Rotational and thermal effects on Rayleigh waves speed in anisotropic materials

By

Abdul Rehman

DEPARTMENT OF MATHEMATICS

QUAID-I-AZAM UNIVERSITY

ISLAMABAD, PAKISTAN

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Certificate

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Dedication

I dedicate this humble work to my esteemed teacher Dr. Faiz Ahmad who has ever been a source of knowledge, guidance, inspiration and habitual vision of greatness.

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All the praises and thanks are to Almighty **Allah,** the One and the Only Lord for the entire universe, its Creator, Owner, Organizer, Provider, Master, Planer, Sustainer, Cherisher, and Giver of security. Innumerous and uncountable blessings and peace of Allah be upon His Holy Prophet **Muhammad**, who is the greatest reformer of the world and is for ever a torch of guidance for humanity.

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Abstract

In this thesis Rayleigh waves in semi-infinite linear homogeneous transversely isotropic and orthotropic materials are studied.

We have derived secular equation of the Rayleigh waves in transversely isotropic material together with a necessary condition under which the equation is valid. We have shown that the corresponding secular equation of the Rayleigh waves in orthotropic material along with the necessary condition validating the equation can be deduced from the derived equation by replacing the elastic constant c_{44} of transversely isotropic material with the elastic constant c_{55} of the orthotropic material. The rotational effects of the said materials on the Rayleigh waves and consequently on the secular equations as well as the necessary conditions which confirm the validity of these equations is observed.

The Rayleigh waves in rotating incompressible transversely isotropic and orthotropic materials are studied. The corresponding secular equations together with the necessary conditions validating these equations are also derived.

Thermal effects on the Rayleigh waves in transversely isotropic material are taken into consideration. The relative secular equation along with the necessary condition confirming its validity is also derived.

We have calculated the Rayleigh wave speed in different specimens of above mentioned materials in all the cases discussed above and very notable results are obtained.

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Introduction

Rayleigh wave, named after John William Strutt, Lord Rayleigh who mathematically predicted the existence of this kind of wave in 1885, is a type of surface acoustic wave that travels on solids. They are produced on the earth by earthquakes, or by other sources of seismic energy such as explosion or even a sledgehammer impact. They can also be produced in materials by many other mechanisms.

A Rayleigh wave rolls along the ground like a wave rolls across a lake or an ocean. As it rolls, it moves the ground up and down, and side-to-side in the same direction the wave is moving. Most of the shaking felt from an earthquake is due to Rayleigh wave. Since the Rayleigh waves are the surface waves, the amplitude of such waves decreases exponentially with depth normal to the surface.

In terms of particles motion the Rayleigh waves may be described as the types of surface waves in which particles move in an elliptical path within the vertical plane containing the direction of wave propagation. At the top of the elliptical path, particles travel opposite to the direction of propagation, and at the bottom of the path they travel in the direction of propagation. The particles at the surface trace out a counter-clockwise ellipse, while particles at the depth of more than 1/5th of a wavelength trace out clockwise ellipse. As the depth into the solid increases the "width" of the elliptical path decreases, the major axis of the ellipse remains perpendicular to the surface.

Rayleigh waves are generated by the interaction of P- and S-waves at the surface of the earth, and travel with a velocity that is lower than the P-, S-, and Love wave velocities. The Rayleigh waves emanating outward from the epicenter of an earthquake travel along the surface of the earth at about 10 times the speed of sound, in air, that is ~3 km/s. Due to their higher speed, the P- and S-waves generated by an earthquake arrive before the surface waves. However, the particle motion of surface waves is larger than that of body waves, so the surface waves tend to cause more damage.

Rayleigh waves have manifold applications in radar and communications systems and have led to practical importance. In ultrasonic frequency range they are used in non-destructive testing applications for detecting cracks and other imperfections in materials. Low frequency Rayleigh waves generated during earthquakes are used in seismology to characterize the earth's interior. In intermediate ranges, they are used in geophysics and geotechnical engineering for the characterization of soil deposits. They can be detected by many mammals, birds, insects and spiders and some animals seem to use them to communicate. Some biologists theorize that elephants may use vocalizations to generate Rayleigh waves. After the 2004 Indian Ocean Earthquake, some people have speculated that Rayleigh waves served as a warning to animals to seek higher ground, allowing them to escape the more slowly-traveling tsunami.

The Rayleigh waves were first studied by Rayleigh [1] for compressible isotropic elastic material. Some other researchers promoted his work in the same material. For instance, first explicit Rayleigh wave speed formula was obtained by Rahman and Barber [2]. Nkemzi [3] expressed the Rayleigh wave speed formula as a continuous function of the material parameter $\varepsilon = \mu / \lambda + 2\mu$, $\lambda \, and \, \mu$ being Lame constants. Malischewsky [4] obtained a formula for a Rayleigh wave speed by using Cardan's formula from the theory of cubic equations together with the trigonometric formulas for the roots of cubic equations and the computer software Mathematica.

After that a number of researchers extended the surface (Rayleigh) wave analysis to compressible anisotropic elastic materials. Stonely [5] studied Rayleigh wave propagation in an elastic medium with orthorhombic symmetry. Chadwick and Smith [6] determined the foundations of the theory of surface waves in anisotropic elastic materials. Royer and Dieulesaint [7] discussed the Rayleigh wave velocity and displacement in orthorhombic, tetragonal, hexagonal and cubic crystals. Mozhaeve [8] gave some new ideas in the theory of surface acoustic waves in anisotropic media. Destrade [9] found the explicit secular equation for the surface acoustic waves in monoclinic crystals. Ting [10, 11] explained the explicit secular equations for surface waves in an elastic material of general anisotropy and in monoclinic materials with symmetry plane at $x_1 = 0,\ x_2 = 0,\ or\ x_3 = 0$ respectively. Pham and Ogden [12] obtained different Rayleigh wave speed formulas by using different forms of the cubic secular equations. Each formula is expressed as a continuous function of three dimensionless material parameters, which are ratios of certain elastic constants.

Recently, some other researchers worked on elastic and Rayleigh waves propagation in incompressible anisotropic elastic solids. For example, Nair and Sotiropoulos [13, 14] discussed elastic waves in orthotropic incompressible mediums, and reflection from an interface, and interfacial waves in incompressible monoclinic materials with an interlayer respectively. Destrade [15] derived secular equation for surface acoustic waves propagating on an incompressible orthotropic half-space in a direct manner, using the method of first integrals. Ogden and Pham [16] obtained the secular equation for the Rayleigh wave speed in an incompressible orthotropic elastic solid in a form that did not admit spurious solutions of Destrade [15] and showed the inequalities on the material constants that ensure positive definiteness of the strain-energy function guaranteed existence and uniqueness of the Rayleigh wave speed. Finally, Khan and Ahmad [17] observed rotating effects on elastic wave propagation in incompressible transversely isotropic elastic medium and showed that the incompressibility requires that only transverse waves can propagate and also showed that the combined effect of incompressibility and rotation reduce by a great deal the types of waves which can exist in such a medium.

Thermo-elastic waves in an isotropic material have been studied by Deresiewicz [18], Lessen [19], Chadwick and Sneddon [20] and Chadwick [21]. In transversely isotropic medium these waves have been studied by Chadwick and Seet [22]. Khan and Ahmad [23] studied thermal effect on elastic body wave propagation in an incompressible transversely isotropic material and showed that attenuation constant and coupling constant attain their maxima in different directions. They also studied the directional dependence of energy flux vector. Thermo-elastic Rayleigh waves were studied by Chadwick [21] in isotropic material.

In chapter 2 we have derived the secular equation for the Rayleigh wave in transversely isotropic material together with a necessary condition which confirms the validity of the equation. We have observed that the secular equation for Rayleigh wave in orthotropic material together with a necessary condition which confirms the validity of the equation can be deduced from the said derived secular equation by replacing the elastic constant

c_{44} of transversely isotropic material with the elastic constant c_{55} of orthotropic material. More interest has been created by calculating the Rayleigh wave speed in different specimens of both kinds of materials. The work has been published in "Engineering Transactions 54, 4, 323-328, (2006)."

In chapter 3 we have observed the rotational effects of both kinds of materials on the Rayleigh waves and consequently on the corresponding secular equations together with the necessary conditions under which these secular equations are valid. It has been observed that the frequency of rotation can be increased up to a certain limit only. The propagation of Rayleigh waves is valid only if the frequency of rotation is less than the numerical value of the elastic constant c_{11}. The Rayleigh wave speed is calculated in different materials of both kinds under rotation. The effort has been published in "Punjab University Journal of Mathematics 39, 29-33, (2007)"

In chapter 4 rotational effects on Rayleigh wave speed in incompressible transversely isotropic and orthotropic materials are studied. We have derived secular equations under rotation together with necessary conditions which affirm the validity of these equations. We observed that rotation has no effect on the necessary conditions of propagation. The Rayleigh wave speed is calculated in the rotating specimens of both kinds and very interesting results are obtained. It is also observed that only one Rayleigh wave can propagate in both the rotating and non-rotating cases. But in case of elastic body waves the matter is different. For example, two elastic body waves propagate in non-rotating and three elastic body waves in rotating transversely isotropic material [17]. The work has been accepted to publish in "World Applied Science Journal 3(1), (2008)."

In chapter 5 thermal effects on Rayleigh speed in transversely isotropic medium are taken into consideration. We have derived the secular equation for the Rayleigh waves and a necessary condition which confirms the validity of the secular equation. We have observed that the thermal effects modify the secular equation keeping the necessary condition for wave propagation unchanged. The Rayleigh wave speed is calculated in different models of transversely isotropic materials. We have observed that two Rayleigh waves propagate in the material under the thermal effects. One wave propagates with the same speed as that of the wave that propagates without thermal effects but the other wave propagates with some higher speed. The attempt has been again accepted to publish in "World Applied Science Journal 3(1), (2008)."

Chapter 1

Preliminaries

In this chapter the most fundamentals of anisotropy are introduced. The source material is mainly provided from [21, 24].

1.1 The most fundamentals of anisotropy

Consider an anisotropic elastic continuous solid subjected to small deformation. Assume that the solid is free of stress before deformation so that the stress-strain relationship is linear, i.e., it follows the generalized Hooke's Law. If the solid is homogenous, the coefficients in the stress-strain relationship are constant, but if it is inhomogeneous they will vary because the elastic properties at different points in the solid are different; they will be function of coordinates.

We can use the various coordinate systems when studying the stresses and strains in a solid generated by external loading. But in this thesis we shall remain confined to the Cartesian coordinates, (x, y, z) only. The other coordinate systems are not within the scope of this work.

In Cartesian coordinate system, the stress tensor σ_{ij} and strain tensor $\in_{ij}$ can be expressed, respectively, as

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \text{ and } \begin{bmatrix} \in_{xx} & \in_{xy} & \in_{xz} \\ \in_{yx} & \in_{yy} & \in_{yz} \\ \in_{zx} & \in_{zy} & \in_{zz} \end{bmatrix}.$$

In tensor form, they are expressed, respectively, as

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \text{ and } \begin{bmatrix} \in_{11} & \in_{12} & \in_{13} \\ \in_{21} & \in_{22} & \in_{23} \\ \in_{31} & \in_{32} & \in_{33} \end{bmatrix}.$$

Both stress and strain are symmetric tensors of rank two, i.e., $\sigma_{ij} = \sigma_{ji}$ *and* $\in_{ij} = \in_{ji}$ and they follow the following transformation rules:

$$\sigma_{i'j'} = l_{i'i}\, l_{j'j}\ \sigma_{ij}, \tag{1.1}$$

$$\in_{i'j'} = l_{i'i}\, l_{j'j}\ \in_{ij}, \tag{1.2}$$

where σ_{ij} stands for stresses in Cartesian coordinates, (x, y, z), $\sigma_{i'j'}$ represents the stresses in new coordinate system after rotation, (x', y', z'), and $l_{i'i}$ are the direction cosines between two coordinate axes.

On any infinitesimal area in a solid with an external normal $\overline{n}$, if the projections of the stress in x-, y-, and z-directions in a Cartesian coordinate system are p_x, p_y and p_z, then

$$\begin{aligned} p_x &= \sigma_{xx}\cos(n,x) + \sigma_{xy}\cos(n,y) + \sigma_{xz}\cos(n,z), \\ p_y &= \sigma_{yx}\cos(n,x) + \sigma_{yy}\cos(n,y) + \sigma_{yz}\cos(n,z), \\ p_z &= \sigma_{zx}\cos(n,x) + \sigma_{zy}\cos(n,y) + \sigma_{zz}\cos(n,z). \end{aligned} \tag{1.3}$$

1.2 Basic equations

The basic equations of elasticity are geometric equations (strain-displacement relations), equations of motion and constitutive equations (stress-strain relations).

1.2.1 Geometric equations

In tensor form, the geometric equations in Cartesian coordinate system can be written concisely as

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad (i, j = 1,2,3), \tag{1.4}$$

where $$u_{i,j} = \frac{\partial u_i}{\partial x_j}.$$

Therefore, in Cartesian coordinates, Eq. (1.4) gives

$$\begin{aligned}
\epsilon_{xx} &= \frac{\partial u}{\partial x}, \quad \epsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial z}\right), \\
\epsilon_{yy} &= \frac{\partial v}{\partial y}, \quad \epsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \\
\epsilon_{zz} &= \frac{\partial w}{\partial z}, \quad \epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),
\end{aligned} \tag{1.5}$$

where u, v, w denote displacement components in Cartesian coordinates.

1.2.2 Equations of motion

The equations of motion in Cartesian coordinates can be written concisely in tensor form as

$$\sigma_{ij,j} + F_i = \rho \ddot{u}_i, \quad (i, j = 1, 2, 3), \tag{1.6}$$

where a dot indicates a partial differentiation with respect to time t and F_i denotes body force.

