ON FUZZY ORDERED SEMIGROUPS

By

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Certificate

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We accept this thesis as confirming to the required standard.

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Asghar Khan *Islamabad*
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DEDICATED

To

My Parents

Whose prayers have always been a source of strength for me

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PREFACE

The notion of ideals created by Dedekind for the theory of algebraic numbers, was generalized by Emmy Noether for associative rings. The one- and two-sided ideals introduced by her, are still central concepts in ring theory. Since then many papers on ideals for rings and semigroups appeared showing the importance of the concept [A. H. Clifford, L. M. Gluskin, M. P. Schützenberger, S. Lajos, K. Iséki and many others]. Further generalization of ideals by lattice-theoretical methods was given by G. Birkhoff, O. Steinfeld and N. Kehayopulu.

In 1965, Lotfi A. Zadeh introduced the notion of a fuzzy subset of a set as a method for representing uncertainty. It provoked, at first (and as expected), a strong negative reaction from some influential scientists and mathematicians—many of whom turned openly hostile. However, despite the controversy, the subject also attracted the attention of other mathematicians and in the following years, the field grew enormously, finding applications in areas as diverse as washing machines to handwritting recognition. In its trajectory of stupendous growth, it has also come to include the theory of fuzzy algebra and for the past several decades, several researchers have been working on concepts like fuzzy semigroup, fuzzy groups, fuzzy rings, fuzzy modules, fuzzy semirings, fuzzy near-rings and so on. The concepts of fuzzy one- and two-sided ideals in groupoids have been introduced by A. Rosenfeld in [129]. Fuzzy ideals in semigroups have been first studied by N. Kuroki [110−115], later by other authors as well [1,2,3,118,124,131,142,143,146,147,]. Fuzzy ideals in ordered groupoids/ordered semigroups have been introduced by Kehayopulu and Tsingelis in [30]. For a recent work on fuzzy ideals in ordered semigroups see also [24,25,26,27,28,106,132,133,144,145,148]. On the otherhand, Murali [122] proposed a definition of a fuzzy point belonging to a fuzzy subset under a natural equivalence on a fuzzy subset. The idea of a quasicoincidence of a fuzzy point with a fuzzy set, which is mentioned in [7], played a vital role to generate some different types of fuzzy subgroups. It is worth pointing out that Bhakat and Das [8,9,10] gave the concepts of (α, β)-fuzzy subgroups by using the "belongs to" relation ($\in$) and "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroup. In particular, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems of other algebraic structures. With this objective in view, Jun and Song in [19] discussed general forms of fuzzy interior ideals in semigroups. Also, Jun introduced the concept of (α, β)-fuzzy subalgebra of a BCK/BCI-algebra and investigated related results.

This work in algebra is concerned with the fuzzy approach to study some algebraic properties of ordered semigroups in the context of fuzzy subsets. Our approach of study is based on the following points:

Our first approach is to use the fuzzy ideals (left and right) to study the basic properties of some classes (left/right simple, left/right regular and completely regular) ordered semigroups.

Secondly, we use the fuzzy quasi-ideals and study the basic properties of left/right simple, regular and left/right regular ordered semigroups.

Thirdly, we give the concept of fuzzy generalized bi-ideals and study the basic properties of some classes of ordered semigroups. Furthermore, we give some characterizations of different classes of ordered semigroups in terms of fuzzy generalized bi-ideals.

Our fourth objective in this project is to define right pure fuzzy ideals in ordered semigroups and to give the basic properties of this structure by the use of right pure fuzzy ideals. We also give the concept of right pure fuzzy prime ideals and discuss ordered semigroups S in terms of this notion.

The fifth approach is to define prime fuzzy bi-ideals in ordered semigroups and to investigate the basic properties of S, we mainly study those ordered semigroups for which the fuzzy bi-ideals form a chain. We also provide the concept of fuzzy bi-filters, fuzzy bi-ideal subsets and fuzzy prime bi-ideal subsets and study the relation of prime fuzzy bi-ideals and fuzzy bi-filters in ordered semigroups.

Our sixth approach is to provide the characterizations of different classes (regular, intra-regular and right weakly regular) ordered semigroups in terms of fuzzy (left/right) ideals, fuzzy generalized bi-ideals, fuzzy quasi-ideals and fuzzy bi-ideals.

The seventh aim of our study is to define generalized fuzzy ideals and generalized fuzzy bi-ideals and to give some interesting characterization theorems of ordered semigroups in terms of these notions.

Chapterwise study

Throughout this thesis, which contains five chapters, S will denote an ordered semigroup, unless otherwise stated. Chapter one, which is of introductory nature provides basic definitions and reviews of some of the background materials which are needed for subsequent chapters. In chapter two, we give some basic properties of fuzzy left (resp. right) ideals and characterize those ordered semigroups which are semilattice of left (resp. right) simple semigroups, we discuss left(resp. right) regular ordered semigroups in terms of fuzzy left (resp. right) ideals. In this chapter, we give some basic properties of fuzzy quasi-ideals and characterize regular, left and right simple ordered semigroups in terms of fuzzy quasi-ideals. We also give the characterization of ordered semigroups in terms of semiprime fuzzy quasi-ideals. We provide charaterizations of semilattices of ordered semigroups in terms of fuzzy quasi-ideals. We define fuzzy generalized bi-ideals of ordered semigroups and characterize some classes in terms of fuzzy generalized bi-ideals. We also characterize different classes of ordered semigroups in terms of fuzzy ideals (resp. fuzzy bi-ideals, fuzzy quasi-ideals and fuzzy interior ideals). In chapter three, we define prime (resp. semiprime) bi-ideals of ordered semigroups, we give some basic properties of prime (resp. semiprime) bi-ideals, we also define prime(resp. semiprime) fuzzy bi-ideals and give some basic properties of prime(resp. semiprime) fuzzy bi-ideals. In this chapter, we also define fuzzy bi-filters (resp. fuzzy left, fuzzy right filters and fuzzy bi-ideal subsets) and give the relations of these notions. In chapter four, we define right pure (fuzzy) ideals and provide the main theorems of ordered semigroups in terms of right pure (fuzzy) ideals. In chapter five, we define generalized fuzzy left (resp. right) ideals and generalized fuzzy bi-ideals and discuss the main characterizations of ordered semigroups in terms of generalized fuzzy left (resp. right and bi-) ideals.

Research Profile

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Chapter 1

PRELIMINARIES

In this introductory chapter we shall define basic concepts of ordered semigroups and fuzzy ordered semigroups and review some of the background materials that will be of value for our later persuits. The main results of this chapter are taken from [25], [26], [27], [28], [30], and [106].

1.1 Basic Concepts in Ordered Semigroups

By an *ordered semigroup* (or *po-semigroup*) we mean a *structure* $(S, \cdot, \leq)$ in which the following are satisfied:
$(OS1)$ $(S, \cdot)$ is a semigroup,
$(OS2)$ $(S, \leq)$ is a poset,
$(OS3)$ $a \leq b \longrightarrow ax \leq bx$ and $xa \leq xb$ for all $a, b, x \in S$.

For $A \subseteq S$, we denote $(A] := \{t \in S | t \leq h$ for some $h \in A\}$. If $A = \{a\}$, then we write $(a]$ instead of $(\{a\}]$. For $A, B \subseteq S$, we denote,

$$AB := \{ab | a \in A, b \in B\}.$$

A non-empty subset A of an ordered semigroup S is called a subsemigroup of S if $A^2 \subseteq A$.

1.1.1 Lemma (cf. [49,50,51,52]).

Let A, B be subsets of an ordered semigroup S, then

(1) $A \subseteq (A]$ for all $A \subseteq S$.
(2) If $A \subseteq B \subseteq S$ then $(A] \subseteq (B]$ for all $A, B \subseteq S$.
(3) $(A](B] \subseteq (AB]$ for all $A, B \subseteq S$.
(4) $((A]] = (A]$ for all $A \subseteq S$.
(5) $((A](B]] = (AB]$ for all $A, B \subseteq S$.
(6) If A is an ideal (resp. quasi, bi, interior) -ideal, then $(A] = A$.

1.1.2 Definition (cf. [45,47,48]).

A non-empty subset A of an ordered semigroup S is called a $right$ (resp. $left$) ideal of S if:
(i) $AS \subseteq A$ (resp. $SA \subseteq A$) and
(ii) $(\forall a \in A)(\forall b \in S)(b \leq a \longrightarrow b \in A)$.

If A is both a right and a left ideal of S, then it is called an *ideal* of S.

1.1.3 Definition (cf. [53]).

A non-empty subset A of S is called a *quasi-ideal* of S if:
(i) $(AS] \cap (SA] \subseteq A$.
(ii) $(\forall a \in A)(\forall b \in S)(b \leq a \longrightarrow b \in A)$.

1.1.4 Definition (cf. [53]).

A subsemigroup A of S is called a *bi-ideal* of S if:
(i) $ASA \subseteq A$.
(ii) $(\forall a \in A)(\forall b \in S)(b \leq a \longrightarrow b \in A)$.

1.1.5 Definition (cf. [13]).

A nonempty subset B of S is called a *bi-ideal subset* of S if
(i) $a \in B$, $x \in S \longrightarrow axa \in B$;
(ii) $(\forall a \in B)(\forall b \in S)(b \leq a \longrightarrow b \in B)$.

1.1.6 Definition (cf. [42]).

A nonempty subset A of S is called an interior ideal of S if
(i) $SAS \subseteq A$.
(ii) $(\forall a \in A)(\forall b \in S)(b \leq a \longrightarrow b \in A)$.

1.1.7 Definition (cf. [77]).

A subsemigroup F of S is called a *filter* of S if:
(i) $(\forall a, b \in S)(ab \in F \longrightarrow a \in F$ and $b \in F)$.
(ii) $(\forall c \in S)(c \geq a \in F \longrightarrow c \in F)$.

Obviously every ideal is an interior ideal but the converse is not true. Also every one-sided ideal is a quasi-ideal, every quasi-ideal is a bi-ideal and every bi-ideal is a bi-ideal subset of S, but the converse is not true. Also every bi-ideal is a generalized bi-ideal of S, but the converse is not true.

1.1.8 Definition (cf. [27,92]).

Let $(S, \cdot, \leq)$ be an ordered semigroup and $\emptyset \neq B \subseteq S$. Then B is called a *prime subset* of S if

$$a, b \in S, ab \in B \text{ implies } a \in B \text{ or } b \in B.$$

Equivalent definition. $A, C \subseteq S$, $AC \subseteq B$ implies $A \subseteq B$ or $C \subseteq B$.

Let B be a bi-ideal subset of S. Then B is called a *prime bi-ideal subset* of S if B is a prime subset of S. A left ideal B of S is called a prime left ideal of S if B is a prime subset of S. B is called a *semiprime subset* if $a \in S$, $a^2 \in B$ implies $a \in B$.

Equivalently, $A \subseteq S$, $A^2 \subseteq B$ implies $A \subseteq B$. B is called a *semiprime bi-ideal subset* (resp. *left ideal*) of S if B is a bi-ideal subset (resp. left ideal) of S.

We denote by $I(a)$ (resp. $B(a)$, $L(a)$ and $Q(a)$) the ideal, (resp. the bi-ideal, the left ideal, and the quasi-ideal) of S generated by a ($a \in S$) respectively. We have

$$\begin{aligned} I(a) &= (a \cup Sa \cup aS \cup SaS], \\ L(a) &= (a \cup Sa], \\ B(a) &= (a \cup a^2 \cup aSa] \\ \text{and } Q(a) &= (a \cup (Sa \cap aS)] \text{ (see [26,53,77]).} \end{aligned}$$

An ordered semigroup $(S, \cdot, \leq)$ is called *regular* (see $[47-52]$) if for every $a \in S$ there exists $x \in S$, such that $a \leq axa$ or, equivalently if $a \in (aSa]$ for all $a \in S$, and $A \subseteq (ASA]$ for all $A \subseteq S$. An ordered semigroup S is called *intra-regular* (see [55]) if for every $a \in S$ there exist $x, y \in S$, such that $a \leq xa^2y$ or, equivalently, if $a \in (Sa^2S]$ for all $a \in S$, and $A \subseteq (SA^2S]$ for all $A \subseteq S$. An ordered semigroup S is called *left* (resp. *right*) *regular* (see [48, 49]) if for every $a \in S$ there exists $x \in S$, such that $a \leq xa^2$ (resp. $a \leq a^2x$) or, equivalently $a \in (Sa^2]$ (resp. $a \in (a^2S]$) for every $a \in S$ and $A \subseteq (SA^2]$ (resp. $A \subseteq (A^2S]$) for all $A \subseteq S$. An ordered semigroup S is called *completely regular* if it is left regular, right regular and regular (see [45]). For $x \in S$, we denote by $N(x)$ the filter of S generated by x (that is the smallest filter with respect to inclusion relation containing x). $\mathcal{N}$ denotes the equivalence relation on S defined by $\mathcal{N} := \{(x, y) \in S \times S | N(x) = N(y)\}$ (see [94]). Let S be an ordered semigroup. An equivalence relation σ on S is called *congruence* if $(a, b) \in \sigma$ implies $(ac, bc) \in \sigma$ and $(ca, cb) \in \sigma$ for every $c \in S$. A congruence σ on S is called *semilattice congruence* if $(a^2, a) \in \sigma$ and $(ab, ba) \in \sigma$ for each $a, b \in S$ (see [94]). If σ is a semilattice congruence on S then the σ-class $(x)_\sigma$ of S containing x is a subsemigroup of S for every $x \in S$. An ordered semigroup S is called a *semilattice of left and right simple semigroups* if there exists a semilattice congruence σ on S such that the *σ-class* $(x)_\sigma$ of S containing x is a left and right simple subsemigroup of S for every $x \in S$ or, equivalently, there exists a semilattice Y and a family $\{S_\alpha\}_{\alpha \in Y}$ of left and right simple subsemigroups of S such that

(i) $S_\alpha \cap S_\beta = \emptyset$ for all $\alpha, \beta \in Y, \quad \alpha \neq \beta$,

(ii) $S = \bigcup_{\alpha \in Y} S_\alpha$,

(iii) $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for all $\alpha, \beta \in Y$.

1.1.9 Lemma (cf. [28, Lemma 3]).

An ordered semigroup S is left (resp. right) simple if and only if $(Sa] = S$ (resp. $(aS] = S$) for every $a \in S$.

1.1.10 Lemma (cf. [90]).

An ordered semigroup S is completely regular if and only if $A \subseteq (A^2SA^2]$ for every $A \subseteq S$ or, equivalently, if $a \in (a^2Sa^2]$ for every $a \in S$.

1.1.11 Lemma (cf. [28, Lemma 6]).

Let S be an ordered semigroup. Then the following are equivalent:

(i) $(x)_{\mathcal{N}}$ is a left (resp. right) simple subsemigroup of S, for every $x \in S$.

(ii) Every left(resp. right) ideal is a right(resp. left) ideal of S and semiprime.

1.1.12 Lemma (cf. [28, Lemma 8]).

An ordered semigroup S is a semilattice of left and right simple semigroups if and only for all bi-ideals A, B of S, we have

$$(A^2] = A \text{ and } (AB] = (BA].$$

1.1.13 Lemma (cf. [26]).

Let $(S, \cdot, \leq)$ be an ordered semigroup. Then S is regular if and only if for every right ideal R and every left ideal L of S we have, $R \cap L = (RL]$, equivalently, $R \cap L \subseteq (RL]$.

1.1.14 Lemma (cf. [45]).

Let $(S, \cdot, \leq)$ be an ordered semigroup. The following are equivalent:

(1) S is intra-regular.

(2) $R \cap L \subseteq (LR]$ for every right ideal R and every left ideal L of S.

(3) $R(a) \cap L(a) \subseteq (L(a)R(a)]$ for every $a \in S$.

1.2 Fuzzy ideals

In this section, we give the definitions and results of ordered semigroups in terms of fuzzy left (resp. right) ideals. The results given here are taken from [30].

Let $(S, \cdot, \leq)$ be an ordered semigroup. By a *fuzzy subset* f of S, we mean a mapping $f : S \longrightarrow [0, 1]$.

If f and g are fuzzy subsets of S then the fuzzy subsets $f \wedge g$ anf $f \vee g$ are defined as

$$\begin{aligned}(f \wedge g)(x) &= f(x) \wedge g(x)\\ (f \vee g)(x) &= f(x) \vee g(x) \text{ for all } x \in S.\end{aligned}$$

More generally, if $\{f_i : i \in I\}$ is a family of fuzzy subsets of S, then $\bigwedge_{i\in I} f_i$ and $\bigvee_{i\in I} f_i$ are defined as

$$\left(\bigwedge_{i\in I} f_i\right)(x) = \bigwedge_{i\in I} f_i(x)$$
$$\left(\bigvee_{i\in I} f_i\right)(x) = \bigvee_{i\in I} f_i(x).$$

If $(S,\cdot,\leq)$ is an ordered semigroup and $A \subseteq S$, the *characteristic function* f_A of A is a fuzzy subset of S, defined as follows:

$$f_A : S \longrightarrow [0,1],\ x \longmapsto f_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

1.2.1 Definition (cf. [30, Definition 1]).

Let $(S,\cdot,\leq)$ be an ordered semigroup and f a fuzzy subset of S. Then f is called a *fuzzy subsemigroup* of S if

$$(\forall x, y \in S)(f(xy) \geq \min\{f(x), f(y)\}).$$

1.2.2 Definition (cf. [28, Definition 1]).

Let $(S,\cdot,\leq)$ be an ordered semigroup. A fuzzy subset f of S is called a *fuzzy left* (resp. *right*) ideal of S if:

(1) $(\forall x, y \in S)(x \leq y \longrightarrow f(x) \geq f(y))$.

(2) $(\forall x, y \in S)(f(xy) \geq f(y)$ (resp. $f(xy) \geq f(x)))$.

If f is both a fuzzy left ideal and a fuzzy right ideal of S, then f is called a *fuzzy ideal* of S or a fuzzy two sided ideal of S.

Equivalently:

(1) $(\forall x, y \in S)(x \leq y \longrightarrow f(x) \geq f(y))$.

(2) $(\forall x, y \in S)(f(xy) \geq f(x) \vee f(y))$.

1.2.3 Lemma (cf. [30, Remark 1]).

If $(S,\cdot,\leq)$ is an ordered semigroup and $\emptyset \neq A \subseteq S$, the characteristic mapping $f_{(A]}$ of $(A]$ is a fuzzy subset of S satisfying the condition

$$(\forall x, y \in S)(x \leq y \longrightarrow f_{(A]}(x) \geq f_{(A]}(y)).$$

1.2.4 Lemma (cf. [30, Proposition 1]).

Let $(S,\cdot,\leq)$ be an ordered semigroup and $\emptyset \neq A \subseteq S$. Then $A = (A]$ if and only if the fuzzy subset f_A of S has the property

$$(\forall x, y \in S)(x \leq y \longrightarrow f_A(x) \geq f_A(y)).$$

1.2.5 Lemma (cf. [30, Proposition 2]).

Let $(S,\cdot,\leq)$ be an ordered semigroup and $\emptyset \neq A \subseteq S$. Then A is a subsemigroup (resp. left or right) ideal of S if and only if f_A is a fuzzy subsemigroup (resp. left or right) ideal of S.

For a fuzzy subset f of S and $t \in (0,1]$, the set

$$U(f;t) = \{x \in S | f(x) \geq t\}$$

is called the *level subset* of f.

1.2.6 Lemma (cf. [30]).

Let $(S,\cdot,\leq)$ be an ordered semigroup. A fuzzy subset f of S is a fuzzy left (resp. right) ideal of S if and only if for every $t \in (0,1]$, $U(f;t) \neq \emptyset$ is a left (resp. right) ideal of S.

1.2.7 Example (cf. [55]).

Let $S = \{a, b, c, d, e, f\}$ be an ordered semigroup defined by the multiplication and the order below:

$\cdot$	a	b	c	d	e	f
a	a	a	a	d	a	a
b	a	b	b	d	b	b
c	a	b	c	d	e	e
d	a	a	d	d	d	d
e	a	b	c	d	e	e
f	a	b	c	d	e	f

$$\leq := \{(a,a),(b,b),(c,c),(d,d),(e,e),(f,e),(f,f)\}$$

Right ideals of S are: $\{a,d\}, \{a,b,d\}$ and S. Left ideals of S are: $\{a\}$, $\{d\}$, $\{a,b\}$, $\{a,d\}$, $\{a,b,d\}$, $\{a,b,c,d\}$, $\{a,b,d,e,f\}$ and S.

Define $f : S \longrightarrow [0,1]$ by $f(a) = 0.8,\ f(b) = 0.5,\ f(d) = 0.6,\ f(c) = f(e) = f(f) = 0.4$.

Then

$$U(f;t) := \begin{cases} S \text{ if } & t \in (0, 0.4] \\ \{a,b,d\} & \text{if } t \in (0.4, 0.5] \\ \{a,d\} & \text{if } t \in (0.5, 0.6] \\ \{a\} & \text{if } t \in (0.6, 0.8] \\ \emptyset & \text{if } t \in (0.8, 1] \end{cases}$$

and $U(f;t)$ are right ideals of S. So by Lemma 1.2.5, f is a fuzzy right ideal of S.

1.2.8 Definition (cf. [30, Definition 2]).

Let $(S,\cdot,\leq)$ be an ordered semigroup. A fuzzy subset f of S is called a *fuzzy filter* of S if

(1) $(\forall x,y\in S)(x\leq y\longrightarrow f(x)\geq f(y))$.
(2) $(\forall x,y\in S)(f(xy)=\min\{f(x),f(y)\})$.

1.2.9 Lemma (cf. [30, Proposition 4]).

Let $(S,\cdot,\leq)$ be an ordered semigroup and $\emptyset\neq F\subseteq S$. Then F is a filter of S if and only if the fuzzy subset f_F is a fuzzy filter of S.

1.2.10 Definition (cf. [30]).

Let $(S,\cdot,\leq)$ be an ordered semigroup. A nonempty fuzzy subset f of S is called a prime (resp. semiprime) fuzzy subset of S if $f(xy)\leq\max\{f(x),f(y)\}$ (resp. $f(x^2)\leq f(x)$) for all $x,y\in S$. If f is a fuzzy bi-ideal subset of S, then f is called a prime fuzzy bi-ideal subset of S if f is a prime fuzzy subset of S. A fuzzy left ideal f of S is called a prime fuzzy left ideal of S if f is a prime fuzzy subset of S.

1.2.11 Definition (cf. [30, Definition 3]).

Let S be an ordered semigroup and f a fuzzy subset of S. The mapping

$$f':S\longrightarrow[0,1]\text{ defined }f'(x)=1-f(x)\text{ for all }x\in S,$$

is a fuzzy subset of S called the *complement* of f in S.

1.2.12 Lemma (cf. [30, Proposition 5]).

Let $(S,\cdot,\leq)$ be an ordered semigroup and f a fuzzy subset of S. Then f is a fuzzy filter of S if and only if the complement f' of f is a prime fuzzy ideal of S.

1.2.13 Proposition

Let $(S,\cdot,\leq)$ be an ordered semigroup, f and g are fuzzy ideals of S. Then $f\wedge g$ is a fuzzy ideal of S.

Proof. Straightforward. □

1.2.14 Theorem

Let $(S, \cdot, \leq)$ be an ordered semigroup, $\emptyset \neq A \subseteq S$ and $s, t \in [0,1]$ such that $s < t$. Define $f : S \longrightarrow [0,1]$ by

$$f(x) := \begin{cases} t \text{ if } x \in A, \\ s \text{ if } x \notin A. \end{cases}$$

Then f is a fuzzy right (resp. left) ideal of S if and only if A is a right(resp. left) ideal of S.

Proof. $(\longrightarrow)$ Let A be a right ideal of S and $x, y \in S$ such that $x \leq y$. If $y \in A$ then $x \in A$ and so $f(x) = f(y) = t$. If $y \notin A$ then $f(y) = s \leq f(x)$. Hence $f(x) \geq f(y)$.

Let $a, b \in S$. If $a \notin A$ then $f(a) = s \leq f(ab)$. If $a \in A$ then $ab \in A$ and so $f(a) = f(ab) = t$. Hence $f(ab) \geq f(a)$. Thus f is a fuzzy right ideal of S.

$(\longleftarrow)$ Assume that f is a fuzzy right ideal of S. Let $x, y \in S$ such that $x \leq y$. If $y \in A$ then $f(y) = t \leq f(x) \longrightarrow f(x) = t \longrightarrow x \in A$.

If $a \in A$ and $b \in S$, then as $f(ab) \geq f(a) = t \longrightarrow f(ab) = t \longrightarrow ab \in A$. Hence A is a right ideal of S. □

1.3 Fuzzy bi-ideals

In this section, we give definitions and results of ordered semigroups, relating to fuzzy bi-ideals. The results of this section are taken from [28].

1.3.1 Lemma

Let $(S, \cdot, \leq)$ be an ordered semigroup. The intersection of any family of bi-ideals of S is either empty or a bi-ideal of S.

Proof. Straightforward. □

1.3.2 Lemma

Let $(S, \cdot, \leq)$ be an ordered semigroup and B_1, B_2 bi-ideals of S. Then $(B_1B_2]$ and $(B_2B_1]$ are bi-ideals of S.

Proof. Straightforward. □

1.3.3 Definition (cf. [28, Definition 2]).

A fuzzy subset f of S is called a *fuzzy bi-ideal* of S if:

(1) $(\forall x, y \in S)(x \leq y \longrightarrow f(x) \geq f(y))$.

(2) $(\forall x, y \in S)(f(xy) \geq \min\{f(x), f(z)\})$.

(3) $(\forall x, y, z \in S)(f(xyz) \geq \min\{f(x), f(z)\})$.

1.3.4 Lemma (cf. [28, Theorem 1]).

A non-empty subset A of an ordered semigroup S is a bi-ideal of S if and only if f_A is a fuzzy bi-ideal of S.

1.3.5 Lemma

Every one-sided ideal of an ordered semigroup S is a bi-ideal of S.

Proof. Starightforward. □

1.3.6 Lemma

Every fuzzy one-sided ideal of an ordered semigroup S is a fuzzy bi-ideal of S.

Proof. Starightforward. □

1.3.7 Lemma

A fuzzy subset of an ordered semigroup S is a fuzzy bi-ideal of S if and only if for every $t \in (0,1]$, $U(f;t) \neq \emptyset$ is a fuzzy bi-ideal of S.

Proof. Straightforward. □

1.3.8 Example (cf. [85,96]).

Let $S = \{a, b, c, d, f\}$ be an ordered semigroup with the following multiplication,

$\cdot$	a	b	c	d	f
a	a	a	a	a	a
b	a	b	a	d	a
c	a	f	c	c	f
d	a	b	d	d	b
f	a	f	a	c	a

We define the order "$\leq$" as follows

$$\leq := \{(a,a),(a,b),(a,c),(a,d),(a,f),(b,b),(c,c),(d,d),(f,f)\}$$

Bi-ideals of S are: $\{a\}, \{a,c\}, \{a,c,d\}$ and S. Define $f : S \longrightarrow [0,1]$ by $f(a) = 0.8$, $f(c) = 0.7$, $f(d) = 0.6$ $f(b) = f(f) = 0.5$.

Then

$$U(f;t) := \begin{cases} S & \text{if } t \in (0, 0.5] \\ \{a,c,d\} & \text{if } t \in (0.5, 0.6] \\ \{a,c\} & \text{if } t \in (0.6, 0.7] \\ \{a\} & \text{if } t \in (0.7, 0.8] \\ \emptyset & \text{if } t \in (0.8, 1] \end{cases}$$

Then $U(f;t)$ is a bi-ideal and by Lemma 1.3.8, f is a fuzzy bi-ideal of S.

1.3.9 Definition (cf. [26])

For $a \in S$, define

$$A_a := \{(y,z) \in S \times S | a \le yz\}.$$

For any two fuzzy subsets f and g of S, define

$$f \circ g : S \longrightarrow [0,1], a \longmapsto (f \circ g)(a) = \begin{cases} \bigvee\limits_{(y,z) \in A_a} \min\{f(y), g(z)\} & \text{if } A_a \neq \emptyset \\ 0 & \text{if } A_a = \emptyset \end{cases}$$

We denote by $F(S)$ the set of all fuzzy subsets of S and define the order relation " $\preceq$ " on $F(S)$ as follows:

$$f \preceq g \text{ if and only if } f(x) \le g(x) \text{ for all } x \in S.$$

Clearly $(F(S), \circ, \preceq)$ is an ordered semigroup.

For an ordered semigroup S, the fuzzy subsets "0" and "1" of S are defined as follows (see [26]):

$$0 : S \longrightarrow [0,1],\ x \longmapsto 0(x) := 0,$$

$$1 : S \longrightarrow [0,1],\ x \longmapsto 1(x) := 1.$$

Clearly, the fuzzy subset "0" (resp. "1") of S is the least (resp. the greatest) element of the ordered set $(F(S), \preceq)$. The fuzzy subset "0" is the zero element of $(F(S), \circ, \preceq)$ (that is, $f \circ 0 = 0 \circ f = 0$ and $0 \preceq f$ for every $f \in F(S)$).

1.3.10 Proposition (cf. [26]).

Let $(S, \cdot, \le)$ be an ordered semigroup, $A, B \subseteq S$. Then

(i) $f_A \circ f_B = f_{(AB]}$.
(ii) $f_A \wedge f_B = f_{A \cap B}$.
(iii) $f_A \vee f_B = f_{A \cup B}$.

1.3.11 Proposition (cf. [26]).

Let $(S, \cdot, \le)$ be an ordered semigroup and f (resp. g) a fuzzy right (resp. left) ideal of S. Then $f \circ 1 \preceq f$ (resp. $1 \circ g \preceq g$).

1.3.12 Proposition (cf. [26]).

Let S be an ordered semigroup and f (resp. g) a fuzzy right (resp. fuzzy left) ideal of S. Then $f \circ f \preceq f$ (resp. $g \circ g \preceq g$).

1.3.13 Proposition (cf. [26]).

Let $(S, \cdot, \leq)$ be a regular ordered semigroup and f (resp. g) a fuzzy right (resp. fuzzy left) ideal of S. Then $f \preceq f \circ f$ (resp. $g \preceq g \circ g$).

1.3.14 Proposition

Let S be an ordered semigroup and f a fuzzy bi-ideal of S. Then $f \circ f \preceq f$.

Proof. Let S be an ordered semigroup and f a fuzzy bi-ideal of S. Then for each $a \in S$ we have

$$(f \circ f)(a) \leq f(a).$$

In fact: If $A_a = \emptyset$, then $(f \circ f)(a) := 0 \leq f(a)$. Let $A_a \neq \emptyset$, then

$$(f \circ f)(a) := \bigvee_{(y,z) \in A_a} \min\{f(y), f(z)\}$$

As f is a fuzzy subsemigroup of S, we have $f(yz) \geq \min\{f(y), f(z)\}$ for all $y, z \in S$. As $a \leq yz$ and f is a fuzzy bi-ideal of S we have $f(a) \geq f(yz)$. Hence $f(a) \geq f(yz) \geq \min\{f(y), f(z)\}$. Thus we have

$$(f \circ f)(a) := \bigvee_{(y,z) \in A_a} \min\{f(y), f(z)\} \leq \bigvee_{(y,z) \in A_a} f(yz) \leq \bigvee_{(y,z) \in A_a} f(a) = f(a)$$

Hence $(f \circ f)(a) \leq f(a)$. □

1.3.15 Lemma

Let S be an ordered semigroup, f and g fuzzy subsets of S. Then $f \circ g \preceq f \circ 1$ (resp. $f \circ g \preceq 1 \circ g$).

Proof. Let $a \in S$. If $A_a = \emptyset$, then $(f \circ g)(a) = 0 \leq (f \circ 1)(a)$. Let $A_a \neq \emptyset$. Then

$$(f \circ g)(a) := \bigvee_{(y,z) \in A_a} \min\{f(y), g(z)\}.$$

As $g(z) \leq 1(z)$ for all $z \in S$. Thus $(f \circ g)(a) := \bigvee_{(y,z) \in A_a} \min\{f(y), g(z)\} \leq \bigvee_{(y,z) \in A_a} \min\{f(y), 1(z)\} = (f \circ 1)(a)$. Therefore $f \circ g \preceq f \circ 1$. Similarly, we can prove that $f \circ g \preceq 1 \circ g$. □

1.3.16 Propoition

Let $(S, \cdot, \leq)$ be an ordered semigroup, f a fuzzy bi-ideal of S. Then

$$f \circ 1 \circ f \preceq f.$$

Proof. Let $a \in S$. If $A_a = \emptyset$, then $(f \circ 1 \circ f)(a) := 0 \leq f(a)$. Let $A_a \neq \emptyset$, then

$$\begin{aligned}(f \circ 1 \circ f)(a) \ :&= \bigvee_{(y,z)\in A_a} \min\{f(y), (1 \circ f)(z)\} \\ &= \bigvee_{(y,z)\in A_a} \min\{f(y), \bigvee_{(p,q)\in A_z} \min\{1(p), f(q)\}\} \\ &= \bigvee_{(y,z)\in A_a} \bigvee_{(p,q)\in A_z} \min\{f(y), \min\{1, f(q)\}\} \\ &= \bigvee_{(y,z)\in A_a} \bigvee_{(p,q)\in A_z} \min\{f(y), f(q)\}.\end{aligned}$$

As $(y, z) \in A_a \longrightarrow a \leq yz$ and $(p, q) \in A_z \longrightarrow z \leq pq$. Thus $a \leq yz \leq ypq$. Since f is a fuzzy bi-ideal of S we have,

$$f(a) \geq f(ypq) \geq \min\{f(y), f(q)\}.$$

Thus we have

$$\bigvee_{(y,z)\in A_a} \bigvee_{(p,q)\in A_z} \min\{f(y), f(q)\} \leq \bigvee_{(y,z)\in A_a} \bigvee_{(p,q)\in A_z} f(a) = f(a).$$

Therefore, $(f \circ 1 \circ f)(a) \leq f(a)$. □

1.3.17 Lemma

Let S be an ordered semigroup, f and g be fuzzy bi-ideals of S. Then $f \circ g$ is a fuzzy bi-ideal of S.

Proof. Let $a \in S$ and f, g be any fuzzy bi-ideals of S. If $A_a = \emptyset$, then

$$((f \circ g) \circ (f \circ g))(a) := 0 = (f \circ g)(a).$$

Let $A_a \neq \emptyset$, then

$$\begin{aligned}
((f\circ g)\circ(f\circ g))(a) \quad &:= \bigvee_{(y,z)\in A_a} [(f\circ g)(y) \wedge (f\circ g)(z)] \\
&= \bigvee_{(y,z)\in A_a} [\bigvee_{(p_1,q_1)\in A_y} \{f(p_1)\wedge g(q_1)\} \wedge \bigvee_{(p_2,q_2)\in A_z} \{f(p_2)\wedge g(q_2)\}] \\
&= \bigvee_{(y,z)\in A_a} \bigvee_{(p_1,q_1)\in A_y} \bigvee_{(p_2,q_2)\in A_z} [\{f(p_1)\wedge g(q_1)\} \wedge \{f(p_2)\wedge g(q_2)\}] \\
&= \bigvee_{(y,z)\in A_a} \bigvee_{(p_1,q_1)\in A_y} \bigvee_{(p_2,q_2)\in A_z} [\{f(p_1)\wedge f(p_2)\wedge g(q_1)\} \wedge g(q_2)] \\
&\leq \bigvee_{(y,z)\in A_a} \bigvee_{(p_1,q_1)\in A_y} \bigvee_{(p_2,q_2)\in A_z} [\{f(p_1)\wedge f(p_2)\} \wedge g(q_2)].
\end{aligned}$$

As $(y,z)\in A_a \longrightarrow a \leq yz, (p_1,q_1)\in A_y \longrightarrow y \leq p_1q_1$ and $(p_2,q_2)\in A_z \longrightarrow z \leq p_2q_2$. Thus $a \leq yz \leq (p_1q_1)(p_2q_2) = (p_1q_1p_2)q_2$ and we have $(p_1q_1p_2, q_2)\in A_a$. Hence

$$\begin{aligned}
&\bigvee_{(y,z)\in A_a} \bigvee_{(p_1,q_1)\in A_y} \bigvee_{(p_2,q_2)\in A_z} [\{f(p_1)\wedge f(p_2)\} \wedge g(q_2)] \\
\leq &\bigvee_{(p_1q_1p_2,q_2)\in A_a} [\{f(p_1)\wedge f(p_2)\} \wedge g(q_2)].
\end{aligned}$$

Since f is a fuzzy bi-ideal of S, we have

$$f(p_1q_1p_2) \geq \min\{f(p_1), f(p_2)\}.$$

Thus

$$\begin{aligned}
&\bigvee_{(p_1q_1p_2,q_2)\in A_a} [\{f(p_1)\wedge f(p_2)\} \wedge g(q_2)] \\
\leq &\bigvee_{(p_1q_1p_2,q_2)\in A_a} [f(p_1q_1p_2) \wedge g(q_2)] \\
= &\bigvee_{(p,q)\in A_a} [f(p)\wedge g(q)] = (f\circ g)(a).
\end{aligned}$$

Therefore, $((f\circ g)\circ(f\circ g))(a) \leq (f\circ g)(a)$.

Let $x, y, z \in S$. Then

$$\begin{aligned}
(f \circ g)(x) \wedge (f \circ g)(z) &= \left[\bigvee_{(a,b)\in A_x} \{f(a) \wedge g(b)\}\right] \wedge \left[\bigvee_{(c,d)\in A_z} \{f(c) \wedge g(d)\}\right] \\
&= \bigvee_{(a,b)\in A_x} \bigvee_{(c,d)\in A_z} [\{f(a) \wedge g(b)\} \wedge \{f(c) \wedge g(d)\}] \\
&= \bigvee_{(a,b)\in A_x} \bigvee_{(c,d)\in A_z} [\{f(a) \wedge f(c)\} \wedge \{g(b) \wedge g(d)\}] \\
&= \bigvee_{(a,b)\in A_x} \bigvee_{(c,d)\in A_z} [\{f(a) \wedge f(c) \wedge g(b)\} \wedge g(d)] \\
&\leq \bigvee_{(a,b)\in A_x} \bigvee_{(c,d)\in A_z} [\{f(a) \wedge f(c)\} \wedge g(d)],
\end{aligned}$$

as $(a, b) \in A_x \longrightarrow x \leq ab$ and $(c, d) \in A_z \longrightarrow z \leq cd$. Then $xyz \leq (ab)y(cd) = (a(by)c)d \longrightarrow (a(by)c, d) \in A_{xyz}$. Thus

$$\bigvee_{(a,b)\in A_x} \bigvee_{(c,d)\in A_z} [\{f(a) \wedge f(c)\} \wedge g(d)] = \bigvee_{(a(by)c,d)\in A_{xyz}} [\{f(a) \wedge f(c)\} \wedge g(d)].$$

As f is a fuzzy bi-ideal of S, we have $f(a(by)c) \geq f(a) \wedge f(c)$. Hence

$$\begin{aligned}
&\bigvee_{(a(by)c,d)\in A_{xyz}} [\{f(a) \wedge f(c)\} \wedge g(d)] \\
\leq &\bigvee_{(a(by)c,d)\in A_{xyz}} [f(a(by)c) \wedge g(d)] \\
\leq &\bigvee_{(e,f)\in A_{xyz}} [f(e) \wedge g(f)] = (f \circ g)(xyz).
\end{aligned}$$

Thus $(f \circ g)(xyz) \geq (f \circ g)(x) \wedge (f \circ g)(z)$.

Let $x, y \in S$ such that $x \leq y$. If $(p, q) \in A_y$ then $pq \geq y \longrightarrow pq \geq x \longrightarrow (p, q) \in A_x$. Hence $Ay \subseteq Ax$.

If $A_x = \emptyset$ then $A_y = \emptyset$ and so $(f \circ g)(x) = 0 = (f \circ g)(y)$. If $A_y \neq \emptyset$ then $A_x \neq \emptyset$, so

$$\begin{aligned}
(f \circ g)(y) \; : &= \bigvee_{(p,q)\in A_y} \min\{f(p), g(q)\} \\
&\leq \bigvee_{(c,d)\in A_x} \min\{f(p), f(q)\} \\
&= (f \circ g)(x)
\end{aligned}$$

Thus $(f \circ g)(x) \geq (f \circ g)(y)$. Therefore $f \circ g$ is a fuzzy bi-ideal of S. □

1.3.18 Theorem (cf. [28, Theorem 2]).

Let S be an ordered semigroup. Then the following are equivalent:
(i) S is left and right simple.
(ii) $S = (aSa]$ for all $a \in S$.
(iii) S is regular, left and right simple.
(iv) Every fuzzy bi-ideal of S is a constant mapping.

1.3.19 Theorem (cf. [28]).

An ordered semigroup S is completely regular if and only if for each fuzzy bi-ideal f of S, we have

$$f(a) = f(a^2) \text{ for every } a \in S.$$

1.3.20 Theorem (cf. [28, Theorem 5]).

An ordered semigroup S is a semilattice of left and right simple semigroups if and only if for every fuzzy bi-ideal f of S, we have

$$f(a) = f(a^2) \text{ and } f(ab) = f(ba) \text{ for all } a, b \in S.$$

1.4 Fuzzy quasi-ideals

In this section, we discuss fuzzy quasi-ideals of ordered semigroups. The results are taken from [26].

1.4.1 Definition (cf. [26]).

A fuzzy subset f of S is called a *fuzzy quasi-ideal* of S if:
(1) $(f \circ 1) \wedge (1 \circ f) \preceq f$.
(2) $x \leq y$, then $f(x) \geq f(y)$ for all $x, y \in S$.

1.4.2 Lemma (cf. [26]).

Let $(S, \cdot, \leq)$ be an ordered semigroup. A non-empty subset A of S is a quasi-ideal of S if and only if f_A is a fuzzy quasi-ideal of S.

1.4.3 Lemma

Let $(S, \cdot, \leq)$ be an ordered semigroup. A fuzzy subset f of S is a fuzzy quasi-ideal of S if and only if for every $t \in (0, 1]$, $U(f; t) \neq \emptyset$ is a quasi-ideal of S.

Proof. Assume that f is a fuzzy quasi-ideal of S. Let $x, y \in S$ be such that $x \leq y$ and $y \in U(f; t)$. Then $f(y) \geq t$. Since $x \leq y \longrightarrow f(x) \geq f(y)$ we have $f(x) \geq t$ and so $x \in U(f; t)$.

Suppose that $x \in S$ be such that $x \in (U(f;t)S] \cap (SU(f;t)]$. Then $x \in (U(f;t)S]$ and $x \in (SU(f;t)]$. Thus $x \leq yz$ and $x \leq y^{'}z^{'}$ for some $y, z^{'} \in U(f;t)$ and $z, y^{'} \in S$. Then $(y,z) \in A_x$ and $(y^{'}, z^{'}) \in A_x$. Since $A_x \neq \emptyset$, so by hypothesis

$$\begin{aligned} f(x) &\geq ((f \circ 1) \wedge (1 \circ f))(x) \\ &= \min[(f \circ 1)(x), (1 \circ f)(x)] \\ &= \min\left[\bigvee_{(p,q)\in A_x} \min\{f(p), 1(q)\}, \bigvee_{(p_1,q_1)\in A_x} \min\{1(p_1), f(q_1)\}\right] \\ &\geq \min[\min\{f(y), 1(z)\}, \min\{1(y^{'}), f(z^{'})\}] \\ &= \min[\min\{f(y), 1\}, \min\{1, f(z^{'})\}] \\ &= \min[f(y), f(z^{'})]. \end{aligned}$$

Since $y, z^{'} \in U(f;t)$ we have $f(y) \geq t$ and $f(z^{'}) \geq t$, therefore

$$f(x) \geq \min[f(y), f(z^{'})] \geq t,$$

and so $x \in U(f;t)$. Hence $(U(f;t)S] \cap (SU(f;t)] \subseteq U(f;t)$.

