory. In generic terms, it is a recognized function of a complicated variable with a host of secluded singularities denoting various fundamental physical processes. In case of scalar terms, it may be possible to achieve through Cauchy's integral theorem. However, for a matrix kernel, the expansion of this technique possibly covers only a limited number of matrices. Gigantic efforts have been made in the not-so-distant past to cover more and more matrices and to discover new constructive methodologies for calculating matrix disintegrations, especially vis-à-vis specific physical problems. These specific problems were spelled out by Rawlins [11], Hurd and Prezezdziecki [25], Prezezdziecki and Hurd [26], Abrahams [27], Hurd [28], Daniele [29], Kharapkov [30] and Jones [31] in their work. Extension of these techniques and other related research can be found in Rawlins and Williams [32], Williams [33] and Rawlins [34]. The nonfigurative theory of matrix factorization in the aforementioned context has been the centre of discussion in the aftermath of Göhberg and Krein [35] theorem especially by Speck [36], Meister [37] and Göhberg and Kashock [38].

The Wiener-Hopf technique is applicable to a wide variety of applications in research areas, including the diffraction of acoustics, elastic and electromagnetic waves, crystal growth [39], fracture mechanics, flow problems [40], diffusion models [41] , geophysical applications [42] and mathematical finance. In science literature one can easily find the many thousands of articles that have employed Wiener-Hopf technique.

Antipov and Willis [43] highlight constant crack growth in a viscoelastic material, with emphasis on the field around the crack tip. Wiener-Hopf technique has proved to be an immensely useful tool, in general, in solid mechanics and particularly, in fracture mechanics. This is because the field quantities are strongly dominated by points of rapid or instantaneous change in boundary conditions, in the vicinity of which Wiener-Hopf or related techniques are ideally suited. This point of view is reinforced in the article by Norris and Abrahams [44] in which crack-growth is again studied. In this case the crack growth model is somewhat simpler but the solution method employs matched asymptotic expansions as well as the Wiener-Hopf technique in order to examine the stability of running waves along the propagating crack tip.

Greens' et al. [45] addressed an important problem in mathematical finance, namely the pricing of discretely monitored barrier options. This article is indicative of the substantial interest in the Wiener-Hopf technique in this new and burgeoning area of applied mathematics. Aero-acoustics is another topic in which researchers have made excellent use of the Wiener-Hopf technique over some three or more decades.

Diffraction theory is another area in which the field variables are dominated by points where the geometry or boundary conditions change abruptly, and this may perhaps be the reason why it has found the most application for the Wiener-Hopf technique. Computational methods to extract the very small sound field from a dominant background flow field are fraught with difficulties, and so analytical methods have proved extremely useful [46].

Although the scattering/diffraction of line/point sources by half planes of different nature are the problems of routine nature. Numerous past investigations have been devoted to the study of classical problems of line source and point source diffractions of electromagnetic and acoustic waves by various types of half planes. To name a few only, e.g., the diffraction by a step in a perfectly conducting plane constitutes a canonical problem for the GTD (geometrical theory of diffraction) analysis of scattering by metallic tapes on panelled compact range reflectors [47] and the line source diffraction of electromagnetic waves by a perfectly conducting half plane was investigated by Jones' [48]. Later on Jones

[49] considered the problem of line source diffraction of acoustic waves by a hard half plane attached to a wake in still air as well as when the medium is convective. Rawlins [50] then considered the line source diffraction of acoustic waves by an absorbing barrier , line source diffraction by an acoustically penetrable or an electromagnetically dielectric half plane whose width is small as compared to the incident wave length [51] and line source diffraction of sound waves by an absorbent semi-infinite plane such that the two faces of half plane have different impedances [24]. Recently Ahmad [52] considered the line source diffraction of acoustic waves by an absorbing half plane using Myres' condition and Ayub et al. [53] have considered the line source and point source scattering of acoustic waves by the junction of transmissive and soft-hard half planes.

The scattering of surface waves by junction of two semi-infinite planes joined together by a step was first introduced by Johansen [54]. Later, Büyükaksoy and Birbir [55, 56] treated the same geometry in more general case of plane wave incidence and when the material properties of the half planes and the step are simulated by constant but different surface impedances which was important for predicting the scattering caused by an abrupt change in material as well as in the geometrical properties of a surface. Büyükaksoy extended [55, 56] to a case where the two half planes with different surface impedances are joined by a reactive step [57]. A similar work is done [59] where the half plane is kept conducting.

The second and important aspect of present work is to discuss the diffraction from a strip. Diffraction by strips of diverse nature is also considered as an important task that have been discussed in the literature with a variety of aspects. The scattering of sound and electromagnetic waves has been studied extensively since the half plane problems were investigated by Poincare [1] and Sommerfeld [2]. The Wiener-Hopf (WH) technique [13] proves to be a powerful tool to tackle, not only, the problems of diffraction by a single half plane but it may further be extended to the case of parallel half planes. However, there were problems in dealing with other configurations and mixed boundary value problems appearing in the diffraction theory. Diffraction from a strip is a well known studied phenomenon in the diffraction theory and is applicable to a variety of physical problems. Many scientists worked on diffraction problem related to strip geometry. Various methods of solution have been given in literature, e.g., Morse and Rubenstein [60] studied the problem of diffraction of acoustic waves from strip/slit using the method of separation of variables, some authors [61 − 66] followed Noble's approach [13], whereas some of them [67 − 70] adopted the method of successive approximations to study the diffraction from a strip, Castro and Kapanadze [71] employed the theory of Bessel potential spaces and Imran et al. [72] used the Kobayashi's potential method to study the diffraction of acoustic/electromagnetic waves by a strip.

Another aspect in the diffraction by strips is to discuss it in context to Beltrami fields. Beltrami flows were first introduced in the late 19th century [75]. There was no significant work on Beltrami flows for next 60 years. However, in 1950s and onwards, it gained wide application in fluid mechanics and other related areas. Chandrasekhar [76], reintroduced Beltrami flows and worked on force free magnetic fields. Lakhtakia [77] compiled a catalogue on contemporary works.

A Beltrami field is proportional to its own curl everywhere in a source-free region and can be either left-handed or right-handed. For the analysis of time-harmonic electromagnetic fields in isotropic chiral and bi-isotropic media, Bohren [78] was the pioneer and his work was enhanced by Lakhtakia [79]. Lakhtakia [80], and Lakhtakia and Weiglhofer [81] worked on the application of Beltrami field to time dependent electromagnetic field. On chiral wedges, Fisanov [82] and Przezdziecki [83] did exceptional job. Asghar and Lakhtakia [84] showed that the concept of Beltrami fields can be exploited to calculate the diffraction of only one scalar field and the rest can be obtained thereof.

A Beltrami magnetostatic field exerts no Lorentz force on an electrically charged particle, and for this reason the concept has been extensively used in astrophysics as well as magnetohydrodynamics [85, 86]. Beltrami fields also occur as the circularly polarized plane waves in electromagnetic theory [87]. Although circularly polarized plane waves in free space and natural, optically active media [88, 89] have been known since the time of Fresenel, their theoretical value is best expressed in bi-isotropic media [90 – 95]. In recent years, propagation of plane waves with negative phase velocity and its related applications in isotropic chiral materials can be found in [96 – 99].

Chapter 2

Preliminaries

In this chapter some mathematical preliminaries, used in the subsequent chapters, are presented. These include Decomposition theorem, Factorization theorem, Fourier transform, Wiener-Hopf technique and the Jones' method. The Steepest Descent method for the asymptotic analysis of certain integrals appearing in the diffraction theory is also discussed. Basics of Bi-isotropic medium and relevant consitutive relations are discussed as well.

2.1 Decomposition Theorem[13]

Let $f(\alpha)$ be an analytic function of $\alpha = \sigma + i\tau$ and regular in the strip $\tau_- < \tau < \tau_+$. Within this strip, $|f(\alpha)| \to 0$ uniformly as $|\sigma| \to \infty$. Then $f(\alpha)$ can be decomposed such that

$$f(\alpha) = f_+(\alpha) + f_-(\alpha), \tag{2.1.1}$$

where

$$f_+(\alpha) = \frac{1}{2\pi i} \int_{-\infty+ic}^{\infty+ic} \frac{f(\xi)}{\xi - \alpha} d\xi, \qquad \tau_- < c < \tau < \tau_+ \tag{2.1.2}$$

Fourier transform technique can also be used to solve problems involving semi-infinite domain.

2.3.1 Fourier Transform

The Fourier transform of a function $f(x)$ which is bounded in the given domain is defined for all x as follows

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x} dx, \tag{2.3.1}$$

provided that the above integral exists where α is a Fourier transform parameter and is usually taken to be real but in Wiener-Hopf technique it is taken as complex, *i.e.*, $\alpha = \sigma + i\tau$, $F(\alpha)$ refers to the Fourier transform of $f(x)$.

The most important relationship connecting $F(\alpha)$ and $f(x)$ is provided by inverse theorem which is stated as

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) e^{-i\alpha x} d\alpha. \tag{2.3.2}$$

The Fourier transform integral in above equation is frequently expressed as a sum of the two integrals each defined in semi-infinite ranges as follows

$$F(\alpha) = F_{+}(\alpha) + F_{-}(\alpha), \tag{2.3.3}$$

where

$$F_{-}(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{0} f(x)\, e^{i\alpha x} dx, \tag{2.3.4}$$

and

$$F_{+}(\alpha) = \frac{1}{2\pi} \int_{0}^{\infty} f(x)\, e^{i\alpha x} dx, \tag{2.3.5}$$

depending upon the behavior of $f(x)$ as $x \to \infty$ and $x \to -\infty$. It is possible that one of the integral exists while the other will vanish.

2.4 The Wiener-Hopf Technique[13]

This technique was first employed in a joint study by N. Wiener and E. Hopf [13] to solve singular integral equations. An important modification was done by Jones' [48] for application of this method to boundary value problems.

One of the remarkable features of the mathematical description of natural phenomena by means of partial differential equations is the comparative case with which solution can be obtained by certain geometrical shapes such as circle and infinite strips, by the method of separation of variables. In contrast considerable difficulty is usually encountered in finding solutions for shapes not covered by the method of separation of variables. The Wiener-Hopf method provides a significant extension of the range of problems that can be solved by Fourier, Laplace and Mellin transforms.

2.4.1 General Scheme Of Wiener Hopf Technique

Wiener-Hopf (WH) technique is used for solving certain integral equations and various boundary value problems of mathematical physics by means of integral transformation. In this technique, we require to determine the unknown function $\psi_+(\alpha)$ and $\psi_-(\alpha)$ of a complex variable α from the equation given below. The function $\psi_+(\alpha)$ and $\psi_-(\alpha)$ are analytic, respectively, in the half planes Im $\alpha > \tau_-$ and Im $\alpha < \tau_+$ tend to zero as $|\alpha| \longrightarrow \infty$ in both domains of analyticity and satisfy in the strip $\tau_- < \tau < \tau_+$,

$$D_1(\alpha)\psi_+(\alpha) + D_2(\alpha)\psi_-(\alpha) + D_3(\alpha) = 0, \tag{2.4.1}$$

where the functions $D_1(\alpha)$, $D_2(\alpha)$ and $D_3(\alpha)$ are known functions regular in the strip $\tau_- < \tau < \tau_+$ and $D_1(\alpha)$ and $D_2(\alpha)$ are non zero in the strip. The basic idea in this technique for solution of this equation is to substitute

$$\frac{D_1(\alpha)}{D_2(\alpha)} = \frac{T_+(\alpha)}{T_-(\alpha)}, \tag{2.4.2}$$

where the function $T_+(\alpha)$ and $T_-(\alpha)$ are analytic and different from zero, respectively, in $\tau > \tau_-$ and $\tau < \tau_+$. By using Eqs.(2.4.1) and (2.4.2), we obtain

$$T_+(\alpha)\psi_+(\alpha) + T_-(\alpha)\psi_-(\alpha) + T_-(\alpha)\frac{D_3(\alpha)}{D_2(\alpha)} = 0. \tag{2.4.3}$$

The last term of the above equation can be written as

$$T_-(\alpha)\frac{D_3(\alpha)}{D_2(\alpha)} = D_+(\alpha) + D_-(\alpha), \tag{2.4.4}$$

where the functions $D_+(\alpha)$ and $D_-(\alpha)$ are analytic in the half planes $\tau > \tau_-$ and $\tau < \tau_+$. Thus Eq.(2.4.3) takes the form

$$T_+(\alpha)\psi_+(\alpha) + D_+(\alpha) = -T_-(\alpha)\psi_-(\alpha) - D_-(\alpha). \tag{2.4.5}$$

We observe that LHS of Eq. (2.4.5) is regular in $\tau > \tau_-$ while the RHS is regular in $\tau < \tau_+$. Thus, both sides are equal to a certain integral function (polynomial) $J(\alpha)$ in the strip $\tau_- < \tau < \tau_+$

$$J(\alpha) = T_+(\alpha)\psi_+(\alpha) + D_+(\alpha) = -T_-(\alpha)\psi_-(\alpha) - D_-(\alpha). \tag{2.4.6}$$

By using analytic continuation, we can determine $J(\alpha)$ which is regular in the whole complex $\alpha - plane$. Let

$$|\, T_+(\alpha)\,\psi_+(\alpha) + D_+(\alpha)\,| < |\,\alpha\,|^p \; as \;\; \alpha \longrightarrow \infty, \; \tau > \tau_-, \tag{2.4.7}$$

$$|\, -T_-(\alpha)\,\psi_-(\alpha) - D_-(\alpha)\,| < |\,\alpha\,|^p \; as \;\; \alpha \longrightarrow \infty, \; \tau < \tau_+. \tag{2.4.8}$$

The extended Liouville's theorem states that :

"If $f(\alpha)$ is an integral function such that $|f(\alpha)| \leq M\ |\alpha|^p$ where M, p are constants, then $f(\alpha)$ is a polynomial of degree less than or equal to $[p]$, where $[p]$ is the integral part of p". Thus, by extended Liouville's theorem $J(\alpha)$ is a polynomial which can be determined from the given edge conditions of the problem.

2.5 Jones' method[13]

In Jones' method, Fourier transform is applied directly to the wave equation and that all boundary conditions are applied in the transformed domain. This is in contrast to the integral equation formulation, in which the boundary conditions are applied before the Fourier transform is taken.

Jones' method is similar and more straight forward than the integral equation formulation. Let us define Fourier transform of a certain function $u(x, y)$ as:

$$\Psi_1(\alpha, y) = \Psi_{1+}(\alpha, y) + \Psi_{1-}(\alpha, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, y)\, e^{i\alpha x} dx, \tag{2.5.1}$$

where

$$\Psi_{1+}(\alpha, y) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} u(x, y)\, e^{i\alpha x} dx, \tag{2.5.2}$$

$$\Psi_{1-}(\alpha, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} u(x, y)\, e^{i\alpha x} dx. \tag{2.5.3}$$

Let us consider the two dimensional steady state wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0, \tag{2.5.4}$$

where k has a positive imaginary part, i.e., $k = k_1 + ik_2$, $k_1, k_2 > 0$. Applying Fourier transform on Eq. (2.5.4), we obtain

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{i\alpha x} dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial y^2} e^{i\alpha x} dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k^2 u e^{i\alpha x} dx, \tag{2.5.4(a)}$$

$$(-i\alpha)\Psi_1^2(\alpha, y) + \frac{d_1^2 \Psi(\alpha, y)}{dy^2} + k\Psi_1^2(\alpha, y) = 0, \tag{2.5.4(b)}$$

$$\frac{d^2 \Psi_1(\alpha, y)}{dy^2} - \gamma^2 \Psi_1(\alpha, y) = 0, \tag{2.5.5}$$

with

$$\gamma = \left(\alpha^2 - k^2\right)^{1/2}. \tag{2.5.6}$$

The Eq. (2.5.5) has the solution

$$\Psi_1(\alpha, y) = \begin{cases} A_1(\alpha) e^{-\gamma y} + B_1(\alpha) e^{\gamma y} & y \geq 0 \\ A_2(\alpha) e^{-\gamma y} + B_2(\alpha) e^{\gamma y} & y \leq 0, \end{cases} \tag{2.5.6(a)}$$

where A_1, B_1, A_2 and B_2 are functions of α only. In Eq. (2.5.5), the real part of γ is always posivitive in $-k_2 < \tau < k_2$ and therefore in Eq. (2.5.6(a)), we must take $B_1 = A_2 = 0$ to satisfy the radiation conditions. Since, $\frac{\partial \Psi_1}{\partial y}$ is continuous across $y = 0$. Hence, $\frac{d\Psi_1}{dy}$ is continuous across $y = 0$ and we write

$$A_1(\alpha) = -B_2(\alpha) = A(\alpha) \qquad \text{say} \tag{25.6(b)}$$

Hence,

$$\Psi_1(\alpha, y) = \begin{cases} A(\alpha)\, e^{-\gamma y} & y \geq 0 \\ -A(\alpha)\, e^{\gamma y} & y \leq 0 \end{cases}. \tag{2.5.7}$$

We also have

$$\Psi'_{1+}(\alpha, y) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{\partial u(x, y)}{\partial y} e^{i\alpha x} dx. \tag{2.5.8}$$

$$\Psi'_{1-}(\alpha, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \frac{\partial u(x, y)}{\partial y} e^{i\alpha x} dx, \tag{2.5.9}$$

where prime denotes derivative with respect to 'y'. Using Eqs. (2.5.2), (2.5.3), (2.5.8), and (2.5.9) in Eq. (2.5.7), we obtain

$$\Psi_{1+}(\alpha, 0) + \Psi_{1-}\left(\alpha, 0^+\right) = A(\alpha). \tag{2.5.10}$$

$$\Psi_{1+}(\alpha, 0) + \Psi_{1-}\left(\alpha, 0^-\right) = -A(\alpha). \tag{2.5.11}$$

$$\Psi'_{1+}(\alpha, 0) + \Psi'_{1-}(\alpha, 0) = -\gamma A(\alpha). \tag{2.5.12}$$

By adding Eqs. (2.5.10) and (2.5.11), we obtain

$$2\Psi_{1+}(\alpha, 0) = -\Psi_{1-}\left(\alpha, 0^+\right) - \Psi_{1-}\left(\alpha, 0^-\right). \tag{2.5.13}$$

By subtracting Eqs. (2.5.10) from (2.5.11), and eliminating $A(\alpha)$, we obtain

$$A(\alpha) = \frac{1}{2}\left[\Psi_{1-}\left(\alpha, 0^+\right) - \Psi_{1-}\left(\alpha, 0^-\right)\right]. \tag{2.5.14}$$

$$\Psi'_{1+}(\alpha, 0) + \Psi'_{1-}(\alpha, 0) = -\frac{1}{2}\gamma\left[\Psi_{1-}\left(\alpha, 0^+\right) - \Psi_{1-}\left(\alpha, 0^-\right)\right]. \tag{2.5.15}$$

In above equation $\Psi'_{1-}(\alpha, 0)$ is known which will be explained in subsequent chapters given to be

$$\Psi'_{1-}(\alpha, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{i\alpha x} \left(ik \sin\theta e^{-ikx\cos\theta}\right) dx,$$

so that

$$\Psi'_{1-}(\alpha, 0) = \frac{k \sin\theta}{\sqrt{2\pi} (\alpha - k\cos\theta)}. \tag{2.5.16}$$

Let

$$\Psi_{1-}(\alpha, 0^+) - \Psi_{1-}(\alpha, 0^-) = 2\Psi^*_-(\alpha), \tag{2.5.17}$$

and

$$\Psi_{1-}(\alpha, 0^+) + \Psi_{1-}(\alpha, 0^-) = 2\Phi^*_-(\alpha). \tag{2.5.18}$$

From above two Eqs. (2.5.13) and (2.5.15), we obtain

$$\Psi_{1+}(\alpha, 0) = -\Phi^*_-(\alpha). \tag{2.5.19}$$

Using Eqs. (2.5.16) and (2.5.17) in Eq. (2.5.15), we obtain

$$\Psi'_{1+}(\alpha, 0) + \frac{k \sin\theta}{\sqrt{2\pi} (\alpha - k\cos\theta)} = -\gamma\Psi^*_-(\alpha). \tag{2.5.20}$$

Eqs. (2.5.19), and (2.5.20) exist in the strip $-k_2 < \tau < k_2 \cos\theta$ and there are four unknown functions, i.e., $\Psi_{1+}(\alpha, 0)$, $\Psi'_{1+}(\alpha, 0)$, $\Phi^*_-(\alpha)$ and $\Psi^*_-(\alpha)$. From Eq. (2.5.6), we obtain

$$\gamma = \left(\alpha^2 - k^2\right)^{1/2} = (\alpha + k)^{1/2} (\alpha - k)^{1/2}.$$

The factor $(\alpha + k)^{1/2}$ is regular and nonzero in $\tau > -k_2$. Therefore, dividing Eq. (2.5.20), by $(\alpha+k)^{1/2}$, we obtain

$$\frac{\Psi'_{1+}(\alpha,0)}{\sqrt{\alpha+k}}+\frac{k\sin\theta}{\sqrt{2\pi}\sqrt{\alpha+k}\,(\alpha-k\cos\theta)}=-\sqrt{\alpha-k}\Psi^{*}_{-}(\alpha). \tag{2.5.21}$$

The first term on LHS is analytic in $\tau>-k_2$ and the term on RHS is regular in $\tau<k_2\cos\theta$. The second term on the LHS is analytic in the strip $-k_2<\tau<k_2\cos\theta$, we can split it as:

$$\frac{k\sin\theta}{\sqrt{2\pi}\sqrt{\alpha+k}\,(\alpha-k\cos\theta)}=P_{-}(\alpha)+P_{+}(\alpha),$$

where

$$P_{+}(\alpha)=\frac{k\sin\theta}{\sqrt{2\pi}\,(\alpha-k\cos\theta)}\left\{\frac{1}{\sqrt{\alpha+k}}-\frac{1}{\sqrt{k+k\cos\theta}}\right\} \tag{2.5.22}$$

is regular in $\tau>-k_2$, and

$$P_{-}(\alpha)=\frac{k\sin\theta}{\sqrt{2\pi}\sqrt{k+k\cos\theta}\,(\alpha-k\cos\theta)} \tag{2.5.23}$$

$P_{-}(\alpha)$ is regular in $\tau<k_2\cos\theta$.

Let us write a function $J(\alpha)$ which is regular in the whole $\alpha-plane$

$$J_1(\alpha)=\begin{cases}\frac{\Psi'_{1+}(\alpha,0)}{\sqrt{\alpha+k}}+P_{+}(\alpha)\\ -\sqrt{\alpha-k}\Psi^{*}_{-}(\alpha)-P_{-}(\alpha)\end{cases}$$

By extended Liouville's theorem, we may have

$$J_1(\alpha)\longrightarrow 0 \quad as \quad \alpha\longrightarrow\infty,$$

Thus,

$$\Psi'_{1+}(\alpha,0)=-\sqrt{\alpha+k}P_{+}(\alpha), \tag{2.5.24}$$

and

$$\Psi^{*}_{-}(\alpha) = -\frac{P_{-}(\alpha)}{\sqrt{\alpha - k}}. \tag{2.5.25}$$

Using Eqs. (2.5.7) and (2.5.25) and Fourier inversion formula we obtain the integral equation which is solved by steepest descent method giving the required solution.

2.6 The Method of Steepest Descent [107]

The method of steepest descents consists essentially in choosing a path of integration with a particular geometrical property. We can find the asymptotic expension, when ν is very large and positive, of a function defined by an integral of the form

$$\int e^{\nu w_1(z)} \phi(z) dz, \tag{2.6.1}$$

where the path of integration is an arc or a closed curve in the z plane. The functions $w_1(z)$ and $\phi(z)$ are independent of ν and are analytic functions of z, regular in a domain which contains the path of integration. The idea is to deform the path of integration to satisfy the following conditions:

i) the path passes through a zero z_0 of $w_1'(z)$;

ii) the imaginary part of $w_1(z)$ is constant on the path.

If we write

$$z = x + iy, \qquad w_1(z) = u(x,y) + iv(x,y), \tag{2.6.2}$$

where x, y, u, v are real, the equation of the new path of integration is $v(x,y) = v(x_0, y_0)$, and the intgrand becomes

$$e^{i\nu(x_0,y_0)} e^{\nu u(x,y)} \phi(x + iy).$$

Since ν and u are real and ϕ does not depend on ν, the integrand does not oscillate rapidly on the new path for large values of ν.

To obtain a geometrical picture of the new path, consider the surface S whose equation is $u = u(x, y)$ in rectangular Cartesian co-ordinates, the axis of u being vertically upwards. By the Cauchy-Riemann conditions,

$$w_1'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}.$$

Hence, if z_0 is a zero of $w_1'(z)$, the tangent plane to S at the corresponding point (x_0, y_0, u_0) is horizontal. But, since

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

this point must be a saddle-point, not a maximum or minimum.

The shape of the surface S can be represented on the $(x, y) - plane$ by drawing the contour lines or level curves on which u is constant; a saddle point is evedently a multiple point on a particular level curve. The level curve through a saddle point separates the nearby part of S of into valleys below the saddle-point and rising ground above the saddle-point. The curves $v = constant$ are the orthogonal trajectories of the level curves, and so are the maps on the $(x, y) - plane$ of the steepest curves on S. Thus, a Debye curve is a steepest path through a saddle point. In the simplest case, when the saddle-point is a double point of the corresponding level curve, there are two paths of steepest descent from it and two paths of steepest ascent.

On a steepest path through a saddle-point z_0, we have

$$w_1(z) = w_1(z_0) - \tau_1, \tag{2.6.3}$$

where τ_1 is real, and so, if s is the arc of the path, $\frac{d\tau_1}{ds} = \pm |w_1'(z)|$. Therefore, $\frac{d\tau_1}{ds}$ can only change sign if the path goes through another saddle-point or a singularity of $w_1'(z)$, but it is rare for this to happen. The variable τ_1 is usually monotonic on a steepest path from a saddle-point and either increases to $+\infty$ or decreases to $-\infty$. But since the integrand is

$$e^{\nu w_1(z_0) - \nu\tau_1}\phi(z),$$

a path on which $\tau_1 \to -\infty$ would lead to a divergent integral. Hence, we try to choose paths of integration on which τ_1 is positive – these are the paths of steepest descent from the saddle-point. If it is possible to deform the path of integration and express the integral as the sum of integrals along paths of steepest descent from a saddle-point, all that remains is to consider the asymptotic behaviour of integrals of the form

$$e^{\nu w_1(z_0)} \int_0^\infty e^{-\nu\tau_1} \phi \frac{dz}{d\tau_1} d\tau_1, \tag{2.6.4}$$

where ν is very large and positive, and, to each of these, we can usually apply Watson's lemma.

Generally, it is rather difficult to get from (2.6.3) a parametric equation for each of the paths of steepest descent from a saddle-point, though it is quite easy to find the form near the saddle-point. Let us consider the case when $w_1'(z)$ has a zero of order $m-1$ at a saddle-point z_0 so that

$$w_1(z) = w_1(z_0) - (z - z_0)^m f(z), \tag{2.6.5}$$

where $f(z)$ is regular in some neighbourhood of z_0 and $f(z_0) = ae^{-\alpha_1 i}$ where $a > 0$. Near the saddle-point the level curves and steepest curves will be roughly the same as for

$$w_1(z) = w_1(z_0) - ae^{-\alpha_1 i}(z - z_0)^m.$$

Hence, if $z = z_0 + re^{\theta i}$,

$$u = u_0 - ar^m \cos(m\theta - \alpha_1), \qquad v = v_0 - ar^m \sin(m\theta - \alpha_1).$$

The level curves $u = u_0$ are approximately $\theta = \frac{\{(n+\frac{1}{2})\pi + \alpha_1\}}{m}$ and the steepest curves $v = v_0$ are $\theta = \frac{n\pi + \alpha_1}{m}$, where $n = 1, 2, 3, ..., 2m$. These are thus $2m$ steepest directions from z_0, m directions of steepest ascent and m directions of steepest descent. The level curves near z_0 divide S into m valleys below the saddle-point and m hills above the saddle point. The paths of integration are in the valleys.

By (2.6.3) and (2.6.5) we have to solve the equation

$$(z - z_0)^m f(z) = \tau_1, \tag{2.6.6}$$

where τ_1 is positive on a path of steepest descent. It is convenient to regard τ_1 as a complex variable and use the theory of analytic functions to find the m solutions of (2.6.6) which have the value z_0 when $\tau_1 = 0$.

By hypothesis, $f(z)$ can be expanded as a convergent power series

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ...,$$

where $a_0 \neq 0$. By continuity, there is a neighbourhood of z_0 in which $f(z)$ does not vanish. The principal value $g(z)$ of the m *th* root of $f(z)$ is therefore analytic in some neighbourhood of z_0 and can be expanded as a power series

$$g(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)^2 + ...,$$

where b_0 is the principal value of $a_0^{\frac{1}{m}}$. The equation (2.6.6) then becomes

$$(z - z_0)g(z) = t,$$

where t is an mth root of τ_1. But by the inverse function theorem for analytic function this equation has a unique solution

$$z = z_0 + c_1 t + c_2 t^2 + c_3 t^3 + ..., \tag{2.6.7}$$

which takes the value z_0 when $t = 0$ and is regular in a neighbourhood of $t = 0$. This gives one solution of (2.6.6); the others are obtained by replacing t by $\omega_1 t$, where ω_1 is a complex $m\ th$ root of unity. When t is real, (2.6.7) gives the parametric equation of a steepest path through the saddle-point; when $t > 0$, it is a path of steepest descent.

In order to find a complete asymptotic expansion, it is necessary first to find the coefficients A_n in the power series

$$\phi(z)\frac{dz}{dt} = \sum_0^\infty A_n t^n, \tag{2.6.8}$$

it being assumed that $\phi(z)$ is regular in a neighbourhood of z_0. Evidently, using the Cauchy Residue theorem, the branch cut integration gives

$$\begin{aligned} A_n &= \frac{1}{2\pi i}\int^{(0+)} \phi(z)\frac{dz}{dt}\frac{dt}{t^{n+1}} \\ &= \frac{1}{2\pi i}\int^{(z_0+)} \frac{\phi(z)}{t^{n+1}}dz \\ &= \frac{1}{2\pi i}\int^{(0+)} \frac{\phi(z)}{(w_0 - w_1)^{\frac{(n+1)}{m}}}dz \end{aligned}$$

with an appropriate branch of $(w_0 - w_1)^{\frac{1}{m}}$. Thus A_n is the residue of $\frac{\phi(z)}{(w_0-w_1)^{\frac{(n+1)}{m}}}$ at z_0. In particular, nc_n is the residue of $\frac{1}{(w_0-w_1)^{\frac{n}{m}}}$. The actual determination of two coefficients may prove rather complicated in particular problems; it may only be possible to find the first few explicitly and not to get a general formula.

At z_0 we have

$$\frac{dz}{dt} = c_1 = \frac{A_0}{\phi(z_0)};$$

and the phase of $\frac{dz}{dt}$ at z_0 gives the direction of corresponding path of steepest descent.

This amount of Debye's method of steepest descents is necessarily merely a description, not a rigrous discussion; the fundamental idea will be made clearer by a consideration of particular examples. It should be noted that, in (2.6.4), we assumed ν large and positive. But it usually happens that the integrals obtained in this way converge when $|\nu|$ is large and $|ph\ \nu| < \alpha$, say; the method then gives an asymptotic expansion for large complex values of ν even though the path is then not a path of steepest descent.

2.7 Constitutive Relations [110]

On the macroscopic level, the description of bi-isotropic media (in short, BI media) is contained in the material parameters of the constitutive relations, written as:

$$\mathbf{D} = \varepsilon \mathbf{E} + (\chi - jk)\sqrt{\mu_o \varepsilon_o}\, \mathbf{E}, \quad (2.7.1)$$

$$\mathbf{B} = \mu \mathbf{H} + (\chi + jk)\sqrt{\mu_o \varepsilon_o}\, \mathbf{E}. \quad (2.7.2)$$

with B, D are respectively magnetic induction and electric displacement and E and H are electric and magnetic field strengths. The dielectric response of the material is contained in the permittivity ε, and the permeability μ is the corresponding magnetic parameter. But the essence of bi-isotropic media are the magnetoelectric material parameters χ and

k, which are dimensionless in the representations (2.7.1, 2.7.2) as the free space constant $\sqrt{\mu_o \varepsilon_o}$ is separated. The imaginary unit i emphasizes the frequency domain character of the equations, and comes from the time harmonic convention $exp(-i\omega t)$.

The chirality parameter k measures the degree of handedness of the material, and for racemic media (Media made of chiral elements of both handedness in equal number so that they do not show any macroscopic chiral properties) this parameter vanishes. A change in the sign of k means taking the mirror image of the material. The other parameter χ describes the magnetoelectric effect. Materials with $\chi \neq 0$ are non-reciprocal. Depending on the values of these parameters, following table can be written to classify BI media.

	non chiral ($k = 0$)	chiral ($k \neq 0$)
reciprocal ($\chi = 0$)	simple isotropic medium	Pasteur medium
non-reciprocal ($\chi \neq 0$)	Tellegen medium	general bi-isotropic medium

In the modern electromagnetic literature, notations other than these constitutive relations are also used in the characterization of chiral media. In relations, the electric and magnetic displacements are represented as functions of the field strengths. A choice, much used in the analysis of resiprocal chiral media, are the Durde-Born-Fedorov [106] relations

$$\begin{aligned} \mathbf{D} &= \varepsilon(\mathbf{E} + \beta \nabla \times \mathbf{E}), \\ \mathbf{B} &= \mu(\mathbf{H} + \beta \nabla \times \mathbf{H}). \end{aligned}$$

Chapter 3

Magnetic Line Source Diffraction by an Impedance Step

This chapter deals with the diffraction of a magnetic line source by an impedance step joined by two half planes where the half planes and step are characterized by different surface impedances. The problem is solved using Wiener Hopf technique and Fourier transform. The scattered field in the far zone is determined by the method of steepest descent [107]. Graphical results for the solution has also been presented. It is observed that if the source is shifted to a large distance these results differ from those of [55] by a multiplicative factor to the part of the scattered field containing the effects of incident and reflected waves.

3.1 Mathematical formulation

Consider the scattering due to a magnetic line source located at (x_0, y_0) which illuminates two half planes $S_1 = \{x \leq 0,\ y = a,\ z \in (-\infty, \infty)\}$ and $S_2 = \{x \geq 0,\ y = 0,\ z \in (-\infty, \infty)\}$ with relative surface impedance η_{1i} joined together by a step of height "a"

with relative surface impedance η_{2i}. The geometry of the line source diffraction problem is depicted in *Fig.* (3.1).

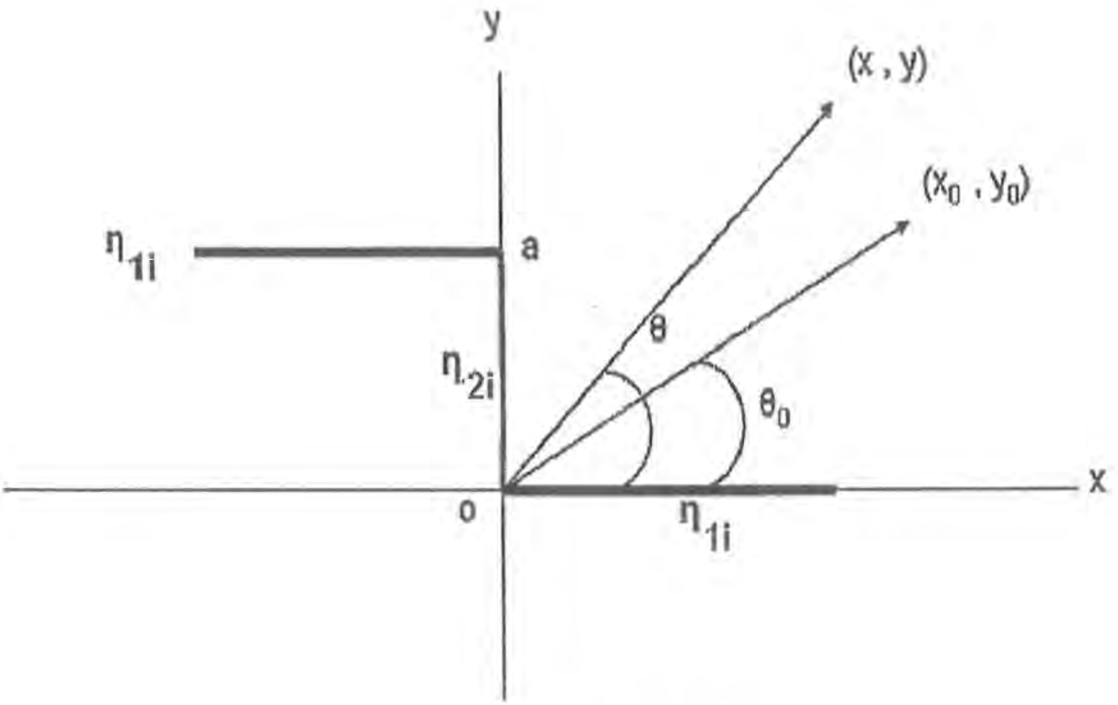


Fig. 3.1: Geometry of the problem

For harmonic vibrations of time dependence $e^{-i\omega t}$, the solution of the following wave equation is required

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2)\Psi_t(x, y) = \delta(x - x_0)\delta(y - y_0), \tag{3.1}$$

where Ψ_t is the total field. For the analysis purpose, it is convenient to express the total field $\Psi_t(x,y)$ as follows:

$$\Psi_t(x,y) = \begin{cases} \Psi^i(x,y) + \Psi_1^r(x,y) + \Psi_1(x,y), & y > a, \\ \Psi_2(x,y), & 0 < y < a, \end{cases} \tag{3.2}$$

well supported by the boundary conditions at two half planes and a step given by:

$$(1 + \frac{\eta_{1i}}{ik}\frac{\partial}{\partial y})\Psi_1(x,a) = 0, \qquad x \in (-\infty, 0), \tag{3.3}$$

$$(1 + \frac{\eta_{2i}}{ik}\frac{\partial}{\partial x})\Psi_t(0,y) = 0, \qquad y \in (0, a), \tag{3.4}$$

$$(1 + \frac{\eta_{1i}}{ik}\frac{\partial}{\partial y})\Psi_t(x,0) = 0, \qquad x \in (0, \infty), \tag{3.5}$$

$$\Psi_t(x,a) - \Psi_2(x,a) = 0, \qquad x \in (0, \infty), \tag{3.6}$$

$$\frac{\partial}{\partial y}\Psi_t(x,a) - \frac{\partial}{\partial y}\Psi_2(x,a) = 0, \quad x \in (0, \infty). \tag{3.7}$$

where $k = \frac{w}{c}$ is the wave number and a time factor e^{-iwt} has been assumed and suppressed hereafter. It is assumed that the wave number k has positive imaginary part. The lossless case can be obtained by making $Im\ k \to 0$ in the final expressions. $\Psi_1(x,y)$ and $\Psi_2(x,y)$ are the scattered fields, $\Psi_1^r(x,y)$ is the field reflected from the plane located at $y = a$ with relative surface impedance η_{1i} and $\Psi^i(x,y)$ is the incident field satisfying the equation:

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2)\Psi^i(x,y) = \delta(x - x_0)\delta(y - y_0). \tag{3.8}$$

The scattered field $\Psi_1(x,y)$ satisfies the Helmholtz equation

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2)\Psi_1(x,y) = 0, \quad x \in (-\infty, \infty). \tag{3.9}$$

It is appropriate to define Fourier transform as follows:

$$\overline{\Psi}_{1\pm}(\alpha,y)=\pm\frac{1}{\sqrt{2\pi}}\int_{0}^{\pm\infty}\Psi_{1}(x,y)e^{i\alpha x}dx, \tag{3.10}$$

$$\Psi_{1}(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overline{\Psi}_{1}(\alpha,y)e^{-i\alpha x}d\alpha, \tag{3.11}$$

$$\overline{\Psi}_{2}(\alpha,y)=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\Psi_{2}(x,y)e^{i\alpha x}dx, \tag{3.12}$$

$$\overline{\Psi}^{i}(\alpha,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi^{i}(x,y)e^{i\alpha x}d\alpha, \tag{3.13}$$

$$\overline{\Psi}_{1}^{r}(\alpha,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi_{1}^{r}(x,y)e^{i\alpha x}d\alpha, \tag{3.14}$$

and

$$\overline{\Psi}_{0}(\alpha,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[\Psi^{i}(x,y)+\Psi_{1}^{r}(x,y)]e^{i\alpha x}dx. \tag{3.15}$$

Using Eq. (3.10), the Eq. (3.9) can be written as

$$\frac{d^{2}\overline{\Psi}_{1}}{dy^{2}}+\gamma^{2}\overline{\Psi}_{1}(\alpha,y)=0, \tag{3.16}$$

where $\gamma(\alpha)=\sqrt{k^{2}-\alpha^{2}}$. The square root function $\gamma(\alpha)$ is defined such that $\gamma(0)=k$ in the complex $\alpha-plane$ cut as shown in $Fig.$ 3.2.

The solution of Eq. (3.16) satisfying the radiation condition for $y>a$ can be written as:

$$\overline{\Psi}_{1}(\alpha,y)=\overline{\Psi}_{1_{+}}(\alpha,y)+\overline{\Psi}_{1_{-}}(\alpha,y)=A(\alpha)e^{i\gamma(\alpha)|y-a|}, \tag{3.17}$$

where $A(\alpha)$ is the unknown coefficient to be determine and $\overline{\Psi}_{1}(\alpha,y)$ is divided into $\overline{\Psi}_{1_{+}}(\alpha,y)$ and $\overline{\Psi}_{1_{-}}(\alpha,y)$ as in [13, 58] .

Taking the Fourier transform of the equation Eq.(3.8), we obtain

$$\left[\frac{d^2}{dy^2} + \gamma^2(\alpha)\right] \overline{\Psi}^i(\alpha, y) = \frac{\delta(y - y_0)\, e^{i\alpha x_0}}{\sqrt{2\pi}}. \tag{3.18}$$

This equation can be solved by using the Greens' function method. Let $G(\alpha, y, y_o)$ be the Greens' function, so that

$$G(\alpha, \infty, y_0) = G(\alpha, -\infty, y_0) = 0. \tag{3.19}$$

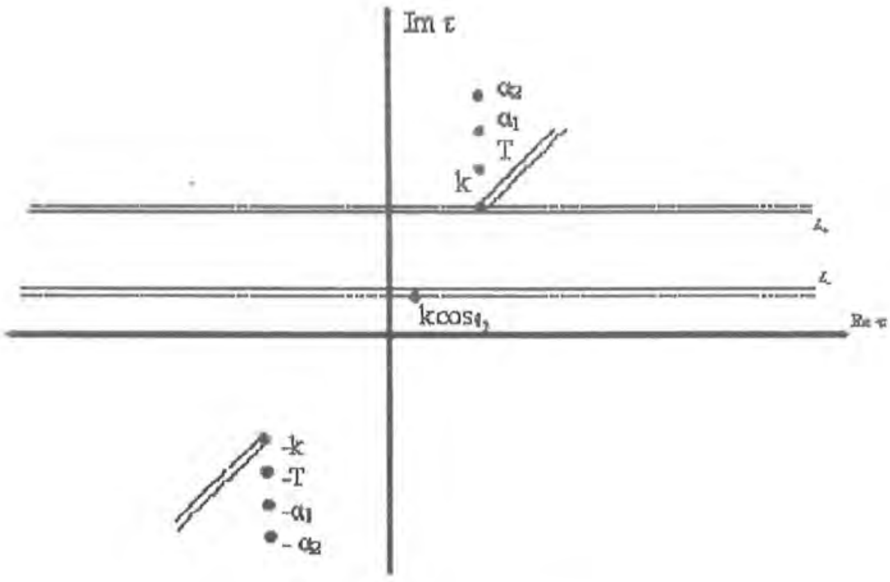


Fig. 3.2: Complex α- plane

Now, since

$$\delta(y - y_0) = \begin{cases} 0 \;\; if \;\; y \neq y_0 \\ \infty \;\; if \;\; y = y_0, \end{cases} \tag{3.20}$$

such that

$$\int_{-\infty}^{\infty} \delta(y-y_0)\,dy = 1. \tag{3.21}$$

Then for $y \neq y_0$ the Eq. (3.18) becomes

$$\left[\frac{d^2}{dy^2} + \gamma^2(\alpha)\right] G(\alpha, y, y_0) = 0, \tag{3.22}$$

which implies that

$$G(\alpha, y, y_0) = \begin{cases} A_1(\alpha)\, e^{i\gamma(\alpha)y} + B_1(\alpha)\, e^{-i\gamma(\alpha)y} & -\infty < y < y_o, \\ A_2(\alpha)\, e^{i\gamma(\alpha)y} + B_2(\alpha)\, e^{-i\gamma(\alpha)y} & y_o < y < \infty. \end{cases} \tag{3.23}$$

Using the condition (3.19), we get

$$G(\alpha, y, y_0) = \begin{cases} B_1(\alpha)\, e^{-i\gamma(\alpha)y} & -\infty < y < y_o, \\ A_2(\alpha)\, e^{i\gamma(\alpha)y} & y_o < y < \infty. \end{cases} \tag{3.24}$$

At $y = y_0$, $\Psi^i(x, y)$ is continuous. Therefore, $G(\alpha, y, y_o)$ must be continuous at $y = y_0$, i.e.,

$$A_2(\alpha) = B_1(\alpha)\, e^{-2i\gamma(\alpha)y_0}. \tag{3.25}$$

Then the Eq. (3.24) becomes

$$G(\alpha, y, y_0) = \begin{cases} B_1(\alpha)\, e^{-i\gamma(\alpha)y}, & -\infty < y < y_o, \\ B_1(\alpha)\, e^{i\gamma(\alpha)y - 2i\gamma(\alpha)y_0}, & y_o < y < \infty. \end{cases} \tag{3.26}$$

To evaluate $B_1(\alpha)$ let us drive another condition by considering

$$\int_{-\infty}^{\infty} \left[\frac{d^2}{dy^2} + \gamma^2(\alpha)\right] G(\alpha, y, y_0)\,dy = \int_{-\infty}^{\infty} \delta(y-y_0)\, e^{i\alpha x_0}\,dy. \tag{3.27}$$

Using Eq. (3.21) , we get

$$\int_{-\infty}^{\infty}\left[\frac{d^2}{dy^2}+\gamma^2(\alpha)\right]G(\alpha,y,y_0)\,dy=e^{i\alpha x_0}. \qquad (3.28)$$

Then from Eq. (3.26), we have

$$\lim_{\epsilon\to 0}\frac{dG(\alpha,y,y_0)}{dy}\Big|_{y_0-\epsilon}^{y_0+\epsilon}=e^{i\alpha x_0}. \qquad (3.29)$$

or

$$\lim_{\epsilon\to 0}\left[\frac{dG(\alpha,y_0+\epsilon,y_0)}{dy}-\frac{dG(\alpha,y_0-\epsilon,y_0)}{dy}\right]=e^{i\alpha x_0}. \qquad (3.30)$$

Thus, we observe that the derivative of the Greens' function at $y=y_0$ is discontinuous. Using the first order derivative w.r.t. y of Eq. (3.26) in Eq. (3.29), we obtain

$$B_1(\alpha)=\frac{1}{4\pi i\gamma(\alpha)}e^{i\alpha x_0+i\gamma(\alpha)y_0}. \qquad (3.31)$$

Hence, $G(\alpha,y,y_0)$ becomes

$$G(\alpha,y,y_o)=\begin{cases}\frac{1}{4\pi i\gamma(\alpha)}e^{i\alpha x_0}e^{-i\gamma(y-y_0)}, & -\infty<y<y_o,\\ \frac{1}{4\pi i\gamma(\alpha)}e^{i\alpha x_0}e^{i\gamma(y-y_0)}, & y_o<y<\infty.\end{cases} \qquad (3.32)$$

Thus, by the method of Greens' function, one can write incident and the corresponding reflected field as follows:

$$\overline{\Psi}^i(\alpha,y)=\frac{1}{4\pi i\gamma}e^{i\alpha x_0+i\gamma(\alpha)|y-y_0|}, \qquad (3.33)$$

and

$$\overline{\Psi}_1^r(\alpha,y)=\frac{\eta_{1i}\sin\theta_0-1}{\eta_{1i}\sin\theta_0+1}.\frac{1}{4\pi i\gamma}e^{i\alpha x_0+i\gamma(\alpha)|(y-2b)+y_0|}. \qquad (3.34)$$

The reflection coefficient, appearing in Eq. (3.34), is because of the properties of the half-plane, e.g., this coefficient would be 1 if the half-plane is hard and -1 if it is soft which can be achieved by putting $\eta_{1i}=\infty$ and 0, respectively [13, $pp.83,203$]. The effect of the line source position (x_0, y_0) is evident from the exponential function. Therefore, both the situations of line source and plane wave will differ only in exponential function not in the reflection coefficient.