Therefore, in Cartesian coordinates, Eq. (1.6) can be expressed as

$$
\begin{aligned}
&\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}+\frac{\partial \sigma_{xz}}{\partial z}+F_x=\rho\frac{\partial^2 u}{\partial t^2},\\
&\frac{\partial \sigma_{yx}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \sigma_{yz}}{\partial z}+F_y=\rho\frac{\partial^2 v}{\partial t^2},\\
&\frac{\partial \sigma_{zx}}{\partial x}+\frac{\partial \sigma_{zy}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z}+F_z=\rho\frac{\partial^2 w}{\partial t^2}.
\end{aligned}
\tag{1.7}
$$

1.2.3 Constitutive equations

The constitutive equations may be divided into two parts.

Case 1 Constitutive equations for general anisotropic material

The Constitutive equations in linear elasticity are represented by generalized Hooke's law. If the state of vanishing strain corresponds to zero stress, then in Cartesian coordinates the generalized Hooke's law can be concisely written as

$$\sigma_{ij}=c_{ijkl}\in_{kl}, \tag{1.8}$$

where c_{ijkl} are components of a fourth rank tensor, representing the properties of a material, which generally vary from one point to another in the material. If c_{ijkl} don't change across a material, it is called a homogeneous material. In this thesis we shall consider homogeneous and elastic materials whose c_{ijkl} are independent of coordinates.

Since σ_{ij} and $\in_{ij}$ are symmetric, it can be proved very easily that

$$c_{ijkl} = c_{jikl} = c_{ijlk}.$$

Thus among the total 81 components of c_{ijkl}, the maximum number of independent ones is 36.

To avoid the double summation over k and l in Eq. (1.8), we introduce the following notations

$$\sigma_{11} = \sigma_1,\ \sigma_{22} = \sigma_2,\ \sigma_{33} = \sigma_3,\ \sigma_{23} = \sigma_4,\ \sigma_{31} = \sigma_5,\ \sigma_{12} = \sigma_6,$$
$$\in_{11} = \in_1,\ \in_{22} = \in_2,\ \in_{33} = \in_3,\ 2\in_{23} = \in_4,\ 2\in_{31} = \in_5,\ 2\in_{12} = \in_6,$$

where $\in_4, \in_5, \in_6$ are engineering strains.

Therefore, Eq. (1.8) can be rewritten as

$$\sigma_1 = c_{11} \in_1 + c_{12} \in_2 + c_{13} \in_3 + c_{14} \in_4 + c_{15} \in_5 + c_{16} \in_6,$$
$$\sigma_2 = c_{21} \in_1 + c_{22} \in_2 + c_{23} \in_3 + c_{24} \in_4 + c_{25} \in_5 + c_{26} \in_6,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\sigma_6 = c_{61} \in_1 + c_{62} \in_2 + c_{63} \in_3 + c_{64} \in_4 + c_{65} \in_5 + c_{66} \in_6,$$

or in a more concise form

$$\sigma_i = c_{ij} \in_j, \quad (i, j = 1, 2, \ldots, 6).$$

The corresponding matrix form is

$$\{\sigma\} = [c]\{\in\}, \tag{1.9}$$

where $\{\sigma\}$ and $\{\in\}$ are vectors of stress and engineering strain respectively.
In Eq. (1.9), $[c] = [c_{ij}]$ should be non-singular and reversible matrix, i.e., det $[c] \neq 0$. Hence Eq. (1.9) can also be written as

$$\{\in\} = [s]\{\sigma\}, \tag{1.10}$$

where $[s] = [s_{ij}]$ is the inverse of $[c]$, given in the above Eq (1.9), c_{ij} are called the elastic stiffnesses (or moduli) of a material, having the dimension of stress $(\frac{F}{L^2})$ because strains are dimensionless, and s_{ij} are called compliances of the material with the dimension of $\frac{L^2}{F}$.
If there exists a strain energy density function defined as

$$W = \frac{1}{2} c_{ij} \in_i \in_j, \tag{1.11}$$

then
$$\frac{\partial^2 w}{\partial \in_i \partial \in_j} = \frac{1}{2} c_{ij}, \quad \frac{\partial^2 w}{\partial \in_j \partial \in_i} = \frac{1}{2} c_{ji}.$$

Therefore, $c_{ij} = c_{ji}$, since the order of differentiation is immaterial. This indicates that the number of elastic stiffnesses c_{ij} is further reduced from 36 to 21. Similarly we have $s_{ij} = s_{ij}$.

Thus for a general anisotropic elastic material, there are 21 independent elastic stiffness constants or elastic compliance coefficients. Because the strain energy density W is always non-negative and becomes zero only when $\in_i = 0$, $(i = 1,2,...,6)$, it is clear that the stiffness matrix $[c]$ and its inverse, the compliance matrix $[s]$, are both positive definite.

In a different coordinate system (x', y', z'), the constitutive equations will have the same form as Eq. (1.9) or Eq. (1.10), i.e.,

$$\{\sigma'\} = [c']\{\in'\}, \tag{1.12}$$

or
$$\{\in'\} = [s']\{\sigma'\}. \tag{1.13}$$

Using Eqs. (1.1) and (1.2) we can transform Eq. (1.12) into linear relationship between $\{\sigma\}$ and $\{\in\}$. Then by comparing it with Eq. (1.9), we can easily obtain the transformation formula between $[c']$ and $[c]$.

Case 2 Constitutive equations for materials possessing certain symmetries

As mentioned above for a general anisotropic material, $[c]$ or $[s]$ has 21 independent elements and hence the application of the constitutive Eq. (1.9) or (1.10) will bring about tremendous difficulties in solving a problem. Fortunately, the equation can be much simplified when the elastic properties of a material posses certain symmetries. We will now introduce the simplified constitutive equations for various materials with special properties.

1-Plane of elastic symmetry

At any point in a solid, if there exists a plane about which the elastic properties are symmetrical, the number of independent elements in $[c]$ will be reduced to 13. The direction perpendicular to this plane of elastic symmetry is often called principal elastic direction or the principal direction of the material.

Consider a substance elastically symmetric with respect to the xoy coordinate plane. The symmetry is expressed by the statement that $[c]$ is invariant under the transformation $x' = x,\ y' = y,\ z' = -z$, thus from Eq. (1.12), we have

$$\{\sigma'\}=[c]\{\in'\}. \qquad (1.14)$$

For this transformation, we have

$$l_{1'1} = l_{2'2} = 1,\ l_{3'3} = -1,\ l_{1'2} = l_{1'3} = l_{2'3} = l_{2'1} = l_{3'1} = l_{3'2} = 0.$$

Substituting these direction cosines into Eqs. (1.1) and (1.2) we get

$$\sigma_{x'} = \sigma_x,\ \sigma_{y'} = \sigma_y,\ \sigma_{z'} = \sigma_z,\ \sigma_{y'z'} = -\sigma_{yz},\ \sigma_{z'x'} = -\sigma_{zx},\ \sigma_{x'y'} = \sigma_{xy},$$

$$\in_{x'} = \in_x,\ \in_{y'} = \in_y,\ \in_{z'} = \in_z,\ \in_{y'z'} = -\in_{yz},\ \in_{z'x'} = -\in_{zx},\ \in_{x'y'} = \in_{xy}.$$

Accordingly, Eqs. (1.14) and (1.9) give

$$c_{14} = c_{15} = c_{24} = c_{25} = c_{34} = c_{35} = c_{64} = c_{65} = 0. \tag{1.15}$$

Thus, the generalized Hooke's law of Eq. (1.9) can be written as

$$\begin{aligned}
\sigma_x &= c_{11} \in_x + c_{12} \in_y + c_{13} \in_z + 2c_{16} \in_{xy}, \\
\sigma_y &= c_{12} \in_x + c_{22} \in_y + c_{23} \in_z + 2c_{26} \in_{xy}, \\
\sigma_y &= c_{13} \in_x + c_{23} \in_y + c_{33} \in_z + 2c_{36} \in_{xy}, \\
\sigma_{yz} &= 2c_{44} \in_{yz} + 2c_{45} \in_{zx}, \\
\sigma_{zx} &= 2c_{45} \in_{yz} + 2c_{55} \in_{zx}, \\
\sigma_{xy} &= c_{16} \in_x + c_{26} \in_y + c_{36} \in_z + 2c_{66} \in_{xy}.
\end{aligned} \tag{1.16}$$

2. Orthotropic material

If there exist three orthogonal planes of elastic symmetry at any point in a solid, then there are nine independent elements in [c], and the material is said to be orthotropic. Let the three coordinate planes of a Cartesian system, xoy, xoz and yoz, coincide with these planes of symmetry. Performing the similar transformation with respect to each co-ordinate plane, we will get, in addition to Eq. (1.15),

$$\begin{aligned} c_{14} = c_{16} = c_{24} = c_{26} = c_{34} = c_{36} = c_{54} = c_{56} = 0, \\ c_{15} = c_{16} = c_{25} = c_{26} = c_{35} = c_{36} = c_{45} = c_{46} = 0. \end{aligned} \tag{1.17}$$

Therefore, Eq.(1.16) simplifies to

$$\begin{aligned} &\sigma_x = c_{11} \in_x + c_{12} \in_y + c_{13} \in_z, \\ &\sigma_y = c_{12} \in_x + c_{22} \in_y + c_{23} \in_z, \\ &\sigma_z = c_{13} \in_x + c_{23} \in_y + c_{33} \in_z, \\ &\sigma_{yz} = 2c_{44} \in_{yz}, \ \sigma_{zx} = 2c_{55} \in_{zx}, \ \sigma_{xy} = 2c_{66} \in_{xy}. \end{aligned} \tag{1.18}$$

In this case, all the coordinate axes are in the principal directions of the material. Equation (1.18) shows that in orthotropic material, normal stresses depend only on normal strains; a shear stress on a plane depends only on shear strain on the same plane. This makes the stress and deformation analysis on orthotropic solid much easier than that of a general anisotropic material.

We can also express the strains in Eq. (1.18) in terms of the stresses using the compliance matrix $[s]=[c]^{-1}$ as

$$\begin{aligned} &\in_x = s_{11}\sigma_x + s_{12}\sigma_y + s_{13}\sigma_z, \\ &\in_y = s_{21}\sigma_x + s_{22}\sigma_y + s_{23}\sigma_z, \\ &\in_z = s_{31}\sigma_x + s_{32}\sigma_y + s_{33}\sigma_z, \\ &2\in_{yz} = s_{44}\sigma_{yz}, \ 2\in_{zx} = s_{55}\sigma_{zx}, \ 2\in_{xy} = s_{66}\sigma_{xy}. \end{aligned} \tag{1.19}$$

3- Transversely isotropic material

If at any point there is an axis of symmetry such that the elastic properties in any direction within a plane perpendicular to the axis are all the same, the total number of independent elements in $[c]$ will be reduced to five. The plane is called isotropic plane and the material is called a transversely isotropic material. The hexagonal crystals, like Cadmium and Zinc, are transversely isotropic.

If we take the coordinate plane xoy to coincide with the isotropic plane, the z-axis is the axis of symmetry. Taking a new Cartesian system such that $x' = y,\ y' = -x,\ z' = z$, then we have

$$l_{1'2} = l_{3'3} = 1,\ l_{2'1} = -1,\ l_{1'1} = l_{1'3} = l_{2'2} = l_{2'3} = l_{3'1} = l_{3'2} = 0.$$

Substituting into Eqs. (1.1) and (1.2) we find

$$\sigma_{x'} = \sigma_y,\ \sigma_{y'} = \sigma_x,\ \sigma_{z'} = \sigma_z, \sigma_{y'z'} = -\sigma_{zx},\ \sigma_{z'x'} = -\sigma_{yz},\ \sigma_{x'y'} = -\sigma_{xy},$$
$$\in_{x'} = \in_y,\ \in_{y'} = \in_x,\ \in_{z'} = \in_z,\ \in_{y'z'} = -\in_{zx},\ \in_{z'x'} = -\in_{yz},\ \in_{x'y'} = -\in_{xy}.$$

Using these relations as well as Eqs. (1.15) and (1.17), we get from Eqs. (1.14) and (1.18) that

$$c_{11} = c_{22},\ c_{13} = c_{23},\ c_{44} = c_{55}. \tag{1.20}$$

Now taking another transformation by rotating the original coordinate system 45 degree about the z-axis, we have the relations

$$l_{1'1} = l_{1'2} = l_{2'2} = 1/\sqrt{2},\ l_{2'1} = -1/\sqrt{2},\ l_{3'3} = 1,\ l_{1'3} = l_{2'3} = l_{3'1} = l_{3'2} = 0.$$

Substitution of these direction cosines into Eqs. (1.1) and (1.2) yields

$$\begin{aligned}
&\sigma_{x'} = (\sigma_x + \sigma_y + 2\sigma_{xy})/2,\quad \sigma_{y'} = (\sigma_x + \sigma_y - 2\sigma_{xy})/2,\quad \sigma_{z'} = \sigma_z,\\
&\sigma_{y'z'} = (\sigma_{yz} - \sigma_{zx})/\sqrt{2},\ \sigma_{z'x'} = (\sigma_{yz} + \sigma_{zx})/\sqrt{2},\ \sigma_{x'y'} = (\sigma_y - \sigma_x)/2,\\
&\in_{x'} = (\in_x + \in_y + 2\in_{xy}),\quad \in_{y'} = (\in_x + \in_y - 2\in_{xy}),\quad \in_{z'} = \in_z,\\
&\in_{y'z'} = (\in_{yz} - \in_{zx})/\sqrt{2},\ \in_{z'x'} = (\in_{yz} + \in_{zx})/\sqrt{2},\ \in_{x'y'} = (\in_y - \in_x)/2.
\end{aligned}$$

Using these relations as well as Eqs. (1.15), (1.17), and (1.20) we get from Eqs. (1.14) and (1.18)

$$2c_{66} = c_{11} - c_{22}. \tag{1.21}$$

Due to which Eq. (1.18) becomes

$$\begin{aligned}
&\sigma_x = c_{11}\in_x + c_{12}\in_y + c_{13}\in_z,\\
&\sigma_y = c_{12}\in_x + c_{11}\in_y + c_{13}\in_z,\\
&\sigma_z = c_{13}\in_x + c_{13}\in_y + c_{33}\in_z,\\
&\sigma_{yz} = 2c_{44}\in_{yz},\quad \sigma_{zx} = 2c_{44}\in_{zx},\ \sigma_{xy} = (c_{11} - c_{12})/2\ \in_{xy}.
\end{aligned} \tag{1.22}$$