Conversely, assume that $U(f;t)$ is a quasi-ideal of S for all $t \in (0,1]$. Let $x, y \in S$ such that $x \leq y$ and $f(x) < f(y)$. Then there exists $t \in (0,1]$ such that $f(x) < t \leq f(y)$. Thus $y \in U(f;t)$ but $x \notin U(f;t)$. This is a contradiction. Hence $f(x) \geq f(y)$ for all $x \leq y$. Let $x \in S$ be such that

$$f(x) < ((f \circ 1) \wedge (1 \circ f))(x),$$

then there exists $t \in (0,1]$ such that

$$f(x) < t \leq ((f \circ 1) \wedge (1 \circ f))(x) = \min[(f \circ 1)(x), (1 \circ f)(x)].$$

and hence $(f \circ 1)(x) \geq t$ and $(1 \circ f)(x) \geq t$. Thus $x \in (U(f;t)S]$ and $x \in (SU(f;t)]$ and we have $x \in (U(f;t)S] \cap (SU(f;t)]$. By hypothesis, $(U(f;t)S] \cap (SU(f;t)] \subseteq U(f;t)$ and so $x \in U(f;t)$. Thus $f(x) \geq t$. This is a contradiction. Thus $f(x) \geq ((f \circ 1) \wedge (1 \circ f))(x)$. □

1.4.4 Example (cf. [96]).

Let $S = \{a, b, c, d, f\}$ be an ordered semigroup with the following multiplication,

$\cdot$	a	b	c	d	f
a	a	a	a	a	a
b	a	b	a	d	a
c	a	f	c	c	f
d	a	b	d	d	b
f	a	f	a	c	a

We define the order "$\leq$" as follows

$$\leq := \{(a,a),(a,b),(a,c),(a,d),(a,f),(b,b),(c,c),(d,d),(f,f)\}$$

Quasi-ideals of S are:

$$\{a\},\{a,b\},\{a,c\},\{a,d\},\{a,f\},\{a,b,d\},$$
$$\{a,c,d\},\{a,b,f\},\{a,c,f\} \text{ and } S.$$

Define $f : S \longrightarrow [0,1]$ by

$$f(a)=0.8, \quad f(b)=0.7, \quad f(d)=0.6 \quad f(c)=f(f)=0.5.$$

Then

$$U(f;t) := \begin{cases} S & \text{if } t \in (0,0.5] \\ \{a,b,d\} & \text{if } t \in (0.5,0.6] \\ \{a,b\} & \text{if } t \in (0.6,0.7] \\ \{a\} & \text{if } t \in (0.7,0.8] \\ \emptyset & \text{if } t \in (0.8,1] \end{cases}$$

Then $U(f;t)$ is a quasi-ideal and by Lemma 1.4.3, f is a fuzzy quasi-ideal of S.

1.4.5 Definition (cf. [26]).

A subset A of S is called *idempotent* if $(A^2] = A$.

1.4.6 Definition (cf. [26]).

A fuzzy subset f of S is called *idempotent* if $f \circ f = f$.

1.4.7 Proposition (cf. [26, Proposition 1]).

If $(S,\cdot,\leq)$ is an ordered semigroup and f_1, f_2, g_1, g_2 are fuzzy subsets of S, such that $f_1 \preceq g_1$ and $f_2 \preceq g_2$, then

$$f_1 \circ f_2 \preceq g_1 \circ g_2.$$

1.4.8 Proposition (cf. [26]).

Let $(S,\cdot,\leq)$ be an ordered semigroup, f a fuzzy right ideal and g a fuzzy left ideal of S. Then $f \circ g \preceq f \wedge g$.

1.4.9 Theorem (cf. [26]).

An ordered semigroup S is regular if and only if for every fuzzy right ideal f and every fuzzy left ideal g of S we have $f \wedge g \preceq f \circ g$, equivalently, $f \wedge g = f \circ g$.

1.4.10 Proposition (cf. [26]).

If S is a regular ordered semigroup, then the fuzzy right and the fuzzy left ideals of S are idempotent.

1.4.11 Proposition (cf. [26]).

Let $(S, \cdot, \leq)$ be a regular ordered semigroup, f a fuzzy right ideal of S and g a fuzzy left ideal of S. Then $f \circ g$ is a fuzzy quasi-ideal of S.

1.4.12 Proposition (cf. [26]).

Let S be an ordered semigroup. A fuzzy right (resp. left) ideal f of S is idempotent if and only if $f \preceq f \circ f$ (resp. $g \preceq g \circ g$).

1.4.13 Theorem (cf. [26]).

An ordered semigroup S is regular if and only if the fuzzy right and the fuzzy left ideals of S are idempotent and for each fuzzy right ideal f and each fuzzy left ideal g of S, the fuzzy set $f \circ g$ is a fuzzy quasi-ideal of S.

1.5 Fuzzy interior ideals

In this section, we give the definitions and results of ordered semigroup in terms of fuzzy interior ideals. The results of this section are taken from [27].

1.5.1 Definition (cf. [27]).

Let $(S, \cdot, \leq)$ be an ordered semigroup. A fuzzy subset f of S is called a *fuzzy interior ideal* of S, if:

(i) $f(xay) \geq f(a)$ for all $x, a, y \in S$ and

(ii) If $x \leq y$, then $f(x) \geq f(y)$.

1.5.2 Lemma (cf. [27, Proposition 2.3]).

Let S be an ordered semigroup and $\emptyset \neq A \subseteq S$. Then A is an interior ideal of S if and only if f_A is a fuzzy interior ideal of S.

1.5.3 Lemma (cf. [27, Proposition 2.4]).

Every fuzzy ideal of an ordered semigroup S is a fuzzy interior ideal of S.

1.5.4 Lemma (cf. [27, Proposition 2.5]).

Let S be a regular ordered semigroup and f a fuzzy interior ideal of S. Then f is a fuzzy ideal of S.

1.5.5 Lemma (cf. [27, Proposition 2.7]).

Let S be an intra-regular ordered semigroup and f a fuzzy interior ideal of S. Then f is a fuzzy ideal of S.

The proof of the following Lemma is easy, so omitted.

1.5.6 Lemma

Let S be an ordered semigroup and f a fuzzy subset of S. Then f is a fuzzy interior ideal of S if and only if for every $t \in (0,1]$, $U(f;t)(\neq \emptyset)$ is an interior ideal of S.

1.5.7 Example (cf. [96]).

Let $S = \{a, b, c, d, e\}$ be a set with the following multiplication table and order relation "$\leq$"

$\cdot$	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

$\leq := \{(a,a),(a,c),(a,d),(a,e),(b,b),(b,d),(b,e),(c,c),(c,e),(d,d),(d,e),(e,e)\}$

Interior ideals of S are: $\{a,b,d\}, \{a,c,d,e\}, S$. Define $f : S \longrightarrow [0,1]$ by $f(a) = 0.8$, $f(c) = 0.7$, $f(e) = 0.6$, $f(d) = 0.5$, $f(b) = 0.3$. Then

$$U(f;t) := \begin{cases} S & \text{if } 0 < t \leq 0.3 \\ \{a,c,d,e\} & \text{if } 0.3 < t \leq 0.5 \\ \emptyset & \text{if } t \leq 1. \end{cases}$$

Then by Lemma 1.5.6, f is a fuzzy interior ideal of S.

Chapter 2

Fuzzy ideals in ordered semigroups

Ideals play an important role in studying the structure of ordered semigroups. In [25] ordered semigroups are characterized by the properties of their fuzzy left (resp. right), fuzzy quasi- (resp. bi-) ideals. In this chapter, we characterize some classes of ordered semigroups by the properties of their fuzzy left (resp. right) ideals. We prove that: A regular ordered semigroup S is left simple if and only if every fuzzy left ideal of S is a constant function. We also show that an ordered semigroup S is left (resp. right) regular if and only if for every fuzzy left (resp. right) ideal f of S we have, $f(a) = f(a^2)$ for every $a \in S$. In section 2.2, we characterize semilattices of ordered semigroups in terms of fuzzy left (resp. right) ideals. In this respect, we prove that an ordered semigroup S is a semilattice of left (resp. right) simple semigroups if and only if for every fuzzy left (resp. right) ideal f of S we have, $f(a) = f(a^2)$ and $f(ab) = f(ba)$ for all $a, b \in S$. Results given in this chapter are part of our published and submitted papers [106], [109], [133], [135], [136] and [137].

2.1 Characterizations of regular ordered semigroups

In this section we characterize regular ordered semigroups in terms of fuzzy left (resp. right) ideals and prove that a regular ordered semigroup is left (resp. right) simple if and only if every fuzzy left (resp. right) ideal is a constant mapping.

2.1.1 Theorem

A regular ordered semigroup S is left simple if and only if every fuzzy left ideal of S is a constant mapping.

Proof. Let S be a regular, left simple ordered semigroup, f a fuzzy left ideal of S and $a \in S$. We consider the set,

$$E_S := \{e \in S | e^2 \geq e\}.$$

Then $E_S \neq \emptyset$. In fact, since S is regular and $a \in S$, there exists $x \in S$ such that $a \leq axa$. It follows from $(OS3)$ that

$$(ax)^2 = (axa)x \geq ax,$$

and so $ax \in E_S$ and hence $E_S \neq \emptyset$.

(1) Let $t \in E_S$, then $f(e) = f(t)$ for every $e \in E_S$. Indeed, since S is left simple and $t \in S$ we have $(St] = S$. Since $e \in S$, therefore $e \in (St]$ and so there exists $z \in S$

such that $e \le zt$. Hence $e^2 \le (zt)(zt) = (ztz)t$. Since f is a fuzzy left ideal of S, we have

$$f(e^2) \ge f((ztz)t) \ge f(t).$$

Since $e \in E_S$, we have $e^2 \ge e$. Therefore $f(e) \ge f(e^2)$ and we have $f(e) \ge f(t)$. Similarly, $f(t) \ge f(e)$, because $e, t \in E_S$. Hence $f(t) = f(e)$, that is, f is constant on E_S.

(2) Let $a \in S$, then $f(a) = f(t)$ for every $t \in E_S$. Indeed, since S is regular there exists $x \in S$ such that $a \le axa$. We consider the element xa of S. Then it follows from $(OS3)$ that,

$$(xa)^2 = x(axa) \ge xa.$$

Hence $xa \in E_S$ and so by (1), we have $f(xa) = f(t)$. Since, f is a fuzzy left ideal of S, we have $f(xa) \ge f(a)$. Then $f(t) \ge f(a)$. On the other hand, since S is left simple and $t \in S$, therefore $S = (St]$. Since $a \in S$, we have $a \le st$ for some $s \in S$. Since f is fuzzy left ideal of S, we have $f(a) \ge f(st) \ge f(t)$. Thus $f(t) = f(a)$, that is, f is constant on S.

Conversely, let $a \in S$. Then the set $(Sa]$ is a left ideal of S. By Lemma 1.2.5, the characteristic mapping

$$f_{(Sa]} : S \longrightarrow \{0,1\},\ x \longmapsto f_{(Sa]}(x) := \begin{cases} 1 \text{ if } x \in (Sa], \\ 0 \text{ if } x \notin (Sa], \end{cases}$$

is a fuzzy left ideal of S. By hypothesis $f_{(Sa]}$ is a constant mapping, that is, there exists $c \in \{0,1\}$ such that

$$f_{(Sa]}(x) = c \text{ for every } x \in S.$$

Since $(Sa]$ is non-empty, so $c \neq 0$. Hence $c = 1$, that is, $(Sa] = S$. □

From left–right dual of Theorem 2.1.1, we have the following:

2.1.2 Theorem

A regular ordered semigroup S is right simple if and only if every fuzzy right ideal of S is a constant mapping.

2.1.3 Theorem

An ordered semigroup $(S, \cdot, \le)$ is left regular if and only if for each fuzzy left ideal f of S, we have $f(a) = f(a^2)$ for all $a \in S$.

Proof. Suppose that f is a fuzzy left ideal of S and let $a \in S$. Since S is left regular, there exists $x \in S$ such that $a \le xa^2$. As f is a fuzzy left ideal of S, we have

$$f(a) \ge f(xa^2) \ge f(a^2) \ge f(a).$$

Thus $f(a) = f(a^2)$.

Conversely, let $a \in S$. We consider the left ideal $L(a^2) = (a^2 \cup Sa^2]$ of S, generated by a^2. Then by Lemma 1.2.5, the characteristic function $f_{L(a^2)}$ of $L(a^2)$ is a fuzzy left ideal of S. By hypothesis we have $f_{L(a^2)}(a) = f_{L(a^2)}(a^2)$. Since $a^2 \in L(a^2)$, we have

$$f_{L(a^2)}(a^2) = 1$$

and so $f_{L(a^2)}(a) = 1$. Thus $a \in L(a^2) = (a^2 \cup Sa^2]$ and so $a \leq y$ for some $y \in a^2 \cup Sa^2$. If $y = a^2$, then $a \leq y = a^2 = aa \leq aa^2$. If $y = xa^2$ for some $x \in S$, then $a \leq y = xa^2$. Thus a is left regular. □

From left-right dual of Theorem 2.1.3, we have the following:

2.1.4 Theorem

An ordered semigroup $(S, \cdot, \leq)$ is right regular if and only if for each fuzzy right ideal f of S, we have $f(a) = f(a^2)$ for all $a \in S$.

An ordered semigroup $(S, \cdot, \leq)$ is called *left* (resp. *right*) *duo* if every left (resp. right) ideal of S is a two-sided ideal of S, and *duo* if it is both left and right duo.

2.1.5 Definition

An ordered semigroup $(S, \cdot, \leq)$ is called *fuzzy left* (resp. *right*) *duo* if every fuzzy left (resp. right) ideal of S is a fuzzy two-sided ideal of S. An ordered semigroup S is called *fuzzy duo* if it is both fuzzy left and fuzzy right duo.

2.1.6 Theorem

A regular ordered semigroup is left (right) duo if and only if it is fuzzy left (right) duo.

Proof. Let S be a left duo ordered semigroup and f a fuzzy left ideal of S. Let $a, b \in S$. Then the set $(Sa]$ is a left ideal of S generated by a, because S is regular. Since S is left duo, therefore $(Sa]$ is a two-sided ideal of S. Thus for each $b \in S$, we have $ab \leq xa$ for some $x \in S$. Since f is a fuzzy left ideal of S, we have

$$f(ab) \geq f(xa) \geq f(a).$$

Let $x, y \in S$ be such that $x \leq y$. Then $f(x) \geq f(y)$, because f is a fuzzy left ideal of S. Thus f is a fuzzy right ideal of S. Therefore S is fuzzy left duo.

Conversely, if S is a fuzzy left duo and A a left ideal of S. Then the characteristic function f_A of A is a fuzzy left ideal of S. By hypothesis f_A is a fuzzy right ideal of S and by Lemma 1.2.5, A is a right ideal of S. Thus S is left duo. □

2.1.7 Theorem

In a regular ordered semigroup every bi-ideal is a right (left) ideal if and only if its fuzzy bi-ideal is a fuzzy right (left) ideal.

Proof. Let f be a fuzzy bi-ideal of S and $a, b \in S$. Then $(aSa]$ is a bi-ideal of S, generated by a, because S is regular. Since $(aSa]$ is a bi-ideal of S, by hypothesis $(aSa]$ is a right ideal of S. Thus for each $b \in S$, we have $ab \in (aSa]$ and so $ab \leq aza$ for some $z \in S$. Since f is a fuzzy bi-ideal of S, we have

$$f(ab) \geq f(aza) \geq \min\{f(a), f(a)\} = f(a).$$

Let $x, y \in S$ be such that $x \leq y$. Then $f(x) \geq f(y)$ because f is a fuzzy bi-ideal of S. Thus f is a fuzzy right ideal of S.

Conversely, if A is a bi-ideal of S. Then by Lemma 1.2.5, f_A is a fuzzy bi-ideal of S. By hypothesis f_A is a fuzzy right ideal of S. By Lemma 1.2.5, A is a right ideal of S. □

2.1.8 Definition

Let $(S, \cdot, \leq)$ be an ordered semigroup and f a fuzzy subsemigroup of S. Then f is called a *fuzzy* $(1,2)$*-ideal* of S if:

(i) $x \leq y \longrightarrow f(x) \geq f(y)$,

(ii) $f(xa(yz)) \geq \min\{f(x), f(y), f(z)\}$

for all $x, y, z, a \in S$.

2.1.9 Proposition

Every fuzzy bi-ideal of an ordered semigroup S is a fuzzy $(1,2)$-ideal of S.

Proof. Let f be a fuzzy bi-ideal of S and let $x, y, z, a \in S$. Then

$$\begin{aligned} f(xa(yz)) &= f((xay)z) \geq \min\{f(xay), f(z)\} \\ &\geq \min\{\min\{f(x), f(y)\}, f(z)\} = \min\{f(x), f(y), f(z)\}. \end{aligned}$$

Now, let $x, y \in S$ be such that $x \leq y$. Then $f(x) \geq f(y)$, because f is a fuzzy bi-ideal of S. □

2.1.10 Corollary

Every fuzzy left (resp. right) ideal f of an ordered semigroup S is a fuzzy $(1,2)$-ideal of S.

If S is a regular ordered semigroup then we have:

2.1.11 Proposition

A fuzzy $(1,2)$*-ideal of a regular ordered semigroup is a fuzzy bi-ideal.*

Proof. Assume that S is a regular ordered semigroup and f be a fuzzy $(1,2)$-ideal of S. Let $x, y, a \in S$. Since S is regular and $(xSx]$ is a bi-ideal of S, so a right ideal of S, by Theorem 2.1.7. Thus

$$xa \leq (xsx)a \in (xSx)S \subseteq (xSx]S \subseteq (xSx],$$

whence $xa \leq xyx$ for some $y \in S$. Thus $xay \leq (xyx)y$ and we have

$$\begin{aligned} f(xay) &\geq f((xyx)y) = f(xy(xy)) \geq \min\{f(x), f(xy)\} \\ &\geq \min\{f(x), \min\{f(x), f(y)\}\} = \min\{f(x), f(y)\}. \end{aligned}$$

Let $x, y \in S$ be such that $x \leq y$. Then $f(x) \geq f(y)$, because f is a fuzzy $(1,2)$-ideal of S. Thus f is a fuzzy bi-ideal of S. □

2.2 Semilattices of left simple ordered semigroups

2.2.1 Lemma

An ordered semigroup $(S, \cdot, \leq)$ *is a semilattice of left simple semigroups if and only if for all left ideals* A*,* B *of* S *we have*

$$(A^2] = A \text{ and } (AB] = (BA].$$

Proof. $(\longrightarrow)$ Let S be a semilattice of left simple semigroups and A, B are left ideals of S. Then there exists a semilattice Y and a family $\{S_\alpha\}_{\alpha \in Y}$ of left simple subsemigroups of S satisfying all conditions mentioned in the definition of a semilattice of left simple semigroups.

Let $a \in A$. Since $a \in S = \bigcup_{\alpha \in Y} S_\alpha$, there exists $\alpha \in Y$ such that $a \in S_\alpha$. Since S_α is left simple, we have

$$S_\alpha = (S_\alpha b] = \{c \in S | \exists x \in S_\alpha : c \leq xb\}$$

for all $b \in S_\alpha$. Since $a \in S_\alpha$, we have $S_\alpha = (S_\alpha a]$ that is $a \leq xa$ for some $x \in S_\alpha$. Since $x \in S_\alpha = (S_\alpha a]$, we have $x \leq ya$ for some $y \in S_\alpha$. Thus we have $a \leq xa \leq (ya)a \in (SA)A \subseteq AA = A^2$ and $a \in (A^2]$. Hence $A \subseteq (A^2]$. On the other hand, since A is a subsemigroup of S, hence $A^2 \subseteq A$ and we have $(A^2] \subseteq (A] = A$. Let $x \in (AB]$, then $x \leq ab$ for some $a \in A$ and $b \in B$. Since $a, b \in S = \bigcup_{\alpha \in Y} S_\alpha$, there exist $\alpha, \beta \in Y$ such that $a \in S_\alpha$, $b \in S_\beta$. Then $ab \in S_\alpha S_\beta \subseteq S_{\alpha\beta}$ and $ba \in S_\beta S_\alpha \subseteq S_{\beta\alpha} = S_{\alpha\beta}$ (since $\alpha, \beta \in Y$ and Y is a semilattice). Since $S_{\alpha\beta}$ is left simple, we have $S_{\alpha\beta} = (S_{\alpha\beta} c]$ for each $c \in S_{\alpha\beta}$. Then $ab \in (S_{\alpha\beta}ba]$ and $ab \leq yba$ for some $y \in S_{\alpha\beta}$. Since B is a left ideal of S, we have $yba \in (SB)A \subseteq BA$. Thus $x \in (BA]$. Hence $(AB] \subseteq (BA]$. By symmetry we have $(BA] \subseteq (AB]$.

($\longleftarrow$) Since $\mathcal{N}$ is a semilattice congruence on S, which is equivalent to the fact that $(x)_{\mathcal{N}}$ $\forall x \in S$, is a left simple subsemigrup of S. By Lemma 1.1.11, it is enough to prove that every left ideal is right ideal and semiprime. Let L be a left ideal of S. Then

$$LS \subseteq (LS] = (SL] \subseteq (L] = L.$$

If $x \in L$, $S \ni y \leq x \in L$, then $y \in L$, since L is a left ideal of S. Thus L is a right ideal of S. Let $x \in S$ be such that $x^2 \in L$. We consider the bi-ideal $B(x)$ of S generated by x. Then

$$\begin{aligned} B(x)^2 &= (x \cup x^2 \cup xSx](x \cup x^2 \cup xSx] \\ &\subseteq ((x \cup x^2 \cup xSx)(x \cup x^2 \cup xSx)] \\ &= (x^2 \cup x^3 \cup xSx^2 \cup x^4 \cup xSx^3 \cup x^2Sx \cup x^3Sx \cup xSx^2Sx]. \end{aligned}$$

Since $x^2 \in L$, $x^3 \in SL \subseteq L, (xS)x^2 \subseteq SL \subseteq L, x^4 \in SL \subseteq L$. Thus

$$B(x)^2 \subseteq (L \cup LS] = (L] = L.$$

Thus $(B(x)^2] \subseteq (L] = L$ and $x \in L$. Hence L is semiprime. □

2.2.2 Theorem

An ordered semigroup $(S, \cdot, \leq)$ is a semilattice of left (right) simple semigroups if and only if for every fuzzy left (right) ideal f of S, we have

$$f(a^2) = f(a) \text{ and } f(ab) = f(ba) \text{ for all } a, b \in S.$$

Proof. Let S be a semilattice of left simple semigroups. By hypothesis, there exists a semilattice Y and a family $\{S_\alpha\}_{\alpha \in Y}$ of left simple subsemigroups of S such that:

(1) $S_\alpha \cap S_\beta = \emptyset$ $\forall \alpha, \beta \in Y$, $\alpha \neq \beta$,

(2) $S = \bigcup_{\alpha \in Y} S_\alpha$,

(3) $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ $\forall \alpha, \beta \in Y$.

Let f be a fuzzy left ideal of S and $a \in S$. Then $f(a) = f(a^2)$. In fact, by Theorem 2.1.3, it is enough to prove that $a \in (Sa^2]$ for every $a \in S$. Let $a \in S$, then there exists $\alpha \in Y$ such that $a \in S_\alpha$. Since S_α is left simple, we have $S_\alpha = (S_\alpha a]$ and $a \leq xa$ for some $x \in S_\alpha$.

Since $x \in S_\alpha$, we have $x \in (S_\alpha a]$ and $x \leq ya$ for some $y \in S_\alpha$. Thus we have

$$a \leq xa \leq (ya)a = ya^2,$$

this implies $a \in (Sa^2]$. Let $a, b \in S$. Then, we have

$$f(ab) = f((ab)^2) = f(a(ba)b) \geq f(ba).$$

By symmetry we can prove that $f(ba) \geq f(ab)$. Hence $f(ab) = f(ba)$.

Conversely, assume that for every fuzzy left ideal f of S, we have

$$f(a^2) = f(a) \text{ and } f(ab) = f(ba)$$

for all $a, b \in S$. Then by Theorem 2.2.2, we have that S is left regular. Let A be a left ideal of S and let $a \in A$. Then $a \in S$, since S is left regular there exists $x \in S$ such that

$$a \leq xa^2 = (xa)a \in (SA)A \subseteq AA = A^2.$$

Hence $a \in (A^2]$ and $A \subseteq (A]$. On the other hand, since A is a left ideal of S, we have $A^2 \subseteq SA \subseteq A$, then $(A^2] \subseteq (A] = A$. Let A and B be left ideals of S and let $x \in (BA]$ then $x \leq ba$ for some $a \in A$ and $b \in B$. We consider the left ideal $L(ab)$ generated by ab. That is, the set $L(ab) = (ab \cup Sab]$. By Lemma 1.2.5, the characteristic function $f_{L(ab)}$ of $L(ab)$ is a fuzzy left ideal of S. By hypothesis, we have $f_{L(ab)}(ab) = f_{L(ab)}(ba)$. Since $ab \in L(ab)$, we have $f_{L(ab)}(ab) = 1$ and $f_{L(ab)}(ba) = 1$ and hence $ba \in L(ab) = (ab \cup Sab]$. Thus $ba \leq ab$ or $ba \leq yab$ for some $y \in S$. If $ba \leq ab$ then $x \leq ab \in AB$ and $x \in (AB]$. If $ba \leq yab$ then $x \leq yab \in (SA)B \subseteq AB$ and $x \in (AB]$. Thus $(BA] \subseteq (AB]$. By symmetry we can prove that $(AB] \subseteq (BA]$. Therefore $(AB] = (BA]$ and by Lemma 2.2.1, it follows that S is a semilattice of left simple semigroups. □

2.2.3 Proposition

Let $(S, \cdot, \leq)$ be an ordered semigroup and f a fuzzy left(resp. right) ideal of S, $a \in S$ such that $a \leq a^2$. Then $f(a) = f(a^2)$.

Proof. Since $a \leq a^2$ and f is a fuzzy left ideal of S, we have

$$f(a) \geq f(a^2) = f(aa) \geq f(a),$$

and so $f(a) = f(a^2)$. □

2.3 Fuzzy quasi-ideals in ordered semigroups

In this section, we prove that an ordered semigroup S is regular, left and right simple if and only if every fuzzy quasi-ideal of S is a constant function. We also prove that S is completely regular if and only if for every fuzzy quasi-ideal f of S we have $f(a) = f(a^2)$ for every $a \in S$. We define semiprime fuzzy quasi-ideal of ordered semigroups and prove that an ordered semigroup S is completely regular if and only if every fuzzy quasi-ideal f of S is semiprime. We characterize semilattices of left and right simple ordered semigroups in terms of fuzzy quasi-ideals of S. We prove that an ordered semigroup S is a semilattice of left and right simple ordered semigroups if and only if for every fuzzy quasi-ideal f of S we have, $f(a) = f(a^2)$ and $f(ab) = f(ba)$, for all $a, b \in S$. In this section we also discuss ordered semigroups having the property $a \leq a^2$ for all $a \in S$ and prove that an ordered semigroup S (having the property $a \leq a^2 \ \forall a \in S$) is a semilattice of left and right simple ordered semigroups if and only if for every fuzzy quasi-ideal f of S we have $f(ab) = f(ba)$, for all $a, b \in S$.

2.3.1 Lemma

Let S be an ordered semigroup. Then every quasi-ideal Q of S is a bi-ideal of S.

Proof. In fact, $Q^2 \subseteq QS \cap SQ \subseteq (QS] \cap (SQ] \subseteq Q$, and $QSQ \subseteq QS, QSQ \subseteq SQ$. Hence $QSQ \subseteq QS \cap SQ \subseteq (QS] \cap (SQ] \subseteq Q$. If $a \in Q$, $S \ni b \leq a$, then $b \in Q$, since Q is a quasi-ideal of S. Thus Q is a bi-ideal of S. □

2.3.2 Lemma

Every fuzzy quasi-ideal of an ordered semigroup $(S, \cdot, \leq)$ is a fuzzy bi-ideal of S.

Proof. Let f be a fuzzy quasi-ideal of S. Let $x, y \in S$. Then $(x, y) \in A_{xy}$. Since $A_{xy} \neq \emptyset$, we have

$$\begin{aligned}
f(xy) &\geq ((f \circ 1) \wedge (1 \circ f))(xy) \\
&= \min[(f \circ 1)(xy), (1 \circ f)(xy)] \\
&= \min\left[\bigvee_{(p,q) \in A_{xy}} \min\{f(p), 1(q)\}, \bigvee_{(p_1,q_1) \in A_{xy}} \min\{1(p_1), f(q_1)\}\right] \\
&\geq \min[\min\{f(x), 1(y)\}, \min\{1(x), f(y)\}] \\
&= \min[\min\{f(x), 1\}, \min\{1, f(y)\}] \\
&= \min[f(x), f(y)].
\end{aligned}$$

Let $x, y, z \in S$. Then $(xy)z = x(yz)$ and we have $(xy, z), (x, yz) \in A_{xyz}$. Since $A_{xyz} \neq \emptyset$, we have

$$\begin{aligned}
f(xyz) &\geq ((f \circ 1) \wedge (1 \circ f))(xyz) \\
&= \min[(f \circ 1)(xyz), (1 \circ f)(xyz)] \\
&= \min\left[\bigvee_{(p,q) \in A_{xyz}} \min\{f(p), 1(q)\}, \bigvee_{(p_1,q_1) \in A_{xyz}} \min\{1(p_1), f(q_1)\}\right] \\
&\geq \min[\min\{f(x), 1(yz)\}, \min\{1(xy), f(z)\}] \\
&= \min[\min\{f(x), 1\}, \min\{1, f(z)\}] \\
&= \min[f(x), f(z)].
\end{aligned}$$

Let $x, y \in S$ be such that $x \leq y$. Then $f(x) \geq f(y)$, because f is a fuzzy quasi-ideal of S. Thus f is a fuzzy bi-ideal of S. □

2.3.3 Remark

The converse of Lemma 2.3.2, is not true in general.

2.3.4 Example

Consider the ordered semigroup $S = \{a, b, c, d\}$

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

$$\leq := \{(a,a), (b,b), (c,c), (d,d), (a,b)\}[132].$$

Then $\{a, d\}$ is a bi-ideal but not a quasi-ideal of S. Define a fuzzy set $f : S \longrightarrow [0,1]$ by

$$f(a) = f(d) = 0.7, \qquad f(b) = f(c) = 0.4$$

Then

$$U(f;t) := \begin{cases} S & \text{if } t \in (0, 0.4] \\ \{a, d\} & \text{if } t \in (0.4, 0.7] \\ \emptyset & \text{if } \ t \in (0.7, 1] \end{cases}$$

Then $U(f;t)$ is a bi-ideal of S and by Lemma 1.3.4, f is a fuzzy bi-ideal of S for all $t \in (0,1]$. Further more, $U(f;t)$ is a bi-ideal of S for $t \in (0.4, 0.7]$ but not a quasi-ideal of S. Thus f is a fuzzy bi-ideal of S but not a fuzzy quasi-ideal of S for $t \in (0.4, 0.7]$.

2.4 Characterizations of left, right and completely regular ordered semigroups in terms of fuzzy quasi-ideals

In this section, we prove that an ordered semigroup S is regular, left and right simple if and only if every fuzzy quasi-ideal f of S is a constant function. We define semiprime fuzzy quasi-ideals of ordered semigroups and prove that an ordered semigroup $(S, \cdot, \leq)$ is completely regular if and only if every fuzzy quasi-ideal f of S is a semiprime fuzzy quasi-ideal of S.

2.4.1 Theorem

An ordered semigroup $(S, \cdot, \leq)$ is regular, left and right simple if and only if every fuzzy quasi-ideal of S is a constant mapping.

Proof. Let S be regular, left and right simple ordered semigroup and let f be a fuzzy quasi-ideal of S. Consider the set,

$$E_S := \{e \in S | e^2 \geq e\}.$$

Then E_S is non-empty, because S is regular.

(1) Let $t \in E_S$. Then $f(e) = f(t)$ for every $e \in E_S$. Indeed, since S is left and right simple, we have $(St] = S$ and $(tS] = S$. Since $e \in S$, then $e \in (St]$ and $e \in (tS]$ so there exist $x, y \in S$ such that, $e \leq xt$ and $e \leq ty$. Hence

$$e^2 = ee \leq (xt)(xt) = (xtx)t,$$

and we have $(xtx, t) \in A_{e^2}$. If $e \leq ty$ then

$$e^2 = ee \leq (ty)(ty) = t(yty),$$

so $(t, yty) \in A_{e^2}$. Since $A_{e^2} \neq \emptyset$, and f is a fuzzy quasi-ideal of S, we have

$$\begin{aligned} f(e^2) &\geq ((f \circ 1) \wedge (1 \circ f))(e^2) \\ &= \min[(f \circ 1)(e^2), (1 \circ f)(e^2)] \\ &= \min\left[\bigvee_{(y_1,z_1) \in A_{e^2}} \min\{f(y_1), 1(z_1)\}, \bigvee_{(y_2,z_2) \in A_{e^2}} \min\{1(y_2), f(z_2)\}\right] \\ &\geq \min[\min\{f(t), 1(yty)\}, \min\{1(xtx), f(t)\}] \\ &= \min[\min\{f(t), 1)\}, \min\{1, f(t)\}] \\ &= \min[f(t), f(t)] = f(t). \end{aligned}$$

Since $e \in E_S$ we have $e^2 \geq e$ and as f is a fuzzy quasi-ideal of S, we have $f(e) \geq f(e^2)$. Thus $f(e) \geq f(t)$. Since S is left and right simple and $e \in S$ we have, $(Se] = S$ and $(eS] = S$. As $t \in S$, we have $t \leq ze$ and $t \leq es$ for some $z, s \in S$. If $t \leq ze$ then

$$t^2 = tt \leq (ze)(ze) = (zez)e,$$

then $(zez, e) \in A_{t^2}$. If $t \leq es$ then

$$t^2 = tt \leq (es)(es) = e(ses),$$

and we have $(e, ses) \in A_{t^2}$. Since $A_{t^2} \neq \emptyset$, we have

$$\begin{aligned} f(t^2) &\geq ((f \circ 1) \wedge (1 \circ f))(t^2) \\ &= \min[(f \circ 1)(t^2), (1 \circ f)(t^2)] \\ &= \min\left[\bigvee_{(y_1,z_1)\in A_{t^2}} \min\{f(y_1), 1(z_1)\}, \bigvee_{(y_2,z_2)\in A_{t^2}} \min\{1(y_2), f(z_2)\}\right] \\ &\geq \min[\min\{f(e), 1(ses)\}, \min\{1(zez), f(e)\}] \\ &= \min[\min\{f(e), 1\}, \min\{1, f(e)\}] \\ &= \min[f(e), f(e)] = f(e) \end{aligned}$$

Since $t \in E_S$ we have $t^2 \geq t$ and since f is a fuzzy quasi-ideal of S, we have $f(t) \geq f(t^2)$. Thus $f(t) \geq f(e)$.

(2) Let $a \in S$ then $f(t) = f(a)$ for every $t \in E_S$. Since $a \in S$ and S is regular, therefore there exists $x \in S$ such that $a \leq axa$. Then by $(OS3)$ it follows that,

$$(ax)^2 = (axa)x \geq ax \text{ and } (xa)^2 = x(axa) \geq xa,$$

Then $ax, xa \in E_S$. Then by (1) we have $f(ax) = f(t)$ and $f(xa) = f(t)$. Since $(ax)(axa) \geq axa \geq a$ and $(axa)(xa) \geq axa \geq a$ and so $(ax, axa), (axa, xa) \in A_a$. Since $A_a \neq \emptyset$, and f is a fuzzy quasi-ideal of S, we have

$$\begin{aligned} f(a) &\geq ((f \circ 1) \wedge (1 \circ f))(a) \\ &= \min[(f \circ 1)(a), (1 \circ f)(a)] \\ &= \min\left[\bigvee_{(y_1,z_1)\in A_a} \min\{f(y_1), 1(z_1)\}, \bigvee_{(y_2,z_2)\in A_a} \min\{1(y_2), f(z_2)\}\right] \\ &\geq \min[\min\{f(ax), 1(axa)\}, \min\{1(axa), f(xa)\}] \\ &= \min[\min\{f(ax), 1\}, \min\{1, f(xa)\}] \\ &= \min[f(ax), f(xa)] = \min[f(t), f(t)] = f(t). \end{aligned}$$

Since S is left and right simple we have $(Sa] = S$, and $(aS] = S$. Since $t \in E_S$, we have $t \in (Sa]$ and $t \in (aS]$. Then $t \leq pa$ and $t \leq aq$ for some $p, q \in S$. Thus $(p, a) \in A_t$ and $(a, q) \in A_t$. Since $A_t \neq \emptyset$, we have

$$\begin{aligned} f(t) &\geq ((f \circ 1) \wedge (1 \circ f))(t) \\ &= \min[(f \circ 1)(t), (1 \circ f)(t)] \\ &= \min\left[\bigvee_{(y_1,z_1)\in A_t} \min\{f(y_1), 1(z_1)\}, \bigvee_{(y_2,z_2)\in A_t} \min\{1(y_2), f(z_2)\}\right] \\ &\geq \min[\min\{f(a), 1(q)\}, \min\{1(p), f(a)\}] \\ &= \min[\min\{f(a), 1\}, \min\{1, f(a)\}] \\ &= \min[f(a), f(a)\}] = f(a). \end{aligned}$$

Conversely, let $a \in S$. Then the set $(aS]$ is a quasi-ideal of S. By hypothesis, $f_{(aS]}$ is a constant function, that is, there exists $c \in \{0,1\}$ such that

$$f_{(aS]}(x) = c \text{ for every } x \in S.$$

Let $(aS] \subset S$ and x be an element of S such that $x \notin (aS]$, then $f_{(aS]}(x) = 0$. On the other hand, since $a^2 \in (aS]$ therefore $f_{(aS]}(a^2) = 1$. A contradiction to the fact that $f_{(aS]}$ is a constant function. Thus $(aS] = S$. By symmetry we can prove that $(Sa] = S$. Since $a \in S$ and $S = (aS] = (Sa]$, we have $a \in (aS] = (a(Sa]] \subseteq (aSa]$, and hence S is regular. □

If S is an ordered semigroup and $\emptyset \neq A \subseteq S$, then the set $(A \cup (AS \cap SA)]$ is the quasi-ideal of S generated by A. If $A = \{x\}$ $(x \in S)$, then we write $(x \cup (xS \cap Sx)]$ instead of $(\{x\} \cup (\{x\}S \cap S\{x\})]$.

2.4.2 Theorem

Let $(S, \cdot, \leq)$ be an *ordered semigroup and* f *a fuzzy quasi-ideal of* S. *Then the following are equivalent:*

(1) $a \in (a^2S] \cap (Sa^2]$ for all $a \in S$.

(2) $f(a) = f(a^2)$ for all $a \in S$.

Proof. (1)$\longrightarrow$(2). Let $a \in S$ and f a fuzzy quasi-ideal of S. Since $a \in (a^2S]$ and $a \in (Sa^2]$, we have $a \leq xa^2$ and $a \leq a^2y$ for some $x, y \in S$. Then (x, a^2), $(a^2, y) \in A_a$. Since $A_a \neq \emptyset$, and f a fuzzy quasi-ideal, we have

$$\begin{aligned}
f(a) &\geq ((f \circ 1) \wedge (1 \circ f))(a) \\
&= \min\left[\bigvee_{(y,z)\in A_a} \min\{f(y), 1(z)\}, \bigvee_{(y,z)\in A_a} \min\{1(y), f(z)\}\right] \\
&\geq \min[\min\{f(a^2), 1(y)\}, \min\{1(x), f(a^2)\}] \\
&= \min[\min\{f(a^2), 1\}, \min\{1, f(a^2)\}] \\
&= \min[f(a^2), f(a^2)] = f(a^2) \\
&= f(aa) \geq \min\{f(a), f(a)\} = f(a).
\end{aligned}$$

Thus $f(a) = f(a^2)$.

Conversely, let $a \in S$. We consider the quasi-ideal $Q(a^2)$ of S, generated by $a^2 (a \in S)$. That is, the set $Q(a^2) = (a^2 \cup (a^2S \cap Sa^2)]$. By Lemma 1.4.13, the characteristic function $f_{Q(a^2)}$ is a fuzzy quasi-ideal of S. By hypothesis

$$f_{Q(a^2)}(a) = f_{Q(a^2)}(a^2).$$

Since $a^2 \in Q(a^2)$, we have $f_{Q(a^2)}(a^2) = 1$ then $f_{Q(a^2)}(a) = 1$ and $a \in Q(a^2) = (a^2 \cup (a^2S \cap Sa^2)]$. Then $a \leq y$ for some $y \in a^2 \cup (a^2S \cap Sa^2)$. Thus $a \leq a^2$ or $a \leq a^2x$

and $a \leq ya^2$ for some $x, y \in S$. If $a \leq a^2$ then $a \leq a^2 = aa \leq a^2a^2 \in a^2S$, Sa^2 and so $a \in (a^2S], (Sa^2] \longrightarrow a \in (a^2S] \cap (Sa^2]$. If $a \leq a^2x$ and $a \leq ya^2$ then $a \leq a^2x \in a^2S$ and $a \leq ya^2 \in Sa^2$ and so $a \in (a^2S], (Sa^2] \longrightarrow a \in (a^2S] \cap (Sa^2]$. □

2.4.3 Theorem

A fuzzy quasi-ideal f of S is semiprime if and only if for all

($\longleftarrow$) Let f be a fuzzy quasi-ideal of S such that $f(a) \geq f(a^2)$ for all $a \in S$. We consider the quasi-ideal $Q(a^2)$ generated by $a^2 (a \in S)$. Then by Lemma 1.4.13, $f_{Q(a^2)}$ is a fuzzy quasi-ideal of S. By hypothesis

$$f_{Q(a^2)}(a) \geq f_{Q(a^2)}(a^2).$$

Since $a^2 \in Q(a^2)$, we have $f_{Q(a^2)}(a^2) = 1$ and $f_{Q(a^2)}(a) = 1 \longrightarrow a \in Q(a^2)$. Then $a \leq a^2$ or $a \leq a^2p$ and $a \leq qa^2$ for some $p, q \in S$. If $a \leq a^2$ then $a \leq a^2 = aa \leq a^2a^2 \in a^2S$, Sa^2 and so $a \in (a^2S], (Sa^2] \longrightarrow a \in (a^2S] \cap (Sa^2]$. □

2.5 Semilattices of left and right simple ordered semigroups in terms of fuzzy quasi-ideals

In this section, we characterize semilattices of left and right simple ordered semigroups in terms of fuzzy quasi-ideals of S. We prove that an ordered semigroup S is a semilattice of left and right simple ordered semigroups if and only if for every fuzzy quasi-ideal f of S we have, $f(a) = f(a^2)$ and $f(ab) = f(ba)$, for all $a, b \in S$. We also discuss the semilattice of ordered semigroups having the property $a \leq a^2$ for all $a \in S$ and prove that an ordered semigroup S (having the property $a \leq a^2$ $\forall a \in S$) is a semilattice of left and right simple ordered semigroups if and only if for every fuzzy quasi-ideal f of S we have $f(ab) = f(ba)$, for all $a, b \in S$.

2.5.1 Theorem

An ordered semigroup $(S, \cdot, \leq)$ is a semilattice of left and right simple semigroups if and only if for every fuzzy quasi-ideal f of S, we have

$$f(a) = f(a^2) \text{ and } f(ab) = f(ba) \text{ for all } a, b \in S.$$

Proof. **Proof.** ($\longrightarrow$) Suppose that S is a semilattice of left and right simple semigroups. Then by hypothesis, there exists a semilattice Y and a family $\{S_\alpha\}_{\alpha \in Y}$ of left and right simple subsemigroups of S such that

(i) $S_\alpha \cap S_\beta = \emptyset$ $\forall \alpha, \beta \in Y$, $\alpha \neq \beta$,

(ii) $S = \bigcup_{\alpha \in Y} S_\alpha$,

(iii) $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ $\forall \alpha, \beta \in Y$.