The unknown coefficient $A(\alpha)$ appearing in Eq. (3.17) is to be determined with the help of boundary conditions (3.3) and (3.5 − 3.7) which after taking Fourier transform give:

$$(1+\frac{\eta_{1i}}{ik}\frac{\partial}{\partial y})\overline{\Psi}_{1-}(\alpha,a)=0, \qquad x\in(-\infty,0), \tag{3.35}$$

$$(1+\frac{\eta_{1i}}{ik}\frac{\partial}{\partial y})\overline{\Psi}_{2}(\alpha,0)=0, \qquad x\in(0,\infty), \tag{3.36}$$

$$\Psi_0(\alpha,a)+\overline{\Psi}_1(\alpha,a)-\overline{\Psi}_2(\alpha,a)=0, \qquad x\in(0,\infty), \tag{3.37}$$

$$\frac{\partial}{\partial y}[\Psi_0(\alpha,a)+\overline{\Psi}_1(\alpha,a)]-\frac{\partial}{\partial y}\overline{\Psi}_2(\alpha,a)=0, \qquad x\in(0,\infty). \tag{3.38}$$

Eq. (3.35) can be written as:

$$\overline{\Psi}_{1-}(\alpha,a)+\frac{\eta_{1i}}{ik}\overline{\Psi}'_{1-}(\alpha,a)=0, \tag{3.39}$$

where prime "$\prime$" denotes differentiation with respect to y. The differentiation of Eq. (3.17) w.r.t "y" yields

$$\overline{\Psi}'_{1_+}(\alpha,y)+\overline{\Psi}'_{1_-}(\alpha,y)=i\gamma(\alpha)A(\alpha)e^{i\gamma(\alpha)|y-a|}. \tag{3.40}$$

By putting $y=a$ in Eqs. (3.17) and (3.40) and then using Eq. (3.35), we get

$$R_+(\alpha) = A(\alpha)\left((1+\frac{\eta_{1i}}{k}\gamma(\alpha)\right), \tag{3.41}$$

where

$$R_+(\alpha) = \overline{\Psi}_{1_+}(\alpha,a) + \frac{\eta_{1i}}{ik}\overline{\Psi}'_{1_+}(\alpha,a). \tag{3.42}$$

In the region $y \in (0,a)$, $x > 0$, the scattered field $\Psi_2(x,y)$ satisfies the Helmholtz equation

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2)\Psi_2(x,y) = 0, \qquad x > 0. \tag{3.43}$$

Taking the Fourier transform, we obtain

$$[\frac{d^2}{dy^2} + \gamma^2(\alpha)]\overline{\Psi}_2(\alpha,y) = \frac{1}{\sqrt{2\pi}}[\frac{\partial}{\partial x}\Psi_2(0,y) - i\alpha\Psi_2(0,y)]. \tag{3.44}$$

Using the relation (3.4), Eq. (3.44) reduces to

$$[\frac{d^2}{dy^2} + \gamma^2(\alpha)]\overline{\Psi}_2(\alpha,y) = -\frac{i}{\sqrt{2\pi}}[\frac{k}{\eta_{2i}} + \alpha]\Psi_2(0,y). \tag{3.45}$$

The multiplication of both sides of Eq. (3.45) by $(\frac{k}{\eta_{2i}} - \alpha)$ yields

$$[\frac{d^2}{dy^2} + \gamma^2(\alpha)][\frac{k}{\eta_{2i}} - \alpha]\overline{\Psi}_2(\alpha,y) = -\frac{i}{\sqrt{2\pi}}[(\frac{k}{\eta_{2i}})^2 - \alpha^2]\Psi_2(0,y). \tag{3.46}$$

Changing the sign of α in Eq. (3.46) gives

$$[\frac{d^2}{dy^2} + \gamma^2(\alpha)][\frac{k}{\eta_{2i}} + \alpha]\overline{\Psi}_2(-\alpha,y) = -\frac{i}{\sqrt{2\pi}}[(\frac{k}{\eta_{2i}})^2 - \alpha^2]\Psi_2(0,y). \tag{3.47}$$

By subtracting Eq. (3.47) from Eq. (3.46), we obtain the following homogenous differential equation:

$$[\frac{d^2}{dy^2}+\gamma^2(\alpha)]\left[(\frac{k}{\eta_{2i}}-\alpha)\overline{\Psi}_2(\alpha,y)-(\frac{k}{\eta_{2i}}+\alpha)\overline{\Psi}_2(-\alpha,y)\right]=0, \tag{3.48}$$

whose general solution is

$$\left[(\frac{k}{\eta_{2i}}-\alpha)\overline{\Psi}_2(\alpha,y)-(\frac{k}{\eta_{2i}}+\alpha)\overline{\Psi}_2(-\alpha,y)\right]=B\cos\gamma y+C\sin\gamma y. \tag{3.49}$$

Using the boundary condition (3.36), we obtain

$$\left[(\frac{k}{\eta_{2i}}-\alpha)\overline{\Psi}_2(\alpha,y)-(\frac{k}{\eta_{2i}}+\alpha)\overline{\Psi}_2(-\alpha,y)\right]=B[\cos\gamma y-\frac{ik}{\eta_{1i}}\frac{\sin\gamma y}{\gamma}]. \tag{3.50}$$

By elimination of B between Eq. (3.50) and its derivative w.r.t "y", we obtain after setting $y=a$

$$\frac{(\frac{k}{\eta_{2i}}-\alpha)\overline{\Psi}_2(\alpha,a)-(\frac{k}{\eta_{2i}}+\alpha)\overline{\Psi}_2(-\alpha,a)}{\left(\cos\gamma a-\frac{ik}{\eta_{1i}}\frac{\sin\gamma a}{\gamma}\right)}=-\frac{(\frac{k}{\eta_{2i}}-\alpha)\overline{\Psi}_2'(\alpha,a)-(\frac{k}{\eta_{2i}}+\alpha)\overline{\Psi}_2'(-\alpha,a)}{\left(\gamma\sin\gamma a+\frac{ik}{\eta_{1i}}\cos\gamma a\right)}. \tag{3.51}$$

Consider the Fourier transformed continuity relations (3.37) and (3.38), *i.e.*,

$$\overline{\Psi}_0(\alpha,y)+\overline{\Psi}_{1_+}(\alpha,a)=\overline{\Psi}_2(\alpha,a), \tag{3.52}$$

and

$$\overline{\Psi}_0'(\alpha,y)+\overline{\Psi}_{1_+}'(\alpha,a)=\overline{\Psi}_2'(\alpha,a). \tag{3.53}$$

Adding Eq. (3.52) $\frac{\eta_{1i}}{ik}$ times of the Eq. (3.53), we obtain

$$R_+(\alpha)=\overline{\Psi}_2(\alpha,a)+\frac{\eta_{1i}}{ik}\overline{\Psi}_2'(\alpha,a)-[\overline{\Psi}_0(\alpha,y)+\frac{\eta_{1i}}{ik}\overline{\Psi}_0'(\alpha,y)], \tag{3.54}$$

and

$$R_+(-\alpha) = \overline{\Psi}_2(-\alpha, a) + \frac{\eta_{1i}}{ik}\overline{\Psi}_2'(-\alpha, a) - [\overline{\Psi}_0(-\alpha, y) + \frac{\eta_{1i}}{ik}\overline{\Psi}_0'(-\alpha, y)]. \tag{3.55}$$

From Eqs. (3.17), (3.40) and (3.41), we get

$$\overline{\Psi}_{1_+}(\alpha, a) = \frac{R_+(\alpha)}{(1 + \frac{\eta_{1i}}{k}\gamma)} - \overline{\Psi}_{1_-}(\alpha, a), \tag{3.56}$$

and

$$\overline{\Psi}_{1_+}'(\alpha, a) = \frac{i\gamma(\alpha)R_+(\alpha)}{(1 + \frac{\eta_1}{k}\gamma)} - \overline{\Psi}_{1_-}'(\alpha, a). \tag{3.57}$$

Replacing $\overline{\Psi}_{1_+}(\alpha, a)$ and $\overline{\Psi}_{1_+}'(\alpha, a)$ appearing in Eqs. (3.52) and (3.53) by their expression given in Eqs. (3.56) and (3.57), respectively, we obtain

$$\overline{\Psi}_2(\alpha, a) = \overline{\Psi}_0(\alpha, y) + \frac{R_+(\alpha)}{1 + \frac{\eta_{1i}}{k}\gamma} - \overline{\Psi}_{1_-}(\alpha, a), \tag{3.58}$$

and

$$\overline{\Psi}_2'(\alpha, a) = \overline{\Psi}_0'(\alpha, y) + \frac{i\gamma(\alpha)R_+(\alpha)}{(1 + \frac{\eta_{1i}}{k}\gamma)} + \frac{ik}{\eta_{1i}}\overline{\Psi}_{1_-}(\alpha, a). \tag{3.59}$$

By putting the values of $\overline{\Psi}_2(\alpha, a), \overline{\Psi}_2'(\alpha, a)$ and $\overline{\Psi}_2'(-\alpha, a)$ from Eqs. (3.58), (3.59) and (3.55) in Eq. (3.51), respectively, yields the following modified Wiener-Hopf equation of second kind valid in the strip $\text{Im}(k\cos\theta_0) < \text{Im}\,\alpha < \text{Im}\,k$:

$$\begin{aligned}
&\frac{(\frac{k}{\eta_{2i}}-\alpha)R_+(\alpha)}{(\alpha^2-\breve{T}^2)\hat{G}(\alpha)} - i\frac{\eta_{1i}}{k}\hat{S}_-(\alpha) \\
=& -\frac{(\frac{k}{\eta_{2i}}-\alpha)\frac{\eta_{1i}}{ik}}{(\alpha^2-\breve{T}^2)\frac{\sin\gamma a}{\gamma}}\left[\begin{array}{c}\overline{\Psi}_0(\alpha,y)\left(\gamma\sin\gamma a+\frac{ik}{\eta_{1i}}\cos\gamma a\right)\\ +\overline{\Psi}_0'(\alpha,y)\left(\cos\gamma a-\frac{ik}{\eta_{1i}}\frac{\sin\gamma a}{\gamma}\right)\end{array}\right] \\
&+\frac{(\frac{k}{\eta_{2i}}+\alpha)}{(\alpha^2-\breve{T}^2)\frac{\sin\gamma a}{\gamma}}\left[\overline{\Psi}_0(-\alpha,y)+\frac{\eta_{1i}}{ik}\overline{\Psi}_0'(-\alpha,y)\right] \\
&+\frac{(\frac{k}{\eta_{2i}}+\alpha)R_+(-\alpha)}{(\alpha^2-\breve{T}^2)\frac{\sin\gamma a}{\gamma}}\left(\cos\gamma a-\frac{ik}{\eta_{1i}}\frac{\sin\gamma a}{\gamma}\right),
\end{aligned} \tag{3.60}$$

where $\hat{S}_-(\alpha)$, $\hat{G}(\alpha)$ and $\breve{T}$ stand for

$$\hat{S}_-(\alpha) = (\frac{k}{\eta_{2i}}-\alpha)\overline{\Psi}_{1-}(\alpha,a) + (\frac{k}{\eta_{2i}}+\alpha)\overline{\Psi}_{1-}(\alpha,a), \tag{3.61}$$

$$\hat{G}(\alpha) = e^{i\gamma a}\frac{\sin\gamma a}{\gamma}, \tag{3.62}$$

$$\breve{T} = k\sqrt{1-\frac{1}{\eta_{1i}^2}}. \tag{3.63}$$

Using the factorization

$$\hat{G}(\alpha) = \hat{G}_+(\alpha)\hat{G}_-(\alpha), \tag{3.64}$$

and

$$\alpha^2-\breve{T}^2 = (\alpha+\breve{T})(\alpha-\breve{T}), \tag{3.65}$$

where the explicit expression of $\hat{G}_\pm(\alpha)$ are given in [13, 111] as

$$\hat{G}_+(\alpha) = \sqrt{a}e^{\{(\frac{\gamma a}{\pi})\ln[\frac{\alpha+i\gamma}{k}]\}}.e^{[(\frac{i\alpha a}{\pi})(1-C+\ln(\frac{2\pi}{ka})+i\frac{\pi}{2}]}\prod_{n=1}^{\infty}\left[\sqrt{1-(\frac{ka}{n\pi})^2}-\frac{i\alpha a}{n\pi}\right]e^{\frac{i\alpha a}{n\pi}}, \tag{3.66}$$

with

$$\hat{G}_-(\alpha) = \hat{G}_+(-\alpha). \tag{3.67}$$

Equation (3.60) can be written as:

$$
\begin{aligned}
& \frac{R_+(\alpha)}{(\alpha+\check{T})\hat{G}_+(\alpha)} - \frac{\frac{\eta_{1i}}{ik}}{\frac{(k^2-\gamma^2\eta_{1i}^2)}{\eta_{1i}^2}\frac{\sin\gamma a}{\gamma}} \times \\
& \left[\begin{array}{c} \Psi_0(\alpha)\left(\gamma\sin\gamma a + \frac{ik}{\eta_{1i}}\cos\gamma a\right) \\ +\Psi_0'(\alpha)\left(\cos\gamma a - \frac{ik}{\eta_{1i}}\frac{\sin\gamma a}{\gamma}\right) \end{array} \right] (\alpha-\check{T})\hat{G}_-(\alpha) \\
& -\frac{(\frac{k}{\eta_{2i}}+\alpha)}{(\alpha+\check{T})(\frac{k}{\eta_{2i}}-\alpha)\frac{\sin\gamma a}{\gamma}}\left[\Psi_0(-\alpha)+\frac{\eta_{1i}}{ik}\Psi_0'(-\alpha)\right]\hat{G}_-(\alpha) \\
& -\frac{(\frac{k}{\eta_{2i}}+\alpha)R_+(-\alpha)}{(\alpha+\check{T})\frac{\sin\gamma a}{\gamma}(\frac{k}{\eta_{2i}}-\alpha)}\left(\cos\gamma a - \frac{ik}{\eta_{1i}}\frac{\sin\gamma a}{\gamma}\right)\hat{G}_-(\alpha) \\
= \; & i\frac{\eta_{1i}}{k}\frac{\hat{S}_-(\alpha)(\alpha-\check{T})\hat{G}_-(\alpha)}{(\frac{k}{\eta_{2i}}-\alpha)}. \qquad (3.68)
\end{aligned}
$$

To make the second, third, fourth and fifth terms on the LHS of Eq. (3.68) regular in the upper and lower half planes, we shall apply theorem B $[13, pp14]$ as follows:

$$
\begin{aligned}
\hat{H}(\alpha) \; = \; & -\frac{\frac{\eta_{1i}}{ik}}{\frac{(k^2-\gamma^2\eta_{1i}^2)}{\eta_{1i}^2}\frac{\sin\gamma a}{\gamma}} \left[\begin{array}{c} \Psi_0(\alpha)\left(\gamma\sin\gamma a + \frac{ik}{\eta_{1i}}\cos\gamma a\right) \\ +\Psi_0'(\alpha,y)\left(\cos\gamma a - \frac{ik}{\eta_{1i}}\frac{\sin\gamma a}{\gamma}\right) \end{array} \right] (\alpha-\check{T})\hat{G}_-(\alpha) \\
& -\frac{(\frac{k}{\eta_{2i}}+\alpha)}{(\alpha+\check{T})(\frac{k}{\eta_{2i}}-\alpha)\frac{\sin\gamma a}{\gamma}}\left[\Psi_0(-\alpha)+\frac{\eta_{1i}}{ik}\Psi_0'(-\alpha,y)\right]\hat{G}_-(\alpha) \\
& -\frac{(\frac{k}{\eta_{2i}}+\alpha)R_+(-\alpha)}{(\alpha+\check{T})\frac{\sin\gamma a}{\gamma}(\frac{k}{\eta_{2i}}-\alpha)}.\left(\cos\gamma a - \frac{ik}{\eta_{1i}}\frac{\sin\gamma a}{\gamma}\right)\hat{G}_-(\alpha) \\
& = \hat{H}_+(\alpha) + \hat{H}_-(\alpha), \qquad (3.69)
\end{aligned}
$$

where

$$
\hat{H}_\pm(\alpha) = \pm\frac{1}{2\pi i}\int\frac{\hat{H}(\xi)}{\xi-\alpha}d\xi. \qquad (3.70)
$$

Using Eq. (3.69) in Eq. (3.68) and separating into positive and negative portions, we arrive at:

$$\frac{R_+(\alpha)}{(\alpha+\check{T})\hat{G}_+(\alpha)}+\hat{H}_+(\alpha)=i\frac{\eta_{1i}}{k}\frac{\hat{S}_-(\alpha)(\alpha-\check{T})\hat{G}_-(\alpha)}{(\frac{k}{\eta_{2i}}-\alpha)}+\hat{H}_-(\alpha). \tag{3.71}$$

The LHS of Eq. (3.71) is regular in upper half plane and the RHS is regular in lower half plane. Using the extended Liouville's theorem, both sides of Eq. (3.71) are equal to a constant zero. Therefore, we can write

$$\frac{R_+(\alpha)}{(\alpha+\check{T})\hat{G}_+(\alpha)}+\hat{H}_+(\alpha)=0. \tag{3.72}$$

Eq. (3.72) can be written as follows:

$$\frac{R_+(\alpha)}{(\alpha+\check{T})\hat{G}_+(\alpha)}=-\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{1}{\xi-\alpha}\left[\left\{-\frac{\eta_{1i}^3\gamma}{ik(k^2-\gamma^2\eta_{1i}^2)\sin\gamma a}\Psi_0(\xi)\left(\gamma\sin\gamma a+\frac{ik}{\eta_{1i}}\cos\gamma a\right)\right.\right.$$

$$\left.+\Psi_0'(\xi)\left(\cos\gamma a-\frac{ik}{\eta_{1i}}\frac{\sin\gamma a}{\gamma}\right)(\xi-\check{T})\hat{G}_-(\xi)\right\}$$

$$-\frac{\gamma(\frac{k}{\eta_{2i}}+\xi)R_+(-\xi)}{(\xi+\check{T})\sin\gamma a(\frac{k}{\eta_{2i}}-\xi)}\left(\cos\gamma a-\frac{ik}{\eta_{1i}}\frac{\sin\gamma a}{\gamma}\right)\hat{G}_-(\xi)$$

$$\left.-\frac{(\frac{k}{\eta_{2i}}+\xi)\gamma}{(\xi+\check{T})(\frac{k}{\eta_{2i}}-\xi)\sin\gamma a}\left(\Psi_0(-\xi)+\frac{\eta_{1i}}{ik}\Psi_0'(-\xi)\right)\hat{G}_-(\xi)\right]d\xi. \tag{3.73}$$

The above integral is solved by method of additive decomposition method [13, $pp-13$].

The solution of Wiener-Hopf Eq. (3.73) reads as

$$\begin{aligned}\frac{R_+(\alpha)}{(\alpha+\check{T})\hat{G}_+(\alpha)} &= \left(\frac{i}{8\pi^2}\right)\left(\frac{i\eta_{1i}}{k}\frac{(k\cos\theta_0-\check{T})\hat{G}_-(k\cos\theta_0)}{(\alpha-k\cos\theta_0)}\frac{2\eta_{1i}\sin\theta_0}{(1+\eta_{1i}\sin\theta_0)}e^{-ikr_0+i\frac{\pi}{4}}(\frac{2\pi}{kr_0})\right)\\ &+\frac{k}{\eta_{1i}}\frac{(\frac{k}{\eta_{2i}}-\check{T})R_+(\check{T})\hat{G}_+(\check{T})e^{\frac{ika}{\eta_{1i}}}}{\frac{\sin ka}{\eta_{1i}}(\alpha+\check{T})}+\sum_{n=1}^{\infty}\frac{(\frac{k}{\eta_{2i}}-\alpha_n)R_+(\alpha_n)\hat{G}_+(\alpha_n)(\frac{n\pi}{a})^2}{a\alpha_n(\check{T}-\alpha_n)(\alpha+\alpha_n)}\end{aligned} \quad (3.74)$$

The function $R_+(\alpha)$ depends upon the unknown series of constants $R_+(\check{T})$, $R_+(\alpha_1)$, $R_+(\alpha_2)$, $R_+(\alpha_3)$.... To find an approximate value for $R_+(\alpha)$, substitute $\alpha = \check{T}, \alpha_1, \alpha_{2,\ldots,}\alpha_m$ in Eq. (3.73) to get $m+1$ equations in $m+1$ unknowns. The simultaneous solution of these equations yields approximate solutions for $R_+(\check{T})$, $R_+(\alpha_1)$, $R_+(\alpha_2), \ldots R_+(\alpha_m)$.

3.2 The Far Field Solution

The scattered field in the region $y > a$ can be obtained by evaluating the value of $A(\alpha)$ using Eq. (3.74) in Eq. (3.41) and then putting the resulting equation into Eq. (3.17) and finally taking inverse Fourier transform of equation, the final expression for the diffracted field comes out to be:

$$\Psi_1(x,y) = \frac{1}{\sqrt{2\pi}}\int_L \frac{R_+(\alpha)}{(1+\frac{\eta_{1i}}{k}\gamma(\alpha))}e^{i\gamma(\alpha)(y-a)}e^{-i\alpha x}d\alpha, \quad (3.75)$$

where L is a straight line parallel to the real axis, lying in the strip $\operatorname{Im} k\cos\theta_0 < \operatorname{Im}\alpha < \operatorname{Im} k$.

To determine the far field behavior of the scattered field, introducing the following substitutions

$$x = r\cos\theta,\ y-a = r\sin\theta, \quad (3.76)$$

and the transformation

$$\alpha = -k\cos(\theta + it_1), \quad (3.77)$$

where t_1, given in Eq. (3.77), is real. The contour of integration over α in Eq. (3.75) goes into the branch of hyperbola around $-ik$ if $\frac{\pi}{2} < \theta < \pi$. We further observe that in deforming the contour into a hyperbola the pole $\alpha = \xi$ may be crossed. If we also make the transformation $\xi = k\cos(\theta_0 + it_2)$ the contour over ξ also goes into a hyperbola. The two hyperbolae will not cross each other if $\theta < \theta_0$. However, if the inequality is reversed there will be a contribution from pole which, in fact, cancels the incident wave in the shadow region. Omitting the details of calculations, the asymptotic evaluation of the integral in equation (3.75), using the method of steepest descent [107], the field due to a line source at a large distance from the edge is given by:

$$\begin{aligned}
\Psi_1(x,y) \;=\; & \Bigg[\Bigg\{ \left(\frac{1}{4\pi i}\right)\frac{i\eta_{1i}}{k} e^{-ika\sin\theta_0} \sin\frac{\theta}{2}\sin\frac{\theta_0}{2}.\left\{\mathcal{F}\left(\sqrt{2kr}\cos\frac{\theta_0-\theta}{2}\right) + \mathcal{F}\left(\sqrt{2kr}\cos\frac{\theta_0+\theta}{2}\right)\right\} \\
& \frac{(k\cos\theta_0 - \check{T})\hat{G}_-(k\cos\theta_0)}{k} . \frac{2\eta_{1i}}{(1+\eta_{1i}\sin\theta_0)(1+\eta_{1i}\sin\theta)} .. \frac{e^{ikr_0 + i\frac{\pi}{4}}}{\sqrt{r_0}} \Bigg\} \\
& + \frac{k}{\eta_{1i}} . \frac{(\frac{k}{\eta_{2i}} - \check{T})R_+(\check{T})\hat{G}_+(\check{T})e^{-\frac{ika}{\eta_{1i}}}}{(\frac{k}{\eta_{2i}} + \check{T})\sin(\frac{ka}{\eta_{1i}})(\check{T} - k\cos\theta)} + \sum_{n=1}^{\infty} \frac{(\frac{k}{\eta_{2i}} - \alpha_n)R_+(\alpha_n)\hat{G}_+(\alpha_n)(\frac{n\pi}{a})^2}{(\frac{k}{\eta_{2i}} + \alpha_n)a\alpha_n(\check{T} - \alpha_n)(\alpha_n - k\cos\theta)} \Bigg] \\
& . \frac{e^{ikr - i\frac{\pi}{4}}}{\sqrt{kr_0}} (\check{T} - k\cos\theta)\hat{G}_-(k\cos\theta), \qquad (3.78)
\end{aligned}$$

where $\mathcal{F}(z)$ stands for the Fresnel function as defined in [50, 13].

$$\mathcal{F}(z) = e^{-iz^2} \int_z^{\infty} e^{it^2} dt. \qquad (3.79)$$

3.3 The Numerical Results

In this section some graphical results have been presented highlighting the effects of impedance parameters η_{1i} and η_{2i}, step height a and line source parameter r_0 on the diffraction phenomenon. Figs. 3.3 and 3.4 show the variation of diffracted field versus the observation angle for different impedance values of η_{1i} while the step height which is purely capacitive with constant impedance $\eta_{2i} = 0.5i$ is fixed, the angle of incident ray is taken to be $\pi/4$ with $k = r = 1$, the impedance of the half planes are chosen purely capacitive and inductive for $r_0 = 1$, respectively.

A comparison of Figs. 3.3 and 3.4 shows that the scattered field amplitude corresponding to the case where the impedances of half planes are capacitive are weaker than those related to the case where the impedances of half planes are inductive. From Fig. 3.5 one can conclude that the scattering effects of the step can be reduced if its impedance is inductive. On comparing the graphs in Figs. 3.3 and 3.4 with the graphs in Figs. 4 (a & b) drawn in [56], given here as Figs 3.10 and 3.11, one can see that the behaviour of the graph is exactly the same and it differs only by the location along vertical-axis because of the multiplicative factor to the part of the scattered field containing the effects of line source. This is a counter check of the validation of our claim that if the source is shifted to a large distance these results differ from those of [55, 56] by a multiplicative factor to the part of the scattered field containing the effects of incident and reflected waves. Similarly, the graphs in Fig. 3.5 plotted are in close tie with the graph plotted in Fig. 5 of [56], given here as Fig. 3.12. Figs. 3.6 and 3.7 represent the variation of the scattered field versus the observation angle with $k = r = 1$ and by fixing $\eta_{1i} = \eta_{2i} = 0.5i$ and $\eta_{1i} = \eta_{2i} = -0.5i$, i.e., capacitive and inductive, respectively, for $r_0 = 1$. It is observed that the amplitude of the diffracted field increases with increase in step height which is an obvious result. Figs. 3.8 and 3.9 represents the variation of the scattered field versus observation angle with $k = r = 1$ and by fixing $\eta_{1i} = \eta_{2i} = 0.5i$ and $\eta_{1i} = \eta_{2i} = -0.5i$, i.e., capacitive and inductive, respectively, for $a = 1$, these graphs show that the amplitude of the diffracted field decreases as the source is taken away from the origin, which is a natural phenomenon and verifying the results.

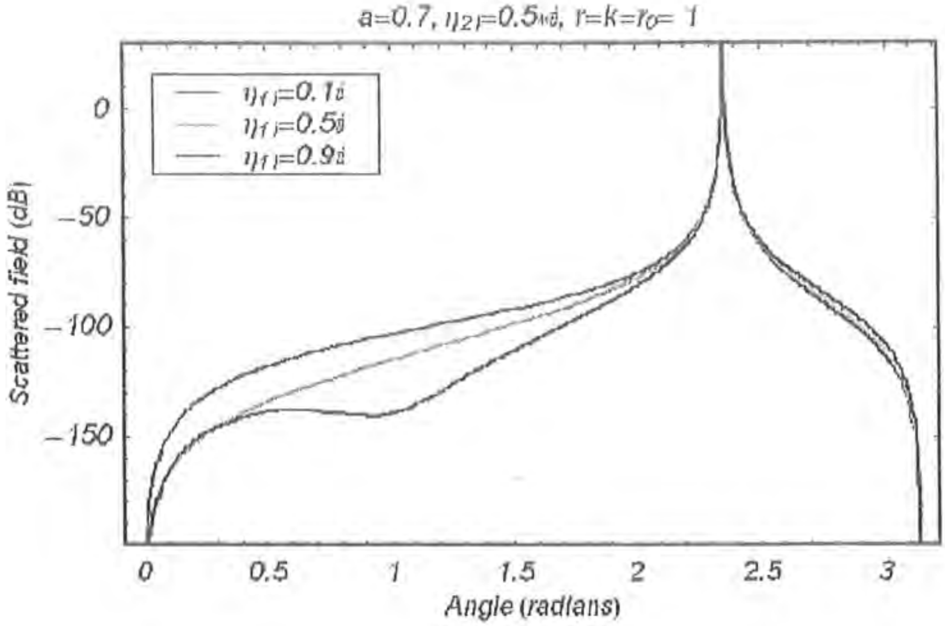


Fig. 3.3: Scattered field versus the observation angle for different values of the half-plane impedances "η_{1i}" when they are of capacitive nature.

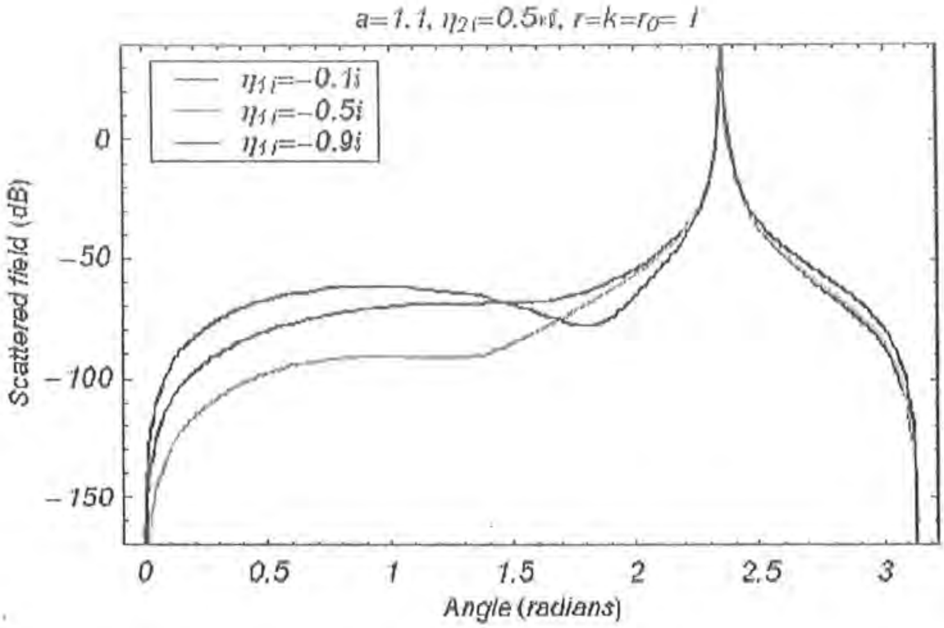


Fig. 3.4: Scattered field versus the observation angle for different values of the half-plane impedances "η_{1i} " when they are of inductive nature.

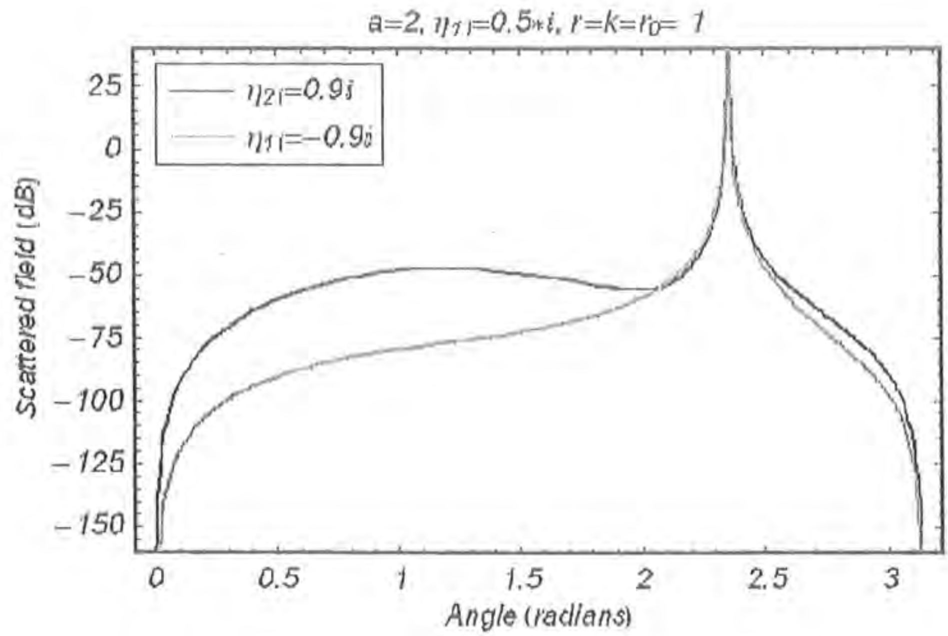


Fig. 3.5: Scattered field versus the observation angle for different values of step impedances"η_{2i} ".

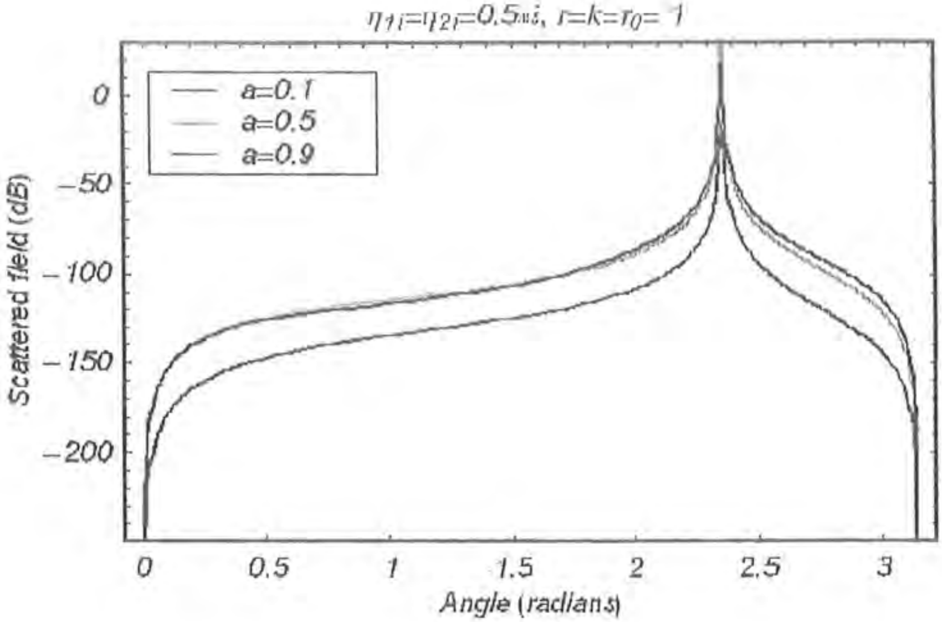


Fig. 3.6: Scattered field versus the observation angle for different values of the step height "a" when half planes and step have impedances of capacitive nature.

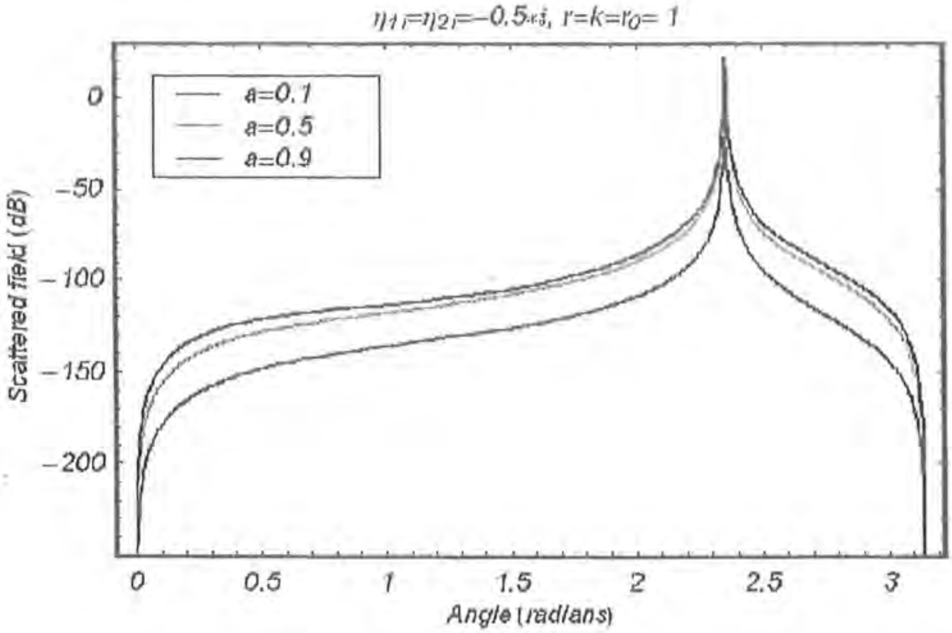


Fig. 3.7: Scattered field versus the observation angle for different values of the step height "a" when half planes and step have impedances of inductive nature.

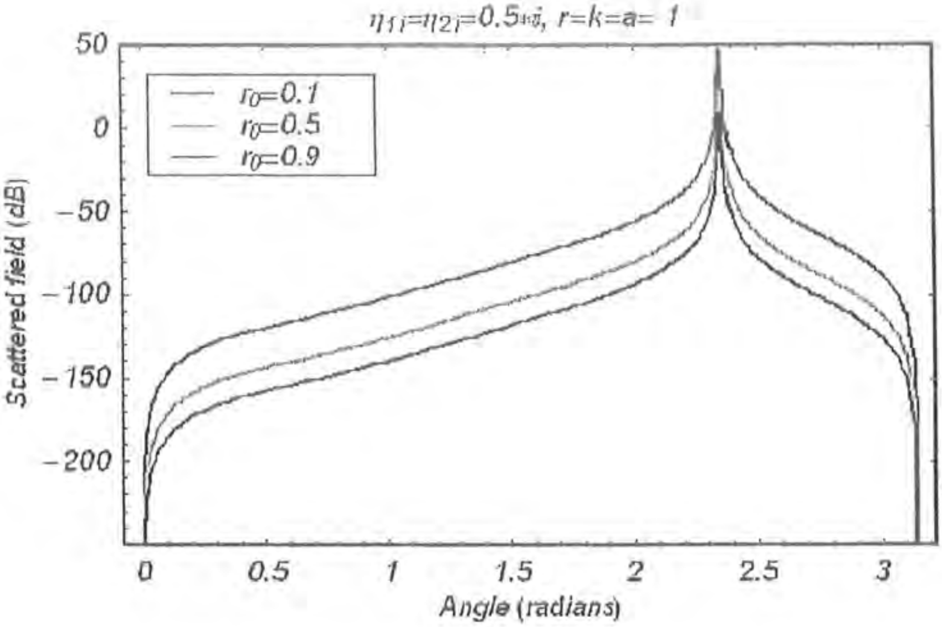


Fig. 3.8: Scattered field versus the observation angle for different values of "r_0" when half planes and step have impedances of capacitive nature.

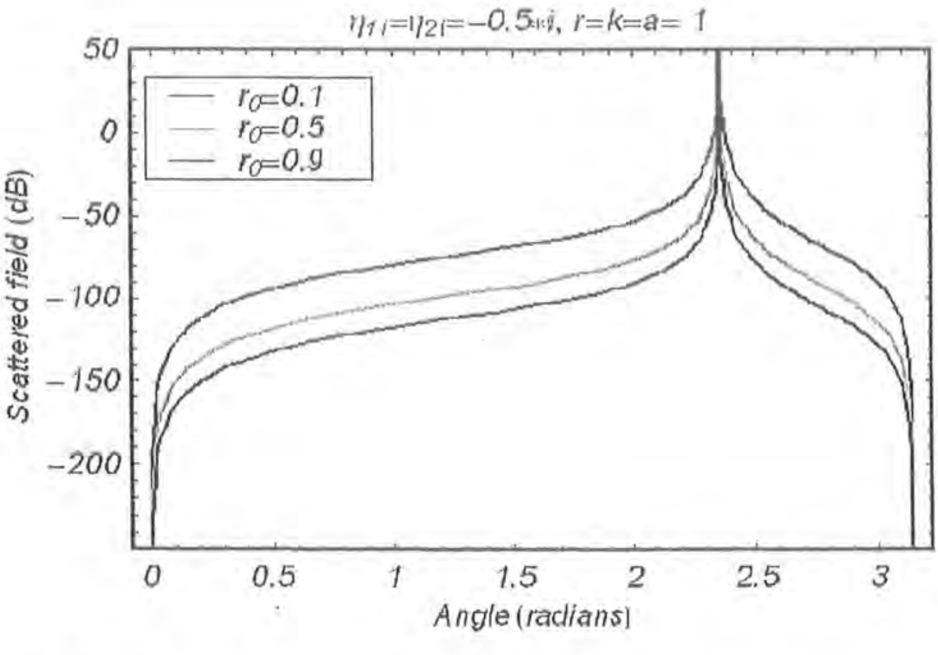
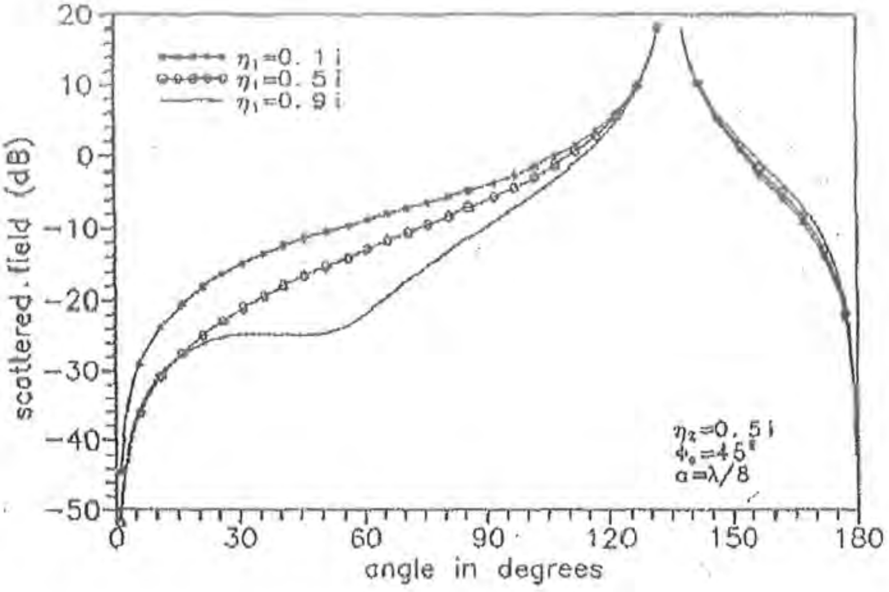
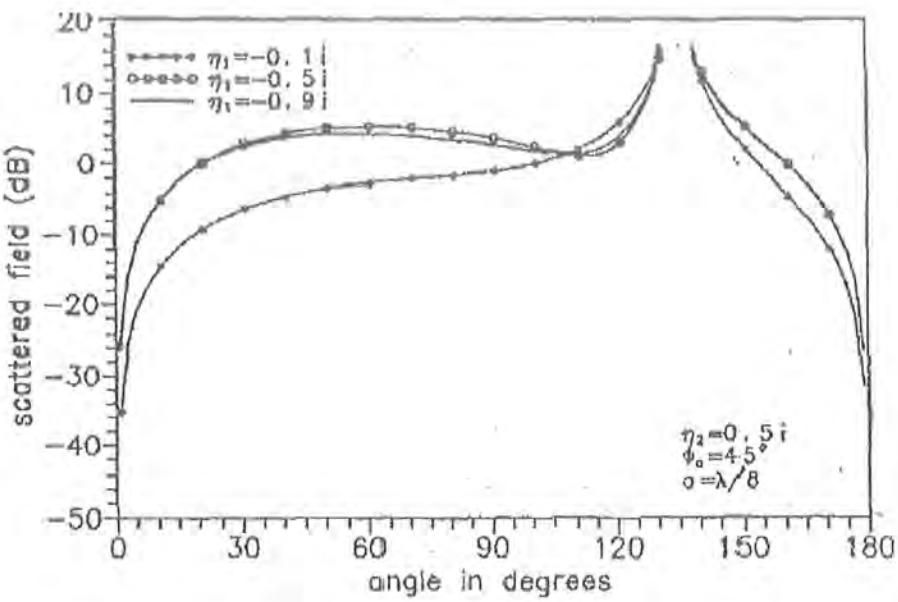


Fig. 3.9: Scattered field versus the observation angle for different values of "r_0" when half planes and step have impedances of inductive nature.



(a)

Fig. 3.10: Fig. 4. Scattered field versus the observation angle ϕ for different values of the half-plane impedances. (a) Half-plane impedances are capacitive, and (b) half-plane impedances are inductive.



(b)

Fig.3.11: Fig. 4. Scattered field versus the observation angle ϕ for different values of the half-plane impedances. (a) Half-plane impedances are capacitive, and (b) half-plane impedances are inductive.

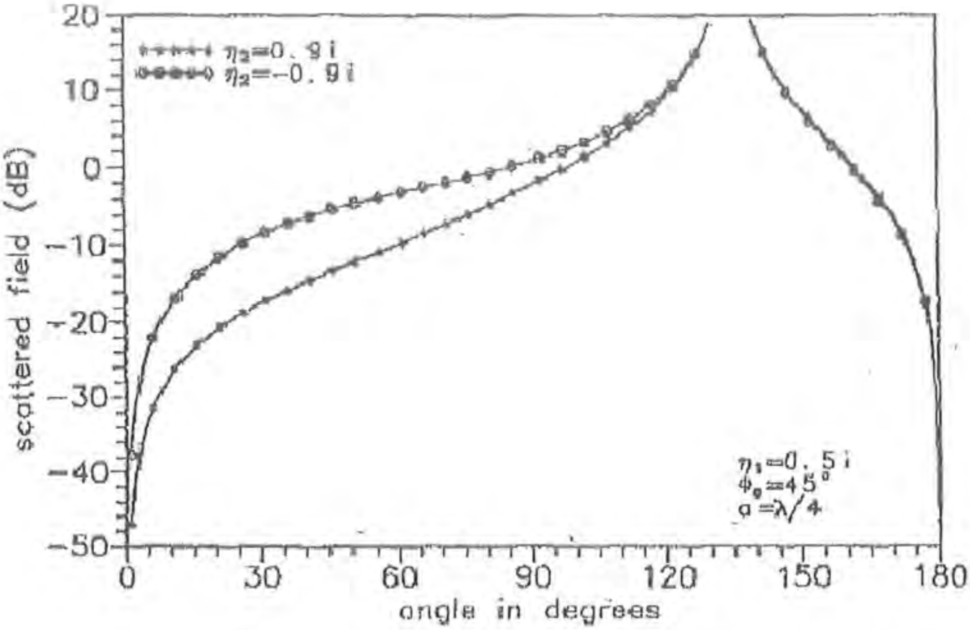


Fig. 3.12: Fig. 5. Scattered field versus the observation angle ϕ for different values of the step impedances.

Chapter 4

Line Source and Point Source Diffraction by a Reactive Step

The diffraction of a line and a point source by a reactive step joined by two half planes where the each half plane and step are characterized by different surface reactances have been studied. The problem is solved by using Wiener Hopf technique and the Fourier transform. The scattered field in the far zone is determined by the method of steepest descent [107]. Graphical results for the line source are also presented. It is observed that if the source is shifted to a large distance the results of the line source differ from those of [57] by a multiplicative factor to the part of the scattered field containing the effects of incident and reflected waves. Subsequently, the point source diffraction is examined using the results obtained for a line source diffraction

4.1 The Line Source Scattering

4.1.1 Formulation of the Problem

Consider the scattering due to a line source, located at (x_0, y_0), which illuminates two half planes $S_1 = \{x < 0,\ y = a,\ z \in (-\infty, \infty)\}$ and $S_2 = \{x > 0,\ y = 0,\ z \in (-\infty, \infty)\}$ with relative surface reactances η_{1r} and η_{3r} , respectively, and these are joined together by a step of height "a" having relative surface impedance η_{2i}. The geometry of the line source diffraction problem is depicted in $Fig.$ (4.1).

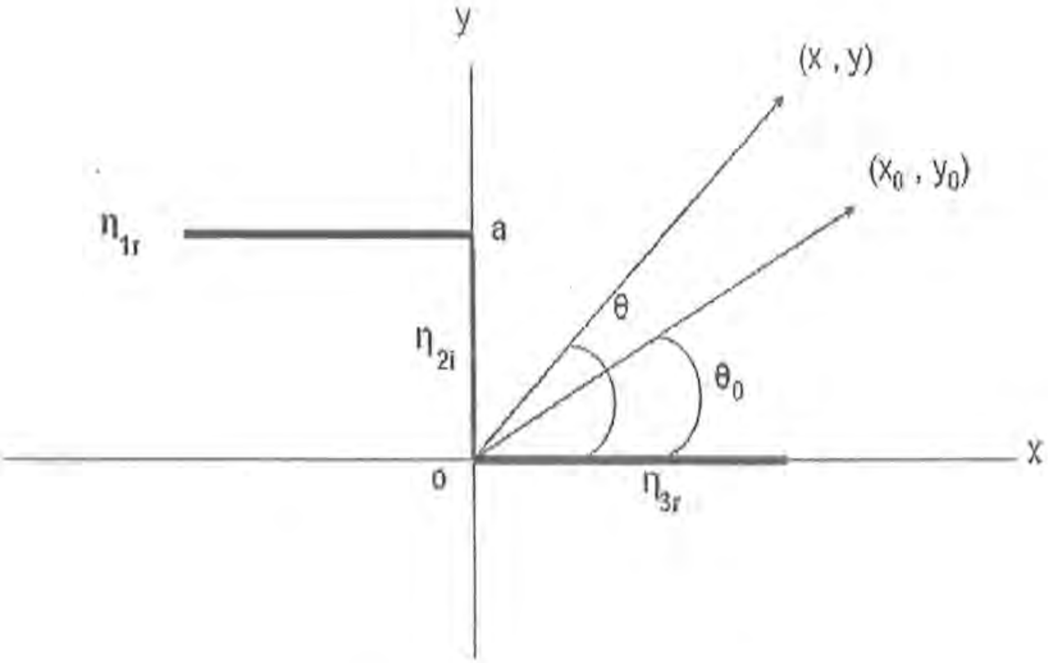


Fig. 4.1: Geometry of the problem

For harmonic vibrations of time dependence $e^{-i\omega t}$, we require the solution of the wave equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)\Psi_t(x,y)=\delta(x-x_0)\delta(y-y_0), \tag{4.1}$$

where Ψ_t is the total field. For the analysis purpose, the total field $\Psi_t(x,y)$ may be expressed as follows:

$$\Psi_t(x,y)=\begin{cases}\Psi^i(x,y)+\Psi_1^r(x,y)+\Psi_1(x,y), & y>a,\\ \Psi_2(x,y), & 0<y<a.\end{cases} \tag{4.2}$$

The boundary conditions at two half planes and a step are given by:

$$(1+\frac{\eta_{1r}}{k}\frac{\partial}{\partial y})\Psi_1(x,a)=0, \qquad x\in(-\infty,0), \tag{4.3a}$$

$$(1+\frac{\eta_{2i}}{ik}\frac{\partial}{\partial x})\Psi_2(0,y)=0, \qquad y\in(0,a), \tag{4.3b}$$

$$(1+\frac{\eta_{3r}}{k}\frac{\partial}{\partial y})\Psi_2(x,0)=0, \qquad x\in(0,\infty), \tag{4.3c}$$

$$\Psi_t(x,a)-\Psi_2(x,a)=0, \qquad x\in(0,\infty), \tag{4.3d}$$

$$\frac{\partial}{\partial y}\Psi_t(x,a)-\frac{\partial}{\partial y}\Psi_2(x,a)=0, \qquad x\in(0,\infty). \tag{4.3e}$$

where $k=\frac{w}{c}$ is the wave number and a time factor $e^{-i\omega t}$ is suppressed hereafter. It is assumed that the wave number k has positive imaginary part. The lossless case can be obtained by making $Imk\to 0$ in the final expressions. $\Psi_1(x,y)$ and $\Psi_2(x,y)$ are the scattered fields, $\Psi_1^r(x,y)$ is the field reflected from the plane located at $y=a$ and $\Psi^i(x,y)$ is the incident field satisfying the equation:

$$(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2)\Psi^i(x,y)=\delta(x-x_0)\delta(y-y_0). \tag{4.4}$$

The scattered field $\Psi_1(x,y)$ satisfies the Helmholtz equation

$$(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2)\Psi_1(x,y)=0, \qquad x\in(-\infty,\infty). \tag{4.5}$$

Fourier transform of Eq. (4.5) can be written as

$$(\frac{d^2}{dy^2}+K^2(\alpha))\overline{\Psi}_1(\alpha,y)=0, \tag{4.6}$$

with $\overline{\Psi}_1(\alpha, y)$ is divided into $\overline{\Psi}_{1+}(\alpha, y)$ and $\overline{\Psi}_{1-}(\alpha, y)$ as [13, 58] :

$$\overline{\Psi}_1(\alpha, y) = \overline{\Psi}_{1+}(\alpha, y) + \overline{\Psi}_{1-}(\alpha, y), \tag{4.6a}$$

where

$$\overline{\Psi}_{1\pm}(\alpha, y) = \pm\frac{1}{\sqrt{2\pi}}\int_0^{\pm\infty} \Psi_1(x, y)e^{i\alpha x}dx, \tag{4.6b}$$

and $K(\alpha) = \sqrt{k^2 - \alpha^2}$. The square root function $K(\alpha)$ is defined in the complex $\alpha-plane$ cut such that $K(0) = k$ [13] already given in Fig. (3.2):

The solution of Eq. (4.5) satisfying the radiation condition for $y > a$ can be written as:

$$\overline{\Psi}_1(\alpha, y) = \overline{\Psi}_{1+}(\alpha, y) + \overline{\Psi}_{1-}(\alpha, y) = A(\alpha)e^{iK(\alpha)|y-a|}, \tag{4.7}$$

where $A(\alpha)$ is the unknown coefficient to be determine. Following the same foot steps as given in last chapter, one can write incident and the corresponding reflected field using the method of Greens' function as follows:

$$\overline{\Psi}^i(\alpha, y) = \frac{1}{4\pi i K(\alpha)}e^{i\alpha x_0 + iK(\alpha)|y-y_0|}, \tag{4.8a}$$

and

$$\overline{\Psi}_1^r(\alpha, y) = \frac{i\eta_{1r}\sin\theta_0 - 1}{i\eta_{1r}\sin\theta_0 + 1}.\frac{1}{4\pi i K(\alpha)}e^{i\alpha x_0 + iK(\alpha)|(y-2a)+y_0|}, \tag{4.8b}$$

where θ_0 is the angle of incident wave.

The unknown coefficient $A(\alpha)$ appearing in Eq.(4.7) is to be determined with the help of boundary conditions $(4.3a)$ and $(4.3c - 4.3e)$ which after taking Fourier transform give:

$$(1 + \frac{\eta_{1r}}{k}\frac{\partial}{\partial y})\overline{\Psi}_{1-}(\alpha, a) = 0, \qquad x \in (-\infty, 0), \tag{4.9a}$$

$$(1 + \frac{\eta_{3r}}{k}\frac{\partial}{\partial y})\overline{\Psi}_2(\alpha, 0) = 0, \qquad x \in (0, \infty), \tag{4.9b}$$

$$\overline{\Psi}_t(\alpha,a)-\overline{\Psi}_2(\alpha,a)=0,, \qquad x\in(0,\infty), \tag{4.9c}$$

$$\frac{\partial}{\partial y}[\overline{\Psi}_t(\alpha,a)-\overline{\Psi}_2(\alpha,a)]=0, \qquad x\in(0,\infty), \tag{4.9d}$$

where $\overline{\Psi}_2$ is the Fourier transform of Ψ_2. Eq. $(4.9a)$ can be written as:

$$\overline{\Psi}_{1-}(\alpha,a)+\frac{\eta_{1r}}{k}\overline{\Psi}'_{1-}(\alpha,a)=0, \tag{4.10}$$

and prime "$\prime$ "denotes differentiation with respect to y. The differentiation of Eq. (4.7) w.r.t "y" at $y=a$ yields

$$\overline{\Psi}'_{1+}(\alpha,a)+\overline{\Psi}'_{1-}(\alpha,a)=iK(\alpha)A(\alpha). \tag{4.11}$$

By using Eqs. (4.10) and (4.11), we can write

$$\mathbb{R}_+(\alpha)=\frac{K(\alpha)}{k}\frac{A(\alpha)}{\chi(\eta_{1r},\alpha)}, \tag{4.12}$$

where

$$\mathbb{R}_+(\alpha)=\overline{\Psi}_1(\alpha,a)+\frac{\eta_{1r}}{k}\overline{\Psi}_2(\alpha,a), \tag{4.12a}$$

and

$$\chi(\eta_{1r},\alpha)=[i\eta_{1r}+\frac{k}{K(\alpha)}]^{-1}. \tag{4.12b}$$

Consider now the region $x>0$, $y\in(0,a)$. In this region $\Psi_2(x,y)$ satisfies the Helmholtz equation in the range $x\in(0,\infty)$:

$$(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2)\Psi_2(x,y)=0, \qquad x\in(0,\infty). \tag{4.13}$$

Multiplying Eq. (4.13) by $\frac{e^{i\alpha x}}{\sqrt{2\pi}}$ and integrating w.r.t "x" from 0 to ∞, we obtain

$$(\frac{d^2}{dy^2}+K^2(\alpha))\overline{\Psi}_2(\alpha,y)=(k+\eta_{2i}\alpha)f(y), \qquad (4.14)$$

with

$$\overline{\Psi}_2(\alpha,y)=\frac{1}{\sqrt{2\pi}}\int_0^\infty \Psi_2(x,y)e^{i\alpha x}dx, \qquad (4.14a)$$

while deriving Eq. (4.14), we have used Eq. (4.3b) and

$$f(y)=\frac{1}{k\sqrt{2\pi}}\frac{\partial}{\partial x}\Psi_2(0,y). \qquad (4.14b)$$

The solution of the inhomogeneous differential equation (4.14) can easily be obtained as follows:

$$\overline{\Psi}_2(\alpha,y)=B_1(\alpha)\cos Ky+C_1(\alpha)\sin Ky+\frac{(k+\eta_{2i}\alpha)}{K(\alpha)}\int_0^y f(t)\sin K(y-t)dt. \qquad (4.15)$$

By using the Eq. (4.9b), i.e.,

$$\overline{\Psi}_2(\alpha,y)+\frac{\eta_{3r}}{k}\overline{\Psi}_2'(\alpha,0)=0. \qquad (4.16)$$

The Eq. (4.15) reduces to

$$\overline{\Psi}_2(\alpha,y)=C_1(\alpha)[\sin Ky-\frac{\eta_{3r}}{k}K\cos Ky]+\frac{(k+\eta_{2i}\alpha)}{K(\alpha)}\int_0^y f(t)\sin K(y-t)dt. \qquad (4.17)$$

The Eq. (4.9c) can be written as:

$$\int_0^\infty [\Psi^i(x,a)+\Psi_1^r(x,a)+\Psi_1(x,a)-\Psi_2(\alpha,a)]e^{i\alpha x}dx=0, \qquad (4.18)$$

which can further be expressed as:

$$\overline{\Psi}_0(\alpha) + \overline{\Psi}_{1_+}(\alpha) - \overline{\Psi}_2(\alpha, a) = 0, \tag{4.19}$$

where

$$\overline{\Psi}_0(\alpha) = \int_0^{\infty} [\Psi^i(x,a) + \Psi_1^r(x,a)] e^{i\alpha x} dx. \tag{4.20}$$

Similarly, Eq. (4.9d) can be reproduced as:

$$\overline{\Psi}_0'(\alpha) + \overline{\Psi}_{1_+}'(\alpha) - \overline{\Psi}_2'(\alpha, a) = 0, \tag{4.21}$$

where

$$\overline{\Psi}_0'(\alpha) = \int_0^{\infty} \frac{\partial}{\partial y} [\Psi^i(x,a) + \Psi_1^r(x,a)] e^{i\alpha x} dx. \tag{4.22}$$

The addition of Eq. (4.19) with $\frac{\eta_{1r}}{k}$ times Eq. (4.21) gives

$$\mathbb{R}_+(\alpha) + [\overline{\Psi}_0(\alpha) + \frac{\eta_{1r}}{k}\overline{\Psi}_0'(\alpha)] = \overline{\Psi}_2(\alpha, a) + \frac{\eta_{1r}}{k}\overline{\Psi}_2'(\alpha, a). \tag{4.23}$$

Substituting Eq. (4.17) and its derivative w.r.t to "y" into Eq. (4.23) enables one to solve $C_1(\alpha)$ as follows:

$$C_1(\alpha) = -\frac{k^2}{\eta_{1r}\eta_{3r} K M(\alpha)} \left\{ \begin{array}{c} \mathbb{R}_+(\alpha) + \left(\overline{\Psi}_0(\alpha) + \frac{\eta_{1r}}{k}\overline{\Psi}_0'(\alpha)\right) \\ -(k + \eta_{2i}\alpha) \int_0^a f(t) \left[\frac{\sin K(a-t)}{K} + \frac{\eta_{1r}}{k} \cos K(a-t)\right] dt \end{array} \right\}, \tag{4.24}$$

with

$$M(\alpha) = (\alpha^2 - \check{T}^2)\frac{\sin Ka}{K} - k(\frac{1}{\eta_{3r}} - \frac{1}{\eta_{1r}}) \cos Ka, \tag{4.24a}$$

and

$$T = k\sqrt{1 + \frac{1}{\eta_{1r}\eta_{3r}}}. \tag{4.24b}$$

Placing Eq. (4.24) into Eq. (4.17), we obtain

$$\begin{aligned}
\overline{\Psi}_2(\alpha, y) &= -\frac{k^2}{\eta_{1r}\eta_{3r}} \frac{\frac{\sin Ky}{K} - \frac{\eta_{3r}}{k}\cos Ky}{M(\alpha)} \\
&\quad \times \left\{ \begin{array}{c} \mathbb{R}_+(\alpha) + \left(\overline{\Psi}_0(\alpha) + \frac{\eta_{1r}}{k}\overline{\Psi}_0'(\alpha)\right) \\ -(k + \eta_{2i}\alpha)\int_0^a f(t)\left[\frac{\sin K(a-t)}{K} + \frac{\eta_1}{k}\cos K(a-t)\right] dt \end{array} \right\} \\
&\quad + \frac{(k + \eta_{2i}\alpha)}{K(\alpha)} \int_0^y f(t) \sin K(y-t) dt.
\end{aligned} \tag{4.25}$$

The LHS of Eq. (4.25) is regular in the upper half of the complex α plane. RHS is also regular in the upper half plane except at the zeros of

$$M(\alpha_m) = 0, \qquad m = 1, 2..., \quad \mathrm{Im}(\alpha_m) > \mathrm{Im}(k). \tag{4.26}$$

To make the RHS regular in the upper half plane, the residues at $\alpha = \alpha_m$ must be zero. That is,

$$\mathbb{R}_+(\alpha_m) + \left(\overline{\Psi}_0(\alpha_m) + \frac{\eta_{1r}}{k}\overline{\Psi}_0'(\alpha_m)\right) = (k + \eta_{2i}\alpha_m) \int_0^a f(t) \left[\frac{\sin K_m(a-t)}{K_m} + \frac{\eta_{1r}}{k}\cos K_m(a-t)\right] dt, \tag{4.27}$$

with

$$K_m = K(\alpha_m). \tag{4.27a}$$

The Eq. (4.24a), at $\alpha = \alpha_m$, gives

$$\frac{\sin K_m a}{K_m} + \frac{\eta_{1r}}{k}\cos K_m a = \frac{\eta_{3r}}{k}\left[\cos K_m a - \frac{\eta_{1r}}{k}\sin K_m a\right]. \tag{4.28}$$

Now, the Eq. (4.27) can be written as

$$\mathbb{R}_+(\alpha_m) = -(k + \eta_{2i}\alpha_m)\left[\cos K_m a - \frac{\eta_{1r}}{k}\sin K_m a\right] Q_1^2 f_m - \left[\overline{\Psi}_0(\alpha_m) + \frac{\eta_{1r}}{k}\overline{\Psi}_0'(\alpha_m)\right], \tag{4.29}$$

with

$$f_m = \frac{1}{Q_1^2}\int_0^a f(t)\left[\frac{\sin K_m t}{K_m} - \frac{\eta_{3r}}{k}\cos K_m t\right] dt, \tag{4.29a}$$

and

$$Q_1^2 = \int_0^a \left[\frac{\sin K_m t}{K_m} - \frac{\eta_{3r}}{k}\cos K_m t\right]^2 dt,$$

or

$$Q_1^2 = \frac{\eta_{1r}\eta_{3r}}{k^2}\frac{\dot{M}(\alpha_m)}{2\alpha_m}\left[\cos K_m a + \frac{\eta_{3r}}{k}\sin K_m a\right]. \tag{4.29b}$$

In Eq. (4.29*b*), $\dot{M}(\alpha)$ denotes the derivative of $M(\alpha)$ w.r.t α. From Eqs. (4.7), (4.12) and (4.19), we have

$$-\overline{\Psi}_{1_-}^{'}(\alpha, a) + \frac{\mathbb{R}_+(\alpha)}{1 + i\eta_{1r}\frac{K(\alpha)}{k}} = \overline{\Psi}_2(\alpha, a) + \frac{\left(\overline{\Psi}_0(\alpha) + \frac{\eta_{1r}}{k}\overline{\Psi}_0'(\alpha)\right)}{1 + i\eta_{1r}\frac{K(\alpha)}{k}} - \overline{\Psi}_0(\alpha). \tag{4.30}$$

By using Eq. (4.25) and the fact that Eq. (4.24*a*) can also be written as

$$M(\alpha) = -\frac{k^2}{\eta_{1r}\eta_{3r}}\left[\left\{\frac{\sin Ka}{K} + \frac{\eta_{1r}}{k}\cos Ka\right\} - \frac{\eta_{3r}}{k}\left\{\cos Ka - \frac{\eta_{1r}}{k}K\sin Ka\right\}\right] \tag{4.31}$$

Eq. (4.30) reduces to

$$-\frac{k^2}{\eta_{1r}\eta_{3r}}\frac{\mathbb{R}_+(\alpha)\chi(\eta_{1r},\alpha)}{N(\alpha)\chi(\eta_{3r},\alpha)} + \overline{\Psi}_{1_-}'(\alpha, a) = \overline{\Psi}_0'(\alpha) - (ik\chi(\eta_{1r},\alpha))\left[\overline{\Psi}_0(\alpha) + \frac{\eta_{1r}}{k}\overline{\Psi}_0'(\alpha)\right]$$

$$+\frac{k^2(k + \eta_2\alpha)}{\eta_{1r}\eta_{3r}M(\alpha)}\int_0^a \left[\frac{\sin Kt}{K} - \frac{\eta_{3r}}{k}\cos Kt\right] dt, \tag{4.32}$$

with

$$N(\alpha) = e^{iKa}M(\alpha). \tag{4.32a}$$

Let us consider the following eigen function expansion of $f(t)$:

$$f(t) = \sum_{m=1}^{\infty} f_m \left[\frac{\sin K_m t}{K_m} - \frac{\eta_{3r}}{k} \cos K_m t \right], \tag{4.33}$$

with f_m being given by Eq. (4.29a). Invoking Eq. (4.33) into Eq. (4.32), we obtain, after term by term integration

$$-\frac{k^2}{\eta_{1r}\eta_{3r}} \frac{\mathbb{R}_+(\alpha)\chi(\eta_{1r},\alpha)}{N(\alpha)\chi(\eta_{3r},\alpha)} + \overline{\Psi}'_{1_-}(\alpha,a) = \overline{\Psi}'_0(\alpha) - (ik\chi(\eta_{1r},\alpha)) \left[\overline{\Psi}_0(\alpha) + \frac{\eta_{1r}}{k}\overline{\Psi}'_0(\alpha) \right]$$

$$+(k+\eta_{2i}\alpha) \sum_{m=1}^{\infty} f_m \frac{\left[\frac{\cos K_m t}{K_m} - \frac{\eta_{3r}}{k} \sin K_m t \right]}{\alpha^2 - \alpha_m^2}, \tag{4.34}$$

which may be considered as a *modified Wiener-Hopf equation of second kind* valid in the strip $Im(k\cos\theta_0) < Im(\alpha) < Im(k)$.