Similarly, the expressions for strain in terms of stresses given by Eq. (1.19), become

$$\begin{aligned}
&\in_x = s_{11}\sigma_x + s_{12}\sigma_y + s_{13}\sigma_z, \\
&\in_y = s_{12}\in_x + s_{11}\sigma_y + s_{13}\sigma_z, \\
&\in_z = s_{13}\sigma_x + s_{13}\sigma_y + s_{33}\sigma_z, \\
&\in_z = s_{44/2}\,\sigma_{yx},\ \in_{zx} = s_{44/2}\,\sigma_{zx},\ \in_z = (s_{11} - s_{12})\sigma_{xy}.
\end{aligned} \tag{1.23}$$

4-Isotropic material

If any plane in the material is a plane of symmetry, then the material is isotropic, and has only two independent elastic constants, which can be shown by a new co-ordinate transformation; $x' = z,\ y' = x,\ z' = y$, which gives

$$l_{1'3} = l_{2'1} = l_{3'2} = 1,\ l_{1'1} = l_{1'2} = l_{2'2} = l_{2'3} = l_{3'1} = l_{3'3} = 0.$$

Substituting these direction cosines into Eqs. (1.1) and (1.2) to yield

$$\begin{aligned}
&\sigma_{x'} = \sigma_z,\ \sigma_{y'} = \sigma_x,\ \sigma_{z'} = \sigma_y,\ \sigma_{y'z'} = \sigma_{xy},\ \sigma_{z'x'} = \sigma_{yz},\ \sigma_{x'y'} = \sigma_{zx}, \\
&\in_{x'} = \in_z,\ \in_{y'} = \in_x,\ \in_{z'} = \in_y,\ \in_{y'z'} = \in_{xy},\ \in_{z'x'} = \in_{yz},\ \in_{x'y'} = \in_{zx}.
\end{aligned}$$

Using these relations as well as Eqs. (1.15), (1.17), and (1.20) we get from Eqs. (1.14) and (1.22)

$$c_{12} = c_{13},\ c_{11} = c_{33},\ c_{44} = c_{66}.$$

Thus, only two independent elastic constants are involved in an isotropic material. Now by introducing the Lame' constants defined by

$$\lambda = c_{12} = c_{13},\ \mu = c_{44} = c_{66},\ c_{33} = c_{11} = \lambda + 2\mu,$$

into Eq. (1.22), we get

$$\begin{aligned}
\sigma_x &= (\lambda+2\mu)\in_x + \lambda \in_y + \lambda \in_z, \\
\sigma_y &= \lambda \in_x + (\lambda+2\mu)\in_y + \lambda \in_z, \\
\sigma_z &= \lambda \in_x + \lambda \in_y + (\lambda+2\mu)\in_z, \\
\sigma_{yz} &= 2\mu\in_{yz},\ \sigma_{yz} = 2\mu\in_{zx},\ \sigma_{xy} = 2\mu\in_{xy}.
\end{aligned} \tag{1.24}$$

1.3 Incompressible orthotropic and transversely isotropic materials

Incompressible orthotropic material

Let the constraint of incompressibility i.e., $\in_{11} + \in_{22} + \in_{33} = 0$, be applied to an orthotropic material and let x_1, x_2, and x_3 be the axes of symmetry, then from Eq. (1.18) the linear stress-strain relations for the material can be written as

$$\begin{aligned}
\sigma_{11} &= -p + c_{11} \in_{11} + c_{12} \in_{22} + c_{13} \in_{33}, \\
\sigma_{33} &= -p + c_{13} \in_{11} + c_{23} \in_{22} + c_{33} \in_{33}, \\
\sigma_{23} &= 2c_{44} \in_{23},\ \sigma_{13} = 2c_{55} \in_{13},\ \sigma_{12} = 2c_{66} \in_{12},
\end{aligned} \tag{1.25}$$

where p is the hydrostatic pressure applied to maintain the incompressibility constraint.

Assuming the plane strain and incompressibility in x_1x_3-plane,

$$\in_{12}=\in_{22}=\in_{23}=0, \quad \in_{11}+\in_{33}=0, \tag{1.26}$$

the orthotropic stress-strain relations (1.25) reduce to

$$\begin{aligned}
\sigma_{11} &= -p+(c_{11}-c_{13})\in_{11}, \\
\sigma_{33} &= -p+(c_{13}-c_{33})\in_{11}, \\
\sigma_{13} &= 2c_{55}\in_{13},
\end{aligned} \tag{1.27}$$

which may be written concisely as

$$\begin{aligned}
\sigma_{ij} = -p\delta_{ij} &+ (c_{11}-c_{13})\in_{ij} + \\
&(2c_{55}-c_{11}+c_{13})(\delta_{i3}\in_{j3+}\delta_{j3}\in_{i3}) + \\
&\quad (c_{11}+c_{33}-2c_{13}-4c_{55})\in_{33}\delta_{i3}\delta_{j3}.
\end{aligned} \tag{1.28}$$

For positive definite energy density function, the elastic constants in Eq. (1.25) or Eq. (1.27) must satisfy [16]

$$c_{11}+c_{33}-2c_{13}\geq 0, \quad c_{55}\geq 0. \tag{1.29}$$

Incompressible transversely isotropic material

Proceeding as above the linear stress-strain relations for incompressible transversely isotropic material may be expressed as [17]

$$\sigma_{ij} = -p\delta_{ij} + 2\mu_T \in_{ij} + 2(\mu_L - \mu_T)(\delta_{i3} \in_{j3} + \delta_{j3} \in_{i3}) + 4(\mu_E - \mu_L) \in_{33} \delta_{i3}\delta_{j3}, \tag{1.30}$$

where μ_L and μ_T are longitudinal and transverse shear moduli respectively and

$$\mu_E = \frac{E_L}{E_T}\mu_T,$$

where E_L and E_T are longitudinal and transverse Young's moduli respectively. Following [16] the elastic constants in (1.30) must satisfy the following inequalities

$$2\mu_E + \mu_T \geq \mu_L, \qquad \mu_L \geq 0. \tag{1.31}$$

1.4 Rotational effects on elastic bodies

If a body is rotating with constant angular velocity $\underline{\Omega}$ relative to a Newtonian frame of reference S, then the rate of change of a vector u_i with respect to time is estimated by an observer in S is $\dot{\underline{u}} + \underline{\Omega} \times \underline{u}$. In tensor notation this expression may be written as $(\dot{u} + \varepsilon_{ijk}\Omega_j u_k)$, where ε_{ijk} is the Levi-Civita tensor. Similarly second derivative with respect to time of u_i becomes [37]

$$\ddot{u}_i + \Omega_j u_j \Omega_i - \Omega^2 u_i + 2\varepsilon_{ijk}\Omega_j \dot{u}_k .$$

Thus equation of motion, $\sigma_{ij},_j = p\ddot{u}_i,$ in the absence of body forces in a rotating medium can be written as follows [35]

$$\sigma_{ij},_j = p\{\ddot{u}_i + \Omega_j u_j \Omega_i - \Omega^2 u_i + 2\varepsilon_{ijk}\Omega_j \dot{u}_k\}. \tag{1.32}$$

Hence when the body undergoes dynamical deformation, the acceleration at any point with position vector $\underline{r}$ (with respect to the origin of a system of axes rotating with the body) consists of two additional parts that do not appear in a rotating body; (i) the time dependent part of the centripetal acceleration $\underline{\Omega} \times (\underline{\Omega} \times \underline{u})$, and (ii) the Coriolis acceleration $2\underline{\Omega} \times \dot{\underline{u}}$.

1.5 Thermo-elasticity

The study of influence of the temperature of an elastic solid upon the distribution of stress and strain, and of the inverse effect of the deformation upon the temperature distribution is the subject of the theory of thermo-elasticity.

Under the influence of temperature change $\theta = |T - T_0| << T_0$, T_0 being the reference temperature, the generalized Hooke's law is given by [17]

$$\sigma_{ij} = c_{ijkl} \in_{kl} - \beta_{ij}\theta, \tag{1.33}$$

and for transversely isotropic material the temperature is governed by the equation

$$\frac{\partial}{\partial x_i}(\kappa_{ij}\frac{\partial T}{\partial x_j}) = \rho c_v \frac{\partial T}{\partial t} + T_0 \beta_{ij} \frac{\partial \in_{ij}}{\partial t}, \tag{1.34}$$

where κ_{ij}, β_{ij}, c_V, and ρ are the conductivity tensor, thermal moduli and specific heat at constant deformation and ρ is the density of the material respectively, and κ_{ij} *and* β_{ij} are defined as

$$\kappa_{ij} = \kappa_1 \delta_{ij} + (\kappa_2 - \kappa_1)\delta_{i3}\delta_{j3},$$
$$\beta_{ij} = \beta_1 \delta_{ij} + (\beta_2 - \beta_1)\delta_{i3}\delta_{j3},$$

in which β_1 and β_2 are the thermal moduli with respect to the plane of isotropy and the axis of symmetry respectively, given by

$$\beta_1 = \alpha_1(c_{11} + c_{12}) + \alpha_2 c_{13},$$
$$\beta_2 = 2\alpha_1 c_{13} + \alpha_2 c_{33},$$

with α_1, α_2 are the coefficients of thermal expansions with respect to the plane of isotropy and axis of symmetry respectively.

We shall use uncoupled theory of thermo-elasticity only according to which if the thermal and the mechanical fields are independent of time the coupling in the heat equation (1.34) between temperature and deformation vanishes. Even for fields that do vary with time the coupling term is often neglected and so the heat equation (1.34) may be written as

$$\kappa_1\theta_{,11} + \kappa_2\theta_{,33} = \rho c_v \dot{\theta}. \qquad (1.35)$$

The comma notation is used for partial differentiation with respect to x_1, x_3 and a dot denotes a partial differentiation with respect to the time variable t.

1.6 Boundary conditions

The boundary conditions on the surface S of a solid consists of mechanical and a thermal condition.

1.6.1 Mechanical boundary conditions

These conditions are commonly divided into two distinct types.

(1) Stress boundary conditions

If the whole surface, S, of a solid is subjected to known external stresses $(\overline{p}_x, \overline{p}_y, \overline{p}_z)$, where the subscripts $x, y,$ and z indicate the directions of the surface stresses applied, then Eq. (1.3) gives the stress boundary conditions

$$\begin{aligned}
\sigma_{xx}\cos(n,x)+\sigma_{xy}\cos(n,y)+\sigma_{xz}\cos(n,z) &= \overline{p}_x, \\
\sigma_{yx}\cos(n,x)+\sigma_{yy}\cos(n,y)+\sigma_{yz}\cos(n,z) &= \overline{p}_y, \qquad (1.36)\\
\sigma_{zx}\cos(n,x)+\sigma_{zy}\cos(n,y)+\sigma_{zz}\cos(n,z) &= \overline{p}_z,
\end{aligned}$$

on S where n stands for the direction of $\bar{n}$, the external normal to S. If we use n_i, $(i = x, y, z)$ to denote the direction cosines of $\bar{n}$ with respect to the coordinates axes, the boundary conditions can be written as

$$\sigma_{ij} n_j = \bar{p}_i, \qquad \text{on S.} \tag{1.37}$$

(2) Displacement boundary conditions

If the whole surface S of the solid has given displacements, $(\bar{u}, \bar{v}, \bar{w})$, the boundary conditions can be specified as

$$u = \bar{u},\ v = \bar{v},\ w = \bar{w}, \qquad \text{on S,} \tag{1.38}$$

or in a more concise form,

$$u_i = \bar{u}_i, \qquad \text{on S.} \tag{1.39}$$

The mechanical boundary condition may of course be mixed, different conditions holding on different parts of S.

1.6.2 Thermal boundary conditions

The thermal conditions which occur most frequently are:

(1) θ given as a function of time and position on S, (1.40)

(2) $\partial\theta / \partial n$ given as a function of time and position on S, (1.41)

(3) $\partial\theta / \partial n + h(\theta - g(t)) = 0$ at all points on S, (1.42)

where $\partial/\partial n$ is the derivative taken along the vector $\bar{n}$ normal to S and, in the cooling law (1.42), h is a non-negative constant and g a given function of time. Again these conditions may be mixed.

1.7 Secular equation

When solving many problems in physics, the linear equation approximation involves systems of homogeneous linear equations. An important example of such linear homogeneous system is provided by the equation for the eigenvectors u_j and the eigenvalues λ of a tensor A_{ij} $(i, j = 1,2,3)$:

$$A_{ij}\, u_j = \lambda u_i, \tag{1.43}$$

or

$$(A_{ij} - \lambda\delta_{ij})u_j = 0. \tag{1.44}$$

For non-trivial solution of Eqs. (1.44) the determinant of coefficients vanishes i-e.

$$\left|A_{ij} - \lambda\delta_{ij}\right| = 0 \tag{1.45}$$

The last equation is called secular equation (of degree 3) in λ.

Chapter 2

Rayleigh wave propagation in transversely isotropic and orthotropic materials

2.1 Introduction

A number of researchers [1, 2, 3, 4] worked on Rayleigh wave speed formula by different techniques in isotropic material. After this a number of other investigators [5, 6, 7, 8] extended the Rayleigh wave analysis to anisotropic elastic materials. Recently, Pham and Ogden [12] obtained different Rayleigh wave speed formulas for orthotropic material.

In this chapter we observe the Rayleigh wave propagation in transversely isotropic material first time. Chadwick and Seet [22] and others [17, 23] used this material for elastic body waves only. We derive the secular equation and the necessary condition for Rayleigh wave propagation in the said material and deduce the corresponding secular equation and necessary condition for the Rayleigh wave propagation in orthotropic material.

In order to create more interest, the Rayleigh wave speed is calculated in different types of transversely isotropic and orthotropic materials and the conclusions are given at the end.