(1) Let f be a fuzzy quasi-ideal of S and $a \in S$. By Lemma 1.1.10, it is enough to prove that $a \in (a^2Sa^2]$ for every $a \in S$. Since $a \in S = \bigcup_{\alpha \in Y} S_\alpha$, then there exists $\alpha \in Y$ such that $a \in S_\alpha$. Since S_α is left and right simple we have $S_\alpha = (S_\alpha a]$ and $S_\alpha = (aS_\alpha]$. Thus we have $(aS_\alpha] = (a(S_\alpha a]](a(S_\alpha a)]] = (aS_\alpha a]$. Since $a \in S_\alpha$ we have $a \in (aS_\alpha a]$ then there exists $x \in S_\alpha$ such that $a \leq axa$. Since $x \in (aS_\alpha a]$ there exists $y \in S_\alpha$ such that, $x \leq aya$. Thus $a \leq axa \leq a(aya)a = a^2ya^2$. Since $y \in S_\alpha$, we have $a^2ya^2 \in a^2S_\alpha a^2 \subseteq a^2Sa^2$ and $a \in (a^2Sa^2]$.

(2) Let $a, b \in S$. By (1), we have

$$f(ab) = f((ab)^2) = f((ab)^4).$$

Also we have

$$\begin{aligned}
(ab)^4 &= (aba)(babab) \in Q(aba)Q(babab) \\
&\subseteq (Q(aba)Q(babab)] \\
&= (Q(babab)Q(aba)] (\text{by Lemma 1.1.12}) \\
&= ((babab \cup (bababS \cap Sbabab)](aba \cup (abaS \cap Saba)]] \\
&= ((babab \cup (bababS \cap Sbabab)(aba \cup (abaS \cap Saba)] (\text{as } (A](B] \subseteq (AB]) \\
&\subseteq ((babab \cup bababS)(aba \cup Saba)] \\
&\subseteq ((baS)(Sba)] = (baSba] \\
&= ((baSba]] \\
&= ((baS] \cap (Sba]] \\
&\quad (\text{since } (baS] \text{ is a right ideal and } (Sba] \text{ is a left ideal of } S).
\end{aligned}$$

Then $(ab)^4 \leq (ba)x$ and $(ab)^4 \leq y(ba)$ for some $x, y \in S$. Then $(ba, x) \in A_{(ab)^4}$ and $(y, ba) \in A_{(ab)^4}$. Since $A_{(ab)^4} \neq \emptyset$, we have

$$\begin{aligned}
f((ab)^4) &\geq ((f \circ 1) \wedge (1 \circ f))((ab)^4) \\
&= \min[(f \circ 1)(ab)^4, (1 \circ f)(ab)^4] \\
&= \min\left[\bigvee_{(y_1,z_1)\in A_{(ab)^4}} \min\{f(y_1), 1(z_1)\}, \bigvee_{(y_2,z_2)\in A_{(ab)^4}} \min\{1(y_2), f(z_2)\}\right] \\
&\leq \min[\min\{f(ba), 1(x)\}, \min\{1(y), f(ba)\}] \\
&= \min[f(ba), f(ba)] = f(ba).
\end{aligned}$$

By symmetry we can prove that $f(ba) \leq f((ab)^4) = f(ab)$.

($\longleftarrow$) Assume that conditions (1) and (2) are true. Then by condition (1) and Theorem 2.2.2, S is completely regular. Let A be a quasi-ideal of S and let $a \in A$.

Since S is completely regular and $a \in S$, there exists $x \in S$ such that $a \leq a^2xa^2$. Then

$$\begin{aligned} a &\leq a^2xa^2 \in (a^2Sa^2) = a(a(Sa)a) \subseteq a(aSa) \\ &\subseteq a(aSa] \\ &= a((aS] \cap (Sa]) \\ &\quad \text{(since } (aS] \text{ is a right ideal and } (Sa] \text{ a left ideal of } S). \\ &\subseteq A((AS] \cap (SA]) \\ &\subseteq AA, \end{aligned}$$

and so $A \subseteq AA \subseteq (A^2]$. On the other hand, since A is a subsemigroup of S, we have $A^2 \subseteq A \Longrightarrow (A^2] \subseteq (A] = A$.

Let A and B be any quasi-ideals of S and let $x \in (AB]$, then $x \leq ab$ for some $a \in A$ and $b \in B$. We consider the quasi-ideal $Q(ab)$ generated by ab. Then by Lemma 1.4.13, the characteristic function $f_{Q(ab)}$ of $Q(ab)$ is a fuzzy quasi-idealof S. By hypthesis

$$f_{Q(ab)}(ba) = f_{Q(ab)}(ab).$$

Since $ab \in Q(ab)$, we have $f_{Q(ab)}(ab) = 1$ and $f_{Q(ab)}(ba) = 1 \longrightarrow ba \in Q(ab) = (ab \cup (abS \cap Sab)]$. Then $ba \leq ab$ or $ba \leq abx$ and $ba \leq yab$. If $ba \leq ab$ then $ba \leq ab \in AB$ and $x \in (AB]$. If $ba \leq abx$ and $ba \leq yab$, then $ba \leq (abx)(yab) = (abxya)b \in (ASA)B \subseteq (AS \cap SA)B \subseteq ((AS] \cap (SA])B \subseteq AB$ and so $ba \in (AB] \longrightarrow BA \subseteq (AB] \longrightarrow (BA] \subseteq ((AB]] = (AB]$. By symmetry we can prove that $(AB] \subseteq (BA]$. Thus $(AB] = (BA]$. □

2.5.2 Lemma

Let $(S, \cdot, \leq)$ be an ordered semigroup such that $a \leq a^2$ for all $a \in S$. Then for every fuzzy quasi-ideal f of S we have,

$$f(a) = f(a^2) \text{ for every } a \in S.$$

Proof. Let $a \in S$ such that $a \leq a^2$. Let f be a fuzzy quasi-ideal of S. Then f is a fuzzy subsemigroup of S. Then

$$f(a) \geq f(a^2) \geq \min\{f(a), f(a)\} = f(a).$$

□

2.5.3 Theorem

Let S be an ordered semigroup and $a \in S$ such that $a \leq a^2$ for all $a \in S$. Then the following are equivalent:

(i) $ab \in (baS] \cap (Sba]$ for each $a, b \in S$.

(ii) For every fuzzy quasi-ideal f of S, we have,

$$f(ab) = f(ba) \text{ for every } a, b \in S.$$

Proof. (i)$\longrightarrow$(ii). Let f be a fuzzy quasi-ideal of S. Since $ab \in (baS] \cap (Sba]$, then $ab \in (baS]$ and we have $ab \le (ba)x$ for some $x \in S$. By (i), we have $(ba)x \in (xbaS] \cap (Sxba]$. Then $(ba)x \in (Sxba]$ and we have $(ba)x \le (yx)(ba)$, so $ab \le (yx)(ba) \longrightarrow (yx, ba) \in A_{ab}$. Again, since $ab \in (Sba]$, then $ab \le z(ba)$ for some $z \in S$ and by (i) we have $z(ba) \in (bazS]$. Thus $z(ba) \le (ba)(zt)$ for some $t \in S$. So we have $ab \le (ba)(zt) \longrightarrow (ba, zt) \in A_{ab}$. Since f is a fuzzy quasi-ideal of S and $A_{ab} \neq \emptyset$, therefore

$$\begin{aligned} f(ab) &\ge ((f \circ 1) \wedge (1 \circ f))(ab) \\ &= \min[(f \circ 1)(ab), (1 \circ f)(ab)] \\ &= \min\left[\bigvee_{(y_1,z_1)\in A_{ab}} \min\{f(y_1), 1(z_1)\}, \bigvee_{(y_2,z_2)\in A_{ab}} \min\{1(y_2), f(z_2)\}\right] \\ &\ge \min[\min\{f(ba), 1(zt)\}, \min\{1(yx), f(ba)\}] \\ &= \min[\min\{f(ba), 1\}, \min\{1, f(ba)\}] \\ &= \min[f(ba), f(ba)] = f(ba). \end{aligned}$$

By symmetry we can prove that $f(ba) \le f(ab)$.

(ii)$\longrightarrow$(i). Let f be a fuzzy quasi-ideal of S. Since $a \le a^2$ for all $a \in S$, by Lemma 2.5.2, we have $f(a) = f(a^2)$. By (ii), we have $f(ba) = f(ab)$ for each $a, b \in S$. By Theorem 2.5.1, it follows that S, is a semilattice of left and right simple semigroups. Thus by hypothesis, there exists a semilattice Y and a family $\{S_\alpha\}_{\alpha\in Y}$ of left and right simple subsemigroups such that

(i) $S_\alpha \cap S_\beta = \emptyset$ for all $\alpha, \beta \in Y$ and $\alpha \neq \beta$,

(ii) $S = \bigcup_{\alpha\in Y} S_\alpha$, and

(iii) $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for all $\alpha, \beta \in Y$.

Let $a, b \in S$, we have to show that $a \in (baS] \cap (Sba]$. Let $\alpha, \beta \in Y$ be such that $a \in S_\alpha$ and $b \in S_\beta$. Then $ab \in S_\alpha S_\beta \subseteq S_{\alpha\beta}$ and $ba \in S_\beta S_\alpha \subseteq S_{\beta\alpha} = S_{\alpha\beta}$. Since $S_{\alpha\beta}$ is left and right simple we have $S_{\alpha\beta} = (S_{\alpha\beta}c]$ and $S_{\alpha\beta} = (cS_{\alpha\beta}]$ for each $c \in S_{\alpha\beta}$. Since $ab, ba \in S_{\alpha\beta}$, we have $ab \in (baS_{\alpha\beta}] \cap (S_{\alpha\beta}ba] \subseteq (baS] \cap (Sba]$. This complete the proof. □

2.6 Fuzzy generalized bi-ideals

In this section, we define fuzzy generalized bi-ideal in ordered semigroup and characterize different classes of ordered semigroups by the properties of their fuzzy generalized bi-ideals.

2.6.1 Definition

Let $(S, \cdot, \leq)$ be an ordered semigroup. A non-empty subset B of S is called a *generalized bi-ideal* of S if:

(1) $BSB \subseteq B$,

(2) $a \in B$, $S \ni b \leq a \longrightarrow b \in B$.

Obviously, every bi-ideal of S is a generalized bi-ideal of S, but the converse is not true.

2.6.2 Example

Consider the ordered semigroup $S = \{a, b, c, d\}$

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

$$\leq := \{(a,a), (b,b), (c,c), (d,d), (a,b)\}.$$

Its subsemigroups are: $\{a\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}$ and $\{a,b,c,d\}$.

All subsemigroups are bi-ideals.

Its generalized bi-ideals are: $\{a\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}$ and $\{a,b,c,d\}$. But $\{a,c\}, \{a,d\}$ and $\{a,c,d\}$ are not bi-ideals.

2.6.3 Definition

Let S be an ordered semigroup. A fuzzy subset f of S is called a *fuzzy generalized bi-ideal* of S if:

(1) $x \leq y \Longrightarrow f(x) \geq f(y)$.

(2) $f(xyz) \geq \min\{f(x), f(z)\}$ for all $x, y, z \in S$.

2.6.4 Proposition

Let $(S, \cdot, \leq)$ be an ordered semigroup and $\emptyset \neq B \subseteq S$. Then the following are equivalent:

(1) B is a generalized bi-ideal of S.

(2) The characteristic function f_B of B is a fuzzy generalized bi-ideal of S.

Proof. Straightforward. □

2.6.5 Proposition

Let $(S, \cdot, \leq)$ be an ordered semigroup. Then every fuzzy bi-ideal of S is a fuzzy generalized bi-ideal of S.

Proof. Straightforward. □

However the following example shows that the converse of the above Proposition is not true in general.

2.6.6 Example

Consider the ordered semigroup S as given in example 2.6.2 and define a fuzzy subset $f : S \longrightarrow [0,1]$. Then all fuzzy subsets f of S which satisfies

(i) $f(a) \geq f(x)$ for all $x \in S$, (ii) $f(b) \geq f(c)$ and (iii) $f(b) \geq f(d)$ are fuzzy bi-ideals.

All fuzzy subsets f of S which satisfies

$f(a) \geq f(x)$ for all $x \in S$ are fuzzy generalized bi-ideals.

The fuzzy generalized bi-ideals are not fuzzy bi-ideals. For example the fuzzy subset f defined by

$$f(a) = 0.5, \quad f(b) = 0, \quad f(c) = 0.2, \quad f(d) = 0$$

is a fuzzy generalized bi-ideal of S but not a fuzzy bi-ideal of S, because

$$0 = f(b) = f(cc) \ngeq f(c) \wedge f(c) = 0.2.$$

2.7 Regular and completely regular ordered semigroups

In this section, we prove that the concepts of fuzzy generalized bi-ideal and fuzzy bi-ideal in regular ordered semigroup coincide. We also characterize regular and completely regular ordered semigroups in terms of their fuzzy generalized bi-ideals.

2.7.1 Proposition

Let $(S, \cdot, \leq)$ be a regular ordered semigroup and B a generalized bi-ideal of S. Then B is a bi-ideal of S.

Proof. Let S be a regular ordered semigroup and B be a generalized bi-ideal of S. Let $a, b \in B$. Since S is regular, so there exists $x \in S$, such that $b \leq bxb \longrightarrow ab \leq a(bxb) = a(bx)b \in BSB \subseteq B \longrightarrow ab \in (B] = B$.

Thus B is a subsemigroup of S and hence a bi-ideal of S. □

The above Proposition shows that in a regular ordered semigroup the concepts of bi-ideal and generalized bi-ideal coincide.

2.7.2 Proposition

Let $(S,\cdot,\leq)$ be a regular ordered semigroup. Then every fuzzy generalized bi-ideal of S is a fuzzy bi-ideal of S.

Proof. Let f be a fuzzy generalized bi-ideal of a regular ordered semigroup S. Let $a, b \in S$. Since S is regular, so there exists $x \in S$, such that $b \leq bxb \longrightarrow ab \leq a(bxb)$. Thus we have,

$$f(ab) \geq f(a(bxb)) = f(a(bx)b) \geq \min\{f(a), f(b)\}$$
$$\text{(because } f \text{ is a fuzzy generalized bi-ideal of } S\text{)}$$

Thus f is a fuzzy subsemigroup of S and so, f is a fuzzy bi-ideal of S. □

2.7.3 Remark

In regular ordered semigroup the concepts of fuzzy generalized bi-ideal and fuzzy bi-ideal coincide.

2.7.4 Proposition

Let $(S,\cdot,\leq)$ be a regular ordered semigroup and let f be a fuzzy generalized bi-ideal of S. Then we have

$$\textit{for all } a \in S \textit{ such that } a \leq a^2 \longrightarrow f(a) = f(a^2).$$

Proof. Let $a \in S$ be such that $a \leq a^2$. Then

$$f(a) \geq f(a^2) = f(aa) \geq \min\{f(a), f(a)\} = f(a).$$

Hence $f(a) = f(a^2)$. □

2.7.5 Proposition

Let $(S,\cdot,\leq)$ be an ordered semigroup such that
(i) $(\forall x \in S)(x \leq x^2)$,
(ii) $(\forall\ a, b \in S)(ab \in (baS] \cap (Sba])$.
Then every fuzzy generalized bi-ideal f of S satisfies the following condition

$$\textit{for all } a, b \in S,\ f(ab) = f(ba).$$

Proof. Since $ab \in (baS] \cap (Sba]$, we have $ab \in (baS]$ and so $ab \leq bax$ for some $x \in S$. Using (ii), we get $(ba)x \in (xbaS] \cap (Sxba]$ and thus $bax \leq yxba$ for some $y \in S$. It follows from (i) that

$$ab \leq (ba)x \leq (ba)^2x = ba(bax) \leq ba(yxba)$$

so

$$\begin{aligned} f(ab) &\geq f(ba(yxba)) = f(ba(yx)ba) \\ &\geq \min\{f(ba), f(ba)\} = f(ba). \end{aligned}$$

By symmetry we can prove that $f(ba) \geq f(ab)$. Hence $f(ab) = f(ba)$. □

2.7.6 Theorem

Let $(S, \cdot, \leq)$ be an ordered semigroup. If S is regular, left and right simple, then every fuzzy generalized bi-ideal of S is constant.

Proof. Since S is regular, so every fuzzy generalized bi-ideal of S is a fuzzy bi-ideal of S and the proof follows from Theorem 1.3.18. □

In this section, denote by $B(a) = (a \cup aSa]$, the generalized bi-ideal of S, generated by a.

2.7.7 Theorem

Let $(S, \cdot, \leq)$ be an ordered semigroup. If every fuzzy generalized bi-ideal f of S satisfies $f(x) = f(x^2)$ for all $x \in S$, then S is completely regular.

Proof. Assume that every fuzzy generalized bi-ideal f of S satisfies $f(x) = f(x^2)$ for all $x \in S$. Let $a \in S$. Note that $B := (a^2 \cup a^2Sa^2]$ is the generalized bi-ideal of S generated by a^2. Thus, by Proposition 2.6.4, $f_{B(a^2)}$ is a fuzzy generalized bi-ideal of S. By hypothesis, $f_{B(a^2)}(a) = f_{B(a^2)}(a^2)$. Since $a^2 \in B(a^2)$ we have $f_{B(a^2)}(a^2) = 1$ and so $f_{B(a^2)}(a) = 1$. Thus $a \in B := (a^2 \cup a^2Sa^2]$, which implies that $a \leq x$ for some $x \in a^2 \cup a^2Sa^2$. If $x = a^2$, then

$$a \leq x = a^2 = aa \leq a^2a^2 = aaa^2 \leq a^2aa^2,$$

and so $a \in (a^2Sa^2]$.

If $x \in a^2Sa^2$ then obviously $a \in (a^2Sa^2]$. Hence S is completely regular. □

2.7.8 Lemma

Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the following are equivalent:

(1) S is regular.

(2) $B \cap L \subseteq (BL]$ for every generalized bi-ideal B and every left ideal L of S.

(3) $B(a) \cap L(a) \subseteq (B(a)L(a)]$ for every $a \in S$.

Proof. (1) $\longrightarrow$ (2). Let S be a regular ordered semigroup, B a generalized bi-ideal, and L a left ideal of S then $B \cap L \subseteq (BL]$. In fact: If $a \in B \cap L$, then $a \in B$ and $a \in L$. Since S is regular, there exists $x \in S$ such that $a \leq a(xa) \in B(SL) \subseteq BL$. Thus $a \in (BL]$.

$(2) \longrightarrow (3)$. Obvious
$(3) \longrightarrow (1)$. Let $a \in S$. Then

$$\begin{aligned} a &\in B(a) \cap L(a) \subseteq (B(a)L(a)] \\ &= ((a \cup aSa](a \cup Sa]] \\ &\subseteq (((a \cup aSa)(a \cup Sa)]] \\ &= ((a \cup aSa)(a \cup Sa)] \\ &= (a^2 \cup aSa \cup aSa^2 \cup aSaSa] \\ &= (a^2 \cup aSa]. \end{aligned}$$

This implies that $a \leq a^2$ or $a \leq axa$ for some $x \in S$. This shows that S is a regular ordered semigroup. □

2.7.9 Theorem

An ordered semigroup S is regular if and only if for every fuzzy generalized bi-ideal f and every fuzzy left ideal g of S, we have,

$$f \wedge g \preceq f \circ g.$$

Proof. Let S be a regular ordered semigroup, f a fuzzy generalized bi-ideal and g a fuzzy left ideal of S. Then for each $a \in S$, there exists $x \in S$, such that $a \leq a(xa)$. Then $(a, xa) \in A_a$. Since $A_a \neq \emptyset$, we have

$$\begin{aligned} (f \circ g)(a) := & \bigvee_{(y,z) \in A_a} \min\{f(y), g(z)\} \\ &\geq \min\{f(a), g(xa)\}. \end{aligned}$$

Since g is a fuzzy left ideal of S, we have $g(xa) \geq g(a)$. Thus

$$\begin{aligned} (f \circ g)(a) &\geq \min\{f(a), g(xa)\} \\ &\geq \min\{f(a), g(a)\} \\ &= (f \wedge g)(a). \end{aligned}$$

Therefore, $(f \wedge g)(a) \leq (f \circ g)(a)$ for all a in S. Hence $f \wedge g \preceq f \circ g$.

Conversely, assume that $f \wedge g \preceq f \circ g$ for every fuzzy generalized bi-ideal f and every fuzzy left ideal g of S. Then S is regular. In fact: By Lemma 2.7.8, it is enough to prove that

$$B(a) \cap L(a) \subseteq (B(a)L(a)] \text{ for all } a \in S.$$

Let $b \in B(a) \cap L(a)$. Then $b \in (B(a)L(a)]$. Indeed: Since $B(a)$ is the generalized bi-ideal and $L(a)$ the left ideal of S, generated by a, respectively. Then $f_{L(a)}$ is a fuzzy left ideal and $f_{B(a)}$ a fuzzy generalized bi-ideal of S. Then by hypothesis,

$$(f_{B(a)} \wedge f_{L(a)})(b) \leq (f_{B(a)} \circ f_{L(a)})(b).$$

As $b \in B(a)$ and $b \in L(a)$, therefore $f_{B(a)}(b) := 1$, and $f_{L(a)}(b) := 1$. Thus we have,

$$\min\{f_{B(a)}(b), f_{L(a)}(b)\} = 1$$

and so,

$$(f_{B(a)} \circ f_{L(a)})(b) = 1.$$

By Proposition 1.3.10 part (i)

$$f_{B(a)} \circ f_{L(a)} = f_{(B(a)L(a)]}.$$

Thus, $f_{(B(a)L(a)]}(b) = 1 \longrightarrow b \in (B(a)L(a)]$. Hence $B(a) \cap L(a) \subseteq (B(a)L(a)]$. □

2.7.10 Lemma

Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the following are equivalent:

(1) S is regular.

(2) $B \cap I = (BIB]$ for every generalized bi-ideal B and every ideal I of S.

(3) $B(a) \cap I(a) = (B(a)I(a)B(a)]$ for every $a \in S$.

Proof. (1) $\longrightarrow$ (2). Let S be a regular ordered semigroup, B a generalized bi-ideal and I an ideal of S. Then $B \cap I = (BIB]$. In fact: Let $a \in B \cap I$, then $a \in B$ and $a \in I$. Since S is regular, there exists $x \in S$, such that

$$\begin{aligned} a &\leq axa \leq ax(axa) = a(xax)a \in B(SIS)B \subseteq BIB. \\ &\longrightarrow a \in (BIB]. \end{aligned}$$

On the other hand $(BIB] \subseteq (BSB] \subseteq (B] = B$, and $(BIB] \subseteq (SIS] \subseteq (I] = I$. Thus $B \cap I = (BIB]$.

(2) $\longrightarrow$ (3). Obvious

(3) $\longrightarrow$ (1). Let $a \in S$, $B(a)$ be the generalized bi-ideal and $I(a)$ be the ideal of S generated by a, respectively. Then

$$\begin{aligned} a &\in B(a) \cap I(a) = (B(a)I(a)B(a)] \\ &= ((a \cup aSa](a \cup Sa \cup aS \cup SaS](a \cup aSa]] \\ &\subseteq (((a \cup aSa)(a \cup Sa \cup aS \cup SaS)(a \cup aSa)]] \\ &= ((a \cup aSa)(a \cup Sa \cup aS \cup SaS)(a \cup aSa)] \\ &= ((a^2 \cup aSa \cup a^2S \cup aSaS \cup aSa^2 \cup aSaSa \cup aSa^2S \cup aSaSaS)(a \cup aSa)] \\ &= ((a^2 \cup aSa \cup a^2S \cup aSaS)(a \cup aSa)] \\ &= (a^3 \cup a^3Sa \cup aSa^2 \cup aSa^2Sa \cup a^2Sa \cup a^2SaSa \cup aSaSa \cup aSaSaSa] \\ &= (a^3 \cup aSa]. \end{aligned}$$

Then $a \leq a^3 = aaa$ or $a \leq axa$ for some $x \in S$. Thus S is regular. □

2.7.11 Proposition

Let $(S, \cdot, \leq)$ be an ordered semigroup, f a fuzzy generalized bi-ideal and g a fuzzy ideal of S. Then we have,

$$f \circ g \circ f \preceq g \wedge f.$$

Proof. Let f be a fuzzy generalized bi-ideal and g a fuzzy ideal of S. Let $a \in S$. Then $(f \circ g \circ f)(a) \leq (g \wedge f)(a)$. In fact: If $A_a = \emptyset$, then $(f \circ g \circ f)(a) := 0$. Since $(f \wedge g)(a) \geq 0$, so $(f \circ g \circ f)(a) \leq (g \wedge f)(a)$. Let $A_a \neq \emptyset$, then

$$\begin{aligned}(f \circ g \circ f)(a) := & \bigvee_{(p,q)\in A_a} \min\{f(p), (g \circ f)(q)\} \\ = & \bigvee_{(p,q)\in A_a} \min\{f(p), \bigvee_{(p_1,q_1)\in A_q} \min\{g(p_1), f(q_1)\}\} \\ = & \bigvee_{(p,q)\in A_a} \bigvee_{(p_1,q_1)\in A_q} \min\{f(p), g(p_1), f(q_1)\} \qquad (1)\end{aligned}$$

For each $(p, q) \in A_a$ and $(p_1, q_1) \in A_q$ we have

$$a \leq pq \text{ and } q \leq p_1 q_1. \text{ Thus } a \leq p p_1 q_1.$$

Since f is a fuzzy generalized bi-ideal of S, so we have

$$f(a) \geq f(pp_1q_1) \geq \min\{f(p), f(q_1)\}$$

Since g is a fuzzy ideal of S, so we have

$$g(a) \geq g(pp_1q_1) \geq g(p_1).$$

Thus

$$\begin{aligned}\min\{f(a), g(a)\} \geq & \min\{\min\{f(p), f(q_1)\}, g(p_1)\}\} \\ = & \min\{f(p), g(p_1), f(q_1)\}.\end{aligned}$$

Therefore, from (1) we get

$$f \circ g \circ f \preceq g \wedge f.$$

□

2.7.12 Theorem

An ordered semigroup S is regular if and only if for every fuzzy generalized bi-ideal f and every fuzzy ideal g of S we have,

$$f \wedge g = f \circ g \circ f.$$

Proof. Let S be a regular ordered semigroup. Let f be a fuzzy generalized bi-ideal, and g a fuzzy ideal of S. Then for each $a \in S$, there exists $x \in S$, such that

$$a \leq axa \leq (axa)xa.$$

Thus $(axa, xa) \in A_a$. Hence

$$\begin{aligned}(f \circ g \circ f)(a) := & \bigvee_{(p,q) \in A_a} \min\{f(p), (g \circ f)(q)\} \\ \geq & \min\{f(axa), (g \circ f)(xa)\} \\ = & \min\{f(axa), \bigvee_{(p_1,q_1) \in A_{xa}} \min\{g(p_1), f(q_1)\}\} \\ \geq & \min\{f(axa), \min\{g(xax), f(axa)\}\} \\ (\text{since } xa \leq & \; x(axa) \leq (xax)(axa)) \\ = & \min\{f(axa), g(xax)\}\end{aligned}$$

As f is a fuzzy generalized bi-ideal and g a fuzzy ideal of S, so we have, $f(axa) \geq f(a)$, $g(xax) \geq g(a)$. Thus,

$$\min\{f(axa), g(xax)\} \geq \min\{f(a), g(a)\} = (f \wedge g)(a).$$

Therefore, $f \wedge g \preceq f \circ g \circ f$.
On the other hand, by Proposition 2.7.11, we have, $f \circ g \circ f \preceq f \wedge g$. Thus,

$$f \wedge g = f \circ g \circ f.$$

Conversely, assume that $f \wedge g = f \circ g \circ f$ for every fuzzy generalized bi-ideal f and every fuzzy ideal g of S. Then S is regular. In fact: By Lemma 2.7.10, it is enough to show that

$$B(a) \cap I(a) = (B(a)I(a)B(a)] \text{ for all } a \in S.$$

Let $b \in B(a) \cap I(a)$. Then $b \in (B(a)I(a)B(a)]$. Indeed: Since $B(a)$ is the generalized bi-ideal and $I(a)$ the ideal of S, generated by a, respectively. Then $f_{I(a)}$ is a fuzzy ideal and $f_{B(a)}$ a fuzzy generalized bi-ideal of S. Thus by hypothesis,

$$(f_{B(a)} \wedge f_{I(a)})(b) = (f_{B(a)} \circ f_{I(a)} \circ f_{B(a)})(b).$$

As $b \in B(a)$ and $b \in I(a)$, we have, $f_{B(a)}(b) := 1$ and $f_{I(a)}(b) := 1$. Thus,

$$\min\{f_{B(a)}(b), f_{I(a)}(b)\} = 1$$

and so,

$$(f_{B(a)} \circ f_{I(a)} \circ f_{B(a)})(b) = 1$$

By Proposition 1.3.10,

$$f_{B(a)} \circ f_{I(a)} \circ f_{B(a)} = f_{(B(a)I(a)B(a)]}$$

Hence,

$$f_{(B(a)I(a)B(a)]}(b) = 1 \longrightarrow b \in (B(a)I(a)B(a)].$$

Therefore

$$B(a) \cap I(a) \subseteq (B(a)I(a)B(a)]$$

But $(B(a)I(a)B(a)] \subseteq B(a) \cap I(a)$ always true. Thus,

$$B(a) \cap I(a) = (B(a)I(a)B(a)]$$

□

2.7.13 Lemma

Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the following are equivalent:

(1) S is regular.

(2) $R \cap B \cap L \subseteq (RBL]$ for every right ideal R, every generalized bi-ideal B and every left ideal L of S.

(3) $R(a) \cap B(a) \cap L(a) \subseteq (R(a)B(a)L(a)]$ for every $a \in S$.

Proof. (1) $\longrightarrow$ (2). Let S be a regular ordered semigroup. Then $R \cap B \cap L \subseteq (RBL]$ for every right ideal R, every generalized bi-ideal B and every left ideal L of S. In fact: Let $a \in R \cap B \cap L$, then $a \in R$, $a \in B$ and $a \in L$. Since S is regular, there exists $x \in S$, such that

$$\begin{aligned} a &\leq axa \leq ax(axa) \leq (axa)x(axa) \\ &= ax(axa)xa \in (RS)(BSB)(SL) \\ &\subseteq RBL \\ &\Longrightarrow a \in (RBL]. \end{aligned}$$

(2) $\longrightarrow$ (3). Obvious.

(3) $\longrightarrow$ (1). Let $a \in S$, $R(a)$ be the right ideal, $B(a)$ be the generalized bi-ideal and $L(a)$ be the left ideal of S generated by a, respectively. Then

$$\begin{aligned} a &\in R(a) \cap B(a) \cap L(a) \subseteq (R(a)B(a)L(a)] \\ &\subseteq ((R(a)S)L(a)] \\ &\subseteq (R(a)L(a)] \\ &= ((a \cup aS](a \cup Sa]] \\ &= (((a \cup aS)(a \cup Sa)]] \\ &= ((a \cup aS)(a \cup Sa)] \\ &= (a^2 \cup aSa \cup aS^2a] \\ &= (a^2 \cup aSa] \end{aligned}$$

Then $a \leq a^2$ or $a \leq axa$. If $a \leq a^2$, then $a \leq a^2 = aa \leq aa^2 = aaa$. Thus S is regular. □

2.7.14 Theorem

An ordered semigroup S is regular if and only if for every fuzzy right ideal f, every fuzzy generalized bi-ideal g and every fuzzy left ideal h of S, we have,

$$f \wedge g \wedge h \preceq f \circ g \circ h.$$

Proof. Suppose S is a regular ordered semigroup, f a fuzzy right, g a fuzzy generalized bi-ideal and h a fuzzy left ideal of S. Then for every $a \in S$, there exists $x \in S$, such that $a \leq (ax)a$. Then $(ax, a) \in A_a$. Since $A_a \neq \emptyset$, we have

$$\begin{aligned}
(f \circ g \circ h)(a) \quad : &= \bigvee_{(p,q) \in A_a} \min\{f(p), (g \circ h)(q)\} \\
&\geq \min\{f(ax), (g \circ h)(a)\} \\
&= \min\{f(ax), \bigvee_{(p_1,q_1) \in A_a} \min\{g(p_1), h(q_1)\} \\
&\geq \min\{f(ax), \min\{g(axa), h(xa)\} \text{ (since } a \leq a(xa) \leq (axa)xa \text{)} \\
&= \min\{f(ax), g(axa), h(xa)\}
\end{aligned}$$

As f is a fuzzy right ideal, g a fuzzy generalized bi-ideal and h a fuzzy left ideal of S, we have, $f(ax) \geq f(a)$, $g(axa) \geq g(a)$ and $h(xa) \geq h(a)$. Thus

$$\min\{f(ax), g(axa), h(xa)\} \geq \min\{f(a), g(a), h(a)\} = (f \wedge g \wedge h)(a)$$

Therefore,

$$f \wedge g \wedge h \preceq f \circ g \circ h.$$

Conversely, assume, that $f \wedge g \wedge h \preceq f \circ g \circ h$ for every fuzzy right ideal f, every fuzzy generalized bi-ideal g and every fuzzy left ideal h of S. Then S is regular. In fcat: By Lemma 2.7.13, we have to prove that

$$R(a) \cap B(a) \cap L(a) \subseteq (R(a)B(a)L(a)] \text{ for every } a \in S.$$

Let $b \in R(a) \cap B(a) \cap L(a)$. Then $b \in (R(a)B(a)L(a)]$. Indeed: Since $R(a)$ is the right ideal, $B(a)$ the generalized bi-ideal and $L(a)$ the left ideal of S generated by a, respectively, therefore by Lemmas 1.2.5 and Proposition 2.6.4, $f_{R(a)}$, $f_{L(a)}$ and $f_{B(a)}$ are fuzzy right ideal, fuzzy left ideal and fuzzy generalized bi-ideal of S. By hypothesis, we have,

$$(f_{R(a)} \wedge f_{B(a)} \wedge f_{L(a)})(b) \leq (f_{R(a)} \circ f_{B(a)} \circ f_{L(a)})(b).$$

As, $b \in R(b)$, $b \in B(a)$, and $b \in L(a)$, we have, $f_{R(a)}(b) := 1, f_{B(a)}(b) := 1$ and $f_{L(a)}(b) := 1$. Thus,

$$\min\{f_{R(a)}(b), f_{B(a)}(b), f_{L(a)}(b)\} = 1$$

and so

$$(f_{R(a)} \circ f_{B(a)} \circ f_{L(a)})(b) = 1.$$

By Proposition 1.3.10 (i), $f_{R(a)} \circ f_{B(a)} \circ f_{L(a)} = f_{(R(a)B(a)L(a)]}$. Thus,

$$f_{(R(a)B(a)L(a)]}(b) = 1 \longrightarrow b \in (R(a)B(a)L(a)].$$

Therefore, $R(a) \cap B(a) \cap L(a) \subseteq (R(a)B(a)L(a)]$. □

2.8 Chararacterizations of regular and intra-regular ordered semigroups in terms of fuzzy right ideals and fuzzy left ideals

In this section, we prove that an ordered semigroup S is intra-regular if and only if for each fuzzy right ideal f and each fuzzy left ideal g of S, we have $f \wedge g \preceq g \circ f$. We also prove that an ordered semigroup $(S, \cdot, \leq)$ is both regular and intra-regular if and only if for every fuzzy right ideal f and every fuzzy left ideal g of S, $f \wedge g \preceq f \circ g \wedge g \circ f$. The results given in this section are part of our submitted papers [135,136,137]. In this section by $B(a)$ we mean a bi-ideal of S generated by a.

2.8.1 Theorem

An ordered semigroup S is intra-regular if and only if for every fuzzy right ideal f and every fuzzy left ideal g of S, we have $f \wedge g \preceq g \circ f$.

Proof. Let S be an intra-regular ordered semigroup, f a fuzzy right ideal, and g a fuzzy left ideal of S. Then, $(f \wedge g)(a) \leq (g \circ f)(a)$ for each $a \in S$. In fact, since S is intra-regular, there exist $x, y \in S$ such that $a \leq xa^2y = (xa)(ay)$. Then $(xa, ay) \in A_a$. Since $A_a \neq \emptyset$, we have

$$\begin{aligned}(g \circ f)(a) \;:&= \bigvee_{(t,z)\in A_a} \min\{g(t), f(z)\} \\ &\geq \min\{g(xa), f(ay)\}.\end{aligned}$$

As g is a fuzzy left ideal and f a fuzzy right ideal of S, we have $g(xa) \geq g(a)$ and $f(ay) \geq f(a)$. Thus ,

$$\begin{aligned}(g \circ f)(a) &\geq \min\{g(xa), f(ay)\} \\ &\geq \min\{g(a), f(a)\} \\ &= (f \wedge g)(a).\end{aligned}$$

Therefore,

$$f \wedge g \preceq g \circ f.$$

Conversely, assume that, $f \wedge g \preceq g \circ f$ for every fuzzy right ideal f and every fuzzy left ideal g of S. Then S is intra-regular. In fact: By Lemma 1.1.13, it is enough to prove that

$$R(a) \cap L(a) \subseteq (L(a)R(a)] \text{ for all } a \in S.$$

Let $b \in R(a) \cap L(a)$. Since $R(a)$ is a right ideal and $L(a)$ is a left ideal of S, generated by a, so by Proposition 1.2.5, $f_{R(a)}$ is a fuzzy right ideal and $f_{L(a)}$ is a fuzzy left ideal of S. Then by hypothesis,

$$\left(f_{R(a)} \wedge f_{L(a)}\right)(b) \leq \left(f_{L(a)} \circ f_{R(a)}\right)(b).$$

As $b \in R(a)$ and $b \in L(a)$, we have $f_{R(a)}(b) := 1$ and $f_{L(a)}(b) := 1$. Thus we have

$$\min\{f_{R(a)}(b), f_{L(a)}(b)\} = 1, \text{ and so } \left(f_{L(a)} \circ f_{R(a)}\right)(b) = 1.$$

By Proposition 1.3.10, we have $f_{L(a)} \circ f_{R(a)} = f_{(L(a)R(a)]}$. Thus $f_{(L(a)R(a)]}(b) = 1$, and hence, $b \in (L(a)R(a)]$. □

2.8.2 Lemma

Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the following are equivalent:

(1) S is both regular and intra-regular.

(2) $R \cap L \subseteq (RL] \cap (LR]$ for every right ideal R and every left ideal L of S.

(3) $R(a) \cap L(a) \subseteq (R(a)L(a)] \cap (L(a)R(a)]$ for every $a \in S$.

Proof. Follows from Lemma 1.1.13, and Lemma 1.1.14. □

2.8.3 Theorem

An ordered semigroup S is both regular and intra-regular if and only if for every fuzzy right ideal f and every fuzzy left ideal g of S, we have $f \wedge g \preceq g \circ f \wedge f \circ g$.

Proof. Follows from Theorem 1.4.9, and Theorem 2.8.1. □

2.8.4 Lemma

Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the following are equivalent:

(1) S is regular.

(2) $B \cap I \cap L \subseteq (BIL]$ for every bi-ideal B, every ideal I and every left ideal L of S.

(3) $B(a) \cap I(a) \cap L(a) \subseteq (B(a)I(a)L(a)]$, for every $a \in S$.

Proof. (1) $\longrightarrow$ (2). Let S be a regular ordered semigroup, B a bi-ideal, I an ideal and L a left ideal of S. Let $a \in B \cap I \cap L$ then, $a \in B$, $a \in I$ and $a \in L$. Since S is regular, there exists $x \in S$ such that $a \leq axa \leq axaxa = a(xa)(xa) \in B(SI)(SL) \subseteq BIL$. Thus $a \in (BIL]$.

(2) $\longrightarrow$ (3). Obvious.

(3) $\longrightarrow$ (1). Let $a \in S$, $B(a)$ be the bi-ideal, $L(a)$ the left ideal and $I(a)$ the ideal of S generated by a, respectively. Then

$$\begin{aligned} a &\in B(a) \cap I(a) \cap L(a) \\ &\subseteq (B(a)I(a)L(a)] \\ &\subseteq (B(a)SL(a)] \\ &= (B(a)(SL(a))] \\ &\subseteq (B(a)L(a)] \\ &= ((a \cup a^2 \cup aSa](a \cup Sa]] \\ &= (((a \cup a^2 \cup aSa)(a \cup Sa)]] \\ &= ((a \cup a^2 \cup aSa)(a \cup Sa)] \\ &= (a^2 \cup aSa \cup a^3 \cup a^2Sa \cup aSa^2 \cup aSaSa] \\ &= (a^2 \cup aSa]. \end{aligned}$$

Thus $a \leq a^2$ or $a \leq axa$ for some $x \in S$. Hence S is regular. □

2.8.5 Theorem

An ordered semigroup S is regular if and only if for every fuzzy bi-ideal f, every fuzzy ideal g and every fuzzy left ideal h of S, we have

$$f \wedge g \wedge h \preceq f \circ g \circ h.$$

Proof. Let S be a regular ordered semigroup and f a fuzzy bi-ideal, g a fuzzy ideal and h a fuzzy left ideal of S. Then $f \wedge g \wedge h \preceq f \circ g \circ h$. In fact: For $a \in S$, there exists $x \in S$ such that

$$a \leq a(xa) \leq (axa)(xa) \leq (axa)(xaxa).$$

Then $(axa, xaxa) \in A_a$. Since $A_a \neq \emptyset$, we have,

$$\begin{aligned} (f \circ g \circ h)(a) \ &:= \bigvee_{(y,z) \in A_a} \min\{f(y), (g \circ h)(z)\} \\ &\geq \min\{f(axa), (g \circ h)(xaxa)\} \\ &= \min\{f(axa), \bigvee_{(s,t) \in A_{xaxa}} \min\{g(s), h(t)\}\} \\ &\geq \min\{f(axa), \min\{g(xa), h(xa)\}\} \end{aligned}$$

As f is a fuzzy bi-ideal, g a fuzzy ideal and h a fuzzy left ideal of S, so we have

$$f(axa) \geq f(a), g(xa) \geq g(a) \text{ and } h(xa) \geq h(a).$$

Thus

$$\begin{aligned}(f \circ g \circ h)(a) &\geq \min\{f(axa), \min\{g(xa), h(xa)\}\} \\ &\geq \min\{f(a), \min\{g(a), h(a)\}\} \\ &= (f \wedge g \wedge h)(a).\end{aligned}$$

Therefore,

$$f \wedge g \wedge h \preceq f \circ g \circ h.$$

Conversely, assume that, $f \wedge g \wedge h \preceq f \circ g \circ h$, for every fuzzy bi-ideal f, every fuzzy ideal g and every fuzzy left ideal h of S. Then S is regular. In fact, by Lemma 2.8.4, it is enough to prove that

$$B(a) \cap I(a) \cap L(a) \subseteq (B(a)I(a)L(a)] \text{ for all } a \in S.$$

Let $b \in B(a) \cap I(a) \cap L(a)$. Then $b \in (B(a)I(a)L(a)]$. Indeed, since $B(a)$ is the bi-ideal, $I(a)$ the ideal and $L(a)$ the left ideal of S generated by a. Then $f_{B(a)}$ is a fuzzy bi-ideal, $f_{I(a)}$ a fuzzy ideal and $f_{L(a)}$ a fuzzy left ideal of S. Hence by hypothesis,

$$(f_{B(a)} \wedge f_{I(a)} \wedge f_{L(a)})(b) \leq (f_{B(a)} \circ f_{I(a)} \circ f_{L(a)})(b).$$

Since $b \in B(a)$, $b \in I(a)$ and $b \in L(a)$, we have $f_{B(a)}(b) := 1, f_{I(a)}(b) := 1$ and $f_{L(a)}(b) := 1$. Thus

$$(f_{B(a)} \wedge f_{I(a)} \wedge f_{L(a)})(b) = 1$$

and so

$$(f_{B(a)} \circ f_{I(a)} \circ f_{L(a)})(b) = 1.$$

By Proposition 1.3.10,

$$(f_{B(a)} \circ f_{I(a)} \circ f_{L(a)})(b) = f_{(B(a)I(a)L(a)]}(b).$$

Thus,

$$f_{(B(a)I(a)L(a)]}(b) = 1$$

and we have $b \in (B(a)I(a)L(a)]$. □

2.8.6 Lemma

Let $(S,\cdot,\leq)$ be an ordered semigroup. Then the following are equivalent:
(1) S is regular.
(2) $B\cap L\subseteq (BL]$, for every bi-ideal B and every left ideal L of S
(3) $B\cap R\subseteq (RB]$ for every bi-ideal B and every right ideal R of S.
(4) $B(a)\cap L(a)\subseteq (B(a)L(a)]$ for every $a\in S$.
(5) $B(a)\cap R(a)\subseteq (R(a)B(a)]$ for every $a\in S$.