To solve Eq. (4.34), we can use the classical Wiener-Hopf procedure to obtain

$$\begin{aligned} \frac{\mathbb{R}_+(\alpha)\chi_+(\eta_{1r},\alpha)}{N_+(\alpha)\chi_+(\eta_{3r},\alpha)} &= -\eta_{1r}\eta_{3r} \frac{1}{4\pi^2 k^2} \frac{k\sin\theta_0}{(i\eta_{1r}\sin\theta_0 + 1)} \frac{e^{-ika\sin\theta_0}}{(k\cos\theta_0 - \alpha)} \\ &\quad \frac{N_-(k\cos\theta_0)\chi_-(\eta_{3r},k\cos\theta_0)}{\chi_-(\eta_{1r},k\cos\theta_0)} \frac{\sqrt{2\pi}}{\sqrt{k\rho_0}} \\ &\quad + \sum_{m=1}^{\infty} \frac{\mathbb{R}_+(\alpha_m)(k-\eta_{2i}\alpha_m)}{\acute{M}(\alpha_m)(k+\eta_{2i}\alpha_m)} \frac{N_+(\alpha_m)\chi_+(\eta_{3r},\alpha_m)}{(\alpha+\alpha_m)\chi_+(\eta_{1r},\alpha_m)} \\ &\quad \cdot \left[\frac{(k^2+\eta_{3r}K_m^2)}{(k^2+\eta_{1r}\eta_{3r}K_m^2)} \right] \cos K_m a. \end{aligned} \tag{4.35}$$

In Eq.(4.35), $N_+(\alpha)$ [resp. $N_-(\alpha)$] and $\chi_+(\eta,\alpha)$ [resp. $\chi_-(\eta,\alpha)$] are the split functions, regular and free of zeros in the upper half-plane $Im(\alpha) > Im(-k)$ [resp. in the lower half-plane $Im(\alpha) > Im(-k)$], resulting from the Wiener-Hopf factorization Eq. (4.32a)

and Eq. (4.12b) as

$$N(\alpha) = N_+(\alpha)N_-(\alpha), \qquad N_-(\alpha) = N_+(-\alpha), \tag{4.36}$$

and

$$\chi(\eta_{1r}, \alpha) = \chi_+(\eta_{1r}, \alpha)\chi_-(\eta_{1r}, \alpha), \qquad \chi_-(\eta_{1r}, \alpha) = \chi_+(\eta_{1r}, -\alpha). \tag{4.37}$$

By following the method described in [13], the explicit expressions of $N_+(\alpha)$ can be obtained [57]:

$$\begin{aligned} N_+(\alpha) &= \left[k(\frac{1}{\eta_{1r}} - \frac{1}{\eta_{3r}})\cos ka - \hat{T}^2 \frac{\sin ka}{k}\right]^{1/2} \times \exp\left\{\frac{Ka}{\pi}\ln\left(\frac{\alpha + iK}{k}\right)\right\} \\ &\quad \exp\left\{\frac{i\alpha a}{\pi}\left(1 - C + \ln\left(\frac{2\pi}{ka}\right) + i\frac{\pi}{2}\right)\right\} \\ &\quad \times \prod_{n=1}^{\infty}\left(1 + \frac{\alpha}{\alpha_n}\right)\exp\left(\frac{i\alpha a}{n\pi}\right). \end{aligned} \tag{4.38}$$

The split function $\chi_+(\eta, \alpha)$ can be expressed explicitly in terms of Maliuzhinetz function [100]

$$\begin{aligned} \chi_+(\eta, k\cos\theta) &= 2^{3/2}\sqrt{\frac{2}{\eta}}\sin\frac{\theta}{2} \times \\ &\quad \left\{\frac{\{M_\pi(3\pi/2 - \theta - \phi)\, M_\pi(\pi/2 - \theta + \phi)\}}{M_\pi^2(\pi/2)}\right\}^2 \\ &\quad \times \left\{\begin{array}{c} \left[1 + \sqrt{2}\cos\left(\frac{\pi/2 - \theta - \phi}{2}\right)\right] \\ \times\left[1 + \sqrt{2}\cos\left(\frac{3\pi/2 - \theta - \phi}{2}\right)\right] \end{array}\right\}^{-1}, \end{aligned} \tag{4.39}$$

with

$$\sin\phi = \frac{1}{\eta}, \tag{4.40}$$

$$M_\pi(z) = \exp\left\{\frac{-1}{8\pi}\int_0^z \frac{\pi\sin u - 2\sqrt{2}\pi\sin\left(\frac{u}{2}\right) + 2u}{\cos u}du\right\}. \tag{4.41}$$

From Eq. (4.35), we observe that $\mathbb{R}_+(\alpha)$ depends upon the unknown series of constants $\mathbb{R}_+(\alpha_1)$, $\mathbb{R}_+(\alpha_2)$, $\mathbb{R}_+(\alpha_3)$.... To find an approximate value for $\mathbb{R}_+(\alpha)$, substitute $\alpha = \alpha_1, \alpha, ..., \alpha_n$ to get n equations in n unknowns. The simultaneous solution of these equations yields approximate solutions for $\mathbb{R}_+(\alpha_m)$, $m = 1, 2, ..., n$.

4.1.2 Analysis of the Field

The scattered field in the region $y > a$ can be obtained by evaluating the value of $A(\alpha)$ using Eq. (4.35) in Eq. (4.12) and then putting the resulting equation into Eq. (4.7) and finally taking inverse Fourier transform of equation, the final expression for the diffracted field comes out to be:

$$\Psi_1(x,y) = \frac{1}{\sqrt{2\pi}} \int_L \frac{\mathbb{R}_+(\alpha)}{(1 + \frac{i\eta_{1r}}{k} K(\alpha))} e^{iK(\alpha)(y-a)} e^{-i\alpha x} d\alpha, \qquad y > a, \tag{4.42}$$

where L is a straight line parallel to the real axis, lying in the strip $\mathrm{Im}(k \cos\theta_0) < \mathrm{Im}\,\alpha < \mathrm{Im}\,k$.

To determine the far field behavior of the scattered field we introduce the following substitutions

$$x = \rho\cos\theta,\ y - a = \rho\sin\theta, \tag{4.42a}$$

and the transformation

$$\alpha = -k\cos(\theta + it_1), \tag{4.42b}$$

where t_1, given in Eq. (42b) is real. The contour of integration over α in Eq. (42) goes into the branch of hyperbola around $-ik$ if $\frac{\pi}{2} < \theta < \pi$. We further observe that in deforming the contour into a hyperbola the pole $\alpha = \xi$ may be crossed. If we also make the transformation $\xi = k\cos(\theta_0 + it_2)$ the contour over ξ also goes into a hyperbola. The two hyperbolae will not cross each other if $\theta < \theta_0$. However, if the inequality is reversed there will be a contribution from pole which, in fact, cancels the incident wave in the shadow region. Omitting the details of calculations, the asymptotic evaluation of the integral in Eq. (4.42), using the method of steepest descent [104], the field due to a line source at a large distance from the edge is given by:

$$\Psi_1(x,y) \sim \Psi_{11}^a(x,y) + \Psi_{12}^a(x,y), \tag{4.43}$$

with

$$\begin{aligned}\Psi_{11}^a(\rho,\theta) &\sim (\frac{\sqrt{2\pi}}{4\pi^2})\frac{i\eta_{1r}\eta_{3r}}{k}e^{-ika\sin\theta_0}.\frac{e^{ik\rho_0+i\frac{\pi}{2}}}{\sqrt{k\rho_0}};\frac{e^{ik\rho}}{\sqrt{k\rho}}\times\frac{\chi_-(\eta_{3r},k\cos\theta_0)\chi_-(\eta_{3r},k\cos\theta)}{\chi_-(\eta_{1r},k\cos\theta_0)\chi_-(\eta_{1r},k\cos\theta)}\\ &\times\frac{N_-(k\cos\theta_0)N_-(k\cos\theta)}{(i\eta_{1r}\sin\theta_0+1)(i\eta_{1r}\sin\theta+1)}\times\frac{\sin\theta\sin\theta_0}{\cos\theta+\cos\theta_0},\end{aligned} \tag{4.44}$$

and

$$\begin{aligned}\Psi_{12}^a(\rho,\theta) &\sim e^{-i\frac{\pi}{4}}\frac{k\sin\theta}{(i\eta_{1r}\sin\theta+1)}N_-(k\cos\theta)\frac{\chi_-(\eta_{3r},k\cos\theta)}{\chi_-(\eta_{1r},k\cos\theta)}\frac{e^{ik\rho}}{\sqrt{k\rho}}\\ &\times\sum_{m=1}^{\infty}\frac{\mathbb{R}_+(\alpha_m)(k-\eta_{2i}\alpha_m)}{\acute{M}(\alpha_m)(k+\eta_{2i}\alpha_m)}\frac{N_+(\alpha_m)}{(\alpha_m-k\cos\theta_0)}\frac{\chi_+(\eta_{3r},\alpha_m)}{\chi_+(\eta_{1r},\alpha_m)}\\ &\left[\frac{(k^2+\eta_{3r}K_m^2)}{(k^2+\eta_{1r}\eta_{3r}K_m^2)}\right]\cos K_m a.\end{aligned} \tag{4.45}$$

4.1.3 Numerical Solution

Now, we will present some graphical results showing the effects of reactance and impedance parameters η_{1r} , η_{3r} and η_{2i}, respectively, step height a and line source parameter ρ_0 on the diffraction phenomenon.

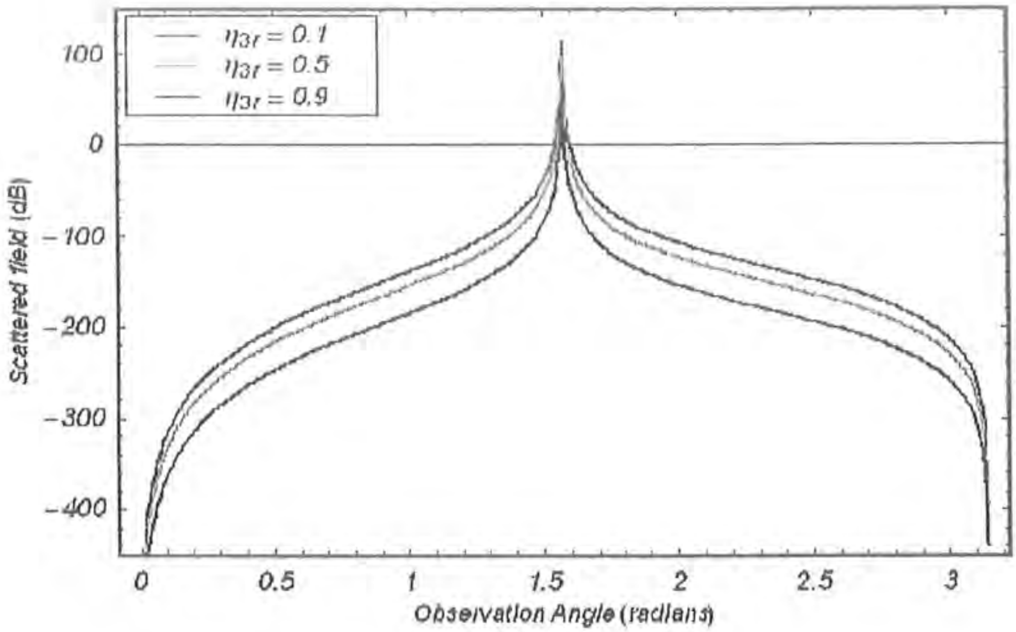


Fig. 4.2: Scattered field (dB) versus the observation angle (radians) for,

$a = 0.1\lambda, \eta_{1r} = 0.2, \eta_{2i} = 0.3i, \rho = \rho_0 = k = 1.$

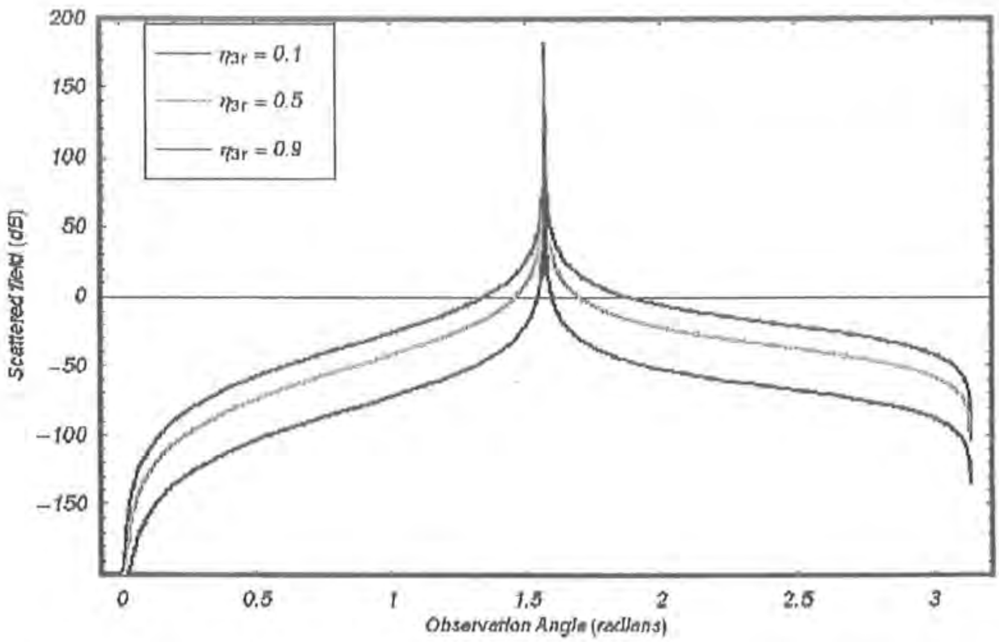


Fig.4.3: Scattered field(dB) versus the observation angle(radians) for,

$a = 0.25\lambda, \theta_0 = \frac{\pi}{2}, \eta_{1r} = 0.2, \eta_{2i} = 0.3i, \rho = \rho_0 = k = 1.$

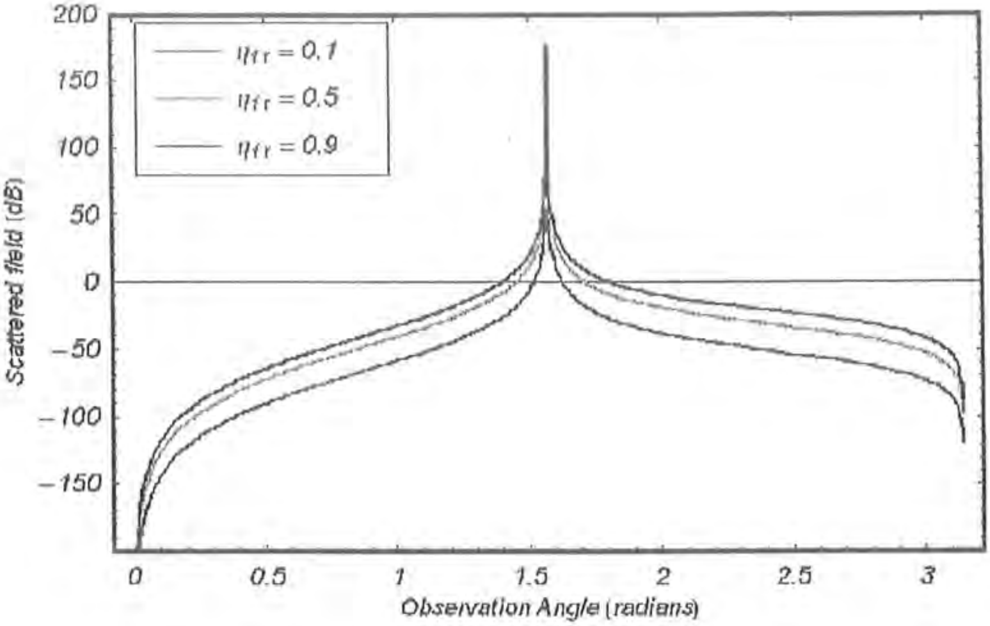


Fig. 4.4: Scattered field (dB) versus the observation angle (radians) for,

$a = 0.1, \lambda, \theta_0 = \frac{\pi}{2}, \eta_{3r} = 0.2, \eta_{2i} = 0.3i, \rho = \rho_0 = k = 1.$

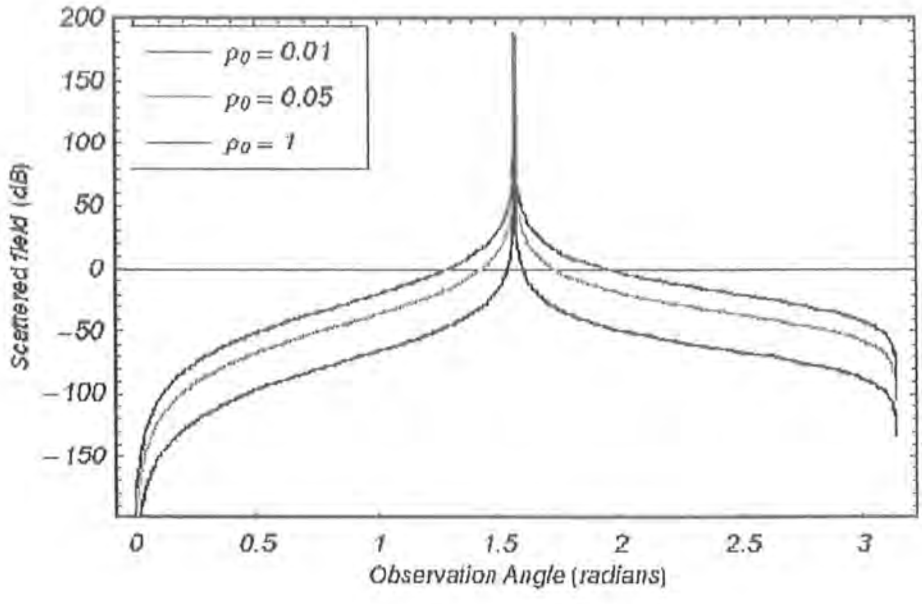


Fig. 4.5: Scattered field (dB) versus the observation angle (radians) for,

$a = 0.1\lambda, \theta_0 = \frac{\pi}{2}, \eta_{1r} = \eta_{3r} = 0.2, \eta_{2i} = 0.3i, \rho = k = 1.$

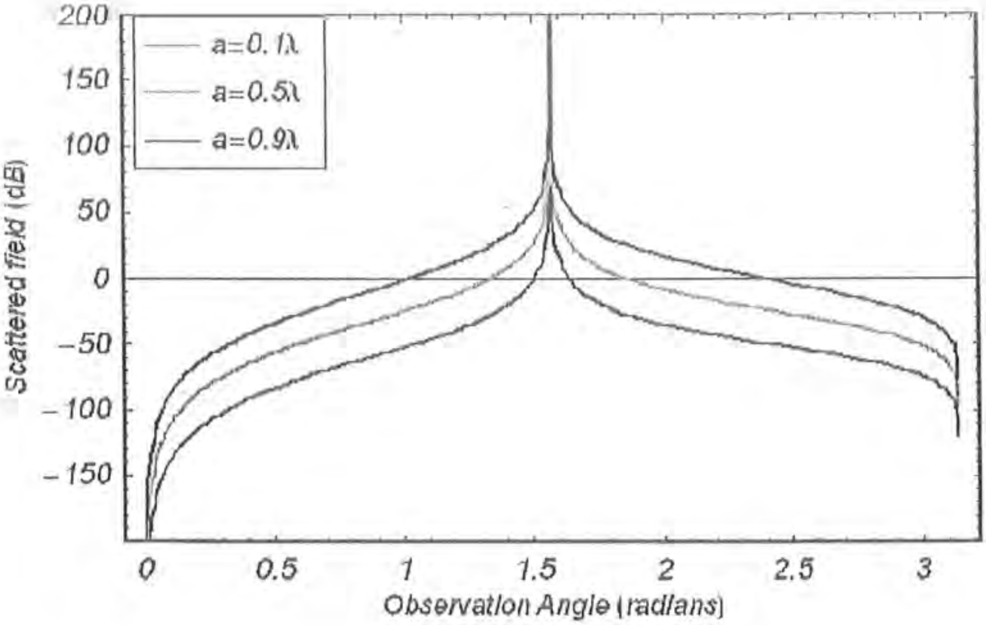


Fig. 4.6: Scattered field (dB) versus the observation angle (radians) for,

$\theta_0 = \frac{\pi}{2}, \eta_{1r} = \eta_{3r} = 0.2\ , \eta_{2i} = 0.3i, \rho = k = \rho_0 = 1.$

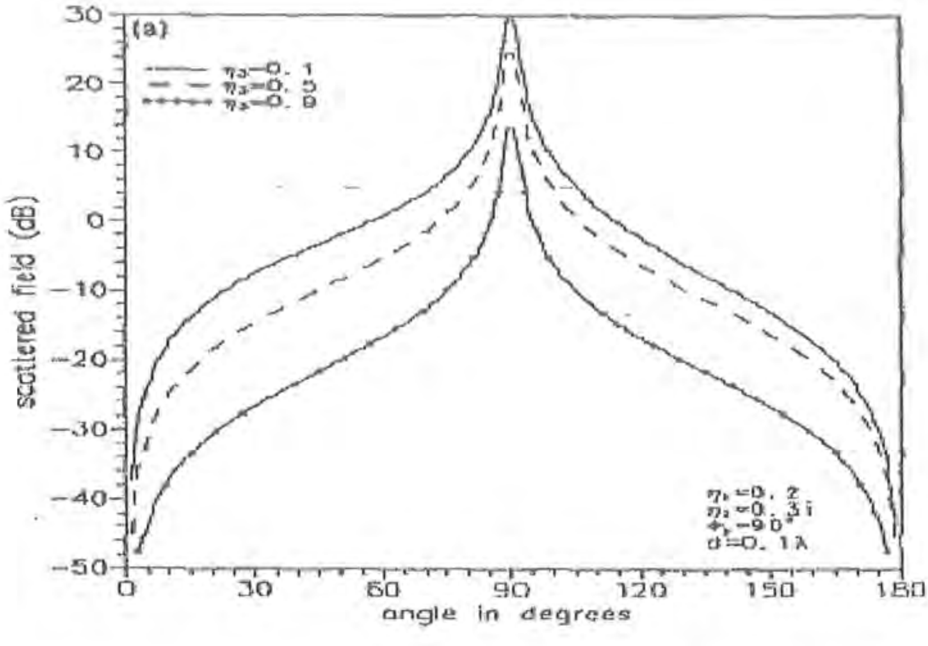


Fig.4.7: Fig. 3. Scattered field versus the observation angle ϕ for different values of half-plane reactance η_3: (a) $d = 0.1\lambda$; (b) $d = 0.25\lambda$.

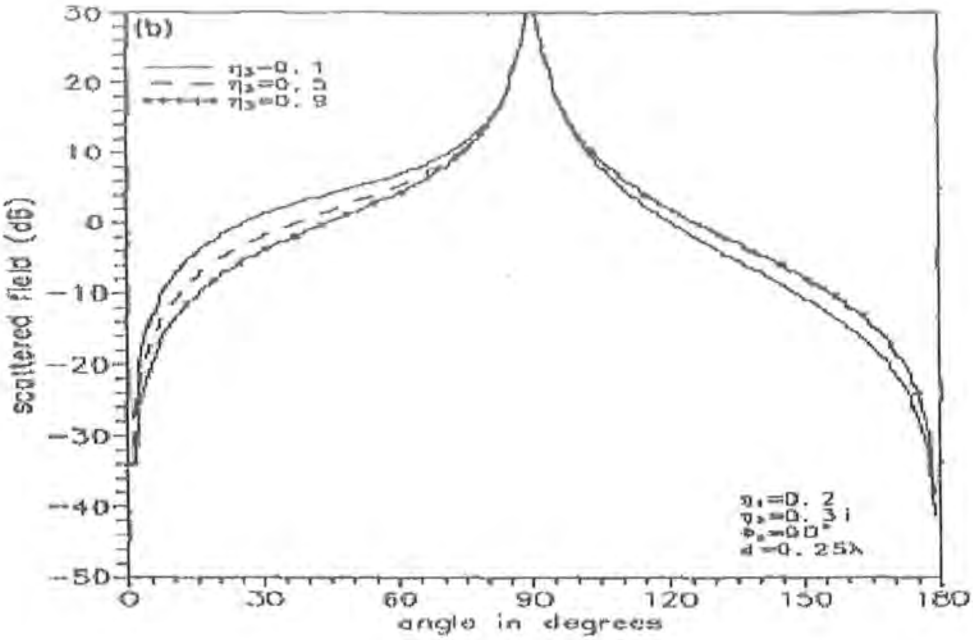


Fig.4.8: Fig. 3. Scattered field versus the observation angle ϕ for different values of half-plane reactance η_3: (a) $d = 0.1\lambda$; (b) $d = 0.25\lambda$.

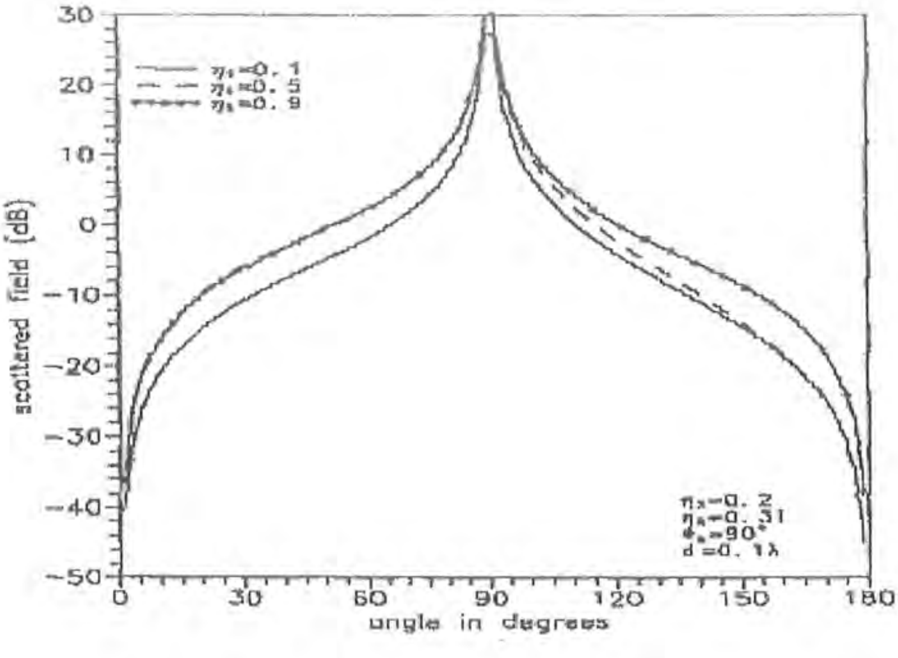


Fig.4.9: Fig. 4. Scattered field versus the observation angle ϕ for different values of half-plane reactance η_1.

On comparing the graphs in Fig. 4.2 and 4.3 with the graphs in Fig. *3a* and *3b* drawn (shown here as Figs. 4.7 and 4.8) in [57], one can see that the behavior of the graph is exactly the same and it differs only by the location along vertical-axis because of the multiplicative factor to the part of the scattered field containing the effects of incident and reflected waves. This is a counter check of the validation of our claim that if the source is shifted to a large distance these results differ from those of [57] by a multiplicative factor to the part of the scattered field containing the effects of incident and reflected waves. Similarly, the graphs in Fig. 4.4 plotted are in close tie with the graph plotted in Fig. 4 of (shown here as Fig. 4.9) [57].

Fig 4.5 shows that the amplitude of the diffracted field decreases as the source is taken away from the origin which is a natural phenomenon and verifying the results. In Fig. 4.6, it is observed that the amplitude of the diffracted field increases with increase in step height which is an obvious result.

Although the calculations were made for the E-polarization case, relying upon the duality principle, the results related to H- polarization can easily be obtained.

4.2 The Point Source Scattering

4.2.1 Formulation of the Problem

For the case of point source scattering, we consider the source occupying the position (x_0, y_0, z_0). Thus, we require the solution of the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\Psi_t(x, y, z) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0), \tag{4.46}$$

subject to the following boundary conditions

$$(1+\frac{\eta_{1r}}{k}\frac{\partial}{\partial y})\Psi_1(x,a,z)=0, \qquad x\in(-\infty,0), \tag{4.47a}$$

$$(1+\frac{\eta_{2i}}{ik}\frac{\partial}{\partial x})\Psi_2(0,y,z)=0, \qquad y\in(0,a), \tag{4.47b}$$

$$(1+\frac{\eta_{3r}}{k}\frac{\partial}{\partial y})\Psi_2(x,0,z)=0, \qquad x\in(0,\infty), \tag{4.47c}$$

$$\Psi_t(x,a,z)-\Psi_2(x,a,z)=0, \qquad x\in(0,\infty), \tag{4.47d}$$

$$\frac{\partial}{\partial y}\Psi_t(x,a,z)-\frac{\partial}{\partial y}\Psi_2(x,a,z)=0, \qquad x\in(0,\infty). \tag{4.47e}$$

where Ψ_t is the same as defined in Eq. (4.2). Also, the incident wave is defined as follows:

$$(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+k^2)\Psi^i(x,y,z)=\delta(x-x_0)\delta(y-y_0)\delta(z-z_0). \tag{4.48}$$

Defining the Fourier transform pair w.r.t the variable z as follows:

$$\widetilde{\Psi}_1(x,y,w)=\sqrt{\frac{k}{2\pi}}\int_{-\infty}^{\infty}\Psi_1(x,y,z)e^{ikwz}dz, \tag{4.49a}$$

$$\Psi_1(x,y,z)=\sqrt{\frac{k}{2\pi}}\int_{-\infty}^{\infty}\widetilde{\Psi}_1(x,y,w)e^{-ikwz}dw. \tag{4.49b}$$

Taking the Fourier transform of Eq. (4.46), Eq. (4.47a) and (4.47c − 4.47e), the governing equation with boundary conditions in transformed domain w takes the following form:

$$(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\gamma_1^2)\widetilde{\Psi}_1(x,y,w)=0, \tag{4.50}$$

with $\gamma_1 = \sqrt{1-w^2}$, and

$$(1+\frac{\eta_{1r}}{k}\frac{\partial}{\partial y})\widetilde{\Psi}_1(x,a,w)=0, \qquad x\in(-\infty,0), \tag{4.51a}$$

$$(1+\frac{\eta_{2i}}{k}\frac{\partial}{\partial x})\widetilde{\Psi}_2(x,a,w)=0, \qquad y\in(0,a), \tag{4.51b}$$

$$(1+\frac{\eta_{3r}}{k}\frac{\partial}{\partial y})\widetilde{\Psi}_2(x,0,w)=0, \qquad x\in(0,\infty), \tag{4.51c}$$

$$\widetilde{\Psi}_t(x,a,w)-\widetilde{\Psi}_2(x,a,w)=0, \qquad x\in(0,\infty), \tag{4.51d}$$

$$\frac{\partial}{\partial y}[\widetilde{\Psi}_t(x,a,w)-\widetilde{\Psi}_2(x,a,w)]=0, \qquad x\in(0,\infty). \tag{4.51e}$$

4.2.2 Solution of the Problem

Thus, we see that the Eq. (4.50) together with the boundary conditions $(4.51a-e)$ in the transformed domain w is the same as in the case of two dimensions formulated in the above section except that $k^2\gamma_1^2$ replaces k^2. Therefore, making use of Eq (4.43), we calculate the scattered field due to a point source as follows:

$$\begin{aligned}\widetilde{\Psi}_1(\rho,\theta,w) \sim\ & (\frac{\sqrt{2\pi}}{4\pi^2})\frac{i\eta_{1r}\eta_{3r}}{k\gamma_1}e^{-ik\gamma_1 a\sin\theta_0}.\frac{e^{ik\gamma_1\rho_0+i\frac{\pi}{2}}}{\sqrt{k\gamma_1\rho_0}}.\frac{e^{ik\gamma_1\rho}}{\sqrt{k\gamma_1\rho}}.\frac{\chi_-(\eta_{3r},k\gamma_1\cos\theta_0)\chi_-(\eta_{3r},k\gamma_1\cos\theta)}{\chi_-(\eta_{1r},k\gamma_1\cos\theta_0)\chi_-(\eta_{1r},k\gamma_1\cos\theta)}\\ &\times\frac{N_-(k\gamma_1\cos\theta_0)N_-(k\gamma_1\cos\theta)}{(i\eta_{1r}\sin\theta_0+1)(i\eta_{1r}\sin\theta+1)}\times\frac{\sin\theta\sin\theta_0}{\cos\theta+\cos\theta_0}.e^{ikwz_0}\\ &+e^{-i\frac{\pi}{4}}\frac{k\gamma_1\sin\theta}{(i\eta_{1r}\sin\theta+1)}N_-(k\gamma_1\cos\theta)\frac{\chi_-(\eta_{3r},k\gamma_1\cos\theta)}{\chi_-(\eta_{1r},k\gamma_1\cos\theta)}\frac{e^{ik\gamma_1\rho}}{\sqrt{k\gamma_1\rho}}.e^{ikwz_0}\\ &\times\sum_{m=1}^{\infty}\frac{\mathbb{R}_+(\alpha_m)(k\gamma_1-\eta_{2i}\alpha_m)}{\acute{M}(\alpha_m)(k\gamma_1+\eta_{2i}\alpha_m)}\frac{N_+(\alpha_m)}{(\alpha_m-k\gamma_1\cos\theta_0)}\\ &\times\frac{\chi_+(\eta_{3r},\alpha_m)}{\chi_+(\eta_{1r},\alpha_m)}\left[\frac{(k^2\gamma_1^2+\eta_{3r}K_m^2)}{(k^2\gamma_1^2+\eta_{1r}\eta_{3r}K_m^2)}\right]\cos K_m a.\end{aligned} \tag{4.52}$$

The scattered field in the spatial domain can now be obtained by taking the inverse Fourier transform of Eq. (4.52). Thus we obtain

$$\Psi_1(\rho,\theta,z) \sim \Psi^a_{11}(\rho,\theta,z) + \Psi a^a_{12}(\rho,\theta,z), \tag{4.53}$$

where

$$\begin{aligned}\Psi^a_{11}(\rho,\theta,z) \sim & \int_{-\infty}^{\infty} [\frac{e^{i\frac{\pi}{2}}}{4\pi^2}\frac{i\eta_{1r}\eta_{3r}}{k\gamma_1}e^{-ik\gamma_1 a\sin\theta_0}.\frac{\chi_-(\eta_{3r},k\gamma_1\cos\theta_0)\chi_-(\eta_{3r},k\gamma_1\cos\theta)}{\chi_-(\eta_{1r},k\gamma_1\cos\theta_0)\chi_-(\eta_{1r},k\gamma_1\cos\theta)}\\ & \times\frac{N_-(k\gamma_1\cos\theta_0)N_-(k\gamma_1\cos\theta)}{(i\eta_{1r}\sin\theta_0+1)(i\eta_{1r}\sin\theta+1)}\times\frac{\sin\theta\sin\theta_0}{\cos\theta+\cos\theta_0}\\ & \times\frac{e^{ik\gamma_1(\rho+\rho_0)+ikw(z_0-z)}}{\gamma_1\sqrt{k\rho\rho_0}}]dw,\end{aligned} \tag{4.53a}$$

and

$$\begin{aligned}\Psi^a_{12}(\rho,\theta,z) \sim & \int_{-\infty}^{\infty} [\frac{e^{-i\frac{\pi}{4}}}{\sqrt{2\pi}}\frac{k\gamma_1\sin\theta}{(i\eta_{1r}\sin\theta+1)}N_-(k\gamma\cos\theta)\frac{\chi_-(\eta_{3r},k\gamma_1\cos\theta)}{\chi_-(\eta_{1r},k\gamma_1\cos\theta)}\frac{e^{ik\gamma_1\rho+ikw(z_0-z)}}{\sqrt{\gamma_1\rho}}.\\ & \times\sum_{m=1}^{\infty}\frac{\mathbb{R}_+(\alpha_m)(k\gamma_1-\eta_{2i}\alpha_m)}{\acute{M}(\alpha_m)(k\gamma_1+\eta_{2i}\alpha_m)}\frac{N_+(\alpha_m)}{(\alpha_m-k\gamma_1\cos\theta_0)}\frac{\chi_+(\eta_{3r},\alpha_m)}{\chi_+(\eta_{1r},\alpha_m)}\\ & (\frac{(k^2\gamma_1^2+\eta_{3r}K_m^2)}{(k^2\gamma_1^2+\eta_{1r}\eta_{3r}K_m^2)})\cos K_m a]dw.\end{aligned} \tag{4.53b}$$

In order to solve Eq. (4.53a) and (4.53b), we have used the following substitutions:

$$\begin{aligned} w &= \cos\beta, \qquad \gamma_1=\sqrt{1-w^2}=\sin\beta, \qquad z-z_0=R_1\cos v,\\ \rho+\rho_0 &= R_1\sin v, \quad R_1=\sqrt{(z-z_0)^2+(\rho+\rho_0)^2},\end{aligned} \tag{4.54a}$$

for $\Psi_{11}^a(\rho, \theta, z)$ and

$$w = \cos\beta, \qquad \gamma_1 = \sqrt{1-w^2}, = \sin\beta, \qquad z - z_0 = R_2 \cos v,$$
$$\rho = R_2 \sin v, \quad R_2 = \sqrt{(z-z_0)^2 + (\rho)^2}, \tag{54b}$$

for $\Psi_{12}^a(\rho, \theta, z)$, respectively.

By using the method of steepest descent [104] and omitting the details of calculations, the final form of the field is given below:

$$\Psi_1(\rho, \theta, z) \sim \Psi_{11}^a(\rho, \theta, z) + \Psi_{12}^a(\rho, \theta, z), \tag{4.55}$$

where

$$\begin{aligned}\Psi_{11}^a(\rho, \theta, z) \sim\ & \frac{e^{i\frac{\pi}{2}}}{4\pi^2} . \frac{i\eta_{1r}\eta_{3r}}{-k\sin v} e^{ika\sin v \sin\theta_0} . \frac{\chi_-(\eta_{3r}, -k\sin v\cos\theta_0)\chi_-(\eta_{3r}, -k\gamma_1\cos\theta)}{\chi_-(\eta_{1r}, -k\sin v\cos\theta_0)\chi_-(\eta_{1r}, -k\sin v\cos\theta)} \\ & \times \frac{N_-(-k\sin v\cos\theta_0)N_-(-k\sin v\cos\theta)}{(i\eta_{1r}\sin\theta_0 + 1)(i\eta_{1r}\sin\theta + 1)} \\ & \times \frac{\sin\theta\sin\theta_0}{\cos\theta + \cos\theta_0} . \frac{e^{-(ikR_1 + i\frac{\pi}{4})}}{k.} . \sqrt{\frac{2\pi}{\rho\rho_0 R_1}},\end{aligned} \tag{4.55a}$$

and

$$\begin{aligned}\Psi_{12}^a(\rho, \theta, z) \sim\ & e^{-i\frac{3\pi}{4}} . \frac{(-k\sin v\sin\theta)}{(i\eta_{1r}\sin\theta + 1)} N_-(-k\sin v\cos\theta) \frac{\chi_-(\eta_{3r}, -k\sin v\cos\theta)}{\chi_-(\eta_{1r}, -k\sin v\cos\theta)} \frac{e^{-(ikR_2 + i\frac{\pi}{4})}}{kR_2\sin v.} \\ & \times \sum_{m=1}^{\infty} \frac{\mathbb{R}_+(\alpha_m)(-k\sin v - \eta_{2i}\alpha_m)}{\acute{M}(\alpha_m)(-k\sin v + \eta_{2i}\alpha_m)} \frac{N_+(\alpha_m)}{(\alpha_m + k\sin v\cos\theta_0)} \\ & \frac{\chi_+(\eta_{3r}, \alpha_m)}{\chi_+(\eta_{1r}, \alpha_m)} (\frac{(k^2\sin^2 v + \eta_{3r}K_m^2)}{(k^2\sin^2 v + \eta_{1r}\eta_{3r}K_m^2)}) \cos K_m a.\end{aligned} \tag{4.55b}$$

Chapter 5

Acoustic diffraction by an Oscillating strip

In this chapter, we have studied the problem of diffraction of a plane acoustic wave from an oscillating rigid strip. The problem is solved by using the temporal and spatial integral transforms and the Wiener-Hopf technique. The scattered field in the far zone is determined by the method of steepest descent. It has been observed that the diffracted field is the sum of the fields produced by the two edges of the strip and an interaction field. The significance of the present analysis is that it recovered the results when a strip is widened to a half plane.

5.1 Formulation of the problem

We consider the scattering of plane acoustic wave from an oscillating strip occupying the space $-l < x < 0$ at $y = 0$ and is oscillating in a direction perpendicular to the screen with velocity $u_0 f_1(t)$, where $f_1(t)$ is a periodic function of time t such that $t \in [0, \infty)$

whose generalized Fourier series is given by

$$f_1(t) = \sum_{n=-\infty}^{n=\infty} C_n e^{in\omega_0 t}. \tag{5.1}$$

In Eq. (5.1), the Fourier coefficients C_n are given by

$$C_n = \frac{1}{T_0} \int_{-\infty}^{\infty} f_1(t) e^{-in\omega_0 t} dt, \tag{5.2}$$

with non zero fundamental frequency

$$\omega_0 = \frac{2\pi}{T_0} (\neq 0). \tag{5.3}$$

We assume the continuity of the velocity across the boundary $y = 0$, $x < 0$, that is [73]:

$$\frac{\partial \phi_t}{\partial y} = u_0 f_1(t), \qquad -l < x < 0, \tag{5.4}$$

where the total velocity potential ϕ_t satisfies the wave equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi_t = \frac{1}{c^2} \frac{\partial^2 \phi_t}{\partial t^2}. \tag{5.5}$$

For convenience, we write

$$\phi_t = \phi + \phi_i, \tag{5.6}$$

where ϕ is the diffracted field and ϕ_i is the incident field given by

$$\phi_i = \exp[-ik_1(x \cos\theta_0 + y \sin\theta_0) - i\omega_1 t], \quad 0 < \theta_0 < \pi, \tag{5.7}$$

where $k_1 = \frac{\omega_1}{c}$, ω_1 is the frequency and c is the speed of the sound. Thus, we have to solve the following boundary value problem

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi=\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}, \tag{5.8}$$

$$\frac{\partial}{\partial y}\phi(x,0,t)-ik_1\sin\theta_0 e^{-ik_1 x\cos\theta_0-i\omega_1 t}=u_0 f_1(t) \qquad -l<x<0, \tag{5.9}$$

and

$$\left.\begin{array}{r}\phi(x,0^+,t)=\phi(x,0^-,t),\\ \frac{\partial}{\partial y}\phi(x,0^+,t)=\frac{\partial}{\partial y}\phi(x,0^-,t),\end{array}\right\},\quad -\infty<x<-l,\ x>0,\ y=0. \tag{5.10}$$

5.2 Solution of the problem

Define the temporal Fourier transform pair as

$$\left\{\begin{array}{l}\psi(x,y,\omega)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\phi(x,y,t)e^{i\omega t}dt,\\ \phi(x,y,t)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\psi(x,y,\omega)e^{-i\omega t}d\omega.\end{array}\right. \tag{5.11}$$

Taking the temporal Fourier transform of Eqs. $(5.8-5.10)$, we obtain

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)\psi(x,y,\omega)=0, \tag{5.12}$$

and

$$\frac{\partial}{\partial y}\psi(x,0,\omega)-2\pi ik_1\sin\theta_0 e^{-ik_1 x\cos\theta_0}\delta(\omega-\omega_1)=u_0\overline{f}_1(\omega) \qquad -l<x<0, \tag{5.13}$$

$$\left\{\begin{array}{l}\psi(x,0^+,\omega)=\psi(x,0^-,\omega),\\ \frac{\partial}{\partial y}\psi(x,0^+,\omega)=\frac{\partial}{\partial y}\psi(x,0^-,\omega),\end{array}\right. \qquad -\infty<x<-l,\ x>0,\ y=0, \tag{5.14}$$

where $k = \frac{\omega}{c} = k_1 + ik_2$ with $k_2 > 0$ and

$$\overline{f}_1(\omega) = 2\pi \sum_{n=-\infty}^{n=\infty} C_n \delta(\omega - n\omega_0). \tag{5.15}$$

Now, we define the spatial Fourier transform over the variable x as follows

$$\begin{cases} \overline{\psi}(\alpha, y, \omega) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \psi(x, y, \omega) e^{i\alpha x} dx, \\ \psi(x, y, \omega) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \overline{\psi}(\alpha, y, \omega) e^{-i\alpha x} d\alpha, \end{cases} \tag{5.16}$$

with

$$\overline{\psi}(\alpha, y, \omega) = \overline{\psi}_+(\alpha, y, \omega) + \overline{\psi}_1(\alpha, y, \omega) + e^{-i\alpha l}\overline{\psi}_-(\alpha, y, \omega), \tag{5.17}$$

where

$$\overline{\psi}_+(\alpha, y, \omega) = \int\limits_0^\infty \psi(x, y, \omega) e^{i\alpha x} dx, \tag{5.18}$$

$$\overline{\psi}_1(\alpha, y, \omega) = \int\limits_{-l}^0 \psi(x, y, \omega) e^{i\alpha x} dx, \tag{5.19}$$

and

$$\overline{\psi}_-(\alpha, y, \omega) = \int\limits_{-\infty}^0 \psi(x, y, \omega) e^{i\alpha x} dx. \tag{5.20}$$

Transforming Eqs. $(5.12 - 5.14)$ into α-plane, result in:

$$\left(\frac{d^2}{dy^2} - \gamma^2\right) \overline{\psi}(\alpha, y, \omega) = 0, \tag{5.21}$$

with

$$\left.\begin{array}{r} \overline{\psi}'_1(\alpha, 0, \omega) - \frac{2\pi k_1 \sin\theta_0 \delta(\omega - \omega_1)}{(\alpha - k_1 \cos\theta_0)} \left[1 - \exp[-i(\alpha - k_1 \cos\theta_0) l]\right] \\ = \frac{u_0 \overline{f}(\omega)}{i\alpha}(1 - \exp[-i\alpha l]) \end{array}\right\}, \quad -l < x < 0, \tag{5.22}$$

and

$$\begin{cases} \overline{\psi}_{\pm}(x,0^+,\omega) = \overline{\psi}_{\pm}(x,0^-,\omega), \\ \frac{\partial}{\partial y}\overline{\psi}_{\pm}(x,0^+,\omega) = \frac{\partial}{\partial y}\overline{\psi}_{\pm}(x,0^-,\omega), \end{cases} \quad -\infty < x < -l,\ x > 0,\ y = 0, \tag{5.23}$$

where $\kappa^2 = \alpha^2 - k^2$ with $\mathrm{Re}\gamma > 0$ in the strip $-\operatorname{Im} k < \operatorname{Im}\alpha < \operatorname{Im} k$.

The solution of Eq. (5.21) after using the continuity of $\overline{\psi}'$ across $y = 0$ is given by

$$\overline{\psi}(\alpha, y, \omega) = \begin{cases} A(\alpha,\omega)e^{-\kappa y}, & if\ y \geq 0, \\ -A(\alpha,\omega)e^{\kappa y}, & if\ y < 0, \end{cases} \tag{5.24}$$

Now using Eqs. (5.17) and (5.24), we have

$$A(\alpha) = \overline{\psi}_+(\alpha, 0^+, \omega) + \overline{\psi}_1(\alpha, 0^+, \omega) + e^{-i\alpha l}\overline{\psi}_-(\alpha, 0^+, \omega), \tag{5.25}$$

and

$$-A(\alpha) = \overline{\psi}_+(\alpha, 0^-, \omega) + \overline{\psi}_1(\alpha, 0^-, \omega) + e^{-i\alpha l}\overline{\psi}_-(\alpha, 0^-, \omega). \tag{5.26}$$

Adding and subtracting Eqs. (5.25) and (5.26), we get

$$2S_- e^{-i\alpha l} + 2\overline{\psi}_+(\alpha, 0, \omega) + 2J_1(\alpha, 0, \omega) = 0, \tag{5.27}$$

where

$$2S_- = \overline{\Psi}_-(\alpha, 0^+, \omega) + \overline{\Psi}_-(\alpha, 0^-, \omega), \tag{5.28}$$

$$2J_1(\alpha, 0, \omega) = \overline{\psi}_1(\alpha, 0^+, \omega) + \overline{\psi}_1(\alpha, 0^-, \omega), \tag{5.29}$$

$$2\overline{\psi}_+(\alpha, 0, \omega) = 2\overline{\psi}_+(\alpha, 0^+, \omega) = \overline{\psi}_+(\alpha, 0^+, \omega) + \overline{\psi}_+(\alpha, 0^-, \omega), \tag{5.30}$$

and

$$A(\alpha) = J_2(\alpha, 0, \omega), \tag{5.31}$$

where

$$J_2(\alpha, 0, \omega) = \frac{1}{2}[\overline{\psi}_1(\alpha, 0^+, \omega) - \overline{\psi}_1(\alpha, 0^-, \omega)]. \tag{5.32}$$

With the help of Eqs. (5.17) and (5.24), we have

$$-\kappa A(\alpha) = \overline{\psi}'_+(\alpha, 0, \omega) + \overline{\psi}'_1(\alpha, 0, \omega) + e^{-i\alpha l}\overline{\psi}'_-(\alpha, 0, \omega). \tag{5.33}$$

Using the continuity of $\overline{\psi}'$ across $y = 0$ and Eqs. (5.22) and (5.31) in Eq. (5.33) will yield

$$-\kappa J_2(\alpha, 0, \omega) = \overline{\psi}'_+(\alpha, 0, \omega) + \frac{2\pi k_1 \sin\theta_0 \delta(\omega - \omega_1)}{(\alpha - k_1 \cos\theta_0)}[1 - \exp[-i(\alpha - k_1 \cos\theta_0)l]]$$

$$+ \frac{u_0 \overline{f}_1(\omega)}{i\alpha}[1 - \exp[-i\alpha l]] + e^{-i\alpha l}\overline{\psi}'_-(\alpha, 0, \omega). \tag{5.34}$$

Eq. (5.34) can be re-arranged to give

$$e^{-i\alpha l}\overline{\psi}'_-(\alpha, 0, \omega) + \kappa J_2(\alpha, 0, \omega) + \overline{\psi}'_+(\alpha, 0, \omega)$$

$$= -2\pi k_1 \sin\theta_0 \delta(\omega - \omega_1)\frac{[1 - \exp[-i(\alpha - k_1 \cos\theta_0)l]]}{(\alpha - k_1 \cos\theta_0)} - \frac{u_0 \overline{f}_1(\omega)}{i\alpha}[1 - \exp[-i\alpha l]]. \tag{5.35}$$

Equations (5.27) and (5.35) are the standard Wiener-Hopf equations. In order to solve the problem, we have to solve Eq. (5.35), using Wiener-Hopf procedure [13].