2.2 Boundary value problem for transversely isotropic material

Consider a semi-infinite stress- free surface of transversely isotropic material. We choose the rectangular co-ordinate system in such a way that $x_3 - axis$ is normal to the boundary and the body occupies the region $x_3 \leq 0$.

By following Pham & Ogden [12] we consider the plane harmonic waves in x_1-direction in x_1x_3-plane with displacement components (u_1, u_2, u_3) such that

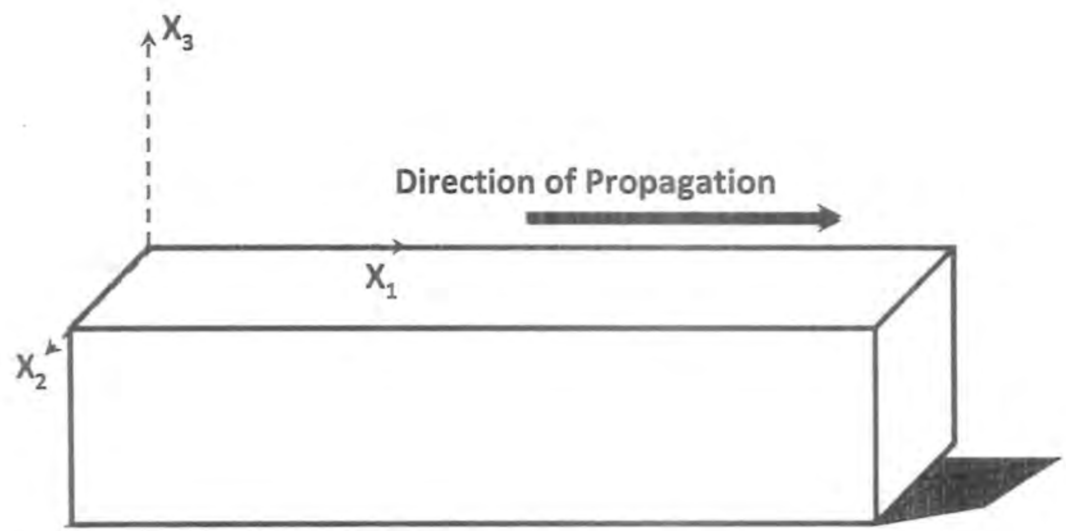
$$u_i = u_i(x_1, x_3, t), \quad i = 1,3, \quad u_2 \equiv 0. \tag{2.1}$$

Generalized Hook's law for transversely isotropic body may be written as

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{c_{11} - c_{12}}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix}, \tag{2.2}$$

where ϵ_{ij} is the strain tensor such that

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i, j = 1, 2, 3 \tag{2.3}$$



Geometry of the Problem

and σ_{ij} is the stress tensor where $c_{ii} > 0,\ i = 1,3,4,\quad c_{11}c_{33} - c_{13}^2 > 0$ which are the necessary and sufficient conditions for the strain energy of the material to be positive definite.

By using the above equations one can write

$$\begin{aligned}\sigma_{11} &= c_{11}u_{1,1} + c_{13}u_{3,3},\\ \sigma_{33} &= c_{13}u_{1,1} + c_{33}u_{3,3},\\ \sigma_{13} &= c_{44}(u_{1,3} + u_{3,1}).\end{aligned} \tag{2.4}$$

Equations of motion for infinitesimal deformation in the absence of body forces may be written as follows (see, Eq. (1.6))

$$\sigma_{ij,j} = \rho \ddot{u}_i. \tag{2.5}$$

In components form the Eqs. (2.5) may be written as

$$\begin{aligned}\sigma_{11,1} + \sigma_{13,3} &= \rho \ddot{u}_1,\\ \sigma_{31,1} + \sigma_{33,3} &= \rho \ddot{u}_3.\end{aligned} \tag{2.6}$$

Using Eqs. (2.4), Eqs. (2.6) may be written in terms of displacements as

$$c_{11}u_{1,11} + c_{44}u_{1,33} + (c_{13} + c_{44})u_{3,31} = \rho \ddot{u}_1,$$
$$c_{44}u_{3,11} + c_{33}u_{3,33} + (c_{13} + c_{44})u_{1,13} = \rho \ddot{u}_3. \tag{2.7}$$

The boundary conditions of zero traction are

$$\sigma_{3i} = 0, \quad i = 1,3, \quad \text{on the plane } x_3 = 0. \tag{2.8}$$

Usual requirements that the displacements and the stress components decay away from the boundary imply

$$u_i \to 0, \ \sigma_{ij} \to 0 \ (i, j = 1,3) \ \text{as } x_3 \to -\infty. \tag{2.9}$$

2.3 Solution of the problem

Considering a harmonic wave propagating in x_1-direction in x_1x_3-plane, then by following Pham & Ogden [12] we assume that

$$u_j = \varphi_j(y)\exp[ik(x_1 - ct)], \quad j = 1,3, \tag{2.10}$$

where $\qquad y = kx_3.$

k is the wave number and c is the wave speed and φ_j, $j=1,3$, are the functions to be determined

Substituting Eq. (2.10) into Eq. (2.7) we obtain

$$\begin{aligned}(c_{11}-\rho c^2)\phi_1-c_{44}\phi_1''-i(c_{44}+c_{13})\varphi_3'=0,\\(c_{44}-\rho c^2)\phi_3-c_{33}\phi_3''-i(c_{44}+c_{13})\phi_1'=0,\end{aligned}\qquad(2.11)$$

where a prime on φ_1 *or* φ_3 indicates differentiation with respect to y.

In terms of φ_j; $j=1,3$ after taking into consideration Eqs.(2.4) and (2.10) the boundary condition (2.8) become on the plane, $x_3=0$

$$\begin{aligned}ic_{13}\phi_1+c_{33}\phi_3'=0,\\\varphi_1'+i\varphi_3=0,\end{aligned}\qquad(2.12)$$

where as from Eq. (2.9)

$$\phi_j,\phi_j'\to 0\quad as\ x_3\to-\infty.\qquad(2.13)$$

Applying Laplace transform to Eqs. (2.11) and using Eqs. (2.12) we have

$$\begin{aligned}(c_{44}s^2+\rho c^2-c_{11})\overline{\phi}_1(s)+i(c_{44}+c_{13})s\overline{\phi}_3(s)\\ =c_{44}\{s\phi_1(0)+\phi_1'(0)\}+i(c_{44}+c_{13})\phi_3(0),\\ i(c_{13}+c_{44})s\overline{\phi}_1(s)+(c_{33}s^2-c_{44}+\rho c^2)\overline{\phi}_3(s)\\ =i(c_{44}+c_{13})\phi_1(0)+c_{33}\{\phi_3(0)s+\phi_3'(0)\}.\end{aligned} \tag{2.14}$$

This implies

$$\overline{\varphi}_1(s)=\frac{\begin{vmatrix}c_{44}\{s\phi_1(0)+\phi_1'(0)\}+i(c_{44}+c_{13})\phi_3(0) & i(c_{44}+c_{13})s\\ i(c_{44}+c_{13})\phi_1(0)+c_{33}\{\phi_3(0)s+\phi_3'(0)\} & (c_{33}s^2-c_{44}+\rho c^2)\end{vmatrix}}{c_{33}c_{44}s^4+\{c_{33}(\rho c^2-c_{11})+c_{44}(\rho c^2-c_{44})+(c_{44}+c_{13})^2\}]s^2+(\rho c^2-c_{44})(\rho c^2-c_{11})}. \tag{2.15}$$

Let s_1^2, s_2^2 be the roots (having real parts positive) of

$$\begin{aligned}c_{33}c_{44}s^4+\{c_{33}(\rho c^2-c_{11})+c_{44}(\rho c^2-c_{44})+(c_{44}+c_{13})^2\}s^2\\ +(\rho c^2-c_{44})(\rho c^2-c_{11})=0.\end{aligned} \tag{2.16}$$

which implies

$$\overline{\phi}_1(s)=\frac{A_1}{s-s_1}+\frac{A_2}{s-s_2}+\frac{A_3}{s+s_1}+\frac{A_4}{s+s_2}, \tag{2.17}$$

where A_1, A_2, A_3, and A_4 are the constants to be determined.

Taking inverse Laplace transform of Eq. (2.17) and applying the conditions (2.13)

$$\varphi_1(y) = A_1 \exp[s_1 y] + A_2 \exp[s_2 y], \tag{2.18}$$

where s_1 and s_2 have positive real parts.

In view of Eqs. (2.18) and (2.11)

$$\varphi_3(y) = \alpha_1 A_1 \exp[s_1 y] + \alpha_2 A_2 \exp[s_2 y], \tag{2.19}$$

where $$\alpha_j = i\frac{c_{44}s_j^{\,2} + (\rho c^2 - c_{11})}{(c_{13} + c_{44})}., \; j = 1,2 \tag{2.20}$$

From Eq. (2.16) we find

$$\begin{aligned} s_1^2 + s_2^2 &= -\frac{\{c_{33}(\rho c^2 - c_{11}) + c_{44}(\rho c^2 - c_{44}) + (c_{44} + c_{13})^2\}}{c_{33}c_{44}}, \\ s_1^2 s_2^2 &= \frac{(\rho c^2 - c_{44})(\rho c^2 - c_{11})}{c_{33}c_{44}}. \end{aligned} \tag{2.21}$$

2.4 Necessary condition for Rayleigh wave propagation

It is evident from Eqs. (2.18) and (2.21) that if $s_1^{\,2}$ and $s_2^{\,2}$ are real, they must be positive in order to ensure that s_1 and s_2 should have positive real parts. But if s_1^2 and s_2^2 are complex, they must be complex conjugate. In both the cases the product $s_1^2 s_2^2$ must be positive.

Therefore, it implies from Eqs. (2.21b) and the condition, $c_{ii} > 0$, $i = 1,3,4$, from Eq. (2.3) that either $0 < \rho c^2 < \min\{c_{11}, c_{44}\}$ or $\rho c^2 > \max\{c_{11}, c_{44}\}$. But if the latter inequality holds, then it is evident that right-hand side of Eq. (2.21a) will be negative and so Eq. (2.16) will have two negative real roots s_1^2 and s_2^2. This contradicts the requirement that s_1 and s_2 should have positive real parts. Therefore, the Rayleigh wave speed must satisfy the following inequality

$$0 < \rho c^2 < \min\{c_{11}, c_{44}\}, \tag{2.22}$$

which is the necessary condition for Rayleigh wave propagation in transversely isotropic material.

2.5 Secular equation

Substituting Eqs. (2.18), and (2.19) into Eq. (2.12) we have

$$\begin{aligned} &(ic_{13} + c_{33}\alpha_1 s_1)A_1 + (ic_{13} + c_{33}\alpha_2 s_2)A_2 = 0, \\ &(s_1 + i\alpha_1)A_1 + (s_2 + i\alpha_2)A_2 = 0. \end{aligned} \tag{2.23}$$

For non trivial solution of above linear homogeneous system of equations, the determinant of coefficient must vanish i.e.,

$$(ic_{13} + c_{33}\alpha_1 s_1)(s_2 + i\alpha_2) - (ic_{13} + c_{33}\alpha_2 s_2)(s_1 + i\alpha_1) = 0\,. \qquad (2.24)$$

Simplifying Eq. (2.24) and making the use of Eqs (2.20) and (2.21) we get

$$\begin{aligned}(c_{44} - \rho c^2)\{c_{13}{}^2 - c_{33}(c_{11} - \rho c^2)\} + \\ \rho c^2 \sqrt{c_{33}c_{44}}\,\sqrt{(c_{11} - \rho c^2)(c_{44} - \rho c^2)} = 0,\end{aligned} \qquad (2.25)$$

which is the required secular equation for Rayleigh wave propagation in transversely isotropic material.

2.6 Necessary condition and secular equation for Rayleigh wave propagation in orthotropic material

If we replace c_{44} by c_{55} in Eqs. (2.22) and (2.25) we obtain the following restriction

$$0 < \rho c^2 < \min\{c_{11}, c_{55}\}, \qquad (2.26)$$

and

$$(c_{55}-\rho c^2)\{c_{13}{}^2-c_{33}(c_{11}-\rho c^2)\}+ \rho c^2\sqrt{c_{33}c_{55}}\sqrt{(c_{11}-\rho c^2)(c_{55}-\rho c^2)}=0. \tag{2.27}$$

The inequality (2.26) and Eq. (2.27) are the necessary condition and the secular equation for Rayleigh wave propagation in orthotropic material (see, Pham and Ogden [12]).

2.7 Practical work for both types of materials

Substituting the values of elastic constants and density from the following table-2.1 of transversely isotropic materials [25] into the Eq. (2.25) and using the computer software Mathematica and the condition (2.22) we obtain the Rayleigh wave speeds of different transversely isotropic materials as is shown in table-2.2.

Table-2.1

Material	Stiffness $\times 10^{11}\ N/m^2$					Density kg/m^3
	c_{11}	c_{12}	c_{13}	c_{33}	c_{44}	ρ
Cobalt	2.59	1.59	1.11	3.35	0.71	8900
Cadmium	1.16	0.42	0.41	0.509	0.196	8642
Titanium boride	6.90	4.10	3.20	4.40	2.50	4500
Zinc	1.628	0.362	0.508	0.627	0.385	7140
Magnesium	0.5974	0.2624	0.217	0.617	0.1639	1740

Table-2.2

Material	Cobalt	Cadmium	Titanium boride	Zinc	Magnesium
Rayleigh wave Speed m/s	2811.11	1404.77	5983.28	2045.01	2894.65

Similarly substituting the values of elastic constants of orthotropic materials from Table-2.3 (see [26]) into the Eq. (2.27) and using the computer software Mathematica and the condition (2.26) we obtain the Rayleigh wave speed of different orthotropic materials as is shown in Table-2.4 .