Proof. (1) $\longrightarrow$ (2). Let S be a regular orderd semigroup, B a bi-ideal and L a left ideal of S. Let $a\in B\cap L$, then $a\in B$ and $a\in L$. Since S is regular, there exists $x\in S$ such that $a\leq axa=a(xa)\in B(SL)\subseteq BL\longrightarrow a\in (BL]$. Hence $B\cap L\subseteq (BL]$.

(2) $\longrightarrow$ (4). Obvious.

(4) $\longrightarrow$ (1). Let $a\in S$, $B(a)$ be the bi-ideal and $L(a)$ be the left ideal of S generated by a, respectively. Then

$$\begin{aligned} a &\in B(a)\cap L(a)\subseteq (B(a)L(a)] \\ &= ((a\cup a^2\cup aSa](a\cup Sa]] \\ &\subseteq (((a\cup a^2\cup aSa)(a\cup Sa))]] \\ &= ((a\cup a^2\cup aSa)(a\cup Sa)] \\ &= (a^2\cup aSa\cup a^3\cup a^2Sa\cup aSa^2\cup aSaSa] \\ &= (a^2\cup aSa] \end{aligned}$$

Then $a\leq a^2$ or $a\leq axa$ for some $x\in S$. Thus S is regular.
Similarly, we can show that (1) $\longrightarrow$ (3) $\longrightarrow$ (5) $\longrightarrow$ (1). □

2.9 Characterizations of weakly regular ordered semigroups in terms of fuzzy right ideal and fuzzy two-sided ideal

In [11], Brown and McCoy considered the notion of weakly regular rings. These rings were later studied by Ramamurthy [128] and others. Adopting this notion to semigroup Ahsan, et al. [3], considered weakly regular semigroups. In the following we define right weakly regular ordered semigroup and characterize these ordered semigroups by the properties of their fuzzy ideals.

2.9.1 Definition

An ordered semigroup $(S,\cdot,\leq)$ is called *right weakly regular* if for every $a\in S$ there exist $x,y\in S$, such that $a\leq axay$ or, equivalently, (1) $a\in ((aS)^2]$ for every $a\in S$ and (2) $A\subseteq ((AS)^2]$ for every $A\subseteq S$.

2.9.2 Proposition

Let $(S, \cdot, \leq)$ be a right weakly regular ordered semigroup and B a generalized bi-ideal of S. Then B is a bi-ideal of S.

Proof. Let S be a right weakly regular ordered semigroup and B a generalized bi-ideal of S. Let $a, b \in B$. Since S is a right weakly regular, therefore there exist $x, y \in S$ such that $b \leq bxby \longrightarrow ba \leq (bxby)a = b(xby)a \in BSB \subseteq B \longrightarrow ba \in (B] = B$. Thus B is a subsemigroup and hence a bi-ideal of S. □

The above Proposition shows that in a right weakly regular ordered semigroup the concepts of generalized bi-ideal and bi-ideal coincide.

2.9.3 Proposition

Let S be a right weakly regular ordered semigroup and f a fuzzy generalized bi-ideal of S. Then f is a fuzzy bi-ideal of S.

Proof. Let S be a right weakly regular ordered semigroup and f be a fuzzy generalized bi-ideal of S. Let $a, b \in S$. Since S is left weakly regular, therefore there exist $x, y \in S$ such that

$$b \leq bxby \longrightarrow ba \leq (bxby)a = b(xby)a.$$

Since f is a fuzzy generalized bi-ideal of S, we have,

$$f(ba) \geq f(b(xby)a) \geq \min\{f(b), f(a)\}.$$

It follows that f is a fuzzy subsemigroup of S. Thus f is a fuzzy bi-ideal of S. □

By Propositions 2.6.5, and 2.9.3, we have the following:

2.9.4 Remark

In right weakly regular ordered semigroup the concepts of fuzzy generalized bi-ideal and fuzzy bi-ideal coincide.

2.9.5 Lemma

Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the following are equivalent:

(1) S is right weakly regular.

(2) $B \cap I \subseteq (BI]$ for every generalized bi-ideal B and every ideal I of S.

(3) $B(a) \cap I(a) \subseteq (B(a)I(a)]$ for every $a \in S$.

Proof. (1) $\longrightarrow$ (2). Let S be a right weakly regular ordered semigroup, B a generalized bi-ideal and I an ideal of S. Then $B \cap I \subseteq (BI]$. In fact: Let $a \in B \cap I$, then $a \in B$ and $a \in I$. Since S is right weakly regular, so there exist $x, y \in S$, such that

$$a \leq a(xay) \in B(SIS) \subseteq BI \longrightarrow a \in (BI].$$

(2) $\longrightarrow$ (3). Obvious.

(3) $\longrightarrow$ (1). Let $a \in S$. Then

$$\begin{aligned} a &\in B(a) \cap I(a) \subseteq (B(a)I(a)] \\ &= ((a \cup aSa](a \cup Sa \cup aS \cup SaS]] \\ &\subseteq (((a \cup aSa)(a \cup Sa \cup aS \cup SaS)]] \\ &= ((a \cup aSa)(a \cup Sa \cup aS \cup SaS)] \\ &= (a^2 \cup aSa \cup a^2S \cup aSaS \cup aSa^2 \cup aSaSa \cup aSa^2S \cup aSaSaS] \\ &= (a^2 \cup aSa \cup a^2S \cup aSaS]. \end{aligned}$$

Then $a \leq a^2$ or $a \leq axa$ or $a \leq a^2x$ or $a \leq axay$ for some $x, y \in S$. If $a \leq a^2$, then $a \leq a^2 = aa \leq a^2a^2 = aaaa$. If $a \leq axa$, then $a \leq axa \leq (axa)xa = axa(xa) = axay$,where $y = xa \in S$.

If $a \leq a^2x$, then $a \leq a^2x \leq a(a^2x)x = aa(x^2) = a(a)ay$, where $y = x^2 \in S$. Thus S is right weakly regular. □

2.9.6 Theorem

An ordered semigroup S is right weakly regular if and only if for every fuzzy generalized bi-ideal f and every fuzzy ideal g of S we have,

$$f \wedge g \preceq f \circ g.$$

Proof. Let S be a right weakly regular ordered semigroup, f a fuzzy generalized bi-ideal and g a fuzzy ideal of S. Then for each $a \in S$, there exist $x, y \in S$, such that

$$a \leq axay \leq axayxay = (axa)(yxay).$$

Then $(axa, yxay) \in A_a$. Since $A_a \neq \emptyset$, we have,

$$\begin{aligned} (f \circ g)(a) \; &:= \bigvee_{(p,q) \in A_a} \min\{f(p), g(q)\} \\ &\geq \min\{f(axa), g(yxay)\} \end{aligned}$$

Since f is a fuzzy generalized bi-ideal and g a fuzzy ideal of S, we have, $f(axa) \geq f(a)$ and $g(yxay) = g(yx(ay)) \geq g(ay) \geq g(a)$. Thus

$$\begin{aligned}(f \circ g)(a) &\geq \min\{f(axa), g(yxay)\} \\ &\geq \min\{f(a), g(a)\} \\ &= (f \wedge g)(a)\end{aligned}$$

Therefore, $f \wedge g \preceq g \circ f$.

Conversely, assume that $f \wedge g \preceq f \circ g$ for every fuzzy generalized bi-ideal f and every fuzzy ideal g of S. Then S is right weakly regular. In fact: By Lemma 2.9.5, we have to prove that

$$B(a) \cap I(a) \subseteq (B(a)I(a)] \text{ for all } a \in S.$$

Let $b \in B(a) \cap I(a)$. Since $I(a)$ is the ideal and $B(a)$ is the generalized bi-ideal of S generated by a, respectively. Then by Lemma 1.2.5 and Proposition 2.6.4, $f_{I(a)}$ is a fuzzy ideal and $f_{B(a)}$ a fuzzy generalized bi-ideal of S. By hypothesis, we have,

$$(f_{B(a)} \wedge f_{I(a)})(b) \leq (f_{B(a)} \circ f_{I(a)})(b).$$

As $b \in B(a)$ and $b \in I(a)$, we have $f_{B(a)} := 1$ and $f_{I(a)} := 1$. Thus,

$$\min\{f_{B(a)}(b), f_{I(a)}(b)\} = 1$$

and so,

$$(f_{B(a)} \circ f_{I(a)})(b) = 1.$$

By Proposition 1.3.10(i),

$$f_{B(a)} \circ f_{I(a)} = f_{(B(a)I(a)]}.$$

Thus, $f_{(B(a)I(a)]}(b) = 1 \longrightarrow b \in (B(a)I(a)]$. Therefore,

$$B(a) \cap I(a) \subseteq (B(a)I(a)].$$

□

2.9.7 Lemma

Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the following are equivalent:

(1) S is right weakly regular.

(2) $R \cap I \subseteq (RI]$ for every right ideal R and two-sided ideal I of S.

(3) $R(a) \cap I(a) \subseteq (R(a)I(a)]$ for every $a \in S$.

Proof. (1) $\longrightarrow$ (2). Let S be a right weakly regular ordered semigroup. Then for each right ideal R and each ideal I of S, we have $R \cap I \subseteq (RI]$. In fact, let $a \in R \cap I$, then $a \in R$ and $a \in I$. Since S is a right weakly regular, there exist $x, y \in S$ such that

$$a \leq axay = (ax)(ay) \in (RS)(IS) \subseteq RI.$$

Thus $a \in (RI]$.

(2) $\longrightarrow$ (3). Obvious.

(3) $\longrightarrow$ (1). Let $a \in S$, $R(a)$ be the right ideal and $I(a)$ the ideal of S generated by a, respectively. Then,

$$\begin{aligned}
a &\in R(a) \cap I(a) \subseteq (R(a)I(a)] \\
&= ((a \cup aS](a \cup aS \cup Sa \cup SaS]] \\
&= (((a \cup aS)(a \cup aS \cup Sa \cup SaS)]] \\
&= ((a \cup aS)(a \cup aS \cup Sa \cup SaS)] \\
&= (a^2 \cup a^2S \cup aSa \cup aSaS \cup aSa \cup aSaS \cup aSSa \cup aSSaS] \\
&= (a^2 \cup a^2S \cup aSa \cup aSaS]
\end{aligned}$$

Then $a \leq a^2$ or $a \leq a^2x$ or $a \leq axa$ or $a \leq axay$ for some $x, y \in S$.

If $a \leq a^2$ then $a \leq a^2 \leq a^3 \leq a^4$. If $a \leq a^2x$ then $a \leq a^2x = a(ax) \leq a^2x(ax) = a(ax)ax$. If $a \leq axa$ then $a \leq axa \leq (axa)(xa) = axa(xa)$. Hence S is right weakly regular. □

2.9.8 Theorem

An ordered semigroup S is right weakly regular if and only if for every fuzzy right ideal f and every fuzzy ideal g of S, we have,

$$f \wedge g \preceq f \circ g.$$

Proof. Let S be a right weakly regular ordered semigroup, f a fuzzy right ideal and g a fuzzy ideal of S. Then we have, $f \wedge g \preceq f \circ g$. In fact: For $a \in S$, there exist $x, y \in S$ such that $a \leq (ax)(ay)$. Then $(ax, ay) \in A_a$. Since $A_a \neq \emptyset$, we have

$$\begin{aligned}
(f \circ g)(a) \;\; : &= \bigvee_{(t,z) \in A_a} \min\{f(t), g(z)\} \\
&\geq \min\{f(ax), g(ay)\} \\
&\geq \min\{f(a), g(a)\} \\
&\quad \text{(since } f \text{ is a fuzzy right ideal and } g \text{ a fuzzy ideal of } S) \\
&\geq (f \wedge g)(a).
\end{aligned}$$

Therefore, $f \wedge g \preceq f \circ g$.

Conversely, assume that $f \wedge g \preceq f \circ g$ for every fuzzy right ideal f and every fuzzy ideal g of S. Then S is right weakly regular. In fact, by Lemma 2.9.7, it is enough to prove that

$$R(a) \cap I(a) \subseteq (R(a)I(a)] \text{ for all } a \in S.$$

Let $b \in R(a) \cap I(a)$. Then $b \in (R(a)I(a)]$. Indeed: Since $R(a)$ is the right ideal and $I(a)$ is the ideal of S generated by a, so $f_{R(a)}$ is a fuzzy right ideal and $f_{I(a)}$ a fuzzy ideal of S. Then by hypothesis,

$$\left(f_{R(a)} \wedge f_{I(a)}\right)(b) \leq \left(f_{R(a)} \circ f_{I(a)}\right)(b).$$

Since $b \in R(a)$ and $b \in I(a)$, we have $f_{R(a)}(b) := 1$ and $f_{I(a)}(b) := 1$. Thus

$$\min\{f_{R(a)}(b), f_{I(a)}(b)\} = 1,$$

and so

$$\left(f_{R(a)} \circ f_{I(a)}\right)(b) = 1.$$

By Proposition 1.3.10, we have

$$f_{R(a)} \circ f_{I(a)} = f_{(R(a)I(a)]}.$$

Thus, we have $f_{(R(a)I(a)]}(b) = 1$. Hence $b \in (R(a)I(a)]$. □

2.9.9 Lemma

Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the following are equivalent:

(1) S is right weakly regular.

(2) $B \cap I \cap R \subseteq (BIR]$ for every bi-ideal B, every right ideal R and every ideal I of S.

(3) $B(a) \cap I(a) \cap R(a) \subseteq (B(a)I(a)R(a)]$ for every $a \in S$.

Proof. (1) $\longrightarrow$ (2). Let S be a right weakly regular ordered semigroup, B a bi-ideal, I an ideal and R a right ideal of S. Let $a \in B \cap I \cap R$, then $a \in B$, $a \in I$ and $a \in R$. Since S is right weakly regular, there exist $x, y \in S$ such that $a \leq axay$. Thus

$$a \leq axay \leq axaxayy = a(xax)(ayy) \in B(SIS)(RS) \subseteq BIR.$$

Hence $a \in (BIR]$.

(2) $\longrightarrow$ (3). Obvious.

(3) $\longrightarrow$ (1). Let $a \in S$, $B(a)$ be the bi-ideal, $I(a)$ be the ideal and $R(a)$ be the right ideal of S generated by a, respectively. Then

$$\begin{aligned} a &\in B(a) \cap I(a) \cap R(a) \\ &\subseteq (B(a)I(a)R(a)] \\ &= ((a \cup a^2 \cup aSa](a \cup Sa \cup aS \cup SaS](a \cup aS]] \\ &\subseteq (a^3 \cup a^3S \cup aSa \cup aSaS]. \end{aligned}$$

Then $a \le a^3$ or $a \le a^3x$ or $a \le axa$ or $a \le axay$ for some x, $y \in S$. In all these cases we can show that there exist u, v in S such $a \le auav$. Hence S is right weakly regular. □

2.9.10 Theorem

An ordered semigroup S is right weakly regular if and only if for every fuzzy bi-ideal f, every fuzzy ideal g and every fuzzy right ideal h of S, we have

$$f \wedge g \wedge h \preceq f \circ g \circ h.$$

Proof. Let S be a right weakly regular ordered semigroup, f a fuzzy bi-ideal, g a fuzzy ideal and h a fuzzy right ideal of S. Then $f \wedge g \wedge h \preceq f \circ g \circ h$. In fact: For $a \in S$ there exist $x, y \in S$ such that $a \le a(xay) \le (axa)(xayy)$. Then $(axa, xay^2) \in A_a$. Since, $A_a \neq \emptyset$, we have

$$\begin{aligned}
(f \circ g \circ h)(a) \quad &:= \bigvee_{(y,z)\in A_a} \min\{f(y), (g \circ h)(z)\} \\
&\ge \min\{f(axa), (g \circ h)(xay^2)\} \\
&= \min\{f(axa), \bigvee_{(s,t)\in A_{xay^2}} \min\{g(s), h(t)\}\} \\
&\ge \min\{f(axa), \min\{g(xax), h(ay^3)\}\} \\
(\text{because } xay^2 &\le x(axay)y^2 = xaxay^3).
\end{aligned}$$

As f is a fuzzy bi-ideal, g a fuzzy ideal and h a fuzzy left ideal of S, so we have

$$f(axa) \ge f(a),\ g(xax) \ge g(a) \text{ and } h(ay^3) \ge h(a).$$

Thus

$$\begin{aligned}
(f \circ g \circ h)(a) &\ge \min\{f(axa), \min\{g(xax), h(ay^3)\}\} \\
&\ge \min\{f(a), \min\{g(a), h(a)\}\} \\
&= \min\{f(a), g(a), h(a)\} \\
&= (f \wedge g \wedge h)(a).
\end{aligned}$$

Therefore

$$f \wedge g \wedge h \preceq f \circ g \circ h.$$

Conversely, assume that $f \wedge g \wedge h \le f \circ g \circ h$, for every fuzzy bi-ideal f, every fuzzy ideal g and every fuzzy right ideal h of S. Then S is right weakly regular. In fact, by Lemma 2.9.9, it is enough to prove that

$$B(a) \cap I(a) \cap R(a) \subseteq (B(a)I(a)R(a)] \text{ for all } a \in S.$$

Let $b \in B(a) \cap I(a) \cap R(a)$. Then $b \in (B(a)I(a)R(a)]$. Indeed, since $B(a)$ is the bi-ideal, $I(a)$ the ideal and $R(a)$ the left ideal of S, generated by a. Then $f_{B(a)}$ is a fuzzy bi-ideal, $f_{I(a)}$ a fuzzy ideal and $f_{R(a)}$ a fuzzy left ideal of S. Then by hypothesis,

$$(f_{B(a)} \wedge f_{I(a)} \wedge f_{R(a)})(b) \leq (f_{B(a)} \circ f_{I(a)} \circ f_{R(a)})(b).$$

As $b \in B(a)$, $b \in I(a)$ and $b \in R(a)$, we have $f_{B(a)}(b) := 1$, $f_{I(a)}(b) := 1$ and $f_{R(a)}(b) := 1$. Thus

$$\min\{f_{B(a)}(b), f_{I(a)}(b), f_{R(a)}(b)\} = 1$$

and so

$$(f_{B(a)} \circ f_{I(a)} \circ f_{R(a)})(b) := 1.$$

On the other hand, by Proposition 1.3.10 (i),

$$f_{B(a)} \circ f_{I(a)} \circ f_{R(a)} = f_{(B(a)I(a)R(a)]}.$$

Thus

$$f_{(B(a)I(a)R(a)]}(b) = 1 \longrightarrow b \in (B(a)I(a)R(a)].$$

□

2.9.11 Lemma

Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the following are equivalent:

(1) S is right weakly regular.

(2) $B \cap I \subseteq (BI]$ for every bi-ideal B and every ideal I of S.

(3) $B(a) \cap I(a) \subseteq (B(a)I(a)]$ for every $a \in S$.

Proof. (1) $\longrightarrow$ (2). Let S be a right weakly regular ordered semigroup and $a \in B \cap I$, then $a \in B$ and $a \in I$, where B is a bi-ideal and I is an ideal of S. Since S is right weakly regular, there exist $x, y \in S$, such that $a \leq axay = a(xay) \in BI$. Hence $a \in (BI]$.

(2) $\longrightarrow$ (3). Obvious.

(3) $\longrightarrow$ (1). Let $a \in S$, $B(a)$ be the bi-ideal and $I(a)$ be the ideal of S generated by a, respectively. Then

$$\begin{aligned}
a &\in B(a) \cap I(a) \subseteq (B(a)I(a)] \\
&= ((a \cup a^2 \cup aSa](a \cup Sa \cup aS \cup SaS]] \\
&= (((a \cup a^2 \cup aSa)(a \cup Sa \cup aS \cup SaS)]] \\
&= ((a \cup a^2 \cup aSa)(a \cup Sa \cup aS \cup SaS)] \\
&= (a^2 \cup aSa \cup a^2S \cup aSaS \cup a^3 \cup a^2Sa \cup a^3S \cup a^2SaS \cup aSa^2 \cup aSaSa \cup aSa^2S \cup aSaSaS] \\
&= (a^2 \cup aSa \cup a^2S \cup aSaS].
\end{aligned}$$

Thus $a \leq a^2$ or $a \leq axa$ or $a \leq a^2x$ or $a \leq axay$ for some $x, y \in S$.

If $a \leq a^2$ then $a \leq a.a \leq a.a^2 \leq a^4$. If $a \leq axa$ then $a \leq ax(axa) = axa(xa)$. If $a \leq a^2x = a.ax \leq a(a^2x)x = aaa(xx)$. Hence in each case $a \leq axay$ for some $x, y \in S$. Thus S is right weakly regular. □

2.9.12 Theorem

An ordered semigroup S is right weakly regular if and only if for every fuzzy bi-ideal f and every fuzzy ideal g of S, we have

$$f \wedge g \preceq f \circ g.$$

Proof. Let S be a right weakly regular ordered semigroup, f a fuzzy bi-ideal and g a fuzzy ideal of S. Then $f \wedge g \preceq f \circ g$. In fact : For $a \in S$, there exist $x, y \in S$ such that

$$a \leq axay \leq axaxayy = (axa)(xay^2).$$

Thus $(axa, xay^2) \in A_a$. Since $A_a \neq \emptyset$, we have

$$\begin{aligned}(f \circ g)(a) \; &:= \bigvee_{(y,z) \in A_a} \min\{f(y), g(z)\} \\ &\geq \min\{f(axa), g(xay^2)\} \\ &\geq \min\{f(a), g(a)\} \\ &\quad \text{(since } f \text{ is a fuzzy bi-ideal and } g \text{ a fuzzy ideal of } S\text{)} \\ &= (f \wedge g)(a).\end{aligned}$$

Therefore, $f \wedge g \preceq f \circ g$.

Conversely, assume that $f \wedge g \preceq f \circ g$ for every fuzzy bi-ideal f, every fuzzy ideal g of S. As each fuzzy right ideal of S is a fuzzy bi-ideal of S, so by hypothesis $f \wedge g \preceq f \circ g$ for every fuzzy right ideal f and every fuzzy ideal g of S. Hence S is right weakly regular. □

2.9.13 Lemma

Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the following are equivalent:

(1) S is right weakly-regular.

(2) $(R^2] = R$ for every right ideal R(resp. left ideal L) of S.

(3) $(R(a)^2] = R(a)$ for every $a \in S$.

Proof. (1) $\longrightarrow$ (2). Let R be a right ideal of a right weakly-regular ordered semigroup S. Then $(R^2] \subseteq (RS] \subseteq (R] = R$. For the reverse inclusion let $a \in R$. Since S is right weakly-regular, there exist $x, y \in S$ such that $a \leq axay \in (RS)(RS) \subseteq RR \subseteq (R^2]$. Thus $(R^2] = R$.

(2) $\longrightarrow$ (3). Obvious.

(3) $\longrightarrow$ (1). Let $a \in S$. Then,

$$\begin{aligned} a &\in R(a) = (R(a)^2] \\ &= ((a \cup aS](a \cup aS]] \\ &\subseteq (((a \cup aS)(a \cup aS))] \\ &= ((a \cup aS)(a \cup aS)] \\ &= (a^2 \cup a^2S \cup aSa \cup aSaS]. \end{aligned}$$

Then $a \le a^2$ or $a \le a^2x$ or $a \le axa$ or $a \le axay$ for some $x, y \in S$.
If $a \le a^2$ then

$$a \le a^2 = a.a \le a^2a^2 = aaaa.$$

If $a \le a^2x$ then

$$a \le a^2x = aax \le a^2xax = aaxax = a(ax)ax = ayax, \text{ where } y = ax \in S.$$

If $a \le axa$ then

$$a \le axa \le axa(xa) = axay, \text{ where } y = xa \in S.$$

Thus S is right weakly regular. □

2.9.14 Proposition

Let $(S, \cdot, \le)$ be an ordered semigroup and f a fuzzy subset of S. Then $1 \circ f$ (resp. $f \circ 1$) is a fuzzy left (resp. right) ideal of S.

Proof. Let f be a fuzzy subset of S and $x, y \in S$. If $A_y = \emptyset$, then $(1 \circ f)(y) := 0 \le (1 \circ f)(xy)$. Let $A_y \ne \emptyset$. Then for each $(a, b) \in A_y$, $ab \ge y \longrightarrow (xa)b \ge xy \longrightarrow (xa, b) \in A_{xy}$.

Also,

$$\{1(a), f(b)\} = \{1(xa), f(b)\}, \text{ because } 1(a) = 1 = 1(xa).$$

Hence

$$\begin{aligned} (1 \circ f)(y) &:= \bigvee_{(a,b) \in A_y} \min\{1(a), f(b)\} \le \bigvee_{(c,d) \in A_{xy}} \min\{1(c), f(d)\} \\ &= (1 \circ f)(xy). \end{aligned}$$

Let $x, y \in S$ such that $x \le y$. If $(a, b) \in A_y$ then $ab \ge y \longrightarrow ab \ge x \longrightarrow (a, b) \in A_x$. Hence $A_y \subseteq A_x$.

If $A_x = \emptyset$ then $A_y = \emptyset$ and so $(1 \circ f)(x) = 0 = (1 \circ f)(y)$. If $A_y \neq \emptyset$ then $A_x \neq \emptyset$, and $A_y \subseteq A_x$ so

$$\begin{aligned}(1 \circ f)(y) \quad : &= \bigvee_{(a,b)\in A_y} \min\{1(a), f(b)\} \\ &\leq \bigvee_{(c,d)\in A_x} \min\{1(c), f(d)\} \\ &= (1 \circ f)(x).\end{aligned}$$

Thus $1 \circ f$ is a fuzzy left ideal of S. Similarly we can prove that $f \circ 1$ is a fuzzy right ideal of S. □

2.9.15 Corollary

Let $(S, \cdot, \leq)$ be an ordered semigroup with identity element 1. Let f be a fuzzy subset of S. Then $1 \circ f$ (resp. $f \circ 1$) is the smallest fuzzy left (resp. right) ideal of S containing f.

Proof. By Proposition 2.9.14, $1 \circ f$ is a fuzzy left ideal of S. Let $x \in S$ then $(1, x) \in A_x$.

$$\begin{aligned}(1 \circ f)(x) \quad : &= \bigvee_{(a,b)\in A_x} \min\{1(a), f(b)\} \\ &\geq \min\{1(1), f(x)\} \\ &= f(x).\end{aligned}$$

Hence $f \prec 1 \circ f$.

Let g be a fuzzy left ideal of S such that $f \prec g$. Then $1 \circ f \preceq 1 \circ g = g$, by Proposition 2.9.14. Hence $1 \circ f$ is the smallest fuzzy left ideal of S containing f. □

2.9.16 Theorem

An ordered semigroup S is right weakly regular if and only if for every fuzzy right ideal f of S, we have $f \circ f = f$.

Proof. Let S be a right weakly regular ordered semigroup, f a fuzzy right ideal of S and $a \in S$. Then

$$(f \circ f)(a) = f(a).$$

Since, S is right weakly regular, there exist $x, y \in S$ such that $a \leq (ax)(ay)$. Then $(ax, ay) \in A_a$. Since $A_a \neq \emptyset$, we have,

$$\begin{aligned}(f \circ f)(a) := &\bigvee_{(p,q)\in A_a} \min\{f(p), f(q)\} \\ &\geq \min\{f(ax), f(ay)\}.\end{aligned}$$

Since f is a fuzzy left ideal of S, we have $f(ax) \geq f(a)$ and $f(ay) \geq f(a)$. Thus

$$\begin{aligned}(f \circ f)(a) &\geq \min\{f(ax), f(ay)\} \geq \min\{f(a), f(a)\} = f(a),\\ \text{and so } f &\preceq f \circ f.\end{aligned}$$

For the reverse inclusion, since f is a fuzzy right ideal of S, so it follows that $f \circ f \preceq f$. Thus $f \circ f = f$.

Conversely, assume that $f \circ f = f$, for every fuzzy right ideal of S. Then S is right weakly-regular. In fact: By Lemma 2.9.13, it is enough to prove that

$$R(a) = (R(a)^2] \text{ for all } a \in S.$$

Let $a \in S$, $b \in R(a)$. Then $b \in (R(a)^2]$. Indeed: Since $R(a)$ is a right ideal of S generated by a. Then $f_{R(a)}$ is a fuzzy right ideal of S. Then by hypothesis,

$$(f_{R(a)} \circ f_{R(a)})(b) = f_{R(a)}(b).$$

Since $b \in R(a)$, we have $f_{R(a)}(b) := 1$. Then it follows that $(f_{R(a)} \circ f_{R(a)})(b) = 1$. But by Proposition 1.3.10, we have, $f_{R(a)} \circ f_{R(a)} = f_{(R(a)^2]}$. Thus $f_{(R(a)^2]}(b) = 1 \longrightarrow b \in (R(a)^2]$. Thus, $R(a) \subseteq (R(a)^2]$. On the other hand, $(R(a)^2] \subseteq R(a)$ always true. Thus, $(R(a)^2] = R(a)$. □

2.9.17 Lemma

Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the following are equivalent:

(1) S is weakly-regular.

(2) $Q = (QS]^2 \cap (SQ]^2$ for every quasi-ideal Q of S.

Proof. (1) $\longrightarrow$ (2). Let S be a weakly regular ordered semigroup and Q a quasi-ideal of S. Then the left ideal $(SQ]$ and the right ideal $(QS]$ of S are idempotents, by Lemma 3.1. Thus we have

$$(QS]^2 \cap (SQ]^2 = (QS] \cap (SQ] \subseteq Q \text{ (since } Q \text{ is a quasi-ideal of } S)$$

For the reverse inclusion, let $a \in Q$. Since, S is left weakly-regular, there exist $x, y \in S$ such that $a \leq xaya \in (SQ)(SQ) \subseteq (SQ]^2$. Similarly, we can prove that $a \in (QS]^2$. Thus, $a \in (QS]^2 \cap (SQ]^2$. Therefore, $Q \subseteq (QS]^2 \cap (SQ]^2$. Hence we have $Q = (QS]^2 \cap (SQ]^2$.

(2) $\longrightarrow$ (1). Let R be any right ideal of S. Then R is a quasi-ideal of S. By (2), we have,

$$\begin{aligned}R &= (RS]^2 \cap (SR]^2\\ &\subseteq (RS]^2\\ &\subseteq (R]^2\\ &\subseteq (R^2]\\ &\subseteq (R] = R.\end{aligned}$$

Thus $(R^2] = R$, and so S is a right weakly regular ordered semigroup. On the same way we can prove that S is left weakly regular. □

2.9.18 Theorem

An ordered semigroup S is weakly regular if and only if for every fuzzy quasi-ideal f of S we have,

$$f = (f \circ 1)^2 \wedge (1 \circ f)^2.$$

Proof. Let S be a weakly regular ordered semirgoup, f a fuzzy quasi-ideal of S. Since f is fuzzy quasi-ideal of S, so by Proposition 2.9.14, $1 \circ f$ is a fuzzy left ideal and $f \circ 1$ is a fuzzy right ideal of S. Since S is weakly regular, by Theorems 2.9.16 and 2.9.17, $1 \circ f$ and $f \circ 1$ are idempotents. Hence

$$(f \circ 1)^2 \wedge (1 \circ f)^2 = (f \circ 1) \wedge (1 \circ f) \preceq f \text{ (since } f \text{ is a fuzzy quasi-ideal of } S).$$

In order to prove the reverse inclusion, let $a \in S$. Since S is right weakly regular, there exist $x, y \in S$ such that $a \leq (ax)(ay)$. Hence $(ax, ay) \in A_a$. Since $A_a \neq \emptyset$, we have,

$$\begin{aligned}
(f \circ 1)^2(a) \quad : &= \bigvee_{(p,q) \in A_a} \min\{(f \circ 1)(p), (f \circ 1)(q)\} \\
&\geq \min\{(f \circ 1)(ax), (f \circ 1)(ay)\} \\
&= \min\{\bigvee_{(u,v) \in A_{ax}} \min\{f(u), 1(v)\}, \bigvee_{(u,v) \in A_{ay}} \{f(u), 1(v)\}\} \\
&\geq \min\{\min\{f(a), 1(x)\}, \min\{f(a), 1(y)\}\} \\
&= \min\{\min\{f(a), 1\}, \min\{f(a), 1\}\} \\
&= \min\{f(a), f(a)\} \\
&= f(a),
\end{aligned}$$

and we have $f \preceq (f \circ 1)^2$. Similarly, we can show that $f \preceq (1 \circ f)^2$. Thus,

$$f \preceq (f \circ 1)^2 \wedge (1 \circ f)^2.$$

Hence,

$$f = (f \circ 1)^2 \wedge (1 \circ f)^2.$$

Conversely, assume that, f is a fuzzy right ideal of S so, f is a fuzzy quasi-ideal of S. By (2), we have,

$$\begin{aligned}
f &= (f \circ 1)^2 \wedge (1 \circ f)^2 \preceq (f \circ 1)^2 \preceq f \circ f \preceq f, \\
\text{(since } f \circ 1 &\preceq f),
\end{aligned}$$

and hence, $f = f \circ f$. Thus by Theorem 2.9.16, S is right weakly regular. By the same way we can prove that S is a left weakly regular. □

2.10 Characterizations of intra-regular and right weakly regular ordered semigroups in terms of fuzzy ideals

In this section we characterize intra-regular and right weakly regular ordered semigroups in terms of their fuzzy left, right, quasi-ideals and bi-ideals.

2.10.1 Lemma

Let $(S, \cdot, \leq)$ be an ordered semigroup with identity element 1. Then the following are equivalent:

(1) S is both intra-regular and right weakly regular.

(2) $L \cap R \cap Q \subseteq (QLR]$ for every quasi-ideal Q, every left ideal L and every right ideal R of S.

(3) $L(a) \cap R(a) \cap Q(a) \subseteq (Q(a)L(a)R(a)]$ for every $a \in S$.

Proof. (1) $\longrightarrow$ (2). Let S be both intra-regular and right weakly regular ordered semigroup. Then for each quasi-ideal Q, left ideal L and right ideal R of S we have,

$$L \cap R \cap Q \subseteq (QLR].$$

In fact: Let $a \in L \cap R \cap Q$ then $a \in L$, $a \in R$ and $a \in Q$. Since S is intra-regular, there exist $x, y \in S$ such that $a \leq xa^2y$, and S is right weakly regular, there exist $u, v \in S$ such that $a \leq auav$. Hence,

$$\begin{aligned} a &\leq auav \leq au(xaay)v \\ &= (a(ux)a)(a(yv)) \in Q(SL)(RS) \\ &\subseteq QLR \subseteq (QLR]. \end{aligned}$$

(2) $\longrightarrow$ (3). Obvious.

(3) $\longrightarrow$ (1). Let $a \in S$. Then

$$\begin{aligned} a &\in L(a) \cap R(a) \cap Q(a) \\ &\subseteq (Q(a)L(a)R(a)] \\ &\subseteq (SL(a)R(a)] \\ &\subseteq (L(a)R(a)] \\ &= ((Sa](aS]] \\ &= (((Sa)(aS)]] \\ &= ((Sa)(aS)] \\ &= (Sa^2S]. \end{aligned}$$

Thus, S is intra-regular.

Again,

$$\begin{aligned} a &\in L(a)\cap R(a)\cap Q(a)\\ &\subseteq (Q(a)L(a)R(a)]\\ &= ((Sa\cap aS](Sa](aS]]\\ &\subseteq (((Sa\cap aS)(Sa)(aS)]]\\ &= ((Sa\cap aS)(Sa)(aS)]\\ &= ((Sa\cap aS)(Sa^2S)]\\ &\subseteq ((aS)(Sa^2S)]\\ &= (aS^2a^2S]\\ &\subseteq (aSaS]. \end{aligned}$$

Thus S is right weakly regular. □

2.10.2 Theorem

An ordered semigroup S with identity element 1, is both intra-regular and right weakly regular if and only if for every fuzzy quasi-ideal f, every fuzzy left ideal g and every fuzzy right ideal h of S, we have,

$$f\wedge g\wedge h\preceq f\circ g\circ h.$$

Proof. Let S be both intra-regular and right weakly-regular ordered semigroup. Let f be a fuzzy left ideal, g a fuzzy right ideal and h a fuzzy quasi-ideal of S. Then for each $a\in S$, we have,

$$(f\wedge g\wedge h)(a)\preceq(f\circ g\circ h)(a).$$

Since S is intra-regular, there exist $x,y\in S$ such that $a\leq xa^2y$. Since S is right weakly regular, there exist $u,v\in S$, such that $a\leq auav$. Thus,

$$\begin{aligned} a &\leq auav\leq au(xaay)v\\ &= (a(ux)a)(a(yv)). \end{aligned}$$

Then $(a(ux)a,a(yv))\in A_a$. Since $A_a\neq\emptyset$, we have

$$\begin{aligned} (f\circ g\circ h)(a) &:= \bigvee_{(p,q)\in A_a}\min\{(f\circ g)(p),h(q)\}\\ &\geq \min\{(f\circ g)(a(ux)a),h(a(yv))\}\\ &= \min\{\bigvee_{(p_1,q_1)\in A_{(a(ux)a)}}\min\{f(p_1),g(q_1))\},\ h(a(yv))\}\\ &\geq \min\{\min\{f(a),g((ux)a)\},h(a(yv))\}\\ &\geq \min\{f(a),g((ux)a),h(a)\}. \end{aligned}$$

Since g is a fuzzy left ideal of S, we have $g((ux)a) \geq g(a)$, h a fuzzy right ideal of S, we have, $h(a(yv)) \geq h(a)$. Thus we have,

$$\min\{f(a), g((ux)a), h(a(yv))\} \geq \min\{f(a), g(a), h(a)\}.$$

Hence we have,

$$\begin{aligned} (f \circ g \circ h)(a) \ :&= \bigvee_{(p,q)\in A_a} \min\{(f \circ g)(p), h(q)\} \\ &\geq \min\{f(a), g((ux)a), h(a(yv))\} \\ &\geq (f \wedge g \wedge h)(a). \end{aligned}$$

Conversely, assume that $f \wedge g \wedge h \preceq f \circ g \circ h$, for every fuzzy quasi-ideal f, every fuzzy left ideal g and every fuzzy right ideal g of S. Then S is both intra-regular and right weakly regular ordered semigroup. In fact: By Lemma 2.10.1, it is enough to prove that

$$L(a) \cap R(a) \cap Q(a) \subseteq (Q(a)L(a)R(a)] \text{ for all } a \in S.$$

Let $a \in S$, and $b \in L(a) \cap R(a) \cap Q(a)$. Then $b \in (Q(a)L(a)R(a)]$. Indeed: Since $L(a)$ is a left ideal, $R(a)$ a right ideal and $Q(a)$ a quasi-ideal of S generated by a respectively. Then $f_{L(a)}$ is a fuzzy left ideal, $f_{R(a)}$ a fuzzy right ideal and $f_{Q(a)}$ is a fuzzy quasi-ideal of S. Thus by hypothesis,

$$\left(f_{L(a)} \wedge f_{R(a)} \wedge f_{Q(a)}\right)(b) \leq \left(f_{Q(a)} \circ f_{L(a)} \circ f_{R(a)}\right)(b).$$

Since $\left(f_{L(a)} \wedge f_{R(a)} \wedge f_{Q(a)}\right)(b) := \min\{f_{L(a)}(b), f_{R(a)}(b), f_{Q(a)}(b)\}$.
We have

$$\min\{f_{L(a)}(b), f_{R(a)}(b), f_{Q(a)}(b)\} \leq \left(f_{Q(a)} \circ f_{L(a)} \circ f_{R(a)}\right)(b).$$

Since $b \in L(a)$, $b \in R(a)$, and $b \in Q(a)$, hence, $f_{L(a)}(b) := 1$, $f_{R(a)}(b) := 1$ and $f_{Q(a)}(b) := 1$, then we have,

$$\begin{aligned} \min\{f_{L(a)}(b), f_{R(a)}(b), f_{Q(a)}(b)\} &= 1 \\ \text{and so } \left(f_{Q(a)} \circ f_{L(a)} \circ f_{R(a)}\right)(b) &= 1. \end{aligned}$$

But from Proposition 1.3.10, it follows that

$$f_{Q(a)} \circ f_{L(a)} \circ f_{R(a)} = f_{(Q(a)L(a)R(a)]}.$$

Thus,

$$f_{(Q(a)L(a)R(a)]}(b) = 1 \longrightarrow b \in (Q(a)L(a)R(a)].$$

Hence by Lemma 2.10.1, it follows that S is both intra-regular and right weakly regular. □

2.10.3 Lemma

Let $(S, \cdot, \leq)$ be an ordered semigroup with identity element 1. Then the following are equivalent:

(1) S is both intra-regular and right weakly regular.

(2) $L \cap R \cap B \subseteq (BLR]$ for every left ideal L, every right ideal R and every bi-ideal B of S.

(3) $L(a) \cap R(a) \cap B(a) \subseteq (B(a)L(a)R(a)]$ for every $a \in S$.

Proof. (1) $\longrightarrow$ (2). Let S be both intra-regular and right weakly regular ordered semigroup. Then $L \cap R \cap B \subseteq (BLR]$ for every left ideal L, right ideal R and bi-ideal B of S. In fact: Let $a \in L \cap R \cap B$ then $a \in L$, $a \in R$ and $a \in B$. Since S is intra-regular, there exist $x, y \in S$ such that $a \leq xa^2y$, and since S is right weakly regular, there exist $u, v \in S$ such that $a \leq auav$. Hence,

$$\begin{aligned} a &\leq auav \leq au(xaay)v \\ &= (a(ux)a)(a(yv)) \in B(SL)(RS) \\ &\subseteq BLR \subseteq (BLR] \end{aligned}$$

(2) $\longrightarrow$ (3). Obvious.

(3) $\longrightarrow$ (1). Let $a \in S$. Then

$$\begin{aligned} a &\in L(a) \cap R(a) \cap B(a) \\ &\subseteq (B(a)L(a)R(a)] \\ &\subseteq (SL(a)R(a)] \\ &\subseteq (L(a)R(a)] \\ &= ((Sa](aS]] \\ &= ((Sa)(aS)] \\ &= (Sa^2S]. \end{aligned}$$

Thus S is intra-regular ordered semigroup.