5.3 Solution of the Wiener-Hopf equations

For the solution of the Wiener-Hopf equations (5.35), one can use the following factorizations [13] :

$$\kappa(\alpha) = \kappa_+(\alpha)\kappa_-(\alpha), \tag{5.36}$$

where $\kappa_+(\alpha)$ is regular for $\operatorname{Im}\alpha > -\operatorname{Im} k$, i.e., for upper half plane and $\kappa_-(\alpha)$ is regular for $\operatorname{Im}\alpha < \operatorname{Im} k$, i.e., for the lower half plane. Using Eq. (5.36) in Eq. (5.35), we get

$$e^{-i\alpha l}\overline{\psi}'_-(\alpha,0,\omega) + \kappa_+(\alpha)\kappa_-(\alpha)J_2(\alpha,0,\omega) + \overline{\psi}'_+(\alpha,0,\omega)$$

$$= -2\pi k_1 \sin\theta_0 \delta(\omega-\omega_1)\frac{[1-\exp[-i(\alpha-k_1\cos\theta_0)l]]}{(\alpha-k_1\cos\theta_0)} - \frac{u_0\overline{f}(\omega)}{i\alpha}1 - [\exp(-i\alpha l)]. \tag{5.37}$$

Dividing Eq. (5.37) by $\kappa_+(\alpha)$, we get

$$\frac{\overline{\psi}'_+(\alpha,0,\omega)}{\kappa_+(\alpha)} + \kappa_-(\alpha)J_2(\alpha,0,\omega) + \frac{e^{-i\alpha l}\overline{\psi}'_-(\alpha,0,\omega)}{\kappa_+(\alpha)}$$

$$= -\frac{2\pi k_1 \sin\theta_0\delta(\omega-\omega_1)}{\kappa_+(\alpha)}\frac{[1-\exp[-i(\alpha-k_1\cos\theta_0)l]]}{(\alpha-k_1\cos\theta_0)} - \frac{u_0\overline{f}_1(\omega)}{i\alpha\kappa_+(\alpha)}[1-[\exp(-i\alpha l)]], \tag{5.38}$$

or

$$\frac{\overline{\psi}'_+(\alpha,0,\omega)}{\kappa_+(\alpha)} + \kappa_-(\alpha)J_2(\alpha,0,\omega) + \frac{e^{-i\alpha l}\overline{\psi}'_-(\alpha,0,\omega)}{\kappa_+(\alpha)}$$

$$= \frac{\widetilde{A}[1-\exp[-i(\alpha-k_1\cos\theta_0)l]]}{\kappa_+(\alpha)(\alpha-k_1\cos\theta_0)} + \frac{iu_0\overline{f}_1(\omega)}{\alpha\kappa_+(\alpha)}[1-[\exp(-i\alpha l)]], \tag{5.39}$$

where

$$\widetilde{A} = -2\pi k_1\delta(\omega-\omega_1)\sin\theta_0.$$

Adding and subtracting the pole contribution

$$\frac{\overline{\psi}'_{+}(\alpha,0,\omega)}{\kappa_{+}(\alpha)} + \kappa_{-}(\alpha)J_2(\alpha,0,\omega) + \frac{e^{-i\alpha l}\overline{\psi}'_{-}(\alpha,0,\omega)}{\kappa_{+}(\alpha)}$$

$$= \frac{\widetilde{A}}{(\alpha - k_1\cos\theta_0)}\left[\frac{1}{\kappa_{+}(\alpha)} - \frac{1}{\kappa_{+}(k_1\cos\theta_0)} + \frac{1}{\kappa_{+}(k_1\cos\theta_0)}\right]$$

$$+ [1 - [\exp(-i\alpha l)]]\frac{iu_0\overline{f}(\omega)}{\alpha\kappa_{+}(\alpha)}\left[\frac{1}{\kappa_{+}(\alpha)} - \frac{1}{\kappa_{+}(0)} + \frac{1}{\kappa_{+}(0)}\right]. \tag{5.40}$$

Writing

$$\frac{e^{-i\alpha l}\overline{\psi}'_{-}(\alpha,0,\omega)}{\kappa_{+}(\alpha)} = U_{+}(\alpha) + U_{-}(\alpha), \tag{5.41}$$

$$\frac{\widetilde{A}[1 - \exp[-i(\alpha - k_1\cos\theta_0)l]]}{\kappa_{+}(\alpha)(\alpha - k_1\cos\theta_0)} = V_{+}(\alpha) + V_{-}(\alpha), \tag{5.42}$$

$$\frac{iu_0\overline{f}_1(\omega)}{\alpha\kappa_{+}(\alpha)}[1 - [\exp(-i\alpha l)]] = W_{+}(\alpha) + W_{-}(\alpha). \tag{5.43}$$

Using Eqs. (5.41) to (5.43) in Eq. (5.40) and separating into positive and negative parts, we obtain

$$\frac{\overline{\psi}'_{+}(\alpha,0,\omega)}{\kappa_{+}(\alpha)} + U_{+}(\alpha) - \frac{\widetilde{A}}{(\alpha - k_1\cos\theta_0)}\left[\frac{1}{\kappa_{+}(\alpha)} - \frac{1}{\kappa_{+}(k_1\cos\theta_0)}\right]$$
$$+V_{+}(\alpha) - \frac{iu_0\overline{f}_1(\omega)}{\alpha}\left[\frac{1}{\kappa_{+}(\alpha)} - \frac{1}{\kappa_{+}(0)}\right] + W_{+}(\alpha)$$

$$= -\kappa_{-}(\alpha)J_2(\alpha,0,\omega) - U_{-}(\alpha) + \frac{\widetilde{A}}{(\alpha - k_1\cos\theta_0)\kappa_{+}(k_1\cos\theta_0)} - V_{-}(\alpha) + \frac{iu_0\overline{f}_1(\omega)}{\alpha\kappa_{+}(0)} - W_{-}(\alpha). \tag{5.44}$$

Similarly, multiplying Eq. (5.37) by $\frac{\exp(i\alpha l)}{\gamma_{-}(\alpha)}$ and after simplification, we get

$$\frac{\exp(i\alpha l)\overline{\psi}'_+(\alpha,0,\omega)}{\kappa_-(\alpha)} + \exp(i\alpha l)\kappa_+(\alpha)J_2(\alpha,0,\omega) + \frac{\overline{\psi}'_-(\alpha,0,\omega)}{\kappa_-(\alpha)}$$

$$= -\frac{\widetilde{A}\exp(i\alpha l)}{\kappa_-(\alpha)(\alpha - k_1\cos\theta_0)} - \frac{\widetilde{A}\exp(ik_1\cos\theta_0)l}{\kappa_-(\alpha)(\alpha - k_1\cos\theta_0)} + \frac{iu_0\overline{f}_1(\omega)\exp(i\alpha l)}{\alpha\kappa_-(\alpha)} - \frac{iu_0\overline{f}_1(\omega)}{\alpha\kappa_-(\alpha)}. \quad (5.45)$$

Writing

$$\frac{\exp(i\alpha l)\overline{\psi}'_+(\alpha,0,\omega)}{\kappa_-(\alpha)} = R_+(\alpha) + R_-(\alpha), \quad (5.46)$$

$$\frac{-\widetilde{A}\exp(i\alpha l)}{\kappa_-(\alpha)(\alpha - k_1\cos\theta_0)} = S_+(\alpha) + S_-(\alpha), \quad (5.47)$$

and

$$\frac{iu_0\overline{f}_1(\omega)\exp(i\alpha l)}{\alpha\kappa_-(\alpha)} = T_+(\alpha) + T_-(\alpha). \quad (5.48)$$

Using Eqs. (5.46) to (5.48) in Eq. (5.45) and separating into positive and negative parts, we obtain

$$\frac{\overline{\psi}'_-(\alpha,0,\omega)}{\kappa_-(\alpha)} + R_-(\alpha) + S_-(\alpha) - T_-(\alpha) + \frac{\widetilde{A}\exp(ik_1\cos\theta_0)l}{\kappa_-(\alpha)(\alpha - k_1\cos\theta_0)} + \frac{iu_0\overline{f}(\omega)}{\alpha\kappa_-(\alpha)}$$

$$= -R_+(\alpha) - \exp(i\alpha l)\kappa_+(\alpha)J_2(\alpha,0,\omega) - S_+(\alpha) + T_+(\alpha). \quad (5.49)$$

Using extended Liouvillies theorem, Eqs. (5.44) and (5.49) will be equal to zero

$$\frac{\overline{\psi}'_+(\alpha,0,\omega)}{\kappa_+(\alpha)} + U_+(\alpha) - \frac{\widetilde{A}}{(\alpha - k_1\cos\theta_0)}\left[\frac{1}{\kappa_+(\alpha)} - \frac{1}{\kappa_+(k_1\cos\theta_0)}\right]$$

$$+V_+(\alpha) - \frac{iu_0\overline{f}(\omega)}{\alpha}\left[\frac{1}{\kappa_+(\alpha)} - \frac{1}{\kappa_+(0)}\right] + W_+(\alpha) = 0, \quad (5.50)$$

and

$$\frac{\overline{\psi}'_-(\alpha,0,\omega)}{\kappa_-(\alpha)} + R_-(\alpha) + S_-(\alpha) - T_-(\alpha) + \frac{\widetilde{A}\exp(ik_1\cos\theta_0)l}{\kappa_-(\alpha)(\alpha - k_1\cos\theta_0)} + \frac{iu_0\overline{f}_1(\omega)}{\alpha\kappa_-(\alpha)} = 0. \quad (5.51)$$

Let us introduce

$$\overline{\psi}'_+(\alpha,0,\omega) - \frac{\widetilde{A}}{(\alpha - k_1\cos\theta_0)} = \overline{\phi}^{*\prime}_+(\alpha,0,\omega), \tag{5.52}$$

$$\overline{\psi}'_-(\alpha,0,\omega) + \frac{\widetilde{A}\exp(ik_1\cos\theta_0)l}{\kappa_-(\alpha)(\alpha - k_1\cos\theta_0)} + \frac{iu_0\overline{f}_1(\omega)}{\alpha\kappa_-(\alpha)} = \overline{\phi}'_-(\alpha,0,\omega). \tag{5.53}$$

Therefore, Eqs. (5.50) and (5.51) can be written as

$$\frac{\overline{\phi}^{*\prime}_+(\alpha,0,\omega)}{\kappa_+(\alpha)} + U_+(\alpha) - \frac{\widetilde{A}}{(\alpha - k_1\cos\theta_0)\kappa_+(k_1\cos\theta_0)} + V_+(\alpha) + \frac{iu_0\overline{f}_1(\omega)}{\alpha\kappa_+(0)} + W_+(\alpha) = 0, \tag{5.54}$$

$$\frac{\overline{\phi}'_-(\alpha,0,\omega)}{\kappa_-(\alpha)} + R_-(\alpha) + S_-(\alpha) - T_-(\alpha) = 0. \tag{5.55}$$

The Eqs. (5.54) and (5.55) can further be refined as:

$$\frac{\overline{\phi}^{*\prime}_+(\alpha,0,\omega)}{\kappa_+(\alpha)} + \frac{\widetilde{A}}{(\alpha - k_1\cos\theta_0)\kappa_+(k_1\cos\theta_0)} + \frac{iu_0\overline{f}_1(\omega)}{\alpha\kappa_+(0)} + \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} \frac{e^{-i\zeta l}\phi'_-(\zeta,0,\omega)}{\kappa_+(\zeta)(\zeta-\alpha)}d\zeta = 0, \tag{5.56}$$

and

$$\frac{\overline{\phi}'_-(\alpha,0,\omega)}{\kappa_-(\alpha)} - \frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id} \frac{e^{i\zeta l}\overline{\phi}^{*\prime}_+(\zeta,0,\omega)}{\kappa_-(\zeta)(\zeta-\alpha)}d\zeta = 0, \tag{5.57}$$

where

$$U_+(\alpha) + V_+(\alpha) + W_+(\alpha) = \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} \frac{[U(\zeta)+V(\zeta)+W(\zeta)]}{(\zeta-\alpha)}d\zeta, \tag{5.58}$$

with

$$U_+(\zeta)+V_+(\zeta)+W_+(\zeta) = \frac{e^{-i\zeta l}}{\kappa_+(\zeta)}[\overline{\Psi}'_-(\alpha,0,\omega) + \frac{\widetilde{A}\exp(ik_1\cos\theta_0)l}{(\zeta - k_1\cos\theta_0)} + \frac{iu_0\overline{f}_1(\omega)}{\zeta}] = \frac{e^{-i\zeta l}}{\kappa_+(\zeta)}\overline{\phi}'_-(\zeta,0,\omega). \tag{5.59}$$

Similarly,

$$R_{-}(\alpha)+S_{-}(\alpha)-T_{-}(\alpha)=-\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id}\frac{[R(\zeta)+S(\zeta)-T(\zeta)]}{(\zeta-\alpha)}d\zeta, \tag{5.60}$$

with

$$R(\zeta)+S(\zeta)-T(\zeta)=\frac{e^{i\zeta l}}{\kappa_{-}(\zeta)}[\overline{\psi}'_{+}(\zeta,0,\omega)-\frac{\widetilde{A}}{(\zeta-k_1\cos\theta_0)}-\frac{iu_0\overline{f}_1(\omega)}{\zeta}]=\frac{e^{i\zeta l}}{\kappa_{-}(\zeta)}\overline{\phi}^{*\prime}_{+}(\zeta,0,\omega). \tag{5.61}$$

Replacing ζ by $-\zeta$ in Eq. (5.56) and α by $-\alpha$ in Eq. (5.57) and using $\gamma_{+}(-\zeta)=\gamma_{-}(\zeta)$, $d=c=b$, and then adding and subtracting the resulting equations, we get

$$\frac{S^{*}_{+}(\alpha,0,\omega)}{\kappa_{+}(\alpha)}+\frac{\widetilde{A}}{(\alpha-k_1\cos\theta_0)\kappa_{+}(k_1\cos\theta_0)}+\frac{iu_0\overline{f}_1(\omega)}{\alpha\kappa_{+}(0)}-\frac{1}{2\pi i}\int_{-\infty+ib}^{\infty+ib}\frac{e^{i\zeta l}S^{*}_{+}(\zeta,0,\omega)}{\kappa_{-}(\zeta)(\zeta+\alpha)}d\zeta=0, \tag{5.62}$$

where

$$S^{*}_{+}(\alpha,0,\omega)=\overline{\phi}^{*\prime}_{+}(\alpha,0,\omega)+\overline{\phi}'_{-}(-\alpha,0,\omega), \tag{5.63}$$

and

$$\frac{D^{*}_{+}(\alpha,0,\omega)}{\kappa_{+}(\alpha)}+\frac{\widetilde{A}}{(\alpha-k_1\cos\theta_0)\kappa_{+}(k_1\cos\theta_0)}+\frac{iu_0\overline{f}_1(\omega)}{\alpha\kappa_{+}(0)}+\frac{1}{2\pi i}\int_{-\infty+ib}^{\infty+ib}\frac{e^{i\zeta l}D^{*}_{+}(\zeta,0,\omega)}{\kappa_{-}(\zeta)(\zeta+\alpha)}d\zeta=0, \tag{5.64}$$

where

$$D^{*}_{+}(\alpha,0,\omega)=\overline{\phi}^{*\prime}_{+}(\alpha,0,\omega)-\overline{\phi}'_{-}(-\alpha,0,\omega). \tag{5.65}$$

Eqs. (5.62) and (5.64) are of same type except $S^{*}_{+}(\alpha,0,\omega)$ and $D^{*}_{+}(\alpha,0,\omega)$. Thus, according to Jones method, we can set $S^{*}_{+}(\alpha,0,\omega)=D^{*}_{+}(\alpha,0,\omega)=F^{*}_{+}(\alpha,0,\omega)$ in Eqs. (5.62) and (5.64) to get

$$\frac{F_+^*(\alpha,0,\omega)}{\kappa_+(\alpha)}+\frac{\widetilde{A}}{(\alpha-k_1\cos\theta_0)\kappa_+(k_1\cos\theta_0)}+\frac{iu_0\overline{f}_1(\omega)}{\alpha\kappa_+(0)}+\frac{\lambda}{2\pi i}\int_{-\infty+ib}^{\infty+ib}\frac{e^{i\zeta l}F_+^*(\zeta,0,\omega)}{\kappa_-(\zeta)(\zeta+\alpha)}d\zeta=0, \tag{5.66}$$

where $\lambda=\pm1, i.e.,\ \lambda=-1$ for $S_+^*(\alpha,0,\omega)$ and $\lambda=+1$ for $D_+^*(\alpha,0,\omega)$. To calculate the value of $F_+^*(\alpha,0,\omega)$, we proceed as follows:

Substituting the values of $\overline{\phi}_+^{*\prime}(\alpha,0,\omega)$ and $\overline{\phi}'_-(-\alpha,0,\omega)$ from Eqs. (5.52) and (5.53) in Eqs. (5.63) and (5.65) respectively and defining

$$F_+(\alpha,0,\omega)=\overline{\psi}'_+(\alpha,0,\omega)-\lambda\overline{\psi}'_-(-\alpha,0,\omega), \tag{5.67}$$

we get

$$F_+^*(\alpha,0,\omega)=F_+(\alpha,0,\omega)-\frac{\widetilde{A}}{(\alpha-k_1\cos\theta_0)}+\frac{\lambda\widetilde{A}\exp(ik_1\cos\theta_0)l}{(\alpha+k_1\cos\theta_0)}-\frac{iu_0\overline{f}_1(\omega)}{\alpha}+\frac{\lambda iu_0\overline{f}_1(\omega)}{\alpha}. \tag{5.68}$$

Using this value of $F_+^*(\alpha,0,\omega)$ in Eq. (5.66), we obtain

$$\frac{1}{\kappa_+(\alpha)}\left[F_+(\alpha,0,\omega)-\frac{\widetilde{A}}{(\alpha-k_1\cos\theta_0)}+\frac{\lambda\widetilde{A}\exp(ik_1\cos\theta_0)l}{(\alpha+k_1\cos\theta_0)}-\frac{iu_0\overline{f}_1(\omega)}{\alpha}+\frac{\lambda iu_0\overline{f}_1(\omega)}{\alpha}\right]+$$

$$\frac{\widetilde{A}}{(\alpha-k_1\cos\theta_0)\kappa_+(k_1\cos\theta_0)}+\frac{iu_0\overline{f}_1(\omega)}{\alpha\kappa_+(0)}$$

$$+\frac{\lambda}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{e^{i\zeta l}}{\kappa_-(\zeta)(\zeta+\alpha)}\left[\begin{array}{c}F_+(\alpha,0,\omega)-\frac{\widetilde{A}}{(\alpha-k_1\cos\theta_0)}+\frac{\lambda\widetilde{A}\exp(ik_1\cos\theta_0)l}{(\alpha+k_1\cos\theta_0)}\\ -\frac{iu_0\overline{f}_1(\omega)}{\alpha}+\frac{\lambda iu_0\overline{f}_1(\omega)}{\alpha}\end{array}\right]d\zeta=0. \tag{5.69}$$

Now, Consider the integral term in Eq.(5.69), *i.e.*,

$$\frac{\lambda}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{e^{i\zeta l}}{\kappa_-(\zeta)(\zeta+\alpha)}\left[\begin{array}{c}F_+(\alpha,0,\omega)-\frac{\widetilde{A}}{(\alpha-k_1\cos\theta_0)}+\frac{\lambda\widetilde{A}\exp(ik_1\cos\theta_0)l}{(\alpha+k_1\cos\theta_0)}\\ -\frac{iu_0\overline{f}_1(\omega)}{\alpha}+\frac{\lambda iu_0\overline{f}_1(\omega)}{\alpha}\end{array}\right]d\zeta,$$

with

$$I_1 = \int_{-\infty+ia}^{\infty+ia} \frac{e^{i\zeta l} F_+(\alpha,0,\omega)}{\kappa_-(\zeta)(\zeta+\alpha)} d\zeta \sim E_{-1}W_{-1}\{-i(k+\alpha)l\}F_+(\alpha,0,\omega) = 2\pi i T(\alpha)F_+(\alpha,0,\omega), \tag{5.70}$$

$$I_2 = \overline{A} \int_{-\infty+ia}^{\infty+ia} \frac{e^{i\zeta l}}{\kappa_-(\zeta)(\zeta+\alpha)(\zeta - k_1\cos\theta_0)} d\zeta \sim 2\pi i \left\{ \frac{\widetilde{A}\exp(ik_1\cos\theta_0)l}{\kappa_-(k_1\cos\theta_0)(\alpha + k_1\cos\theta_0)} + N_2(\alpha) \right\}, \tag{5.71}$$

$$I_3 = \lambda\widetilde{A}\exp(ik_1\cos\theta_0)l \int_{-\infty+ia}^{\infty+ia} \frac{e^{i\zeta l}}{\gamma_-(\zeta)(\zeta+\alpha)(\zeta+k_1\cos\theta_0)} d\zeta \sim \lambda\widetilde{A}\exp(ik_1\cos\theta_0)l.2\pi i N_1(\alpha), \tag{5.72}$$

$$I_4 = iu_0\overline{f}_1(\omega) \int_{-\infty+ia}^{\infty+ia} \frac{e^{i\zeta l}}{\kappa_-(\zeta)(\zeta+\alpha)(\zeta)} d\zeta \sim \frac{iu_0\overline{f}_1(\omega)}{k} E_{-1}W_{-1}\{-i(k+\alpha)l\}, \tag{5.73}$$

and

$$I_5 = \lambda iu_0\overline{f}_1(\omega) \int_{-\infty+ia}^{\infty+ia} \frac{e^{i\zeta l}}{\kappa_-(\zeta)(\zeta+\alpha)(\zeta)} d\zeta \sim \frac{\lambda iu_0\overline{f}_1(\omega)}{k} E_{-1}W_{-1}\{-i(k+\alpha)l\}. \tag{5.74}$$

Using these values of I_1, I_2, I_3, I_4 and I_5 in Eq. (5.69) and simplifying

$$\frac{F_+(\alpha,0,\omega)}{\kappa_+(\alpha)} = \frac{\widetilde{A}}{(\alpha - k_1\cos\theta_0)}\left[\frac{1}{\kappa_+(\alpha)} - \frac{1}{\kappa_+(k_1\cos\theta_0)}\right] - \frac{\lambda\widetilde{A}\exp(ik_1\cos\theta_0)l}{(\alpha + k_1\cos\theta_0)}\left[\frac{1}{\kappa_+(\alpha)} - \frac{1}{\kappa_-(k_1\cos\theta_0)}\right]$$

$$+\frac{iu_0\overline{f}_1(\omega)}{\alpha}\left[\frac{1}{\kappa_+(\alpha)} - \frac{1}{\kappa_+(0)}\right] - \frac{\lambda iu_0\overline{f}_1(\omega)}{\alpha\kappa_+(\alpha)} - \lambda T(\alpha)F_+(k,0,\omega) + \frac{\lambda T(\alpha)iu_0\overline{f}_1(\omega)}{k} - \frac{T(\alpha)iu_0\overline{f}_1(\omega)}{k}$$

$$+\lambda\widetilde{A}N_2(\alpha) - \widetilde{A}\exp(ik_1\cos\theta_0)lN_1(\alpha). \tag{5.75}$$

Suppose

$$P_1(\alpha) = \frac{1}{(\alpha - k_1\cos\theta_0)}\left[\frac{1}{\kappa_+(\alpha)} - \frac{1}{\kappa_+(k_1\cos\theta_0)}\right], \tag{5.76}$$

and

$$P_2(\alpha) = \frac{1}{(\alpha + k_1\cos\theta_0)}\left[\frac{1}{\kappa_+(\alpha)} - \frac{1}{\kappa_-(k_1\cos\theta_0)}\right]. \tag{5.77}$$

Then, Eq. (5.75) reduces to

$$\frac{F_+(\alpha,0,\omega)}{\kappa_+(\alpha)} = \widetilde{A}P_1(\alpha) - \lambda\widetilde{A}\exp(ik_1\cos\theta_0)lP_2(\alpha) + \frac{iu_0\overline{f}_1(\omega)}{\alpha}\left[\frac{1}{\kappa_+(\alpha)} - \frac{1}{\kappa_+(0)}\right]$$

$$-\frac{\lambda iu_0\overline{f}_1(\omega)}{\alpha\kappa_+(\alpha)} - \lambda T(\alpha)F_+(k,0,\omega) + \frac{\lambda T(\alpha)iu_0\overline{f}_1(\omega)}{k} - \frac{T(\alpha)iu_0\overline{f}_1(\omega)}{k}$$
$$+\lambda\widetilde{A}N_2(\alpha) - \widetilde{A}\exp(ik_1\cos\theta_0)lN_1(\alpha). \tag{5.78}$$

Define

$$G_1(\alpha) = P_1(\alpha) - \exp(ik_1\cos\theta_0)lN_1(\alpha), \tag{5.79}$$

and

$$G_2(\alpha) = \exp(ik_1\cos\theta_0)lP_2(\alpha) - N_2(\alpha).$$

Using it in Eq. (5.78), we get

$$\frac{F_+(\alpha,0,\omega)}{\kappa_+(\alpha)} = \widetilde{A}G_1(\alpha) - \lambda\widetilde{A}G_2(\alpha) + \frac{iu_0\overline{f}_1(\omega)}{\alpha}\left[\frac{1}{\kappa_+(\alpha)} - \frac{1}{\kappa_+(0)}\right] - \frac{\lambda iu_0\overline{f}_1(\omega)}{\alpha\kappa_+(\alpha)}$$

$$-\lambda T(\alpha)F_+(k,0,\omega) + \frac{\lambda T(\alpha)iu_0\overline{f}_1(\omega)}{k} - \frac{T(\alpha)iu_0\overline{f}_1(\omega)}{k}. \tag{5.80}$$

Next, we need to find the value of $F_+(k,0,\omega)$ and using this value in Eq. (5.80), we obtain

$$\frac{F_+(\alpha,0,\omega)}{\kappa_+(\alpha)} = \widetilde{A}\left[G_1(\alpha) - \lambda G_2(\alpha)\right] - \frac{\lambda T(\alpha)\widetilde{A}\kappa_+(k)}{1+\lambda T(\alpha)\kappa_+(k)}\left[G_1(\alpha) - \lambda G_2(\alpha)\right]$$

$$+\frac{iu_0\overline{f}_1(\omega)}{\alpha}\left[\frac{1}{\kappa_+(\alpha)} - \frac{1}{\kappa_+(0)}\right] - \frac{\lambda iu_0\overline{f}_1(\omega)}{\alpha\kappa_+(\alpha)} - \lambda T(\alpha)\left[\frac{iu_0\overline{f}_1(\omega)}{k}\left\{\frac{1}{\kappa_+(k)} - \frac{1}{\kappa_+(0)}\right\} - \frac{\lambda iu_0\overline{f}_1(\omega)}{k\kappa_+(k)}\right.$$
$$\left.+\frac{(\lambda-1)T(k)iu_0\overline{f}_1(\omega)}{k}\right]\frac{\kappa_+(k)}{1+\lambda T(\alpha)\kappa_+(k)} + \frac{(\lambda-1)T(\alpha)iu_0\overline{f}_1(\omega)}{k}. \tag{5.81}$$

Now, substituting $\lambda = -1$ and $\lambda = +1$ in Eq. (5.67) and then using the value of

$F_+(\alpha, 0, \omega)$ in Eq. (5.81) and then adding and subtracting the resulting equations, we get the values of $\overline{\psi}'_+(\alpha, 0, \omega)$ and $\overline{\psi}'_-(\alpha, 0, \omega)$ as follows:

$$\overline{\psi}'_+(\alpha, 0, \omega) = \widetilde{A}\kappa_+(\alpha)G_1(\alpha) + \widetilde{A}T(\alpha)\kappa_+(\alpha)\kappa_+(k)C_1 + \frac{iu_0\overline{f}_1(\omega)\kappa_+(\alpha)}{\alpha}\left[\frac{1}{\kappa_+(\alpha)} - \frac{1}{\kappa_+(0)}\right]$$

$$+\frac{T(\alpha)T(k)\kappa_+(k)\kappa_+(\alpha)iu_0\overline{f}_1(\omega)}{k[1-\kappa_+^2(k)T^2(k)]}\left[\frac{1}{\kappa_+(k)} - \frac{1}{\kappa_+(0)}\right] - \frac{T(\alpha)T(k)\kappa_+(k)\kappa_+(\alpha)iu_0\overline{f}_1(\omega)}{k[1-\kappa_+^2(k)T^2(k)]}, \tag{5.82}$$

and

$$\overline{\psi}'_-(\alpha, 0, \omega) = \widetilde{A}\kappa_+(-\alpha)G_2(-\alpha) + \widetilde{A}T(-\alpha)\kappa_+(-\alpha)\kappa_+(K)C_2 - \frac{iu_0\overline{f}_1(\omega)\kappa_+(-\alpha)}{\alpha}\left[\frac{1}{\kappa_+(-\alpha)} - \frac{1}{\kappa_+(0)}\right]$$

$$+\frac{T(-\alpha)T(k)\kappa_+(k)\kappa_+(-\alpha)iu_0\overline{f}_1(\omega)}{k[1-\kappa_+^2(k)T^2(k)]}\left[\frac{1}{\kappa_+(k)} - \frac{1}{\kappa_+(0)}\right] - \frac{T(-\alpha)T(k)\kappa_+(k)\kappa_+(-\alpha)iu_0\overline{f}_1(\omega)}{k[1-\kappa_+^2(k)T^2(k)]}, \tag{5.83}$$

where

$$\widetilde{A} = -2\pi k_1\delta(\omega - \omega_1)\sin\theta_0 \tag{5.84}$$

$$G_1(\alpha) = \frac{1}{(\alpha - k_1\cos\theta_0)}\left[\frac{1}{\kappa_+(\alpha)} - \frac{1}{\kappa_+(k_1\cos\theta_0)}\right] - e^{ilk_1\cos\theta_0}N_1(\alpha), \tag{5.85}$$

$$G_2(\alpha) = \frac{e^{ilk_1\cos\theta_0}}{(\alpha + k_1\cos\theta_0)}\left[\frac{1}{\kappa_+(\alpha)} - \frac{1}{\kappa_+(k_1\cos\theta_0)}\right] - N_2(\alpha), \tag{5.86}$$

$$C_1 = \left[\frac{G_2(k) + \kappa_+(k)G_1(k)T(k)}{1-\kappa_+^2(k)T^2(k)}\right], \tag{5.87}$$

and

$$C_2 = \left[\frac{G_1(k) + \kappa_+(k)G_2(k)T(k)}{1-\kappa_+^2(k)T^2(k)}\right], \tag{5.88}$$

$$N_{1,2}(\alpha) = \frac{E_{-1}[W_{-1}\{-i(k \pm k\cos\theta)l\} - W_{-1}\{-i(k+\alpha)l\}]}{2\pi i(\alpha \mp k\cos\theta_0)}, \tag{5.89}$$

$$T(\alpha) = \frac{1}{2\pi i}E_{-1}W_{-1}\{-i(k+\alpha)l\}, \tag{5.90}$$

$$E_{-1} = 2e^{i\frac{\pi}{4}}e^{ikl}\ (l)^{-1}\ (i)^{\frac{-1}{2}}\ h_{-1}, \tag{5.91}$$

where $h_{-1} = e^{i\frac{\pi}{4}}$

$$W_{n-\frac{1}{2}}(z) = \int_0^\infty \frac{u^n e^{-u}}{u+z} du = \Gamma(n+1) e^{\frac{z}{2}} z^{\frac{n}{2}-\frac{1}{2}} W_{-\frac{1}{2}(n+1),\frac{n}{2}}(z), \tag{5.92}$$

where $z = -i(k+\alpha)l$ and $W_{m,n}$ is known as a Whittaker function.

Now, substituting Eqs. (5.82) and (5.83) in Eq. (5.33), we get

$$\begin{aligned} A(\alpha) &= -\frac{1}{\kappa}[\frac{\widetilde{A}\kappa_+(-\alpha)\exp[-i(\alpha - k_1\cos\theta_0)l]}{\kappa_-(k_1\cos\theta_0)(\alpha - k_1\cos\theta_0)} - \widetilde{A}\kappa_+(-\alpha)N_2(-\alpha)\exp[-i\alpha l] \\ &+ \exp[-i\alpha l]\widetilde{A}\kappa_+(-\alpha)\kappa_+(k)T(-\alpha)C_2 + \frac{iu_0\overline{f}_1(\omega)\kappa_+(-\alpha)\exp[-i\alpha l]}{\alpha\gamma_+(0)} \\ &+ \frac{\exp[-i\alpha l]T(-\alpha)T(k)\kappa_+(k)\kappa_+(-\alpha)iu_0\overline{f}_1(\omega)}{k[1-\kappa_+^2(k)T^2(k)]}\left[\frac{1}{\kappa_+(k)} - \frac{1}{\kappa_+(0)}\right] \\ &- \frac{\exp[-i\alpha l]T(-\alpha)T(k)\kappa_+(k)\kappa_+(-\alpha)iu_0\overline{f}_1(\omega)}{k[1-\kappa_+^2(k)T^2(k)]} - \frac{\widetilde{A}\kappa_+(\alpha)}{\kappa_+(k_1\cos\theta_0)(\alpha - k_1\cos\theta_0)} \\ &- \widetilde{A}\kappa_+(\alpha)e^{ilk_1\cos\theta_0}N_1(\alpha) - \frac{iu_0\overline{f}_1(\omega)\kappa_+(\alpha)}{\alpha\kappa_+(0)} \\ &+ \frac{T(\alpha)T(k)\kappa_+(k)\kappa_+(\alpha)iu_0\overline{f}_1(\omega)}{k[1-\kappa_+^2(k)T^2(k)]}\left[\frac{1}{\kappa_+(k)} - \frac{1}{\kappa_+(0)}\right] \\ &- \frac{T(\alpha)T(k)\kappa_+(k)\kappa_+(\alpha)iu_0\overline{f}_1(\omega)}{k[1-\kappa_+^2(k)T^2(k)]} + \widetilde{A}T(\alpha)\kappa_+(\alpha)\kappa_+(k)C_1]. \end{aligned} \tag{5.93}$$

The function $\psi(x,y,\omega)$ can be obtained by taking the inverse Fourier transform of Eq. (5.16) as follows:

$$\psi(x,y,\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{\psi}(x,y,\omega)e^{-i\alpha x}d\alpha = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(\alpha,\omega)e^{-\kappa|y|-i\alpha x}d\alpha. \tag{5.94}$$

Now, $\psi(x,y,\omega)$ can be broken up as follows:

$$\psi(x,y,\omega) = \psi^{sep}(x,y,\omega) + \psi^{int}(x,y,\omega), \tag{5.95}$$

where

$$\psi^{sep}(x,y,\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{-1}{\kappa}\left[\frac{\widetilde{A}\kappa_{+}(-\alpha)\exp[-i(\alpha-k_1\cos\theta_0)l]}{\kappa_{-}(k_1\cos\theta_0)(\alpha-k_1\cos\theta_0)}-\frac{\widetilde{A}\gamma_{+}(\alpha)}{\kappa_{+}(k_1\cos\theta_0)(\alpha-k_1\cos\theta_0)}\right.$$

$$\left.+\frac{iu_0\overline{f}_1(\omega)\kappa_{+}(-\alpha)\exp[-i\alpha l]}{\alpha\kappa_{+}(0)}-\frac{iu_0\overline{f}_1(\omega)\kappa_{+}(\alpha)}{\alpha\kappa_{+}(0)}\right]e^{-\kappa|y|-i\alpha x}d\alpha, \tag{5.96}$$

and

$$\psi^{int}(x,y,\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{-1}{\kappa}\left[-\widetilde{A}\gamma_{+}(-\alpha)N_2(-\alpha)\exp[-i\alpha l]+\widetilde{A}\kappa_{+}(-\alpha)\kappa_{+}(k)T(-\alpha)C_2\exp[-i\alpha l]\right.$$

$$+\frac{T(-\alpha)T(k)\kappa_{+}(k)\kappa_{+}(-\alpha)iu_0\overline{f}_1(\omega)\exp[-i\alpha l]}{k[1-\kappa_{+}^2(k)T^2(k)]}\left[\frac{1}{\kappa_{+}(k)}-\frac{1}{\kappa_{+}(0)}\right]$$

$$-\frac{T(-\alpha)T(k)\kappa_{+}(k)\kappa_{+}(-\alpha)iu_0\overline{f}_1(\omega)\exp[-i\alpha l]}{k[1-\kappa_{+}^2(k)T^2(k)]}$$

$$-\widetilde{A}\kappa_{+}(\alpha)e^{ilk_1\cos\theta_0}N_1(\alpha)+\frac{T(\alpha)T(k)\kappa_{+}(k)\kappa_{+}(\alpha)iu_0\overline{f}_1(\omega)}{k[1-\kappa_{+}^2(k)T^2(k)]}\left[\frac{1}{\kappa_{+}(k)}-\frac{1}{\kappa_{+}(0)}\right]$$

$$\left.-\frac{T(\alpha)T(k)\kappa_{+}(k)\kappa_{+}(\alpha)iu_0\overline{f}_1(\omega)}{k[1-\kappa_{+}^2(k)T^2(k)]}+\widetilde{A}T(\alpha)\kappa_{+}(\alpha)\kappa_{+}(k)C_1\right]e^{-\kappa|y|-i\alpha x}d\alpha. \tag{5.97}$$

Here $\psi^{sep}(x,y,\omega)$ consists of two parts each representing the diffracted field produced by the edges at $x=0$ and $x=-l$, respectively, as though the other edge were absent while $\psi^{int}(x,y,\omega)$ gives the interaction of one edge upon the other.

5.4 Far field approximation

The far field may now be calculated by evaluating the integral in Eqs. (5.96) and (5.97) asymptotically [107]. For that we substitute $x=r\cos\theta$, $|y|=r\sin\theta$ and deform the contour by the transformation $\alpha=-k\cos\beta$, where $\beta=\mu+i\nu$, with $0\leq\mu\leq\pi$, $-\infty<\nu<\infty$ [13].

Hence for large kr, Eqs. (5.96) and (5.97) becomes

$$\psi^{sep}(x,y,\omega) = \left[\frac{-k_1 \sin\theta_0 \sqrt{k}\delta(\omega-\omega_1)\sin\theta}{[k\cos\theta + k_1\cos\theta_0]} \begin{pmatrix} \frac{e^{-i[k\cos\theta + k_1\cos\theta_0]l}}{\sqrt{1-\cos\theta}\sqrt{k_1\cos\theta_0 - k}} \\ +\frac{1}{i\sqrt{1+\cos\theta}\sqrt{k_1\cos\theta_0+k}} \end{pmatrix} \right.$$

$$\left. + \frac{iu_0 \sin\theta \sum_{n=-\infty}^{n=\infty} C_n \delta(\omega - n\omega_0)}{k\cos\theta} \begin{pmatrix} \frac{e^{ilk\cos\theta}}{\sqrt{1-\cos\theta}} \\ +\frac{1}{i\sqrt{1+\cos\theta}} \end{pmatrix} \right] e^{ikr - i\frac{\pi}{4}} \left(\frac{2\pi}{kr}\right)^{\frac{1}{2}} \tag{5.98}$$

and

$$\psi^{int}(x,y,\omega) = \left[\frac{\widetilde{A}e^{ikl\cos\theta}}{\kappa_+(-k\cos\theta)} \left(N_2(k\cos\theta) - T(k\cos\theta)\gamma_+(k)C_2 \right) \right.$$

$$+ \frac{e^{ikl\cos\theta} T(k\cos\theta) iu_0 \overline{f}_1(\omega) T(k)\kappa_+(k)}{k[1 - T^2(k)\kappa_+^2(k)]\kappa_+(-k\cos\theta)} \left(1 - \frac{1}{\sqrt{2k}} + \frac{1}{\sqrt{k}}\right)$$

$$+ \frac{\widetilde{A}e^{ik_1 l\cos\theta_0}}{\kappa_-(k\cos\theta)} [N_1(-k\cos\theta) - C_1 T(-k\cos\theta)\kappa_+(k)]$$

$$\left. + \frac{T(-k\cos\theta) iu_0 \overline{f}_1(\omega) T(k)\kappa_+(k)}{k[1 - T^2(k)\kappa_+^2(k)]\kappa_-(-k\cos\theta)} \left(1 - \frac{1}{\sqrt{2k}} + \frac{1}{\sqrt{k}}\right) \right] |k\sin\theta e^{ikr - i\frac{\pi}{4}} \left(\frac{2\pi}{kr}\right)^{\frac{1}{2}} \tag{5.99}$$

Now taking the inverse temporal Fourier transform by using

$$\int_{-\infty}^{\infty} f_1(t)\delta(t - t_0)dt = f(t_0),$$

we get

$$\phi^{sep}(x,y,t) = -\frac{\sin\theta\sin\theta_0}{2\pi[\cos\theta + \cos\theta_0]} \begin{pmatrix} \frac{e^{-ik_1 l[\cos\theta + \cos\theta_0]}}{\sqrt{1-\cos\theta}\sqrt{\cos\theta_0 - 1}} \\ +\frac{1}{i\sqrt{1+\cos\theta}\sqrt{\cos\theta_0+1}} \end{pmatrix} e^{ik_1 r - i\frac{\pi}{4} - i\omega_1 t} \left(\frac{2\pi}{k_1 r}\right)^{\frac{1}{2}}$$

$$+ \frac{iu_0 \sin\theta \sum_{-\infty}^{\infty} C_n}{2\pi\overline{k}\cos\theta} \begin{pmatrix} \frac{e^{il\overline{k}\cos\theta}}{\sqrt{1-\cos\theta}} \\ +\frac{1}{i\sqrt{1+\cos\theta}} \end{pmatrix} e^{i\overline{k}r - i\frac{\pi}{4} - in\omega_0 t} \left(\frac{2\pi}{\overline{k}r}\right)^{\frac{1}{2}}, \tag{5.100}$$

where $\theta \neq \pi - \theta_0$ and the interacted term is given by

$$\phi^{int}(x,y,t) = \left[\left\{\frac{e^{ik_1 l\cos\theta}(-k_1\sin\theta_0)}{\kappa_+(-k_1\cos\theta)}\left(N_2(k_1\cos\theta) - T(k_1\cos\theta)\kappa_+(k_1)C_2\right)k_1\sin\theta e^{ik_1 r - i\frac{\pi}{4}}\left(\frac{2\pi}{k_1 r}\right)^{\frac{1}{2}}\right\}\right.$$

$$+\left\{\frac{e^{il\overline{k}\cos\theta}T(\overline{k}\cos\theta)iu_0\sum_{-\infty}^{\infty}C_nT(\overline{k})\kappa_+(\overline{k})}{\overline{k}[1-T^2(\overline{k})\kappa_+^2(\overline{k})]\kappa_+(-\overline{k}\cos\theta)}\left(1-\frac{1}{\sqrt{2\overline{k}}}+\frac{1}{\sqrt{\overline{k}}}\right)\overline{k}\sin\theta e^{i\overline{k}r - i\frac{\pi}{4}}\left(\frac{2\pi}{\overline{k}r}\right)^{\frac{1}{2}}\right\}$$

$$+\left\{\frac{e^{ik_1 l\cos\theta_0}(-k_1\sin\theta_0)}{\kappa_-(k_1\cos\theta)}\left(N_1(-k_1\cos\theta) - T(-k_1\cos\theta)\kappa_+(k_1)C_1\right)k_1\sin\theta e^{ik_1 r - i\frac{\pi}{4}}\left(\frac{2\pi}{k_1 r}\right)^{\frac{1}{2}}\right\}$$

$$\left.+\left\{\frac{T(-\overline{k}\cos\theta)iu_0\sum_{-\infty}^{\infty}C_nT(\overline{k})\kappa_+(\overline{k})}{\overline{k}[1-T^2(\overline{k})\kappa_+^2(\overline{k})]\kappa_-(-\overline{k}\cos\theta)}\left(1-\frac{1}{\sqrt{2\overline{k}}}+\frac{1}{\sqrt{\overline{k}}}\right)\overline{k}\sin\theta e^{i\overline{k}r - i\frac{\pi}{4}}\left(\frac{2\pi}{\overline{k}r}\right)^{\frac{1}{2}}\right\}\right]. \tag{5.101}$$

Remarks:

Mathematically we can derive the results of the half plane problem [74] in the following manner:

For the analysis purpose, in Eq. (5.93), it is assumed that the wave number k_1 has positive imaginary part and using the De L Hopital rule successively, the value of E_{-1}, reduces to $\lim_{l\to\infty}(\frac{\sqrt{l}}{e^l})$ which becomes zero and in turn result the quantities $T(\alpha)$, $N_{1,2}(\alpha)$, C_1 and $G_2(\alpha)$ in zero. The third term in Eqs. (5.85) also becomes zero. Using the substitution $\widetilde{A} = -2\pi k_1\delta(\omega-\omega_1)\sin\theta_0$, $A(\alpha)$ reduces to

$$A(\alpha) = -\frac{1}{\kappa}\left[\frac{2\pi k_1\sin\theta_0\delta(\omega-\omega_1)\kappa_+(\alpha)}{\kappa_+(k_1\cos\theta_0)(\alpha-k_1\cos\theta_0)} - \frac{iu_0\overline{f}_1(\omega)\kappa_+(\alpha)}{\alpha\sqrt{k}}\right].$$

Using the factorization

$$\kappa(\alpha) = \kappa_+(\alpha)\kappa_+(\alpha),$$

the above result reduces to $A(\alpha)$ of the Half Plane [74]. Subsequently, Eq. (5.97), i.e., the interacted field vanishes by adopting the same procedure as in case of Eq. (5.93), while the separated field results into the diffracted field [74] as the strip is widened to half plane by taking the limit $l \to \infty$.

5.5 Numerical Results and Discussion

A computer program MATHEMATICA has been used for obtaining the graphical results. In Fig. 5.1, one can see that as we increase the angle of incidence, the amplitude of the separated field also increases [109]. Fig. 5.2 depicts that the wave with the higher frequency obtains higher amplitude of the separated field as compared to the other wave frequencies which is an obvious result. Fig. 5.3 shows that the amplitude of the separated field with higher strip frequency will be at the top and the rest will gradually be positioned accordingly which is according to the physics of the problem. The effect of strip length l, for its different values, can be seen in Figs. 5.3, 5.4 and 5.5. We observe that as l tends to its highest value the graphs for the strip (Fig.5.5) and the half plane (Fig. 5.6) agree each other which can be considered as numerical proof of our claim that the strip reduces to half plane [74] as $l \to \infty$. This can be considered as a check of validity of the analysis numerically as well in this paper.

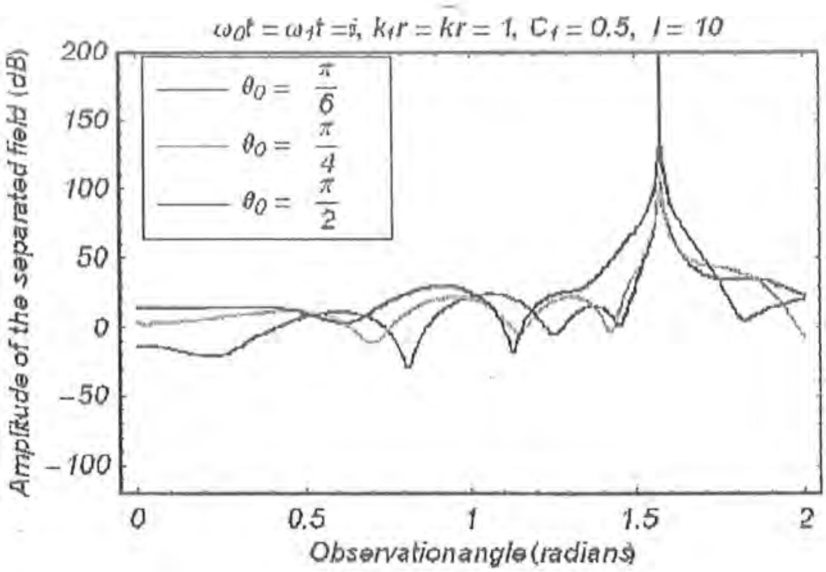


Fig. 5.1: Amplitude of the separated field for different values of angle of incidence

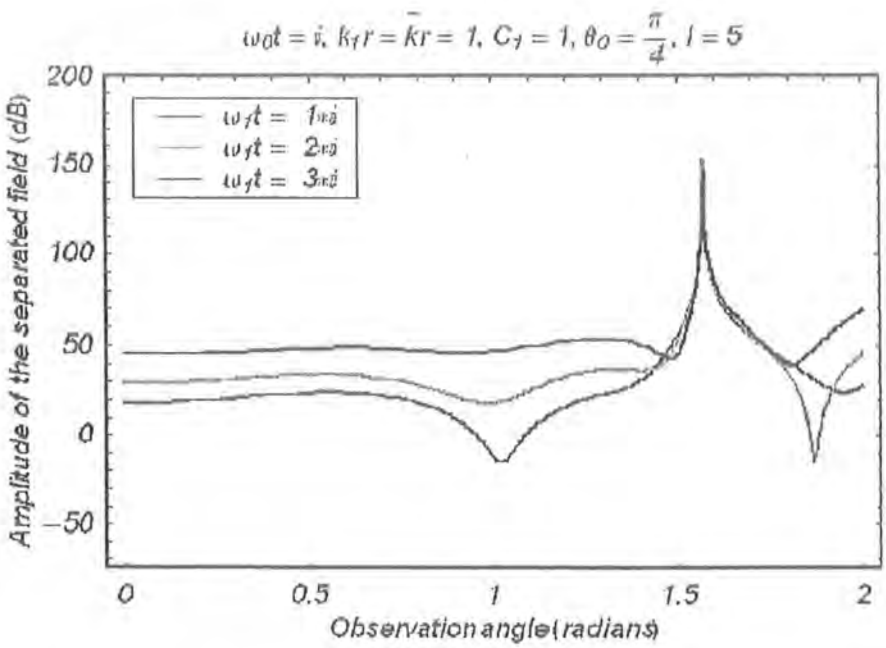


Fig. 5.2: Amplitude of the separated field for different values of wave frequency

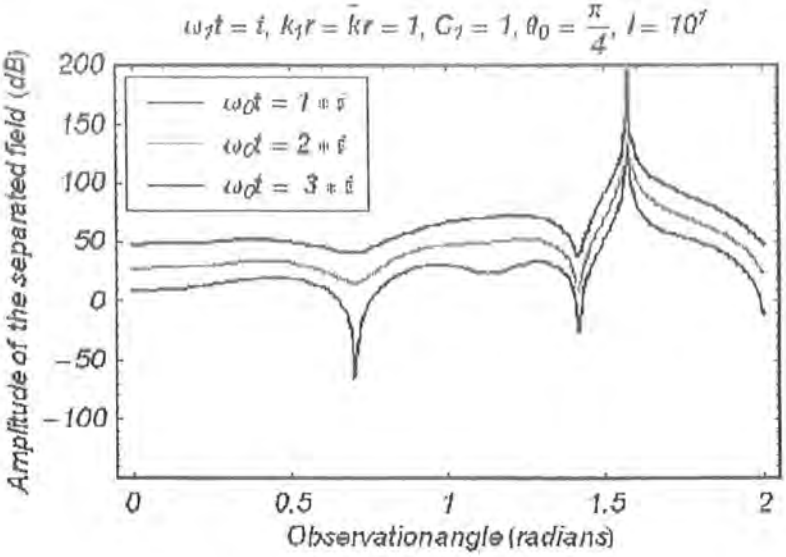


Fig. 5.3: Amplitude of the separated field for different values of strip frequency for $l = 10^1$.

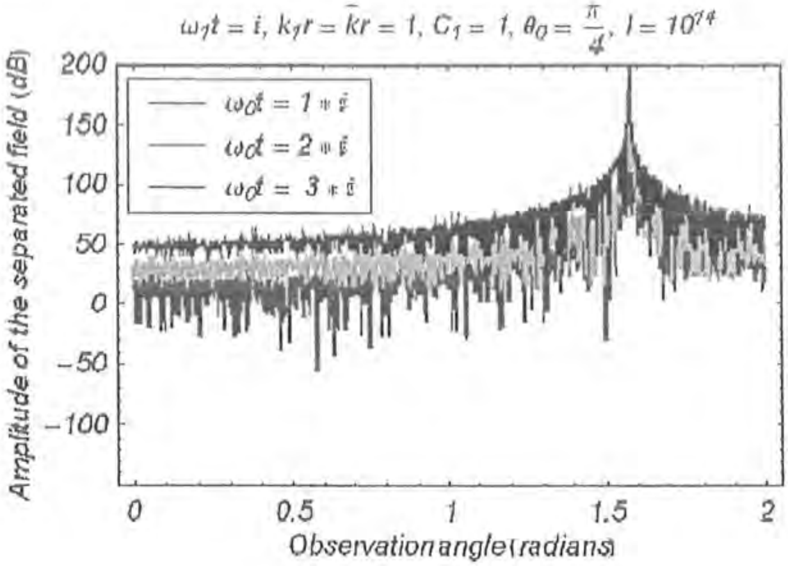


Fig. 5.4: Amplitude of the separated field for different values of strip frequency for $l = 10^{14}$

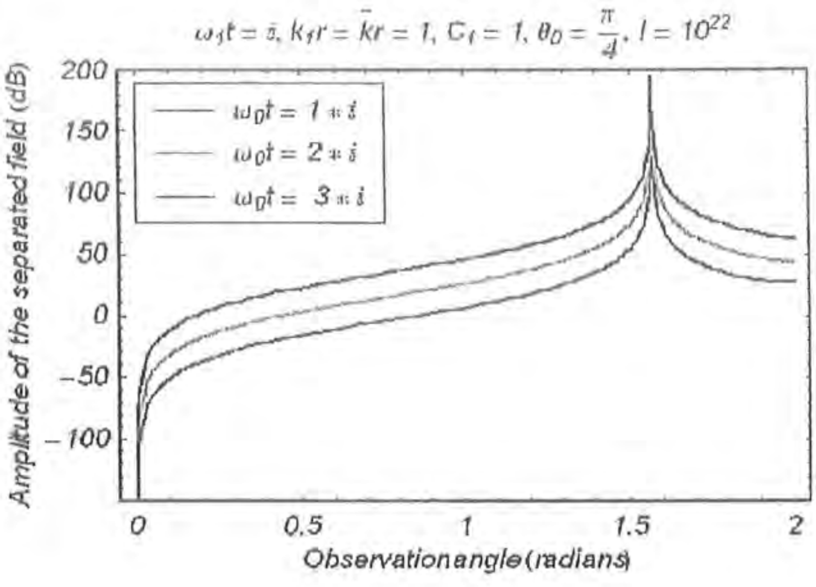


Fig. 5.5: Amplitude of the separated field for different values of strip frequency for $l = 10^{22}$.

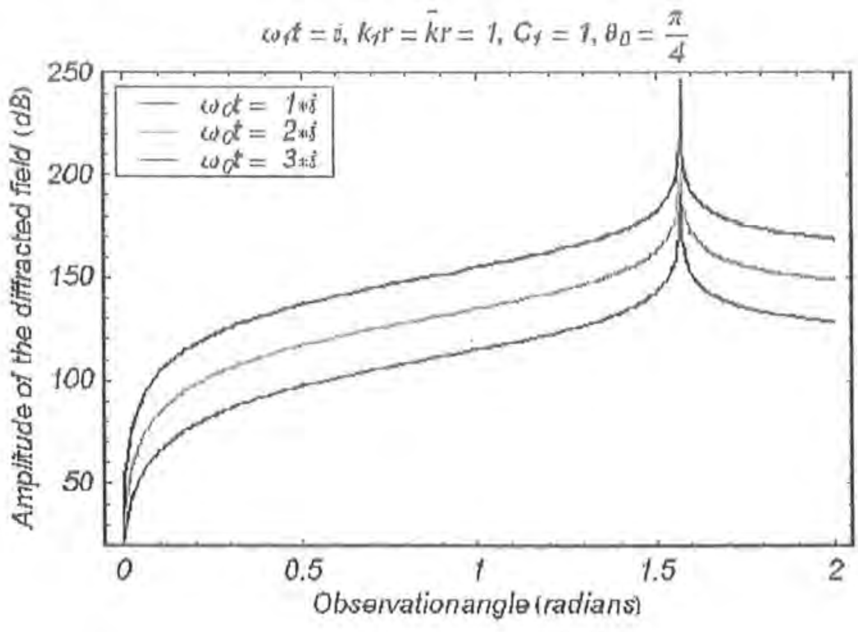


Fig. 5.6: Amplitude of the diffracted field for different values of half plane frequency

Chapter 6

A note on plane wave diffraction by a perfectly conducting strip in a homogeneous bi-isotropic medium

In this chapter, we studied the problem of diffraction of an electromagnetic plane wave by a perfectly conducting finite strip in a homogeneous bi-isotropic medium and obtained some improved results for [104] which were presented both mathematically and graphically. The problem was solved by using the Wiener-Hopf technique and Fourier transform. The scattered field in the far zone was determined by the method of steepest descent. The significance of present analysis is that it recovered the results when a strip is widened into a half plane.

6.1 Formulation of the problem

Let us assume the scattering of a plane electromagnetic wave with the assumption that all space is occupied by a homogeneous bi-isotropic medium except for a perfectly conducting strip $z = 0$, $-l \leq x \leq 0$. In the Drude-Born-Fedorov representation [79, 106], the bi-isotropic medium is characterized by the following equations

$$\mathbf{D} = \varepsilon\mathbf{E} + \varepsilon\alpha_{BF}\nabla \times \mathbf{E}, \tag{6.1}$$

$$\mathbf{B} = \mu\mathbf{H} + \mu\beta_{BF}\nabla \times \mathbf{H}, \tag{6.2}$$

where ε and μ are the permittivity and the permeability scalars, respectively, while α_{BF} and β_{BF} are the bi-isotropy scalars. **D** is the electric displacement, **H** is the magnetic field strength, **B** is the magnetic induction, and **E** is the electric field strength. The bi-isotropic medium with $\alpha_{BF} = \beta_{BF}$ is reciprocal and is then called a chiral medium. Recently, it has been proved [101] that non-reciprocal bi-isotropic media are not permitted by the structure of modern electromagnetic theory. Certainly in the MHz-PHz regime, this statement has not been experimentally challenged yet, although in the < 1 kHZ regime there is some experimental evidence to the contrary which has not been independently confirmed [105]. However, in the mathematical study the case $\alpha_{BF} \neq \beta_{BF}$ may also be considered for generality.