Table-2.3

Material	Stiffness ($10^{10}\ N/m^2$)				Density
	c_{11}	c_{13}	c_{33}	c_{55}	(Kg/m^3)
Iodic acid HIO_3	3.01	1.11	4.29	2.06	4.64
Barium sodium niobate $Ba_2NaNb_5O_{15}$	23.9	5.00	13.5	6.60	5.30

Table-2.4

Material	Speed (Km/s)
Iodic acid	53.44
Bariumsodium niobate	102.55

2.8 Concluding remarks

Only one Rayleigh wave can propagate in different types of transversely isotropic and orthotropic materials with different speeds depending upon elastic constants and densities, and satisfying the conditions (2.22) and (2.26) respectively. These speeds of Rayleigh waves are shown in tables-2.2 and 2.4 for transversely isotropic and orthotropic materials respectively.

Chapter 3

Rotational effects on Rayleigh waves in transversely isotropic and orthotropic mediums

3.1 Introduction

Elastic wave propagation in a transversely isotropic medium has been discussed in detail in a series of papers by Chadwick [29-34]. Chadwick made use of the theory of Barnett and Lothe [27, 28] concerning conditions for the existence and propagation of the surface (Rayleigh) waves in anisotropic material.

Propagation of elastic waves in a rotating isotropic medium was studied by Shoenberg and Censor [35]. Khan and Ahmad [39] discussed the propagation of longitudinal and transverse waves in transversely isotropic material rotating about the axis of symmetry.

In this chapter we derive the secular equations and necessary conditions for Rayleigh wave propagation in transversely isotropic and orthotropic materials under rotational effect. We calculate the Rayleigh wave speed in both types of different materials under different frequencies of rotation and the conclusions are given at the end.

3.2 Problem formulation for transversely isotropic material under rotational effect

We assume the same transversely isotropic material as in chapter 2 such that the body is rotating about x_3-axis which is also the axis of symmetry. Then the equations of motion in the absence of body forces from Eq. (1.32) are given as

$$\sigma_{ij,j} = \rho\{\ddot{u}_i + \Omega_j u_j \Omega_i - \Omega^2 u_i + 2\varepsilon_{ijk}\Omega_j \dot{u}_k\},$$

where $\underline{\Omega} = \Omega(0,0,1).$

In components form the above equations may be written as

$$\begin{aligned} \sigma_{11,1} + \sigma_{13,3} &= \rho(\ddot{u}_1 - \Omega^2 u_1), \\ \sigma_{31,1} + \sigma_{33,3} &= \rho\ddot{u}_3. \end{aligned} \tag{3.1}$$

Substituting Eq. (2.4) into Eq. (3.1) we have

$$\begin{aligned} c_{11}u_{1,11} + c_{13}u_{3,31} + c_{44}(u_{1,33} + u_{3,13}) &= \rho(\ddot{u}_1 - \Omega^2 u_1), \\ c_{44}(u_{1,31} + u_{3,11}) + c_{13}u_{1,13} + c_{33}u_{3,33} &= \rho\ddot{u}_3. \end{aligned} \tag{3.2}$$

3.3 Solution of the problem

Following the same geometry and the wave propagating in the same direction as in chapter 2, we substitute Eq. (2.10) into Eq. (3.2) to get

$$c_{44}\varphi_1'' + i(c_{44}+c_{13})\varphi_3' + \{\rho c^2 - (c_{11} - \frac{\rho\Omega^2}{k^2})\}\varphi_1 = 0,$$
$$c_{33}\varphi_3'' + i(c_{44}+c_{13})\varphi_1' + \{\rho c^2 - c_{44})\varphi_3 = 0. \tag{3.3}$$

Applying Laplace transform with the boundary conditions (2.12) and (2.13) we may assume (see, [12])

$$\varphi_1(y) = B_1 \exp(s_1 y) + B_2 \exp(s_2 y),$$
$$\varphi_3(y) = B_1\gamma_1 \exp(s_1 y) + B_2\gamma_2 \exp(s_2 y), \tag{3.4}$$

where $$\gamma_j = \frac{i[c_{44}s_j^{\,2} + \{\rho c^2 - (c_{11} - \frac{\rho\Omega^2}{k^2})\}]}{(c_{44}+c_{13})s_j}, \quad j = 1,2,$$

and $s_1^{\,2}$, $s_2^{\,2}$ (where s_1 and s_2 are assumed to have positive real parts) are the roots of the quadratic equation (in s^2) given below

$$c_{33}c_{44}s^4 + [(c_{44}+c_{13})^2 + c_{44}(\rho c^2 - c_{44}) + c_{33}\{\rho c^2 - (c_{11} - \frac{\rho\Omega^2}{k^2})\}]s^2 + (\rho c^2 - c_{44})\{\rho c^2 - (c_{11} - \frac{\rho\Omega^2}{k^2})\} = 0, \tag{3.5}$$

which implies

$$s_1^{\,2} + s_2^{\,2} = -\frac{[(c_{44}+c_{13})^2 + c_{44}(\rho c^2 - c_{44}) + c_{33}\{\rho c^2 - (c_{11} - \frac{\rho\Omega^2}{k^2})\}]}{c_{33}c_{44}},$$
$$s_1^{\,2} s_2^{\,2} = \frac{(\rho c^2 - c_{44})\{\rho c^2 - (c_{11} - \frac{\rho\Omega^2}{k^2})\}}{c_{33}c_{44}}. \tag{3.6}$$

3.4 Necessary condition for Rayleigh propagation in transversely isotropic material under rotational effect

Following the same steps as in case of deriving the inequality (2.23) from Eqs. (2.21), we can derive the following inequality from Eqs. (3.6) that

$$0 < \rho c^2 < \min\{c_{11} - \rho\frac{\Omega^2}{k^2}, c_{44}\}. \tag{3.7}$$

Inequality (3.7) is the required necessary condition for Rayleigh wave propagation in transversely isotropic material.

3.5 Secular Equation for Rayleigh wave propagation in transversely isotropic material under rotational effect

Substituting the values of Eq. (3.4) into Eqs. (2.12) we get

$$\begin{aligned}(ic_{13}+c_{33}\gamma_1 s_1)B_1+(ic_{13}+c_{33}\gamma_2 s_1)B_2&=0,\\(s_1+i\gamma_1)B_1+(s_2+i\gamma_2)B_2&=0.\end{aligned}\tag{3.8}$$

For non-trivial solution of the homogeneous system of Eqs.(3.8) the determinant of coefficients must be vanished i.e.,

$$(ic_{13}+c_{33}\gamma_1 s_1)(s_2+i\gamma_2)-(ic_{13}+c_{33}\gamma_2 s_2)(s_1+i\gamma_1)=0.\tag{3.9}$$

Substituting the values from Eqs. (3.4), and (3.6) into Eq.(3.9) and simplifying we get

$$\begin{aligned}&(\rho c^2-c_{44})[c_{13}{}^2+c_{33}\{\rho c^2-(c_{11}-\frac{\rho\Omega^2}{k^2})\}]-\\&\qquad\rho c^2\sqrt{c_{33}c_{44}}\sqrt{\{\rho c^2-(c_{11}-\frac{\rho\Omega^2}{k^2})\}(\rho c^2-c_{44})}=0.\end{aligned}\tag{3.10}$$

Equation (3.10) is the required secular equations of Rayleigh waves for transversely isotropic material under rotational effect.

3.6 Necessary condition and secular equation for Rayleigh propagation in orthotropic material under rotational effect

If we substitute the values of stress components for orthotropic material into the equations of motion (3.1) and proceed as above for transversely isotropic material we shall obtain the necessary condition for propagation and the secular equation respectively as

$$0 < \rho c^2 < \min\{c_{11} - \rho \frac{\Omega^2}{k^2}, c_{55}\}, \tag{3.11}$$

$$(\rho c^2 - c_{55})[c_{13}{}^2 + c_{33}\{\rho c^2 - (c_{11} - \frac{\rho\Omega^2}{k^2})\}] - \rho c^2 \sqrt{c_{33}c_{55}} \sqrt{\{\rho c^2 - (c_{11} - \frac{\rho\Omega^2}{k^2})\}(\rho c^2 - c_{55})} = 0. \tag{3.12}$$

3.7 Practical Applications for both types of materials under rotation

In order to find the rotational effects on Rayleigh speed in transversely isotropic and orthotropic materials we substitute the values of elastic constants and densities from Tables-2.1 and 2.2 into the Eqs. (3.10) and (3.12) respectively and using computer software Mathematica we obtain the Tables-3.1, 3.2 showing the rotational effects on the wave speed in both of the materials respectively. Noting that we take only those values of speeds in the given types of materials which satisfy the condition (3.7) and (3.11) respectively.

Table-3.1

Rayleigh wave speed (m/s) in different transversely isotropic materials under different rotations					
$\rho\frac{\Omega^2}{k^2}$	Cadmium	Cobalt	Titanium boride	Zinc	Magnesium
0	1404.75.	2685.55	5983.22	2045.01	2894.65
10	1404.75.	2685.55	5983.22	2045.01	2894.65
10^2	1404.75.	2685.55	5983.22	2045.01	2894.65
10^3	1404.75.	2685.55	5983.22	2045.01	2894.65
10^4	1404.75.	2685.55	5983.22	2045.01	2894.65
10^5	1404.65.	2685.55	5983.22	2045.01	2894.65
10^6	1404.65.	2685.55	5983.22	2045.01	2894.65
10^7	1404.63	2685.54	5983.19	2044.98	2894.61
10^8	1404.49	2685.47	5982.9	2044.76	2894.22
10^9	1403.05	2684.7	5980.05	2042.51	2890.26
10^{10}	1386.39	2676.55	5951.00	2018.03	2838.99
10^{11}	–	2501.08	5601.01	1209.91	–

Table-3.2

Rayleigh wave speed (m/s) in different orthotropic materialsunder different rotations.		
$\rho\frac{\Omega^2}{k^2}$	Iodic acid HIO_3	Barium sodium niobate $Ba_2NaNb_5O_{15}$
0	53497.1	102554.0
10	53497.1	102554.0
10^2	53497.1	102554.0
10^3	53497.1	102554.0
10^4	53497.1	102554.0
10^5	53497.0	102554.0
10^6	53496.6	102554.0
10^7	53492.6	102554.0
10^8	53451.4	102550.0
10^9	53027.2	102510.0
10^{10}	47068.0	102094.0
10^{11}	–	94456.2

3.8 Conclusions

It is evident from Table-3.1 that if $0 \le \rho \frac{\Omega^2}{k^2} \le 10^{10}$ then the speed of the Rayleigh wave in the given transversely isotropic materials decreases slightly with the increase of rotation. If $\rho \frac{\Omega^2}{k^2} = 10^{11}$ then the Rayleigh wave speed does not exist in Cadmium and Magnesium but in Cobalt, Titanium and Zinc the speed exists. If $\rho \frac{\Omega^2}{k^2} > 10^{11}$ then the speed does not exist for any given transversely isotropic materials.

Similarly it is clear from Table-3.2 that the Rayleigh wave speed decreases slightly within the above range of rotation in the orthotropic materials also. If $\rho \frac{\Omega^2}{k^2} = 10^{11}$ then the speed does not exist in Iodic acid, but it does exist in Barium sodium niobate. For $\rho \frac{\Omega^2}{k^2} > 10^{11}$ the speed does not exist in any of the given orthotropic materials.

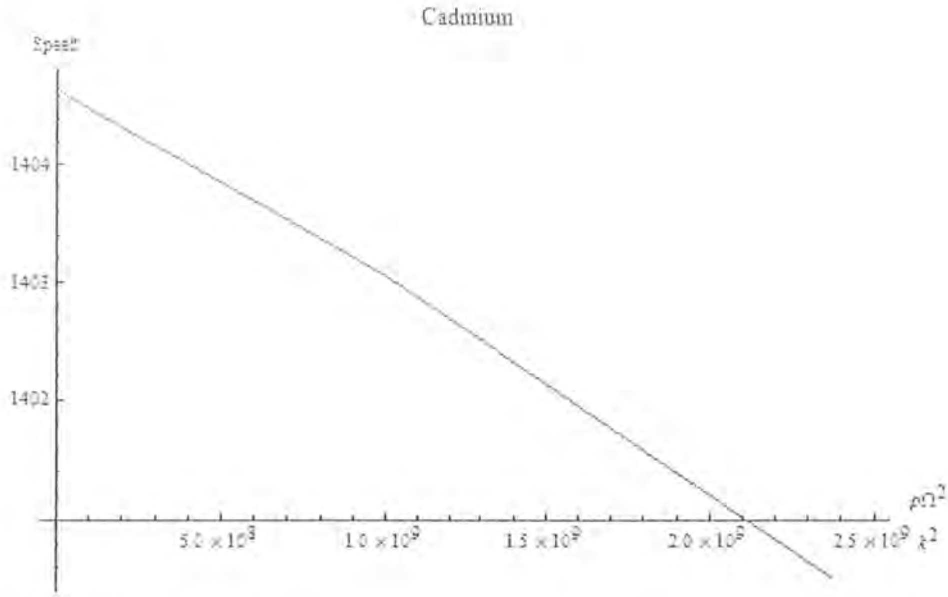


Figure 3.1: Speed plotted against frequency of rotation for Cadmium.

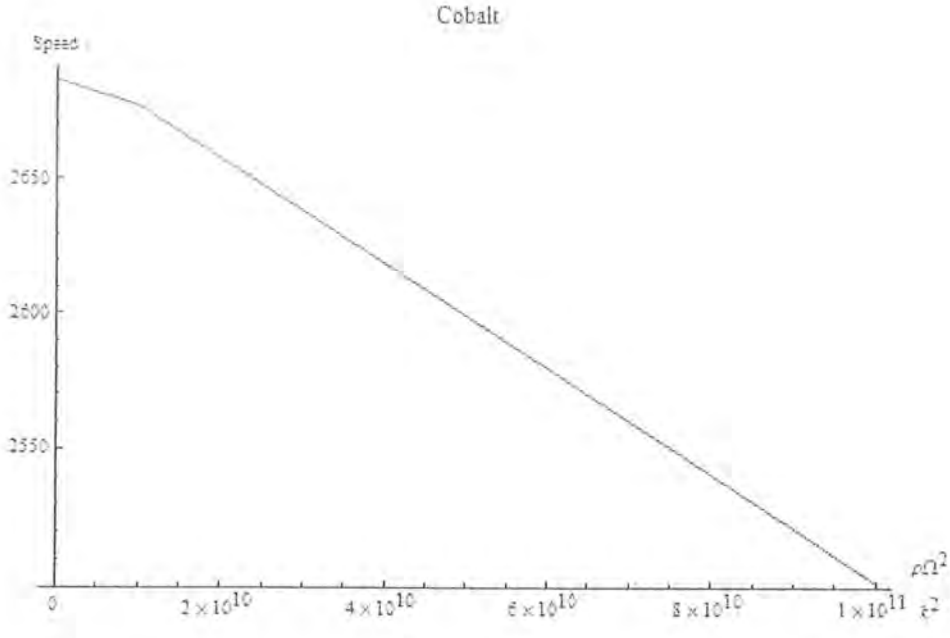


Figure 3.2: Speed plotted against frequency of rotation for Cobalt.