Also,

$$\begin{aligned} a &\in L(a) \cap R(a) \cap B(a) \\ &\subseteq (B(a)L(a)R(a)] \\ &= ((aSa](Sa](aS]] \\ &= (((aSa)(Sa)(aS))]] \\ &= ((aSa)(Sa)(aS)] \\ &= ((aSa)(Sa^2S)] \\ &\subseteq (aSaSa^2S] \\ &\subseteq (aSaS]. \end{aligned}$$

Hence, S is right weakly regular. □

2.10.4 Theorem

An ordered semigroup S is both intra-regular and right weakly regular if and only if for every fuzzy left ideal f, every fuzzy right ideal g and every fuzzy bi-ideal h of S we have,

$$f \wedge g \wedge h \preceq h \circ f \circ g.$$

Proof. Let S be both intra-regular and left weakly regular ordered semigroup. Let f be a fuzzy left ideal, g a fuzzy right ideal and h a fuzzy bi-ideal of S. Then for each $a \in S$, we have,

$$(f \wedge g \wedge h)(a) \leq (h \circ f \circ g)(a).$$

Since S is intra-regular, there exist $x, y \in S$ such that $a \leq xa^2y$. Since S is right weakly regular, there exist $u, v \in S$, such that $a \leq auav$. Thus,

$$\begin{aligned} a &\leq auav \leq au(xa^2y)v \\ &= (a(ux)a)(a(yv)). \end{aligned}$$

Then $(a(ux)a, a(yv)) \in A_a$. Since $A_a \neq \emptyset$, we have

$$\begin{aligned} (f \circ g \circ h)(a) \ &: = \bigvee_{(p,q)\in A_a} \min\{(h \circ f)(p), g(q)\} \\ &\geq \min\{(h \circ f)(a(ux)a), g(a(yv))\} \\ &= \min\{ \bigvee_{(p_1,q_1)\in A_{(a(ux)a)}} \min\{h(p_1), f(q_1)\}, g(a(yv))\} \\ &\geq \min\{\min\{h(a), f((ux)a)\}, g(a(yv))\} \\ &= \min\{h(a), f((ux)a)\}, g(a(yv))\}. \end{aligned}$$

Since f is a fuzzy left ideal of S, we have $f((ux)a) \geq f(a)$, g a fuzzy right ideal of S, we have, $g(a(yv)) \geq g(a)$. Thus we have,

$$\min\{h(a), f((ux)a)\}, g(a(yv))\} \geq \min\{f(a), g(a), h(a)\}$$

Hence we have,

$$\begin{aligned} (h \circ f \circ g)(a) \ &: = \bigvee_{(p,q)\in A_a} \min\{(h \circ f)(p), g(q)\} \\ &\geq \min\{h(a), f((ux)a)\}, g(a(yv))\} \\ &\geq (f \wedge g \wedge h)(a). \end{aligned}$$

Conversely, assume that $f \wedge g \wedge h \leq h \circ f \circ g$ for every fuzzy left ideal f, every fuzzy right ideal g and every fuzzy bi-ideal h of S. Then S is both intra-regular and right weakly regular. In fact: By Lemma 2.10.3, it is enough to prove that

$$L(a) \cap R(a) \cap B(a) \subseteq (B(a)L(a)R(a)] \text{ for all } a \in S.$$

Let $a \in S$, $b \in L(a) \cap R(a) \cap B(a)$. Then $b \in (B(a)L(a)R(a)]$. Since, $L(a)$ is a left ideal, $R(a)$ a right ideal and $B(a)$ a bi-ideal of S generated by a respectively. Then $f_{L(a)}$ is a fuzzy left ideal, $f_{R(a)}$ a fuzzy right ideal and $f_{B(a)}$ is a fuzzy bi-ideal of S. Thus by hypothesis,

$$\left(f_{L(a)} \wedge f_{R(a)} \wedge f_{B(a)}\right)(b) \leq \left(f_{B(a)} \circ f_{L(a)} \circ f_{R(a)}\right)(b).$$

Since $\left(f_{L(a)} \wedge f_{R(a)} \wedge f_{B(a)}\right)(b) := \min\{f_{L(a)}(b), f_{R(a)}(b), f_{B(a)}(b)\}$.

We have

$\min\{f_{L(a)}(b), f_{R(a)}(b), f_{B(a)}(b)\} \leq \left(f_{B(a)} \circ f_{L(a)} \circ f_{R(a)}\right)(b)$.

Since $b \in L(a)$, $b \in R(a)$, and $b \in B(a)$, hence, $f_{L(a)}(b) := 1$, $f_{R(a)}(b) := 1$ and $f_{B(a)}(b) := 1$, thus we have,

$$\min\{f_{L(a)}(b), f_{R(a)}(b), f_{B(a)}(b)\} = 1$$

and so $\left(f_{B(a)} \circ f_{L(a)} \circ f_{R(a)}\right)(b) = 1$.

But from Proposition 1.3.10, it follows that

$f_{B(a)} \circ f_{L(a)} \circ f_{R(a)} = f_{(B(a)L(a)R(a)]}$. Thus,

$f_{(B(a)L(a)R(a)]}(b) = 1 \longrightarrow b \in (B(a)L(a)R(a)]$.

Thus by Lemma 2.10.3, it follows that S is both intra-regular and right weakly regular. □

2.11 Characterizations of semisimple oredered semigroups in terms of fuzzy ideals

In [3] J. Ahsan and others studied fuzzy semisimple semigroups. Adopting this notion we study fuzzy semisimple ordered semigroups in terms of fuzzy left (resp. right, two-sided and interior) ideal of ordered semigroups. In this section, we prove that an ordered semigroup S is semisimple if and only if for every fuzzy two-sided ideal f of S we have, $f \circ f = f$.

2.11.1 Lemma

Let $(S, \cdot, \leq)$ be a semisimple ordered semigroup, f a fuzzy interior ideal of S. Then f is a fuzzy two-sided ideal of S.

Proof. Let f be a fuzzy interior ideal of S. Let $a, b \in S$. Since S is semisimple, there exist $x, y, z \in S$ such that $a \leq xayaz$. Thus $ab \leq xayazb = (xay)a(zb)$. Since f is a fuzzy interior ideal of S, so we have $f(ab) \geq f((xay)a(zb)) \geq f(a)$. Thus f is a fuzzy left ideal of S. Similarly we can prove that f is a fuzzy right ideal of S. Thus f is a fuzzy ideal of S. □

The following Proposition is a special case of Lemma 2.11.1.

2.11.2 Proposition

Let $(S, \cdot, \leq)$ be a semisimple ordered semigroup, I an interior ideal of S. Then I is a two-sided ideal of S.

2.11.3 Lemma

Let $(S, \cdot, \leq)$ be an ordered semigroup with identity element 1. Then the following are equivalent:

(1) S is semisimple.
(2) $I_1 \cap I_2 = (I_1 I_2]$ for all ideals I_1, I_2 of S.
(3) $I = (I^2]$ for every ideal I of S.
(4) $I(a) = (I(a)^2]$ for every $a \in S$.

Proof. $(1) \longrightarrow (2)$. Let I_1, and I_2 be ideals of S, and $a \in I_1 \cap I_2$. Then $a \in I_1$ and $a \in I_2$. Since S is semisimple, there exist $x, y, z \in S$ such that $a \leq xayaz$ Thus

$$a \in (SaSaS] \subseteq ((SI_1)(SI_2S)] \subseteq (I_1 I_2].$$

On the other hand, $(I_1 I_2] \subseteq I_1 \cap I_2$ always true. Thus,

$$I_1 \cap I_2 = (I_1 I_2].$$

$(2) \longrightarrow (3)$. Take $I_1 = I_2 = I$, then $I = I_1 \cap I_2 = (I_1 I_2] = (I^2]$.
$(3) \longrightarrow (4)$. Obvious.
$(4) \longrightarrow (1)$. Let $a \in S$. Then

$$\begin{aligned} a &\in I(a) = (I(a)^2] \\ &= (((SaS)(SaS)]] \\ &= ((SaS)(SaS)] \\ &\subseteq (SaSaS]. \end{aligned}$$

Thus S is a semisimple ordered semigroup. □

2.11.4 Theorem

An ordered semigroup S with identity element 1 is semisimple if and only if for every fuzzy two-sided ideal f of S we have,

$$f \circ f = f.$$

Proof. Let S be a semisimple ordered semigroup, and $a \in S$. Then $(f \circ f)(a) = f(a)$. In fact: Since S is semisimple, there exist $x, y, z \in S$ such that $a \leq (xay)(az)$. Then $(xay, az) \in A_a$. Since $A_a \neq \emptyset$, we have

$$\begin{aligned}(f \circ f)(a) \ : &= \bigvee_{(p,q)\in A_a} \min\{f(p), f(q)\} \\ &\geq \min\{f(xay), f(az)\}\end{aligned}$$

Since f is a fuzzy two-sided ideal of S, we have, $f(xay) \geq f(a)$, and $f(az) \geq f(a)$. Thus we have,

$$\begin{aligned}(f \circ f)(a) &\geq \min\{f(a), f(a)\} \\ &= f(a)\end{aligned}$$

For the reverse inclusion, since f is a fuzzy ideal of S so $f \circ f \leq f$ always hold. Thus $f \circ f = f$.

Conversely, assume that $f \circ f = f$ for every fuzzy two-sided ideal f of S. Then S is semisimple. In fact, by Lemma 2.11.13, it is enough to prove that

$$(I(a)^2] = I(a) \text{ for all } a \in S.$$

Let $b \in I(a)$. Then $b \in (I(a)^2]$. Indeed: Since $I(a)$ is an ideal of S, generated by a. Then $f_{I(a)}$ is a fuzzy ideal of S. By hypothesis,

$$(f_{I(a)} \circ f_{I(a)})(b) = f_{I(a)}(b).$$

Since $b \in I(a)$, we have, $f_{I(a)}(b) := 1$. Hence we have, $(f_{I(a)} \circ f_{I(a)})(b) = 1$. By Proposition 1.3.10, we have,

$$f_{I(a)} \circ f_{I(a)} = f_{(I(a)^2]}$$

Thus, $f_{(I(a)^2]}(b) = 1 \longrightarrow b \in (I(a)^2]$. Thus $I(a) \subseteq (I(a)^2]$. On the other hand, $(I(a)^2] \subseteq I(a)$ always true. Therefore, $(I(a)^2] = I(a)$ and S is semisimple. □

2.11.5 Lemma

Let $(S, \cdot, \leq)$ be an ordered semigroup with identity element 1. Then the following are equivalent:

(1) S is semisimple.

(2) $R \cap I \subseteq (IR]$ for each right ideal R and two-sided ideal I of S.

(3) $R(a) \cap I(a) \subseteq (I(a)R(a)]$ for every $a \in S$.

(4) $L \cap I \subseteq (LI]$ for ecah left ideal L and two-sided ideal I of S.

(5) $L(a) \cap I(a) \subseteq (L(a)I(a)]$ for every $a \in S$.

Proof. (1) $\longrightarrow$ (2). Let S be a semisimple ordered semigroup. Let $a \in R \cap I$. Then $a \in R$ and $a \in I$. Since $a \in S$ and S is semisimple, there exist, $x, y, z \in S$, such that $a \leq (xay)(az) \in ((SIS)(RS)] \subseteq (IR]$.

(2) $\longrightarrow$ (3). Obvious.

(3) $\longrightarrow$ (1). Let $a \in S$. Then,

$$\begin{aligned} a &\in R(a) \cap I(a) \subseteq (I(a)R(a)] \\ &= ((SaS](aS]] \\ &\subseteq ((SaS)(aS)]] \\ &= (SaSaS]. \end{aligned}$$

Thus S is semisimple.

Similarly, (1) $\longleftrightarrow$ (4) $\longleftrightarrow$ (5). □

2.11.6 Theorem

An ordered semigroup S with identity element 1. Then S is semisimple if and only if for each fuzzy right ideal f and fuzzy two-sided ideal g of S, we have,

$$f \wedge g \preceq f \circ g$$

Proof. Let S be a semisimple ordered semigroup, $a \in S$. Since S is semisimple, there exist x, y, z such that $a \leq (xa)(yaz)$. Then $(xa, yaz) \in A_a$. Since $A_a \neq \emptyset$, we have,

$$\begin{aligned} (f \circ g)(a) \ &: = \bigvee_{(p,q) \in A_a} \min\{f(p), g(q)\} \\ &\geq \min\{f(xa), g(yaz)\}, \end{aligned}$$

since f is a fuzzy left ideal and g a fuzzy two-sided ideal of S, we have, $f(xa) \geq f(a)$ and $g(yaz) \geq g(a)$. Thus we have $\min\{f(xa), g(yaz)\} \geq \min\{f(a), g(a)\} = (f \wedge g)(a)$. Hence,

$$(f \wedge g)(a) \leq (f \circ g)(a).$$

Conversely, assume that $f \wedge g \preceq f \circ g$, for every fuzzy left ideal f and fuzzy two-sided ideal g of S. Let $a \in S$. Then S is semisimple. Indeed, by Lemma 2.11.5, it is enough to prove that

$$R(a) \cap I(a) \subseteq (I(a)R(a)] \text{ for all } a \in S.$$

Let $a \in S$ and $b \in R(a) \cap I(a)$. Then $b \subset (I(a)R(a)]$. Indeed, since $L(a)$ is the left ideal, and $R(a)$, the right ideal of S generated by a respectively then $f_{L(a)}$ is a fuzzy left ideal, and $f_{I(a)}$ a fuzzy two-sided ideal of S. By hypothesis,

$$\left(f_{L(a)} \wedge f_{I(a)}\right)(b) \leq \left(f_{L(a)} \circ f_{I(a)}\right)(b).$$

Since $\left(f_{L(a)} \wedge f_{I(a)}\right)(b) := \min\{f_{L(a)}(b), f_{I(a)}(b)\}$.
We have

$$\min\{f_{L(a)}(b), f_{I(a)}(b)\} \leq \left(f_{L(a)} \circ f_{I(a)}\right)(b)$$

Since $b \in L(a)$, and $b \in I(a)$, hence $f_{L(a)}(b) := 1$, and $f_{I(a)}(b) := 1$ then we have,

$$\begin{aligned} \min\{f_{L(a)}(b), f_{R(a)}(b), f_{Q(a)}(b)\} &= 1 \\ \text{and hence, } \left(f_{L(a)} \circ f_{R(a)}\right)(b) &= 1. \end{aligned}$$

But from Proposition 1.3.10, it follows that

$$f_{L(a)} \circ f_{I(a)} = f_{(L(a)I(a)]}.$$

Thus,

$$f_{(L(a)I(a)]}(b) = 1 \longrightarrow b \in (L(a)I(a)].$$

Thus by Lemma 2.11.3, it follows that S is semisimple. $\square$

Chapter 3

PRIME AND SEMIPRIME FUZZY BI-IDEALS

In [139], the authors introduced the concept of prime bi-ideals in semigroups, motivated by their work, we define prime, strongly prime and semiprime bi-ideals (resp. fuzzy bi-ideals) in ordered semigroups and characterize those ordered semigroups in which each bi-ideal (resp. fuzzy bi-ideal) is semiprime. We also characterize those ordered semigroups in which each bi-ideal (resp. fuzzy bi-ideal) is strongly prime. Results in this chapter are part of our submitted paper [137].

3.1 Ordered semigroups in which each fuzzy bi-ideal is idempotent

In this section, we prove that an ordered semigroup S is both regular and intra-regular if and only if for each fuzzy bi-ideal f of S we have, $f \circ f = f$. We also prove that an ordered semigroup S is both regular and intra-regular if and only if for all fuzzy bi-ideals f and g of S, we have, $f \wedge g = f \circ g \wedge g \circ f$.

3.1.1 Lemma

Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the following are equivalent:
(1) S is regular.
(2) $B = (BSB]$ for every bi-ideal B of S.
(3) $B(a) = (B(a)SB(a)]$ for every $a \in S$.

Proof. (1) $\longrightarrow$ (2). Let S be a regular ordered semigroup, B a bi-ideal of S and $a \in B$. Since S is regular there exists $x \in S$ such that

$$a \leq axa \in BSB \longrightarrow a \in (BSB].$$

On the other hand since B is a bi-ideal of S, we have

$$BSB \subseteq B \longrightarrow (BSB] \subseteq (B] = B.$$

Thus $B = (BSB]$.
(2) $\longrightarrow$ (3). Obvious.

(3) $\longrightarrow$ (1). Let $B(a)$ be the bi-ideal of S generated by a. Then

$$\begin{aligned} a &\in B(a) = (B(a)SB(a)] \\ &= ((a \cup a^2 \cup aSa](S](a \cup a^2 \cup aSa]] \\ &\subseteq (((a \cup a^2 \cup aSa)S(a \cup a^2 \cup aSa)]] \\ &= ((a \cup a^2 \cup aSa)S(a \cup a^2 \cup aSa)] \\ &= ((aS \cup a^2S \cup aSaS)(a \cup a^2 \cup aSa)] \\ &= (aSa \cup aSa^2 \cup aSaSa \cup a^2Sa \cup a^2Sa^2 \cup a^2SaSa \cup aSaSa \cup aSaSa^2 \cup aSaSaSa] \\ &\subseteq (aSa]. \end{aligned}$$

Thus $a \leq axa$ for some $x \in S$. Hence S is regular. □

3.1.2 Theorem

An ordered semigroup S is regular if and only if for every fuzzy bi-ideal f of S, we have

$$f \circ 1 \circ f = f.$$

Proof. ($\longrightarrow$) Let S be a regular ordered semigroup, f a fuzzy bi-ideal of S and $a \in S$. Since S is regular, there exists $x \in S$ such that $a \leq axa \leq a(xaxa) \longrightarrow (a, xaxa) \in A_a$. Since $A_a \neq \emptyset$, therefore

$$\begin{aligned} (f \circ 1 \circ f)(a) \quad &:= \bigvee_{(y,z) \in A_a} \min\{f(y), (1 \circ f)(z)\} \\ &\geq \min\{f(a), (1 \circ f)(xaxa)\}. \end{aligned}$$

Now,

$$\begin{aligned} (1 \circ f)(xaxa) &= \bigvee_{(p,q) \in A_{xaxa}} \min\{1(p), f(q)\} \\ &\geq \min\{1(xax), f(a)\} \\ &= \min\{1, f(a)\} \text{ (as } 1(xax) = 1 \text{ for all } x, a \in S) \\ &= f(a). \end{aligned}$$

Thus

$$(f \circ 1 \circ f)(a) \geq \min\{f(a), f(a)\} = f(a).$$

On the other hand, by Proposition 1.3.16, for every fuzzy bi-ideal f of S, we have $f \circ 1 \circ f \preceq f$. Thus $f \circ 1 \circ f = f$.

($\longleftarrow$) Assume that $f \circ 1 \circ f = f$ for every fuzzy bi-ideal f of S. Then S is regular. Infact, by Lemma 3.1.1, it is enough to prove that

$$B(a) = (B(a)SB(a)] \text{ for all } a \in S.$$

Let $y \in B(a)$. Then $y \in (B(a)SB(a)]$. Indeed: Since $B(a)$ is the bi-ideal of S generated by a. By Lemma 1.3.4, $f_{B(a)}$ is a fuzzy bi-ideal of S. By hypothesis,

$$(f_{B(a)} \circ 1 \circ f_{B(a)})(y) = f_{B(a)}(y).$$

As $y \in B(a)$, we have $f_{B(a)}(y) = 1$. Hence $(f_{B(a)} \circ 1 \circ f_{B(a)})(y) = 1$. By Proposition 1.3.10, $f_{B(a)} \circ 1 \circ f_{B(a)} = f_{((Ba)SB(a)]}$. Thus $f_{(B(a)SB(a)]}(y) = 1 \longrightarrow y \in (B(a)SB(a)]$. On the other hand, $(B(a)SB(a)] \subseteq (B(a)] = B(a)$ always. Therefore $(B(a)SB(a)] = B(a)$. □

3.1.3 Lemma

Let S be an ordered semigroup. Then the following are equivalent:

(1) S is both regular and intra-regular.

(2) $B = (B^2]$ for every bi-ideal B of S.

(3) $B_1 \cap B_2 = (B_1B_2] \cap (B_2B_1]$ for all bi-ideals B_1, B_2 of S.

(4) $R \cap L = (RL] \cap (LR]$ for every right ideal R and every left ideal L of S.

(5) $R(a) \cap L(a) = (R(a)L(a)] \cap (L(a)R(a)]$ for every $a \in S$.

Proof. (1) $\longrightarrow$ (2). Assume that S is both regular and intra-regular ordered semigroup. Let B be a bi-ideal of S. Then $B = (B^2]$. Indeed: Since $B^2 \subseteq B \longrightarrow (B^2] \subseteq (B] = B$. For the reverse inclusion let $a \in B$. Since S is both regular and intra-regular, there exist $x, y, z \in S$ such that $a \leq axa$ and $a \leq ya^2z$. Then we have

$$\begin{aligned} a &\leq axa \leq axaxa \leq ax(ya^2z)xa = (axya)(azxa) \in (BSB)(BSB) \subseteq B^2 \\ &\longrightarrow a \in (B^2]. \end{aligned}$$

Thus $B = (B^2]$.

(2) $\longrightarrow$ (3). Let B_1, B_2 be bi-ideals of S. Then $B_1 \cap B_2$ is a bi-ideal of S. By (2),

$$B_1 \cap B_2 = ((B_1 \cap B_2)^2] = ((B_1 \cap B_2)(B_1 \cap B_2)] \subseteq (B_1B_2].$$

Similarly, we can prove that $B_1 \cap B_2 \subseteq (B_2B_1]$. Thus $B_1 \cap B_2 \subseteq (B_1B_2] \cap (B_2B_1]$. On the other hand, by Lemma 1.3.2, $(B_1B_2]$ and $(B_2B_1]$ are bi-ideals of S, so $(B_1B_2] \cap (B_2B_1]$ is a bi-ideal of S. By (2),

$$\begin{aligned} (B_1B_2] \cap (B_2B_1] &= (((B_1B_2] \cap (B_2B_1])^2] \\ &= ((B_1B_2] \cap (B_2B_1])(B_1B_2] \cap (B_2B_1])] \\ &\subseteq ((B_1B_2](B_2B_1]] \subseteq ((B_1B_2B_2B_1]] \\ &= (B_1B_2B_2B_1] \subseteq (B_1SB_1] \subseteq (B_1] = B_1. \end{aligned}$$

Similarly, we can prove that $(B_1B_2] \cap (B_2B_1] \subseteq B_2$. Therefore $B_1 \cap B_2 = (B_1B_2] \cap (B_2B_1]$.

(3) $\longrightarrow$ (4). Let R and L be right and left ideal of S, respectively. Then by Lemma 1.1.5., these are bi-ideals of S. The assertion follows by (3).

(4) $\longrightarrow$ (5). This is obvious.

(5) $\longrightarrow$ (1). The assertion follows from Lemma 1.1.13. □

3.1.4 Theorem

Let S be an ordered semigroup. Then the following are equivalent:

(1) S is both regular and intra-regular.

(2) $f \circ f = f$ for every fuzzy bi-ideal f of S.

(3) $f \wedge g = f \circ g \wedge g \circ f$ for all fuzzy bi-ideals f and g of S.

Proof. (1) $\longrightarrow$ (2). Suppose S is both regular and intra-regular ordered semigroup and f a fuzzy bi-ideal of S. Then for each $a \in S$, we have $(f \circ f)(a) \geq f(a)$. Indeed: Since S is regular and intra-regular therefore there exist $x, y, z \in S$ such that $a \leq axa$ and $a \leq ya^2z$. Thus

$$a \leq axa \leq axaxa = ax(ya^2z)xa = (axya)(azxa).$$

Then $(axya, azxa) \in A_a$. Since $A_a \neq \emptyset$, we have

$$\begin{aligned} (f \circ f)(a) \;:&= \bigvee_{(y,z)\in A_a} \min\{f(y), f(z)\} \\ &\geq \min\{f(axya), f(azxa)\}. \end{aligned}$$

As f is a fuzzy bi-ideal of S we have $f(axya) \geq$min$\{f(a), f(a)\} = f(a)$, and $f(ayxa) \geq$min$\{f(a), f(a)\} = f(a)$. Thus

$$\begin{aligned} (f \circ f)(a) &\geq \min\{f(axya), g(azxa)\} \\ &\geq \min\{f(a), f(a)\} = f(a). \end{aligned}$$

Thus $f \preceq f \circ f$. By Lemma 1.1.12, we have $f \circ f \preceq f$. Thus $f = f \circ f$.

(2) $\longrightarrow$ (3). Let f and g be fuzzy bi-ideals of S. Then $f \wedge g$ is a fuzzy bi-ideal of S. By (2),

$$f \wedge g = (f \wedge g) \circ (f \wedge g) \preceq f \circ g.$$

Similarly, we can prove that $f \wedge g \preceq g \circ f$. Thus $f \wedge g \preceq f \circ g \wedge g \circ f$. For the reverse inclusion, by Lemma 1.3.17, $f \circ g$ and $g \circ f$ are fuzzy bi-ideals of S and so, $f \circ g \wedge g \circ f$ is a fuzzy bi-ideal of S. By (2), we have

$$\begin{aligned} f \circ g \wedge g \circ f &= (f \circ g \wedge g \circ f) \circ (f \circ g \wedge g \circ f) \\ &\preceq f \circ g \circ g \circ f = f \circ (g \circ g) \circ f \\ &= f \circ g \circ f \text{ (as } g \circ g = g \text{ by (2) above)} \\ &\preceq f \circ 1 \circ f \text{ (as } f \circ g \preceq f \circ 1 \text{ by Lemma 1.3.15)} \\ &\preceq f \text{ (as } f \circ 1 \circ f = f \text{ by Theorem 3.1.2)}. \end{aligned}$$

Hence $f \circ g \wedge g \circ f \preceq f$. Similarly, we can prove that $f \circ g \wedge g \circ f \preceq g$. Thus $f \circ g \wedge g \circ f \preceq f \wedge g$. Therefore $f \circ g \wedge g \circ f = f \wedge g$.

(3) $\longrightarrow$ (1). Let f be a fuzzy right ideal and g a fuzzy left ideal of S, then f, g are fuzzy bi-ideals of S. Hence by hypothesis, $f \wedge g = f \circ g \wedge g \circ f$. Thus by Theorem 2.8.3, S is both regular and intra-regular. □

3.1.5 Lemma

Let S be an ordered semigroup. Then the following are equivalent:

(1) S is both regular and intra-regular.

(2) $B \cap L \subseteq (BLB]$ for every bi-ideal B and every left ideal L of S.

(3) $Q \cap L \subseteq (QLQ]$ for every quasi-ideal Q and every left ideal L of S.

(4) $Q(a) \cap L(a) \subseteq (Q(a)L(a)Q(a)]$ for every $a \in S$.

Proof. $(1) \longrightarrow (2)$. Let S be both regular and intra-regular ordered semigroup, B a bi-ideal and L a left ideal of S. Let $a \in B \cap L$, then $a \in B$ and $a \in L$. Since S is regular, there exists $x \in S$ such that $a \leq axa \leq axaxa$. Also S is intra-regular, there exist $y, z \in S$ such that $a \leq ya^2z$. Thus

$$\begin{aligned} a &\leq ax(ya^2z)xa = a(xya)(azxa) \in B(SL)(BSB) \subseteq BLB \\ &\longrightarrow a \in (BLB]. \end{aligned}$$

Thus $B \cap L \subseteq (BLB]$.

$(2) \longrightarrow (3) \longrightarrow (4)$. These assertions are obvious

$(4) \longrightarrow (1)$. Let $Q(a)$ be the quasi-ideal and $L(a)$ the left ideal of S generated by a, respectively. Then

$$\begin{aligned} a &\in Q(a) \cap L(a) \subseteq (Q(a)L(a)Q(a)] \\ &= (((a \cup (Sa \cap aS)](a \cup Sa](a \cup (Sa \cap aS)]] \\ &\subseteq (((a \cup (Sa \cap aS))(a \cup Sa)(a \cup (Sa \cap aS))]] \\ &= ((a \cup (Sa \cap aS)(a \cup Sa)(a \cup (Sa \cap aS)] \\ &= ((a \cup Sa) \cap (a \cup aS))(a \cup Sa)((a \cup Sa) \cap (a \cup aS))] \\ &\subseteq ((a \cup aS)(a \cup Sa)(a \cup Sa)] \\ &= (a^3 \cup aSa]. \end{aligned}$$

Then $a \leq a^3$ or $a \leq axa$ for some $x \in S$. Thus S is regular.

Again

$$\begin{aligned} a &\in Q(a) \cap L(a) \subseteq (Q(a)L(a)Q(a)] \\ &= ((a \cup Sa) \cap (a \cup aS))(a \cup Sa)((a \cup Sa) \cap (a \cup aS))] \\ &\subseteq ((a \cup Sa)(a \cup Sa)(a \cup aS)] \\ &= (a^3 \cup a^3S \cup aSa^2 \cup aSa^2S \cup Sa^3 \cup Sa^3S \cup SaSa^2 \cup SaSa^2S] \\ &\subseteq (a^3 \cup a^3S \cup aSa^2 \cup Sa^3 \cup Sa^2S]. \end{aligned}$$

Then $a \leq a^3$ or $a \leq a^3x$ or $a \leq axa^2$ or $a \leq xa^3$ or $a \leq xa^2y$ for some $x, y \in S$. If $a \leq a^3$ then $a \leq a^3 = aa^2 \leq a^3a^2 = (a)a^2(aa)$. If $a \leq a^3x$ then $a \leq a^3x = aa^2x$. If $a \leq axa^2$ then $a \leq axa^2 = axa^2xa^2 = (ax)a^2(xa^2)$. In every case S is an intra-regular ordered semigroup. □

3.1.6 Theorem

An ordered semigroup S is both regular and intra-regular if and only if for every fuzzy bi-ideal f and every fuzzy left ideal g of S, we have

$$f \wedge g \preceq f \circ g \circ f.$$

Proof. ($\longrightarrow$) Suppose that S is both regular and intra-regular ordered semigroup, f a fuzzy bi-ideal and g a fuzzy left ideal of S. Then for each $a \in S$, we have $(f \wedge g)(a) \leq (f \circ g \circ f)(a)$. In fact: Since S is regular and intra-regular ordered semigroup, there exist $x, y, z \in S$ such that $a \leq axa$ and $a \leq ya^2z$. Thus

$$a \leq axa \leq axaxa \leq ax(ya^2z)xa = (axya)(azxa) = a((xya)(azxa)).$$

Then $(a, (xya)(azxa)) \in A_a$. Since $A_a \neq \emptyset$, we have

$$\begin{aligned}(f \circ g \circ f)(a) \ :&= \bigvee_{(y,z)\in A_a} \min\{f(y), (g \circ f)(z)\} \\ &\geq \min\{f(a), (g \circ f)((xya)(azxa))\} \\ &= \min\{f(a), \bigvee_{(p,q)\in A_{(xya)(azxa)}} \min\{g(p), f(q)\}\} \\ &\geq \min\{f(a), \min\{g(xya), f(azxa)\}\}\end{aligned}$$

As g is a fuzzy left ideal and f a fuzzy bi-ideal of S, so we have, $g(xya) \geq g(a)$ and $f(azxa) \geq \min\{f(a), f(a)\} = f(a)$. Thus

$$\begin{aligned}(f \circ g \circ f)(a) &\geq \min\{f(a), \min\{g(xya), f(azxa)\}\} \\ &\geq \min\{f(a), \min\{g(a), f(a)\}\} \\ &= \min\{f(a), g(a)\} = (f \wedge g)(a).\end{aligned}$$

Thus $(f \wedge g)(a) \leq (f \circ g \circ f)(a)$.

($\longleftarrow$) Assume that $f \wedge g \preceq f \circ g \circ f$, for every fuzzy bi-ideal and every fuzzy left ideal g of S. Then S is both regular and intra-regular. In deed: By Lemma 3.1.5, it is enough to prove that

$$Q(a) \cap L(a) \subseteq (Q(a)L(a)Q(a)] \text{ for all } a \in S.$$

Let $y \in Q(a) \cap L(a)$. Then $y \in (Q(a)L(a)Q(a)]$. In fact: Since $Q(a)$ is the quasi-ideal and $L(a)$ the left ideal of S, generated by a, respectively. Then $f_{Q(a)}$ is a fuzzy quasi-ideal and $f_{L(a)}$ is a fuzzy left ideal of S. By hypothesis, we have

$$(f_{Q(a)} \wedge f_{L(a)})(y) \leq (f_{Q(a)} \circ f_{L(a)} \circ f_{Q(a)})(y).$$

As $y \in Q(a)$ and $y \in L(a)$, we have $f_{Q(a)}(y) := 1$ and $f_{L(a)}(y) := 1$. Thus

$$(f_{Q(a)} \circ f_{L(a)} \circ f_{Q(a)})(y) = 1.$$

Thus by Proposition 1.3.10,

$$f_{Q(a)} \circ f_{L(a)} \circ f_{Q(a)} = f_{(Q(a)L(a)Q(a)]}.$$

Thus $f_{(Q(a)L(a)Q(a)]}(y) = 1 \longrightarrow y \in (Q(a)L(a)Q(a)]$. □

3.2 Prime and semiprime bi-ideals

In this section we define prime, strongly prime and semiprime bi-ideals (resp. fuzzy bi-ideals) and characterize those ordered semigroups in which each bi-ideal (resp. fuzzy bi-ideal) is semiprime. We also characterize those ordered semigroups in which each bi-ideal (resp fuzzy bi-ideal) of an ordered semigroup S is strongly prime.

3.2.1 Definition

Let S be an ordered semigroup. A bi-ideal B of S is called *prime* (resp. *semiprime*) if:

$B_1B_2 \subseteq B$ (resp. $B_1^2 \subseteq B$) implies $B_1 \subseteq B$ or $B_2 \subseteq B$ (resp. $B_1 \subseteq B$) for all bi-ideals B_1, B_2 of S.

3.2.2 Definition

Let S be an ordered semigroup. A bi-ideal B of S is called *strongly prime* if $(B_1B_2] \cap (B_2B_1] \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ for all bi-ideals B_1, B_2 of S.

3.2.3 Definition

Let $(S, \cdot, \leq)$ be an ordered semigroup, f a fuzzy bi-ideal of S. f is called *prime* (*strongly prime, semiprime*) if $g \circ h \preceq f$ (resp. $g \circ h \wedge h \circ g \preceq f$, $g \circ g \preceq f$) implies $g \preceq f$ or $h \preceq f$ (resp. $g \preceq f$ or $h \preceq f$, $g \preceq f$) for all fuzzy bi-ideals g and h of S.

3.2.4 Proposition

Let S be an ordered semigroup. A subset B of S is a prime bi-ideal of S, if and only if the characteristic function f_B of B is a prime fuzzy bi-ideal of S.

Proof. Suppose B is a prime bi-ideal of an ordered semigroup S and f_B the characteristic function of B. By Lemma 1.3.4, f_B is a fuzzy bi-ideal of S. Let g, h be any fuzzy bi-ideals of S such that $g \circ h \preceq f_B$, with $g \npreceq f_B$ and $h \npreceq f_B$. Then there exist $x, y \in S$ such that $g(x) \neq 0$ and $h(y) \neq 0$ but $f_B(x) = 0$ and $f_B(y) = 0$. Then $x \notin B$ and $y \notin B$. Since B is a prime bi-ideal of S, we have $B(x)B(y) \nsubseteq B$. Hence there exists $a \in B(x)B(y)$ such that $a \notin B$. So we have $f_B(a) = 0$ and hence $(g \circ h)(a) = 0$. Since $g(x) \neq 0$ and $h(y) \neq 0$ we have,

$$\min\{g(x), h(y)\} \neq 0.$$

Since $a \in B(x)B(y)$, then $a \leq x_1y_1$ for some $x_1 \in B(x)$ and $y_1 \in B(y)$, so $(x_1, y_1) \in A_a$ and $A_a \neq \emptyset$. Thus

$$\begin{aligned}(g \circ h)(a) &= \bigvee_{(p,q)\in A_a} \min\{g(p), h(q)\} \\ &\geq \min\{g(x_1), h(y_1)\}.\end{aligned}$$

Since $x_1 \in B(x) = (x \cup x^2 \cup xSx]$, then $x_1 \leq x$ or $x_1 \leq x^2$ or $x_1 \leq xzx$ for some $z \in S$. If $x_1 \leq x$ then since g is a fuzzy bi-ideal of S, we have $g(x_1) \geq g(x)$. If $x_1 \leq x^2$ then $g(x_1) \geq g(x^2) \geq \min\{g(x), g(x)\} = g(x)$. If $x_1 \leq xzx$ then $g(x_1) \geq \min\{g(x), g(x)\} = g(x)$. Also $y_1 \in B(y) = (y \cup y^2 \cup ySy]$ implies that $y_1 \leq y$ or $y_1 \leq y^2$ or $y_1 \leq yty$ for some $t \in S$. If $y_1 \leq y$ then $h(y_1) \geq h(y)$ because h is a fuzzy bi-ideal of S. If $y_1 \leq y^2$ then $h(y_1) \geq h(y^2) \geq \min\{h(y), h(y)\} = h(y)$. If $y_1 \leq yxy$ then $h(y_1) \geq h(yty) \geq \min\{h(y), h(y)\} = h(y)$. Thus

$$\begin{aligned}\min\{g(x_1), h(y_1)\} &\geq \min\{g(x), h(y)\} \neq 0 \\ &\Longrightarrow (g \circ h)(a) > 0, \text{ which is a contradiction.}\end{aligned}$$

Therefore for any fuzzy bi-ideals g, h of S, $g \circ h \preceq f_B$ implies $g \preceq f_B$ or $h \preceq f_B$.

Conversely, assume that B is a bi-ideal of S and B_1, B_2 are any bi-ideals of S such that $B_1B_2 \subseteq B$. Then $(B_1B_2] \subseteq (B] = B$, and we have $f_{(B_1B_2]} \preceq f_B$. Since $f_{(B_1B_2]} = f_{B_1} \circ f_{B_2}$ by Proposition 1.4.5. Then we have $f_{B_1} \circ f_{B_2} \preceq f_B$. As f_B is a prime fuzzy bi-ideal of S, we have $f_{B_1} \preceq f_B$ or $f_{B_2} \preceq f_B$. Hence, $B_1 \subseteq B$ or $B_2 \subseteq B$. □

3.2.5 Proposition

Let S be an ordered semigroup and B a subset of S then B is strongly prime bi-ideal if and only if the characteristic function f_B is strongly prime fuzzy bi-ideal.

Proof. Similar to the proof of Proposition 3.2.4. □

3.2.6 Proposition

Let S be an ordered semigroup. A subset B of S is a semiprime bi-ideal of S if and only if the characteristic function f_B of B is a semiprime fuzzy bi-ideal of S.

Proof. Similar to the proof of Proposition 3.2.4. □

The proof of the following Lemma is straightforward.

3.2.7 Lemma

Let S be an ordered semigroup, $\{B_i | i \in I\}$ a family of prime bi-ideals of S. Then $\bigcap_{i \in I} B_i$ is a semiprime bi-ideal of S.

3.2.8 Proposition

Let S be an ordered semigroup, $\{f_i | i \in I\}$ a family of prime fuzzy bi-ideals of S. Then $\bigwedge_{i \in I} f_i$ is a semiprime fuzzy bi-ideal of S.

Proof. Proof is straightforward. □

3.2.9 Definition

Let S be an ordered semigroup and B a bi-ideal of S. Then B is called an *irreducible* (resp. *strongly irreducible*) if for any bi-ideals B_1, B_2 of S we have, $B_1 \cap B_2 = B$ (resp. $B_1 \cap B_2 \subseteq B$) implies $B_1 = B$ or $B_2 = B$ (resp. $B_1 \subseteq B$ or $B_2 \subseteq B$).

Note that every strongly irreducible bi-ideal of an ordered semigroup S is irreducible.

3.2.10 Lemma

Let $(S, \cdot, \leq)$ be an ordered semigroup, B a bi-ideal of S and $a \in S$ such that $a \notin B$. Then there exists an irreducible bi-ideal A of S such that $B \subseteq A$ and $a \notin A$.

Proof. Let $\mathcal{A}$ be the collection of all bi-ideals of the ordered semigroup S which contains B and does not contain a. Then $\mathcal{A} \neq \emptyset$, because $B \in \mathcal{A}$. The collection $\mathcal{A}$ is a partially ordered set under inclusion. As every totally ordered subset of $\mathcal{A}$ is bounded above, so by Zorn's Lemma there exists a maximal element say A in $\mathcal{A}$. We show that A is an irreducible bi-ideal of S. Let C, D be any two bi-ideals of S such that $A = C \cap D$. If both C and D properly contains A then $a \in C$ and $a \in D$. Hence $a \in C \cap D = A$. This contradicts the fact that $a \notin A$. Thus $A = C$ or $A = D$. □

3.2.11 Proposition

Let $(S, \cdot, \leq)$ be an ordered semigroup. Then every strongly irreducible semiprime bi-ideal of S is a strongly prime bi-ideal of S.

Proof. Let B be a strongly irreducible semiprime bi-ideal of S. Let B_1, B_2 be any bi-ideals of S such that $(B_1B_2] \cap (B_2B_1] \subseteq B$. Since $(B_1 \cap B_2)^2 \subseteq B_1B_2 \subseteq (B_1B_2]$ and $(B_1 \cap B_2)^2 \subseteq B_2B_1 \subseteq (B_2B_1]$. Thus $(B_1 \cap B_2)^2 \subseteq (B_1B_2] \cap (B_2B_1] \subseteq B$. Since B is semiprime bi-ideal of S, we have $B_1 \cap B_2 \subseteq B$. Since B is strongly irreducible, we have $B_1 \subseteq B$ or $B_2 \subseteq B$. Thus B is strongly prime bi-ideal of S. □

3.2.12 Definition

Let $(S, \cdot, \leq)$ be an ordered semigroup, f a fuzzy bi-ideal of S. f is called *irreducible* (resp. *strongly irreducible*) *fuzzy bi-ideal* of S, if $g \wedge h = f$ (resp. $g \wedge h \preceq f$) implies $g = f$ or $h = f$ (resp. $g \preceq f$ or $h \preceq f$) for every fuzzy bi-ideals g, h of S.

3.2.13 Proposition

Let S be an ordered semigroup. Then every strongly irreducible, semiprime fuzzy bi-ideal of S is strongly prime.

Proof. Let f be a strongly irreducible semiprime fuzzy bi-ideal of S. Let g, h be any fuzzy bi-ideals of S such that $g \circ h \wedge h \circ g \preceq f$. As $g \wedge h$ is a fuzzy bi-ideal of S and $(g \wedge h) \circ (h \wedge g) \preceq g \circ h$, $(g \wedge h) \circ (h \wedge g) \preceq h \circ g$. Thus $(g \wedge h) \circ (h \wedge g) \preceq g \circ h \wedge h \circ g \preceq f$. Since f is semiprime, we have $g \wedge h \preceq f$. Since f is strongly irreducible we have $g \preceq f$ or $h \preceq f$. Thus f is strongly prime. □

3.2.14 Proposition

Let $(S, \cdot, \leq)$ be an ordered semigroup, f a fuzzy bi-ideal of S with $f(a) = t$, where $a \in S$ and $t \in (0, 1]$, then there exists an irreducible fuzzy bi-ideal g of S such that $f \preceq g$ and $g(a) = t$.

Proof. Let $X = \{h | h$ is a fuzzy bi-ideal of S, $h(a) = t$ and $f \preceq h\}$. Then $X \neq \emptyset$, because $f \in X$. The collection X is a partially ordered set under inclusion. If Y is any totally ordered subset of X, say $Y = \{h_i | i \in I\}$. Then $\bigvee_{i \in I} h_i$ is a fuzzy bi-ideal of S containing f. Indeed: Let $x, y, z \in S$.

$$\begin{aligned}
(\bigvee_{i \in I} h_i)(xy) &= \bigvee_{i \in I}(h_i(xy)) \\
&\geq \bigvee_{i \in I}(h_i(x) \wedge h_i(y)) \\
&= \bigvee_{i \in I}(h_i(x)) \wedge \bigvee_{i \in I}(h_i(y)) \\
&= \bigvee_{i \in I}(h_i)(x) \wedge \bigvee_{i \in I}(h_i)(y).
\end{aligned}$$

Hence $\bigvee_{i \in I} h_i$ is a fuzzy subsemigroup of S.

Also

$$
\begin{aligned}
(\bigvee_{i\in I} h_i)(xyz) &= \bigvee_{i\in I}(h_i(xyz)) \\
&\geq \bigvee_{i\in I}(h_i(x)\wedge h_i(z)) \\
&= \bigvee_{i\in I}(h_i(x))\wedge \bigvee_{i\in I}(h_i(z)) \\
&= \bigvee_{i\in I}(h_i)(x)\wedge \bigvee_{i\in I}(h_i)(z).
\end{aligned}
$$

Let $x, y \in S$ such that $x \leq y$. Then

$$
\begin{aligned}
(\bigvee_{i\in I} h_i)(x) &= \bigvee_{i\in I}(h_i(x)) \geq \bigvee_{i\in I}(h_i(y)) \text{ (since } h_i \text{ are fuzzy bi-ideals of } S) \\
&= (\bigvee_{i\in I} h_i)(y).
\end{aligned}
$$

Hence $\bigvee_{i\in I} h_i$ is a fuzzy bi-ideal of S.