Let us assume the time dependence of Beltrami fields to be of the form $\exp(-i\omega t)$, where ω is the angular frequency. Here, Beltrami fields are preferred because there are two methods to uncouple the Maxwell's equations namely:

i) Field Decomposition Method as given in the book by Lindell and Shivola [110].

ii) To introduce the concept of Beltrami fields as proposed in the book by Lakhtakia [79].

The approach (ii) is more straight forward and we followed it in our analysis. The source free Maxwell curl postulates in the bi-isotropic medium can be set up as

$$\nabla \times \mathbf{Q}_1 = \gamma_1\mathbf{Q}_1, \tag{6.3}$$

$$\nabla \times \mathbf{Q}_2 = -\gamma_2 \mathbf{Q}_2. \tag{6.4}$$

The two wave numbers $\gamma_1$and γ_2 are given by

$$\gamma_1 = \frac{k_{BF}}{(1 - k_{BF}^2 \alpha_{BF} \beta_{BF})} \left\{ \sqrt{1 + \frac{k_{BF}^2 (\alpha_{BF} - \beta_{BF})^2}{4}} + \frac{k_{BF}(\alpha_{BF} + \beta_{BF})}{2} \right\}, \tag{6.5}$$

and

$$\gamma_2 = \frac{k_{BF}}{(1 - k_{BF}^2 \alpha_{BF} \beta_{BF})} \left\{ \sqrt{1 + \frac{k_{BF}^2 (\alpha_{BF} - \beta_{BF})^2}{4}} - \frac{k_{BF}(\alpha_{BF} + \beta_{BF})}{2} \right\}, \tag{6.6}$$

where Beltrami fields in terms of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$, as given in [102], are :

$$\mathbf{Q}_1 = \frac{\eta_{1_{BF}}}{\eta_{1_{BF}} + \eta_{2_{BF}}} (\mathbf{E} + i\eta_{2_{BF}} \mathbf{H}), \tag{6.7}$$

and

$$\mathbf{Q}_2 = \frac{i}{\eta_{1_{BF}} + \eta_{2_{BF}}} (\mathbf{E} - i\eta_{1_{BF}} \mathbf{H}), \tag{6.8}$$

where $\mathbf{Q}_1$ is the left-handed Beltrami field and $\mathbf{Q}_2$ is the right-handed Beltrami field. In Eqs. (6.7) and (6.8), the two impedances $\eta_{1_{BF}}$ and $\eta_{2_{BF}}$ are given by

$$\eta_{1_{BF}} = \frac{\eta_{BF}}{\sqrt{1 + \frac{k_{BF}^2 (\alpha_{BF} - \beta_{BF})^2}{4}} + \frac{k_{BF}(\alpha_{BF} - \beta_{BF})}{2}}, \tag{6.9}$$

and

$$\eta_{2_{BF}} = \eta_{BF} \left\{ \sqrt{1 + \frac{k_{BF}^2 (\alpha_{BF} - \beta_{BF})^2}{4}} - \frac{k_{BF}(\alpha_{BF} - \beta_{BF})}{2} \right\}, \tag{6.10}$$

where $k_{BF} = \omega\sqrt{\varepsilon\mu}$ and $\eta_{BF} = \sqrt{\frac{\mu}{\varepsilon}}$.

Since we are interested in scattering of electromagnetic waves with a prescribed y variation, therefore, it is appropriate to decompose the Beltrami fields as [108].

$$\mathbf{Q}_1 = \mathbf{Q}_{1t} + \mathbf{j}.Q_{1y}, \tag{6.11}$$

with

$$\mathbf{Q}_{1t} = Q_{1x}\mathbf{i} + Q_{1z}\mathbf{k}. \tag{6.12}$$

and

$$\mathbf{Q}_2 = \mathbf{Q}_{2t} + \mathbf{j}.Q_{2y}. \tag{6.13}$$

where the fields $\mathbf{Q}_{1t}$ and $\mathbf{Q}_{2t}$ lie in the $xz-$plane and $\mathbf{j}$ is a unit vector along the y-axis such that $\mathbf{j}.\mathbf{Q}_{1t} = 0$ and $\mathbf{j}.\mathbf{Q}_{2t} = 0$. Now, the Eq. (6.3) can be written as:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q_{1x} & Q_{1y} & Q_{1z} \end{vmatrix} = \gamma_1(Q_{1x}\mathbf{i} + Q_{1y}\mathbf{j} + Q_{1z}\mathbf{k}). \tag{6.14}$$

Assuming all the field vectors having an explicit $\exp(ik_y y)$ dependence on the variable y and comparing x and z components on both sides of the above equation, we obtain

$$Q_{1x} = \frac{1}{k_{1xz}^2}\left[ik_y\frac{\partial Q_{1y}}{\partial x} - \gamma_1\frac{\partial Q_{1y}}{\partial z}\right], \tag{6.15}$$

and

$$Q_{1z} = \frac{1}{k_{1xz}^2}\left[ik_y\frac{\partial Q_{1y}}{\partial z} + \gamma_1\frac{\partial Q_{1y}}{\partial x}\right], \tag{6.16}$$

where

$$k_{1xz}^2 = \gamma_1^2 - k_y^2. \tag{6.17}$$

Similarly, from Eq. (6.4), with explicit $\exp(ik_y y)$ dependence on the variable y, we may obtain

$$Q_{2x} = \frac{1}{k_{2xz}^2}\left[ik_y\frac{\partial Q_{2y}}{\partial x} + \gamma_2\frac{\partial Q_{2y}}{\partial z}\right], \tag{6.18}$$

$$Q_{2z} = \frac{1}{k_{2xz}^2}\left[ik_y\frac{\partial Q_{2y}}{\partial z} - \gamma_2\frac{\partial Q_{2y}}{\partial x}\right], \tag{6.19}$$

with

$$k_{2xz}^2 = \gamma_2^2 - k_y^2. \tag{6.19a}$$

It is sufficient to explore the scattering of the scalar field Q_{1y} and Q_{2y} because the other components of $\mathbf{Q}_1$ and $\mathbf{Q}_2$ can then be completely determined by using Eqs. (6.15 − 6.19).

Now using the constitutive relations (6.1) and (6.2), the Maxwell curl postulates $\nabla \times \mathbf{E} = i\omega\mathbf{B} - \mathbf{K}$ and $\nabla \times \mathbf{H} = -i\omega\mathbf{D} + \mathbf{J}$ may be written as:

$$\nabla \times \mathbf{Q}_1 - \gamma_1\mathbf{Q}_1 = \mathbf{S}_1, \tag{6.20a}$$

$$\nabla \times \mathbf{Q}_2 - \gamma_2\mathbf{Q}_2 = \mathbf{S}_2, \tag{6.20b}$$

where S_1 and S_2 are the corresponding source densities and are given by

$$\mathbf{S}_1 = \frac{\eta_{1_{BF}}}{\eta_{1_{BF}} + \eta_{2_{BF}}}\left(\frac{i\gamma_1}{\omega\varepsilon}\mathbf{J} - (1 + \alpha_{BF}\gamma_1)\mathbf{K}\right), \tag{6.21a}$$

$$\mathbf{S}_2 = \frac{\eta_{1_{BF}}}{\eta_{1_{BF}} + \eta_{2_{BF}}}\left(-\frac{i\gamma_2}{\omega\mu}\mathbf{K} - (1 + \beta_{BF}\gamma_2)\mathbf{J}\right). \tag{6.21b}$$

In deriving Eqs. (6.21a) and (6.21b), we have used the following relations

$$1 + \omega\varepsilon\alpha_{BF}\eta_{2_{BF}} = (1 - k_{BF}^2\alpha_{BF}\beta_{BF})(1 + \alpha_{BF}\gamma), \tag{6.22}$$

$$1 - \omega\varepsilon\alpha_{BF}\eta_{1_{BF}} = (1 - k_{BF}^2\alpha_{BF}\beta_{BF})\eta_{1_{BF}}\frac{\gamma_2}{\omega\mu}, \tag{6.23}$$

$$\eta_{2_{BF}} + \omega\mu\beta_{_{BF}} = (1 - k^2_{_{BF}}\alpha_{_{BF}}\beta_{_{BF}})\frac{\gamma_1}{\omega\varepsilon}, \tag{6.24}$$

$$\eta_{1_{BF}} - \omega\mu\beta_{_{BF}} = (1 - k^2_{_{BF}}\alpha_{_{BF}}\beta_{_{BF}})\eta_{1_{BF}}(1 - \beta_{_{BF}}\gamma_2). \tag{6.25}$$

Furthermore, $\mathbf{Q}_1$ is $\mathbf{E}$ like and $\mathbf{Q}_2$ is $\mathbf{H}$ like. Similarly $\mathbf{S}_1$ is $\mathbf{K}$ like and $\mathbf{S}_2$ is $\mathbf{J}$ like where $\mathbf{J}$ and $\mathbf{K}$ are the electric and magnetic source current densities, respectively. The boundary condition which is necessary is that the tangential component of the electric field must vanish on perfectly conducting finite plane. This implies that $E_x = E_y = 0$, for $z = 0$, $-l \leq x \leq 0$. Using this fact in Eqs. (6.7) and (6.8), the boundary conditions on the finite plane take the form

$$Q_{1y} - i\eta_{2_{BF}}Q_{2y} = 0,\ z = 0,\ \ -l \leq x \leq 0, \tag{6.26a}$$

and

$$Q_{1x} - i\eta_{2_{BF}}Q_{2x} = 0,\ z = 0,\ \ -l \leq x \leq 0. \tag{6.26b}$$

Using the idea of perfectly conducting surface, we have been able to make $E_x = E_y = 0$, for $z = 0$, $-l \leq x \leq 0$, enabling us to derive (26*a*) and (26*b*). With the help of Eqs. (6.16) and (6.17), Eq. (6.26*b*) becomes

$$\frac{1}{k^2_{1xz}}\left[ik_y\frac{\partial Q_{1y}}{\partial x} - \gamma_1\frac{\partial Q_{1y}}{\partial z}\right] - i\eta_{2_{BF}}\frac{1}{k^2_{2xz}}\left[ik_y\frac{\partial Q_{2y}}{\partial x} + \gamma_2\frac{\partial Q_{2y}}{\partial z}\right] = 0,\ z = 0,\ \ -l \leq x \leq 0. \tag{6.27}$$

Thus the scalar fields Q_{1y} and Q_{2y} satisfy the boundary conditions (26*a*) and (27).

Now, eliminating Q_{2y} from Eqs. (6.26*a*) and (6.27), we obtain

$$\frac{\partial Q_{1y}}{\partial x} \mp \Omega\frac{\partial Q_{1y}}{\partial z} = 0, \quad z = 0^{\pm},\ \ -l \leq x \leq 0, \tag{6.28}$$

where

$$\Omega = \frac{\gamma_2 k_{1xz}^2 + \gamma_1 k_{2xz}^2}{ik_y(k_{2xz}^2 - k_{1xz}^2)}. \tag{6.28a}$$

It is worthwhile to note that the boundary conditions (6.28) are of the same form as impedance boundary conditions [13]. We observe that there is no boundary for $-\infty < x < -l$, $x > 0$, $z = 0$. Therefore the continuity conditions are given by

$$Q_{1y}(x, z^+) = Q_{1y}(x, z^-); \quad -\infty < x < -l,\ x > 0,\ z = 0, \tag{6.29}$$

$$\frac{\partial Q_{1y}(x, z^+)}{\partial z} = \frac{\partial Q_{1y}(x, z^-)}{\partial z}; \quad -\infty < x < -l,\ x > 0,\ z = 0. \tag{6.30}$$

The edge conditions (local properties) on the field that invoke the appropriate physical constraint of finite energy near the edges of the boundary discontinuities require that

$$Q_{1y}(x, 0) = O(1) \text{ and } \frac{\partial Q_{1y}(x, 0)}{\partial z} = O(x^{-\frac{1}{2}}) \text{ as } x \to 0^+, \tag{6.31a}$$

$$Q_{1y}(x, 0) = O(1) \text{ and } \frac{\partial Q_{1y}(x, 0)}{\partial z} = O(x + l)^{-\frac{1}{2}} \text{ as } x \to -l. \tag{6.31b}$$

It is to be noted that the field Q_{2y} also satisfies Eqs. $(6.28 - 6.30)$. Finally, the scattered field must satisfy the radiation conditions in the limit $(x^2 + z^2)^{1/2} \to \infty$. We must also observe at this juncture that, in effect, we need to consider the diffraction of only one scalar field, that is either Q_{1y} or Q_{2y}, at a time, but the presence of the other scalar field is reflected in the complicated nature of the boundary condition (6.28). If we set the incident field to be a plane wave, then

$$Q_{1y}(x, z) = Q_{1y}^{inc}(x, z) + Q_{1y}^{sca}(x, z), \tag{6.32}$$

with

$$Q_{1y}^{inc}(x,y,z) = exp[i(k_y y + k_{1x}x + k_{1z}z)], \tag{6.32a}$$

and scattered field Q_{1y}^{sca} satisfies the following homogeneous Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_{1xz}^2\right) Q_{1y}^{sca} = 0. \tag{6.33}$$

where

$$k_{1xz}^2 = k_{1x}^2 + k_{1z}^2 = \gamma_1^2 - k_y^2. \tag{6.33a}$$

Also the boundary conditions (6.28) to (6.30) will take the following form

$$\left(\frac{\partial}{\partial x} \mp \Omega\frac{\partial}{\partial z}\right) Q_{1y}^{inc} + \left(\frac{\partial}{\partial x} \mp \Omega\frac{\partial}{\partial z}\right) Q_{1y}^{sca} = 0, \quad z = 0^{\pm}, -l \leq x \leq 0, \tag{6.34}$$

and

$$Q_{1y}^{sca}(x,z^+) = Q_{1y}^{sca}(x,z^-); \quad -\infty < x < -l, \ x > 0, \ z = 0, \tag{6.35a}$$

$$\frac{\partial}{\partial z}Q_{1y}^{sca}(x,z^+) = \frac{\partial}{\partial z}Q_{1y}^{sca}(x,z^-); \quad -\infty < x < -l, \ x > 0, \ z = 0. \tag{6.35b}$$

For the solution of Eq. (6.33) subject to the boundary conditions ($6.34 - 6.35b$), we introduce the Fourier transform $w.r.t$ variable x as:

$$\overline{\Phi}(\alpha,z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Q_{1y}^{sca}(x,z)e^{i\alpha x}dx = \overline{\Phi}_+(\alpha,z) + e^{-i\alpha l}\overline{\Phi}_-(\alpha,z) + \overline{\Phi}_1(\alpha,z), \tag{6.36}$$

where

$$\overline{\Phi}_+(\alpha, z) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} Q_{1y}^{sca}(x, z) e^{i\alpha x} dx,$$

$$\overline{\Phi}_-(\alpha, z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-l} Q_{1y}^{sca}(x, z) e^{i\alpha(x+l)} dx,$$

$$\overline{\Phi}_1(\alpha, z) = \frac{1}{\sqrt{2\pi}} \int_{-l}^{0} Q_{1y}^{sca}(x, z) e^{i\alpha x} dx. \tag{6.37}$$

Note that $\overline{\Phi}_-(\alpha, z)$ is regular for $\operatorname{Im}\alpha < \operatorname{Im} k_{1xz}$, and $\overline{\psi}_+(\alpha, z)$ is regular for $\operatorname{Im}\alpha > -\operatorname{Im} k_{1xz}$ and $\overline{\psi}_1(\alpha, z)$ is analytic in the common region $-\operatorname{Im} k_{1xz} < \operatorname{Im}\alpha < \operatorname{Im} k_{1xz}$. The Fourier transform of Eq. (6.32a) in the region $-l \leq x \leq 0$, $z = 0$ gives

$$\overline{\Phi}_0(\alpha, 0) = \frac{i}{\sqrt{2\pi}(k_{1x} + \alpha)} \left[-1 + \exp[-i(k_{1x} + \alpha)l]\right], \tag{6.38}$$

and its derivative is defined as

$$\overline{\Phi}_0'(\alpha, 0) = \frac{k_{1z}}{\sqrt{2\pi}(k_{1x} + \alpha)} \left[-1 + \exp[-i(k_{1x} + \alpha)l]\right]. \tag{6.38a}$$

The Fourier transform of Eqs. (6.34 − 6.35b), respectively, yields

$$\left(\frac{d^2}{dz^2} + \kappa_{BF}^2\right) \overline{\Phi}(\alpha, z) = 0, \tag{6.39}$$

where

$$\kappa_{BF}^2 = k_{1xz}^2 - \alpha^2, \tag{6.39a}$$

$$\overline{\Phi}_1'(\alpha, 0^+) = -\frac{i\alpha}{\Omega}[\overline{\Phi}_1(\alpha, 0^+) + \overline{\Phi}_0(\alpha, 0)] - \overline{\Phi}_0'(\alpha, 0), \tag{6.40}$$

$$\overline{\Phi}_1'(\alpha, 0^-) = \frac{i\alpha}{\Omega}[\overline{\Phi}_1(\alpha, 0^-) + \overline{\Phi}_0(\alpha, 0)] - \overline{\Phi}_0'(\alpha, 0), \tag{6.41}$$

and

$$
\begin{aligned}
\overline{\Phi}_-(\alpha,0^+) &= \overline{\Phi}_-(\alpha,0^-) = \overline{\Phi}_-(\alpha,0),\\
\overline{\Phi}_+(\alpha,0^+) &= \overline{\Phi}_+(\alpha,0^-) = \overline{\Phi}_+(\alpha,0),\\
\overline{\Phi}'_-(\alpha,0^+) &= \overline{\Phi}'_-(\alpha,0^-) = \overline{\Phi}'_-(\alpha,0),\\
\overline{\Phi}'_+(\alpha,0^+) &= \overline{\Phi}'_+(\alpha,0^-) = \overline{\Phi}'_+(\alpha,0).
\end{aligned} \tag{6.42}
$$

The solution of Eq. (6.39) satisfying radiation condition is given by

$$
\overline{\Phi}(\alpha,z) = \begin{cases} A(\alpha)e^{i\kappa_{BF}z} & if\ z>0,\\ C(\alpha)e^{-i\kappa_{BF}z} & if\ z<0. \end{cases} \tag{6.43}
$$

By substituting Eqs. (6.36) and (6.42) to Eq. (6.43), we get

$$
\overline{\Phi}_+(\alpha,0) + e^{-i\alpha l}\overline{\Phi}_-(\alpha,0) + \overline{\Phi}_1(\alpha,0^+) = A(\alpha), \tag{6.44a}
$$

$$
\overline{\Phi}_+(\alpha,0) + e^{-i\alpha l}\overline{\psi}_-(\alpha,0) + \overline{\psi}_1(\alpha,0^-) = C(\alpha), \tag{6.44b}
$$

$$
\overline{\psi}'_+(\alpha,0) + e^{-i\alpha l}\overline{\Phi}'_-(\alpha,0) + \overline{\Phi}'_1(\alpha,0^+) = i\kappa_{BF}A(\alpha), \tag{6.44c}
$$

$$
\overline{\Phi}'_+(\alpha,0) + e^{-i\alpha l}\overline{\Phi}'_-(\alpha,0) + \overline{\Phi}'_1(\alpha,0^-) = -i\kappa_{BF}C(\alpha). \tag{6.44d}
$$

Subtracting Eq. (6.44b) from Eq. (6.44a) and Eq. (6.44d) from Eq. (6.44c) and then by adding and subtracting the resultant equations, we obtain

$$
A(\alpha) = J_1(\alpha,0) + \frac{J_1'(\alpha,0)}{i\kappa_{BF}}, \tag{6.45}
$$

and

$$
C(\alpha) = -J_1(\alpha,0) + \frac{J_1'(\alpha,0)}{i\kappa_{BF}}, \tag{6.46}
$$

where

$$J_1(\alpha,0) = \frac{1}{2}[\overline{\Phi}_1(\alpha,0^+) - \overline{\Phi}_1(\alpha,0^-)], \tag{6.47}$$

and

$$J_1'(\alpha,0) = \frac{1}{2}[\overline{\Phi}_1'(\alpha,0^+) - \overline{\Phi}_1'(\alpha,0^-)]. \tag{6.48}$$

Making use of Eq. (6.44a) in Eq. (6.44c) and Eq. (6.44b) in Eq. (6.44d), we can write

$$\overline{\Phi}_+'(\alpha,0)+e^{-i\alpha l}\overline{\Phi}_-'(\alpha,0)+\overline{\Phi}_1'(\alpha,0^+) = i\kappa_{_{BF}}[\overline{\Phi}_+(\alpha,0)+e^{-i\alpha l}\overline{\Phi}_-(\alpha,0)+\overline{\Phi}_1(\alpha,0^+)], \tag{6.49a}$$

$$\overline{\Phi}_+'(\alpha,0)+e^{-i\alpha l}\overline{\Phi}_-'(\alpha,0)+\overline{\Phi}_1'(\alpha,0^-) = -i\kappa_{_{BF}}[\overline{\Phi}_+(\alpha,0)+e^{-i\alpha l}\overline{\Phi}_-(\alpha,0)+\overline{\Phi}_1(\alpha,0^-)]. \tag{6.49b}$$

By eliminating $\overline{\Phi}_1'(v,0^+)$ from Eqs. (6.49a) and (6.45) and $\overline{\Phi}_1'(\alpha,0^-)$ from Eqs. (6.49b) and (6.46) and then by adding the resultant equations, we get

$$\overline{\Phi}_+'(\alpha,0)+e^{-i\alpha l}\overline{\Phi}_-'(\alpha,0) - i\kappa_{_{BF}}L(\alpha)J_1(\alpha) + \frac{k_{1z}}{\sqrt{2\pi}(k_{1x}+\alpha)}[-1+\exp[-i(k_{1x}+\alpha)l]] = 0. \tag{6.50}$$

In a similar way, by eliminating $\overline{\Phi}_1(\alpha,0^+)$ from Eqs. (6.49a) and (6.40), $\overline{\Phi}_1(\alpha,0^-)$ from Eqs. (6.49b) and (6.41), and then subtracting the resulting equations, we get

$$-i\alpha\overline{\Phi}_+(\alpha,0) - i\alpha e^{-i\alpha l}\overline{\Phi}_-(\alpha,0) + \Omega L(\alpha)J_1'(\alpha) + \frac{k_{1x}}{\sqrt{2\pi}(k_{1x}+\alpha)}[-1+\exp[-i(k_{1x}+\alpha)l]] = 0, \tag{6.51}$$

where

$$L(v) = \left(1 + \frac{\alpha}{\Omega\kappa_{_{BF}}}\right). \tag{6.51a}$$

Eqs. (6.50) and (6.51) are the standard Wiener-Hopf equations. Let us proceed to find the solution for these equations.

6.2 Solution of the Wiener-Hopf equations

For the solution of the Wiener-Hopf equations, one can make use of the following factorization

$$L(\alpha) = \left(1 + \frac{\alpha}{\Omega \kappa_{BF}}\right) = L_+(\alpha)L_-(\alpha), \tag{6.52a}$$

and

$$\kappa_{BF}(\alpha) = \kappa_{BF+}(\alpha)\kappa_{BF-}(\alpha), \tag{6.52b}$$

where $L_+(\alpha)$ and $\kappa_{BF+}(\alpha)$ are regular for $\operatorname{Im}\alpha > -\operatorname{Im} k_{1xz}$, i.e., for upper half plane and $L_-(\alpha)$ and $\kappa_{BF-}(\alpha)$ are regular for $\operatorname{Im}\alpha < \operatorname{Im} k_{1xz}$, i.e., lower half plane. The factorization expression $(6.52a)$ has been accomplished by Asghar et al [103]. By putting the values of $J_1(\alpha, 0)$ and $J_1'(\alpha, 0)$ from Eqs. (6.50) and (6.51) into Eqs. (6.45) and (6.46), we get

$$A(\alpha) = \frac{1}{i\kappa_{BF}L(\alpha)}\{\overline{\Phi}'_+(\alpha,0) + e^{-i\alpha l}\overline{\Phi}'_-(\alpha,0) + \frac{k_{1z}}{\sqrt{2\pi}(k_{1x}+\alpha)}\left[-1+\exp[-i(k_{1x}+\alpha)l]\right]$$

$$+\frac{\alpha\Omega_1}{\kappa_{BF}\ L(\alpha)}\{\overline{\Phi}_+(\alpha,0) + e^{-i\alpha l}\overline{\Phi}_-(\alpha,0) - \frac{k_{1x}}{\sqrt{2\pi}(k_{1x}+\alpha)}\left[-1+\exp[-i(k_{1x}+\alpha)l]\right], \tag{6.53}$$

$$C(\alpha) = -\frac{1}{i\kappa_{BF}L(\alpha)}\{\overline{\Phi}'_+(\alpha,0) + e^{-i\alpha l}\overline{\Phi}'_-(\alpha,0) + \frac{k_{1z}}{\sqrt{2\pi}(k_{1x}+\alpha)}\left[-1+\exp[-i(k_{1x}+\alpha)l]\right]$$

$$+\frac{\alpha\Omega_1}{\kappa_{BF}L(\alpha)}\{\overline{\Phi}_+(\alpha,0) + e^{-i\alpha l}\overline{\Phi}_-(\alpha,0) - \frac{k_{1x}}{\sqrt{2\pi}(k_{1x}+\alpha)}\left[-1+\exp[-i(k_{1x}+\alpha)l]\right], \tag{6.54}$$

where $\Omega_1 = \frac{1}{\Omega}$. In [104], the terms of $O(\Omega_1)$ are neglected while in the present analysis the Ω_1 parameter is taken up to order one so that the results due to semi infinite barrier [84] can be recovered by taking an appropriate limit. To accomplish this, we have to solve both the Wiener-Hopf equations to find the values of unknown functions $A(\alpha)$ and $C(\alpha)$. For this we use Eqs. $(6.52a)$ and $(6.52b)$ in Eqs.(6.50) and (6.51), which gives

$$\overline{\Phi}'_+(\alpha,0)+e^{-i\alpha l}\overline{\Phi}'_-(\alpha,0)+S(\alpha)J_1(\alpha)=\frac{k_{1z}}{\sqrt{2\pi}(k_{1x}+\alpha)}\left[1-\exp[-i(k_{1x}+\alpha)l]\right], \tag{6.55}$$

and

$$-i\alpha\overline{\Phi}_+(\alpha,0)-i\alpha e^{-i\alpha l}\overline{\Phi}_-(\alpha,0)+\Omega L_+(\alpha)L_-(\alpha)J_1'(\alpha)=\frac{k_{1x}}{\sqrt{2\pi}(k_{1x}+\alpha)}\left[1-\exp[-i(k_{1x}+\alpha)l]\right], \tag{6.56}$$

where

$$S(\alpha) = -i\kappa_{BF}(\alpha)L(\alpha) = S_+(\alpha)S_-(\alpha), \tag{6.57}$$

and $S_+(\alpha)$ and $S_-(\alpha)$ are regular in upper and lower half plane, respectively. Equations of types (6.55) and (6.56) have been considered by Noble [13] and a similar analysis may be employed to obtain an approximate solution for large $k_{1xz}r$ ($r = \sqrt{x^2+z^2}$). Thus, following the procedure given in [13] (Sec. 5.5, pp. 196), we obtain

$$\overline{\Phi}'_+(\alpha,0) = \frac{k_{1z}S_+(\alpha)}{\sqrt{2\pi}}\left[G_1(\alpha) + T(\alpha)C_1\right], \tag{6.58}$$

$$\overline{\Phi}'_-(\alpha,0) = \frac{k_{1z}S_-(\alpha)}{\sqrt{2\pi}}\left[G_2(-\alpha) + T(-\alpha)C_2\right], \tag{6.59}$$

$$\overline{\Phi}_+(\alpha,0) = \frac{iL_+(\alpha)}{\sqrt{2\pi}\alpha}\left[G_1'(\alpha) + T(\alpha)C_1'\right], \tag{6.60}$$

and

$$\overline{\Phi}_-(\alpha,0) = \frac{-iL_-(\alpha)}{\sqrt{2\pi}\alpha}\left[G_2'(-\alpha) - T(-\alpha)C_2'\right], \tag{6.61}$$

where

$$S_+(\alpha) = (k_{1xz}+\alpha)^{\frac{1}{2}}L_+(\alpha), \tag{6.62a}$$

and

$$S_-(\alpha) = e^{\frac{-i\pi}{2}}(k_{1xz}-\alpha)^{\frac{1}{2}}L_-(\alpha), \tag{6.62b}$$

$$G_1(\alpha) = \frac{1}{(\alpha+k_{1x})}\left[\frac{1}{S_+(\alpha)} - \frac{1}{S_+(-k_{1x})}\right] - e^{-ilk_{1x}}N_1(\alpha), \tag{6.63}$$

$$G_2(\alpha) = \frac{e^{-ilk_{1x}}}{(\alpha-k_{1x})}\left[\frac{1}{S_+(\alpha)} - \frac{1}{S_+(k_{1x})}\right] - N_2(\alpha), \tag{6.64}$$

$$C_1 = S_+(k_{1xz})\left[\frac{G_2(k_{1xz}) + S_+(k_{1xz})G_1(k_{1xz})\,T(k_{1xz})}{1 - S_+^2(k_{1xz})T^2(k_{1xz})}\right], \tag{6.65}$$

$$C_2 = S_+(k_{1xz})\left[\frac{G_1(k_{1xz}) + S_+(v)G_2(k_{1xz})\,T(k_{1xz})}{1 - S_+^2(k_{1xz})T^2(k_{1xz})}\right], \tag{6.66}$$

$$G_1'(\alpha) = \frac{\alpha}{(\alpha + k_{1x})}\left[\frac{1}{L_+(\alpha)} - \frac{1}{L_+(-k_{1x})}\right] - e^{-ilk_{1x}}N_1(\alpha), \tag{6.67}$$

$$G_2'(\alpha) = \frac{e^{-ilk_{1x}}}{(\alpha - k_{1x})}\left[\frac{\alpha}{L_+(\alpha)} + \frac{k_{1x}}{L_+(k_{1x})}\right] - N_2(\alpha), \tag{6.68}$$

$$C_1' = L_+(k_{1xz})\left[\frac{G_2'(k_{1xz}) + L_+(k_{1xz})G_1'(k_{1xz})\,T(k_{1xz})}{1 - L_+^2(k_{1xz})T^2(k_{1xz})}\right], \tag{6.69}$$

$$C_2' = L_+(k_{1xz})\left[\frac{G_1'(k_{1xz}) + L_+(k_{1xz})G_2'(k_{1xz})\,T(k_{1xz})}{1 - L_+^2(k_{1xz})T^2(k_{1xz})}\right], \tag{6.70}$$

$$N_{1,2}(v) = \frac{E_{-1}[W_{-1}\{-i(k_{1xz} \mp k_{1x})\,l\} - W_{-1}\{-i(k_{1xz} + \alpha)\,l\}]}{2\pi i(v \pm k_{1x})}, \tag{6.71}$$

$$T(\alpha) = \frac{1}{2\pi i}E_{-1}W_{-1}\{-i(k_{1xz} + \alpha)\,l\}, \tag{6.72}$$

$$E_{-1} = 2e^{i\frac{\pi}{4}}e^{ik_{1xz}l}\ (l)^{\frac{1}{2}}\ (i)^{-1}h_{-1}, \tag{6.73}$$

and

$$W_{n-\frac{1}{2}}(p) = \int_0^\infty \frac{u^n e^{-u}}{u + p}du = \Gamma(n+1)\,e^{\frac{z}{2}}z^{\frac{1}{2}n-\frac{1}{2}}W_{-\frac{1}{2}(n+1),\frac{1}{2}n}(p), \tag{6.74}$$

where $p = -i(k_{1xz} + \alpha)\,l$ and $n = \frac{-1}{2}$. $W_{m,n}$ is known as a Whittaker function.

Now, making use of Eqs. (6.58 − 6.61) in Eqs. (6.53) and (6.54), we get

$$\left.\begin{array}{c} A(\alpha) \\ C(\alpha) \end{array}\right\} = \frac{k_{1z}sgn(z)}{\sqrt{2\pi}i\kappa_{BF}L(\alpha)}\left\{\begin{array}{c} S_+(\alpha)G_1(\alpha) + S_+(\alpha)T(\alpha)\,C_1 + e^{-i\alpha l}S_-(\alpha) \\ \times[G_2(-\alpha) + T(-\alpha)\,C_2] - \frac{(1-e^{-il(k_{1x}+\alpha)})}{(k_{1x}+\alpha)} \end{array}\right\}$$

$$+\frac{\alpha\Omega_1}{\sqrt{2\pi}\kappa_{BF}L(\alpha)}\left\{\begin{array}{c} L_+(\alpha)G_1'(\alpha) + T(\alpha)\,L_+(\alpha)C_1' + e^{-i\alpha l} \\ \times[\,(L_-(\alpha)G_2'(-\alpha) + T(-\alpha)\,L_+(\alpha)C_2')\,] - \frac{(1-e^{-il(k_{1x}+\alpha)})}{(k_{1x}+\alpha)} \end{array}\right\}, \tag{6.75}$$

where $A(\alpha)$ corresponds to $z > 0$ and $C(\alpha)$ corresponds to $z < 0$. We can see that the second term in the above equation was altogether missing in Eq. (70) of [104]. This term includes the effect of Ω_1 parameter in it which can be seen from the solution also. Now, $Q_{1y}^{sca}(x,z)$ can be obtained by taking the inverse Fourier transform of Eq. (6.43). Thus

$$Q_{1y}^{sca}(x,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ \begin{array}{c} A(\alpha) \\ C(\alpha) \end{array} \right\} \exp(i\kappa_{BF}|z| - i\alpha x) d\alpha, \tag{6.76}$$

where $A(\alpha)$ and $C(\alpha)$ are given by Eq. (6.75). Substituting the value of $A(\alpha)$ and $C(\alpha)$ from Eq. (6.75) into Eq. (6.76) and using the approximations $(6.63-6.70)$, one can break up the field $\Psi(x,z)$ into two parts

$$Q_{1y}^{sca}(x,z) = \Phi^{sep}(x,z) + \Phi^{int}(x,z), \tag{6.77}$$

where

$$\Phi^{sep}(x,z) = -\frac{k_{1z} sgn(z)}{2\pi} \int_{-\infty}^{\infty} \frac{S_+(\alpha)\exp(i\kappa_{BF}|z| - i\alpha x)}{i\kappa_{BF} L(\alpha) S_+(-k_{1x})(k_{1x}+\alpha)} d\alpha$$

$$+\frac{k_{1z} sgn(z)}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-il(k_{1x}+\alpha)} S_-(\alpha)\exp(i\kappa_{BF}|z| - i\alpha x)}{i\kappa_{BF} L(\alpha) S_+(k_{1x})(k_{1x}+\alpha)} d\alpha$$

$$-\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Omega_1 e^{-il(k_{1x}+\alpha)}\exp(i\kappa_{BF}|z| - i\alpha x)}{\kappa_{BF} L(\alpha)(k_{1x}+\alpha)} d\alpha + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{L_-(\alpha) e^{-il(k_{1x}+\alpha)}\exp(i\kappa_{BF}|z| - i\alpha x)}{\kappa_{BF} L(\alpha)(k_{1x}+\alpha) L_+(k_{1x})} d\alpha$$

$$+\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Omega_1 \exp(i\kappa_{BF}|z| - i\alpha x)}{\kappa_{BF} L(\alpha)(k_{1x}+\alpha)} d\alpha, \tag{6.78}$$

and

$$\Phi^{int}(x,z) = -\frac{k_{1z} sgn(z)}{2\pi} \int_{-\infty}^{\infty} \frac{1}{i\kappa_{BF} L(\alpha)} \left[S_+(\alpha) N_1(\alpha) e^{-ilk_{1x}} - C_1 S_+(\alpha) T(\alpha) \right.$$

$$\left. + S_+(-\alpha) e^{-il\alpha} N_2(-\alpha) - C_2 T(-\alpha) S_+(-\alpha) e^{-il\alpha} \right] \exp(i\kappa_{BF}|z| - i\alpha x) d\alpha$$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Omega_1}{\kappa_{_{BF}}L(\alpha)}\left[T(\alpha)L_+(\alpha)C_1'+T(-\alpha)L_-(\alpha)C_2'-L_+(\alpha)N_1(\alpha)e^{-ilk_{1x}}\right.$$
$$\left.-L_-(\alpha)N_2(-\alpha)e^{-il\alpha}\right]\exp\left(i\kappa_{_{BF}}|z|-i\alpha x\right)d\alpha. \tag{6.79}$$

Here, $\Phi^{sep}(x,z)$ consists of two parts each representing the diffracted field produced by the edges at $x=0$ and $x=-l$, respectively, although the other edge were absent while $\Phi^{int}(x,z)$ gives the interaction of one edge upon the other.

6.3 Far field solution

The far field may now be calculated by evaluating the integrals appearing in Eqs. (6.76), (6.78) and (6.79), asymptotically [107]. For that we put $x=r\cos\vartheta$, $|z|=r\sin\vartheta$ and deform the contour by the transformation $\alpha=-k_{1xz}\cos(\vartheta+i\xi)$, $(0<\vartheta<\pi, -\infty<\xi<\infty)$. Hence, for large $k_{1xz}r$, Eqs. (6.76), (6.78) and (6.79) become

$$Q_{1y}^{sca}(x,z)=\frac{ik_{1xz}}{\sqrt{2\pi}}\left(\frac{\pi}{2k_{1xz}r}\right)^{\frac{1}{2}}\left\{\begin{array}{c}A(-k_{1xz}\cos\vartheta)\\ C(-k_{1xz}\cos\vartheta)\end{array}\right\}\sin(\vartheta)\exp(ik_{1xz}r+i\frac{\pi}{4}), \tag{6.80}$$

$$\begin{aligned}Q_{1y}^{sca(sep)}(x,z) &= -\left[ik_{1z}sgn(z)f_1(-k_{1xz}\cos\vartheta)+g_1(-k_{1xz}\cos\vartheta)\right]\\ &\quad\times\frac{1}{4\pi k_{1xz}}\left(\frac{1}{k_{1xz}r}\right)^{\frac{1}{2}}\exp(ik_{1xz}r+i\frac{\pi}{4}),\end{aligned} \tag{6.81}$$

and

$$\begin{aligned}Q_{1y}^{sca(int)}(x,z) &= -\left[ik_{1z}sgn(z)f_2(-k_{1xz}\cos\vartheta)+g_2(-k_{1xz}\cos\vartheta)\right]\\ &\quad\times\frac{1}{4\pi k_{1xz}}\left(\frac{1}{k_{1xz}r}\right)^{\frac{1}{2}}\exp(ik_{1xz}r+i\frac{\pi}{4}),\end{aligned} \tag{6.82}$$

where $A(-k_{1xz}\cos\vartheta)$ and $C(-k_{1xz}\cos\vartheta)$ can be found from Eq. (6.75), while

$$f_1(-k_{1xz}\cos\vartheta) = \frac{S_+(-k_{1xz}\cos\vartheta)}{L(-k_{1xz}\cos\vartheta)S_+(-k_{1x})(k_{1x}-k_{1xz}\cos\vartheta)} - \frac{e^{-il(k_{1x}-k_{1xz}\cos\vartheta)}S_+(k_{1xz}\cos\vartheta)}{L(-k_{1xz}\cos\vartheta)S_+(k_{1x})(k_{1x}-k_{1xz}\cos\vartheta)}, \tag{6.83}$$

$$g_1(-k_{1xz}\cos\vartheta) = \frac{1}{(k_{1x}-k_{1xz}\cos\vartheta)}\left[\frac{\Omega_1 e^{-il(k_{1x}-k_{1xz}\cos\vartheta)}}{L(-k_{1xz}\cos\vartheta)} - \frac{L_+(k_{1xz}\cos\vartheta)e^{-il(k_{1x}-k_{1xz}\cos\vartheta)}}{L(-k_{1xz}\cos\vartheta)L_+(k_{1x})} - \frac{\Omega_1}{L(-k_{1xz}\cos\vartheta)}\right], \tag{6.84}$$

$$f_2(-k_{1xz}\cos\vartheta) = \frac{1}{L(-k_{1xz}\cos\vartheta)}\left[S_+(-k_{1xz}\cos\vartheta)N_1(-k_{1xz}\cos\vartheta)e^{-ilk_{1x}} + S_+(k_{1xz}\cos\vartheta)e^{ilk_{1xz}\cos\vartheta}N_2(k_{1xz}\cos\vartheta) - C_1 S_+(-k_{1xz}\cos\vartheta)T(-k_{1xz}\cos\vartheta) - C_2 T(k_{1xz}\cos\vartheta)S_+(k_{1xz}\cos\vartheta)e^{ilk_{1xz}\cos\vartheta}\right], \tag{6.85}$$

and

$$g_2(-k_{1xz}\cos\vartheta) = \frac{1}{L(-k_{1xz}\cos\vartheta)}\left[L_+(-k_{1xz}\cos\vartheta)N_1(-k_{1xz}\cos\vartheta)e^{-ilk_{1x}} + L_+(k_{1xz}\cos\vartheta)N_2(k_{1xz}\cos\vartheta)e^{ilk_{1xz}\cos\vartheta} - T(-k_{1xz}\cos\vartheta)L_+(-k_{1xz}\cos\vartheta)C_1' - T(k_{1xz}\cos\vartheta)L_+(k_{1xz}\cos\vartheta)C_2'\right]. \tag{6.86}$$

The expressions (6.84) and (6.86) are additional terms including the effect of Ω_1 parameter, which were altogether missing in the analysis of [104].

Remarks:

Mathematically we can derive the results of the half plane problem in the following manner:

For the analysis purpose, in Eq. (6.75), it is assumed that the wave number k_{1xz} has positive imaginary part and using the L Hopital rule successively, the value of $E_{-1,}$ reduces to $\lim_{l\to\infty}(\frac{e^{ikl}}{\sqrt{lk_{BF}}})$ which becomes zero and in turn result the quantities $T(\alpha)$, $N_{1,2}(\alpha)$, $G_2'(\alpha)$, C_1' and $G_2(\alpha)$ in zero. The third term in Eqs. (6.63), (6.64) and (6.67) also becomes zero as $l\to\infty$. The Eq. (6.75), after these eliminations reduces to

$$A(\alpha)=\frac{1}{\sqrt{2\pi}}[\frac{-k_{1z}\kappa_{BF+}(\alpha)L_+(\alpha)}{i\kappa_{BF}(\alpha)L(\alpha)(k_{1x}+\alpha)\kappa_{BF+}(-k_{1x})L_+(-k_{1x})}+\frac{\alpha\Omega L_+(\alpha)}{\kappa_{BF}(\alpha)L(\alpha)(k_{1x}+\alpha)L_+(-k_{1x})}].$$

Using the factorization

$$L(\alpha)=L_+(\alpha)L_-(\alpha),$$

and

$$\kappa_{BF}(\alpha)=\kappa_{BF+}(\alpha)\kappa_{BF-}(\alpha).$$

and substituting the pole contribution $\alpha=-k_{1x}$, the above result reduces to Eq. (6.26a) of the Half Plane [84]. Subsequently, Eq. (6.82), i.e. , the interacted field vanishes by adopting the same procedure as in case of Eq. (6.75), while the separated field results into the diffracted field [84] as the strip is widened to half plane by taking the limit $l\to\infty$, also well supported by the numerical results discussed above.

6.4 Graphical results

A computer program MATHEMATICA has been used for graphical plotting of the separated field given by the expression (6.81). The values of parameter Ω_1 are taken from 0.2 to 0.4. The following situations are considered:

(i) When the source is fixed in one position (for all values of Ω_1) relative to the finite barrier, ($\theta_0=45^o$, l and θ are allowed to vary).

(ii) When the source is fixed in one position, relative to the infinite barrier ($\theta_0 = 45^o$, l and θ are allowed to vary).

For all the situations, $\theta_0 = 45^o$, the graphs (6.1), (6.2), (6.3), (6.4) and (6.5) show that the field, in the region $0 < \theta \leq \pi$, is most affected by the changes in Ω_1, l and k_{1xz}. The main features of the graphical results, some of which can be seen in graphs (6.1), (6.2), (6.3), (6.4) and (6.5) are as follows:

(a) In graphs (6.1), (6.2) and (6.3) by increasing the value of strip length l and Ω_1, the number of oscillations increases and the amplitude of the separated field decreases,respectively.

(b) The graphs of the diffracted field corresponding to the half plane is given in fig. (5). It is observed that the Figs. (6.1) – (6.4) are in comparison with Fig. (6.5) for various values of the different parameters.

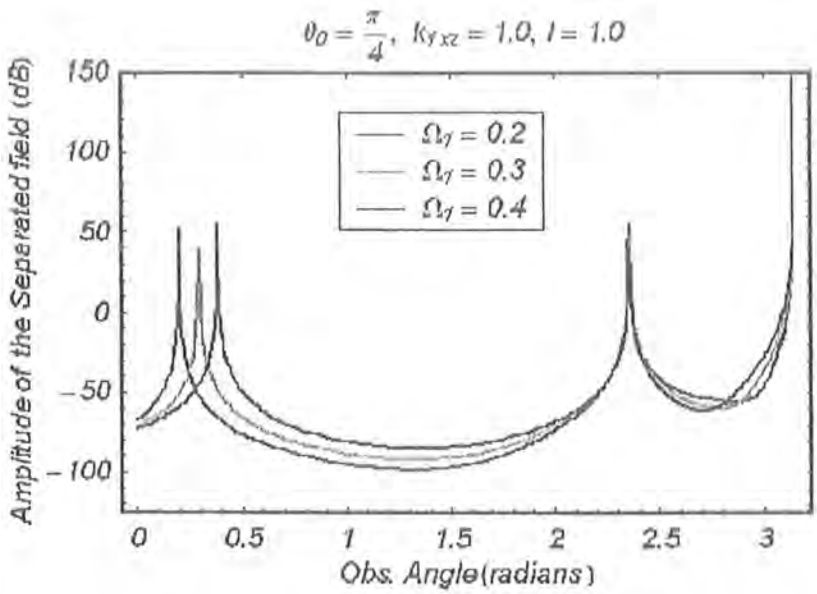


Fig. 6.1: Amplitude of the separated field for different values Ω_1 for $l = 1$.

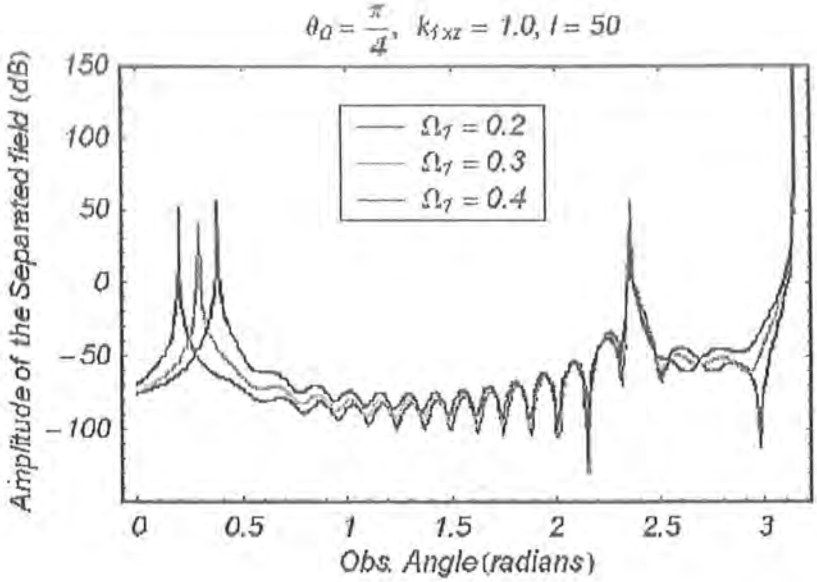


Fig. 6.2: Amplitude of the separated field for different values Ω_1 for $l = 50$.

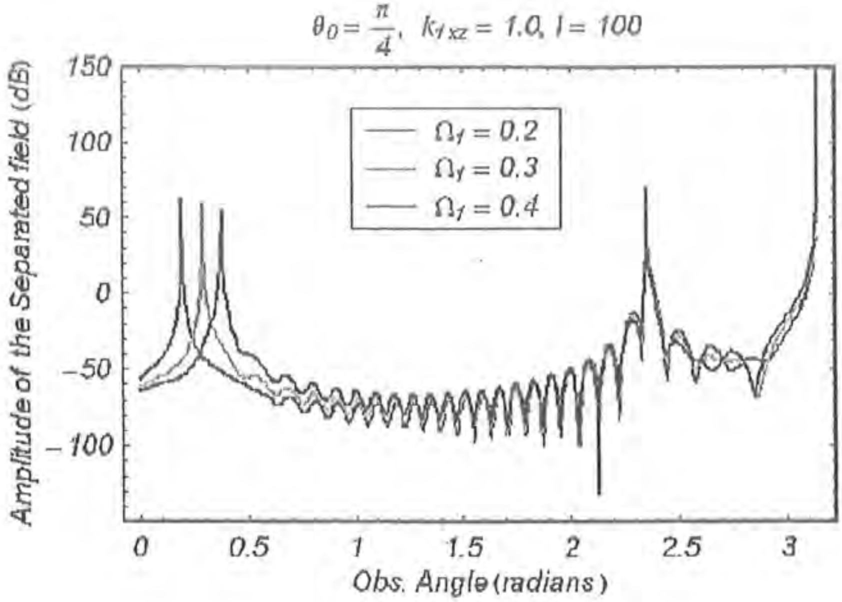


Fig. 6.3: Amplitude of the separated field for different values Ω_1 for $l = 100$.

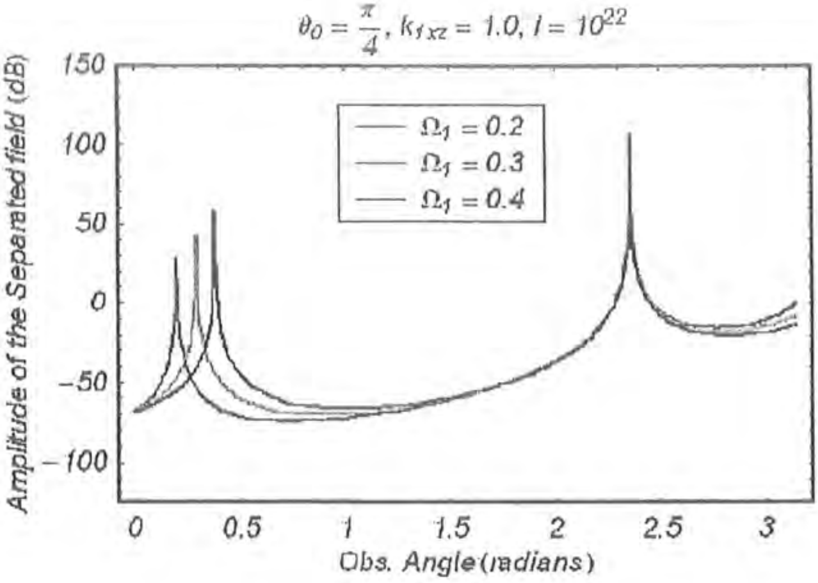


Fig. 6.4: Amplitude of the separated field for different values Ω_1 for $l = 10^{22}$.

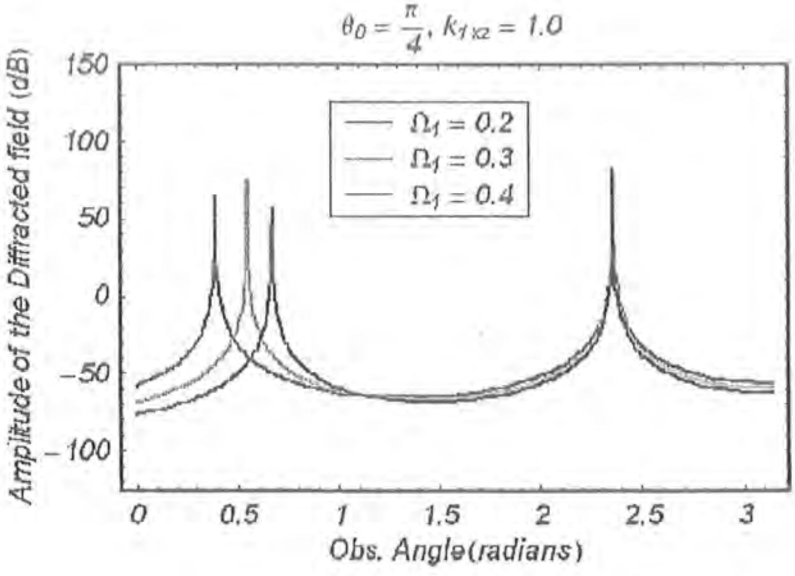


Fig. 6.5: Amplitude of the diffracted field for different values of Ω_1

Chapter 7

Thesis conclusion and suggestions for future work

In this chapter, we are presenting an abridgment of the thesis. The first chapter portraits a synoptic view of Wiener Hopf technique and its application in a diversity of fields including diffraction of acoustic, elastic and electromagnetic waves, crystal growth, fracture mechanics, flow problems, diffusion models, geophysical applications and mathematical finance.

Chapter two provides an overview of the various mathematical and physical issues common to the following four chapters. This includes Decomposition and Factorization theorems, Fourier transform, general Wiener Hopf technique, Jones' method, method of Steepest Descent and constitutive relations in Beltrami field.

Step discontinuity is an important topic in electrical circuits depending upon the nature of the step whether it is impedance, reactive or resistive. In chapter three, the scattering due to a magnetic line source from an impedance step has been studied. The said problem is first reduced to a modified Wiener-Hopf equation of second kind whose solution contains an infinite set of constants satisfying an infinite system of linear algebraic equations. Numerical solution of this system is obtained for various values of the surface impedances and the height of the step, through which the effect of these parameters on the diffraction phenomenon is studied. The possible excitation of the surface waves on the impedance surfaces can be neglected as the observation point is far from the surface while using the steepest descent method and diffracted field dominates [72]. It is observed that if the source is shifted to a large distance these results differ from those of [55] by a multiplicative factor to the part of the scattered field containing the effects of incident and reflected waves.

One can found numerous examples [112 − 115] emphasizing the physical significance of Magnetic line source. Magnetic line source diffraction by an impedance step seems to be first attempt in this gigantic area and will certainly be a useful addition to the existing physical problems. This problem may be extended to the case of point source diffraction by an impedance step.

Chapter four covers the scattering due to a line source and a point source from a reactive step. For a line source, the said problem is first reduced to a modified Wiener-Hopf equation of second kind whose solution contains an infinite set of constants satisfying an infinite system of linear algebraic equations. Numerical solution for the line source is obtained for various values of the surface reactances and the height of the step, through which the effect of these parameters on the diffraction phenomenon is studied. It is observed that if the source is shifted to a large distance, the results of the line source differ from those of [57] by a multiplicative factor to the part of the scattered field containing the effects of incident and reflected waves. Subsequently, the point source diffraction is examined using the results obtained for a line source diffraction.

The line and point sources problem may also be extended to a step discontinuity of resistive nature. A plane wave diffraction by a resistive step has already been discussed by Alinur Büyükaksoy [116].

In chapter five, the problem of diffraction of a plane acoustic wave by an oscillating rigid strip is investigated rigorously. It is very important as the oscillating nature of the strip distinguishes it from the earlier work. This problem is a useful contribution to the existing diffraction theory. The significance of the present analysis is that the results of the half plane [74] can be deduced mathematically and numerically as well by taking an appropriate limit $l \to \infty$. This can be considered as check of the validity of the analysis in this chapter.

The problem of oscillating half plane, strip and slit may be extended to the line and point sources. All these problems may also be defined for time domain and intermediate zones.

Chapter six discusses the problem of diffraction of an electromagnetic wave by a perfectly conducting finite strip in a homogeneous bi-isotropic medium in an improved form. It is found that the two edges of the strip give rise to two diffracted fields (one from each edge) and an interaction field (double diffraction of two edges). This seems to be the first attempt in this direction as we can deduce the results of half plane [84] by taking an appropriate limit. In [104], the Ω parameter was not taken into account which ends up in an equation from which one cannot deduce the results for semi infinite barrier [84]. This can be considered as check of the validity of the analysis in this paper. Thus, the new solution can be regarded as a correct solution for a perfectly conducting barrier.

Similar corrections have also been incorporated for cylindrical [62] and spherical waves [117] by the same authors. The method of widening the strip to a half plane devised by the authors is new in literature can be implemented to the problems of same nature.