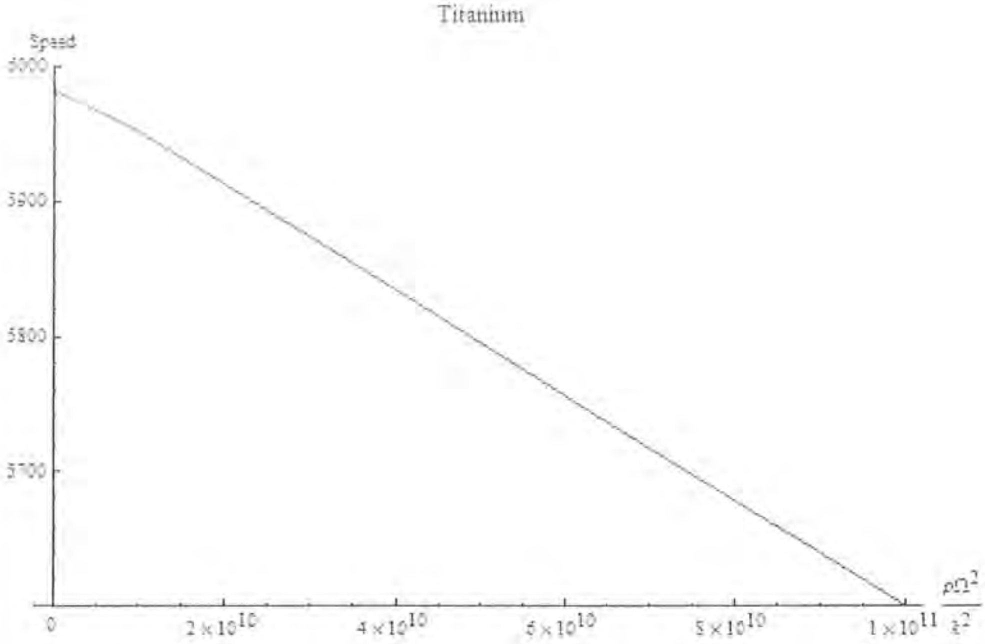


Figure 3.3: Speed plotted against frequency of rotation for Titanium.

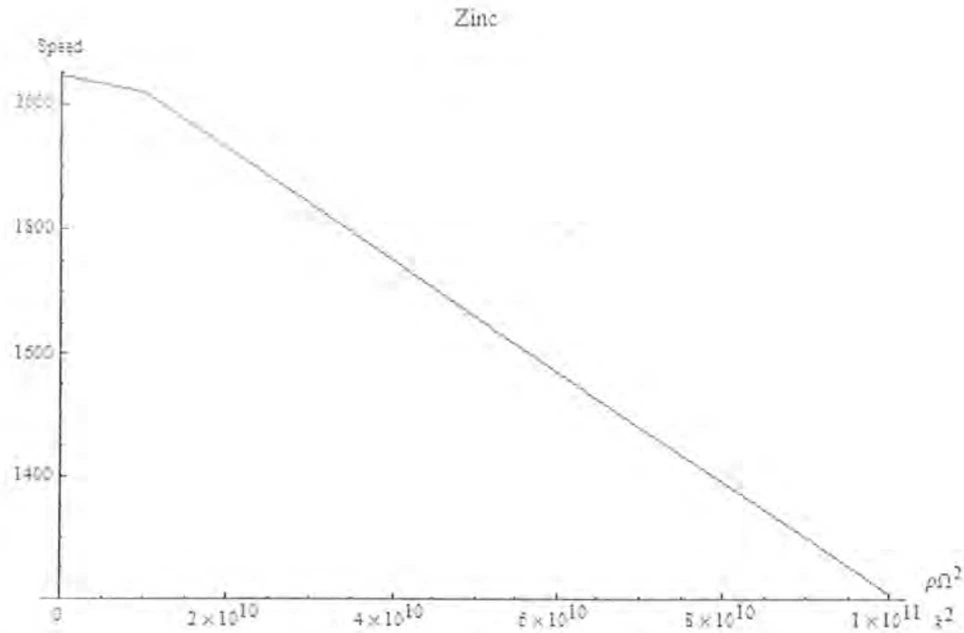


Figure 3.4: Speed plotted against frequency of rotation for Zinc.

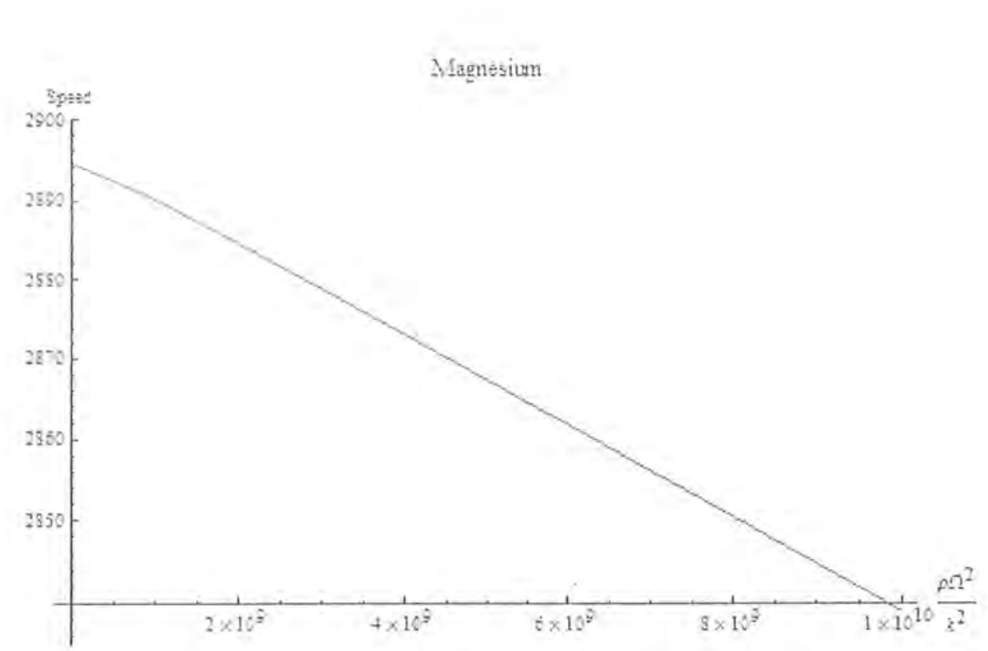


Figure 3.5: Speed plotted against frequency of rotation for Magnesium.

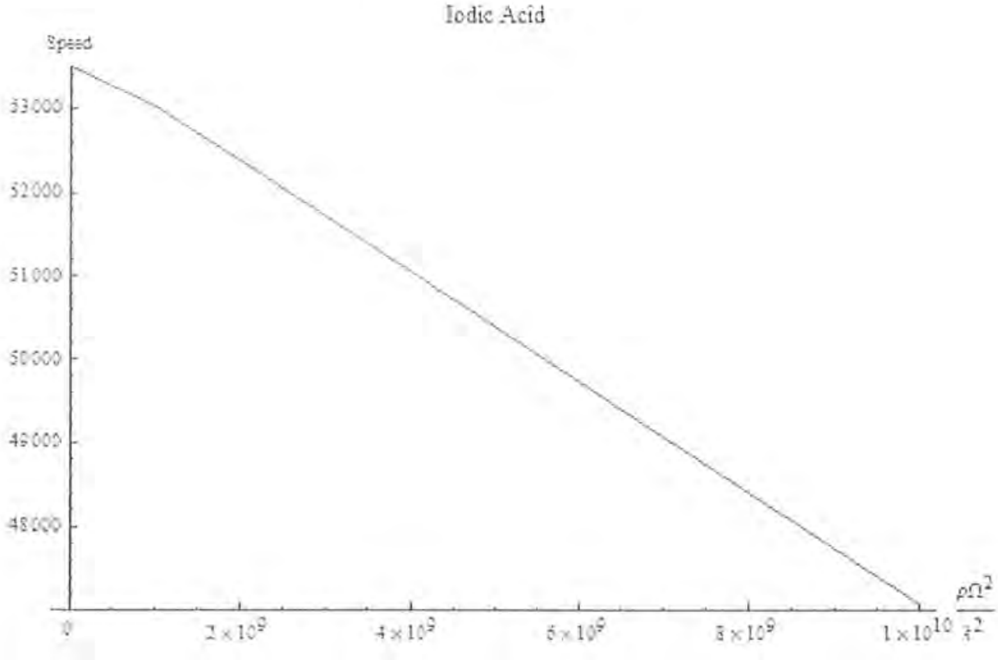


Figure 3.6: Speed plotted against frequency of rotation for Iodic Acid.

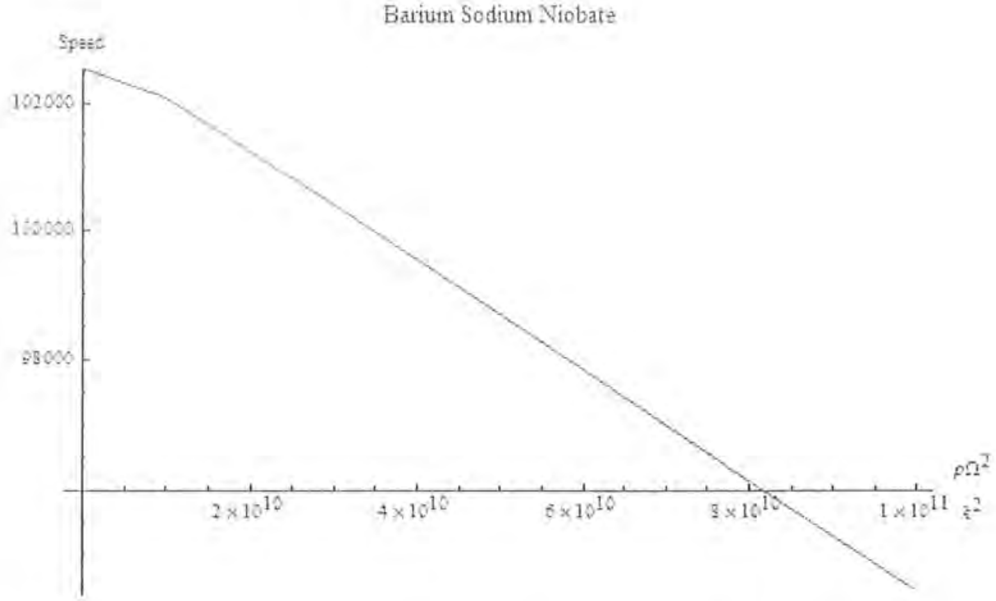


Figure 3.7: Speed plotted against frequency of rotation for Barium Sodium Niobate.

Chapter 4

Rayleigh waves in rotating incompressible transversely isotropic and orthotropic materials

4.1 Introduction

Some researchers like Rogerson [38] and Chadwick [32, 33] studied the elastic wave propagation in transversely isotropic material with the additional constraint of incompressibility. Nair and Sotiropoules [13] discussed elastic waves in incompressible orthotropic materials. Khan and Ahmad [17] found rotational effect on elastic body waves in the same material. Destrade [15] and Ogden and Pham [16] studied Rayleigh wave propagation in incompressible orthotropic material.

In this chapter we discuss the rotational effect on Rayleigh waves in both incompressible transversely isotropic and orthotropic materials. We derive the secular equations of Rayleigh waves under rotation and obtain the necessary conditions for the Rayleigh wave propagation in both of the materials. We choose different model materials of both types and find the Rayleigh wave speed in the materials under different rotations and the conclusions are given at the end.

4.2 Governing problem for Rayleigh wave propagation in incompressible transversely isotropic material under rotational effect

Consider an incompressible transversely isotropic material having the same geometry as that of the transversely isotropic material in chapter 2 and is rotating about x_3-axis which is also the axis of symmetry as that in chapter 3. Now in view of Eq. (1.4) the incompressibility condition (1.26) gives

$$u_{1,1} + u_{3,3} = 0. \tag{4.1}$$

Equation (4.1) implies that there exists a potential function Ψ such that

$$u_1 = \Psi_{,3}, \; u_3 = -\Psi_{,1}. \tag{4.2}$$

From Eq. (1.30) the stress-strain components are

$$\begin{aligned}
\sigma_{11} &= -p + 2\mu_T \in_{11}, \\
\sigma_{33} &= -p + (4\mu_E - 2\mu_T) \in_{33}, \\
\sigma_{13} &= 2\mu_L \in_{13}, \\
\sigma_{22} &= -p,
\end{aligned} \tag{4.3}$$

where from (1.31)

$$2\mu_E + \mu_T \geq \mu_L, \qquad \mu_L \geq 0. \tag{4.4}$$

If the body is rotating about x_3-axis, with a constant angular velocity $\underline{\Omega}$ then equations of motion in the absence of body forces for infinitesimal deformation can be obtained from Eq. (1.32) as

$$\sigma_{ij,j} = \rho\{\ddot{u}_i + \Omega_j u_j \Omega_i - \Omega^2 u_i + 2\varepsilon_{ijk}\Omega_j \dot{u}_k\}, \tag{4.5}$$

where $\qquad \underline{\Omega} = \Omega(0,0,1).$

In view of Eq. (4.5), the relevant components of the equation of motion are as follows

$$\begin{aligned}
&\sigma_{11,1}+\sigma_{13,3}=\rho(\ddot{u}_1-\Omega^2 u_1),\\
&\sigma_{31,1}+\sigma_{33,3}=\rho\ddot{u}_3,\\
&\sigma_{22,2}=2\rho\Omega\dot{u}_1,
\end{aligned} \tag{4.6}$$

where dots denote partial differentiation with respect to t.