As $f \preceq h_i$ for each $i \in I$, so $f \preceq \bigvee_{i\in I} h_i$. Also $(\bigvee_{i\in I} h_i)(a) = \bigvee_{i\in I} h_i(a) = t$. Thus $\bigvee_{i\in I} h_i$ is the least upper bound of Y. By Zorn's Lemma, there exists a fuzzy bi-ideal g of S which is maximal with respect to the property that $f \preceq g$ and $g(a) = t$. We now show that g is an irreducible fuzzy bi-ideal of S. Suppose that $g = g_1 \wedge g_2$ where g_1 or g_2 are fuzzy bi-ideals of S. Thus $g \preceq g_1$ and $g \preceq g_2$. We claim that $g = g_1$ and $g = g_2$. Suppose on the contrary that $g \neq g_1$ and $g \neq g_2$. Since g is maximal with respect to the property that $g(a) = t$ and since $g \precneqq g_1$ and $g \precneqq g_2$, it follows that $g_1(a) \neq t$ and $g_2(a) \neq t$. Hence $t = g(a) = (g_1 \wedge g_2)(a) \neq t$, which is a contradiction. Hence either $g = g_1$ or $g = g_2$. Thus g is an irreducible fuzzy bi-ideal of S. □

3.2.15 Lemma

Let S be an ordered semigroup. Then the following are equivalent:

(1) S is both regular and intra-regular.

(2) $(B^2] = B$ for every bi-ideal B of S.

(3) $B_1 \cap B_2 = (B_1B_2] \cap (B_2B_1]$ for all bi-ideals B_1, B_2 of S.

(4) Each bi-ideal of S is semiprime.

(5) Each bi-ideal of S is the intersection of all irreducible semiprime bi-ideals of S which contain it.

Proof. (1) $\longleftrightarrow$ (2) $\longleftrightarrow$ (3). Follows from Lemma 3.1.3.

(3) $\longrightarrow$ (4). Let B_1, B be any bi-ideals of S such that $B_1^2 \subseteq B$. By hypothesis

$$\begin{aligned} B_1 &= B_1 \cap B_1 \\ &= (B_1^2] \cap (B_1^2] \\ &= (B_1^2]. \end{aligned}$$

Since $B_1^2 \subseteq B \longrightarrow (B_1^2] \subseteq (B] = B$. Thus $B_1 \subseteq B$ and hence every bi-ideal of S is semiprime.

(4) $\longrightarrow$ (5). Let B be a proper bi-ideal of S, then B is contained in the intersection of all irreducible bi-ideals of S which contain B. By Lemma 3.2.7, there exist such irreducible bi-ideals. If $a \notin B$ then there exists an irreducible bi-ideal of S which contains B but does not contain a. Hence B is the intersection of all irreducible bi-ideals of S which contains it. By hypothesis each bi-ideal of S is semiprime, so each bi-ideal of S is the intersection of irreducible semiprime bi-ideals of S which contain it.

(5) $\longrightarrow$ (2). Let B be a proper bi-ideal of S. If $(B^2] = S$ then B is idempotent, that is, $(B^2] = B$. If $(B^2] \neq S$, then $(B^2]$ is a proper bi-ideal of S and by hypothesis,

$$(B^2] = \cap_\alpha \{B_\alpha | B_\alpha \text{ is irreducible semiprime bi-ideal of } S \text{ containing } (B^2]\}$$

This implies that $(B^2] \subseteq B_\alpha$ for all α. Since every B_α is semiprime, therefore $B \subseteq B_\alpha$ for all α and so $B \subseteq \cap_\alpha B_\alpha = (B^2]$. Hence each bi-ideal of S is idempotent. □

3.2.16 Theorem

Let S be an ordered semigroup. Then the following are equivalent:

(1) S is both regular and intra-regular.

(2) $f \circ f = f$ for every fuzzy bi-ideal f of S.

(3) $f \wedge g = f \circ g \wedge g \circ f$ for all fuzzy bi-ideals f, g of S.

(4) Each fuzzy bi-ideal of S is fuzzy semiprime.

(5) Each proper fuzzy bi-ideal of S is the intersection of irreducible fuzzy semiprime bi-ideals of S which contain it.

Proof. (1) $\longleftrightarrow$ (2) $\longleftrightarrow$ (3). (cf. Theorem 3.1.4).

(3) $\longrightarrow$ (4). Let f, g be any fuzzy bi-ideals of S such that $f \circ f \preceq g$. By hypothesis,

$$\begin{aligned} f &= f \wedge f \\ &= f \circ f \wedge f \circ f \\ &= f \circ f. \end{aligned}$$

Thus $f \preceq g$. Hence each fuzzy bi-ideal of S is semiprime.

(4) $\longrightarrow$ (5). Let f be a proper fuzzy bi-ideal of S and $\{f_i | i \in I\}$ be the collection of all irreducible fuzzy bi-ideal of S which contain f. By Proposition 3.2.8, this

collection is non-empty. Hence $f \preceq \wedge_{i\in I} f_i$. Let $a \in S$, then there exists an irreducible fuzzy bi-ideal f_α of S such that $f \preceq f_\alpha$ and $f(a) = f_\alpha(a)$. Thus $f_\alpha \in \{f_i : i \in I\}$. Hence $\wedge_{i\in I} f_i \preceq f_\alpha$. So, $\wedge_{i\in I} f_i(a) = f_\alpha(a) = f(a)$. Since $\wedge_{i\in I} f_i \preceq f_\alpha$. So $\wedge_{i\in I} f_i = f$. By hypothesis each fuzzy bi-ideal is semiprime. So each fuzzy bi-ideal of S is the intersection of all irreducible fuzzy semiprime bi-ideals of S which contain it.

(5) $\longrightarrow$ (2). Let f be a fuzzy bi-ideal of S. Then $f \circ f$ is also a fuzzy bi-ideal of S by Lemma 1.3.17. Since f is a fuzzy subsemigroup of S, so $f \circ f \preceq f$. By hypothesis $f \circ f = \wedge_{i\in I} f_i$ where f_i are irreducible fuzzy semiprime bi-ideals of S. Thus $f \circ f \preceq f_i$ for all $i \in I$. Hence $f \preceq f_i$ for all $i \in I$, because f_i are semiprime. Thus $f \preceq \wedge_{i\in I} f_i = f \circ f$. Hence $f \circ f = f$. □

3.2.17 Proposition

Let S be both regular and intra-regular ordered semigroup. Then the following are equivalent:

(1) Every bi-ideal of S is strongly irreducible.

(2) Every bi-ideal of S is strongly prime.

Proof. By Lemma 3.2.15, S is both regular and intra-regular if and only if $B_1 \cap B_2 = (B_1B_2](B_2B_1]$ for all bi-ideals B_1, B_2 of S. The proof follows from this fact. □

3.2.18 Proposition

Let S be both regular and intra-regular ordered semigroup. Then the following are equivalent:

(1) Every fuzzy bi-ideal of S is strongly irreducible.

(2) Every fuzzy bi-ideal of S is strongly prime.

Proof. The proof follows from Theorem 3.2.16. □

3.2.19 Proposition

Each bi-ideal of an ordered semigroup S is strongly prime if and only if S is both regular and intra-regular and the set of bi-ideals of S is totally ordered under inclusion.

Proof. ($\longrightarrow$) Suppose that each bi-ideal of the ordered semigroup S is strongly prime, then each bi-ideal of S is semiprime. Thus by Lemma 3.2.15, S is both regular and intra-regular. To prove that the set of bi-ideals of S is totally ordered under inclusion, let B_1, B_2 be any bi-ideals of S, then by Lemma 3.1.3, $B_1 \cap B_2 = (B_1B_2] \cap (B_2B_1]$. As each bi-ideal of S is strongly prime, so $B_1 \cap B_2$ is strongly prime. Hence either $B_1 \subseteq B_1 \cap B_2$ or $B_2 \subseteq B_1 \cap B_2$. If $B_1 \subseteq B_1 \cap B_2$ then $B_1 \subseteq B_2$ and if $B_2 \subseteq B_1 \cap B_2$ then $B_2 \subseteq B_1$.

($\longleftarrow$) Assume that S is both regular and intra-regular and the set of bi-ideals of S is totally ordered under inclusion. Then each bi-ideal of S is strongly prime. Indeed: Let B be an arbitrary bi-ideal of S and B_1, B_2 be arbitrary bi-ideals of S such that

$(B_1B_2] \cap (B_2B_1] \subseteq B$. Since S is both regular and intra-regular, so by Lemma 3.1.3, $(B_1B_2] \cap (B_2B_1] = B_1 \cap B_2$. Thus $B_1 \cap B_2 \subseteq B$. Since the set of bi-ideals of S is totally ordered under inclusion, so either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Thus $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$. Therefore either $B_1 \subseteq B$ or $B_2 \subseteq B$ and hence B is strongly prime fuzzy bi-ideal of S. □

3.2.20 Proposition

Each fuzzy bi-ideal of an ordered semigroup S is strongly prime if and only if S is both regular and intra-regular and the set of fuzzy bi-ideals of S is totally ordered under inclusion.

Proof. ($\longrightarrow$) Suppose that each fuzzy bi-ideal of the ordered semigroup S is strongly prime, then each fuzzy bi-ideal of S is semiprime. Thus by Theorem 3.2.16, S is both regular and intra-regular. To prove that the set of fuzzy bi-ideals of S is totally ordered under inclusion, let f, g be any fuzzy bi-ideals of S, then by Theorem 3.2.16, $f \wedge g = f \circ g \wedge g \circ f$. As each fuzzy bi-ideal of S is strongly prime, so $f \wedge g$ is strongly prime. Hence either $f \preceq f \wedge g$ or $g \preceq f \wedge g$. If $f \preceq f \wedge g$ then $f \preceq g$ and if $g \preceq f \wedge g$ then $g \preceq f$.

($\longleftarrow$) Assume that S is both regular and intra-regular and the set of fuzzy bi-ideals of S is totally ordered under inclusion. Then each fuzzy bi-ideal of S is strongly prime. Indeed: Let f be an arbitrary fuzzy bi-ideal of S and g, h be any fuzzy bi-ideals of S such that $g \circ h \wedge h \circ g \preceq f$. Since S is both regular and intra-regular, by Theorem 3.2.16, $g \circ h \wedge h \circ g = g \wedge h$. Thus $g \wedge h \preceq f$. Since the set of fuzzy bi-ideals of S is totally ordered under inclusion, so either $g \preceq h$ or $h \preceq g$. Thus $g \wedge h = g$ or $g \wedge h = h$. Therefore either $g \preceq f$ or $h \preceq f$ and hence f is strongly prime bi-ideal of S. □

3.2.21 Lemma

Let S be an ordered semigroup and the set of bi-ideals of S is totally ordered under inclusion. Then the following are equivalent:

(1) S is both regular and intra-regular.

(2) Each bi-ideal of S is prime.

Proof. (1) $\longrightarrow$ (2). Suppose that S is both regular and intra-regular and B be a bi-ideal of S. Then B is prime. Indeed: Let B_1, B_2 be any arbitrary bi-ideals of S such that $B_1B_2 \subseteq B$. Since the set of bi-ideals of S is totally ordered under inclusion, so either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Let $B_1 \subseteq B_2$, then $B_1B_1 \subseteq B_1B_2 \subseteq B$. By Lemma 3.2.15, B is semiprime, so $B_1 \subseteq B$. Hence B is prime.

(2) $\longrightarrow$ (1). Assume that every bi-ideal of S is prime. Then S is both regular and intra-regular. Indeed: Since the set of bi-ideals of S is totally ordered under inclusion, so the concept of prime and strongly prime bi-ideals coincide. Therefore by Lemma 3.2.15, S is both regular and intra-regular. □

3.2.22 Proposition

If the set of fuzzy bi-ideals of an ordered semigroup S is totally ordered under inclusion then S is both regular and intra-regular if and only if each fuzzy bi-ideal of S is prime.

Proof. ($\longrightarrow$) Suppose that S is both regular and intra-regular ordered semigroup. Let f be any fuzzy bi-ideal of S. Then f is prime. Indeed: Let g, h be any arbitrary fuzzy bi-ideals of S such that $g \circ h \preceq f$. Since the set of fuzzy bi-ideals of S is totally ordered under inclusion, so either $g \preceq h$ or $h \preceq g$. If $g \preceq h$ then $g \circ g \preceq g \circ h \preceq f$. By Theorem 3.2.16, f is semiprime, so $g \preceq f$. Hence f is prime.

($\longleftarrow$) Assume that every fuzzy bi-ideal of S is prime. Since the set of fuzzy bi-ideals of S is totally ordered under inclusion so the concepts of strongly prime fuzzy bi-ideals and prime fuzzy bi-ideals coincide. Thus by Theorem 3.2.16, S is both regular and intra-regular. □

3.2.23 Theorem

Let S be an ordered semigroup. Then the folowing are equivalent:

(1) The set of bi-ideals of S is totally ordered under inclusion.

(2) Each bi-ideal of S is strongly irreducible.

(3) Each bi-ideal of S is irreducible.

Proof. (1) $\longrightarrow$ (2). Let B be a bi-ideal of S and B_1, B_2 be any two bi-ideals of S such that $B_1 \cap B_2 \subseteq B$. Since the set of bi-ideals of S is totally ordered , so either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Thus either $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$. Hence $B_1 \cap B_2 \subseteq B$ implies either $B_1 \subseteq B$ or $B_2 \subseteq B$. This shows that B is strongly irreducible.

(2) $\longrightarrow$ (3). Let B be any arbitrary bi-ideal of S and B_1, B_2 any two bi-ideals of S such that $B_1 \cap B_2 = B$. Then $B \subseteq B_1$ and $B \subseteq B_2$. By hypothesis, either $B_1 \subseteq B$ or $B_2 \subseteq B$. Hence, either $B_1 = B$ or $B_2 = B$. Thus B is irreducible.

(3) $\longrightarrow$ (1). Let B_1 and B_2 be any two bi-ideals of S. Then $B_1 \cap B_2$ is a bi-ideal of S. Also $B_1 \cap B_2 = B_1 \cap B_2$. So by hypothesis, either $B_1 = B_1 \cap B_2$ or $B_2 = B_1 \cap B_2$, that is, either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Hence the set of bi-ideals of S is totally ordered. □

3.2.24 Theorem

Let S be an ordered semigroup. Then the following are equivalent:

(1) Set of fuzzy bi-ideals of S is totally ordered under inclusion.

(2) Each fuzzy bi-ideal of S is strongly irreducible.

(3) Each fuzzy bi-ideal of S is irreducible.

Proof. (1) $\longrightarrow$ (2). Let f be an arbitrary fuzzy bi-ideal of S and g, h be any fuzzy bi-idals of S such that $g \wedge h \preceq f$. Since the set of fuzzy bi-ideals of S is totally ordered, thus either $g \preceq h$ or $h \preceq g$. Therefore $g \wedge h = h$ or $g \wedge h = g$. Hence $g \wedge h \preceq f$ implies either $h \preceq f$ or $g \preceq f$. Hence f is strongly irreducible.

(2) $\longrightarrow$ (3). Let f be an arbitrary fuzzy bi-ideal of S and g, h be any two fuzy bi-ideals of S such that $g \wedge h = f$. Then $f \preceq h$ and $f \preceq g$. By hypothesis, either $g \preceq f$ or $h \preceq f$. So either $g = f$ or $h = f$. Thus f is irreducible.

(3) $\longrightarrow$ (1). Let g, h be any arbitrary fuzzy bi-ideals of S. Then $g \wedge h$ is a fuzzy bi-ideal of S. Also $g \wedge h = g \wedge h$. So by hypothesis, either $g = g \wedge h$ or $h = g \wedge h$, that is either $g \preceq h$ or $h \preceq g$. Therefore the set of fuzzy bi-ideals of S is totally ordered under inclusion. □

3.3 Fuzzy filters

In this section, we define fuzzy bi-ideal subsets and fuzzy bi-filters in ordered semigroups and characterize ordered semigroups in terms of fuzzy bi-ideal subsets and fuzzy bi-filters. We also define fuzzy left filters and fuzzy prime left ideals in ordered semigroups and characterize ordered semigroups in terms of these notions. Results of this section are part of our published paper [133].

3.3.1 Definition

Let $(S, \cdot, \leq)$ be an ordered semigroup. A fuzzy subset f of S is called a fuzzy left (respectively, right) filter of S if

(1) $x \leq y \longrightarrow f(x) \leq f(y)$;

(2) $f(xy) \geq \min\{f(x), f(y)\}$ for all $x, y \in S$;

(3) $f(xy) \leq f(y)$ (respectively, $f(xy) \leq f(x)$) for all $x, y \in S$.

Next we define a fuzzy bi-ideal subset of an ordered semigroup S and a fuzzy bi-filter of S.

3.3.2 Definition

Let $(S, \cdot, \leq)$ be an ordered semigroup. A fuzzy subset f of S called a fuzzy bi-filter of S if

(1) $x \leq y \longrightarrow f(x) \leq f(y)$;

(2) $f(xy) \geq \min\{f(x), f(y)\}$;

(3) $f(xyx) \leq f(x)$ for all $x, y \in S$.

3.3.3 Definition

Let S be an ordered semigroup. A fuzzy subset f of S is called a fuzzy bi-ideal subset of S if

(1) $x \leq y \longrightarrow f(x) \geq f(y)$;

(2) $f(xyx) \geq f(x)$ for all $x, y \in S$.

3.3.4 Lemma

Let S be an ordered semigroup and f a fuzzy subset of S. Then the following are equivalent:

(1) $f(xyx) \leq f(x)$ for all $x, y \in S$,
(2) $f'(xyx) \geq f'(x)$ for all $x, y \in S$.

Proof. (1)$\longrightarrow$(2). Suppose that $f(xyx) \leq f(x)$ for all $x, y \in S$. Then

$$\begin{aligned} 1 - f(xyx) &\geq 1 - f(x), \\ f'(xyx) &\geq f'(x). \end{aligned}$$

(2)$\longrightarrow$(1). Suppose that $f'(xyx) \geq f'(x)$ for all $x, y \in S$. Then

$$\begin{aligned} 1 - f(xyx) &\geq 1 - f(x) \\ &\longrightarrow -f(xyx) \geq -f(x) \\ &\longrightarrow f(xyx) \leq f(x). \end{aligned}$$

□

The proof of the following Lemma is similar to the proof of the above lemma.

3.3.5 Lemma

Let S be an ordered semigroup and f a fuzzy subset of S. Then the following are equivalent:

(1) $f(xy) \leq f(y)$ (resp. $f(xy) \leq f(x)$) for all $x, y \in S$,
(2) $f'(xy) \geq f'(y)$ (respectively, $f'(xy) \leq f'(x)$) for all $x, y \in S$.

3.3.6 Lemma

Let S be an ordered semigroup and f a fuzzy subset of S. Then the following are equivalent:

(1) $f(xy) \leq max\{f(x), f(y)\}$ for all $x, y \in S$;
(2) $f'(xy) \geq min\{f'(x), f'(y)\}$ for all $x, y \in S$.

Proof. (1) $\longrightarrow$ (2). Suppose that $f(xy) \leq \max\{f(x), f(y)\}$ for all $x, y \in S$. Then

$$\begin{aligned} 1 - \max\{f(x), f(y)\} &\leq 1 - f(xy) \\ &= f'(xy). \end{aligned}$$

But

$$\begin{aligned} 1 - \max\{f(x), f(y)\} &= \min\{1 - f(x), 1 - f(y)\} \\ &= \min\{f'(x), f'(y)\}. \end{aligned}$$

Thus,

$$f'(xy) \geq \min\{f'(x), f'(y)\}$$

(2) $\longrightarrow$ (1). Suppose that $f(xy) \geq \min\{f(x), f(y)\}$ for all $x, y \in S$. Then

$$\begin{aligned} 1 - f(xy) &\geq \min\{1 - f(x), 1 - f(y)\} \\ &\longrightarrow f'(xy) \leq \max\{f'(x), f'(y)\}. \end{aligned}$$

□

3.4 Fuzzy prime and semiprime bi-ideal subsets of ordered semigroups

In this section, we study the concept of fuzzy bi-ideal subsets in ordered semigroups and characterize the bi-ideal subsets of an ordered semigroup in terms of fuzzy bi-ideal subsets. We also characterize bi-filters of ordered semigroups in terms of fuzzy bi-filters.

3.4.1 Definition (cf. [13]).

A nonempty subset B of S is called a *bi-ideal subset* of S if
(i) $a \in B$, $x \in S \longrightarrow axa \in B$;
(ii) $(\forall a \in B)(\forall b \in S)(b \leq a \longrightarrow b \in B)$.

3.4.2 Proposition

Let $(S, \cdot, \leq)$ be an ordered semigroup and B a nonempty subset of S. Then B is a bi-ideal subset of S if and only if the characteristic function f_B of B is a fuzzy bi-ideal subset of S.

Proof. ($\longrightarrow$) Suppose that B is a bi-ideal subset of S and f_B the characteristic function of B. Let $x, y \in S$. If $x \notin B$, then $f_B(x) = 0$. So $f_B(xyx) \geq f_B(x)$. If $x \in B$, then $f_B(x) = 1$. Besides, $x \in B$ implies that $xyx \in B$. Hence $f_B(xyx) = 1$. Thus again $f_B(xyx) \geq f_B(x)$. Let x, y be any arbitrary elements of S such that $x \leq y$ and $y \notin B$; then $f_B(y) = 0$. Since $f_B(x) \geq 0$, for all $x \in S$ we have $f_B(x) \geq f_B(y)$. If $y \in B$, then $f_B(y) = 1$. Since B is a bi-ideal subset of S and $x \leq y$, we have $x \in B$; then $f_B(x) = 1$.

$$f_B(x) \geq f_B(y).$$

($\longleftarrow$) Assume that f_B is a fuzzy bi-ideal subset of S. Let $x, y \in S$. If $x \in B$, then $f_B(x) = 1$; since $f_B(xyx) \geq f_B(x)$, so $f_B(xyx) = 1$. Thus $xyx \in B$. Let x, y be any arbitrary elements of S such that $x \leq y$. If $y \in B$, then $f_B(y) = 1$. Since f_B is a fuzzy bi-ideal subset of S and $x \leq y$, we have $f_B(x) \geq f_B(y)$. Thus $f_B(x) = 1 \longrightarrow x \in B$. □

3.4.3 Proposition

Let $(S,\cdot,\leq)$ be an ordered semigroup and F a nonempty subset of S. Then, F is a bi-filter of S if and only if the characteristic function f_F of F is a fuzzy bi-filter of S.

Proof. $(\longrightarrow)$ Suppose that F is a bi-filter of S and f_F the characteristic function of F. Then, by Lemma 1.2.12, f_F is a fuzzy subsemigroup of S. Let $x, y \in S$. If $xyx \notin F$, then

$$f_F(xyx) = 0 \leq f_F(x).$$

If $xyx \in F$, then $f_F(xyx) = 1$. In addition, $xyx \in F$ implies that $x \in F$. So we have $f_F(x) = 1$. Thus again $f_F(xyx) \leq f_F(x)$.

Let x, y be any arbitrary elements of S such that $x \leq y$. If $x \notin F$, then $f_F(x) = 0$. Since $f_F(y) \geq 0$ for all $y \in S$, we have $f_F(x) \leq f_F(y)$. If $x \in F$, then $f_F(x) = 1$. Since $x \leq y$ and F is a bi-filter of S, we have $y \in F$. Then $f_F(y) = 1$, which implies that $f_F(x) \leq f_F(y)$.

$(\longleftarrow)$ Assume that f_F is a fuzzy bi-filter of S and F, a nonempty subset of S. By Lemma 1.2.5, F is a subsemigroup of S. Let $x, y \in S$ be such that $xyx \in F$. Since f_F is a fuzzy bi-filter of S, we have

$$f_F(xyx) \leq f_F(x).$$

Since $xyx \in F$, we have $f_F(xyx) = 1$. Thus $f_F(x) = 1$, and hence $x \in F$. Let $x, y \in S, x \leq y$, and $x \in F$. Then $f_F(x) = 1$. Since $f_F(x) \leq f_F(y)$, we have $f_F(y) = 1$, which implies that $y \in F$. □

3.4.4 Lemma.

Let $(S,\cdot,\leq)$ be an ordered semigroup and f a fuzzy subset of S. Then f is a fuzzy bi-ideal subset of S if and only if $(\forall t \in [0,1])$ $U(f;t) \neq \emptyset$ is a bi-ideal subset of S.

Proof. Straightforward. □

3.4.5 Proposition.

Let $(S,\cdot,\leq)$ be an ordered semigroup and f a fuzzy subset of S. Then f is a prime fuzzy bi-ideal subset of S if and only if $(\forall t \in [0,1])$ $U(f;t) \neq \emptyset$ is a prime bi-ideal subset of S.

Proof. Suppose that f is a prime fuzzy bi-ideal subset of S. Then f is a fuzzy bi-ideal subset of S. By Lemma 1.3.7, $(\forall t \in [0,1])U(f;t) \neq \emptyset$ is a bi-ideal subset of S. Let $x, y \in S$ such that $xy \in f_t$. Then $f(xy) \geq t$. Since f is a prime fuzzy bi-ideal subset of S, we have

$$f(xy) \leq \max\{f(x), f(y)\}.$$

Thus, $\max\{f(x), f(y)\} \geq t$, which implies that $f(x) \geq t$ or $f(y) \geq t$. Thus, $x \in U(f;t)$ or $y \in U(f;t)$.

Conversely, assume that $U(f;t)$ is a prime bi-ideal subset of S for any $t \in [0,1]$. Then f_t is a bi-ideal subset of S. By Lemma 1.3.7, f is a fuzzy bi-ideal subset of S. Let $x, y \in S$ such that $f(xy) = t$. Since $U(f;t) \neq \emptyset$ is a prime bi-ideal subset of S and $xy \in U(f;t)$, we have $x \in U(f;t)$ or $y \in U(f;t)$, which implies that $f(x) \geq t$ or $f(y) \geq t$. □

Chapter 4

RIGHT PURE FUZZY IDEALS IN ORDERED SEMIGROUPS

In this chapter, we introduce the concept of right pure ideal in ordered semigroup and prove that the set of right pure ideal of an ordered semigroup S is a complete distributive lattice. We characterize right weakly regular ordered semigroup in terms of right pure ideals. We extend the concept of right pure ideal of ordered semigroup in fuzzy context and define right pure fuzzy ideals in ordered semigroups and prove that S is right weakly regular if and only if each ideal of S is right pure if and only if each fuzzy ideal of S is right pure. We also define purely prime ideals and purely prime fuzzy ideals of S and construct a topology on the set of purely prime ideals of S. The results of this chapter are part of our submitted paper [138].

4.1 Right pure ideals

In this section we prove that every two-sided ideal I of an ordered monoids S is right pure if and only if for every right ideal R of S, we have $R \cap I = (RI]$. We extend the notion of right pure ideals of ordered monoids in fuzzy context and prove that every fuzzy ideal g of S is a right pure fuzzy ideal of S if and only if for every fuzzy right ideal f of S we have $f \wedge g = f \circ g$.

4.1.1 Definition

An ideal I of an ordered semigroup S is called *right pure*, if for each $x \in I$ there exists $y \in I$ such that $x \leq xy$.

Equivalent Definition: $x \in (xI]$ for every $x \in I$.

4.1.2 Lemma

An ideal I of an ordered semigroup S is right pure if and only if $R \cap I = (RI]$ for every right ideal R of S.

Proof. Suppose that I is an ideal of S and R a right ideal of S. Then

$$(RI] \subseteq (SI] \subseteq (I] = I$$

and

$$(RI] \subseteq (RS] \subseteq (R] = R.$$

Hence $(RI] \subseteq R \cap I$. Let $a \in R \cap I$, then $a \in R$ and $a \in I$. Since I is right pure, so there exists $b \in I$ such that $a \leq ab$. But $ab \in RI$, so $a \in (RI]$. Thus $R \cap I \subseteq (RI]$. Hence $R \cap I = (RI]$.

Conversely, assume that $R \cap I = (RI]$ for every right ideal R of S. Let $a \in I$. Take R, the right ideal of S generated by a, that is, $R = (a \cup aS]$. Then $R \cap I = (RI]$ implies $a \in (RI]$, where $(RI] = ((a \cup aS]I] \subseteq ((a \cup aS)I] \subseteq (aI]$. Thus there exists $b \in I$ such that $a \leq ab$. Hence I is a right pure ideal of S. □

4.1.3 Definition

A fuzzy ideal g of an ordered semigroup S is called *right pure fuzzy ideal* of S if $f \wedge g = f \circ g$, for each fuzzy right ideal f of S.

4.1.4 Theorem

Let A be a non-empty subset of an ordered semigroup S and $t \in (0,1]$. Define $f : S \longrightarrow [0,1]$ by

$$f(x) := \begin{cases} t \text{ if } x \in A \\ 0 \text{ if } x \notin A. \end{cases}$$

Then f is a right pure fuzzy ideal of S if and only if A is a right pure ideal of S.

Proof. By Proposition 1.4.8, it follows that $g \circ f \preceq g \wedge f$ for every fuzzy right ideal g of S.

If $a \notin A$ then $f(a) = 0$ and so

$$(g \wedge f)(a) = 0 \leq (g \circ f)(a).$$

If $a \in A$ then there exists $b \in A$ such that $a \leq ab$. Thus $(a, b) \in A_a$. Hence

$$\begin{aligned} (g \circ f)(a) &= \bigvee_{(y,z) \in A_a} \min\{g(y), f(z)\} \\ &\geq \min\{g(a), f(b)\} \\ &= \min\{g(a), t\} \\ &= \min\{g(a), f(a)\} \\ &= (g \wedge f)(a). \end{aligned}$$

Thus $g \circ f \succeq g \wedge f$. Hence $g \circ f = g \wedge f$, that is, f is a right pure fuzzy ideal of S.

($\longleftarrow$) Assume that A is a right pure fuzzy ideal of S. Then clearly, A is an ideal of S. Let R be a right ideal of S, then f_R, is a fuzzy right ideal of S. Clearly $(RA] \subseteq R \cap A$. Let $x \in R \cap A$ then $x \in R$ and $x \in A$. As $f_R \wedge f = f_R \circ f$ and since $(f_R \wedge f)(x) = t \longrightarrow (f_R \circ f)(x) = t$ implies that there exist $a, b \in S$ such that $x \leq ab$ and $a \in R$, $b \in A$ implies that $x \in (RA]$. That is, $R \cap A \subseteq (RA]$. Hence $R \cap A = (RA]$ and so A is a right pure ideal of S. □

4.1.5 Corollary

An ideal I of an ordered semigroup S is right pure if and only if the characteristic function f_I of I is a right pure fuzzy ideal of S.

4.1.6 Proposition

Let $(S, \cdot, \leq)$ be an ordered monoid with zero 0. Then the following are true:

(1) (0) and S are right pure ideals of S.

(2) Union of right pure ideals of S is a right pure ideal of S.

(3) Finite intersection of right pure ideals of S is a right pure ideal of S.

Proof. (1) (0) and S are obviously right pure ideals of S.

(2) Let $\{I_k : k \in \Lambda\}$ be any family of right pure ideals of S. Then $\bigcup_{k\in\Lambda} I_k$ is a two-sided ideal of S. Suppose $x \in \bigcup_{k\in\Lambda} I_k$, then there exists $k \in \Lambda$ such that $x \in I_k$. Since I_k is right pure, so there exists $y \in I_k$ such that $x \leq xy$. Hence $\bigcup_{k\in\Lambda} I_k$ is a right pure ideal of S.

(3) Let I_1 and I_2 be any two right pure ideals of S. Then $I_1 \cap I_2$ is an ideal of S. Let $x \in I_1 \cap I_2$, then $x \in I_1$ and $x \in I_2$. So there exist $y_1 \in I_1$ and $y_2 \in I_2$ such that $x \leq xy_1$ and $x \leq xy_2$. Now $x \leq xy_2 \leq xy_1y_2$, and $y_1y_2 \in I_1 \cap I_2$, so $I_1 \cap I_2$ is a right pure ideal of S. □

4.1.7 Proposition

Let $(S, \cdot, \leq)$ be an ordered semigroup, f a fuzzy right ideal and g a fuzzy ideal of S. Then $f \circ g$ is a fuzzy right ideal of S.

Proof. Let $a, b \in S$. If $A_a = \emptyset$, then $(f \circ g)(a) = 0 \leq (f \circ g)(ab)$. Let $A_a \neq \emptyset$, then

$$(f \circ g)(a) := \bigvee_{(y,z)\in A_a} \min\{f(y), g(z)\}.$$

Since $a \leq yz$, then $ab \leq (yz)b = y(zb)$ and so $(y, zb) \in A_{ab}$. Since f is a fuzzy right ideal and g a fuzzy ideal of S, we have $f(ab) \geq f(y(zb)) \geq f(y)$ and $g(ab) \geq g(y(zb)) \geq g(zb) \geq g(z)$. Thus

$$\begin{aligned}(f \circ g)(a) \ &:= \bigvee_{(y,z)\in A_a} \min\{f(y), g(z)\} \leq \bigvee_{(y,z)\in A_a} \min\{f(y), g(zb)\} \\ &\leq \bigvee_{(y',z')\in A_{ab}} \min\{f(y'), g(z')\} = (f \circ g)(ab).\end{aligned}$$

Hence $(f \circ g)(ab) \geq (f \circ g)(a)$.

Let $x, y \in S$ with $x \leq y$. Let $(p, q) \in A_y$, then $y \leq pq$. Since $x \leq y$ then $x \leq pq$ and we have $(p, q) \in A_x$, and so $A_y \subseteq A_x$. Now,

$$\begin{aligned}(f \circ g)(x) &= \bigvee_{(p,q)\in A_x} \min\{f(p), g(q)\} \\ &\geq \bigvee_{(p,q)\in A_y} \min\{f(p), g(q)\} \\ &= (f \circ g)(y).\end{aligned}$$

Hence $(f \circ g)(x) \geq (f \circ g)(y)$. Therefore, $f \circ g$ is a fuzzy right ideal of S. □

4.1.8 Proposition

Let $(S, \cdot, \leq)$ be an ordered semigroup with zero element 0. Then the following are true:

(1). The fuzzy subsets "Φ" and "φ" of S, defined respectively, as

$$\Phi : S \longrightarrow [0,1] | x \longrightarrow \Phi(x) := \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

and

$$\varphi : S \longrightarrow \varphi(x) := 1 \text{ for all } x \in S,$$

are right pure fuzzy ideals of S.

(2). If f_1 and f_2 are right pure fuzzy ideals of S, then so is $f_1 \wedge f_2$.

(3). If $\{f_i : i \in I\}$ is a family of right pure fuzzy ideals of S then so is $\bigvee_{i\in I} f_i$.

(4). The fuzzy subset "Ψ" of S, defined by

$$\Psi : S \longrightarrow [0,1] | x \longrightarrow \Psi(x) := \begin{cases} 0 \text{ if } x \neq 0 \\ t \text{ if } x = 0 \end{cases}$$

where $t \in (0, 1]$, is a right pure fuzzy ideal of S.

Proof. (1) Since "Φ" and "φ" are the chracteristic functions of $\{0\}$ and S, respectively, so by Corollary 4.1.5, "Φ" and "φ" are right pure fuzzy ideals of S.

(2) Let f_1 and f_2 be right pure fuzzy ideals of S. We have to show that $f_1 \wedge f_2$ is a right pure fuzzy ideal of S. That is, for each fuzzy right ideal g of S, we have

$$g \circ (f_1 \wedge f_2) = g \wedge (f_1 \wedge f_2).$$

Indeed: Since f_2 is a right pure fuzzy ideal of S, so it follows that

$$f_1 \circ f_2 = f_1 \wedge f_2.$$

Hence,

$$g \circ (f_1 \wedge f_2) = g \circ (f_1 \circ f_2). \tag{i}$$

Also

$$g \wedge (f_1 \wedge f_2) = (g \wedge f_1) \wedge f_2 = (g \circ f_1) \wedge f_2 \tag{ii}$$

(since f_1 is a right pure fuzzy ideal of S)

Since, $g \circ f_1$ is a fuzzy right ideal of S, therefore (ii) gives us

$$\begin{aligned} g \wedge (f_1 \wedge f_2) &= (g \wedge f_1) \wedge f_2 \\ &= (g \circ f_1) \wedge f_2 = (g \circ f_1) \circ f_2 \\ &= g \circ (f_1 \circ f_2) \end{aligned} \tag{iii}$$

(by the associativity of the operation " $\circ$ ").

Thus (i) and (iii) give us, $g \circ (f_1 \wedge f_2) = g \wedge (f_1 \wedge f_2)$.

Therefore $f_1 \wedge f_2$ is a right pure fuzzy ideal of S.

(3) Let $\{f_i : i \in I\}$ be a family of right pure fuzzy ideals of S. We show that $\bigvee_{i \in I} f_i$ is also a right pure fuzzy ideal of S. For this we show that, for each fuzzy right ideal g of S, we have,

$$g \circ \left(\bigvee_{i \in I} f_i \right) = g \wedge \left(\bigvee_{i \in I} f_i \right).$$

Indeed:

$$g \wedge \left(\bigvee_{i \in I} f_i \right) = \bigvee_{i \in I} (g \wedge f_i) = \bigvee_{i \in I} (g \circ f_i)$$

(since f_i are right pure fuzzy ideal of S).

Also

$$\begin{aligned} g \circ \left(\bigvee_{i \in I} f_i \right)(a) &= \bigvee_{(y,z) \in A_a} \{\min(g(y), \left(\bigvee_{i \in I} f_i \right)(z)\} \\ &= \bigvee_{(y,z) \in A_a} \{g(y) \wedge \left(\bigvee_{i \in I} f_i(z) \right)\} \\ &= \bigvee_{(y,z) \in A_a} \{\bigvee_{i \in I} (g(y) \wedge f_i(z)\} \\ &= \bigvee_{i \in I} \{ \bigvee_{(y,z) \in A_a} (g(y) \wedge f_i(z)\} \\ &= \bigvee_{i \in I} (g \circ f_i)(a). \end{aligned}$$

Thus $g \circ \left(\bigvee_{i\in I} f_i\right) = g \wedge \left(\bigvee_{i\in I} f_i\right)$, and so $\bigvee_{i\in I} f_i$ is a right pure fuzzy ideal of S.

(4). Since $\{0\}$ is a right pure ideal of S, so by Proposition 7.2.4, Ψ is a right pure fuzzy ideal of S. □

From Proposition 4.1.8, it follows that the set of right pure ideals of S is a distributive lattice with supremum as "union" and infimum as "intersection". Also from Proposition 4.1.8, we have that the set of right pure fuzzy ideals of S is a distributive lattice with supremum "$\vee$" and infimum "$\wedge$".

4.1.9 Proposition

Let I be any ideal of an ordered semigroup S with 0. Then I contains a largest pure ideal. We call it, the right pure part of I and denote it by $P(I)$.

Proof. Let $P(I)$ be the union of all right pure ideals of S contained in I. Such ideals exist for example $\{0\}$. By Proposition 4.1.6 (2), $P(I)$ is a right pure ideal of S. It is indeed the largest right pure ideal of S contained in I. □

4.1.10 Proposition

Let f be any fuzzy ideal of an ordered semigroup S with 0. Then f contains a largest right pure fuzzy ideal of S. We call it, the pure part of f and denote it by $P(f)$.

Proof. Let $P(f)$ be the union of all right pure fuzzy ideals of S contained in f. Such ideals exit, for example Ψ. By Proposition 4.1.8 (3), $P(f)$ is a right pure fuzzy ideal of S. It is indeed, the largest right pure fuzzy ideal of S contained in f. □

4.1.11 Proposition

Let I, J be two ideals of an ordered semigroup S with 0 and $\{I_k : k \in K\}$ a family of ideals of S. Then

(1) $P(I \cap J) = P(I) \cap P(J)$.

(2) $P\left(\bigcup_{k\in K} I_k\right) \supseteq \bigcup_{k\in K} P(I_k)$.

Proof. (1). Since $P(I) \subseteq I$ and $P(J) \subseteq J$, so $P(I) \cap P(J) \subseteq I \cap J$. But by Proposition 4.1.6 (3), $P(I) \cap P(J)$ is right pure, so $P(I) \cap P(J) \subseteq P(I \cap J)$. On the other hand, $P(I \cap J) \subseteq I \cap J \subseteq I$ and $P(I \cap J)$ is right pure, so $P(I \cap J) \subseteq P(I)$. Similarly, $P(I \cap J) \subseteq P(J)$. Hence $P(I \cap J) \subseteq P(I) \cap P(J)$. Thus $P(I \cap J) = P(I) \cap P(J)$.

(2). Since $P(I_k) \subseteq I_k$ and $P(I_k)$ is right pure for all $k \in K$, so $\bigcup_{k\in K} P(I_k) \subseteq \bigcup_{k\in K} I_k$ and $\bigcup_{k\in K} P(I_k)$ is a right pure ideal of S, by Proposition 4.1.6 (2). By definition of pure part, we have $\bigcup_{k\in K} P(I_k) \subseteq P\left(\bigcup_{k\in K} I_k\right)$. □

4.1.12 Proposition

Let f, g be two fuzzy ideals of an ordered semigroup S with 0 and $\{f_k : k \in K\}$ a family of fuzzy ideals of S. Then

(1) $P(f \wedge g) = P(f) \wedge P(g)$.

(2) $P\left(\bigvee_{i\in I} f_i\right) \succeq \bigvee_{i\in I} P(f_i)$.

Proof. (1) Since $P(f) \preceq f$ and $P(g) \preceq g$, $P(f) \wedge P(g) \preceq f \wedge g$. But by Proposition 4.1.8 (2), $P(f) \wedge P(g)$ is right pure, so $P(f) \wedge P(g) \preceq P(f \wedge g)$. On the other hand, $P(f \wedge g) \preceq f \wedge g \preceq f$ and $P(f \wedge g)$ is pure, so $P(f \wedge g) \preceq P(f)$. Similarly, $P(f \wedge g) \preceq P(g)$. Hence $P(f \wedge g) \preceq P(f) \wedge P(g)$. Thus $P(f \wedge g) = P(f) \wedge P(g)$.

(2) Since $P(f_i) \preceq f_i$ and $P(f_i)$ is right pure for all $i \in I$, so $\bigvee_{i\in I} P(f_i) \preceq \bigvee_{i\in I} f_i$ and $\bigvee_{i\in I} P(f_i)$ is right pure. By the definition of pure part, we have $\bigvee_{i\in I} P(f_i) \preceq P\left(\bigvee_{i\in I} f_i\right)$. □

4.2 Right weakly regular ordered semigroups

In this section we characterize regular and right weakly regular ordered semigroups by their right pure ideals. We prove that the concepts of ideal and right pure ideal as well as fuzzy ideal and right pure fuzzy ideal in regular and in right weakly regular ordered semigroups coincide. Obviously every regular ordered semigroup is right weakly regular. If the ordered semigroup is commutative then the two concept coincide.

4.2.1 Proposition

An ordered semigroup $(S, \cdot, \leq)$ is right weakly regular if and only if every ideal of S is right pure.

Proof. Proof follows from Lemma 4.1.2. □

4.2.2 Corollary

In a regular ordered semigroups, every ideal is right pure.

4.2.3 Corollary

A commutative ordered semigroup S is regular if and only if every ideal of S is right pure.

4.2.4 Proposition

An ordered semigroup $(S, \cdot, \leq)$ is right weakly regular if and only if every fuzzy ideal of S is right pure fuzzy ideal of S.

Proof. Proof follows from Lemma 4.1.3 and Corollary 4.1.5. □

4.2.5 Corollary

In a regular ordered semigroups S every fuzzy ideal of S is right pure.

4.2.6 Corollary

A commutative ordered semigroup S is regular if and only if every fuzzy ideal of S is right pure.

4.3 Purely Prime Ideals

We begin with the following definitions:

4.3.1 Definirtion

A right pure ideal I of an ordered semigroup S is called *purely maximal* if I is a maximal element in the lattice of proper right pure ideals of S.

4.3.2 Definition

A proper right pure ideal I of an ordered semigroup S is called *purely prime* if for any right pure ideals A and B of S, $A \cap B \subseteq I$ implies that $A \subseteq I$ or $B \subseteq I$.

If A, B are right pure ideals of S then $A \cap B = (AB]$. Thus the above definition is equivalent to $AB \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$.

4.3.3 Proposition

Let $(S, \cdot, \leq)$ be an ordered semigroup. Then any purely maximal ideal of S is purely prime.