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Magnetic Line Source Diffraction by an Impedance Step

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***Abstract*—Diffraction of a magnetic line source by an impedance step joined by two half planes is studied in the case where the half planes and step are characterized by different surface impedances. The problem is solved using Wiener Hopf technique and Fourier transform. The scattered field in the far zone is determined by the method of steepest descent. Graphical results for the solution has also been presented. It is observed that if the source is shifted to a large distance these results differ from those of [8] by a multiplicative factor to the part of the scattered field containing the effects of incident and reflected waves.**

***Index Terms*—Far-field approximation, Green's function, line source, saddle point method, scattering, Wiener–Hopf theory.**

I. Introduction

Numerous past investigations have been devoted to the study of classical problems of line source and point source diffractions of electromagnetic and acoustic waves by various types of half planes. To name a few only, e.g., the line source diffraction of electromagnetic waves by a perfectly conducting half plane was investigated by Jones [1]. Later on Jones [2] considered the problem of line source diffraction of acoustic waves by a hard half plane attached to a wake in still air as well as when the medium is convective. Rawlins then considered the line source diffraction of acoustic waves by an absorbing barrier [3], line source diffraction by an acoustically penetrable or an electromagnetically dielectric half plane whose width is small as compared to the incident wave length [4] and line source diffraction of sound waves by an absorbent semi-infinite plane such that the two faces of half plane have different impedances [5]. Recently Ahmad [6] considered the line source diffraction of acoustic waves by an absorbing half plane using Myers' condition and Ayub *et al.* [15] have considered the line source and point source scattering of acoustic waves by the junction of transmissive and soft-hard half planes.

The scattering of surface waves by junction of two semi-infinite planes joined together by a step was first introduced by Johansen [7]. Later, Büyükaksoy and Birbir [8], [9] treated the same geometry in more general case of plane wave incidence and when the material properties of the half planes and the step are simulated by constant but different surface impedances which was important for predicting the scattering caused by an abrupt change in material as well as in the geometrical properties of a surface. Büyükaksoy recently extended [8], [9] to a case where the two half planes with different surface impedances are joined by a reactive step [10]. A similar work is done [13] where the half plane is kept conducting.

We have extended the problem of plane wave scattering [8] to the problem of scattering due to a magnetic line source situated at (x_0, y_0) because the line sources are considered as better substitute than the plane waves. It is perhaps the first attempt to look at the line source geometry with a step discontinuity. The introduction of line source changes the incident field and the method of solution requires a careful analysis in calculating the diffracted field. Using the Fourier transform, the diffraction problem is first reduced to modified Wiener-Hopf equation of second kind whose solution contains infinitely many constants satisfying an infinite system of linear equations. Numerical solution of this system is obtained for various values of surface impedances and the height of the step, from which the effects of these parameters on the diffraction phenomenon are studied. The possible excitation of the surface waves on the impedance surfaces can be neglected as the observation point is far from the surface while using the steepest descent method and the diffracted field dominates [14].

It is found that if the source is shifted to a large distance these results differ from those of [8] by a multiplicative factor to the part of the scattered field containing the effects of incident and reflected waves. To the best of authors' knowledge, this seems to be the first attempt in this direction with the magnetic line source diffraction at a step with impedance boundary conditions well supported by numerical results.

II. Mathematical Formulation of the Problem

Consider the scattering due to a magnetic line source located at (x_0, y_0), illuminates two half planes $S_1 = \{x \leq 0, y = a, z \in (-\infty, \infty)\}$ and $S_2 = \{x \geq 0, y = 0, z \in (-\infty, \infty)\}$ with relative surface impedance $\eta_1$joined together by a step of height "a" with relative surface impedance η_2. The geometry of the line source diffraction problem is depicted in Fig. 1. For harmonic vibrations of time dependence $e^{-i\omega t}$, the solution of the following wave equation is required:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right) u_t(x, y) = \delta(x - x_0)\delta(y - y_0) \qquad (1)$$

where u_t is the total field. For the analysis purpose, it is convenient to express the total field $u_t(x, y)$ as follows:

$$u_t(x, y) = \begin{cases} u^i(x, y) + u_1^r(x, y) + u_1(x, y), & y > a, \\ u_2(x, y), & 0 < y < a, \end{cases} \qquad (2)$$

and is supported by the boundary conditions at two half planes and a step given by

$$\left(1 + \frac{\eta_1}{ik}\frac{\partial}{\partial y}\right) u_1(x, a) = 0, \quad x \in (-\infty, 0) \qquad (3a)$$

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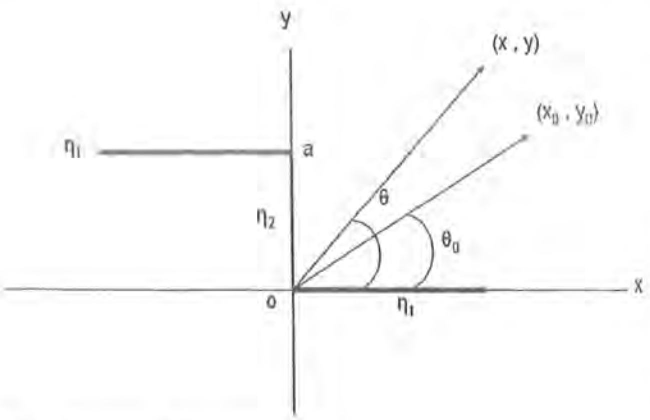

Fig. 1. Geometry of the problem.

$$\left(1+\frac{\eta_2}{ik}\frac{\partial}{\partial x}\right)u_t(0,y)=0,\quad y\in(0,a) \tag{3b}$$

$$\left(1+\frac{\eta_1}{ik}\frac{\partial}{\partial y}\right)u_t(x,0)=0,\quad x\in(0,\infty) \tag{3c}$$

$$u_t(x,a)-u_2(x,a)=0,\quad x\in(0,\infty) \tag{3d}$$

$$\frac{\partial}{\partial y}u_t(x,a)-\frac{\partial}{\partial y}u_2(x,a)=0,\quad x\in(0,\infty). \tag{3e}$$

where $k=w/c$ is the wave number and a time factor $e^{-i\omega t}$ has been assumed and suppressed hereafter. It is assumed that the wave number k has positive imaginary part. The lossless case can be obtained by making $Imk\to 0$ in the final expressions. $u_1(x,y)$ and $u_2(x,y)$ are the scattered fields, $u_1^r(x,y)$ is the field reflected from the plane located at $y=a$ with relative surface impedance η_1 and $u^i(x,y)$ is the incident field satisfying the equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)u^i(x,y)=\delta(x-x_0)\delta(y-y_0). \tag{4}$$

The scattered field $u_1(x,y)$ satisfies the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)u_1(x,y)=0,\quad x\in(-\infty,\infty). \tag{5}$$

It is appropriate to define Fourier transform as follows:

$$\phi_\pm(\alpha,y)=\pm\frac{1}{\sqrt{2\pi}}\int_0^{\pm\infty}u_1(x,y)e^{i\alpha x}dx \tag{6a}$$

$$u_1(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(\alpha,y)e^{-i\alpha x}d\alpha \tag{6b}$$

$$\psi_+(\alpha,y)=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}u_2(x,y)e^{i\alpha x}dx \tag{6c}$$

$$\phi^i(\alpha,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u^i(x,y)e^{i\alpha x}d\alpha \tag{6d}$$

$$\phi^r(\alpha,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u_1^r(x,y)e^{i\alpha x}d\alpha \tag{6e}$$

$$\phi_0(\alpha,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[u^i(x,y)+u_1^r(x,y)\right]e^{i\alpha x}dx. \tag{7}$$

Using (6a), the (5) can be written as

$$\frac{d^2\phi}{dy^2}+\gamma^2\phi(\alpha,y)=0 \tag{8}$$

where $\gamma(\alpha)=\sqrt{k^2-\alpha^2}$. The square root function $\gamma(\alpha)$ is defined in the complex α-plane cut as shown in Fig. 2, [8] such that $\gamma(0)=k$.

The solution of (8) satisfying the radiation condition for $y>a$ can be written as

$$\phi(\alpha,y)=\phi_+(\alpha,y)+\phi_-(\alpha,y)=A(\alpha)e^{i\gamma(\alpha)|y-a|} \tag{9}$$

where $A(\alpha)$ is the unknown coefficient to be determine and $\phi(\alpha,y)$ is divided into $\phi_+(\alpha,y)$ and $\phi_-(\alpha,y)$ as in [11], [12]. By the method of Green's function, one can write incident and the corresponding reflected field as follows:

$$\phi^i(\alpha,y)=\frac{1}{4\pi i\gamma}e^{i\alpha x_0+i\gamma(\alpha)|y-y_0|} \tag{9a}$$

$$\phi^r(\alpha,y)=\frac{\eta_1\sin\theta_0-1}{\eta_1\sin\theta_0+1}\cdot\frac{1}{4\pi i\gamma}e^{i\alpha x_0+i\gamma(\alpha)|(y-2b)+y_0|}. \tag{9b}$$

The unknown coefficient $A(\alpha)$ appearing in (9) is to be determined with the help of boundary conditions (3a)–(e) and the Fourier transform of (3a) and (3c)–(e), i.e.,

$$\left(1+\frac{\eta_1}{ik}\frac{\partial}{\partial y}\right)\phi_-(\alpha,a)=0,\quad x\in(-\infty,0) \tag{10a}$$

$$\left(1+\frac{\eta_1}{ik}\frac{\partial}{\partial y}\right)\psi_+(\alpha,0)=0,\quad x\in(0,\infty) \tag{10b}$$

$$\phi_0(\alpha,a)+\phi(\alpha,a)-\psi_+(\alpha,a)=0,\quad x\in(0,\infty) \tag{10c}$$

$$\frac{\partial}{\partial y}[\phi_0(\alpha,a)+\phi(\alpha,a)]-\frac{\partial}{\partial y}\psi_+(\alpha,a)=0\quad x\in(0,\infty). \tag{10d}$$

Equation (10a) can be written as

$$\phi_-(\alpha,a)+\frac{\eta_1}{ik}\phi_-'(\alpha,a)=0 \tag{11}$$

where "$'$" denotes differentiation with respect to y. The differentiation of (9) w.r.t "y" yields

$$\phi_+'(\alpha,y)+\phi_-'(\alpha,y)=i\gamma(\alpha)A(\alpha)e^{i\gamma(\alpha)|y-a|}. \tag{12}$$

By putting $y=a$ in (9) and (12) and then using (11), we get

$$R_+(\alpha)=A(\alpha)\left(1+\frac{\eta_1}{k}\gamma(\alpha)\right) \tag{13a}$$

where

$$R_+(\alpha)=\phi_+(\alpha,a)+\frac{\eta_1}{ik}\phi_+'(\alpha,a). \tag{13b}$$

To avoid any repetition, the calculations in the region $y\in(0,a)$, $x>0$, are the same as in [8], which results in

$$\frac{\left[\frac{k}{\eta_2}-\alpha\right]\psi_+(\alpha,a)-\left(\frac{k}{\eta_2}+\alpha\right)\psi_+(-\alpha,a)}{\left(\cos\gamma a-\frac{ik}{\eta_1}\frac{\sin\gamma a}{\gamma}\right)}=-\frac{\left[\frac{k}{\eta_2}-\alpha\right]\psi_+'(\alpha,a)-\left(\frac{k}{\eta_2}+\alpha\right)\psi_+'(-\alpha,a)}{\left(\gamma\sin\gamma a+\frac{ik}{\eta_1}\cos\gamma a\right)}. \tag{14}$$

Consider the Fourier transformed continuity relations (10c) and (10d), i.e.,

$$\phi_0(\alpha, y) + \phi_+(\alpha, a) = \psi_+(\alpha, a) \tag{15a}$$

$$\phi_0'(\alpha, y) + \phi_+'(\alpha, a) = \psi_+'(\alpha, a). \tag{15b}$$

Adding (15a) $(\eta_1)/(ik)$ times of the (15b), we obtain

$$\begin{aligned} R_+(\alpha) &= \psi_+(\alpha, a) + \frac{\eta_1}{ik}\psi_+'(\alpha, a) \\ &\quad - \left[\phi_0(\alpha, y) + \frac{\eta_1}{ik}\phi_0'(\alpha, y)\right] \end{aligned} \tag{16a}$$

$$\begin{aligned} R_+(-\alpha) &= \psi_+(-\alpha, a) + \frac{\eta_1}{ik}\psi_+'(-\alpha, a) \\ &\quad - \left[\phi_0(-\alpha, y) + \frac{\eta_1}{ik}\phi_0'(-\alpha, y)\right]. \end{aligned} \tag{16b}$$

From (9), (12) and (13a), we get

$$\phi_+(\alpha, a) = \frac{R_+(\alpha)}{\left(1 + \frac{\eta_1}{k}\gamma\right)} - \phi_-(\alpha, a) \tag{17a}$$

$$\phi_+'(\alpha, a) = \frac{i\gamma(\alpha) R_+(\alpha)}{\left(1 + \frac{\eta_1}{k}\gamma\right)} - \phi_-'(\alpha, a). \tag{17b}$$

Replacing $\phi_+(\alpha, a)$ and $\phi_+'(\alpha, a)$ appearing in (15a) and (15b) by their expression given in (17a) and (17b), respectively, we obtain

$$\psi_+(\alpha, a) = \phi_0(\alpha, y) + \frac{R_+(\alpha)}{1 + \frac{\eta_1}{k}\gamma} - \phi_-(\alpha, a) \tag{18a}$$

$$\psi_+'(\alpha, a) = \phi_0'(\alpha, y) + \frac{i\gamma(\alpha) R_+(\alpha)}{\left(1 + \frac{\eta_1}{k}\gamma\right)} + \frac{ik}{\eta_1}\phi_-(\alpha, a). \tag{18b}$$

By putting the values of $\psi_+(\alpha, a)$, $\psi_+'(\alpha, a)$ and $\psi_+'(-\alpha, a)$ from (18a), (18b) and (16b) in (14), respectively, yields the modified Wiener-Hopf equation of second kind valid in the strip $\text{Im}k \cos\theta_0 < \text{Im}\alpha < \text{Im}k$ as given in (19)

$$\begin{aligned} &\frac{\left(\frac{k}{\eta_2} - \alpha\right) R_+(\alpha)}{(\alpha^2 - T^2) G(\alpha)} + \frac{\left(\frac{k}{\eta_2} - \alpha\right)\left[\phi_0(\alpha, y) + \frac{\eta_1}{ik}\phi_0'(\alpha, y)\right]}{(\alpha^2 - T^2) G(\alpha)} - i\frac{\eta_1}{k} S_-(\alpha) \\ &= -\frac{\left(\frac{k}{\eta_2} - \alpha\right)\frac{\eta_1}{ik}}{(\alpha^2 - T^2)\frac{\sin\gamma a}{\gamma}} \begin{bmatrix} \phi_0(\alpha, y)\left(\gamma \sin\gamma a + \frac{ik}{\eta_1}\cos\gamma a\right) \\ +\phi_0'(\alpha, y)\left(\cos\gamma a - \frac{ik}{\eta_1}\frac{\sin\gamma a}{\gamma}\right) \end{bmatrix} \\ &\quad + \frac{\left(\frac{k}{\eta_2} + \alpha\right)}{(\alpha^2 - T^2)\frac{\sin\gamma a}{\gamma}} \left[\phi_0(-\alpha, y) + \frac{\eta_1}{ik}\phi_0'(-\alpha, y)\right] \\ &\quad + \frac{\left(\frac{k}{\eta_2} + \alpha\right) R_+(-\alpha)}{(\alpha^2 - T^2)\frac{\sin\gamma a}{\gamma}} \left(\cos\gamma a - \frac{ik}{\eta_1}\frac{\sin\gamma a}{\gamma}\right) \end{aligned} \tag{19}$$

with $S_-(\alpha)$, $G(\alpha)$ and T stands for the same as in [8]. The solution of Wiener–Hopf (19) is specified in (20).

$$\begin{aligned} \frac{R_+(\alpha)}{(\alpha + T) G_+(\alpha)} &= \left(\frac{i}{8\pi^2}\right)\left(\frac{i\eta_1}{k}\frac{(k\cos\theta_0 - T) G_-(k\cos\theta_0)}{(\alpha - k\cos\theta_0)}\right. \\ &\quad \left. \times \frac{2\eta_1 \sin\theta_0}{(1 + \eta_1 \sin\theta_0)} e^{-ikr_0 + i\frac{\pi}{4}}\left(\frac{2\pi}{kr_0}\right)\right) \\ &\quad + \frac{k}{\eta_1}\frac{\left(\frac{k}{\eta_2} - T\right) R_+(T) G_+(T) e^{\frac{ika}{\eta_1}}}{\frac{\sin ka}{\eta_1}(\alpha + T)} \\ &\quad + \sum_{n=1}^{\infty} \frac{\left(\frac{k}{\eta_2} - \alpha_n\right) R_+(\alpha_n) G_+(\alpha_n)\left(\frac{n\pi}{a}\right)^2}{a\alpha_n (T - \alpha_n)(\alpha + \alpha_n)} \end{aligned} \tag{20}$$

The values of α_n and $G_+(\alpha)$ are as defined in [8]. The function $R_+(\alpha)$ depends upon the unknown series of constants $R_+(T)$, $R_+(\alpha_1)$, $R_+(\alpha_2)$, $R_+(\alpha_3)\cdots$. To find an approximate value for $R_+(\alpha)$, substitute $\alpha = T, \alpha_1, \alpha_2, \ldots, \alpha_m$ in (21) to get $m + 1$ equations in $m + 1$ unknowns. The simultaneous solution of these equations yields approximate solutions for $R_+(T)$, $R_+(\alpha_1)$, $R_+(\alpha_2), \ldots R_+(\alpha_m)$.

III. The Far Field Solution

The scattered field in the region $y > a$ can be obtained by evaluating the value of $A(\alpha)$ using (20) in (13a) and then putting the resulting equation into (9) and finally taking inverse Fourier transform of equation, the final expression for the diffracted field comes out to be:

$$u_1(x, y) = \frac{1}{\sqrt{2\pi}} \int_L \frac{R_+(\alpha)}{\left(1 + \frac{\eta_1}{k}\gamma(\alpha)\right)} e^{i\gamma(\alpha)(y-a)} e^{-i\alpha x} d\alpha \tag{21}$$

where L is a straight line parallel to the real axis, lying in the strip $\text{Im}k \cos\theta_0 < \text{Im}\alpha < \text{Im}k$.

To determine the far field behavior of the scattered field, introducing the following substitutions

$$x = r\cos\theta, \quad y - a = r\sin\theta \tag{22a}$$

and the transformation

$$\alpha = -k\cos(\theta + it) \tag{22b}$$

where t, given in (22b), is real. The contour of integration over α in (21) goes into the branch of hyperbola around $-ik$ if $(\pi/2) < \theta < \pi$. We further observe that in deforming the contour into a hyperbola the pole $\alpha = \xi$ may be crossed. If we also make the transformation $\xi = k\cos(\theta_0 + it_1)$ the contour over ξ also goes into a hyperbola. The two hyperbolae will not cross each other if $\theta < \theta_0$. However, if the inequality is reversed there will be a contribution from pole which, in fact, cancels the incident wave in the shadow region. Omitting the details of calculations, the asymptotic evaluation of the integral in (21), using the method of steepest descent, the field due to a line source at a large distance from the edge is given by (23)

$$\begin{aligned} u_1(x, y) = &\left[\left\{\left(\frac{1}{4\pi i}\right)\frac{i\eta_1}{k} e^{-ika\sin\theta_0}\frac{e^{ikr_0 + i\frac{\pi}{4}}}{\sqrt{r_0}}\right.\right. \\ &\times \frac{(k\cos\theta_0 - T) G_-(k\cos\theta_0)}{k} \\ &\times \frac{2\eta_1}{(1 + \eta_1\sin\theta_0)(1 + \eta_1\sin\theta)} \\ &\times \sin\frac{\theta}{2}\sin\frac{\theta_0}{2}\left\{F\left(\sqrt{2kr}\cos\frac{\theta_0 - \theta}{2}\right)\right. \\ &\left.\left. + F\left(\sqrt{2kr}\cos\frac{\theta_0 + \theta}{2}\right)\right\}\right\} \\ &+ \frac{k}{\eta_1}\frac{\left(\frac{k}{\eta_2} - T\right) R_+(T) G_+(T) e^{-\frac{ika}{\eta_1}}}{\left(\frac{k}{\eta_2} + T\right)\sin\left(\frac{ka}{\eta_1}\right)(-k\cos\theta + T)} \\ &\left. + \sum_{n=1}^{\infty}\frac{\left(\frac{k}{\eta_2} - \alpha_n\right) R_+(\alpha_n) G_+(\alpha_n)\left(\frac{n\pi}{a}\right)^2}{\left(\frac{k}{\eta_2} + \alpha_n\right) a\alpha_n (T - \alpha_n)(-k\cos\theta + \alpha_n)}\right] \\ &\times \frac{e^{ikr - i\frac{\pi}{4}}}{\sqrt{kr_0}}(-k\cos\theta + T) G_-(k\cos\theta). \end{aligned} \tag{23}$$

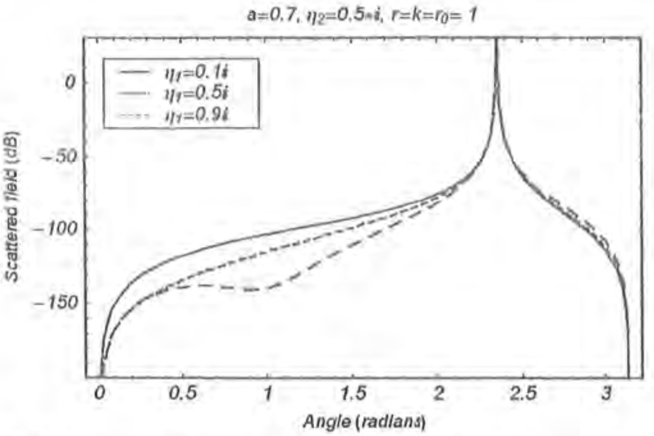

Fig. 2. Scattered field versus the observation angle for different values of the half-plane impedances "η_1" when they are of capacitive nature.

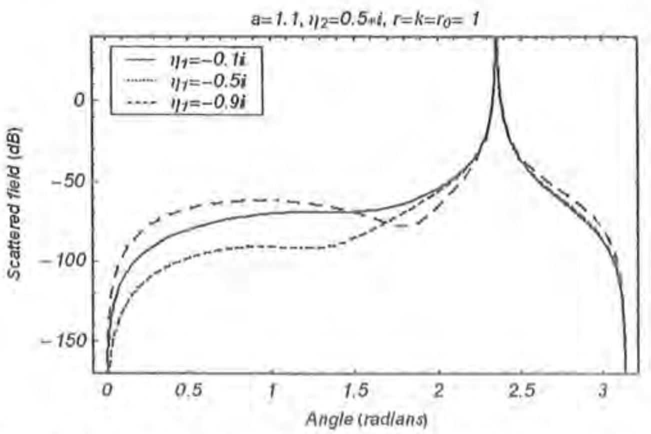

Fig. 3. Scattered field versus the observation angle for different values of the half-plane impedances "η_1" when they are of inductive nature.

with $F(z)$ stands for the Fresnel function as defined in [3], [11] is written in (24):

$$F(z) = e^{-iz^2} \int_z^\infty e^{it^2} dt. \tag{24}$$

IV. The Numerical Results

In this section some graphical results have been presentred highlighting the effects of impedance parameters η_1 and η_2, step height a and line source parameter r_0 on the diffraction phenomenon. Figs. 2 and 3 show the variation of diffracted field versus the observation angle for different impedance values of η_1 while the step height which is purely capacitive with constant impedance $\eta_2 = 0.5i$ is fixed, the angle of incident ray is taken to be $\pi/4$ with $k = r = 1$, the impedance of the half planes are chosen purely capacitive and inductive for $r_0 = 1$, respectively. A comparison of Figs. 2 and 3 show that the scattered field amplitude corresponding to the case where the impedances of half planes are capacitive are weaker than those related to the case where the impedances of half planes are inductive. From Fig. 5 one can conclude that the scattering effects of the step can be reduced if its impedance is inductive.

On comparing the graphs in Figs. 2 and 3 with the graphs in Fig. 4 drawn in [9], one can see that the behavior of the graph is exactly the same and it differs only by the location along vertical axis because of the multiplicative factor to the part of the scattered field containing the effects of line source. This is a counter check of the validation of our claim that if the source is shifted to a large distance these results differ from those of [8], [9] by a multiplicative factor to the part of the scattered field containing the effects of incident and reflected waves. Similarly, the graphs plotted in Fig. 4 are in close tie with the graph plotted in Fig. 5 of [9].

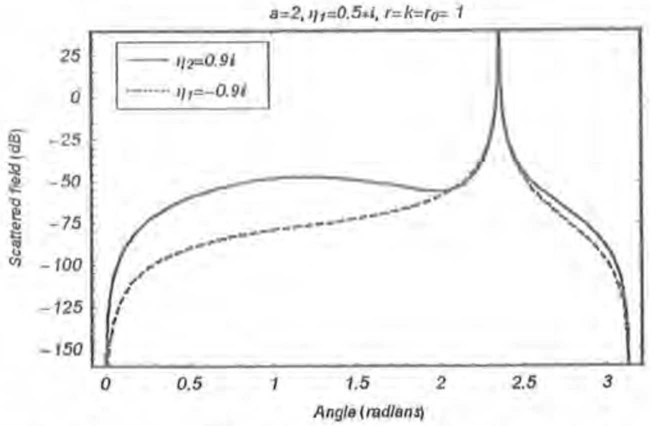

Fig. 4. Scattered field versus the observation angle for different values of step impedances "η_2".

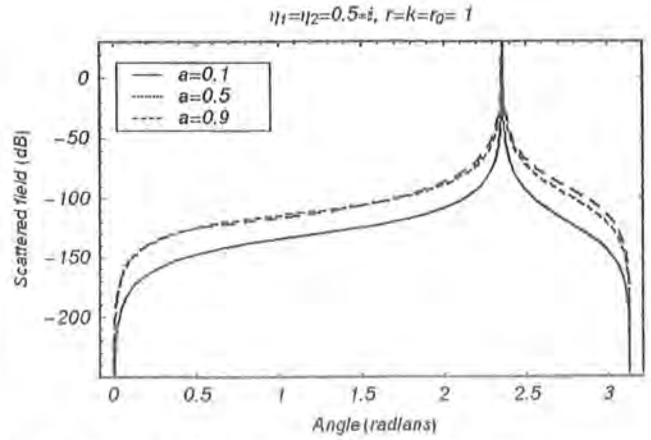

Fig. 5. Scattered field versus the observation angle for different values of the step height "a" when half planes and step have impedances of capacitive nature.

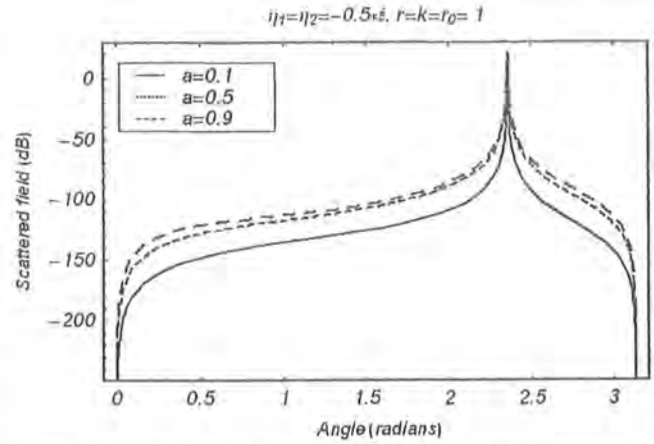

Fig. 6. Scattered field versus the observation angle for different values of the step height "a" when half planes and step have impedances of inductive nature.

Figs. 5 and 6 represent the variation of the scattered field versus the observation angle with $k = r = 1$ and by fixing $\eta_1 = \eta_2 = 0.5i$

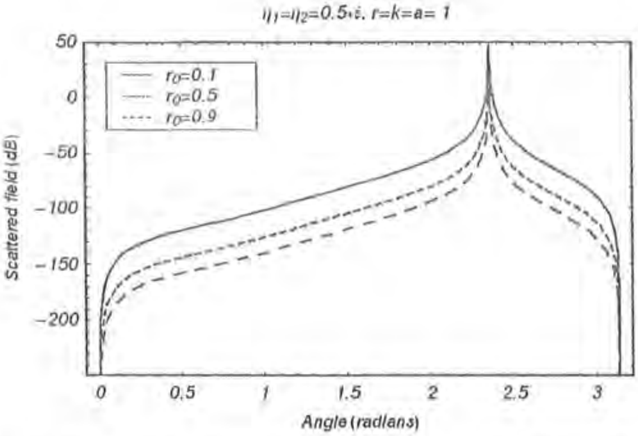

Fig. 7. Scattered field versus the observation angle for different values of "r_0" when half planes and step have impedances of capacitive nature.

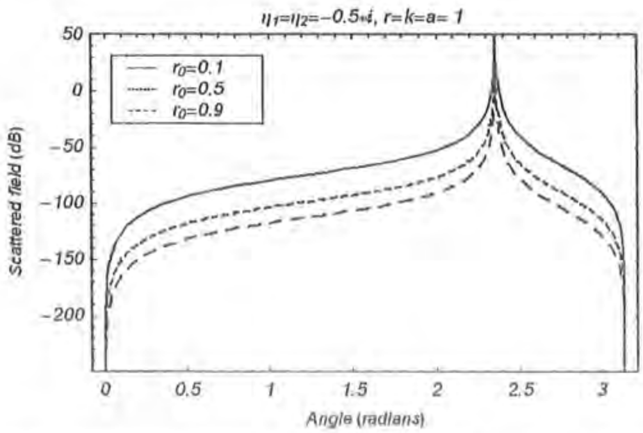

Fig. 8. Scattered field versus the observation angle for different values of "r_0" when half planes and step have impedances of inductive nature.

and $\eta_1 = \eta_2 = -0.5i$, i.e., capacitive and inductive, respectively, for $r_0 = 1$. It is observed that the amplitude of the diffracted field increases with increase in step height which an obvious result. Figs. 7 and 8 represents the variation of the scattered field versus observation angle with $k = r = 1$ and by fixing $\eta_1 = \eta_2 = 0.5i$ and $\eta_1 = \eta_2 = -0.5i$, i.e., capacitive and inductive, respectively, for $a = 1$, these graphs show that the amplitude of the diffracted field decreases as the source is taken away from the origin, which is a natural phenomenon and verifying the results.

V. Conclusion

In this article, the scattering due to a magnetic line source from an impedance step has been studied. The said problem is first reduced to a modified Wiener-Hopf equation of second kind whose solution contains an infinite set of constants satisfying an infinite system of linear algebraic equations. A numerical solution of this system is obtained for various values of the surface impedances and the height of the step, through which the effect of these parameters on the diffraction phenomenon is studied. The possible excitation of the surface waves on the impedance surfaces can be neglected as the observation point is far from the surface while using the steepest descent method and diffracted field dominates [14]. To the best of authors' knowledge, this seems to be the first step in this direction with the magnetic line source diffraction at a step with impedance boundary conditions well supported by numerical proofs. It is observed that if the source is shifted to a large distance these results differ from those of [8] by a multiplicative factor to the part of the scattered field containing the effects of incident and reflected waves.

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Line source and point source diffraction by a reactive step

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Line source and point source diffraction by a reactive step

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The diffraction of a line and a point source by a reactive step joined by two half planes where each half plane and step are characterized by different surface reactances have been studied. The problem is solved by using the Wiener–Hopf technique and the Fourier transform. The scattered field in the far zone is determined by the method of steepest descent. Graphical results for the line source are also presented. It is observed that if the source is shifted to a large distance the results of the line source differ from those of Buyukaksoy and Birbir [*Int. J. Eng. Sci.* **1997**, *35*, 311–319] by a multiplicative factor to the part of the scattered field containing the effects of incident and reflected waves. Subsequently, the point source diffraction is examined using the results obtained for a line source diffraction.

Keywords: diffraction; line source; point source; Wiener–Hopf theory; Green's function; saddle point method; far field approximation

1. Introduction

Numerous past investigations have been devoted to the study of classical problems of line source and point source diffractions of electromagnetic and acoustic waves by various types of half planes. To name a few only, e.g. the line source diffraction of electromagnetic waves by a perfectly conducting half plane was investigated by Jones [1]. Later on Jones [2] considered the problem of line source diffraction of acoustic waves by a hard half plane attached to a wave in still air as well as when the medium is convective. This analysis is further extended to the case of point source diffraction by Balasubramanyam [3] and Rienstra [4] to take into account the effect of the transient nature of the field. Rawlins then considered the line source diffraction of acoustic waves by an absorbing barrier [5], line source diffraction by an acoustically penetrable or an electromagnetically dielectric half plane whose width is small as compared to the incident wave length [6] and line source diffraction of sound waves by an absorbent semi-infinite plane such that the two faces of the half plane have different impedances [7]. Asghar et al. [8,9] studied the problems of point source scattering of acoustic waves by three soft semi-finite parallel plates in a moving fluid and by an absorbing barrier. Ahmad [10] considered the line source diffraction of acoustic waves by an absorbing half plane using Myre's condition. Recently, Ayub et al. [11] have studied the problem of line source and point source scattering of acoustic waves by a junction of transmissive and soft-hard half planes.

The scattering of surface waves by a junction of two semi-infinite planes joined together by a step was first introduced by Johansen [12]. Later, Buyukaksoy and Birbir [13,14] treated the same geometry in the more general case of plane wave incidence and when the material properties of the half planes and the step were simulated by constant but different surface impedances, which was important for predicting the scattering caused by an abrupt change in material as well as in the geometrical properties of a surface. Buyukaksoy and Birbir extended [13,14] to a case where the two half planes with different surface reactances are joined by a reactive step [15].

We have extended the problem of plane wave scattering [15] to the problem of scattering due to a line source and a point source because the line sources are considered as a better substitute than the plane waves. While calculating the point source, we have adopted the technique given in [3]. The introduction of a line and a point source changes the incident field and the method of solution requires a careful analysis in calculating the scattered field. Using the Fourier transform, the diffraction problem is first reduced to a modified Wiener–Hopf equation of the second kind whose solution contains infinitely many constants satisfying an infinite system of linear equations. Numerical solution in the case of a line source is obtained for various values of surface reactances and

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the height of the step, from which the effects of these parameters on the diffraction phenomenon are studied.

The possible excitation of the surface waves on the reactance surfaces can be neglected as the observation point is far from the surface while using the steepest descent method and diffracted field dominates [16]. It is found that if the source is shifted to a large distance these results differ from those of [15] by a multiplicative factor to the part of the scattered field containing the effects of incident and reflected waves in the case of a line source. To the best of authors' knowledge, this seems to be the first step in this direction with the line and point source diffraction at a step with impedance boundary conditions well supported by numerical proofs.

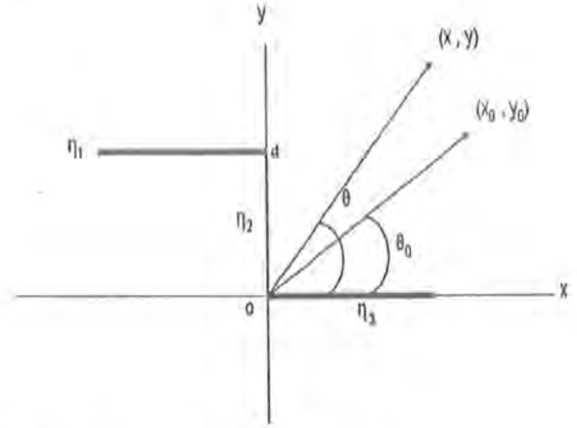

Figure 1. Geometry of the problem.

2. Line source scattering

2.1. Formulation of the problem

Consider the scattering due to a line source, located at (x_0, y_0), which illuminates two half planes $S_1 = \{x < 0, y = d, z \in (-\infty, \infty)\}$ and $S_2 = \{x > 0, y = 0, z \in (-\infty, \infty)\}$ with relative surface reactances $\eta_1$and η_3, respectively, and these are joined together by a step of height d having relative surface impedance η_2. The geometry of the line source diffraction problem is depicted in Figure 1.

For harmonic vibrations of time dependence $\exp(-i\omega t)$, we require the solution of the wave equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)u_t(x, y) = \delta(x - x_0)\delta(y - y_0), \quad (1)$$

where u_t is the total field. For the analysis purpose, the total field $u_t(x, y)$ may be expressed as follow:

$$u_t(x, y) = \begin{cases} u^i(x, y) + u_1^r(x, y) + u_1(x, y), & y > d, \\ u_2(x, y), & 0 < y < d. \end{cases} \quad (2)$$

The boundary conditions at two half planes and a step are given by:

$$\left(1 + \frac{\eta_1}{k}\frac{\partial}{\partial y}\right)u_1(x, d) = 0, \quad x \in (-\infty, 0), \quad (3a)$$

$$\left(1 + \frac{\eta_2}{ik}\frac{\partial}{\partial x}\right)u_2(0, y) = 0, \quad y \in (0, d), \quad (3b)$$

$$\left(1 + \frac{\eta_3}{k}\frac{\partial}{\partial y}\right)u_2(x, 0) = 0, \quad x \in (0, \infty), \quad (3c)$$

$$u_t(x, d) - u_2(x, d) = 0, \quad x \in (0, \infty), \quad (3d)$$

$$\frac{\partial}{\partial y}u_t(x, d) - \frac{\partial}{\partial y}u_2(x, d) = 0, \quad x \in (0, \infty). \quad (3e)$$

where $k = w/c$ is the wave number and a time factor $\exp(-i\omega t)$ is suppressed hereafter. It is assumed that the wave number k has a positive imaginary part. The lossless case can be obtained by making $\mathrm{Im}\, k \to 0$ in the final expressions. $u_1(x, y)$ and $u_2(x, y)$ are the scattered fields, $u_1^r(x, y)$ is the field reflected from the plane located at $y = d$ and $u^i(x, y)$ is the incident field satisfying the equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)u^i(x, y) = \delta(x - x_0)\delta(y - y_0). \quad (4)$$

The scattered field $u_1(x, y)$ satisfies the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)u_1(x, y) = 0, \quad x \in (-\infty, \infty). \quad (5)$$

Fourier transform of Equation (5) can be written as

$$\left(\frac{d^2}{dy^2} + K^2(\alpha)\right)F(\alpha, y) = 0, \quad (6)$$

with $F(\alpha, y)$ is divided into $F_+(\alpha, y)$ and $F_-(\alpha, y)$ as [17,18]:

$$F(\alpha, y) = F_+(\alpha, y) + F_-(\alpha, y), \quad (6a)$$

where

$$F_\pm(\alpha, y) = \pm\frac{1}{(2\pi)^{1/2}}\int_0^{\pm\infty} u_1(x, y)\exp(i\alpha x)dx, \quad (6b)$$

and $K(\alpha) = (k^2 - \alpha^2)^{1/2}$. The square root function $K(\alpha)$ is defined in the complex *α-plane* cut as shown in Figure 2 such that $K(0) = k$ [15].

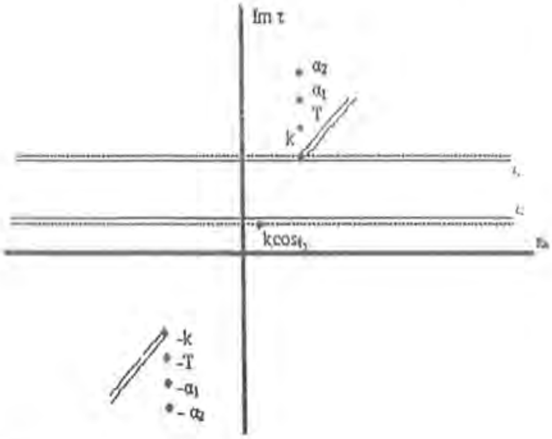

Figure 2. Complex α-plane.

The solution of Equation (5) satisfying the radiation condition for $y > d$ can be written as:

$$F(\alpha, y) = F_+(\alpha, y) + F_-(\alpha, y) = A(\alpha)\exp[\mathrm{i}K(\alpha)|y - d|], \tag{7}$$

where $A(\alpha)$ is the unknown coefficient to be determined. By the method of Green's function, one can write the incident and the corresponding reflected field as follows:

$$u^i(\alpha, y) = \frac{1}{4\pi\mathrm{i}K(\alpha)}\exp[\mathrm{i}\alpha x_0 + \mathrm{i}K(\alpha)|y - y_0|], \tag{8a}$$

and

$$u^r(\alpha, y) = \frac{\mathrm{i}\eta_1 \sin\theta_0 - 1}{\mathrm{i}\eta_1 \sin\theta_0 + 1}\frac{1}{4\pi\mathrm{i}K(\alpha)} \times \exp[\mathrm{i}\alpha x_0 + \mathrm{i}K(\alpha)|(y - 2d) + y_0|], \tag{8b}$$

where θ_0 is the angle of incident wave.

The unknown coefficient $A(\alpha)$ appearing in Equation (7) is to be determined with the help of boundary conditions (3a) and (3c)–(3e) which after taking Fourier transform give:

$$\left(1 + \frac{\eta_1}{k}\frac{\partial}{\partial y}\right)F_-(\alpha, d) = 0, \quad x \in (-\infty, 0), \tag{9a}$$

$$\left(1 + \frac{\eta_3}{k}\frac{\partial}{\partial y}\right)\bar{u}_2(\alpha, 0) = 0, \quad x \in (0, \infty), \tag{9b}$$

$$\bar{u}_t(\alpha, d) - \bar{u}_2(\alpha, d) = 0, \quad x \in (0, \infty), \tag{9c}$$

$$\frac{\partial}{\partial y}[\bar{u}_t(\alpha, d) - \bar{u}_2(\alpha, d)] = 0, \quad x \in (0, \infty), \tag{9d}$$

where $\bar{u}_2$ is the Fourier transform of u_2. Equation (9*a*) can be written as:

$$F_-(\alpha, d) + \frac{\eta_1}{k}F'_-(\alpha, d) = 0, \tag{10}$$

and prime $'$ denotes differentiation with respect to y. The differentiation of Equation (7) w.r.t. y at $y = d$ yields

$$F'_+(\alpha, d) + F'_-(\alpha, d) = \mathrm{i}K(\alpha)A(\alpha). \tag{11}$$

By using Equations (10) and (11), we can write

$$R_+(\alpha) = \frac{K(\alpha)}{k}\frac{A(\alpha)}{\chi(\eta_1, \alpha)}, \tag{12}$$

where

$$R_+(\alpha) = F_+(\alpha, d) + \frac{\eta_1}{k}F'_+(\alpha, d), \tag{12a}$$

and

$$\chi(\eta_1, \alpha) = \left[\mathrm{i}\eta_1 + \frac{k}{K(\alpha)}\right]^{-1}. \tag{12b}$$

Consider now the region $x > 0$, $y \in (0, d)$. In this region $u_2(x, y)$ satisfies the Helmholtz equation in the range $x \in (0, \infty)$:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)u_2(x, y) = 0, \quad x \in (0, \infty). \tag{13}$$

Multiplying Equation (13) by $\exp(\mathrm{i}\alpha x)/(2\pi)^{1/2}$ and integrating w.r.t. x from 0 to ∞, we obtain

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} + K^2(\alpha)\right)G_+(\alpha, y) = (k + \eta_2\alpha)f(y), \tag{14}$$

with

$$G_+(\alpha, y) = \frac{1}{(2\pi)^{1/2}}\int_0^\infty u_2(x, y)\exp(\mathrm{i}\alpha x)\mathrm{d}x; \tag{14a}$$

while deriving Equation (14), we have used Equation (3b) and

$$f(y) = \frac{1}{k(2\pi)^{1/2}}\frac{\partial}{\partial x}u_2(0, y). \tag{14b}$$

The solution of the inhomogeneous differential equation in (14) can easily be obtained as follows:

$$G_+(\alpha, y) = B(\alpha)\cos Ky + C(\alpha)\sin Ky + \frac{(k + \eta_2\alpha)}{K(\alpha)} \times \int_0^y f(t)\sin K(y - t)\mathrm{d}t. \tag{15}$$

By using the Equation (9b), i.e.

$$G_+(\alpha, y) + \frac{\eta_3}{k}G'_+(\alpha, 0) = 0, \tag{16}$$

Equation (15) reduces to

$$G_+(\alpha, y) = C(\alpha)\left[\sin Ky - \frac{\eta_3}{k}K\cos Ky\right] + \frac{(k + \eta_2\alpha)}{K(\alpha)} \times \int_0^y f(t)\sin K(y - t)\mathrm{d}t. \tag{17}$$

Equation (9c) can be written as:

$$\int_0^{\infty}[u^i(x,d)+u_1^r(x,d)+u_1(x,d)-u_2(\alpha,d)]\exp(i\alpha x)dx=0, \tag{18}$$

which can further be expressed as:

$$p(\alpha)+F_+(\alpha)-G_+(\alpha,d)=0, \tag{19}$$

where

$$p(\alpha)=\int_0^{\infty}[u^i(x,d)+u_1^r(x,d)]\exp(i\alpha x)dx. \tag{20}$$

Similarly, Equation (9d) can be reproduced as:

$$r(\alpha)+F'_+(\alpha)-G'_+(\alpha,d)=0, \tag{21}$$

where

$$r(\alpha)=\int_0^{\infty}\frac{\partial}{\partial y}[u^i(x,d)+u_1^r(x,d)]\exp(i\alpha x)dx. \tag{22}$$

The addition of Equation (19) with η_1/k times Equation (21) gives

$$R_+(\alpha)+\left[p(\alpha)+\frac{\eta_1}{k}r(\alpha)\right]=G_+(\alpha,d)+\frac{\eta_1}{k}G'_+(\alpha,d). \tag{23}$$

Substituting Equation (17) and its derivative w.r.t. to y into Equation (23) enables one to solve $C(\alpha)$ as follows:

$$C(\alpha)=-\frac{k^2}{\eta_1\eta_3 KM(\alpha)}\Bigg\{R_+(\alpha)+(p(\alpha)+\frac{\eta_1}{k}r(\alpha)) -(k+\eta_2\alpha)\int_0^d f(t)\left[\frac{\sin K(d-t)}{K}+\frac{\eta_1}{k}\cos K(d-t)\right]dt\Bigg\}, \tag{24}$$

with

$$M(\alpha)=(\alpha^2-T^2)\frac{\sin Kd}{K}-k\left(\frac{1}{\eta_3}-\frac{1}{\eta_1}\right)\cos Kd, \tag{24a}$$

and

$$T=k\left(1+\frac{1}{\eta_1\eta_3}\right)^{1/2}. \tag{24b}$$

Placing Equation (24) into Equation (17), we obtain

$$G_+(\alpha,y)=-\frac{k^2}{\eta_1\eta_3}\frac{(\sin Ky/K)-(\eta_3/k)\cos Ky}{M(\alpha)} \times\Bigg\{R_+(\alpha)+(p(\alpha)+\frac{\eta_1}{k}r(\alpha)) -(k+\eta_2\alpha)\int_0^d f(t)\left[\frac{\sin K(d-t)}{K}+\frac{\eta_1}{k}\cos K(d-t)\right]dt\Bigg\} +\frac{(k+\eta_2\alpha)}{K(\alpha)}\int_0^y f(t)\sin K(y-t)dt. \tag{25}$$

The LHS of Equation (25) is regular in the upper half of the complex α plane. RHS is also regular in the upper half plane except at the zeros of

$$M(\alpha_m)=0,\quad m=1,2\ldots,\quad \mathrm{Im}(\alpha_m)>\mathrm{Im}(k). \tag{26}$$

To make the RHS regular in the upper half plane, the residues at $\alpha=\alpha_m$ must be zero. That is,

$$R_+(\alpha_m)+\left(p(\alpha_m)+\frac{\eta_1}{k}r(\alpha_m)\right) =(k+\eta_2\alpha_m)\int_0^d f(t)\left[\frac{\sin K_m(d-t)}{K_m}+\frac{\eta_1}{k}\cos K_m(d-t)\right]dt, \tag{27}$$

with

$$K_m=K(\alpha_m). \tag{27a}$$

The Equation (24a), at $\alpha=\alpha_m$, gives

$$\frac{\sin K_m d}{K_m}+\frac{\eta_1}{k}\cos K_m d=\frac{\eta_3}{k}\left[\cos K_m d-\frac{\eta_1}{k}\sin K_m d\right]. \tag{28}$$

Now, Equation (27) can be written as

$$R_+(\alpha_m)=-(k+\eta_2\alpha_m)\left[\cos K_m d-\frac{\eta_1}{k}\sin K_m d\right] \times Q^2 f_m-\left[p(\alpha_m)+\frac{\eta_1}{k}r(\alpha_m)\right], \tag{29}$$

with

$$f_m=\frac{1}{Q^2}\int_0^d f(t)\left[\frac{\sin K_m t}{K_m}-\frac{\eta_3}{k}\cos K_m t\right]dt, \tag{29a}$$

and

$$Q^2=\int_0^d\left[\frac{\sin K_m t}{K_m}-\frac{\eta_3}{k}\cos K_m t\right]^2 dt,$$

or

$$Q^2=\frac{\eta_1\eta_3}{k^2}\frac{\dot{M}(\alpha_m)}{2\alpha_m}\left[\cos K_m d+\frac{\eta_3}{k}\sin K_m d\right]. \tag{29b}$$

In Equation (29b), $\dot{M}(\alpha)$ denotes the derivative of $M(\alpha)$ w.r.t α. From Equations (7), (12) and (19), we have

$$-F_-(\alpha,d)+\frac{R_+(\alpha)}{1+i\eta_1[K(\alpha)/k]}=G_+(\alpha,d) +\frac{[p(\alpha)+(\eta_1/k)r(\alpha)]}{1+i\eta_1[K(\alpha)/k]}-p(\alpha). \tag{30}$$

By using Equation (25) and the fact that Equation (24a) can also be written as

$$M(\alpha)=-\frac{k^2}{\eta_1\eta_3}\Bigg\{\left[\frac{\sin Kd}{K}+\frac{\eta_1}{k}\cos Kd\right] -\frac{\eta_3}{k}\left[\cos Kd-\frac{\eta_1}{k}K\sin Kd\right]\Bigg\}. \tag{31}$$

Equation (30) reduces to

$$-\frac{k^2}{\eta_1\eta_3}\frac{R_+(\alpha)\chi(\eta_1,\alpha)}{N(\alpha)\chi(\eta_3,\alpha)}+\acute{F}_-(\alpha,d)=r(\alpha)$$
$$-(\mathrm{i}k\chi(\eta_1,\alpha))\Big[p(\alpha)+\frac{\eta_1}{k}r(\alpha)\Big]$$
$$+(k+\eta_2\alpha)\sum_{m=1}^{\infty}f_m\frac{[\cos K_m d+\frac{\eta_3}{k}K_m\sin K_m d]}{\alpha^2-\alpha_m^2} \quad (32)$$

with

$$N(\alpha)=\exp(\mathrm{i}Kd)M(\alpha). \quad (32\mathrm{a})$$

Let us consider the following eigenfunction expansion of $f(t)$:

$$f(t)=\sum_{m=1}^{\infty}f_m\left[\frac{\sin K_m t}{K_m}-\frac{\eta_3}{k}\cos K_m t\right], \quad (33)$$

with f_m being given by Equation (29a). Invoking Equation (33) into Equation (32), we obtain, after term by term integration

$$-\frac{k^2}{\eta_1\eta_3}\frac{R_+(\alpha)\chi(\eta_1,\alpha)}{N(\alpha)\chi(\eta_3,\alpha)}+\acute{F}_-(\alpha,d)=r(\alpha)$$
$$-(\mathrm{i}k\chi(\eta_1,\alpha))\Big[p(\alpha)+\frac{\eta_1}{k}r(\alpha)\Big]$$
$$+(k+\eta_2\alpha)\sum_{m=1}^{\infty}f_m\frac{[(\cos K_m t/K_m)-(\eta_3/k)\sin K_m t]}{\alpha^2-\alpha_m^2}, \quad (34)$$

which may be considered as a *modified Wiener–Hopf equation of the second kind* valid in the strip $\mathrm{Im}(k\cos\theta_0)<\mathrm{Im}(\alpha)<\mathrm{Im}(k)$.

To solve Equation (34), we can use the classical Wiener–Hopf procedure to obtain

$$\frac{R_+(\alpha)\chi_+(\eta_1,\alpha)}{N_+(\alpha)\chi_+(\eta_3,\alpha)}=-\eta_1\eta_3\frac{1}{4\pi^2k^2}\frac{k\sin\theta_0}{(\mathrm{i}\eta_1\sin\theta_0+1)}\frac{\exp(-\mathrm{i}kd\sin\theta_0)}{(k\cos\theta_0-\alpha)}$$
$$\times\frac{N_-(k\cos\theta_0)\chi_-(\eta_3,k\cos\theta_0)}{\chi_-(\eta_1,k\cos\theta_0)}\frac{(2\pi)^{1/2}}{(k\rho_0)^{1/2}}$$
$$+\sum_{m=1}^{\infty}\frac{R_+(\alpha_m)(k-\eta_2\alpha_m)}{\acute{M}(\alpha_m)(k+\eta_2\alpha_m)}\frac{N_+(\alpha_m)\chi_+(\eta_3,\alpha_m)}{(\alpha+\alpha_m)\chi_+(\eta_1,\alpha_m)}$$
$$\times\left[\frac{(k^2+\eta_3K_m^2)}{(k^2+\eta_1\eta_3K_m^2)}\right]\cos K_m d. \quad (35)$$

In Equation (35), $N_+(\alpha)$ [resp. $N_-(\alpha)$] and $\chi_+(\eta,\alpha)$ [resp. $\chi_-(\eta,\alpha)$] are the split functions, regular and free of zeros in the upper half-plane $\mathrm{Im}(\alpha)>\mathrm{Im}(-k)$ [resp. in the lower half-plane $\mathrm{Im}(\alpha)>\mathrm{Im}(-k)$], resulting from the Wiener–Hopf factorization Equation (32a) and Equation (12b) as

$$N(\alpha)=N_+(\alpha)N_-(\alpha),\quad N_-(\alpha)=N_+(-\alpha), \quad (36)$$

and

$$\chi(\eta_1,\alpha)=\chi_+(\eta_1,\alpha)\chi_-(\eta_1,\alpha),\quad \chi_-(\eta_1,\alpha)=\chi_+(\eta_1,-\alpha). \quad (37)$$

By following the method described in [17], the explicit expressions of $N_+(\alpha)$ can be obtained [15]:

$$N_+(\alpha)=\left[k\left(\frac{1}{\eta_1}-\frac{1}{\eta_3}\right)\cos kd-T^2\frac{\sin kd}{k}\right]^{1/2}$$
$$\times\exp\left\{\frac{Kd}{\pi}\ln\left(\frac{\alpha+\mathrm{i}K}{k}\right)\right\}$$
$$\times\exp\left\{\frac{\mathrm{i}\alpha d}{\pi}\left(1-C+\ln\left(\frac{2\pi}{kd}\right)+\mathrm{i}\frac{\pi}{2}\right)\right\}$$
$$\times\prod_{n=1}^{\infty}\left(1+\frac{\alpha}{\alpha_n}\right)\exp\left(\frac{\mathrm{i}\alpha d}{n\pi}\right). \quad (38)$$

The split function $\chi_+(\eta,\alpha)$ can be expressed explicitly in terms of Maliuzhinetz function [19]

$$\chi_+(\eta,k\cos\theta)=2^{3/2}\left(\frac{2}{\eta}\right)^{1/2}\sin\frac{\theta}{2}$$
$$\times\left\{\frac{\{M(3\pi/2-\theta-\phi)M(\pi/2-\theta+\phi)\}}{M^2(\pi/2)}\right\}^2$$
$$\times\left\{\left[1+\sqrt{2}\cos\left(\frac{\pi/2-\theta-\phi}{2}\right)\right]\right.$$
$$\left.\times\left[1+\sqrt{2}\cos\left(\frac{3\pi/2-\theta-\phi}{2}\right)\right]\right\}^{-1}, \quad (39)$$

with

$$\sin\phi=\frac{1}{\eta}, \quad (40)$$

$$M_\pi(z)=\exp\left\{\frac{-1}{8\pi}\int_0^z\frac{\pi\sin u-2(2^{1/2})\pi\sin\left(\frac{u}{2}\right)+2u}{\cos u}\,\mathrm{d}u\right\}. \quad (41)$$

From Equation (35), we observe that $R_+(\alpha)$ depends upon the unknown series of constants $R_+(\alpha_1)$, $R_+(\alpha_2)$, $R_+(\alpha_3)\dots$ To find an approximate value for $R_+(\alpha)$, substitute $\alpha=\alpha_1,\alpha_2,\dots,\alpha_n$ to get n equations in n unknowns. The simultaneous solution of these equations yields approximate solutions for $R_+(\alpha_m)$, $m=1,2,\dots,n$.