Using Eqs. (1.4), (4.2), and (4.3) in Eqs.(4.6a) and (4.6b), and eliminating p by cross differentiating we get

$$(4\mu_E-2\mu_L)\Psi_{,1133}+\mu_L(\Psi_{,1111}+\Psi_{,3333})=\rho(\ddot{\Psi}_{,11}+\ddot{\Psi}_{,33})-\rho\Omega^2\Psi_{,33}\ , \tag{4.7}$$

and also from Eq. (4.6c)

$$-p_{,2}=2\rho\Omega\dot{\Psi}_{,3}. \tag{4.8}$$

Since the right hand side of Eq. (4.8) is a function of x_1 and x_3 only, p is linear in x_2. This also implies from Eq. (4.8) that there exists a function θ which is linear in x_2 such that

$$-p=\frac{\partial\theta}{\partial x_3}, \tag{4.9}$$

and $2\rho\Omega\dot{\psi}=\dfrac{\partial\theta}{\partial x_2}=\chi$ (say) is a function of x_1 and x_3 only.

This implies from Eq. (4.9b) that

$$\dot{\Psi} = \frac{1}{2\rho\Omega}\chi,$$

$$\Rightarrow \Psi = \frac{1}{2\rho\Omega}\zeta \text{ (say)},$$

where ζ is a function of x_1 and x_3 only. (4.10)

Substituting the value of Ψ from Eq. (4.10) into Eq. (4.7), we obtain

$$(4\mu_E - 2\mu_L)\zeta_{,1133} + \mu_L(\zeta_{,1111} + \zeta_{,3333}) = \rho(\ddot{\zeta}_{,11} + \ddot{\zeta}_{,33}) - \rho\Omega^2\zeta_{,33}. \tag{4.11}$$

The boundary conditions of zero traction can be expressed as

$$\sigma_{3i} = 0, \quad i = 1,3, \quad \text{on the plane } x_3 = 0. \tag{4.12}$$

The usual requirements that the displacements and the stress components decay away from the boundary imply

$$u_i \to 0,\ (i = 1,3),\quad \sigma_{ij} \to 0,\ (i,j = 1,3),\ \text{as } x_3 \to -\infty. \tag{4.13}$$

Using Eqs. (1.4), (4.2), (4.3) and the first of (4.6) and (4.10), the resulting boundary conditions of Eqs. (4.12) can be written as

$$\begin{aligned}&\mu_L(\zeta_{,33}-\zeta_{,11})=0,\\&(4\mu_E-\mu_L)\zeta_{,113}+\mu_L\zeta_{,333}-\rho\ddot{\zeta}_{,3}+\rho\Omega^2\zeta_{,3}=0,\ on\ x_3=0.\end{aligned}\tag{4.14}$$

In view of Eq. (4.2), boundary conditions (4.13) and Eq. (4.10) we find that

$$\zeta(x_1,x_3,t)\to 0,\ as\ x_3\to-\infty.\tag{4.15}$$

4.3 Solution of the problem

To consider harmonic waves in x_1-direction in x_1x_3-plane, we write ζ in the form [16]

$$\begin{aligned}\zeta(x_1,x_3,t)&=\varphi(y)\exp[ik(x_1-ct)],\\y&=kx_3,\end{aligned}\tag{4.16}$$

where k is the wave number, c is the wave speed and the function φ is to be determined.
Substitution of Eq. (4.16) into Eq. (4.11) yields

$$\mu_L\varphi'''' - (4\mu_E - 2\mu_L - \rho c^2 - \rho\frac{\Omega^2}{k^2})\varphi'' + (\mu_L - \rho c^2)\varphi = 0, \tag{4.17}$$

where a prime on φ indicates differentiating with respect to y. In terms of φ the boundary conditions (4.14) become

$$\begin{aligned} &\varphi''(0) + \varphi(0) = 0, \\ &\mu_L\varphi'''(0) - (4\mu_E - \mu_L - \rho c^2 - \rho\frac{\Omega^2}{k^2})\varphi'(0) = 0, \end{aligned} \tag{4.18}$$

in the first of which we have omitted the factor μ_L on the assumption that $\mu_L > 0$.

From Eq. (4.15) we find that

$$\varphi(y) \to 0, \quad as \quad y \to -\infty. \tag{4.19}$$

Thus the problem has reduced to the solution of Eq. (4.17) with the boundary conditions (4.18) and (4.19). The general solution for $\varphi(y)$ that satisfies the condition (4.19) is as follows:

$$\varphi(y) = A\exp(s_1 y) + B\exp(s_2 y), \tag{4.20}$$

where A and B are constants to be determined, while s_1 and s_2 (with positive real parts) are solutions of the equation

$$\mu_L s^4 - (4\mu_E - 2\mu_L - \rho c^2 - \rho\frac{\Omega^2}{k^2})s^2 + (\mu_L - \rho c^2) = 0. \qquad (4.21)$$

From Eq. (4.21) it follows that

$$\begin{aligned} &{s_1}^2 + {s_2}^2 = (4\mu_E - 2\mu_L - \rho c^2 - \rho\frac{\Omega^2}{k^2})/\mu_L, \\ &{s_1}^2{s_2}^2 = (\mu_L - \rho c^2)/\mu_L. \end{aligned} \qquad (4.22)$$

4.4 Necessary condition for Rayleigh wave propagation in incompressible transversely isotropic material

Arguing same as in deriving the inequality (2.22) from Eqs. (2.21) we get from Eqs. (4.22) the following on the assumption $\mu_L > 0$,

$$\rho c^2 < \mu_L, \qquad (4.23)$$

which is the required necessary condition for Rayleigh wave propagation in incompressible transversely isotropic material.

4.5 Secular equation for Rayleigh wave propagation incompressible transversely isotropic material

Substituting Eq. (4.20) into the boundary conditions (4.18), we obtain

$$
\begin{aligned}
&(s_1{}^2+1)A+(s_2{}^2+1)B=0,\\
&\{\mu_L s_1{}^3-(4\mu_E-\mu_L-\rho c^2-\rho\frac{\Omega^2}{k^2})s_1\}A+\\
&\qquad\qquad \{\mu_L s_2{}^3-(4\mu_E-\mu_L-\rho c^2-\rho\frac{\Omega^2}{k^2})s_2\}B=0.
\end{aligned}
\tag{4.24}
$$

For nontrivial solution the determinant of coefficients of the system (4.24) must vanish. This yields after removal of the factor (s_2-s_1) that

$$
\begin{aligned}
&\mu_L\{(s_1{}^2+s_1{}^2)+s_1{}^2s_2{}^2\}+s_1s_2(4\mu_E-\rho c^2-\rho\frac{\Omega^2}{k^2})-\\
&\qquad\qquad (4\mu_E-\mu_L-\rho c^2-\rho\frac{\Omega^2}{k^2})=0.
\end{aligned}
\tag{4.25}
$$

Use of Eqs.(4.22) into Eq. (4.25) leads to

$$u^3 + \sqrt{\mu_L}\,u^2 + (4\mu_E - \mu_l - \rho\Gamma)u - \mu^{3/2} = 0, \tag{4.26}$$

where $u = \sqrt{\mu_L - \rho c^2}$,

$$\Gamma = \frac{\Omega^2}{k^2}.$$

Equation (4.26) is the required secular equation for Rayleigh waves in incompressible transversely isotropic material.

4.6 Necessary condition and secular equation for Rayleigh wave propagation in incompressible orthotropic material

If for the above problem we use the incompressible orthotropic material then we shall use the stress-strain relations from Eq. (1.27) and the relations between elastic constants from inequalities (1.29). Then proceeding all the process explained as above in case of Rayleigh wave in incompressible transversely isotropic material we obtain the necessary condition for propagation and the secular equation for Rayleigh wave as follows:

$$\rho c^2 < c_{66}\,, \tag{4.27}$$

$$u^3 + \sqrt{c_{66}}\,u^2 + (c_{11} + c_{22} - 2c_{12} - c_{66} - \rho\Gamma)u - c_{66}^{\;3/2} = 0,$$

where $u = \sqrt{c_{66} - \rho c^2}$, (4.28)

$$\Gamma = \frac{\Omega^2}{k^2}.$$

4.7 Practical application for incompressible transversely isotropic and orthotropic material

By using the data measured by Markham [37] and referred by Rogerson [36], the values of elastic constants for carbon fibre-epoxy resin composite are as follows:

$$\begin{aligned} \mu_L &= 5.66\times10^9\, NM^{-2}, \\ \mu_T &= 2.46\times10^9\, NM^{-2}, \\ \mu_E &= 59.72\times10^9\, NM^{-2}, \\ \rho &= 1.6\times10^3\, KgM^{-3}. \end{aligned} \tag{4.29}$$

The value of density is taken from Carbon Fibre Profile, Composite Structural Profile System.

Substituting the values of (4.29) into Eq. (4.26) and using the computer software Mathematica and the condition (4.23) we obtain following Table-4.1 for Rayleigh wave speed in the given incompressible transversely isotropic material for different values of rotations. Table-4.2 of elastic constants and density for an incompressible orthotropic material is taken from [26]

Table-4.1

$\Gamma = \dfrac{\Omega^2}{k^2}$ cycle/s	Speed c m/s
0	1880.27
10	1880.27
10^2	1880.27
10^3	1880.27
10^4	1880.27
10^5	1880.27
10^6	1880.26
10^7	1880.19
10^8	1875.27
10^9	1880.81
10^{10}	1880.82
10^{11}	1880.82

Table-4.2

Materials	Stiffness $10^{10}\,N/m^2$									Density 10^3 Kg $/m^3$ ρ
	c_{11}	c_{12}	c_{13}	c_{22}	c_{23}	c_{33}	c_{44}	c_{55}	c_{66}	
Iodic acid HIO_3	3.01	1.61	1.11	5.8	0.80	4.29	1.69	2.06	1.58	4.64

Substituting the values from Table-4.2 into the Eq. (4.27) and using the computer software Mathematica and the condition (4.28) we obtain the following Table-4.3 for Rayleigh wave speed in the incompressible orthotropic material, Iodic acid for different values of rotations.

Table-4.3

$\Gamma = \dfrac{\Omega^2}{k^2}$ cycle/s	Speed c m/s
0	1738.69
10	1738.69
10^2	1738.69
10^3	1738.68
10^4	1738.51
10^5	1736.85
10^6	1718.22
10^7	970.093
10^8	1844.03
10^9	1845.03
10^{10}	1845.31
10^{11}	1845.31
10^{12}	1845.31

4.8 Conclusions

It can be seen from Table-4.1 that at very high frequency of rotation of the order of 10^6 the speed begins to decrease very slightly. This process continues till the frequency becomes of the order of 10^8. At the frequency of the order of 10^9 the speed begins to increase very slightly and attains the maximum value when the frequency becomes of the order of 10^{10}. After this the speed does not change with the increase of frequency of rotation.

Similarly it can be seen from Table-4.3 that at a very high frequency of rotation of the order of 10^3 the speed begins to decrease very slightly. This process continues till the frequency becomes of the order of 10^7. At the frequency of the order of 10^8 the speed begins to increase very slightly and attains the maximum value when the frequency becomes of the order of 10^{10}. After this no matter, however, the frequency may be the speed does not change.

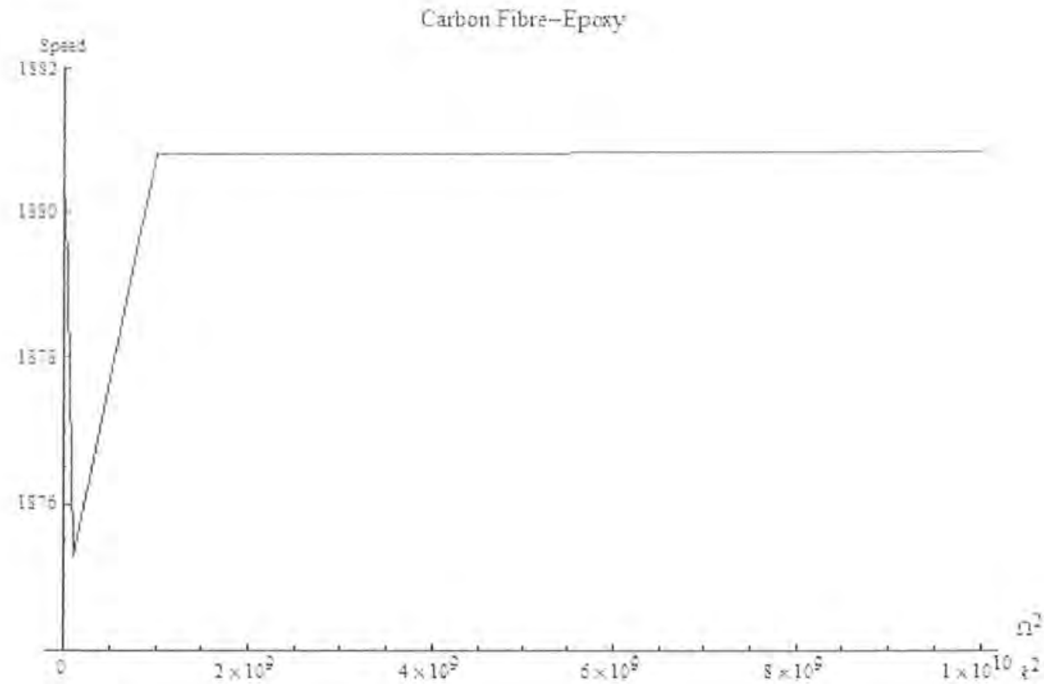


Figure 4.1: Speed plotted against frequency of rotation for Carbon Fibre-Epoxy.