Proof. Let I be any purely maximal ideal of S. Let A and B be right pure ideals of S such that $A \cap B \subseteq I$. Suppose that $A \nsubseteq I$. Since I and A are right pure ideals of S, so by Proposition 4.1.6 (2), $A \cup I$ is a right pure ideal of S. Since I is maximal in the lattice of proper right pure ideals of S, it follows that $I \cup A = S$. Hence

$$B = B \cap S = B \cap (I \cup A) = (B \cap I) \cup (B \cap A) \subseteq I \cup I \subseteq I.$$

Hence I is purely prime. □

4.3.4 Definition

A right pure fuzzy ideal f of an ordered semigroup S is called *purely maximal* if f is a maximal element in the lattice of proper right pure fuzzy ideals of S.

4.3.5 Definition

A right pure fuzzy ideal f of an ordered semigroup S is called *purely prime fuzzy ideal* of S if for any right pure fuzzy ideals f_1 and f_2 of S, $f_1 \wedge f_2 \preceq f$ implies $f_1 \preceq f$ or $f_2 \preceq f$.

If f_1, f_2 are right pure fuzzy ideals of S, then $f_1 \wedge f_2 = f_1 \circ f_2$. Thus the above definition is equivalent to $f_1 \circ f_2 \preceq f$ implies $f_1 \preceq f$ or $f_2 \preceq f$.

4.3.6 Proposition

Let $(S, \cdot, \leq)$ be an ordered semigroup. Then any purely maxiaml fuzzy ideal of S is purely prime fuzzy ideal of S.

Proof. Let f be any purely maximal fuzzy ideal of S. Let g and h be right pure fuzzy ideals of S such that $g \wedge h \preceq f$. Suppose that $g \npreceq f$. Since f and g are right pure fuzzy ideals of S, so $f \vee g$ is a right pure fuzzy ideal of S. Since f is maximal in the lattice of proper right pure fuzzy ideals of S, it follows that $f \vee g = \varphi$. Hence

$$h = h \wedge \varphi = h \wedge (f \vee g) = (h \wedge f) \vee (h \wedge g) \preceq f \vee f \preceq f.$$

Thus f is a purely prime fuzzy ideal of S. □

4.3.7 Proposition

Let $(S, \cdot, \leq)$ be an ordered monoid. Then any right pure ideal of S is contained in a maximal right pure ideal of S.

Proof. Let I be a proper right pure ideal of S. Then the set

$$X = \{J | J \text{ is a proper right pure ideal and } I \subseteq J\}$$

is partially ordered by inclusion.

$X \neq \emptyset$ because $I \in X$. For any $J \in X$, $1 \notin J$ since J is proper. Let $\{J_k\}_{k \in K}$ be any non-empty totally ordered subset of X. Then by Proposition 4.1.4 (2), $\bigcup_{k \in K} J_k$ is a right pure ideal of S such that $1 \notin \bigcup_{k \in K} J_k$ and $I \subseteq \bigcup_{k \in K} J_k$, and we have $\bigcup_{k \in K} J_k \in X$. This shows that X is inductively ordered. Hence by Zorn's Lemma, X contains a maximal element sy J' such that $I \subseteq J'$. Obviously J' is purely maximal ideal of S and contains I. □

4.3.8 Proposition

Let $(S, \cdot, \leq)$ be an ordered monoid. Then any right pure fuzzy ideal of S is contained in a maximal right pure fuzzy ideal of S.

Proof. Straightforward. □

4.3.9 Proposition

Let $(S, \cdot, \leq)$ be an ordered semigroup. If I is a right pure ideal of S and $a \notin I$, then there exists a purely prime ideal J of S such that $I \subseteq J$ and $a \notin J$.

Proof. We consider the set, ordered by inclusion,

$$X = \{J | J \text{ is a proper ideal of } S \text{ and } I \subseteq J,\ a \notin J\}.$$

Then $X \neq \emptyset$ since $I \in X$. For any $J \in X$, $1 \notin J$ since J is proper. Let $\{J_k\}_{k \in K}$ be any non-empty totally ordered subset of X. Then by Proposition 4.1.6 (2), $\bigcup_{k \in K} J_k$ is a right pure ideal of S such that $1 \notin \bigcup_{k \in K} J_k$, $I \subseteq \bigcup_{k \in K} J_k$ and $a \notin \bigcup_{k \in K} J_k$. Thus $\bigcup_{k \in K} J_k \in X$. Hence X is inductively ordered. By Zorn's Lemma X has a maximal element say J such that J is right pure, $I \subseteq J$ and $a \notin J$. We claim that J is purely prime. Suppose that I_1 and I_2 are right pure ideals of S such that $I_1 \nsubseteq J$ and $I_2 \nsubseteq J$. Since I_1, I_2 and J are right pure ideals of S, so $I_1 \cup J$ and $I_2 \cup J$ are right pure ideals of S. We then claim that $a \in I_k \cup J (k = 1, 2)$. Because, if $a \notin I_k \cup J$, then by the maximality of J, we have $I_k \cup J \subseteq J$. This contradicts the assumption $I_k \nsubseteq J$. Hence $a \in (I_1 \cup J) \cap (I_2 \cup J) = (I_1 \cap I_2) \cup J$. Since $a \notin J$, it follows that $a \in I_1 \cap I_2$ and so $I_1 \cap I_2 \nsubseteq J$. Hence by contrapositivity, we conclude that J is purely prime. □

4.3.10 Proposition

Let $(S, \cdot, \leq)$ be an ordered semigroup. If f is a right pure fuzzy ideal of S with $f(a) = t$ where $a \in S$ and $t \in (0, 1]$, then there exists a purely prime fuzzy ideal g of S such that $f \preceq g$ and $g(a) = t$.

Proof. Let $X = \{h | h \text{ is a right pure fuzzy ideal of } S,\ h(a) = t \text{ and } f \preceq h\}$. Then $X \neq \emptyset$, because $f \in X$. The collection X is a partially ordered set under inclusion. If Y is any totally ordered subset of X, say $Y = \{h_i | i \in I\}$. Then $\bigvee_{i \in I} h_i$ is a right pure fuzzy ideal of S, by Proposition 4.1.8 (3). As $f \preceq h_i$ for each $i \in I$, so $f \preceq \bigvee_{i \in I} h_i$. Also $\left(\bigvee_{i \in I} h_i\right)(a) = \bigvee_{i \in I} h_i(a) = t$. Thus $\bigvee_{i \in I} h_i$ is the least upper bound of Y. By Zorn's Lemma, there exists a right pure fuzzy ideal g of S which is maximal with respect to the property $f \preceq g$ and $g(a) = t$. We show that g is purely prime. Suppose that $g_1 \wedge g_2 \preceq g$ but $g_1 \npreceq g$ and $g_2 \npreceq g$, where g_1 and g_2 are right pure fuzzy ideals of S. Since g_i $(i = 1, 2)$ and g are right pure fuzzy ideals of S, so by Proposition 4.1.8 (2), $g_i \vee g$ is a right pure fuzzy ideal of S such that $g \preceq g_i \vee g$. We claim that $(g_i \vee g)(a) \neq t$. Because if $(g_i \vee g)(a) = t$, then by the maximality of g, we have $g_i \vee g \preceq g$, which is a contradiction with our assumption that $g_i \npreceq g$. Hence $((g_1 \vee g) \wedge (g_2 \vee g))(a) = ((g_1 \wedge g_2) \vee g)(a) \neq t$. Since $g(a) = t$, it follows that $(g_1 \wedge g_2)(a) \neq t$ and so $g_1 \wedge g_2 \npreceq g$. Hence by contrapositivity, we conclude that g is a purely prime fuzzy ideal of S. □

4.4 Pure spectrum

In this section, S will denote an ordered monoid with zero. We denote by $RP(S)$ the lattice of right pure ideals of S and $PP(S)$ the set of purely prime ideal of S. For any right pure ideal I of S, we define

$$\Omega_I := \{J \in PP(S) : I \nsubseteq J\} \text{ and } \Im(PP(S)) := \{\Omega_I : I \in RP(S)\}.$$

4.4.1 Theorem

The set $\Im(PP(S))$ forms a topology on the set $PP(S)$.

Proof. Since $\{0\}$ is a right pure ideal of S, we have $\Omega_{\{0\}} = \{J \in PP(S) : \{0\} \nsubseteq J\} = \emptyset$. Thus $\Omega_{\{0\}}$ is the empty subset of $\Im(PP(S))$. On the other hand

$$\Omega_S = \{J \in PP(S) : S \nsubseteq J\} = PP(S).$$

This is true since purely prime ideals of S are proper. So $\Omega_S = PP(S)$ is an element of $\Im(PP(S))$.

Now let $\Omega_{I_1}, \Omega_{I_2} \in \Im(PP(S))$ with I_1, I_2 right pure ideals of S. Then

$$\begin{aligned} \Omega_{I_1} \cap \Omega_{I_2} &= \{J \in PP(S) : I_1 \nsubseteq J \text{ and } I_2 \nsubseteq J\} \\ &= \{J \in PP(S) : I_1 \cap I_2 \nsubseteq J\} \\ &= \Omega_{I_1 \cap I_2}. \end{aligned}$$

This follows from the equivalence $I_1 \cap I_2 \nsubseteq J \Longleftrightarrow I_1 \nsubseteq J$ and $I_2 \nsubseteq J$.

Next, let us consider any family $\{I_k\}_{k \in K}$ of right pure ideals of S. Since

$$\begin{aligned} \bigcup_{k \in K} \Omega_{I_k} &= \bigcup_{k \in K} \{J \in PP(S) : I_k \nsubseteq J\} \\ &= \{J \in PP(S) : \exists k \in K \text{ so that } I_k \nsubseteq J\} \\ &= \left\{J \in PP(S) : \bigcup_{k \in K} I_k \nsubseteq J\right\} \\ &= \Omega_{\bigcup\limits_{k \in K} I_k}. \end{aligned}$$

Since $\bigcup\limits_{k \in K} I_k$ is a right pure ideal of S it follows that $\bigcup\limits_{k \in K} I_k$ is contained in $\Im(PP(S))$. Thus the set $\Im(PP(S))$ of subsets Ω_I with I right pure ideal of S constitutes a topology on the set $PP(S)$. □

4.4.2 Definition

A right pure fuzzy ideal f of an ordered semigroup S is called *normal* if $f(0) = 1$.

Let $RPNF(S)$ denotes the set of all right pure normal fuzzy ideals of an ordered semigroup S and $RPNFP(S)$, the set of all proper purely prime normal fuzzy ideals of S. As remarked earlier that $RPF(S)$ is a lattice with respect to the partial ordering in $[0,1]$ with a least element Ψ and greatest element φ. For any right pure fuzzy ideal f of S, we define:

$$\Theta_f := \{g \in RPFP(S) : f \npreceq g\},$$

thus Θ_f is a subset of $RPFP(S)$ for each right pure fuzzy ideal f of S. We will show that the set $RPFP(S)$, together with the subsets Θ_f $(f \in RPF(S))$ forms a topology on $RPFP(S)$. By $\tau(RPFP(S))$ we mean the set of all subsets Θ_f, defined as open subsets of $RPFP(S)$.

4.4.3 Theorem

The set $\tau(RPFP(S))$, together with the subsets Θ_f ($f \in RPF(S)$), forms a topology on $RPFP(S)$.

Proof. For the right pure fuzzy ideal Φ defined by

$$\Psi : S \longrightarrow [0,1] | x \longrightarrow \Psi(x) = \begin{cases} 0 \text{ if for all } x \in S\backslash\{0\}, \\ t \qquad\qquad \text{for } x = 0, \end{cases}$$

where $t = \inf\{g(0) : g \in RPFP(S)\}$

the subset $\Theta_\Psi = \{g \in RPFP(S) : \Psi \npreceq g\} = \emptyset$ (the classical empty set). Thus the empty subset of $RPFP(S)$, $(\Theta_\Psi) \in \tau(RPFP(S))$. On the other hand, for the right pure fuzzy ideal φ of S defined by

$$\varphi : S \longrightarrow [0,1] | x \longrightarrow \varphi(x) := 1 \text{ for all } x \in S,$$

$\Theta_\varphi = \{g \in RPFP(S) : \varphi \npreceq g\} = RPFP(S)$. This is true, since purely prime fuzzy ideals are proper. Hence the whole set $RPFP(S)(= \Theta_\varphi) \in \tau(RPFP(S))$. Now, let Θ_{f_1} and $\Theta_{f_2} \in \tau(RPFP(S))$ with f_1 and $f_2 \in RPF(S)$. We show that $\Theta_{f_1} \cap \Theta_{f_2} = \Theta_{f_1 \wedge f_2}$. Let $g \in \Theta_{f_1} \cap \Theta_{f_2}$, then $g \in RPFP(S)$ and $f_1 \npreceq g$ and $f_2 \npreceq g$. Suppose that $f_1 \wedge f_2 \preceq g$. Since g is a purely prime fuzzy ideal of S and f_1, f_2 are right pure fuzzy ideals of S, we have $f_1 \preceq g$ or $f \preceq g$, which is a contradiction. Thus $\Theta_{f_1} \cap \Theta_{f_2} \subseteq \Theta_{f_1 \wedge f_2}$. On the other hand, if $g \in \Theta_{f_1 \wedge f_2}$, then $g \in RPFP(S)$ and $f_1 \wedge f_2 \npreceq g$, which implies that $f_1 \npreceq g$ and $f_2 \npreceq g$. Hence $g \in \Theta_{f_1}$ and $g \in \Theta_{f_2} \Longrightarrow g \in \Theta_{f_1} \cap \Theta_{f_2}$ and we have $\Theta_{f_1 \wedge f_2} \subseteq \Theta_{f_1} \cap \Theta_{f_2}$. Therefore, $\Theta_{f_1} \cap \Theta_{f_2} = \Theta_{f_1 \wedge f_2}$.

Let us consider an arbitrary family $\{f_i : i \in I\}$ of right pure fuzzy ideals of S. Since

$$\begin{aligned} \bigcup_{i \in I} \Theta_{f_i} &= \bigcup_{i \in I} \{g \in RPFP(S) | \exists\, i \in I \text{ such that } f_i \npreceq g\} \\ &= \{g \in RPFP(S) | \exists\, i \in I \bigvee_{i \in I} f_i \npreceq g\} \\ &= \Theta_{\bigvee_{i \in I} f_i}. \end{aligned}$$

Now since $\bigvee_{i \in I} f_i$ is a right pure fuzzy ideal of S, by Proposition 4.1.8 (3). It follows that $\bigcup_{i \in I} \Theta_{f_i} \in \tau(RPFP(S))$. Thus the subset Θ_f with $f \in PF(S)$ constitute a topology on $RPFP(S)$. $\square$

Chapter 5

Generalized fuzzy ideals in ordered semigroups

In this chapter, we introduce the concepts of generalized fuzzy left (right) ideals and generalized fuzzy bi-ideals of ordered semigroups and characterize ordered semigroups in terms of these notions. The results of this chapter are part of our accepted paper [20] and submitted paper [107].

5.1 (α, β)-fuzzy ideals

Let S be an ordered semigroup. A fuzzy subset f of S of the form

$$f(y) := \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is called a *fuzzy point* with support x and value t and is denoted by x_t (cf. [122]). A fuzzy point x_t is said to *belong to* (resp. *quasi-coincident* with) a fuzzy set f, written as $x_t \in f$ (resp. $x_t q f$) if $f(x) \geq t$ (resp. $f(x) + t > 1$). If $x_t \in f$ or $x_t q f$, then $x_t \in \vee q f$. The symbol $\overline{\in \vee q}$ means $\in \vee q$ does not hold (cf. [19]).

In what follows let S denote an ordered semigroup and α, β any one of $\in$, q, $\in \vee$q, $\in \wedge$q unless otherwise specified.

Let f be a fuzzy subset of S such that $f(x) \leq 0.5$ for all $x \in S$. Let $x \in S$ and $t \in (0,1]$ be such that $x_t \in \wedge q f$. Then $f(x) \geq t$ and $f(x) + t \geq 1$. It follows that $1 < f(x) + t < f(x) + f(x) = 2f(x)$. This implies that $f(x) > 0.5$. Hence $\{x_t | x_t \in \wedge q f\} = \emptyset$.

Thus the case $\alpha = \in \wedge q$ is omitted in the following definition.

5.1.1 Definition

Let $(S, \cdot, \leq)$ be an ordered semigroup and f a fuzzy subset of S. Then f is called an *(α, β)-fuzzy left (*resp. *right) ideal* of S if for all $t \in (0,1]$ and for all $x, y \in S$, we have

(I_1) $x \leq y$, $y_t \alpha f \longrightarrow x_t \beta f$.

(I_2) $y_t \alpha f \longrightarrow (xy)_t \beta f$ (resp. $(yx)_t \beta f$).

5.1.2 Theorem

For a fuzzy subset f of S, the conditions (I_3), and (I_4) are equivalent to the conditions (I_5), and (I_6), respectively, where (I_3),(I_4), (I_5), and (I_6), are given as follows:

(I_3) $x \leq y \longrightarrow f(x) \geq f(y)$.

(I_4) $f(xy) \geq f(y)$ (resp. $f(xy) \geq f(x)$).

(I_5) $(\forall x, y \in S)(\forall t \in (0,1])(x \leq y, y_t \in f \longrightarrow x_t \in f)$,
(I_6) $(\forall x, y \in S)(\forall t \in (0,1])(x \in S, y_t \in f \longrightarrow (xy)_t \in f(resp.\ (yx)_t \in f))$.

Proof. $(I_3) \longrightarrow (I_5)$. Let $x, y \in S$ and $t \in (0,1]$ be such that $x \leq y$, $y_t \in f$. Then $f(y) \geq t$. Since $x \leq y$, we have $f(x) \geq f(y) \geq t$ by (I_3). Hence $x_t \in f$.

$(I_5) \longrightarrow (I_3)$. Assume that (I_3) is not valid. Then there exist $x, y \in S$ such that $x \leq y$ and $f(x) < f(y)$. Hence $f(x) < t \leq f(y)$ for some $t \in (0,1]$ and so $y_t \in f$ but $x_t \notin f$, a contradiction. Hence (I_3) is valid.

$(I_4) \longrightarrow (I_6)$. Let $x, y \in S$ and $t \in (0,1]$ be such that $y_t \in f$. Then $f(y) \geq t$. By (I_4), we have $f(xy) \geq f(y) \geq t$. It follows that $(xy)_t \in f$.

$(I_6) \longrightarrow (I_4)$. Let $x, y \in S$. Since $y_{f(y)} \in f$. By (I_6) we have $(xy)_{f(y)} \in f$, it follows that $f(xy) \geq f(y)$. □

5.1.3 Example

Consider the ordered semigroup $S = \{a, b, c, d, e\}$ with multiplication "$\cdot$" and order relation "$\leq$" given below:

$\cdot$	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

$$\leq \; := \{(a,a), (a,c), (a,d), (a,e), (b,b), (b,d), (b,e), (c,c), (c,e), (d,d), (d,e), (e,e)\}.$$

The ideals of S are: $\{a, b, d\}$, $\{a, c, d, e\}$ and S. Define a fuzzy subset f of S as follows:

$$f(a) = 0.8, \quad f(c) = 0.7, \quad f(e) = 0.6, \quad f(d) = 0.5, \quad f(b) = 0.3.$$

Then

$$U(f;t) := \begin{cases} S & \text{if } 0 < t \leq 0.3 \\ \{a,c,d,e\} & \text{if } 0.3 < t \leq 0.5 \\ \emptyset & \text{if } t \leq 0.8 \end{cases}$$

Then clearly f is an $(\in, \in \vee q)$-fuzzy ideal of S. But
(i) f is not an $(\in, \in)$-fuzzy ideal of S, since

$$c_{0.68} \in f \text{ but } (ce)_{0.68} = e_{0.68} \overline{\in} f.$$

(ii) f is not a $(q, \in)$-fuzzy ideal of S, since

$$b_{0.88} q f \text{ but } (bd)_{0.88} = d_{0.88} \overline{\in} f.$$

(iii) f is not an $(\in, q)$-fuzzy ideal of S, since

$$b_{0.28} \in f \text{ but } (bd)_{0.28} = d_{0.28}\overline{q}f.$$

(iv) f is not a $(q, \in \wedge q)$-fuzzy ideal of S, since

$$b_{0.76}qf \text{ but } (bd)_{0.76} = d_{0.76}\overline{\in \wedge q}f.$$

(v) f is not an $(\in \vee q, \in \wedge q)$-fuzzy ideal of S, since

$$e_{0.54} \in \vee qf \text{ but } (ed)_{0.54} = d_{0.54}\overline{\in \wedge q}f.$$

(vi) f is not an $(\in \vee q, \in)$-fuzzy ideal of S, since

$$e_{0.54} \in \vee qf \text{ but } (ed)_{0.54} = d_{0.54}\overline{\in}f.$$

(vii) f is not an $(\in, \in \wedge q)$-fuzzy ideal of S, since

$$e_{0.54} \in f \text{ but } (de)_{0.54} = d_{0.54}\overline{\in \wedge q}f.$$

(viii) f is not a $(q, \in \vee q)$-fuzzy ideal of S, since

$$b_{0.76}qf \text{ but } (bd)_{0.76} = d_{0.76}\overline{\in \vee q}f.$$

(ix) f is not a (q, q)-fuzzy ideal of S, since

$$a_{0.32}qf \text{ but } (ad)_{0.32} = d_{0.32}\overline{q}f.$$

5.1.4 Theorem

Every $(\in, \in)$-fuzzy left (resp. right) ideal of S is an $(\in, \in \vee q)$-fuzzy left (resp. right) ideal.

Proof. Straighforward. □

The converse of above Theorem is not true in general as shown in the above example.

5.1.5 Theorem

Every $(\in \vee q, \in \vee q)$-fuzzy left (resp. right) ideal is an $(\in, \in \vee q)$-fuzzy left (resp. right) ideal.

Proof. Let f be an $(\in \vee q, \in \vee q)$-fuzzy left ideal of S. Let $x, y \in S$, and $t \in (0, 1]$ be such that $x \leq y$, $y_t \in f$. Then $y_t \in \vee qf$. Since $x \leq y$ by hypothesis, we have $x_t \in \vee qf$. Let $x, y \in S$ and $t \in (0, 1]$ be such that $y_t \in f$. Then $y_t \in \vee qf$ and hence $(xy)_t \in \vee qf$. Similarly we can prove that $(yx)_t \in \vee qf$. □

5.1.6 Theorem

Let f be a non-zero (α,β)-fuzzy left (resp. right) ideal of S. Then the set $f_0 = \{x \in S | f(x) > 0\}$ is a left (resp. right) ideal of S.

Proof. Let $x, y \in S$, $x \leq y$. If $y \in f_0$, then $f(y) > 0$. Assume that $x \notin f_0$, that is $f(x) = 0$. If $\alpha \in \{\in, \in \vee q\}$ then $y_{f(y)}\alpha f$ but $x_{f(x)}\overline{\beta} f$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Also $y_1 q f$ but $x_1 \overline{\beta} f$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Hence $f(x) > 0$, that is $x \in f_0$. Let $y \in f_0$. Assume that $f(xy) = 0$. If $\alpha \in \{\in, \in \vee \mathrm{q}\}$ then $y_{f(y)}\alpha f$ but $(xy)_{f(xy)}\overline{\beta} f$ for every $\beta \in \{\in, q, \in \vee \mathrm{q}, \in \wedge \mathrm{q}\}$, a contradiction. Note that $y_1 \mathrm{q} f$ but $(xy)_1 \overline{\beta} f$ for every $\beta \in \{\in, q, \in \vee \mathrm{q}, \in \wedge \mathrm{q}\}$, a contradiction. Hence $f(xy) > 0$, that is, $xy \in f_0$. Consequently, f_0 is a left ideal of S. Similarly we can prove that f_0 is a right ideal of S. □

5.1.7 Theorem

Let I be a left (resp. right) ideal and f a fuzzy subset of S defined by

$$f(x) = \begin{cases} 0 \text{ if } x \in S \backslash I \\ \geq 0.5 \text{ if } x \in I \end{cases}.$$

Then
(a) f is a $(q, \in \vee q)$-fuzzy ideal of S.
(b) f is an $(\in, \in \vee q)$-fuzzy ideal of S.

Proof. (a) Let I be a left ideal of S and $x, y \in S$ such that $x \leq y$. Let $t \in (0, 1]$ be such that $y_t q f$. Then $y \in I$ and so $x \in I$. If $t \leq 0.5$, then $f(x) \geq 0.5 \geq t$. Hence $x_t \in f$. If $t > 0.5$, then

$$f(x) + t > 0.5 + 0.5 = 1$$

and so $x_t q f$. It follows that $x_t \in \vee q f$. Let $x, y \in S$ and $t \in (0, 1]$ be such that $y_t q f$. Then $y \in I$ and we have $xy \in I$. If $t \leq 0.5$ then $f(xy) \geq 0.5 \geq t$ and hence $(xy)_t \in f$. If $t > 0.5$, then

$$f(xy) + t > 0.5 + 0.5 = 1$$

and so $(xy)_t \mathrm{q} f$. Therefore $(xy)_t \in \vee \mathrm{q} f$.

(b) Let $x, y \in S$, and $t \in (0, 1]$ be such that $x \leq y$, $y_t \in f$. Then $f(y) \geq t$ and we have $x \leq y \in I$, it follows that $x \in I$. If $t \leq 0.5$, then $f(x) \geq 0.5 \geq t$. Hence $x_t \in f$. If $t > 0.5$, then

$$f(x) + t > 0.5 + 0.5 = 1$$

and so $x_t q f$. It follows that $x_t \in \vee q f$. Let $x, y \in S$ and $t \in (0, 1]$ be such that $y_t \in f$. Then $f(y) \geq t$ and it follows that $y \in I$. Then $xy \in I$. If $t \leq 0.5$, then $f(xy) \geq 0.5 \geq t$. Hence $(xy)_t \in f$. If $t > 0.5$, then

$$f(xy) + t > 0.5 + 0.5 = 1$$

and so $(xy)_t q f$. It follows that $(xy)_t \in \vee q f$. □

5.1.8 Remark

Every $(\in, \in \vee q)$*-fuzzy left (resp. right) ideal of* S *is not a* $(q, \in \vee q)$*-fuzzy left (resp. right) ideal (see example 5.1.3, Part iv).*

5.2 $(\in, \in \vee q)$-fuzzy ideals

5.2.1 Proposition

For a fuzzy subset f *of an ordered semigroup* S*, the conditions* $(I_1)'$*, and* $(I_2)'$ *are equivalent* (I_7) *and* (I_8)*, respectively. Where* $(I_1)'$,$(I_2)'$,(I_7) *and* (I_8) *are as following:*

$(I_5)'$ $x \leq y$, $y_t \in f \longrightarrow x_t \in \vee q f$.

$(I_6)'$ $y_t \in f \longrightarrow (xy)_t \in \vee q f$ (resp. $(yx)_t \in \vee q f$).

(I_7) $(\forall x, y \in S)(x \leq y \longrightarrow f(x) \geq min\{f(y), 0.5\})$.

(I_8) $(\forall x, y \in S)(f(xy) \geq min\{f(y), 0.5\}(resp.\ f(xy) \geq min\{f(x), 0.5\}))$.

Proof. $(I_1)' \longrightarrow (I_7)$. Let $x, y \in S$ such that $x \leq y$. We consider the following cases:

a) $f(y) < 0.5$,

b) $f(y) \geq 0.5$.

Case a: Let $x \leq y$ and $f(y) < 0.5$. Assume that $f(x) < \min\{f(y), 0.5\}$. Then $f(x) < f(y)$. Choose $t \in (0, 1]$ such that $f(x) < t \leq f(y)$, then $f(x) + t < 1$. Thus $y_t \in f$ but $x_t \overline{\in \vee q} f$, a contradiction. Hence $f(x) \geq \min\{f(y), 0.5\}$.

Case b: Let $x \leq y$ and $f(y) \geq 0.5$. If $f(x) < \min\{f(y), 0.5\} = 0.5$, then $y_{0.5} \in f$ but $x_{0.5} \overline{\in \vee q} f$, which is again a contradiction. Therefore $f(x) \geq \min\{f(y), 0.5\}$.

$(I_7) \longrightarrow (I_1)'$. Let $x, y \in S$ such that $x \leq y$. Suppose that $y_t \in f$. Then $f(y) \geq t$ and so

$$f(x) \geq \min\{f(y), 0.5\} \geq \min\{t, 0.5\}.$$

This implies that $f(x) \geq t$ or $f(x) \geq 0.5$, according to $t \leq 0.5$ or $t > 0.5$. Therefore $x_t \in \vee q f$.

$(I_2)' \longrightarrow (I_8)$. Let $x, y \in S$ such that

$$f(xy) < \min\{f(y), 0.5\}.$$

If $f(y) < 0.5$, then $f(xy) < f(y)$. Choose $s_0 \in (0, 1]$ such that $f(xy) < s_0 \leq f(y)$, then $y_{s_0} \in f$, but $(xy)_{s_0} \overline{\in \vee q} f$, a contradiction. If $f(y) \geq 0.5$, then $f(xy) < 0.5$, $y_{0.5} \in f$ but $(xy)_{0.5} \overline{\in \vee q} f$, again a contradiction. Hence $f(xy) \geq \min\{f(y), 0.5\}$ for all $x, y \in S$.

$(I_8) \longrightarrow (I_2)'$. Let $x \in S$, $y_t \in f$, then $f(y) \geq t$. By hypothesis,

$$f(xy) \geq \min\{f(y), 0.5\} \geq \min\{t, 0.5\}.$$

If $t > 0.5$, then $f(xy) \geq 0.5$, which implies that $f(xy) + t > 1$, it follows that $(xy)_t \mathrm{q} f$. If $t \leq 0.5$, then $f(xy) \geq t$. This implies $(xy)_t \in f$. Hence $(xy)_t \in \vee q f$. □

5.2.2 Corollary

A fuzzy subset f of an ordered semigroup S is an $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of S if and only if it satisfies conditions (I_7) and (I_8).

5.2.3 Corollary

Every fuzzy left (resp. right) ideal of an ordered semigroup is an $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of S. But the converse is not true.

5.2.4 Example

Consider the ordered semigroup given in example 5.1.3, and define a fuzzy subset $f : S \longrightarrow [0,1]$ by:

$$f(a) = 0.8, \quad f(c) = 0.7, \quad f(e) = 0.6, \quad f(d) = 0.5, \quad f(b) = 0.3.$$

Then f is an $(\in, \in \vee \mathrm{q})$-fuzzy ideal of S. But $U(f;t) = \{a, c\}$ for all $t \in (0.6, 0.7]$ is not an ideal of S by Lemma 1.2.6, and so f is not a fuzzy ideal of S for all $t \in (0.6, 0.7]$.

Now, we characterize $(\in, \in \vee \mathrm{q})$-fuzzy left (resp. right) ideals by their level sets.

5.2.5 Theorem

Let S be an ordered semigroup and f a fuzzy subset of S. Then f is an $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of S if and only if $U(f;t)(\neq \emptyset)$ is a left (resp. right) ideal of S for all $t \in (0, 0.5]$.

Proof. Let f be an $(\in, \in \vee q)$-fuzzy left ideal of S and $t \in (0, 0.5]$. Let $x, y \in S$ be such that $x \leq y$. If $y \in U(f;t)$ then $f(y) \geq t$. Since

$$f(x) \geq \min\{f(y), 0.5\} \geq \min\{t, 0.5\} = t$$

so $x \in U(f;t)$. Let $x \in S$ and $y \in U(f;t)$. Then $f(y) \geq t$. By hypothesis $f(xy) \geq \min\{f(y), 0.5\} \geq \min\{t, 0.5\} = t$, and so $xy \in U(f;t)$. Thus $U(f;t)$ is a left ideal of S.

Conversely, let $U(f;t) = \{x \in S | f(x) \geq t\} (\neq \emptyset)$ be a left ideal of S for all $t \in (0, 0.5]$. Let $x, y \in S$ with $x \leq y$. Then

$$f(y) \geq \min\{f(y), 0.5\} = t_0.$$

Thus $t_0 \in (0, 0.5]$ and $y \in U(f;t_0)$ since $x \leq y \in U(f;t_0)$ and $U(f;t_0)$ is a left ideal of S, we have $x \in U(f;t_0)$. Hence $f(x) \geq t_0 = \min\{f(y), 0.5\}$. Let $x, y \in S$, then $f(y) \geq \min\{f(y), 0.5\} = t_0$, and so $y \in U(f;t_0)$. Since $U(f;t_0)$ is a left ideal of S, we have $xy \in U(f;t_0)$. Thus $f(xy) \geq t_0 = \min\{f(y), 0.5\}$. Similarly we can prove that f is an $(\in, \in \vee \mathrm{q})$-fuzzy right ideal of S. □

It is clear from Lemma 1.2.6, that a fuzzy subset f of an ordered semigroup S is a fuzzy left (resp. right) ideal of S if and only if $U(f;t)(\neq \emptyset)$ is a left (resp. right) ideal of S for all $t \in (0,1]$ and from Theorem 5.2.5, f is an $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of S if and only if $U(f;t)(\neq \emptyset)$ is a left (resp. right) ideal of S for all $t \in (0.0.5]$.

5.2.6 Remark

Every (α,β)-fuzzy left ideal is not an (α,β)-fuzzy right ideal.

5.2.7 Example

Consider the ordered semigroup, S given in example 5.1.3. Left ideals of S are the sets:

$\{a\}, \{a,c\}, \{a,c,d\}, \{a,b,c,d\}$ *and* S. *Define a fuzzy subset* $f : S \longrightarrow [0,1]$ *by:*

$$f(a) = 0.8, \quad f(b) = 0.4, \quad f(c) = 0.6, \quad f(d) = 0.5, \quad f(e) = 0.1$$

$$U(f;t) := \begin{cases} \{a\} & \text{if } t \in (0.6, 0.8] \\ \{a,c\} & \text{if } t \in (0.5, 0.6] \\ \{a,c,d\} & \text{if } t \in (0.4, 0.5] \\ \{a,b,c,d\} & \text{if } t \in (0.1, 0.4] \\ S & \text{if } t \in (0, 0.1] \end{cases}$$

Clearly, f is an $(\in, \in \vee \mathrm{q})$-fuzzy left ideal of S. But A is not an $(\in, \in \vee \mathrm{q})$-fuzzy right ideal of S. Since

$$c_{0.58} \in f \text{ but } (cb)_{0.58} = d_{0.58} \overline{\in \vee \mathrm{q}} f$$

Using Proposition 5.2.1, we have the following characterization of fuzzy left(resp. right) ideals of ordered semigroups.

5.2.8 Proposition

Let $(S, \cdot, \leq)$ be an ordered semigroup and $\emptyset \neq I \subseteq S$. Then I is a left (resp. right) ideal of S if and only if the characteristic function f_I of I is an $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of S.

In the following Theorem we give a condition for an $(\in, \in \vee \mathrm{q})$-fuzzy left (resp. right) ideal of S to be an $(\in, \in)$-fuzzy left (resp. right) ideal of S.

5.2.9 Theorem

Let f be an $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of S such that $f(x) < 0.5$ for all $x \in S$. Then f is an $(\in, \in)$-fuzzy left (resp. right) ideal of S.

Proof. Let $x, y \in S$ and $t \in (0, 0.5]$ be such that $x \leq y$, $y_t \in f$. Then $f(y) \geq t$. By hypothesis

$$f(x) \geq \min\{f(y), 0.5\} \geq \min\{t, 0.5\} = t$$

hence $x \in f_t$. Let $x, y \in S$ and $t \in (0, 0.5]$ be such that $y_t \in f$. Then $f(y) \geq t$ and we have

$$f(xy) \geq \min\{f(y), 0.5\} \geq \min\{t, 0.5\} = t,$$

and hence $(xy)_t \in f$. Similarly we can prove that $(yx)_t \in f$. □

For any fuzzy subset f of an ordered semigroup S and $t \in (0, 1]$ we denote by:

$$Q(f;t) := \{x \in S | x_t q f\} \text{ and } [f]_t := \{x \in S | x_t \in \vee q f\}.$$

Obviously $[f]_t = U(f;t) \cup Q(f;t)$.

We call $[f]_t$ an ($\in \vee$q)-level set of f and $Q(f;t)$ a q-level set of f.

In Theorem 5.2.5, we give a characterization of $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of S by using level subsets. Now, we give another characterization of ($\in, \in \vee$q)-fuzzy left (resp. right) ideals by using $[f]_t$.

5.2.10 Theorem

Let S be an ordered semigroup and f a fuzzy subset of S. Then f is an $(\in, \in \vee q)$-fuzzy left(resp. right) ideal of S if and only if $[f]_t$ is a left (resp. right) ideal of S for all $t \in (0, 1]$.

Proof. Let f be an ($\in, \in \vee$q)-fuzzy left ideal of S. Let $x, y \in S$ and $t \in (0, 1]$ be such that $x \leq y$, and $y \in [f]_t$. Then $y_t \in \vee q f$, that is, $f(y) \geq t$ or $f(y) + t > 1$. Since f is an ($\in, \in \vee$q)-fuzzy left ideal of S and $x \leq y$ we have $f(x) \geq \min\{f(y), 0.5\}$. We have the following cases:

Case 1 $f(y) \geq t$. If $t > 0.5$, then $f(x) \geq \min\{f(y), 0.5\} = 0.5$ and so

$$f(x) + t > 0.5 + 0.5 = 1,$$

hence $x_t q f$. If $t \leq 0.5$, then $f(x) \geq \min\{f(y), 0.5\} \geq t$, and hence $x_t \in f$.

Case 2 $f(y) + t > 1$. If $t > 0.5$, then

$$f(x) \geq \min\{f(y), 0.5\} > \min\{1 - t, 0.5\} = 1 - t,$$

that is, $f(x) + t > 1$ and thus $x_t q f$. If $t \leq 0.5$, then

$$f(x) \geq \min\{f(y), 0.5\} \geq \min\{1 - t, 0.5\} = 0.5 \geq t,$$

and so $x_t \in f$.

Consequently, $x_t \in \vee qf$. Thus $x \in [f]_t$. Let $x \in S$ and $y \in [f]_t$ for $t \in (0,1]$. Then $y_t \in \vee qf$, that is $f(y) \geq t$ or $f(y) + t > 1$. Since f is an $(\in, \in \vee q)$-fuzzy left ideal of S, we have

$$f(xy) \geq \min\{f(y), 0.5\}.$$

Case 1 Let $f(y) \geq t$. If $t > 0.5$, then $f(xy) \geq \min\{f(y), 0.5\} = 0.5$ and so

$$f(xy) + t > 0.5 + 0.5 = 1,$$

hence $(xy)_t qf$. If $t \leq 0.5$, then $f(xy) \geq \min\{f(y), 0.5\} \geq t$, and hence $(xy)_t \in f$.
Case 2 $f(y) + t > 1$. If $t > 0.5$, then

$$f(xy) \geq \min\{f(y), 0.5\} > \min\{1 - t, 0.5\} = 1 - t,$$

that is, $f(xy) + t > 1$ and thus $(xy)_t qf$. If $t \leq 0.5$, then

$$f(xy) \geq \min\{f(y), 0.5\} \geq \min\{1 - t, 0.5\} = 0.5 \geq t,$$

and so $(xy)_t \in f$. Hence $(xy)_t \in \vee qf$.

Consequently, $(xy)_t \in \vee qf$. Thus $xy \in [f]_t$. Hence $[f]_t$ is a left ideal of S.

Conversely, let f be a fuzzy subset of S and $t \in (0,1]$ be such that $[f]_t$ is a left ideal of S. Let $x, y \in S$ that $x \leq y$. If possible, let $f(x) < t \leq \min\{f(y), 0.5\}$ for some $t \in (0, 0.5]$. Then $y \in U(f;t) \subseteq [f]_t$. Since $x \leq y \in [f]_t$ then $x \in [f]_t$ and we have $f(x) \geq t$ or $f(x) + t > 1$. This is a contradiction. Hence $f(x) \geq \min\{f(y), 0.5\}$ for all $x, y \in S$ with $x \leq y$. Let $x, y \in S$ be such that $f(xy) < t \leq \min\{f(y), 0.5\}$ for some $t \in (0, 0.5]$. Then $y \in U(f;t) \subseteq [f]_t$, and we have $xy \in [f]_t$ so $f(xy) \geq t$ or $f(xy) + t > 1$. This is a contradiction. Hence $f(xy) \geq \min\{f(y), 0.5\}$ for all $x, y \in S$. Thus f is an $(\in, \in \vee q)$-fuzzy left ideal of S. Similarly, we can prove that f is an $(\in, \in \vee q)$-fuzzy right ideal of S. □

From Theorem 5.2.5 and 5.2.10, we see that if f is an $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of S then, $U(f;t)$ and $[f]_t$ are left (resp. right) ideals of S for all $t \in (0, 0.5]$, but $Q(f;t)$ is not a left ideal of S for $t \in (0, 0.5]$ in general. As shown in the following example.

5.2.11 Example

Consider the ordered semigroup given in Example 5.1.3. Define a fuzzy subset f by $f(a) = 0.8, \quad f(b) = 0.6, \quad f(d) = 0.5, \quad f(c) = 0.4, \quad f(e) = 0.2.$

Then

$$U(f;t) = \begin{cases} S \text{ if } 0 < t \leq 0.3 \\ \{a, b, d\} \text{ if } 0.4 < t \leq 0.5 \end{cases}$$

Obviously, f is an $(\in, \in \vee q)$-fuzzy ideal of S. But $Q(f;t) = \{a, b, c, d\}$ for $0.2 < t \leq 0.4$. Since $c_{o.52} qf$ but $(ce)_{0.52} = e_{0.52} \overline{q} f$.

5.2.12 Definition

Let f be a fuzzy subset of an ordered semigroup $(S, \cdot, \leq)$. Then f is called an *$(\in, \in \vee q)$-fuzzy quasi-ideal* of S if for $x, y \in S$, it satisfies:

(1) $x \leq y \longrightarrow f(x) \geq \min\{f(y), 0.5\}$,

(2) $f(x) \geq \min\{((f \circ 1) \cap (1 \circ f))(x), 0.5\}$.

5.2.13 Theorem

Let f be a non-zero $(\in, \in \vee q)$-fuzzy quasi-ideal of S, then the set $f_0 = \{x \in S | f(x) > 0\}$ is a quasi-ideal of S.

Proof. Let $x, y \in S$ be such that $x \leq y$. If $y \in f_0$, then $f(y) > 0$. Since $x \leq y$ we have $f(x) \geq \min\{f(y), 0.5\}$, then $f(x) > 0$ and so $x \in f_0$. Let $a \in ((f_0S] \cap (Sf_0])$, then $a \in (f_0S]$ and $a \in (Sf_0]$ and hence $a \leq xs$ and $a \leq ry$ for some $r, s \in S$ and $x, y \in f_0$. Thus

$$\begin{aligned} f(a) &\geq \min\{(f \circ 1)(a), (1 \circ f)(a), 0.5\} \\ &= \min\left[\left\{\bigvee_{(p,q) \in A_a} \min\{f(p), 1(q)\}, \bigvee_{(p,q) \in A_a} \min\{1(q), f(p)\}\right\}, 0.5\right] \\ &\geq \min\left[\{\min\{f(x), 1\}, \min\{1(r), f(y)\}\}, 0.5\right] \\ &= \min\left[f(x), f(y), 0.5\right] > 0 (\text{since } x, y \in f_0, \text{ so } f(x) > 0, f(y) > 0). \end{aligned}$$

Hence $a \in f_0$ and so $(f_0S] \cap (Sf_0] \subseteq f_0$. □

The proof of following Proposition is easy and so omitted.

5.2.14 Proposition

Every $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of S is an $(\in, \in \vee q)$-fuzzy quasi-ideal of S.

The converse of above Proposition is not true in general.