2.2. *Analysis of the field*

The scattered field in the region $y>d$ can be obtained by evaluating the value of $A(\alpha)$ using Equation (35) in Equation (12) and then putting the resulting equation into Equation (7) and finally taking the inverse Fourier transform of the equation, the final expression for the diffracted field comes out to be:

$$u_1(x,y)=\frac{1}{(2\pi)^{1/2}}\int_L\frac{R_+(\alpha)}{[1+(\mathrm{i}\eta_1/k)K(\alpha)]}$$
$$\exp[\mathrm{i}K(\alpha)(y-d)]\exp(-\mathrm{i}\alpha x)\mathrm{d}\alpha,\quad y>d, \quad (42)$$

where L is a straight line parallel to the real axis, lying in the strip $\mathrm{Im}(k\cos\theta_0) < \mathrm{Im}\,\alpha < \mathrm{Im}\,k$.

To determine the far field behavior of the scattered field we introduce the following substitutions

$$x = \rho\cos\theta, \quad y - d = \rho\sin\theta, \tag{42a}$$

and the transformation

$$\alpha = -k\cos(\theta + \mathrm{i}t), \tag{42b}$$

where t, given in Equation (42b) is real. The contour of integration over α in Equation (42) goes into the branch of a hyperbola around $-\mathrm{i}k$ if $\pi/2 < \theta < \pi$. We further observe that in deforming the contour into a hyperbola the pole $\alpha = \xi$ may be crossed. If we also make the transformation $\xi = k\cos(\theta_0 + \mathrm{i}t_1)$ the contour over ξ also goes into a hyperbola. The two hyperbolae will not cross each other if $\theta < \theta_0$. However, if the inequality is reversed there will be a contribution from pole which, in fact, cancels the incident wave in the shadow region. Omitting the details of calculations, the asymptotic evaluation of the integral in Equation (42), using the method of steepest descent [20], the field due to a line source at a large distance from the edge is given by:

$$u_1(x,y) \sim u_{11}^d(x,y) + u_{12}^d(x,y), \tag{43}$$

with

$$\begin{aligned} u_{11}^d(\rho,\theta) \sim & \left(\frac{(2\pi)^{1/2}}{4\pi^2}\right)\frac{\mathrm{i}\eta_1\eta_3}{k}\exp(-\mathrm{i}kd\sin\theta_0) \\ & \times \frac{\exp[\mathrm{i}k\rho_0 + \mathrm{i}(\pi/2)]\exp(\mathrm{i}k\rho)}{(k\rho_0)^{1/2} \qquad (k\rho)^{1/2}} \\ & \times \frac{\chi_-(\eta_3, k\cos\theta_0)\chi_-(\eta_3, k\cos\theta)}{\chi_-(\eta_1, k\cos\theta_0)\chi_-(\eta_1, k\cos\theta)} \\ & \times \frac{N_-(k\cos\theta_0)N_-(k\cos\theta)}{(\mathrm{i}\eta_1\sin\theta_0 + 1)(\mathrm{i}\eta_1\sin\theta + 1)} \times \frac{\sin\theta\sin\theta_0}{\cos\theta + \cos\theta_0}, \end{aligned} \tag{44}$$

and

$$\begin{aligned} u_{12}^d(\rho,\theta) \sim & \exp\left(-\mathrm{i}\frac{\pi}{4}\right)\frac{k\sin\theta}{(\mathrm{i}\eta_1\sin\theta + 1)} \\ & \times N_-(k\cos\theta)\frac{\chi_-(\eta_3, k\cos\theta)}{\chi_-(\eta_1, k\cos\theta)}\frac{\exp(\mathrm{i}k\rho)}{(k\rho)^{1/2}} \\ & \times \sum_{m=1}^{\infty}\frac{R_+(\alpha_m)(k - \eta_2\alpha_m)}{\acute{M}(\alpha_m)(k + \eta_2\alpha_m)}\frac{N_+(\alpha_m)}{(\alpha_m - k\cos\theta_0)} \\ & \times \frac{\chi_+(\eta_3, \alpha_m)}{\chi_+(\eta_1, \alpha_m)} \\ & \times \left[\frac{(k^2 + \eta_3 K_m^2)}{(k^2 + \eta_1\eta_3 K_m^2)}\right]\cos K_m d. \end{aligned} \tag{45}$$

2.3. Numerical solution

Now, we will present some graphical results showing the effects of reactance and impedance parameters η_1, η_3 and η_2, respectively, step height d and line source parameter ρ_0 on the diffraction phenomenon.

On comparing the graphs in Figures 3 and 4 with the graphs in Figures 3(*a*) and (*b*) drawn in [15], one can see that the behavior of the graph is exactly the same and it differs only by the location along the vertical axis because of the multiplicative factor to the part of the scattered field containing the effects of incident and reflected waves. This is a counter check of the validation of our claim that if the source is shifted to a large distance these results differ from those of [15] by a multiplicative factor to the part of the scattered field containing the effects of incident and reflected waves. Similarly, the graphs plotted in Figure 5 tie in closely with the graph plotted in Figure 4 of [15].

Figure 6 shows that the amplitude of the diffracted field decreases as the source is taken away from the origin, which is a natural phenomenon and verifying the results. In Figure 7, it is observed that the amplitude of the diffracted field increases with the increase in step height which is an obvious result.

Although the calculations were made for the E-polarization case, relying upon the duality principle, the results related to H-polarization can easily be obtained.

3. Point source scattering

3.1. Formulation of the problem

For the case of point source scattering, we consider the scattering occupying the position (x_0, y_0, z_0). Thus, we require the solution of the equation

$$\begin{aligned} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)u_t(x,y,z) = & \ \delta(x - x_0) \\ & \times \delta(y - y_0)\delta(z - z_0), \end{aligned} \tag{46}$$

subject to the following boundary conditions

$$\left(1 + \frac{\eta_1}{k}\frac{\partial}{\partial y}\right)u_1(x, d, z) = 0, \quad x \in (-\infty, 0), \tag{47a}$$

$$\left(1 + \frac{\eta_2}{\mathrm{i}k}\frac{\partial}{\partial x}\right)u_2(0, y, z) = 0, \quad y \in (0, d), \tag{47b}$$

$$\left(1 + \frac{\eta_3}{k}\frac{\partial}{\partial y}\right)u_2(x, 0, z) = 0, \quad x \in (0, \infty), \tag{47c}$$

$$u_t(x, d, z) - u_2(x, d, z) = 0, \quad x \in (0, \infty), \tag{47d}$$

$$\frac{\partial}{\partial y}u_t(x, d, z) - \frac{\partial}{\partial y}u_2(x, d, z) = 0, \quad x \in (0, \infty). \tag{47e}$$

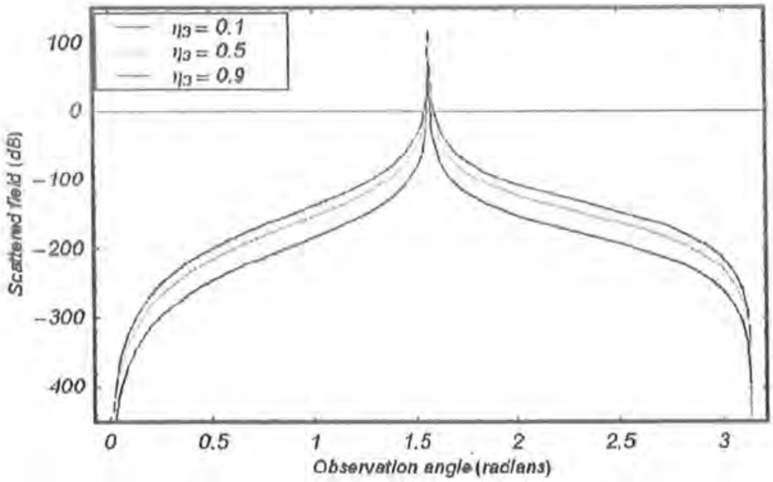

Figure 3. Scattered field (dB) versus the observation angle (radians) for, $d=0.1\lambda$, $\eta_1=0.2$, $\eta_2=0.3\mathrm{i}$, $\rho=\rho_0=k=1$. (The color version of this figure is included in the online version of the journal.)

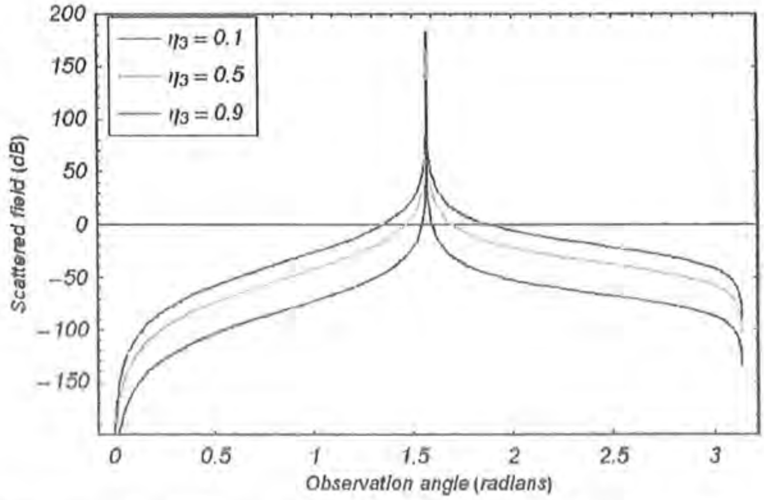

Figure 4. Scattered field (dB) versus the observation angle (radians) for, $d=0.25\lambda$, $\theta_0=\pi/2$, $\eta_1=0.2$, $\eta_2=0.3\mathrm{i}$, $\rho=\rho_0=k=1$. (The color version of this figure is included in the online version of the journal.)

where u_i is the same as defined in Equation (2). Also, the incident wave is defined as follows:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+k^2\right)u^i(x,y,z)=\delta(x-x_0)\,\delta(y-y_0)\delta(z-z_0). \qquad (48)$$

Defining the Fourier transform and the inverse w.r.t. the variable z as follows:

$$\bar{u}_1(x,y,w)=\left(\frac{k}{2\pi}\right)^{1/2}\int_{-\infty}^{\infty}u_1(x,y,z)\exp(\mathrm{i}kwz)\mathrm{d}z, \qquad (49a)$$

$$u_1(x,y,z)=\left(\frac{k}{2\pi}\right)^{1/2}\int_{-\infty}^{\infty}\bar{u}_1(x,y,w)\exp(-\mathrm{i}kwz)\mathrm{d}w. \qquad (49b)$$

Taking the Fourier transform of Equation (46), Equations (47a) and (47c)–(47e), with boundary conditions in the transformed domain w takes the following form:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\gamma^2\right)\bar{u}_1(x,y,w)=0, \qquad (50)$$

with $\gamma=(1-w^2)^{1/2}$ and

$$\left(1+\frac{\eta_1}{k}\frac{\partial}{\partial y}\right)\bar{u}_1(x,d,w)=0, \quad x\in(-\infty,0), \qquad (51a)$$

$$\left(1+\frac{\eta_2}{k}\frac{\partial}{\partial x}\right)\bar{u}_2(x,d,w)=0, \quad y\in(0,d), \qquad (51b)$$

$$\left(1+\frac{\eta_3}{k}\frac{\partial}{\partial y}\right)\bar{u}_2(x,0,w)=0, \quad x\in(0,\infty), \qquad (51c)$$

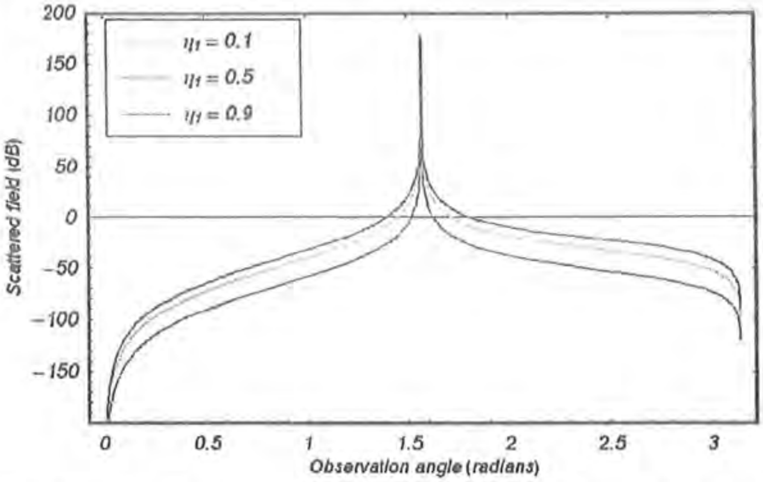

Figure 5. Scattered field (dB) versus the observation angle (radians) for, $d=0.1\lambda$, $\theta_0=\pi/2$, $\eta_3=0.2$, $\eta_2=0.3\text{i}$, $\rho=\rho_0=k=1$. (The color version of this figure is included in the online version of the journal.)

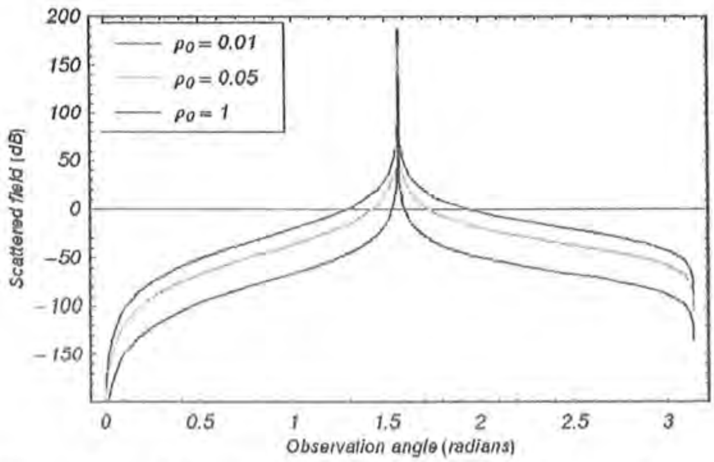

Figure 6. Scattered field (dB) versus the observation angle (radians) for, $d=0.1\lambda$, $\theta_0=\pi/2$, $\eta_1=\eta_3=0.2$, $\eta_2=0.3\text{i}$, $\rho=k=1$. (The color version of this figure is included in the online version of the journal.)

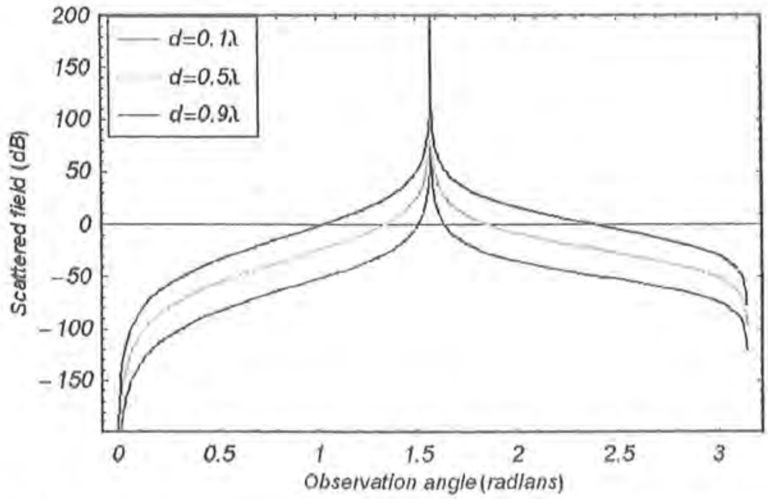

Figure 7. Scattered field (dB) versus the observation angle (radians) for, $\theta_0=\pi/2$, $\eta_1=\eta_3=0.2$, $\eta_2=0.3\text{i}$, $\rho=k=\rho_0=1$. (The color version of this figure is included in the online version of the journal.)

$$\tilde{u}_i(x,d,w)-\tilde{u}_2(x,d,w)=0,\quad x\in(0,\infty), \tag{51d}$$

$$\frac{\partial}{\partial y}[\tilde{u}_i(x,d,w)-\tilde{u}_2(x,d,w)]=0,\quad x\in(0,\infty). \tag{51e}$$

3.2. Solution of the problem

Thus, we see that Equation (50) together with the boundary conditions (51a)–(51e) in the transformed domain w is the same as in the case of two dimensions formulated in the above section except that $k^2\gamma^2$ replaces k^2. Therefore, making use of Equation (43), we calculate the scattered field due to a point source as follows:

$$\begin{aligned}\tilde{u}_1(\rho,\theta,w)\sim&\left(\frac{(2\pi)^{1/2}}{4\pi^2}\right)\frac{i\eta_1\eta_3}{k\gamma}\exp(-ik\gamma d\sin\theta_0)\\&\times\frac{\exp[ik\gamma\rho_0+i(\pi/2)]}{(k\gamma\rho_0)^{1/2}}\\&\times\frac{\exp[ik\gamma\rho]}{(k\gamma\rho)^{1/2}}\frac{\chi_-(\eta_3,k\gamma\cos\theta_0)\chi_-(\eta_3,k\gamma\cos\theta)}{\chi_-(\eta_1,k\gamma\cos\theta_0)\chi_-(\eta_1,k\gamma\cos\theta)}\\&\times\frac{N_-(k\gamma\cos\theta_0)N_-(k\gamma\cos\theta)}{(i\eta_1\sin\theta_0+1)(i\eta_1\sin\theta+1)}\\&\times\frac{\sin\theta\sin\theta_0}{\cos\theta+\cos\theta_0}\exp(ikwz_0)\\&+\exp[-i(\pi/4)]\frac{k\gamma\sin\theta}{(i\eta_1\sin\theta+1)}N_-(k\gamma\cos\theta)\\&\times\frac{\chi_-(\eta_3,k\gamma\cos\theta)\exp(ik\gamma\rho)}{\chi_-(\eta_1,k\gamma\cos\theta)\,(k\gamma\rho)^{1/2}}\exp(ikwz_0)\\&\times\sum_{m=1}^{\infty}\frac{R_+(\alpha_m)(k\gamma-\eta_2\alpha_m)}{\acute{M}(\alpha_m)(k\gamma+\eta_2\alpha_m)}\frac{N_+(\alpha_m)}{(\alpha_m-k\gamma\cos\theta_0)}\\&\times\frac{\chi_+(\eta_3,\alpha_m)}{\chi_+(\eta_1,\alpha_m)}\left[\frac{(k^2\gamma^2+\eta_3K_m^2)}{(k^2\gamma^2+\eta_1\eta_3K_m^2)}\right]\cos K_md.\end{aligned} \tag{52}$$

The scattered field in the spatial domain can now be obtained by taking the inverse Fourier transform of Equation (52). Thus, we obtain

$$u_1(\rho,\theta,z)\sim u_{11}^d(\rho,\theta,z)+u_{12}^d(\rho,\theta,z), \tag{53}$$

where

$$\begin{aligned}u_{11}^d(\rho,\theta,z)\sim\int_{-\infty}^{\infty}\Bigg[&\frac{\exp[i(\pi/2)]}{4\pi^2}\frac{i\eta_1\eta_3}{k\gamma}\exp(-ik\gamma d\sin\theta_0)\\&\times\frac{\chi_-(\eta_3,k\gamma\cos\theta_0)\chi_-(\eta_3,k\gamma\cos\theta)}{\chi_-(\eta_1,k\gamma\cos\theta_0)\chi_-(\eta_1,k\gamma\cos\theta)}\\&\times\frac{N_-(k\gamma\cos\theta_0)N_-(k\gamma\cos\theta)}{(i\eta_1\sin\theta_0+1)(i\eta_1\sin\theta+1)}\times\frac{\sin\theta\sin\theta_0}{\cos\theta+\cos\theta_0}\\&\times\frac{\exp[ik\gamma(\rho+\rho_0)+ikw(z_0-z)]}{\gamma(k\rho\rho_0)^{1/2}}\Bigg]dw,\end{aligned} \tag{53a}$$

and

$$\begin{aligned}u_{12}^d(\rho,\theta,z)\sim\int_{-\infty}^{\infty}\Bigg[&\frac{\exp[-i(\pi/4)]}{(2\pi)^{1/2}}\frac{k\gamma\sin\theta}{(i\eta_1\sin\theta+1)}N_-(k\gamma\cos\theta)\\&\times\frac{\chi_-(\eta_3,k\gamma\cos\theta)\exp[ik\gamma\rho+ikw(z_0-z)]}{\chi_-(\eta_1,k\gamma\cos\theta)\qquad(\gamma\rho)^{1/2}}\\&\times\sum_{m=1}^{\infty}\frac{R_+(\alpha_m)(k\gamma-\eta_2\alpha_m)}{\acute{M}(\alpha_m)(k\gamma+\eta_2\alpha_m)}\frac{N_+(\alpha_m)}{(\alpha_m-k\gamma\cos\theta_0)}\\&\times\frac{\chi_+(\eta_3,\alpha_m)}{\chi_+(\eta_1,\alpha_m)}\\&\times\left(\frac{(k^2\gamma^2+\eta_3K_m^2)}{(k^2\gamma^2+\eta_1\eta_3K_m^2)}\right)\cos K_md\Bigg]dw.\end{aligned} \tag{53b}$$

In order to solve Equations (53a) and (53b), we have used the following substitutions:

$$w=\cos\beta,\ \ \gamma=(1-w^2)^{1/2}=\sin\beta,\ \ z-z_0=R_1\cos\nu,$$
$$\rho+\rho_0=\sin\nu,\ \ R_1=[(z-z_0)^2+(\rho+\rho_0)^2]^{1/2}, \tag{54a}$$

for $u_{11}^d(\rho,\theta,z)$ and

$$w=\cos\beta,\quad \gamma=(1-w^2)^{1/2}=\sin\beta,\quad z-z_0=R_1\cos\nu,$$
$$\rho=R_1\sin\nu,\quad R_1=[(z-z_0)^2+(\rho)^2]^{1/2}, \tag{54b}$$

for $u_{12}^d(\rho,\theta,z)$, respectively.

By using the the method of steepest descent [20] and omitting the details of calculations, the final form of the field is given below:

$$u_1(\rho,\theta,z)\sim u_{11}^d(\rho,\theta,z)+u_{12}^d(\rho,\theta,z), \tag{55}$$

where

$$\begin{aligned}u_{11}^d(\rho,\theta,z)\sim&\frac{\exp[i(\pi/2)]}{4\pi^2}\frac{i\eta_1\eta_3}{-k\sin\nu}\exp(ikd\sin\nu\sin\theta_0)\\&\times\frac{\chi_-(\eta_3,-k\sin\nu\cos\theta_0)\chi_-(\eta_3,-k\gamma\cos\theta)}{\chi_-(\eta_1,-k\sin\nu\cos\theta_0)\chi_-(\eta_1,-k\sin\nu\cos\theta)}\\&\times\frac{N_-(-k\sin\nu\cos\theta_0)N_-(-k\sin\nu\cos\theta)}{(i\eta_1\sin\theta_0+1)(i\eta_1\sin\theta+1)}\\&\times\frac{\sin\theta\sin\theta_0}{\cos\theta+\cos\theta_0}\frac{\exp\{-[ikR_1+i(\pi/4)]\}}{k}\\&\times\left(\frac{2\pi}{\rho\rho_0R_1}\right)^{1/2},\end{aligned} \tag{55a}$$

and

$$\begin{aligned}u_{12}^d(\rho,\theta,z)\sim&\exp[-i(3\pi/4)]\frac{(-k\sin\nu\sin\theta)}{(i\eta_1\sin\theta+1)}N_-(-k\sin\nu\cos\theta)\\&\times\frac{\chi_-(\eta_3,-k\sin\nu\cos\theta)\exp\{-[ikR_1+i(\pi/4)]\}}{\chi_-(\eta_1,-k\sin\nu\cos\theta)\qquad kR_1\sin\nu}\\&\times\sum_{m=1}^{\infty}\frac{R_+(\alpha_m)(-k\sin\nu-\eta_2\alpha_m)}{\acute{M}(\alpha_m)(-k\sin\nu+\eta_2\alpha_m)}\frac{N_+(\alpha_m)}{(\alpha_m+k\sin\nu\cos\theta_0)}\\&\times\frac{\chi_+(\eta_3,\alpha_m)}{\chi_+(\eta_1,\alpha_m)}\left(\frac{(k^2\sin^2\nu+\eta_3K_m^2)}{(k^2\sin^2\nu+\eta_1\eta_3K_m^2)}\right)\cos K_md.\end{aligned} \tag{56}$$

4. Conclusion

In this article, the scattering due to a line source and a point source from a reactive step has been studied. For a line source, the said problem is first reduced to a modified Wiener–Hopf equation of second kind whose solution contains an infinite set of constants satisfying an infinite system of linear algebraic equations. A numerical solution of this system is obtained for various values of the surface reactances and the height of the step, through which the effect of these parameters on the diffraction phenomenon is studied. To the best of the authors' knowledge, this seems to be the first step in this direction with the line source diffraction at a step with impedance boundary conditions well supported by numerical proofs. It is observed that if the source is shifted to a large distance, the results of the line source differ from those of [15] by a multiplicative factor to the part of the scattered field containing the effects of incident and reflected waves. Subsequently, the point source diffraction is examined using the results obtained for a line source diffraction.

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Acoustic diffraction by an oscillating strip

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ABSTRACT

The problem of diffraction of a plane acoustic wave from an oscillating rigid strip is studied. The problem is solved by using the temporal and spatial integral transform and the Wiener-Hopf technique. The scattered field in the far zone is determined by the method of steepest descent. The significance of the present analysis is that it recovered the results when a strip is widened to a half plane. Graphical results for the diffraction problem have also been presented.

1. Introduction

The scattering of sound and electromagnetic waves has been studied extensively since the half plane problems were investigated by Poincar'e [1] and Sommerfeld [2]. The Wiener-Hopf (WH) technique [3] proves to be a powerful tool to tackle, not only, the problems of diffraction by a single half plane but it may further be extended to the case of parallel half planes. However, there were problems in dealing with other configurations and mixed boundary value problems appearing in the diffraction theory. Diffraction from a strip is a well known studied phenomenon in the diffraction theory and is applicable to a variety of physical problems. Many scientists worked on diffraction problem related to strip geometry. Various methods of solution have been given in literature, e.g., Morse and Rubenstein [4] studied the problem of diffraction of acoustic waves from strip/slit using the method of separation of variables, some authors [5–11] followed Noble's approach [3], where as some of them [12–15] adopted the method of successive approximations to study the diffraction from a strip, Castro and Kapanadze [16] employed the theory of Bessel potential spaces and Imran et al. [17] used the Kobayashi's potential method to study the diffraction of acoustic/electromagnetic waves by a strip. Ahmad [18] have considered the problem of oscillating (instead of static) half plane by considering the oscillating plane wave. We have extended the analysis of Ahmad [18] to the problem of an oscillating strip instead of a static strip and go a step further to understand the diffraction phenomenon from the oscillating strip.

In this paper, the diffracted field due to a plane wave by an oscillating rigid strip is obtained by solving two uncoupled Wiener-Hopf equations. It is observed that the diffracted field corresponding to any type of oscillation of the strip can be obtained by just inserting in it the values of the generalized Fourier coefficients corresponding to the oscillation involved [22]. It is found that the two edges of the strip give rise to two diffracted fields (one from each edge) and an interaction field. To the best of authors knowledge, this seems to be the first attempt in this direction (with an oscillating strip) and will be a useful contribution to the existing diffraction theory. An additional trait of the present analysis is that the results of the half plane [18] can be deduced by taking an appropriate limit $l \to \infty$ which is also a mathematical check of the validity of the analysis. Graphical results for the system have also been presented.

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2. Formulation of the problem

We consider the scattering of plane acoustic wave from an oscillating strip occupying the space $-l < x < 0$ at $y = 0$ and is oscillating in a direction perpendicular to the screen with velocity $u_0 f(t)$, where $f(t)$ is a periodic function of time t whose generalized Fourier series is given by

$$f(t) = \sum_{-\infty}^{\infty} C_n e^{in\omega_0 t}, \tag{1}$$

where the Fourier coefficients C_n are given by

$$C_n = \frac{1}{T_0}\int_{-\infty}^{\infty} f(t)e^{-in\omega_0 t}dt \tag{2}$$

and non-zero fundamental frequency

$$\omega_0 = \frac{2\pi}{T_0}(\neq 0). \tag{3}$$

Assume the continuity of the velocity across the boundary $y = 0$, $x < 0$, that is [19]:

$$\frac{\partial \phi_t}{\partial y} = u_0 f(t), \quad -l < x < 0, \tag{4}$$

where the total velocity potential ϕ_t satisfies the wave equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi_t = \frac{1}{c^2}\frac{\partial^2 \phi_t}{\partial t^2}. \tag{5}$$

Writing

$$\phi_t = \phi + \phi_i, \tag{6}$$

where ϕ is the diffracted field and ϕ_i is the incident field given by

$$\phi_i = \exp[-ik_1(x\cos\theta_0 + y\sin\theta_0) - i\omega_1 t], \quad 0 < \theta_0 < \pi, \tag{7}$$

where $k_1 = \frac{\omega_1}{c}$, ω_1 is the frequency and c is the speed of the sound. Thus, we have to solve the following boundary value problem

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}, \tag{8}$$

$$\frac{\partial}{\partial y}\phi(x,0,t) - ik_1\sin\theta_0 e^{-ik_1 x\cos\theta_0 - i\omega_1 t} = u_0 f(t) \quad -l < x < 0, \tag{9}$$

and

$$\left.\begin{array}{l}\phi(x,0^+,t) = \phi(x,0^-,t)\\ \frac{\partial}{\partial y}\phi(x,0^+,t) = \frac{\partial}{\partial y}\phi(x,0^-,t)\end{array}\right\}, \quad -\infty < x < -l, \quad x > 0, \quad y = 0. \tag{10}$$

3. Solution of the problem

Define the temporal Fourier transform pair as

$$\left\{\begin{array}{l}\Psi(x,y,\omega) = \int_{-\infty}^{\infty}\phi(x,y,t)e^{i\omega t}dt,\\ \phi(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi(x,y,\omega)e^{-i\omega t}d\omega.\end{array}\right. \tag{11}$$

Taking the temporal Fourier transform of Eqs. (8)–(10), we obtain

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)\Psi(x,y,\omega) = 0 \tag{12}$$

and

$$\frac{\partial}{\partial y}\Psi(x,0,\omega) - 2\pi i k_1\sin\theta_0 e^{-ik_1 x\cos\theta_0}\delta(\omega - \omega_1) = u_0\bar{f}(\omega), \quad -l < x < 0, \tag{13}$$

$$\begin{cases} \Psi(x,0^+,\omega) = \Psi(x,0^-,\omega), \\ \frac{\partial}{\partial y}\Psi(x,0^+,\omega) = \frac{\partial}{\partial y}\Psi(x,0^-,\omega), \end{cases} \quad -\infty < x < -l, \quad x > 0, \quad y = 0, \tag{14}$$

where $k = \frac{\omega}{c} = k_1 + ik_2$ with $k_2 > 0$ and

$$\bar{f}(\omega) = 2\pi \sum_{-\infty}^{\infty} C_n \delta(\omega - n\omega_0). \tag{15}$$

Now, we define the spatial Fourier transform over the variable x as follows

$$\begin{cases} \overline{\Psi}(\alpha,y,\omega) = \int_{-\infty}^{\infty} \Psi(x,y,\omega) e^{i\alpha x} dx, \\ \Psi(x,y,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\Psi}(\alpha,y,\omega) e^{-i\alpha x} d\alpha, \end{cases} \tag{16}$$

with

$$\overline{\Psi}(\alpha,y,\omega) = \overline{\Psi}_+(\alpha,y,\omega) + \overline{\Psi}_1(\alpha,y,\omega) + e^{-i\alpha l}\overline{\Psi}_-(\alpha,y,\omega), \tag{17}$$

where

$$\overline{\Psi}_+(\alpha,y,\omega) = \int_0^{\infty} \Psi(x,y,\omega) e^{i\alpha x} dx, \tag{18a}$$

$$\overline{\Psi}_1(\alpha,y,\omega) = \int_{-l}^{0} \Psi(x,y,\omega) e^{i\alpha x} dx, \tag{18b}$$

and

$$\overline{\Psi}_-(\alpha,y,\omega) = \int_{-\infty}^{0} \Psi(x,y,\omega) e^{i\alpha x} dx. \tag{18c}$$

Transforming Eqs. (12)–(14) into α-plane, result in:

$$\left(\frac{d^2}{dy^2} - \gamma^2\right)\overline{\Psi}(\alpha,y,\omega) = 0, \tag{19}$$

with

$$\left.\begin{aligned} \overline{\Psi}'_1(\alpha,0,\omega) - \tfrac{2\pi k_1 \sin\theta_0 \delta(\omega-\omega_1)}{(\alpha - k_1\cos\theta_0)}[1 - \exp[-i(\alpha - k_1\cos\theta_0)l]] \\ = \tfrac{u_0 \bar{f}(\omega)}{i\alpha}(1 - \exp[-i\alpha l]) \end{aligned}\right\}, \quad -l < x < 0, \tag{20}$$

and

$$\begin{cases} \overline{\Psi}_\pm(x,0^+,\omega) = \overline{\Psi}_\pm(x,0^-,\omega), \\ \frac{\partial}{\partial y}\overline{\Psi}_\pm(x,0^+,\omega) = \frac{\partial}{\partial y}\overline{\Psi}_\pm(x,0^-,\omega), \end{cases} \quad -\infty < x < -l, \quad x > 0, \quad y = 0, \tag{21}$$

where $\gamma^2 = \alpha^2 - k^2$ with $\mathrm{Re}\,\gamma > 0$ in the strip $-\mathrm{Im}\,k < \mathrm{Im}\,\alpha < \mathrm{Im}\,k$.

The Solution of Eq. (19) after using the continuity of $\overline{\Psi}'$ across $y = 0$ is given by

$$\overline{\Psi}(\alpha,y,\omega) = \begin{cases} A(\alpha,\omega)e^{-\gamma y}, & \text{if } y \geqslant 0, \\ -A(\alpha,\omega)e^{\gamma y}, & \text{if } y < 0, \end{cases} \tag{22}$$

Now using Eqs. (17) and (22), we have

$$A(\alpha) = \overline{\Psi}_+(\alpha,0^+,\omega) + \overline{\Psi}_1(\alpha,0^+,\omega) + e^{-i\alpha l}\overline{\Psi}_-(\alpha,0^+,\omega), \tag{23}$$

and

$$-A(\alpha) = \overline{\Psi}_+(\alpha,0^-,\omega) + \overline{\Psi}_1(\alpha,0^-,\omega) + e^{-i\alpha l}\overline{\Psi}_-(\alpha,0^-,\omega). \tag{24}$$

Adding and subtracting Eqs. (23) and (24), we get

$$2S_- e^{-i\alpha l} + 2\overline{\Psi}_+(\alpha,0,\omega) + 2J_1(\alpha,0,\omega) = 0, \tag{25}$$

where

$$2S_- = \overline{\Psi}_-(\alpha,0^+,\omega) + \overline{\Psi}_-(\alpha,0^-,\omega), \tag{26a}$$

$$2J_1(\alpha,0,\omega) = \overline{\Psi}_1(\alpha,0^+,\omega) + \overline{\Psi}_1(\alpha,0^-,\omega), \tag{26b}$$

$$2\overline{\Psi}_+(\alpha,0,\omega) = 2\overline{\Psi}_+(\alpha,0^+,\omega) = \overline{\Psi}_+(\alpha,0^+,\omega) + \overline{\Psi}_+(\alpha,0^-,\omega), \tag{26c}$$

and

$$A(\alpha) = J_2(\alpha, 0, \omega), \tag{27}$$

where

$$J_2(\alpha, 0, \omega) = \frac{1}{2}\left[\overline{\Psi}_1(\alpha, 0^+, \omega) - \overline{\Psi}_1(\alpha, 0^-, \omega)\right], \tag{28}$$

With the help of Eqs. (17) and (22), we have

$$-\gamma A(\alpha) = \overline{\Psi}'_+(\alpha, 0, \omega) + \overline{\Psi}'_1(\alpha, 0, \omega) + e^{-i\alpha l}\overline{\Psi}'_-(\alpha, 0, \omega). \tag{29}$$

Using the continuity of $\overline{\Psi}'$ across $y = 0$ and Eqs. (20) and (27) in Eq. (29) will yield

$$\begin{aligned} -\gamma J_2(\alpha, 0, \omega) = \overline{\Psi}'_+(\alpha, 0, \omega) + \frac{2\pi k_1 \sin\theta_0 \delta(\omega - \omega_1)}{(\alpha - k_1\cos\theta_0)}[1 - \exp[-i(\alpha - k_1\cos\theta_0)l]] + \frac{u_0\bar{f}(\omega)}{i\alpha}[1 - \exp[-i\alpha l]] \\ + e^{-i\alpha l}\overline{\Psi}'_-(\alpha, 0, \omega). \end{aligned} \tag{30}$$

Eq. (30) can be re-arranged to give

$$e^{-i\alpha l}\overline{\Psi}'_-(\alpha, 0, \omega) + \gamma J_2(\alpha, 0, \omega) + \overline{\Psi}'_+(\alpha, 0, \omega) = -2\pi k_1 \sin\theta_0\delta(\omega - \omega_1)G(\alpha) - \frac{u_0\bar{f}(\omega)}{i\alpha}[1 - \exp[-i\alpha l]], \tag{31}$$

where

$$G(\alpha) = \frac{[1 - \exp[-i(\alpha - k_1\cos\theta_0)l]]}{(\alpha - k_1\cos\theta_0)}.$$

Eqs. (25) and (31) are the standard Wiener-Hopf equations. In order to solve the problem, we shall solve Eq. (31), using Wiener-Hopf procedure [3].

For the solution of the Wiener-Hopf equations (31), one can use the following factorizations [3]

$$\gamma(\alpha) = \gamma_+(\alpha)\gamma_-(\alpha), \tag{32}$$

where $\gamma_+(\alpha)$ is regular for $\operatorname{Im}\alpha > -\operatorname{Im}k$ i.e., for upper half plane and $\gamma_-(\alpha)$ is regular for $\operatorname{Im}\alpha < \operatorname{Im}k$ i.e., for the lower half plane. Using Eq. (32) in Eq. (31), we get

$$e^{-i\alpha l}\overline{\Psi}'_-(\alpha, 0, \omega) + \gamma_+(\alpha)\gamma_-(\alpha)J_2(\alpha, 0, \omega) + \overline{\Psi}'_+(\alpha, 0, \omega) = -2\pi k_1\sin\theta_0\delta(\omega - \omega_1)G(\alpha) - \frac{u_0\bar{f}(\omega)}{i\alpha}[1 - [\exp -i\alpha l]], \tag{33}$$

Equation of type (31) have been considered by Noble [3] and a similar analysis may be employed to obtain an approximate solution for large k. Thus, we follow the procedure given in [3, Section 5.5, pp. 196].

$$\begin{aligned} \overline{\Psi}'_+(\alpha, 0, \omega) = \bar{A}\gamma_+(\alpha)G_1(\alpha) + \bar{A}T(\alpha)\gamma_+(\alpha)\gamma_+(k)C_1 + \frac{iu_0\bar{f}(\omega)\gamma_+(\alpha)}{\alpha}\left[\frac{1}{\gamma_+(\alpha)} - \frac{1}{\gamma_+(0)}\right] \\ + \frac{T(\alpha)T(k)\gamma_+(k)\gamma_+(\alpha)iu_0\bar{f}(\omega)}{k\left[1 - \gamma_+^2(k)T^2(k)\right]}\left[\frac{1}{\gamma_+(k)} - \frac{1}{\gamma_+(0)}\right] - \frac{T(\alpha)T(k)\gamma_+(k)\gamma_+(\alpha)iu_0\bar{f}(\omega)}{k\left[1 - \gamma_+^2(k)T^2(k)\right]} \end{aligned} \tag{34}$$

and

$$\begin{aligned} \overline{\Psi}'_-(\alpha, 0, \omega) = \bar{A}\gamma_+(-\alpha)G_2(-\alpha) + \bar{A}T(-\alpha)\gamma_+(-\alpha)\gamma_+(K)C_2 - \frac{iu_0\bar{f}(\omega)\gamma_+(-\alpha)}{\alpha}\left[\frac{1}{\gamma_+(-\alpha)} - \frac{1}{\gamma_+(0)}\right] \\ + \frac{T(-\alpha)T(k)\gamma_+(k)\gamma_+(-\alpha)iu_0\bar{f}(\omega)}{k\left[1 - \gamma_+^2(k)T^2(k)\right]}\left[\frac{1}{\gamma_+(k)} - \frac{1}{\gamma_+(0)}\right] - \frac{T(-\alpha)T(k)\gamma_+(k)\gamma_+(-\alpha)iu_0\bar{f}(\omega)}{k\left[1 - \gamma_+^2(k)T^2(k)\right]}, \end{aligned} \tag{35}$$

where

$$\bar{A} = -2\pi k_1\sin\theta_0\delta(\omega - \omega_1), \tag{35a}$$

$$G_1(\alpha) = \frac{1}{(\alpha - k_1\cos\theta_0)}\left[\frac{1}{\gamma_+(\alpha)} - \frac{1}{\gamma_+(k_1\cos\theta_0)}\right] - e^{ilk_1\cos\theta_0}R_1(\alpha), \tag{36}$$

$$G_2(\alpha) = \frac{e^{ilk_1\cos\theta_0}}{(\alpha + k_1\cos\theta_0)}\left[\frac{1}{\gamma_+(\alpha)} - \frac{1}{\gamma_+(k_1\cos\theta_0)}\right] - R_2(\alpha), \tag{37}$$

$$C_1 = \left[\frac{G_2(k) + \gamma_+(k)G_1(k)T(k)}{1 - \gamma_+^2(k)T^2(k)}\right], \tag{38}$$

and

$$C_2 = \left[\frac{G_1(k) + \gamma_+(k)G_2(k)T(k)}{1 - \gamma_+^2(k)T^2(k)}\right], \tag{39}$$

$$R_{1,2}(\alpha) = \frac{E_{-1}[W_{-1}\{-i(k \pm k\cos\theta)l\} - W_{-1}\{-i(k+\alpha)l\}]}{2\pi i(\alpha \mp k\cos\theta_0)}, \tag{40}$$

$$T(\alpha) = \frac{1}{2\pi i}E_{-1}W_{-1}\{-i(k+\alpha)l\}, \tag{41}$$

$$E_{-1} = 2e^{i\frac{\pi}{4}}e^{ikl}\,(l)^{-1}\,(i)^{\frac{-1}{2}}h_{-1}, \tag{42}$$

where $h_{-1} = e^{i\frac{\pi}{4}}$

$$W_{n-\frac{1}{2}}(z) = \int_0^\infty \frac{u^n e^{-u}}{u+z}du = \Gamma(n+1)e^{\frac{z}{2}}z^{\frac{n}{2}-\frac{1}{2}}W_{-\frac{1}{2}(n+1),\frac{n}{2}}(z), \tag{43}$$

where $z = -i(k+\alpha)l$ and $W_{m,n}$ is known as a Whittaker function. Now, substituting Eqs. (34) and (35) in Eq. (29), we get

$$\begin{aligned}
A(\alpha) = -\frac{1}{\gamma}\Bigg[&\frac{\bar{A}\gamma_+(-\alpha)\exp[-i(\alpha - k_1\cos\theta_0)l]}{\gamma_-(k_1\cos\theta_0)(\alpha - k_1\cos\theta_0)} - \bar{A}\gamma_+(-\alpha)R_2(-\alpha)\exp[-i\alpha l] + \exp[-i\alpha l]\bar{A}\gamma_+(-\alpha)\gamma_+(k)T(-\alpha)C_2\\
&+\frac{iu_0\bar{f}(\omega)\gamma_+(-\alpha)\exp[-i\alpha l]}{\alpha\gamma_+(0)} + \frac{\exp[-i\alpha l]T(-\alpha)T(k)\gamma_+(k)\gamma_+(-\alpha)iu_0\bar{f}(\omega)}{k[1-\gamma_+^2(k)T^2(k)]}\left[\frac{1}{\gamma_+(k)} - \frac{1}{\gamma_+(0)}\right]\\
&-\frac{\exp[-i\alpha l]T(-\alpha)T(k)\gamma_+(k)\gamma_+(-\alpha)iu_0\bar{f}(\omega)}{k[1-\gamma_+^2(k)T^2(k)]} - \frac{\bar{A}\gamma_+(\alpha)}{\gamma_+(k_1\cos\theta_0)(\alpha - k_1\cos\theta_0)} - \bar{A}\gamma_+(\alpha)e^{ilk_1\cos\theta_0}R_1(\alpha)\\
&-\frac{iu_0\bar{f}(\omega)\gamma_+(\alpha)}{\alpha\gamma_+(0)} + \frac{T(\alpha)T(k)\gamma_+(k)\gamma_+(\alpha)iu_0\bar{f}(\omega)}{k[1-\gamma_+^2(k)T^2(k)]}\left[\frac{1}{\gamma_+(k)} - \frac{1}{\gamma_+(0)}\right]\\
&-\frac{T(\alpha)T(k)\gamma_+(k)\gamma_+(\alpha)iu_0\bar{f}(\omega)}{k[1-\gamma_+^2(k)T^2(k)]} + \bar{A}T(\alpha)\gamma_+(\alpha)\gamma_+(k)C_1\Bigg].
\end{aligned} \tag{44}$$

The function $\Psi(x, y, \omega)$ can be obtained by taking the inverse Fourier transform of Eq. (16) as follows:

$$\Psi(x,y,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi(\alpha,y,\omega)e^{-i\alpha x}d\alpha = \frac{1}{2\pi}\int_{-\infty}^{\infty}A(\alpha,\omega)e^{-\gamma|y|-i\alpha x}d\alpha. \tag{45}$$

Now, $\Psi(x, y, \omega)$ can be broken up as follows:

$$\Psi(x,y,\omega) = \Psi^{sep}(x,y,\omega) + \Psi^{int}(x,y,\omega), \tag{46}$$

where

$$\begin{aligned}
\Psi^{sep}(x,y,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{-1}{\gamma}\Bigg[&\frac{\bar{A}\gamma_+(-\alpha)\exp[-i(\alpha - k_1\cos\theta_0)l]}{\gamma_-(k_1\cos\theta_0)(\alpha - k_1\cos\theta_0)} - \frac{\bar{A}\gamma_+(\alpha)}{\gamma_+(k_1\cos\theta_0)(\alpha - k_1\cos\theta_0)}\\
&+\frac{iu_0\bar{f}(\omega)\gamma_+(-\alpha)\exp[-i\alpha l]}{\alpha\gamma_+(0)} - \frac{iu_0\bar{f}(\omega)\gamma_+(\alpha)}{\alpha\gamma_+(0)}\Bigg]e^{-\gamma|y|-i\alpha x}d\alpha,
\end{aligned} \tag{47}$$

and

$$\begin{aligned}
\Psi^{int}(x,y,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{-1}{\gamma}\Bigg[&-\bar{A}\gamma_+(-\alpha)R_2(-\alpha)\exp[-i\alpha l]\\
&+\bar{A}\gamma_+(-\alpha)\gamma_+(k)T(-\alpha)C_2\exp[-i\alpha l]\begin{matrix}+\frac{T(-\alpha)T(k)\gamma_+(k)\gamma_+(-\alpha)iu_0\bar{f}(\omega)\exp[-i\alpha l]}{k[1-\gamma_+^2(k)T^2(k)]}\left[\frac{1}{\gamma_+(k)} - \frac{1}{\gamma_+(0)}\right]\\ -\frac{T(-\alpha)T(k)\gamma_+(k)\gamma_+(-\alpha)iu_0\bar{f}(\omega)\exp[-i\alpha l]}{k[1-\gamma_+^2(k)T^2(k)]}\end{matrix} - \bar{A}\gamma_+(\alpha)e^{ilk_1\cos\theta_0}R_1(\alpha)\\
&+\frac{T(\alpha)T(k)\gamma_+(k)\gamma_+(\alpha)iu_0\bar{f}(\omega)}{k\left[1-\gamma_+^2(k)T^2(k)\right]}\left[\frac{1}{\gamma_+(k)} - \frac{1}{\gamma_+(0)}\right]\\
&-\frac{T(\alpha)T(k)\gamma_+(k)\gamma_+(\alpha)iu_0\bar{f}(\omega)}{k[1-\gamma_+^2(k)T^2(k)]} + \bar{A}T(\alpha)\gamma_+(\alpha)\gamma_+(k)C_1\Bigg]e^{-\gamma|y|-i\alpha x}d\alpha.
\end{aligned} \tag{48}$$

Here $\Psi^{sep}(x, y, \omega)$ consists of two parts each representing the diffracted field produced by the edges at $x = 0$ and $x = -l$, respectively, as though the other edge were absent while $\Psi^{int}(x, y, \omega)$ gives the interaction of one edge upon the other.

4. Far field approximation

The far field may now be calculated by evaluating the integral in Eqs. (47) and (48) asymptotically [20]. For that we substitute $x = r\cos\theta$, $|y| = r\sin\theta$ and deform the contour by the transformation $\alpha = -k\cos\beta$, where $\beta = \mu + i\nu$, with $0 \leqslant \mu \leqslant \pi$, $-\infty < \nu < \infty$[3].

Hence for large kr, Eqs. (47) and (48) become

$$\Psi^{sep}(x,y,\omega) = \left[\frac{-k_1\sin\theta_0\sqrt{k}\delta(\omega-\omega_1)\sin\theta}{[k\cos\theta + k_1\cos\theta_0]}\left(\begin{array}{l}\frac{e^{-i[k\cos\theta+k_1\cos\theta_0]l}}{\sqrt{1-\cos\theta}\sqrt{k_1\cos\theta_0-k}}\\ +\frac{1}{i\sqrt{1+\cos\theta}\sqrt{k_1\cos\theta_0+k}}\end{array}\right) + \frac{iu_0\sum_{-\infty}^{\infty}\sin\theta C_n\delta(\omega-n\omega_0)}{k\cos\theta}\left(\begin{array}{l}\frac{e^{ilk\cos\theta}}{\sqrt{1-\cos\theta}}\\ +\frac{1}{i\sqrt{1+\cos\theta}}\end{array}\right)\right] e^{ikr-i\frac{\pi}{4}}\left(\frac{2\pi}{kr}\right)^{\frac{1}{2}}. \tag{49}$$

and

$$\Psi^{inc}(x,y,\omega) = \left[\frac{\overline{A}e^{ikl\cos\theta}}{\gamma_+(-k\cos\theta)}(R_2(k\cos\theta) - T(k\cos\theta)\gamma_+(k)C_2) + \frac{e^{ikl\cos\theta}T(k\cos\theta)iu_0\overline{f}(\omega)T(k)\gamma_+(k)}{k[1-T^2(k)\gamma_+^2(k)]\gamma_+(-k\cos\theta)}\left(1-\frac{1}{\sqrt{2k}}+\frac{1}{\sqrt{k}}\right) + \frac{\overline{A}e^{ik_1l\cos\theta_0}}{\gamma_-(k\cos\theta)}[R_1(-k\cos\theta) - C_1T(-k\cos\theta)\gamma_+(k)] + \frac{T(-k\cos\theta)iu_0\overline{f}(\omega)T(k)\gamma_+(k)}{k[1-T^2(k)\gamma_+^2(k)]\gamma_-(-k\cos\theta)}\left(1-\frac{1}{\sqrt{2k}}+\frac{1}{\sqrt{k}}\right)\right]]k\sin\theta e^{ikr-i\frac{\pi}{4}}\left(\frac{2\pi}{kr}\right)^{\frac{1}{2}}. \tag{50}$$

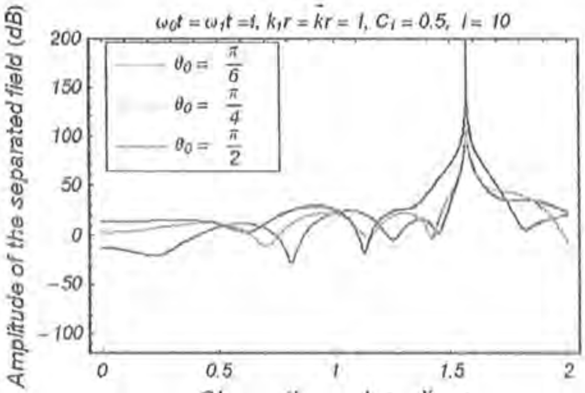

Fig. 1. Amplitude of the Separated field versus the observation angle for different values of incidence wave angle.

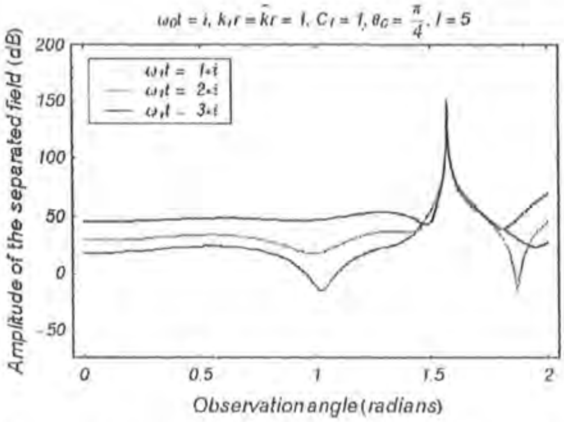

Fig. 2. Amplitude of the Separated field versus the observation angle for different values of incidence wave frequency.

Now taking the inverse temporal Fourier transform by using

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0),$$

we get

$$\phi^{sep}(x,y,t) = -\frac{\sin\theta\sin\theta_0}{2\pi[\cos\theta+\cos\theta_0]}\begin{pmatrix}\frac{e^{-ik_1 l(\cos\theta+\cos\theta_0)}}{\sqrt{1-\cos\theta}\sqrt{\cos\theta_0-1}} \\ +\frac{1}{i\sqrt{1+\cos\theta}\sqrt{\cos\theta_0+1}}\end{pmatrix} e^{ik_1 r - i\frac{\pi}{4} - i\omega_1 t}\left(\frac{2\pi}{k_1 r}\right)^{\frac{1}{2}} + \frac{iu_0\sin\theta\sum_{-\infty}^{\infty} C_n}{2\pi\bar{k}\cos\theta}\begin{pmatrix}\frac{e^{i\bar{k}\cos\theta}}{\sqrt{1-\cos\theta}} \\ +\frac{1}{i\sqrt{1+\cos\theta}}\end{pmatrix} e^{i\bar{k}r - i\frac{\pi}{4} - in\omega_0 t}\left(\frac{2\pi}{\bar{k}r}\right)^{\frac{1}{2}}, \tag{51}$$

where $\theta \neq \pi - \theta_0$ and the interacted term is given by

$$\begin{aligned}\phi^{int}(x,y,t) = \Bigg[&\left\{\frac{e^{ik_1 l\cos\theta}(-k_1\sin\theta_0)}{\gamma_+(-k_1\cos\theta)}(R_2(k_1\cos\theta) - T(k_1\cos\theta)\gamma_+(k_1)C_2)k_1\sin\theta e^{ik_1 r - i\frac{\pi}{4}}\left(\frac{2\pi}{k_1 r}\right)^{\frac{1}{2}}\right\} \\ &+\left\{\frac{e^{i\bar{k}\cos\theta}T(\bar{k}\cos\theta)iu_0\sum_{-\infty}^{\infty} C_n T(\bar{k})\gamma_+(\bar{k})}{\bar{k}[1-T^2(\bar{k})\gamma_+^2(\bar{k})]\gamma_+(-\bar{k}\cos\theta)}\left(1-\frac{1}{\sqrt{2\bar{k}}}+\frac{1}{\sqrt{\bar{k}}}\right)\bar{k}\sin\theta e^{i\bar{k}r - i\frac{\pi}{4}}\left(\frac{2\pi}{\bar{k}r}\right)^{\frac{1}{2}}\right\} \\ &+\left\{\frac{e^{ik_1 l\cos\theta_0}(-k_1\sin\theta_0)}{\gamma_-(k_1\cos\theta)}(R_1(-k_1\cos\theta) - T(-k_1\cos\theta)\gamma_+(k_1)C_1)k_1\sin\theta e^{ik_1 r - i\frac{\pi}{4}}\left(\frac{2\pi}{k_1 r}\right)^{\frac{1}{2}}\right\} \\ &\times\left\{\frac{T(-\bar{k}\cos\theta)iu_0\sum_{-\infty}^{\infty} C_n T(\bar{k})\gamma_+(\bar{k})}{\bar{k}[1-T^2(\bar{k})\gamma_+^2(\bar{k})]\gamma_-(-\bar{k}\cos\theta)}\left(1-\frac{1}{\sqrt{2\bar{k}}}+\frac{1}{\sqrt{\bar{k}}}\right)\bar{k}\sin\theta e^{i\bar{k}r - i\frac{\pi}{4}}\left(\frac{2\pi}{\bar{k}r}\right)^{\frac{1}{2}}\right\}\Bigg].\end{aligned} \tag{52}$$

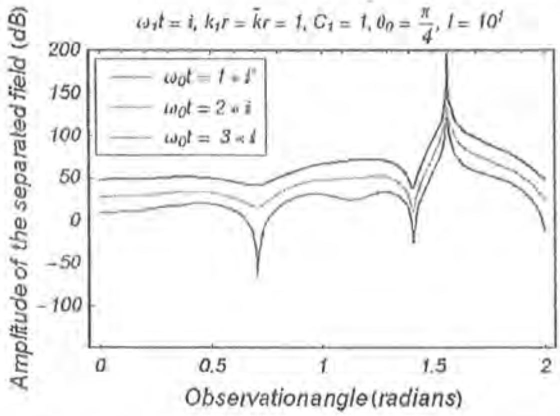

Fig. 3. Amplitude of the Separated field versus the observation angle for different values of oscillating strip frequency for $l = 10^1$.

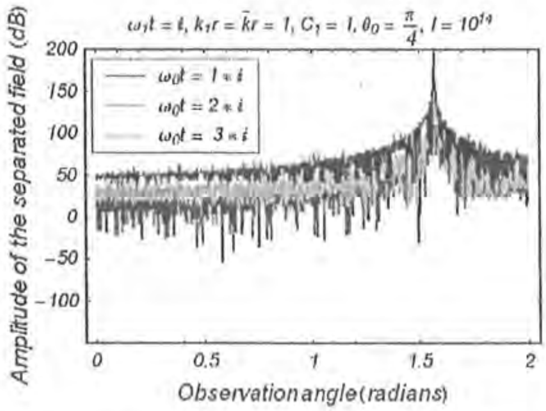

Fig. 4. Amplitude of the Separated field versus the observation angle for different values of oscillating strip frequency for $l = 10^{14}$.

Remarks. Mathematically we can derive the results of the half plane problem [18] in the following manner:

For the analysis purpose, in Eq. (44), it is assumed that the wave number k_1 has positive imaginary part and using the De L Hopital rule successively, the value of E_{-1} reduces to $\lim_{l\to\infty}\left(\frac{\sqrt{l}}{e^{l}}\right)$ which becomes zero and in turn result the quantities $T(\alpha)$, $R_{1,2}(\alpha)$, C_1 and $G_2(\alpha)$ in zero. The third term in Eq. (36) also becomes zero. Using the substitution $\overline{A} = -2\pi k_1 \sin\theta_0 \delta(\omega - \omega_1)$, $A(\alpha)$ reduces to

$$A(\alpha) = -\frac{1}{\gamma}\left[\frac{2\pi k_1 \sin\theta_0 \delta(\omega - \omega_1)\gamma_+(\alpha)}{\gamma_+(k_1\cos\theta_0)(\alpha - k_1\cos\theta_0)} - \frac{iu_0\overline{f}(\omega)\gamma_+(\alpha)}{\alpha\sqrt{k}}\right].$$

Using the factorization

$$\gamma(\alpha) = \gamma_+(\alpha)\gamma_+(\alpha),$$

the above result reduces to $A(\alpha)$ of the Half Plane [18]. Subsequently, Eq. (48), i.e., the interacted field vanishes by adopting the same procedure as in case of Eq. (44), while the separated field results into the diffracted field [18] as the strip is widened to half plane by taking the limit $l \to \infty$.

5. Numerical results and discussion

A computer program MATHEMATICA has been used for obtaining the graphical results. In Fig. 1, one can see that as we increase the angle of incidence, the amplitude of the separated field also increases [21]. Fig. 2 depicts that the wave with the higher frequency obtains higher amplitude of the separated field as compared to the other wave frequencies which is an obvious result. Fig. 3 shows that the amplitude of the separated field with higher strip frequency will be at the top and the rest will gradually be positioned accordingly which is according to the physics of the problem. The effect of strip length

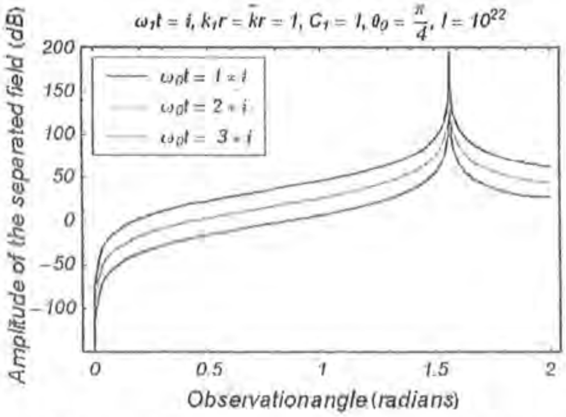

Fig. 5. Amplitude of the Separated field versus the observation angle for different values of oscillating strip frequency for $l = 10^{22}$.

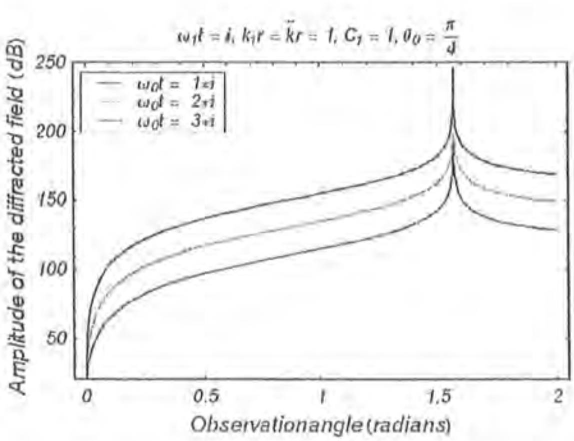

Fig. 6. Amplitude of the Diffracted field versus the observation angle for different values of oscillating strip frequency.

l, for its different values, can be seen in Figs. 3–5. We observe that as l tends to its highest value the graphs for the strip (Fig. 5) and the half plane (Fig. 6) agree each other which can be considered as numerical proof of our claim that the strip reduces to half plane [18] as $l \to \infty$. This can be considered as a check of validity of the analysis numerically as well in this paper.

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A note on plane wave diffraction by a perfectly conducting strip in a homogeneous bi-isotropic medium

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Abstract: We studied the problem of diffraction of an electromagnetic plane wave by a perfectly conducting finite strip in a homogeneous bi-isotropic medium and obtained some improved results which were presented both mathematically and graphically. The problem was solved by using the Wiener-Hopf technique and Fourier transform. The scattered field in the far zone was determined by the method of steepest decent. The significance of present analysis was that it recovered the results when a strip was widened into a half plane.

OCIS codes: (290.5838) Scattering, in-field; (050.1940) Diffraction

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1. Introduction

Beltrami flows were first introduced in the late 19th century [1]. There was no significant work on Beltrami flows for next 60 years. However, in 1950s and onwards it gained wide application in fluid mechanics and other related areas. Chandrasekhar [2], reintroduced Beltrami flows and worked on force free magnetic fields. Lakhtakia [3] compiled a catalogue on contemporary works.

A Beltrami field is proportional to its own curl everywhere in a source-free region and can be either left-handed or right-handed. For the analysis of time-harmonic electromagnetic fields in isotropic chiral and bi-isotropic media, Bohren [4] was the pioneer and his work was enhanced by Lakhtakia [5]. Lakhtakia [6], and Lakhtakia and Weiglhofer [7] worked on the application of Beltrami field to time dependent electromagnetic field. On chiral wedges, Fisanov [8] and Przezdziecki [9] did exceptional job. Asghar and Lakhtakia [10] showed that the concept of Beltrami fields can be exploited to calculate the diffraction of only one scalar field and the rest can be obtained thereof.

A Beltrami magnetostatic field exerts no Lorentz force on an electrically charged particle, and for this reason the concept has been extensively used in astrophysics as well as magnetohydrodynamics [11, 12]. Beltrami fields also occur as the circularly polarized plane waves in electromagnetic theory [13]. Although circularly polarized plane waves in free space and natural, optically active media [14, 15] have been known since the time of Fresenel, their theoretical value is best expressed in biisotropic media [16 – 21]. In recent years, propagation of plane waves with negative phase velocity and its related applications in isotropic chiral materials can be found in [22 – 25].

In this paper, the diffracted field due to a plane wave by a perfectly conducting finite strip in a homogeneous bi-isotropic medium is obtained in an improved form by solving two uncoupled Wiener-Hopf equations. The significance of the present analysis is that the results of half plane [10] can be deduced by taking an appropriate limit $l \to \infty$ whereas this is not possible in [31]. It is found that the two edges of the strip give rise to two diffracted fields (one from each edge) and an interaction field (double diffraction of two edges).

2. Formulation of the problem

Let us assume the scattering of a plane electromagnetic wave with the assumption that all space is occupied by a homogeneous bi-isotropic medium except for a perfectly conducting strip $z = 0$, $-l \leq x \leq 0$. In the Drude-Born-Fedorov representation [5, 34], the bi-isotropic medium is characterized by the following equations

$$\mathbf{D} = \varepsilon\mathbf{E} + \varepsilon\alpha\nabla \times \mathbf{E} \tag{1}$$

$$\mathbf{B} = \mu\mathbf{H} + \mu\beta\nabla \times \mathbf{H} \tag{2}$$

where ε and μ are the permittivity and the permeability scalars, respectively, while α and β are the bi-isotropy scalars. $\mathbf{D}$ is the electric displacement, $\mathbf{H}$ is the magnetic field strength, $\mathbf{B}$ is the magnetic induction, and $\mathbf{E}$ is the electric field strength. The bi-isotropic medium with $\alpha = \beta$ is reciprocal and is then called a chiral medium. Recently, it has been proved [26] that non-reciprocal bi-isotropic media are not permitted by the structure of modern electromagnetic theory. Certainly in the MHz-PHz regime, this statement has not been experimentally challenged yet, although in the ¡ 1 kHz regime there is some experimental evidence to the contrary which has not been independently confirmed [33]. However, in the mathematical study the case $\alpha \neq \beta$ may also be considered for generality.

Let us assume the time dependence of Beltrami fields to be of the form $\exp(-i\omega t)$, where ω is the angular frequency. The source free Maxwell curl postulates in the bi-isotropic medium can be set up as

$$\nabla \times \mathbf{Q}_1 = \gamma_1\mathbf{Q}_1, \tag{3}$$

$$\nabla \times \mathbf{Q}_2 = -\gamma_2\mathbf{Q}_2. \tag{4}$$

The two wave numbers γ_1 and γ_2 are given by

$$\gamma_1 = \frac{k}{(1-k^2\alpha\beta)}\left\{\sqrt{1+\frac{k^2(\alpha-\beta)^2}{4}}+\frac{k(\alpha+\beta)}{2}\right\}, \tag{5}$$

and

$$\gamma_2 = \frac{k}{(1-k^2\alpha\beta)}\left\{\sqrt{1+\frac{k^2(\alpha-\beta)^2}{4}}-\frac{k(\alpha+\beta)}{2}\right\}, \tag{6}$$

where Beltrami fields in terms of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$, as given in [27], are :

$$\mathbf{Q}_1 = \frac{\eta_1}{\eta_1+\eta_2}(\mathbf{E}+i\eta_2\mathbf{H}), \tag{7}$$

and

$$\mathbf{Q}_2 = \frac{i}{\eta_1+\eta_2}(\mathbf{E}-i\eta_1\mathbf{H}), \tag{8}$$

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where $\mathbf{Q}_1$ is the left-handed Beltrami field and $\mathbf{Q}_2$ is the right-handed Beltrami field. In Eqs. (7) and (8), the two impedances η_1 and η_2 are given by

$$\eta_1 = \frac{\eta}{\sqrt{1+\frac{k^2(\alpha-\beta)^2}{4}}+\frac{k(\alpha-\beta)}{2}}, \tag{9}$$

and

$$\eta_2 = \eta\left\{\sqrt{1+\frac{k^2(\alpha-\beta)^2}{4}}-\frac{k(\alpha-\beta)}{2}\right\}, \tag{10}$$

where $k=\omega\sqrt{\varepsilon\mu}$ and $\eta=\sqrt{\frac{\mu}{\varepsilon}}$.

Since we are interested in scattering of electromagnetic waves with a prescribed y-variation, therefore, it is appropriate to decompose the Beltrami fields as [28].

$$\mathbf{Q}_1 = \mathbf{Q}_{1t} + y Q_{1y}, \tag{11}$$

with

$$\mathbf{Q}_{1t} = Q_{1x}\mathbf{i} + Q_{1z}\mathbf{k}. \tag{12}$$

and

$$\mathbf{Q}_2 = \mathbf{Q}_{2t} + y Q_{2y}. \tag{13}$$

where the fields $\mathbf{Q}_{1t}$ and $\mathbf{Q}_{2t}$ lie in the $xz-$plane and $\mathbf{j}$ is a unit vector along the y-axis such that $\mathbf{j}.\mathbf{Q}_{1t}=0$ and $\mathbf{j}.\mathbf{Q}_{2t}=0$. Now, the Eq. (3) can be written as:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q_{1x} & Q_{1y} & Q_{1z} \end{vmatrix} = \gamma_1(Q_{1x}\mathbf{i}+Q_{1y}\mathbf{j}+Q_{1z}\mathbf{k}). \tag{14}$$

Assuming all the field vectors having an explicit $\exp(ik_y y)$ dependence on the variable y and comparing x and z components on both sides of the above equation, we obtain

$$Q_{1x} = \frac{1}{k_{1xz}^2}\left[ik_y\frac{\partial Q_{1y}}{\partial x} - \gamma_1\frac{\partial Q_{1y}}{\partial z}\right], \tag{15}$$

and

$$Q_{1z} = \frac{1}{k_{1xz}^2}\left[ik_y\frac{\partial Q_{1y}}{\partial z} + \gamma_1\frac{\partial Q_{1y}}{\partial x}\right], \tag{16}$$

where

$$k_{1xz}^2 = \gamma_1^2 - k_y^2. \tag{17}$$

Similarly, from Eq. (4), with explicit $\exp(ik_y y)$ dependence on the variable y, we may obtain

$$Q_{2x} = \frac{1}{k_{2xz}^2}\left[ik_y\frac{\partial Q_{2y}}{\partial x} + \gamma_2\frac{\partial Q_{2y}}{\partial z}\right], \tag{18}$$

$$Q_{2z} = \frac{1}{k_{2xz}^2}\left[ik_y\frac{\partial Q_{2y}}{\partial z} - \gamma_2\frac{\partial Q_{2y}}{\partial x}\right], \tag{19}$$

with

$$k_{2xz}^2 = \gamma_2^2 - k_y^2. \tag{19a}$$

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It is sufficient to explore the scattering of the scalar field Q_{1y} and Q_{2y} because the other components of $\mathbf{Q}_1$ and $\mathbf{Q}_2$ can then be completely determined by using Eqs. $(15-19)$.

Now using the constitutive relations (1) and (2), the Maxwell curl postulates $\nabla\times\mathbf{E}=i\omega\mathbf{B}-\mathbf{K}$ and $\nabla\times\mathbf{H}=-i\omega\mathbf{D}+\mathbf{J}$ may be written as:

$$\nabla\times\mathbf{Q}_1-\gamma_1\mathbf{Q}_1=\mathbf{S}_1, \tag{20a}$$

$$\nabla\times\mathbf{Q}_2-\gamma_2\mathbf{Q}_2=\mathbf{S}_2, \tag{20b}$$

where $\mathbf{S}_1$ and $\mathbf{S}_2$ are the corresponding source densities and are given by

$$\mathbf{S}_1=\frac{\eta_1}{\eta_1+\eta_2}\left(\frac{i\gamma_1}{\omega\varepsilon}\mathbf{J}-(1+\alpha\gamma_1)\mathbf{K}\right), \tag{21a}$$

$$\mathbf{S}_2=\frac{\eta_1}{\eta_1+\eta_2}\left(-\frac{i\gamma_2}{\omega\mu}\mathbf{K}-(1+\beta\gamma_2)\mathbf{J}\right). \tag{21b}$$

In deriving Eqs. $(21a)$ and $(21b)$, we have used the following relations

$$1+\omega\varepsilon\alpha\eta_2=(1-k^2\alpha\beta)(1+\alpha\gamma), \tag{22}$$

$$1-\omega\varepsilon\alpha\eta_1=(1-k^2\alpha\beta)\eta_1\frac{\gamma_2}{\omega\mu}, \tag{23}$$

$$\eta_2+\omega\mu\beta=(1-k^2\alpha\beta)\frac{\gamma_1}{\omega\varepsilon}, \tag{24}$$

$$\eta_1-\omega\mu\beta=(1-k^2\alpha\beta)\eta_1(1-\beta\gamma_2). \tag{25}$$

Furthermore, $\mathbf{Q}_1$ is $\mathbf{E}$ like and $\mathbf{Q}_2$ is $\mathbf{H}$ like. Similarly $\mathbf{S}_1$ is $\mathbf{K}$ like and $\mathbf{S}_2$ is $\mathbf{J}$ like where $\mathbf{J}$ and $\mathbf{K}$ are the electric and magnetic source current densities, respectively. The boundary condition which is necessary is that the tangential component of the electric field must vanish on perfectly conducting finite plane. This implies that $E_x=E_y=0$, for $z=0$, $-l\le x\le 0$. Using this fact in Eqs. (7) and (8), the boundary conditions on the finite plane take the form

$$Q_{1y}-i\eta_2Q_{2y}=0,\ z=0,\ -l\le x\le 0, \tag{26a}$$

and

$$Q_{1x}-i\eta_2Q_{2x}=0,\ z=0,\ -l\le x\le 0. \tag{26b}$$

With the help of Eqs. (16) and (17), Eq. $(26b)$ becomes

$$\frac{1}{k_{1xz}^2}\left[ik_y\frac{\partial Q_{1y}}{\partial x}-\gamma_1\frac{\partial Q_{1y}}{\partial z}\right]-i\eta_2\frac{1}{k_{2xz}^2}\left[ik_y\frac{\partial Q_{2y}}{\partial x}+\gamma_2\frac{\partial Q_{2y}}{\partial z}\right]=0,\ z=0,\ -l\le x\le 0. \tag{27}$$

Thus the scalar fields Q_{1y} and Q_{2y} satisfy the boundary conditions $(26a)$ and (27).

Now, eliminating Q_{2y} from Eqs. $(26a)$ and (27), we obtain

$$\frac{\partial Q_{1y}}{\partial x}\mp\delta\frac{\partial Q_{1y}}{\partial z}=0,\quad z=0^{\pm},\ -l\le x\le 0, \tag{28}$$

where

$$\delta=\frac{\gamma_2k_{1xz}^2+\gamma_1k_{2xz}^2}{ik_y(k_{2xz}^2-k_{1xz}^2)}. \tag{28a}$$

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It is worthwhile to note that the boundary conditions (28) are of the same form as impedance boundary conditions [29]. We observe that there is no boundary for $-\infty < x < -l, x > 0, z = 0$. Therefore the continuity conditions are given by

$$Q_{1y}(x,z^+) = Q_{1y}(x,z^-); \quad -\infty < x < -l,\ x > 0,\ z = 0, \tag{29}$$

$$\frac{\partial Q_{1y}(x,z^+)}{\partial z} = \frac{\partial Q_{1y}(x,z^-)}{\partial z}; \quad -\infty < x < -l,\ x > 0,\ z = 0. \tag{30}$$

The edge conditions (local properties) on the field that invoke the appropriate physical constraint of finite energy near the edges of the boundary discontinuities require that

$$Q_{1y}(x,0) = O(1) \text{ and } \frac{\partial Q_{1y}(x,0)}{\partial z} = O(x^{-\frac{1}{2}}) \text{ as } x \to 0^+, \tag{31a}$$

$$Q_{1y}(x,0) = O(1) \text{ and } \frac{\partial Q_{1y}(x,0)}{\partial z} = O(x+l)^{-\frac{1}{2}} \text{ as } x \to -l. \tag{31b}$$

It is to be noted that the field Q_{2y} also satisfies Eqs. $(28-30)$. Finally, the scattered field must satisfy the radiation conditions in the limit $(x^2+z^2)^{1/2} \to \infty$. We must also observe at this juncture that, in effect, we need to consider the diffraction of only one scalar field, that is either Q_{1y} or Q_{2y}, at a time, but the presence of the other scalar field is reflected in the complicated nature of the boundary condition (28). If we set the incident field to be a plane wave, then

$$Q_{1y}(x,z) = Q_{1y}^{inc}(x,z) + Q_{1y}^{sca}(x,z), \tag{32}$$

with

$$Q_{1y}^{inc}(x,y,z) = exp[i(k_y y + k_{1x}x + k_{1z}z)], \tag{32a}$$

and scattered field Q_{1y}^{sca} satisfies the following homogeneous Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_{1xz}^2\right) Q_{1y}^{sca} = 0, \tag{33}$$

where

$$k_{1xz}^2 = k_{1x}^2 + k_{1z}^2 = \gamma_1^2 - k_y^2. \tag{33a}$$

Also the boundary conditions (28) to (30) will take the following form

$$\left(\frac{\partial}{\partial x} \mp \delta\frac{\partial}{\partial z}\right) Q_{1y}^{inc} + \left(\frac{\partial}{\partial x} \mp \delta\frac{\partial}{\partial z}\right) Q_{1y}^{sca} = 0, \quad z = 0^{\pm}, -l \le x \le 0, \tag{34}$$

and

$$Q_{1y}^{sca}(x,z^+) = Q_{1y}^{sca}(x,z^-); \quad -\infty < x < -l,\ x > 0,\ z = 0, \tag{35a}$$

$$\frac{\partial}{\partial z} Q_{1y}^{sca}(x,z^+) = \frac{\partial}{\partial z} Q_{1y}^{sca}(x,z^-); \quad -\infty < x < -l,\ x > 0,\ z = 0. \tag{35b}$$

For the solution of Eq. (33) subject to the boundary conditions $(34-35b)$, we introduce the Fourier transform *w.r.t* variable x as:

$$\overline{\Psi}(\upsilon,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Q_{1y}^{sca}(x,z) e^{i\upsilon x} dx = \overline{\Psi}_+(\upsilon,z) + e^{-i\upsilon l}\overline{\Psi}_-(\upsilon,z) + \overline{\Psi}_1(\upsilon,z), \tag{36}$$

#96994 - $15.00 USD Received 10 Jun 2008; revised 4 Jul 2008; accepted 18 Jul 2008; published 13 Aug 2008

where

$$\overline{\Psi}_+(\upsilon,z)=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}Q_{1y}^{sca}(x,z)e^{i\upsilon x}dx,$$

$$\overline{\Psi}_-(\upsilon,z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-l}Q_{1y}^{sca}(x,z)e^{i\upsilon(x+l)}dx,$$

$$\overline{\Psi}_1(\upsilon,z)=\frac{1}{\sqrt{2\pi}}\int_{-l}^{0}Q_{1y}^{sca}(x,z)e^{i\upsilon x}dx. \tag{37}$$

Note that $\overline{\Psi}_-(\upsilon,z)$ is regular for $Im\upsilon < Imk_{1xz}$, and $\overline{\Psi}_+(\upsilon,z)$ is regular for $Im\upsilon > -Imk_{1xz}$ and $\overline{\Psi}_1(\upsilon,z)$ is analytic in the common region $-Imk_{1xz} < Im\upsilon < Imk_{1xz}$. The Fourier transform of Eq. $(32a)$ in the region $-l \le x \le 0, z=0$ gives

$$\overline{\Psi}_0(\upsilon,0)=\frac{i}{\sqrt{2\pi}(k_{1x}+\upsilon)}\left[-1+\exp[-i(k_{1x}+\upsilon)l]\right], \tag{38}$$

and its derivative is defined as

$$\overline{\Psi}_0^{\prime}(\upsilon,0)=\frac{k_{1z}}{\sqrt{2\pi}(k_{1x}+\upsilon)}\left[-1+\exp[-i(k_{1x}+\upsilon)l]\right]. \tag{38a}$$

The Fourier transform of Eqs. $(34-35b)$, respectively, yields

$$\left(\frac{d^2}{dz^2}+\kappa^2\right)\overline{\Psi}(\upsilon,z)=0, \tag{39}$$

where

$$\kappa^2=k_{1xz}^2-\upsilon^2, \tag{39a}$$

$$\overline{\Psi}_1^{\prime}(\upsilon,0^+)=-\frac{i\upsilon}{\delta}[\overline{\Psi}_1(\upsilon,0^+)+\overline{\Psi}_0(\upsilon,0)]-\overline{\Psi}_0^{\prime}(\upsilon,0), \tag{40}$$

$$\overline{\Psi}_1^{\prime}(\upsilon,0^-)=\frac{i\upsilon}{\delta}[\overline{\Psi}_1(\upsilon,0^-)+\overline{\Psi}_0(\upsilon,0)]-\overline{\Psi}_0^{\prime}(\upsilon,0), \tag{41}$$

and

$$\begin{aligned}
\overline{\Psi}_-(\upsilon,0^+)&=\overline{\Psi}_-(\upsilon,0^-)=\overline{\Psi}_-(\upsilon,0),\\
\overline{\Psi}_+(\upsilon,0^+)&=\overline{\Psi}_+(\upsilon,0^-)=\overline{\Psi}_+(\upsilon,0),\\
\overline{\Psi}_-^{\prime}(\upsilon,0^+)&=\overline{\Psi}_-^{\prime}(\upsilon,0^-)=\overline{\Psi}_-^{\prime}(\upsilon,0),\\
\overline{\Psi}_+^{\prime}(\upsilon,0^+)&=\overline{\Psi}_+^{\prime}(\upsilon,0^-)=\overline{\Psi}_+^{\prime}(\upsilon,0).
\end{aligned} \tag{42}$$

The solution of Eq. (39) satisfying radiation condition is given by

$$\overline{\Psi}(\upsilon,z)=\begin{cases} A(\upsilon)e^{i\kappa z} & if\; z>0,\\ C(\upsilon)e^{-i\kappa z} & if\; z<0.\end{cases} \tag{43}$$

By substituting Eqs. (36) and (42) to Eq. (43), we get

$$\overline{\Psi}_+(\upsilon,0)+e^{-i\upsilon l}\overline{\Psi}_-(\upsilon,0)+\overline{\Psi}_1(\upsilon,0^+)=A(\upsilon), \tag{44a}$$

$$\overline{\Psi}_+(\upsilon,0)+e^{-i\upsilon l}\overline{\Psi}_-(\upsilon,0)+\overline{\Psi}_1(\upsilon,0^-)=C(\upsilon), \tag{44b}$$

$$\overline{\Psi}'_+(\upsilon,0)+e^{-i\upsilon l}\overline{\Psi}'_-(\upsilon,0)+\overline{\Psi}'_1(\upsilon,0^+)=i\kappa A(\upsilon), \tag{44c}$$

$$\overline{\Psi}'_+(\upsilon,0)+e^{-i\upsilon l}\overline{\Psi}'_-(\upsilon,0)+\overline{\Psi}'_1(\upsilon,0^-)=-i\kappa C(\upsilon). \tag{44d}$$

Subtracting Eq. (44*b*) from Eq. (44*a*) and Eq. (44*d*) from Eq. (44*c*) and then by adding and subtracting the resultant equations, we obtain

$$A(\upsilon)=J_1(\upsilon,0)+\frac{J_1'(\upsilon,0)}{i\kappa}, \tag{45}$$

and

$$C(\upsilon)=-J_1(\upsilon,0)+\frac{J_1'(\upsilon,0)}{i\kappa}, \tag{46}$$

where

$$J_1(\upsilon,0)=\frac{1}{2}[\overline{\Psi}_1(\upsilon,0^+)-\overline{\Psi}_1(\upsilon,0^-)], \tag{47}$$

and

$$J_1'(\upsilon,0)=\frac{1}{2}[\overline{\Psi}'_1(\upsilon,0^+)-\overline{\Psi}'_1(\upsilon,0^-)]. \tag{48}$$

Making use of Eq. (44*a*) in Eq. (44*c*) and Eq. (44*b*) in Eq. (44*d*), we can write

$$\overline{\Psi}'_+(\upsilon,0)+e^{-i\upsilon l}\overline{\Psi}'_-(\upsilon,0)+\overline{\Psi}'_1(\upsilon,0^+)=i\kappa[\overline{\Psi}_+(\upsilon,0)+e^{-i\upsilon l}\overline{\Psi}_-(\upsilon,0)+\overline{\Psi}_1(\upsilon,0^+)], \tag{49a}$$

$$\overline{\Psi}'_+(\upsilon,0)+e^{-i\upsilon l}\overline{\Psi}'_-(\upsilon,0)+\overline{\Psi}'_1(\upsilon,0^-)=-i\kappa[\overline{\Psi}_+(\upsilon,0)+e^{-i\upsilon l}\overline{\Psi}_-(\upsilon,0)+\overline{\Psi}_1(\upsilon,0^-)]. \tag{49b}$$

By eliminating $\overline{\Psi}'_1(\upsilon,0^+)$ from Eqs. (49*a*) and (45) and $\overline{\Psi}'_1(\upsilon,0^-)$ from Eqs. (49*b*) and (46) and then by adding the resultant equations, we get

$$\overline{\Psi}'_+(\upsilon,0)+e^{-i\upsilon l}\overline{\Psi}'_-(\upsilon,0)-i\kappa L(\upsilon)J_1(\upsilon)+\frac{k_{1z}}{\sqrt{2\pi}(k_{1x}+\upsilon)}[-1+\exp[-i(k_{1x}+\upsilon)l]]=0. \tag{50}$$

In a similar way, by eliminating $\overline{\Psi}_1(\upsilon,0^+)$ from Eqs. (49*a*) and (40), $\overline{\Psi}_1(\upsilon,0^-)$ from Eqs. (49*b*) and (41), and then subtracting the resulting equations, we get

$$-i\upsilon\overline{\Psi}_+(\upsilon,0)-i\upsilon e^{-i\upsilon l}\overline{\Psi}_-(\upsilon,0)+\delta L(\upsilon)J_1'(\upsilon)+\frac{k_{1x}}{\sqrt{2\pi}(k_{1x}+\upsilon)}[-1+\exp[-i(k_{1x}+\upsilon)l]]=0, \tag{51}$$

where

$$L(\upsilon)=\left(1+\frac{\upsilon}{\delta\kappa}\right). \tag{51a}$$

Eqs. (50) and (51) are the standard Wiener-Hopf equations. Let us proceed to find the solution for these equations.

3. Solution of the Wiener-Hopf equations

For the solution of the Wiener-Hopf equations, one can make use of the following factorization

$$L(\upsilon)=\left(1+\frac{\upsilon}{\delta\kappa}\right)=L_+(\upsilon)L_-(\upsilon), \tag{52a}$$

and

$$\kappa(\upsilon)=\kappa_+(\upsilon)\kappa_-(\upsilon), \tag{52b}$$

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where $L_+(\upsilon)$ and $\kappa_+(\upsilon)$ are regular for $Im\upsilon > -Im\, k_{1xz}$, i.e., for upper half plane and $L_-(\upsilon)$ and $\kappa_-(\upsilon)$ are regular for $Im\ \upsilon < Imk_{1xz}$, i.e., lower half plane. The factorization expression ($52a$) has been accomplished by Asghar et al [30]. By putting the values of $J_1(\upsilon,0)$ and $J_1'(\upsilon,0)$ from Eqs. (50) and (51) into Eqs. (45) and (46), we get

$$A(\upsilon) = \frac{1}{i\kappa\, L(\upsilon)}\{\overline{\Psi}'_+(\upsilon,0) + e^{-i\upsilon l}\overline{\Psi}'_-(\upsilon,0) + \frac{k_{1z}}{\sqrt{2\pi}(k_{1x}+\upsilon)}[-1+\exp[-i(k_{1x}+\upsilon)l]]$$

$$+\frac{\upsilon\delta_1}{\kappa\, L(\upsilon)}\{\overline{\Psi}_+(\upsilon,0) + e^{-i\upsilon l}\overline{\Psi}_-(\upsilon,0) - \frac{k_{1x}}{\sqrt{2\pi}(k_{1x}+\upsilon)}[-1+\exp[-i(k_{1x}+\upsilon)l]], \quad (53)$$

$$C(\upsilon) = -\frac{1}{i\kappa\, L(\upsilon)}\{\overline{\Psi}'_+(\upsilon,0) + e^{-i\upsilon l}\overline{\Psi}'_-(\upsilon,0) + \frac{k_{1z}}{\sqrt{2\pi}(k_{1x}+\upsilon)}[-1+\exp[-i(k_{1x}+\upsilon)l]]$$

$$+\frac{\upsilon\delta_1}{\kappa\, L(\upsilon)}\{\overline{\Psi}_+(\upsilon,0) + e^{-i\upsilon l}\overline{\Psi}_-(\upsilon,0) - \frac{k_{1x}}{\sqrt{2\pi}(k_{1x}+\upsilon)}[-1+\exp[-i(k_{1x}+\upsilon)l]], \quad (54)$$

where $\delta_1 = \frac{1}{\delta}$. In [31], the terms of $O(\delta_1)$ are neglected while in the present analysis the δ_1 parameter is taken up to order one so that the results due to semi infinite barrier [10] can be recovered by taking an appropriate limit. To accomplish this, we have to solve both the Wiener-Hopf equations to find the values of unknown functions $A(\upsilon)$ and $C(\upsilon)$. For this we use Eqs. ($52a$) and ($52b$) in Eqs.(50) and (51), which gives

$$\overline{\Psi}'_+(\upsilon,0)+e^{-i\upsilon l}\overline{\Psi}'_-(\upsilon,0)+S(\upsilon)J_1(\upsilon)=\frac{k_{1z}}{\sqrt{2\pi}(k_{1x}+\upsilon)}[1-\exp[-i(k_{1x}+\upsilon)l]], \quad (55)$$

and

$$-i\upsilon\overline{\Psi}_+(\upsilon,0)-i\upsilon e^{-i\upsilon l}\overline{\Psi}_-(\upsilon,0)+\delta L_+(\upsilon)L_-(\upsilon)J_1'(\upsilon)=\frac{k_{1x}}{\sqrt{2\pi}(k_{1x}+\upsilon)}[1-\exp[-i(k_{1x}+\upsilon)l]], \quad (56)$$

where

$$S(\upsilon) = -i\kappa(\upsilon)L(\upsilon) = S_+(\upsilon)S_-(\upsilon), \quad (57)$$

and $S_+(\upsilon)$ and $S_-(\upsilon)$ are regular in upper and lower half plane, respectively. Equations of types (55) and (56) have been considered by Noble [29] and a similar analysis may be employed to obtain an approximate solution for large $k_{1xz}r$ ($r = \sqrt{x^2+z^2}$). Thus, following the procedure given in [29] (Sec. 5.5, p. 196), we obtain

$$\overline{\Psi}'_+(\upsilon,0) = \frac{k_{1z}S_+(\upsilon)}{\sqrt{2\pi}}[G_1(\upsilon)+T(\upsilon)C_1], \quad (58)$$

$$\overline{\Psi}'_-(\upsilon,0) = \frac{k_{1z}S_-(\upsilon)}{\sqrt{2\pi}}[G_2(-\upsilon)+T(-\upsilon)C_2], \quad (59)$$

$$\overline{\Psi}_+(\upsilon,0) = \frac{iL_+(\upsilon)}{\sqrt{2\pi}\upsilon}[G_1'(\upsilon)+T(\upsilon)C_1'], \quad (60)$$

and

$$\overline{\Psi}_-(\upsilon,0) = \frac{-iL_-(\upsilon)}{\sqrt{2\pi}\upsilon}[G_2'(-\upsilon)-T(-\upsilon)C_2'], \quad (61)$$

where

$$S_+(\upsilon) = (k_{1xz}+\upsilon)^{\frac{1}{2}}L_+(\upsilon), \quad (62a)$$

#96994 - $15.00 USD Received 10 Jun 2008; revised 4 Jul 2008; accepted 18 Jul 2008; published 13 Aug 2008

and

$$S_{-}(\upsilon) = e^{\frac{-i\pi}{4}}(k_{1xz} - \upsilon)^{\frac{1}{2}} L_{-}(\upsilon), \tag{62b}$$

$$G_1(\upsilon) = \frac{1}{(\upsilon + k_{1x})}\left[\frac{1}{S_+(\upsilon)} - \frac{1}{S_+(-k_{1x})}\right] - e^{-ilk_{1x}} R_1(\upsilon), \tag{63}$$

$$G_2(\upsilon) = \frac{e^{-ilk_{1x}}}{(\upsilon - k_{1x})}\left[\frac{1}{S_+(\upsilon)} - \frac{1}{S_+(k_{1x})}\right] - R_2(\upsilon), \tag{64}$$

$$C_1 = S_+(k_{1xz})\left[\frac{G_2(k_{1xz}) + S_+(k_{1xz})G_1(k_{1xz})\,T(k_{1xz})}{1 - S_+^2(k_{1xz})T^2(k_{1xz})}\right], \tag{65}$$

$$C_2 = S_+(k_{1xz})\left[\frac{G_1(k_{1xz}) + S_+(v)G_2(k_{1xz})\,T(k_{1xz})}{1 - S_+^2(k_{1xz})T^2(k_{1xz})}\right], \tag{66}$$

$$G_1'(\upsilon) = \frac{\upsilon}{(\upsilon + k_{1x})}\left[\frac{1}{L_+(\upsilon)} - \frac{1}{L_+(-k_{1x})}\right] - e^{-ilk_{1x}} R_1(\upsilon), \tag{67}$$

$$G_2'(\upsilon) = \frac{e^{-ilk_{1x}}}{(\upsilon - k_{1x})}\left[\frac{\upsilon}{L_+(\upsilon)} + \frac{k_{1x}}{L_+(k_{1x})}\right] - R_2(\upsilon), \tag{68}$$

$$C_1' = L_+(k_{1xz})\left[\frac{G_2'(k_{1xz}) + L_+(k_{1xz})G_1'(k_{1xz})\,T(k_{1xz})}{1 - L_+^2(k_{1xz})T^2(k_{1xz})}\right], \tag{69}$$

$$C_2' = L_+(k_{1xz})\left[\frac{G_1'(k_{1xz}) + L_+(k_{1xz})G_2'(k_{1xz})\,T(k_{1xz})}{1 - L_+^2(k_{1xz})T^2(k_{1xz})}\right], \tag{70}$$

$$R_{1,2}(\upsilon) = \frac{E_{-1}[W_{-1}\{-i(k_{1xz} \mp k_{1x})l\} - W_{-1}\{-i(k_{1xz} + \upsilon)l\}]}{2\pi i(\upsilon \pm k_{1x})}, \tag{71}$$

$$T(\upsilon) = \frac{1}{2\pi i} E_{-1} W_{-1}\{-i(k_{1xz} + \upsilon)l\}, \tag{72}$$

$$E_{-1} = 2e^{i\frac{\pi}{4}} e^{ik_{1xz}l}\,(l)^{\frac{1}{2}}\,(i)^{-1} h_{-1}, \tag{73}$$

and

$$W_{n-\frac{1}{2}}(p) = \int_0^\infty \frac{u^n e^{-u}}{u+p}\,du = \Gamma(n+1)e^{\frac{1}{2}} p^{\frac{1}{2}n-\frac{1}{2}} W_{-\frac{1}{2}(n+1),\frac{1}{2}n}(p), \tag{74}$$

where $p = -i(k_{1xz} + \upsilon)l$ and $n = \frac{-1}{2}$. $W_{m,n}$ is known as a Whittaker function.

Now, making use of Eqs. $(58-61)$ in Eqs. (53) and (54), we get

$$\left.\begin{array}{c} A(\upsilon) \\ C(\upsilon) \end{array}\right\} = \frac{k_{1z}sgn(z)}{\sqrt{2\pi}\, i\kappa L(\upsilon)}\left\{\begin{array}{c} S_+(\upsilon)G_1(\upsilon) + S_+(\upsilon)T(\upsilon)C_1 + e^{-i\upsilon l}S_-(\upsilon) \\ \times[G_2(-\upsilon) + T(-\upsilon)C_2] - \frac{(1-e^{-il(k_{1x}+\upsilon)})}{(k_{1x}+\upsilon)} \end{array}\right\}$$

$$+\frac{\upsilon\delta_1}{\sqrt{2\pi}\kappa L(\upsilon)}\left\{\begin{array}{c} L_+(\upsilon)G_1'(\upsilon) + T(\upsilon)L_+(v)C_1' + e^{-i\upsilon l} \\ \times[(L_-(\upsilon)G_2'(-\upsilon) + T(-\upsilon)L_+(v)C_2')] - \frac{\left(1-e^{-il(k_{1x}+\upsilon)}\right)}{(k_{1x}+\upsilon)} \end{array}\right\}, \tag{75}$$

where $A(v)$ corresponds to $z > 0$ and $C(v)$ corresponds to $z < 0$. We can see that the second term in the above equation was altogether missing in Eq. (70) of [31]. This term includes the effect of δ_1 parameter in it which can be seen from the solution also. Now, $Q_{1y}^{sca}(x,z)$ can be obtained by taking the inverse Fourier transform of Eq. (43). Thus

$$Q_{1y}^{sca}(x,z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{\begin{array}{c} A(\upsilon) \\ C(\upsilon) \end{array}\right\}\exp(i\kappa|z| - i\upsilon x)\,d\upsilon, \tag{76}$$

where $A(\upsilon)$ and $C(\upsilon)$ are given by Eq. (75). Substituting the value of $A(\upsilon)$ and $C(\upsilon)$ from Eq. (75) into Eq. (76) and using the approximations $(63-70)$, one can break up the field $\Psi'(x,z)$ into two parts

$$Q_{1y}^{sca}(x,z)=\Psi^{sep}(x,z)+\Psi^{int}(x,z), \tag{77}$$

where

$$\Psi^{sep}(x,z)=-\frac{k_{1z}sgn(z)}{2\pi}\int_{-\infty}^{\infty}\frac{S_+(\upsilon)\exp(i\kappa|z|-i\upsilon x)}{i\kappa L(\upsilon)S_+(-k_{1x})(k_{1x}+\upsilon)}d\upsilon$$
$$+\frac{k_{1z}sgn(z)}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-il(k_{1x}+\upsilon)}S_-(\upsilon)\exp(i\kappa|z|-i\upsilon x)}{i\kappa L(\upsilon)S_+(k_{1x})(k_{1x}+\upsilon)}d\upsilon$$
$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\delta_1 e^{-il(k_{1x}+\upsilon)}\exp(i\kappa|z|-i\upsilon x)}{\kappa L(\upsilon)(k_{1x}+\upsilon)}d\upsilon+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{L_-(\upsilon)e^{-il(k_{1x}+\upsilon)}\exp(i\kappa|z|-i\upsilon x)}{\kappa L(\upsilon)(k_{1x}+\upsilon)L_+(k_{1x})}d\upsilon$$
$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\delta_1\exp(i\kappa|z|-i\upsilon x)}{\kappa L(\upsilon)(k_{1x}+\upsilon)}d\upsilon, \tag{78}$$

and

$$\Psi^{int}(x,z)=-\frac{k_{1z}sgn(z)}{2\pi}\int_{-\infty}^{\infty}\frac{1}{i\kappa L(\upsilon)}\Big[S_+(\upsilon)R_1(\upsilon)e^{-ilk_{1x}}-C_1S_+(\upsilon)T(\upsilon)$$
$$+S_+(-\upsilon)e^{-il\upsilon}R_2(-\upsilon)-C_2T(-\upsilon)S_+(-\upsilon)e^{-il\upsilon}\Big]\exp(i\kappa|z|-i\upsilon x)d\upsilon$$
$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\delta_1}{\kappa L(\upsilon)}\Big[T(\upsilon)L_+(\upsilon)C_1'+T(-\upsilon)L_-(\upsilon)C_2'-L_+(\upsilon)R_1(\upsilon)e^{-ilk_{1x}}$$
$$-L_-(\upsilon)R_2(-\upsilon)e^{-il\upsilon}\Big]\exp(i\kappa|z|-i\upsilon x)d\upsilon. \tag{79}$$

Here, $\Psi^{sep}(x,z)$ consists of two parts each representing the diffracted field produced by the edges at $x=0$ and $x=-l$, respectively, although the other edge were absent while $\Psi^{int}(x,z)$ gives the interaction of one edge upon the other.

4. Far field solution

The far field may now be calculated by evaluating the integrals appearing in Eqs. (76), (78) and (79), asymptotically [32]. For that we put $x=r\cos\vartheta$, $|z|=r\sin\vartheta$ and deform the contour by the transformation $\upsilon=-k_{1xz}\cos(\vartheta+i\xi)$, $(0<\vartheta<\pi, -\infty<\xi<\infty)$. Hence, for large $k_{1xz}r$, Eqs. (76), (78) and (79) become

$$Q_{1y}^{sca}(x,z)=\frac{ik_{1xz}}{\sqrt{2\pi}}\left(\frac{\pi}{2k_{1xz}r}\right)^{\frac{1}{2}}\left\{\begin{array}{c}A(-k_{1xz}\cos\vartheta)\\ C(-k_{1xz}\cos\vartheta)\end{array}\right\}\sin(\vartheta)\exp(ik_{1xz}r+i\frac{\pi}{4}), \tag{80}$$

$$Q_{1y}^{sca(sep)}(x,z)=-[ik_{1z}sgn(z)f_1(-k_{1xz}\cos\vartheta)+g_1(-k_{1xz}\cos\vartheta)]$$
$$\times\frac{1}{4\pi k_{1xz}}\left(\frac{1}{k_{1xz}r}\right)^{\frac{1}{2}}\exp(ik_{1xz}r+i\frac{\pi}{4}), \tag{81}$$

and

$$Q_{1y}^{sca(int)}(x,z)=-[ik_{1z}sgn(z)f_2(-k_{1xz}\cos\vartheta)+g_2(-k_{1xz}\cos\vartheta)]$$

$$\times \frac{1}{4\pi k_{1xz}}\left(\frac{1}{k_{1xz}r}\right)^{\frac{1}{2}}\exp(ik_{1xz}r+i\frac{\pi}{4}), \tag{82}$$

where $A(-k_{1xz}\cos\vartheta)$ and $C(-k_{1xz}\cos\vartheta)$ can be found from Eq. (75), while

$$f_1(-k_{1xz}\cos\vartheta)=\frac{S_+(-k_{1xz}\cos\vartheta)}{L(-k_{1xz}\cos\vartheta)S_+(-k_{1x})(k_{1x}-k_{1xz}\cos\vartheta)} - \frac{e^{-il(k_{1x}-k_{1xz}\cos\vartheta)}S_+(k_{1xz}\cos\vartheta)}{L(-k_{1xz}\cos\vartheta)S_+(k_{1x})(k_{1x}-k_{1xz}\cos\vartheta)}, \tag{83}$$

$$g_1(-k_{1xz}\cos\vartheta)=\frac{1}{(k_{1x}-k_{1xz}\cos\vartheta)}\left[\frac{\delta_1 e^{-il(k_{1x}-k_{1xz}\cos\vartheta)}}{L(-k_{1xz}\cos\vartheta)}-\frac{L_+(k_{1xz}\cos\vartheta)e^{-il(k_{1x}-k_{1xz}\cos\vartheta)}}{L(-k_{1xz}\cos\vartheta)L_+(k_{1x})} - \frac{\delta_1}{L(-k_{1xz}\cos\vartheta)}\right], \tag{84}$$

$$\begin{aligned} f_2(-k_{1xz}\cos\vartheta)=\frac{1}{L(-k_{1xz}\cos\vartheta)}\Big[&S_+(-k_{1xz}\cos\vartheta)R_1(-k_{1xz}\cos\vartheta)e^{-ilk_{1x}} \\ &+S_+(k_{1xz}\cos\vartheta)e^{ilk_{1xz}\cos\vartheta}R_2(k_{1xz}\cos\vartheta) \\ &-C_1S_+(-k_{1xz}\cos\vartheta)T(-k_{1xz}\cos\vartheta) \\ &-C_2T(k_{1xz}\cos\vartheta)S_+(k_{1xz}\cos\vartheta)e^{ilk_{1xz}\cos\vartheta}\Big], \end{aligned} \tag{85}$$

and

$$\begin{aligned} g_2(-k_{1xz}\cos\vartheta)=\frac{1}{L(-k_{1xz}\cos\vartheta)}\Big[&L_+(-k_{1xz}\cos\vartheta)R_1(-k_{1xz}\cos\vartheta)e^{-ilk_{1x}}. \\ &+L_+(k_{1xz}\cos\vartheta)R_2(k_{1xz}\cos\vartheta)e^{ilk_{1xz}\cos\vartheta} \\ &-T(-k_{1xz}\cos\vartheta)L_+(-k_{1xz}\cos\vartheta)C_1' \\ &-T(k_{1xz}\cos\vartheta)L_+(k_{1xz}\cos\vartheta)C_2'\Big]. \end{aligned} \tag{86}$$

The expressions (84) and (86) are additional terms including the effect of δ_1 parameter, which were altogether missing in the analysis of [31].

5. Remarks

Mathematically we can derive the results of the half plane problem in the following manner: For the analysis purpose, in Eq. (75), it is assumed that the wave number k_{1xz} has positive imaginary part and using the L Hospital rule successively, the value of E_{-1}, reduces to $Lt_{l\to\infty}(\frac{e^{ikl}}{\sqrt{kl}})$ which becomes zero and in turn result the quantities $T(\upsilon)$, $R_{1,2}(\upsilon)$, $G_2'(\upsilon)$, C_1' and $G_2(\upsilon)$ in zero. The third term in Eqs. (63), (64) and (67) also becomes zero as $l\to\infty$. The Eq. (75), after these eliminations reduces to

$$A(\upsilon)=\frac{1}{\sqrt{2\pi}}\left[\frac{-k_{1z}\kappa_+(\upsilon)L_+(\upsilon)}{i\kappa(\upsilon)L(\upsilon)(k_{1x}+\upsilon)\kappa_+(-k_{1x})L_+(-k_{1x})}+\frac{\delta_1\upsilon L_+(\upsilon)}{\kappa(\upsilon)L(\upsilon)(k_{1x}+\upsilon)L_+(-k_{1x})}\right]. \tag{87}$$

Using the factorization

$$L(\upsilon)=L_+(\upsilon)L_-(\upsilon),$$

and

$$\kappa(\upsilon)=\kappa_+(\upsilon)\kappa_-(\upsilon).$$

and substituting the pole contribution $\upsilon = -k_{1x}$, the above result reduces to Eq. (26a) of the Half Plane [10]. Subsequently, Eq. (82), i.e. , the interacted field vanishes by adopting the same procedure as in case of Eq. (75), while the separated field results into the diffracted field [10] as the strip is widened to half plane.

6. Graphical results

A computer program MATHEMATICA has been used for graphical plotting of the separated field given by the expression (81). The values of parameter δ_1 are taken from 0.2 to 0.4. The following situations are considered:

(i) When the source is fixed in one position (for all values of δ_1) relative to the finite barrier, ($\theta_0 = 45^o$, l and θ are allowed to vary).

(ii) When the source is fixed in one position, relative to the infinite barrier ($\theta_0 = 45^o$, l and θ are allowed to vary).

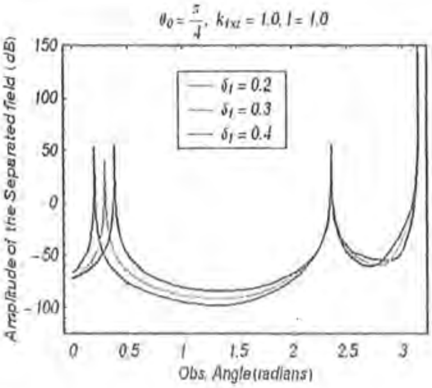

Fig. 1. Variation of the amplitude of separated field versus observation angle ϑ, for different values of δ_1 at $\theta_0 = \frac{\pi}{4}, k_{1xz} = 1, l = 1$.

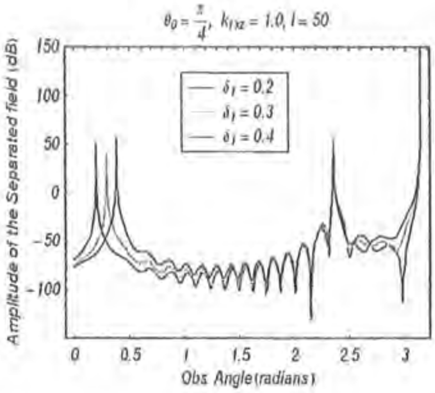

Fig. 2. Variation of the amplitude of separated field versus observation angle ϑ, for different values of δ_1 at $\theta_0 = \frac{\pi}{4}, k_{1xz} = 1, l = 50$.

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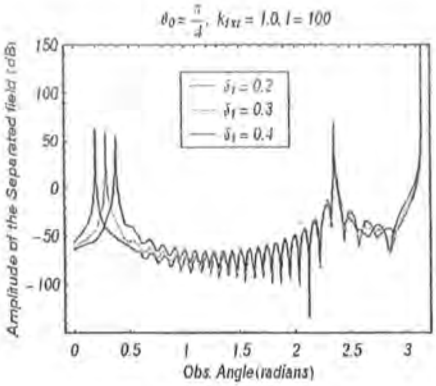

Fig. 3. Variation of the amplitude of separated field versus observation angle ϑ, for different values of δ_1 at $\theta_0 = \frac{\pi}{4}$, $k_{1xz} = 1$, $l = 100$.

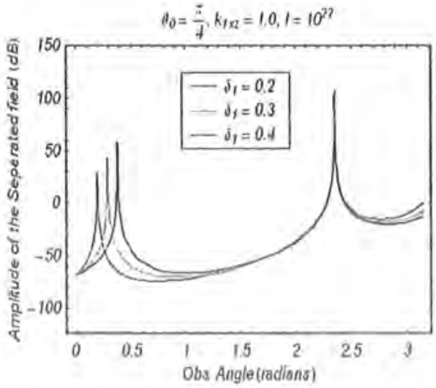

Fig. 4. Variation of the amplitude of separated field versus observation angle ϑ, for different values of δ_1 at $\theta_0 = \frac{\pi}{4}$, $k_{1xz} = 1$, $l = 10^{22}$.

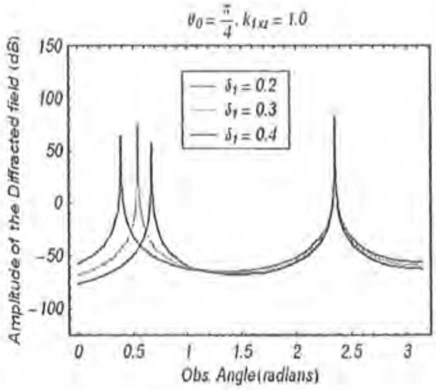

Fig. 5. Variation of the amplitude of diffracted field in the half plane versus observation angle ϑ, for different values of δ_1 at $\theta_0 = \frac{\pi}{4}$, $k_{1xz} = 1$.

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For all the situations, $\theta_0 = 45^o$, the graphs (1), (2), (3), (4) and (5) show that the field, in the region $0 < \theta \leq \pi$, is most affected by the changes in δ_1, l and k_{1xz}. The main features of the graphical results, some of which can be seen in graphs (1), (2), (3), (4) and (5) are as follows:

(a) In graphs (1), (2) and (3) by increasing the value of strip length l and δ_1, the number of oscillations increases and the amplitude of the separated field decreases, respectively.

(b) The graphs of the diffracted field corresponding to the half plane is given in fig. (5). It is observed that the figs. (1)-(4) are in comparison with fig. (5) for various values of the different parameters.

7. Conclusion

The diffracted field due to a plane wave by a perfectly conducting finite strip in a homogeneous bi-isotropic medium is obtained in an improved form. It is found that the two edges of the strip give rise to two diffracted fields (one from each edge) and an interaction field (double diffraction of two edges). This seems to be the first attempt in this direction as we can deduce the results of half plane [10] by taking an appropriate limit. In [31], the δ parameter was not taken into account which ends up in an equation from which one cannot deduce the results for semi infinite barrier [10]. This has been proved mathematically as well as numerically which can be considered as check of the validity of the analysis in this paper. Thus, the new solution can be regarded as a correct solution for a perfectly conducting barrier.

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