5.2.15 Example

Let $S = \{0, 1, 2, 3\}$ be an ordered semigroup with the following multiplication table and order relation:

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	0
2	0	0	0	0
3	0	3	0	0

$$\leq := \{(0,0), (1,1), (2,2), (3,3), (0,1), (0,2), (0,3)\}$$

Then $\{0,1\}$ is a quasi-ideal of S. But not a left (resp. right) ideal of S. Define a fuzzy subset $f : S \longrightarrow [0,1]$ by

$$f(0) = f(1) = 0.5 \quad f(2) = f(3) = 0$$

Then f is a fuzzy quasi-ideal and hence an $(\in, \in \vee q)$-fuzzy quasi-ideal of S. But f is not an $(\in, \in \vee q)$-fuzzy left ideal of S. Because

$$f(3.1) = f(3) = 0 < 0.5 = f(1) \wedge 0.5$$

5.2.16 Definition

Let $(S, \cdot, \leq)$ be an ordered semigroup and f, g are fuzzy subsets of S. Then the 0.5-product of f and g is defined by:

$$(f \circ_{0.5} g)(a) := \begin{cases} \bigvee\limits_{(y,z)\in A_a} \min\{f(y), g(z), 0.5\} & \text{if } A_a \neq \emptyset \\ 0 & \text{if } A_a = \emptyset \end{cases}$$

We also define $f \cap_{0.5} g$ by $(f \cap_{0.5} g)(a) = \min\{f(a), g(a), 0.5\}$ for all $a \in S$.

5.2.17 Proposition

If $(S, \cdot, \leq)$ is an ordered semigroup and f, g, h, k are fuzzy subsets of S such that $f \subseteq h$ and $g \subseteq k$. Then $f \circ_{0.5} g \subseteq h \circ_{0.5} k$.

5.2.18 Lemma

Let S be an ordered semigroup. If f and g are $(\in, \in \vee q)$-fuzzy left (resp. right) ideals of S. Then $f \cap_{0.5} g$ is an $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of S.

Proof. Let f and g be $(\in, \in \vee \mathrm{q})$-fuzzy left ideals of S. Let $x, y \in S$ such that $x \leq y$. Then

$$\begin{aligned} (f \cap_{0.5} g)(x) &= \min\{f(x), g(x), 0.5\} \\ &\geq \min\{\min\{f(y), 0.5\}, \min\{g(y), 0.5\}, 0.5\} \\ &\text{(because } f \text{ and } g \text{ are } (\in, \in \vee \mathrm{q})\text{-fuzzy left ideals of } S) \\ &= \min\{f(y), g(y), 0.5\} \\ &= (f \cap_{0.5} g)(y). \end{aligned}$$

Hence $(f \cap_{0.5} g)(x) \geq (f \cap_{0.5} g)(y)$.
Let $x, y \in S$. Then

$$\begin{aligned} (f \cap_{0.5} g)(xy) &= \min\{f(xy), g(xy), 0.5\} \\ &\geq \min\{\min\{f(y), 0.5\}, \min\{g(y), 0.5\}, 0.5\} \\ &= \min\{f(y), g(y), 0.5\} \\ &= (f \cap_{0.5} g)(y). \end{aligned}$$

Hence $f \cap_{0.5} g$ is an $(\in, \in \vee q)$-fuzzy left ideal of S. Similarly we can prove that $f \cap_{0.5} g$ is an $(\in, \in \vee q)$-fuzzy right ideal of S. □

5.2.19 Lemma

Let $(S, \cdot, \leq)$ be an ordered semigroup. If f is an $(\in, \in \vee q)$-fuzzy right ideal and g an $(\in, \in \vee q)$-fuzzy left ideal of S, respectively. Then $f \circ_{0.5} g \subseteq f \cap_{0.5} g$.

Proof. Let f be an $(\in, \in \vee q)$-fuzzy right ideal and g is an $(\in, \in \vee q)$-fuzzy left ideal of S and $a \in S$. If $A_a = \emptyset$, then $(f \circ_{0.5} g)(a) = 0 \leq (f \cap_{0.5} g)(a)$. Let $A_a \neq \emptyset$, then

$$(f \circ_{0.5} g)(a) := \bigvee_{(y,z) \in Aa} \min\{f(y), g(z), 0.5\}.$$

Since $a \leq yz$, and f is an $(\in, \in \vee q)$-fuzzy right ideal and g an $(\in, \in \vee q)$-fuzzy left ideal of S, we have

$$f(a) \geq \min\{f(yz), 0.5\} \geq \min\{\min\{f(y), 0.5\}, 0.5\} = \min\{f(y), 0.5\}$$

and

$$g(a) \geq \min\{g(yz), 0.5\} \geq \min\{\min\{g(z), 0.5\}, 0.5\} = \min\{g(z), 0.5\}.$$

Thus, $\min\{f(y), g(z), 0.5\} \leq \min\{f(a), g(a), 0.5\} = (f \cap_{0.5} g)(a)$, and so $(f \circ_{0.5} g)(a) \leq (f \cap_{0.5} g)(a)$. □

The proof of the following Lemma is obvious.

5.2.20 Lemma

Let S be an ordered semigroup. Then the following are true.

(i) $A \subseteq B$ if and only if $min\{f_A(a), 0.5\} \leq min\{f_B(a), 0.5\}$ for all $a \in S$.

(ii) $(f_A \cap_{0.5} f_B)(a) \leq min\{f_{A \cap B}(a), 0.5\}$ for all $a \in S$.

(iii) $(f_A \circ_{0.5} f_B)(a) \leq min\{f_{(AB]}(a), 0.5\}$ for all $a \in S$.

In the following Theorem we prove that an ordered semigroup S is regular if and only if for every $(\in, \in \vee q)$-fuzzy right ideal f and every $(\in, \in \vee q)$-fuzzy left ideal g of S we have $f \cap_{0.5} g = f \circ_{0.5} g$.

5.2.21 Theorem

An ordered semigroup S is regular if and only if for every $(\in, \in \vee q)$-fuzzy right ideal f and every $(\in, \in \vee q)$-fuzzy left ideal g of S, we have $f \circ_{0.5} g = f \cap_{0.5} g$.

Proof. Let $a \in S$. Then $(f \cap_{0.5} g)(a) \leq (f \circ_{0.5} g)(a)$. In fact: Since S is regular, so there exists $x \in S$ such that $a \leq axa$. Then $(ax, a) \in A_a$ and we have

$$\begin{aligned}(f \circ_{0.5} g)(a) &= \bigvee_{(y,z) \in A_a} \min\{f(y), g(z), 0.5\} \\ &\geq \min\{f(ax), g(a), 0.5\},\end{aligned}$$

Since f is an $(\in, \in \vee \text{q})$-fuzzy right ideal of S, we have $f(ax) \geq \min\{f(a), 0.5\}$. Thus

$$\begin{aligned}\min\{f(ax), g(a), 0.5\} &\geq \min\{\min\{f(a), 0.5\}, g(a), 0.5\} \\ &= \min\{f(a), g(a), 0.5\} \\ &= (f \cap_{0.5} g)(a).\end{aligned}$$

and we have $(f \cap_{0.5} g)(a) \leq (f \circ_{0.5} g)(a)$. On the other hand, by Lemma 5.2.19, we have $(f \circ_{0.5} g)(a) \leq (f \cap_{0.5} g)(a)$. Therefore $(f \circ_{0.5} g)(a) = (f \cap_{0.5} g)(a)$.

Conversely, assume that $f \circ_{0.5} g = f \cap_{0.5} g$, for every $(\in, \in \vee \text{q})$-fuzzy right ideal f and every $(\in, \in \vee \text{q})$-fuzzy left ideal g of S. Then S is regular. In fact: By Lemma 1.1.13, it is enough to prove that

$$R \cap L = (RL] \text{ for every right ideal } R \text{ and every left ideal } L \text{ of } S.$$

Let $y \in R \cap L$, then $y \in (RL]$. Indeed: Since R is a right ideal and L a left ideal of S, then f_R is a fuzzy right ideal and f_L is a fuzzy left ideal of S and so by Proposition 5.2.8, f_R is an $(\in, \in \vee \text{q})$-fuzzy right ideal and f_L an $(\in, \in \vee \text{q})$-fuzzy left ideal of S. By hypothesis, we have

$$(f_R \circ_{0.5} f_L)(y) = (f_R \cap_{0.5} f_L)(y).$$

Since $y \in R$ and $y \in L$ we have $f_R(y) = 1$, and $f_L(y) = 1$, then $(f_R \cap_{0.5} f_L)(y) = \min\{f_R(y), f_L(y), 0.5\} = 0.5$. It follows that $(f_R \circ_{0.5} f_L)(y) = 0.5$. By Lemma 5.2.20 (iii), $(f_R \circ_{0.5} f_L)(y) = \min\{f_{(RL]}(y), 0.5\}$ then $\min\{f_{(RL]}(y), 0.5\} = 0.5 \longrightarrow y \in (RL]$. Hence $R \cap L \subseteq (RL]$. On the other hand, $(RL] \subseteq R \cap L$, always hold. Therefore, $R \cap L = (RL]$. □

5.3 (α, β)-fuzzy bi-ideals

In this section, we characterize ordered semigroups in terms of generalized fuzzy bi-ideals.

5.3.1 Theorem

For any fuzzy subset f of S, the conditions (B_1), (B_2) and (B_3) are equivalent to the conditions (B_4), (B_5) and (B_6), respectively. Where $(B_1),(B_2),(B_3),(B_4)$,(B_5) and (B_6) are given as follows:

(B_1) $x \leq y \longrightarrow f(x) \geq f(y)$.
(B_2) $f(xy) \geq \min\{f(x), f(y)\}$.
(B_3) $f(xyz) \geq \{f(x), f(z)\})$.
(B_4) $(\forall x, y \in S)$ $(\forall t \in (0,1])(x \leq y,\, y_t \in f \longrightarrow x_t \in f)$.
(B_5) $(\forall x, y \in S)(t, r \in (0,1])(x_t, y_r \in f \longrightarrow (xy)_{\min\{t,r\}} \in f)$.
(B_6) $(\forall x, y, z \in S)(t, r \in (0,1])(y \in S, x_t, z_r \in f \longrightarrow (xyz)_{\min\{t,r\}} \in f)$.

Proof. $(B_1) \longleftrightarrow (B_4)$. Follows from Theorem 5.1.2.

$(B_2) \longrightarrow (B_5)$. Let $x, y \in S$ and $t, r \in (0,1]$ be such that $x_t, y_r \in f$. Then $f(x) \geq t$ and $f(y) \geq r$. By (B_2), we have $f(xy) \geq \min\{f(x), f(y)\} \geq \min\{t, r\}$, it follows that $(xy)_{\min\{t,r\}} \in f$.

$(B_5) \longrightarrow (B_2)$. Let $x, y \in S$. Since $x_{f(x)} \in f$ and $y_{f(y)} \in f$. By (B_5) we have $(xy)_{\min\{f(x),f(y)\}} \in f$, it follows that $f(xy) \geq \min\{f(x), f(y)\}$.

$(B_3) \longrightarrow (B_6)$. Let $x, y, z \in S$ and $t, r \in (0,1]$ be such that x_t, $z_r \in f$. Then $f(x) \geq t$ and $f(z) \geq r$. By (B_3), we have $f(xyz) \geq \min\{f(x), f(z)\} \geq \min\{t, r\}$, it follows that $(xyz)_{\min\{t,r\}} \in f$.

$(B_6) \longrightarrow (B_3)$. Let $x, y, z \in S$. Since $x_{f(x)} \in f$ and $z_{f(z)} \in f$. By (B_5) we have $(xyz)_{\min\{f(x),f(z)\}} \in f$, it follows that $f(xyz) \geq \min\{f(x), f(z)\}$. □

5.3.2 Definition

A fuzzy subset f of S is called an (α, β)-fuzzy bi-ideal of S, where $\alpha \neq \in \wedge q$, if for all $x, y, z \in S$ and for all $t, r \in (0,1]$ it satisfies:

(B_7) $x \leq y$, $y_t \alpha f \longrightarrow x_t \beta f$.
(B_8) $x_t, y_r \alpha f \longrightarrow (xy)_{\min\{t,r\}} \beta f$.
(B_9) $x_t, z_r \alpha f \longrightarrow (xyz)_{\min\{t,r\}} \beta f$.

5.3.3 Example

Consider the set $S = \{a, b, c, d, e\}$ with the following multiplication "." and order relation "$\leq$":

$\cdot$	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

$$\leq \;:= \{(a,a),(a,c),(a,d),(a,e),(b,b),(b,d),$$
$$(b,e),(c,c),(c,e),(d,d),(d,e),(e,e)\}$$

Then $(S,\cdot,\leq)$ is an ordered semigroup and $\{a\}$, $\{a,b,e\}$ and $\{a,b,d,e\}$ are bi-ideals of S. We define a fuzzy subset $f : S \longrightarrow [0,1]$ by:

$$f(a) = 0.8, \quad f(b) = 0.7, \quad f(e) = 0.6, \quad f(d) = 0.5, \quad f(c) = 0.3.$$

Then

$$U(f;t) := \begin{cases} S & \text{if } t \in (0, 0.3], \\ \{a,b,d,e\} & \text{if } t \in (0.3, 0.5], \\ \{a,b,e\} & \text{if } t \in (0.5, 0.6], \\ \{a\} & \text{if } t \in (0.6, 0.8], \\ \emptyset & \text{if } t \in (0.8, 1] \end{cases}$$

Clearly f is an $(\in, \in\vee q)$-fuzzy bi-ideal of S. But
(i) f is not an $(\in,\in)$-fuzzy bi-ideal of S, since $a_{0.78} \in f$ and $b_{0.66} \in f$ but

$$(ab)_{\min\{0.78,0.76\}} = d_{0.76}\overline{\in} f.$$

(ii) f is not a (q,$\in$)-fuzzy bi-ideal of S, since $a_{0.75}qf$ and $b_{0.65}qf$ but

$$(ab)_{\min\{0.75,0.65\}} = d_{0.65}\overline{\in} f.$$

(iii) f is not an ($\in$,q)-fuzzy bi-ideal of S, since $a_{0.30} \in f$ and $b_{0.20} \in f$ but

$$(ab)_{\min\{0.30,0.20\}} = d_{0.20}\overline{q} f.$$

(iv) f is not an $(q, \in\vee q)$-fuzzy bi-ideal of S since $a_{0.65}qf$ and $b_{0.55}qf$ but

$$(ab)_{\min\{0.65,0.55\}} = d_{0.55}\overline{\in\vee q} A.$$

(v) f is not a (q,$\in\wedge$q)-fuzzy bi-ideal of S, since $a_{0.72}qf$ and $b_{0.62}qf$ but

$$(ab)_{\min\{0.72,0.62\}} = d_{0.62}\overline{\in\wedge q} f.$$

(vi) f is not an ($\in\vee$q,$\in\wedge$q)-fuzzy bi-ideal of S, since $a_{0.64} \in\vee qf$ and $b_{0.54} \in\vee qf$ but

$$(ab)_{\min\{0.64,0.54\}} = d_{0.54}\overline{\in} f \text{ and so } d_{0.54}\overline{\in\wedge q} f.$$

(vii) f is not an ($\in\vee$q,$\in$)-fuzzy bi-ideal of S, since $a_{0.63} \in\vee qf$ and $b_{0.53} \in\vee qf$ but

$$(ab)_{\min\{0.63,0.53\}} = d_{0.53}\overline{\in} f.$$

(viii) f is not an $(\in, \in\wedge q)$-fuzzy bi-ideal of S, since $a_{0.62} \in f$ and $b_{0.52} \in f$ but

$$(ab)_{\min\{0.62,0.52\}} = d_{0.52}\overline{\in} f \text{ and so } d_{0.52}\overline{\in\wedge q} f.$$

(xi) f is not a (q,q)-fuzzy bi-ideal of S, since $a_{0.38}\mathrm{q}f$ and $b_{0.48}\mathrm{q}f$ but

$$(ab)_{\min\{0.38,0.48\}} = d_{0.38}\overline{q}f.$$

(x) f is not an ($\in \vee$q, q)-fuzzy bi-ideal of S, since $a_{0.39} \in \vee qf$ and $b_{0.49} \in \vee qf$ but

$$(ab)_{\min\{0.39,0.49\}} = d_{0.39}\overline{q}f.$$

(xi) f is not an ($\in \vee$q, $\in \vee$q)-fuzzy bi-ideal of S, $a_{0.68} \in \vee qf$ and $b_{0.58} \in \vee qf$ but

$$(ab)_{\min\{0.68,0.58\}} = d_{0.58}\overline{\in \vee q}f.$$

5.3.4 Theorem

Every $(\in, \in)$*-fuzzy bi-ideal is an* $(\in, \in \vee q)$*-fuzzy bi-ideal.*

Proof. Straightforward. □

5.3.5 Theorem

Every $(\in \vee q, \in \vee q)$*-fuzzy bi-ideal is* $(\in, \in \vee q)$*-fuzzy bi-ideal.*

Proof. Let f be an ($\in \vee$q,$\in \vee$q)-fuzzy bi-ideal of S. Let $x, y \in S$, $x \leq y$ and $t \in (0,1]$ be such that $y_t \in f$. Then $y_t \in \vee \mathrm{q}f$. Since $x \leq y$ and $y_t \in \vee \mathrm{q}f$ we have $x_t \in \vee \mathrm{q}f$. Let $x, y \in S$ and $t, r \in (0,1]$ be such that $x_t, y_r \in f$. Then $x_t, y_r \in \vee \mathrm{q}f$, which implies $(xy)_{\min\{t,r\}} \in \vee \mathrm{q}f$. Let now, $x, y, z \in S$ and $t, r \in (0,1]$ be such that $x_t, z_r \in f$. Then $x_t, z_r \in \vee \mathrm{q}f$, which implies $(xyz)_{\min\{t,r\}} \in \vee \mathrm{q}f$. □

5.3.6 Theorem

Let f *be a non-zero* (α, β)*-fuzzy bi-ideal of* S*. Then the set* $f_0 := \{x \in S | f(x) > 0\}$ *is a bi-ideal of* S*.*

Proof. The proof is similar to the proof of Theorem 5.1.6. □

5.3.7 Theorem

Let B *be a bi-ideal of an ordered semigroup* S *and* f *a fuzzy subset of* S *defined by*

$$f(x) = \begin{cases} 0 \text{ if } x \in S \backslash B \\ 0.5 \text{ if } x \in B \end{cases}.$$

Then

(a) f *is a* $(q, \in \vee q)$*-fuzzy bi-ideal of* S*.*
(b) f *is an* $(\in, \in \vee q)$*-fuzzy bi-ideal of* S*.*

Proof. Proof is similar to the proof of Theorem 5.1.7. □

From example 5.3.4, we see that an ($\in, \in \vee$q)-fuzzy bi-ideal is not a (q,$\in \vee$q)-fuzzy bi-ideal (Example 5.3.4, Part iv).

5.4 $(\in, \in \vee q)$-fuzzy bi-ideals

In this section we define the notions of $(\in, \in \vee q)$-fuzzy bi-ideals of an ordered semigroup and investigate some of their properties in terms of $(\in, \in \vee q)$-fuzzy bi-ideals.

5.4.1 Lemma

For a fuzzy subset f of an ordered semigroup S, the conditions $(B_7)'$, $(B_8)'$, and $(B_9)'$ are equivalent to the conditions (B_{10}), (B_{11}) and (B_{12}). Where $(B_7)'$, $(B_8)'$, $(B_9)'$, (B_{10}), (B_{11}) and (B_{12}) are as follows:

$(B_7)'$ $x \leq y,\ y_t \in f \longrightarrow x_t \in \vee q f.$
$(B_8)'$ $x_t, y_r \in f \longrightarrow (xy)_{\min\{t,r\}} \in \vee q f.$
$(B_9)'$ $x_t, z_r \in f \longrightarrow (xyz)_{\min\{t,r\}} \in \vee q f.$
(B_{10}) $(\forall x, y \in S)(x \leq y \longrightarrow f(x) \geq min\{f(y), 0.5\}).$
(B_{11}) $(\forall x, y \in S)(f(xy) \geq min\{f(x), f(y), 0.5\}).$
(B_{12}) $(\forall x, y, z \in S)(f(xyz) \geq min\{f(x), f(z), 0.5\}).$

Proof. The proof is similar to the proof of Proposition 5.2.1. □

5.4.2 Remark

A fuzzy subset f of an ordered semigroup S is an $(\in, \in \vee q)$-fuzzy bi-ideal of S if and only if it satisfies conditions (B_{10}), (B_{11}) and (B_{12}) of the above Lemma.

5.4.3 Remark

By the above Remark every fuzzy bi-ideal of an ordered semigroup S is an $(\in, \in \vee q)$-fuzzy bi-ideal of S. Howevere, the converse is not true, in general.

5.4.4 Example

Consider the ordered semigroup given in Example 5.3.4, and define a fuzzy subset $f : S \longrightarrow [0, 1]$ by:

$$f(a) = 0.8, \quad f(b) = 0.7, \quad f(e) = 0.6, \quad f(d) = 0.5, \quad f(c) = 0.3.$$

Clearly f is an $(\in, \in \vee q)$-fuzzy bi-ideal of S. But f is not an (α, β)-fuzzy bi-ideal of S as shown in example 5.3.4.

5.4.5 Proposition

Let $(S, \cdot, \leq)$ be an ordered semigroup and $\emptyset \neq B \subseteq S$. Then B is a bi-ideal of S if and only if the characteristic function f_B of B is an $(\in, \in \vee q)$-fuzzy bi-ideal of S.

In the following Theorem we give a condition for an $(\in, \in \vee q)$-fuzzy bi-ideal to be an $(\in, \in)$-fuzzy bi-ideal of S.

5.4.6 Theorem

Let f be an $(\in, \in \vee q)$-fuzzy bi-ideal of S such that $f(x) < 0.5$ for all $x \in S$. Then f is an $(\in, \in)$-fuzzy bi-ideal of S.

Proof. The proof is similar to the proof of Theorem 5.2.9. □

5.4.7 Theorem

Let S be an ordered semigroup and f a fuzzy subset of S. Then f is an $(\in, \in \vee q)$-fuzzy bi-ideal of S if and only if $U(f;t)(\neq \emptyset)$ is a bi-ideal of S for all $t \in (0, 0.5]$.

Proof. The proof is similar to the proof of Theorem 5.2.5. □

Now we provide another characterization of $(\in, \in \vee \mathrm{q})$-fuzzy bi-ideals by using the set $[f]_t$.

5.4.8 Theorem

Let S be an ordered semigroup and f a fuzzy subset of S. Then f is an $(\in, \in \vee q)$-fuzzy bi-ideal of S if and only if $[f]_t$ is a bi-ideal of S for all $t \in (0, 1]$.

Proof. The proof is similar to the proof of Theorem 5.2.10. □

$U(f;t)$ and $[f]_t$ are bi-ideals of S for all $t \in (0, 0.5]$, but $Q(f;t)$ is not a bi-ideal of S for all $t \in (0, 1]$, in general. As shown in the following Example.

5.4.9 Example

Consider the ordered semigroup as given in Example 5.3.4. Define a fuzzy subset f by

$$f(a) = 0.8, \quad f(b) = 0.7, \quad f(c) = 0.6, \quad f(e) = 0.5, \quad f(d) = 0.4.$$

Then $Q(f;t) = \{a, b, c, e\}$ for $0.4 < t \leq 0.5$. Since $a_{0.36} q f$ and $b_{0.32} q f$ but $(cb)_{\min\{0.36, 0.32\}} = d_{0.32} \overline{q} f$. Hence $Q(f;t)$ is not a bi-ideal of S for all $t \in (0.4, 0.5]$.

5.4.10 Proposition

If f and g are $(\in, \in \vee q)$-fuzzy bi-ideals of S then $f \cap_{0.5} g$ is an $(\in, \in \vee q)$-fuzzy bi-ideal of S.

Proof. Straightforward. □

5.4.11 Lemma

Let S be an ordered semigroup. Then every one-sided $(\in, \in \vee q)$-fuzzy ideal is an $(\in, \in \vee q)$-fuzzy bi-ideal of S.

Proof. Let f be an $(\in, \in \vee q)$-fuzzy left ideal of S and $a, b \in S$. Then

$$f(ab) \geq \min\{f(b), 0.5\} \geq \min\{f(a), f(b), 0.5\}.$$

Hence f is an $(\in, \in \vee q)$-fuzzy subsemigroup of S.
Let $a, b, c \in S$. Then

$$f(abc) = f((ab)c) \geq \min\{f(c), 0.5\} \geq \min\{f(a), f(c), 0.5\}.$$

Let $a, b \in S$ be such that $a \leq b$. Then $f(a) \geq \min\{f(b), 0.5\}$, since f is an $(\in, \in \vee q)$-fuzzy left ideal of S. Hence f is an $(\in, \in \vee q)$-fuzzy bi-ideal of S. Similarly we can prove that if f is an $(\in, \in \vee q)$-fuzzy right ideal of S then f is an $(\in, \in \vee q)$-fuzzy bi-ideal of S. □

5.4.12 Definition

An $(\in, \in \vee q)$-fuzzy bi-ideal of S is called idempotent if $f \circ_{0.5} f = f$.

5.4.13 Proposition

Let S be an ordered semigroup and f an $(\in, \in \vee q)$-fuzzy bi-ideal of S. Then $f \circ_{0.5} f \leq f$.

Proof. Let f be an $(\in, \in \vee q)$-fuzzy bi-ideal of S. Then for each $a \in S$, we have

$$(f \circ_{0.5} f)(a) \leq f(a).$$

In fact: If $A_a = \emptyset$, then $(f \circ_{0.5} f)(a) = 0 \leq f(a)$. If $A_a \neq \emptyset$ then

$$\begin{aligned}(f \circ_{0.5} f)(a) &= \bigvee_{(y,z) \in A_a} \min\{f(y), f(z), 0.5\} \\ &\leq \bigvee_{(y,z) \in A_a} f(yz) \leq \bigvee_{(y,z) \in A_a} f(a) = f(a).\end{aligned}$$

□

5.4.14 Lemma

Let S be an ordered semigroup and f, g are fuzzy subsets of S. Then $f \circ_{0.5} g \subseteq 1 \circ_{0.5} g$ (resp. $f \circ_{0.5} g \subseteq f \circ_{0.5} 1$).

Proof. Straightforward. □

5.4.15 Proposition

Let S be an ordered semigroup and f an $(\in, \in \vee q)$-fuzzy bi-ideal of S. Then $(f \circ_{0.5} 1 \circ_{0.5} f)(a) \subseteq \min\{f(a), 0.5\}$ for all $a \in S$.

Proof. Let $a \in S$. If $A_a = \emptyset$. Then $(f \circ_{0.5} 1 \circ_{0.5} f)(a) = 0 \leq \min\{f(a), 0.5\}$. If $A_a \neq \emptyset$, then

$$\begin{aligned}(f \circ_{0.5} 1 \circ_{0.5} f)(a) &= \bigvee_{(y,z)\in A_a} \min\{f(y), (1 \circ_{0.5} f)(z), 0.5\} \\ &= \bigvee_{(y,z)\in A_a} \min\{f(y), \bigvee_{(t,r)\in A_z} \min\{1(t), f(r), 0.5\}, 0.5\} \\ &= \bigvee_{(y,z)\in A_a} \bigvee_{(t,r)\in A_z} \min\{f(y), 1, f(r), 0.5\} \\ &= \bigvee_{(y,z)\in A_a} \bigvee_{(t,r)\in A_z} \min\{f(y), f(r), 0.5\}.\end{aligned}$$

Since $a \leq yz \leq y(tr)$ and f is an $(\in, \in \vee$q)-fuzzy bi-ideal of S, we have

$$f(a) \geq \min\{f(y), f(r), 0.5\}.$$

Thus

$$\bigvee_{(y,z)\in A_a} \bigvee_{(t,r)\in A_z} \min\{f(y), f(r), 0.5\} \leq \bigvee_{(y,z)\in A_a} \bigvee_{(t,r)\in A_z} \min\{f(a), 0.5\} = \min\{f(a), 0.5\},$$

consequently, $(f \circ_{0.5} 1 \circ_{0.5} f)(a) \leq \min\{f(a), 0.5\}$. □

5.4.16 Theorem

An ordered semigroup S is regular if and only if for every $(\in, \in \vee q)$-fuzzy bi-ideal f of S we have

$$(f \circ_{0.5} 1 \circ_{0.5} f)(a) = f(a) \text{ for all } a \in S \qquad (5.1)$$

Proof. $(\longrightarrow)$ Let S be a regular ordered semigroup and let $a \in S$. Since S is regular there exists $x \in S$ such that $a \leq axa \leq ax(axa) = a(xaxa)$. Then $(a, xaxa) \in A_a$, and we have

$$\begin{aligned}(f \circ_{0.5} 1 \circ_{0.5} f)(a) &= \bigvee_{(y,z)\in A_a} \min\{f(y), (1 \circ_{0.5} f)(z), 0.5\} \\ &\geq \min\{f(a), (1 \circ_{0.5} f)(xaxa), 0.5\} \\ &= \min\{f(a), \bigvee_{(t,r)\in A_{xaxa}} \min\{1(t), f(r), 0.5\}, 0.5\} \\ &\geq \min\{f(a), \min\{1(xax), f(a), 0.5\}, 0.5\} \\ &= \min\{f(a), \min\{1, f(a), 0.5\}, 0.5\} \\ &= \min\{f(a), 0.5\}.\end{aligned}$$

Hence $f(a) \leq (f \circ_{0.5} 1 \circ_{0.5} f)(a)$. On the other hand, by Proposition 5.4.16, we have $(f \circ_{0.5} 1 \circ_{0.5} f)(a) \leq \min\{f(a), 0.5\}$. Therefore $(f \circ_{0.5} 1 \circ_{0.5} f)(a) = \min\{f(a), 0.5\}$.

($\longleftarrow$) Let f be an $(\in, \in \vee q)$-fuzzy bi-ideal of S such that expression (5.1), is satisfied. To prove that S is regular, we will prove that $(BSB] = B$ for all bi-ideals B of S. Let $b \in B$, the by Remark 5.4.3, f_B is an $(\in, \in \vee q)$-fuzzy bi-ideal of S. By hypothesis

$$(f_B \circ_{0.5} 1 \circ_{0.5} f_B)(b) = \min\{f_B(b), 0.5\}$$

Since $b \in B$, then $f_B(b) = 1$ and we have $(f_B \circ_{0.5} 1 \circ_{0.5} f_B)(b) = 0.5$. By Proposition 5.2.20, we have $(f_B \circ_{0.5} 1 \circ_{0.5} f_B)(b) = \min\{f_{(BSB]}(b), 0.5\}$ and hence $f_{(BSB]}(b) = 0.5 \longrightarrow b \in (BSB]$. Thus $B \subseteq (BSB]$. Since B is a bi-ideal of S, we $(BSB] \subseteq (B] = B$. Therefore $(BSB] = B$. □

5.4.17 Lemma

Let f and g be $(\in, \in \vee q)$-fuzzy bi-ideals of S. Then $f \circ_{0.5} g$ is also an $(\in, \in \vee q)$-fuzzy bi-ideal of S.

Proof. Let f and g be $(\in, \in \vee q)$-fuzzy bi-ideals of S. Let $a \in S$. If $A_a = \emptyset$ then

$$((f \circ_{0.5} g) \circ_{0.5} (f \circ_{0.5} g))(a) = 0 \leq (f \circ_{0.5} g)(a).$$

If $A_a \neq \emptyset$ then

$$\begin{aligned}
((f \circ_{0.5} g) \circ_{0.5} (f \circ_{0.5} g))(a) &= \bigvee_{(y,z) \in A_a} \{(f \circ_{0.5} g)(y) \wedge (f \circ_{0.5} g)(z) \wedge 0.5\} \\
&= \bigvee_{(y,z) \in A_a} \left[\begin{array}{c} \bigvee_{(p_1,q_1) \in A_a} \{f(p_1) \wedge g(q_1) \wedge 0.5\} \\ \wedge \bigvee_{(p_2,q_2) \in A_a} \{f(p_2) \wedge g(q_2) \wedge 0.5\} \end{array} \right] \\
&= \bigvee_{(y,z) \in A_a} \bigvee_{(p_1,q_1) \in A_a} \bigvee_{(p_2,q_2) \in A_a} \left[\begin{array}{c} \{f(p_1) \wedge g(q_1) \wedge 0.5\} \\ \wedge \{f(p_2) \wedge g(q_2) \wedge 0.5\} \end{array} \right] \\
&= \bigvee_{(y,z) \in A_a} \bigvee_{(p_1,q_1) \in A_a} \bigvee_{(p_2,q_2) \in A_a} \left[\begin{array}{c} \{f(p_1) \wedge g(p_2) \\ \wedge g(q_1) \wedge g(q_2) \wedge 0.5\} \end{array} \right] \\
&\leq \bigvee_{(y,z) \in A_a} \bigvee_{(p_1,q_1) \in A_a} \bigvee_{(p_2,q_2) \in A_a} \left[\begin{array}{c} \{f(p_1) \wedge f(p_2) \\ \wedge g(q_2) \wedge 0.5\} \end{array} \right].
\end{aligned}$$

Since $a \leq yz$, $y \leq p_1 q_1$ and $z \leq p_2 q_2$. Then $a \leq (p_1 q_1)(p_2 q_2) = p_1(q_1 p_2 q_2)$ and we have $(p_1, q_1p_2q_2) \in A_a$. Then

$$\bigvee_{(y,z)\in A_a} \bigvee_{(p_1,q_1)\in A_a} \bigvee_{(p_2,q_2)\in A_a} [\{f(p_1) \wedge f(p_2) \wedge g(q_2) \wedge 0.5\}]$$
$$\leq \bigvee_{(p_1,q_1p_2q_2)\in A_a} [\{f(p_1) \wedge f(p_2) \wedge g(q_2) \wedge 0.5\}].$$

Since f is an $(\in, \in \vee \mathrm{q})$-fuzzy bi-ideal of S we have

$$f(p_1q_1p_2) \geq \{f(p_1) \wedge f(p_2) \wedge 0.5\}.$$

Then

$$\bigvee_{(p_1,q_1p_2q_2)\in A_a} [\{f(p_1) \wedge f(p_2) \wedge g(q_2) \wedge 0.5\}]$$
$$\leq \bigvee_{(p_1,q_1p_2q_2)\in A_a} [\{f(p_1q_1p_2) \wedge g(q_2) \wedge 0.5\}]$$
$$\leq \bigvee_{(p,q)\in A_a} [\{f(p) \wedge g(q) \wedge 0.5\}] = (f \circ_{0.5} g)(a).$$

Therefore $((f \circ_{0.5} g) \circ_{0.5} (f \circ_{0.5} g))(a) \leq (f \circ_{0.5} g)(a)$, and $f \circ_{0.5} g$ is an $(\in, \in \vee \mathrm{q})$-fuzzy subsemigroup of S. Let $a, b, c \in S$. Then

$$\begin{aligned}
(f \circ_{0.5} g)(a) \wedge (f \circ_{0.5} g)(c) &= \left[\bigvee_{(p,q)\in A_a} \{f(p) \wedge g(q) \wedge 0.5\}\right] \\
&\quad \wedge \left[\bigvee_{(r,s)\in A_c} \{f(r) \wedge g(s) \wedge 0.5\}\right] \\
&= \bigvee_{(p,q)\in A_a} \bigvee_{(r,s)\in A_c} \left[\begin{array}{c} \{f(p) \wedge g(q) \wedge 0.5\} \\ \wedge \{f(r) \wedge g(s) \wedge 0.5\} \end{array}\right] \\
&= \bigvee_{(p,q)\in A_a} \bigvee_{(r,s)\in A_c} \left[\begin{array}{c} \{f(p) \wedge f(r) \wedge g(q)\} \\ \wedge g(s) \wedge 0.5 \end{array}\right] \\
&\leq \bigvee_{(p,q)\in A_a} \bigvee_{(r,s)\in A_c} [\{f(p) \wedge f(r) \wedge g(s) \wedge 0.5\}].
\end{aligned}$$

Since $a \leq pq$, and $c \leq rs$. Then $abc \leq (pq)b(rs) = (p(qb)r)s$ and we have $(p(qb)r, s) \in A_{abc}$. Thus

$$\bigvee_{(p,q)\in A_a} \bigvee_{(r,s)\in A_c} [\{f(p) \wedge f(r) \wedge g(s) \wedge 0.5\}]$$
$$\leq \bigvee_{(p(qb)r,s)\in A_{abc}} [\{f(p) \wedge f(r) \wedge g(s) \wedge 0.5\}].$$

Since f is an $(\in, \in \vee \mathrm{q})$-fuzzy bi-ideal of S, we have

$$f(p(qb)r) \geq \{f(p) \wedge f(r) \wedge 0.5\}.$$

Hence

$$\begin{aligned} & \bigvee_{(p(qb)r,s)\in A_{abc}} [\{f(p) \wedge f(r) \wedge g(s) \wedge 0.5\}] \\ \leq & \bigvee_{(p(qb)r,s)\in A_{abc}} [\{f(p(qb)r) \wedge g(s) \wedge 0.5\}] \\ \leq & \bigvee_{(x,y)\in A_{abc}} [\{f(x) \wedge g(y) \wedge 0.5\}] = (f \circ_{0.5} g)(abc). \end{aligned}$$

Thus $(f \circ_{0.5} g)(abc) \geq (f \circ_{0.5} g)(a) \wedge (f \circ_{0.5} g)(c)$.

Let $x, y \in S$ be such that $x \leq y$. If $(p,q) \in A_y$ then $y \leq pq$ and so $x \leq pq \longrightarrow (p,q) \in A_x \longrightarrow A_y \subseteq A_x$. If $A_x = \emptyset$, then $A_y = \emptyset$ and we have $(f \circ_{0.5} g)(x) = 0 = (f \circ_{0.5} g)(y)$. If $A_x \neq \emptyset$, then $A_y \neq \emptyset$ and we have

$$\begin{aligned} (f \circ_{0.5} g)(y) &= \bigvee_{(p,q)\in A_y} \{f(p) \wedge g(q) \wedge 0.5\} \\ &\leq \bigvee_{(c,d)\in A_x} \{f(c) \wedge g(d) \wedge 0.5\} \\ &= (f \circ_{0.5} g)(x). \end{aligned}$$

Therefore $(f \circ_{0.5} g)(x) \geq (f \circ_{0.5} g)(y)$, consequently $f \circ_{0.5} g$ is an $(\in, \in \vee \mathrm{q})$-fuzzy bi-ideal of S. □

5.4.18 Theorem

Let S be an ordered semigroup. The following are equivalent:

(i) S is both regular and intra-regular.

(ii) $f \circ_{0.5} f = f$ for every $(\in, \in \vee q)$-fuzzy bi-ideal f of S.

(iii) $f \cap_{0.5} f = f \circ_{0.5} g \cap_{0.5} g \circ_{0.5} f$ for all $(\in, \in \vee q)$-fuzzy bi-ideals f and g of S.

Proof. (i)$\longrightarrow$(ii). Let f be an $(\in, \in \vee \mathrm{q})$-fuzzy bi-ideal of S and $a \in S$. Since S is regular and intra-regular there exist $x, y, z \in S$ such that $a \leq axa \leq axaxa$, and $a \leq ya^2z$. Then $a \leq axaxa \leq ax(ya^2z)xa = (axya)(azxa)$ and hence $(axya, azxa) \in A_a$. Then

$$\begin{aligned} (f \circ_{0.5} f)(a) &= \bigvee_{(p,q)\in A_a} \{f(p) \wedge f(q) \wedge 0.5\} \\ &\geq \{f(axya) \wedge f(azxa) \wedge 0.5\} \\ &\geq \left\{ \begin{array}{c} \{f(a) \wedge f(a) \wedge 0.5\} \\ \wedge\{f(a) \wedge f(a) \wedge 0.5\} \wedge 0.5 \end{array} \right\} \\ &= \{f(a) \wedge 0.5\} = f(a). \end{aligned}$$

On the other hand, by Proposition 5.4.14, we have $(f \circ_{0.5} f)(a) \leq f(a)$. Thus $f \circ_{0.5} f = f$.

(ii)$\longrightarrow$(iii). Let f and g be $(\in, \in \vee q)$-fuzzy bi-ideals of S. Then $f \sqcap_{0.5} g$ is an $(\in, \in \vee q)$-fuzzy bi-ideal of S. By (ii)

$$\begin{aligned} f \sqcap_{0.5} g &= (f \sqcap_{0.5} g) \circ_{0.5} (f \sqcap_{0.5} g) \\ &\subseteq f \circ_{0.5} g. \end{aligned}$$

Similarly, $f \sqcap_{0.5} g \preceq g \circ_{0.5} f$. Thus

$$f \sqcap_{0.5} g \preceq f \circ_{0.5} g \sqcap_{0.5} g \circ_{0.5} f.$$

On the other hand, $f \circ_{0.5} g$ and $g \circ_{0.5} f$ are $(\in, \in \vee q)$-fuzzy bi-ideals of S by Lemma 5.4.18. Hence $f \circ_{0.5} g \sqcap_{0.5} g \circ_{0.5} f$ is an $(\in, \in \vee q)$-fuzzy bi-ideal of S. By (ii)

$$\begin{aligned} & f \circ_{0.5} g \sqcap_{0.5} g \circ_{0.5} f \\ = \ & (f \circ_{0.5} g \sqcap_{0.5} g \circ_{0.5} f) \circ_{0.5} (f \circ_{0.5} g \sqcap_{0.5} g \circ_{0.5} f) \\ \subseteq \ & (f \circ_{0.5} g) \circ_{0.5} (g \circ_{0.5} f) = f \circ_{0.5} (g \circ_{0.5} g) \circ_{0.5} f \\ = \ & f \circ_{0.5} g \circ_{0.5} f \text{ (as } g \circ_{0.5} g = g \text{ by (i) above)} \\ \subseteq \ & f \circ_{0.5} 1 \circ_{0.5} f \\ = \ & f \text{ (as } f \circ_{0.5} 1 \circ_{0.5} f = f \text{ by Theorem 5.4.17).} \end{aligned}$$

By a similar way we can prove that $f \circ_{0.5} g \sqcap_{0.5} g \circ_{0.5} f \subseteq g$. Consequently,

$$f \circ_{0.5} g \sqcap_{0.5} g \circ_{0.5} f \subseteq f \sqcap_{0.5} g.$$

Therefore $Af \sqcap_{0.5} g = f \circ_{0.5} g \sqcap_{0.5} g \circ_{0.5} f$.

(iii)$\longrightarrow$(i). To prove that S is regular we prove that $P \cap Q = (PQ] \cap (QP]$ for every bi-ideal P, and Q of S. Let $b \in P \cap Q$. By Proposition 5.4.5, f_P and f_Q are $(\in, \in \vee q)$-fuzzy bi-ideals of S. By (iii) $(f_P \sqcap_{0.5} f_Q)(b) = (f_P \circ_{0.5} f_Q \sqcap_{0.5} f_Q \circ_{0.5} f_P)(b)$. Since $b \in P$ and $b \in Q$, then $f_P(b) = 1$ and $f_Q(b) = 1$. Then $(f_P \sqcap_{0.5} f_Q)(b) = \min\{f_P(b), f_Q(b), 0.5\} = 0.5$. Hence $(f_P \circ_{0.5} f_Q \sqcap_{0.5} f_Q \circ_{0.5} f_P)(b) = 0.5$. By Lemma 5.2.20, $(f_P \circ_{0.5} f_Q \sqcap_{0.5} f_Q \circ_{0.5} f_P)(b) = \min\{f_{(PQ]\cap(QP]}(a), 0.5\}$ and hence $f_{(PQ]\cap(QP]}(b) = 0.5 \longrightarrow b \in (PQ] \cap (QP]$. On the other hand, if $b \in (PQ] \cap (QP]$, then

$$\begin{aligned} 0.5 &= f_{(PQ]\sqcap_{0.5}(QP]}(b) \\ &= (f_{(PQ]} \sqcap_{0.5} f_{(QP]})(b) \\ &= (f_P \circ_{0.5} f_Q \sqcap_{0.5} f_Q \circ_{0.5} f_P)(b) \\ &= (f_P \sqcap_{0.5} f_Q)(b) \text{ (by (iii))} \\ &= \min\{f_{P\cap Q}(b), 0.5\} \end{aligned}$$

hence $b \in P \cap Q$. Therefore $P \cap Q = (PQ] \cap (QP]$, consequently, S is both regular and intra-regular. □

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