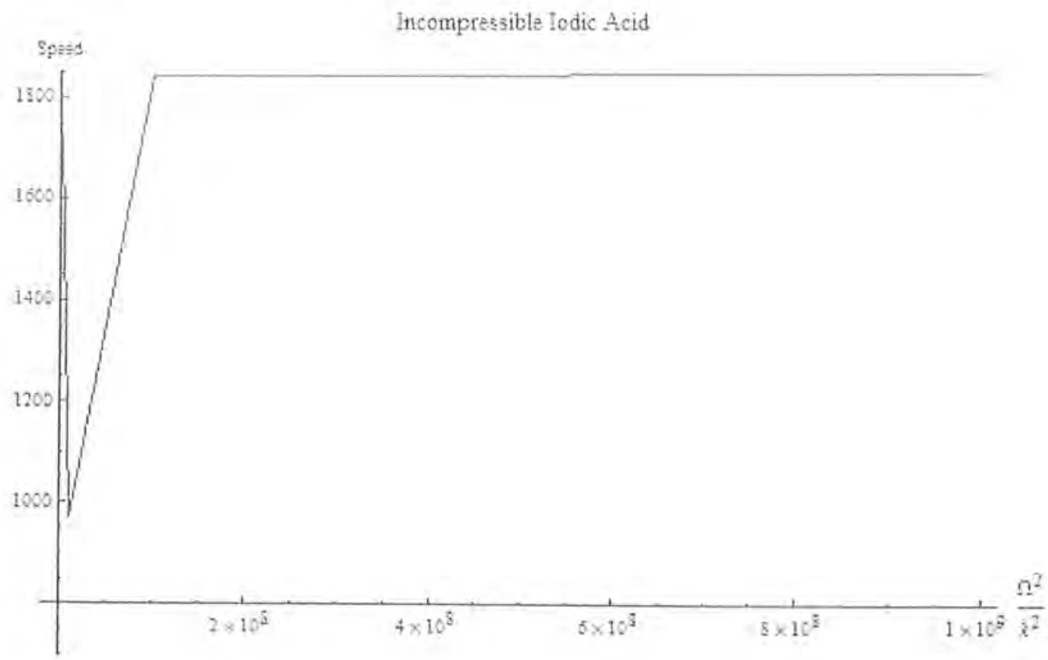


Figure 4.2: Speed plotted against frequency of rotation for incompressible Iodic Acid.

Chapter 5

Thermal effects on Rayleigh wave speed in transversely isotropic material

5.1 Introduction

The theory of irreversible thermodynamics for an elastic material was established by Biot [36]. His coupled equations for thermo-elastic waves in an isotropic material were solved by the researchers [18, 19, 20, 21], and in a transversely isotropic medium by the investigators [22, 23]. Thermo-elastic Rayleigh waves were studied by Chadwick [21] in isotropic material. In this chapter we study the thermo-elastic Rayleigh waves in transversely isotropic conducting material using the uncoupled theory of thermodynamics in non-steady temperature field case only.

We derive the secular equation and the necessary condition for the Rayleigh wave propagation under thermal effect and calculate the Rayleigh wave speed in some of the transversely isotropic materials. We observe that two Rayleigh waves propagate in transversely isotropic material under thermal effect. One wave propagates with the same speed as that of the wave which propagates without thermal effect and the other one propagates with some higher speed.

5.2 Boundary value problem

Assume a semi-infinite stress-free surface of a homogeneous heat-conducting elastic material which is transversely isotropic in both elastic and thermal response and having the same geometry as that of the material defined in chapter 2. Again we consider a plane harmonic wave in x_1-direction in x_1x_3-plane with displacement components (u_1, u_2, u_3) such that

$$u_i = u_i(x_1, x_3, t), \quad i = 1,3, \quad u_2 \equiv 0. \tag{5.1}$$

Then by using Eqs. (1.4), (1.22), (1.33), and (5.1) we have

$$\begin{aligned}
\sigma_{11} &= c_{11}u_{1,1} + c_{13}u_{3,3} - \beta_1\theta, \\
\sigma_{33} &= c_{13}u_{1,1} + c_{33}u_{3,3} - \beta_2\theta, \\
\sigma_{13} &= c_{44}(u_{1,3} + u_{3,1}).
\end{aligned} \tag{5.2}$$

Using equations of motion (1.6) in the absence of body forces and Eq. (5.2) we obtain

$$\begin{aligned}
c_{11}u_{1,11} + c_{13}u_{3,31} + c_{44}(u_{1,33} + u_{3,13}) - \beta_1\theta_{,1} &= \rho\ddot{u}_1, \\
c_{44}(u_{1,31} + u_{3,11}) + c_{13}u_{1,13} + c_{33}u_{3,33} - \beta_2\theta_{,3} &= \rho\ddot{u}_3.
\end{aligned} \tag{5.3}$$

The non-steady temperature field is governed by the Eq. (1.35) of heat conduction which is given by

$$\kappa_1\theta_{,11} + \kappa_2\theta_{,33} = \rho c_v \dot{\theta}. \tag{5.4}$$

It can be proved that $\kappa_1 \geq 0$, $\kappa_2 \geq 0$ and of course, $\rho > 0$, $T_0 > 0$. We assume in addition that $c_V > 0$ and that for positive definite strain energy density function the elastic constants appearing in Eq. (5.2) must satisfy the inequalities [22]

$$c_{11} > 0,\ c_{11}^2 > c_{12}^2,\ c_{44} > 0,\ c_{33}(c_{11} + c_{12}) > 2c_{13}^2. \tag{5.5}$$

The boundary conditions of zero traction are

$$\sigma_{3i} = 0\ ,\ i = 1,3, \quad \text{on the plane } x_3 = 0. \tag{5.6}$$

Usual requirements that the displacement and the stress components decay away from the boundary implies

$$u_i \to 0,\ \ \sigma_{ij} \to 0,\ \ (i, j = 1,3),\ \ as\ x_3 \to -\infty. \tag{5.7}$$

5.3 Solution of the problem

Following Pham & Ogden [12] and Chadwick [21] we may assume

$$\begin{aligned} u_j &= \varphi_j(y)\exp[ik(x_1 - ct)], \quad j = 1,3, \\ \theta &= k\psi(y)\exp[ik(x_1 - ct)], \end{aligned} \tag{5.8}$$

where $y = kx_3$,

in which k is the wave number, c is the wave speed, φ_j, ψ are the functions to be determined.

Substituting Eqs.(5.8) in (5.3) and (5.4) we have

$$\begin{aligned} &(\rho c^2 - c_{11})\varphi_1 + i(c_{13} + c_{44})\varphi_3' + c_{44}\varphi_1'' - i\beta_1\psi = 0, \\ &(\rho c^2 - c_{44})\varphi_3 + i(c_{13} + c_{44})\varphi_1' + c_{33}\varphi_3'' - \beta_2\psi' = 0, \\ &\kappa_2 k\psi'' - (\kappa_1 k - i\rho c_V c)\psi = 0. \end{aligned} \tag{5.9}$$

The boundary conditions (5.6) and (5.7) may be written as

$$\begin{aligned} &ic_{13}\varphi_1 + c_{33}\varphi_3' - \beta_2 k\psi(y) = 0, \\ &\varphi_1' + i\varphi_3 = 0, \end{aligned} \quad \text{on the plane, } y=0 \tag{5.10}$$

and

$$\varphi_j, \varphi_j', \psi, \psi' \to 0,\ j = 1, 3 \quad as\ y \to -\infty . \tag{5.11}$$

Imposing the conditions (5.11), we assume from (5.9) that

$$\begin{aligned} &\varphi_1(y) = A\exp(s_1 y) + B\exp(s_2 y) + C\exp(s_3 y), \\ &\varphi_3(y) = \alpha_1 A\exp(s_1 y) + \alpha_2 B\exp(s_2 y) + \alpha_3 C\exp(s_3 y), \\ &\psi(y) = \gamma_3 C\exp(s_3 y), \end{aligned} \tag{5.12}$$

where s_1, s_2, and s_3 have positive real parts, and

$$\alpha_j = -\frac{i[\{\beta_2(\rho c^2 - c_{11}) + \beta_1(c_{13} + c_{44})\}s_j + c_{44}\beta_2 s_j^{\ 3}]}{\{\beta_1 c_{33} - \beta_2(c_{13} + c_{44})\}s_j^{\ 2} + \beta_1(\rho c^2 - c_{44})}, \quad j = 1,2,3,$$

$$\gamma_j = -\frac{\begin{aligned} c_{33}c_{44}s_3^{\ 4} + [c_{33}(\rho c^2 - c_{11}) + c_{44}(\rho c^2 - c_{44}) + \\ (c_{13} + c_{44})^2]s_3^{\ 2} + (\rho c^2 - c_{11})(\rho c^2 - c_{44}) \end{aligned}}{\{\beta_1 c_{33} - \beta_2(c_{13} + c_{44})\}s_j^{\ 2} + \beta_1(\rho c^2 - c_{44})}.$$

In Eqs. (5.12) $s_1^{\ 2}, s_2^{\ 2}$ are the roots of the equation

$$\begin{aligned} c_{33}c_{44}s^4 + [c_{33}(\rho c^2 - c_{11}) + c_{44}(\rho c^2 - c_{44}) + (c_{13} + c_{44})^2]s^2 + \\ (\rho c^2 - c_{11})(\rho c^2 - c_{44}) = 0, \end{aligned} \tag{5.13}$$

where

$$s_1^{\,2}+s_2^{\,2}=-\frac{c_{33}(\rho c^2-c_{11})+c_{44}(\rho c^2-c_{44})+(c_{13}+c_{44})^{\,2}}{c_{33}c_{44}},$$
$$s_1^{\,2}s_2^{\,2}=\frac{(\rho c^2-c_{11})(\rho c^2-c_{44})}{c_{33}c_{44}},\qquad(5.14)$$

and
$$s_3^{\,2}=\frac{\kappa_1 k-i\rho c_V c}{\kappa_2 k}.\qquad(5.15)$$

5.4 Necessary condition for wave propagation

Arguing the same as that in deriving Eqs. (2.22) from (2.18) and (2.21) we can obtain the following inequality from Eqs. (5.12) and (5.14)

$$0<\rho c^2<\min\{c_{11},c_{44}\}.\qquad(5.16)$$

It is evident from (5.16) and (2.23) that the necessary condition for Rayleigh wave propagation in transversely isotropic material does not change under thermal effect.

5.5 Secular equation

Substituting the solutions (5.12) into the boundary conditions (5.10) and the thermal boundary condition ([21] or using (1.42) by taking g (t) =0)

$$\frac{\partial \theta}{\partial y} + h\theta = 0, \quad \text{on the plane, } y = 0, \tag{5.17}$$

where h is non-negative thermal constant, we obtain

$$\begin{aligned}
&(ic_{13} + c_{33}s_1\alpha_1)A + (ic_{13} + c_{33}s_2\alpha_2)B + (ic_{13} + c_{33}s_3\alpha_3 - \beta_2\gamma_3 k)C = 0,\\
&(s_1 + i\alpha_1)A + (s_2 + i\alpha_2)B = 0,\\
&k(s_3 + h)C = 0.
\end{aligned} \tag{5.18}$$

The last equation of Eqs. (5.18) gives C=0 (because if $C \neq 0$, then absurd values of h and c are obtained.)

Therefore, from Eqs. (5.18) we obtain

$$\begin{aligned}
&(ic_{13} + c_{33}s_1\alpha_1)A + (ic_{13} + c_{33}s_2\alpha_2)B = 0,\\
&(s_1 + i\alpha_1)A + (s_2 + i\alpha_2)B = 0.
\end{aligned} \tag{5.19}$$

The condition of consistency between these two homogeneous equations is

Now substituting the values $s_1^2 s_2^2$, $s_1^2 + s_2^2$, $s_1 s_2$, $s_1^3 s_2^3$ *and* $s_1 s_2 (s_1^2 + s_2^2)$ from Eqs. (5.14) and elastic and thermal constants from Table-5.1 into the Eq. (5.20) and by using the computer software Mathematica we obtain the values of the Rayleigh wave speeds in the given three transversely isotropic materials. We accept only those values of speeds which satisfy the inequality (5.16) and are shown in Table-5.2.

Table-5.1

Basic data for single crystals of three metals

Quantity	Units	Cobalt	Magnesium	Zinc
ρ_0	$kg\,m^{-3}$	8.836×10^{3}	1.74×10^{3}	7.14×10^{3}
T_0	^0K	298	298	296
c_{11}	$N\,m^{-2}$	3.071×10^{11}	5.974×10^{10}	1.628×10^{11}
c_{12}	$N\,m^{-2}$	1.650×10^{11}	2.64×10^{10}	0.362×10^{11}
c_{13}	$N\,m^{-2}$	1.027×10^{11}	2.17×10^{10}	0.508×10^{11}
c_{33}	$N\,m^{-2}$	3.581×10^{11}	6.17×10^{10}	0.627×10^{11}
c_{44}	$N\,m^{-2}$	0.755×10^{11}	1.639×10^{10}	0.385×10^{11}
β_1	$N\,m^{-2}\deg^{-1}$	7.04×10^{6}	2.68×10^{6}	5.75×10^{6}
β_2	$N\,m^{-2}\deg^{-1}$	6.90×10^{6}	2.68×10^{6}	5.17×10^{6}
c_V	$J\,kg^{-1}\deg^{-1}$	4.27×10^{2}	1.04×10^{3}	3.9×10^{2}
κ_1	$W\,m^{-1}\deg^{-1}$	0.690×10^{2}	1.7×10^{2}	1.24×10^{2}
κ_2	$W\,m^{-1}\deg^{-1}$	0.690×10^{2}	1.7×10^{2}	1.24×10^{2}

Table-5.2

Rayleigh wave speed under thermal effect

Material	Rayleigh Wave Speed (m/s)
Cobalt	2811.58, 2923.11
Magnesium	2894.65, 3069.13
Zinc	2045.01, 2322.1

Rayleigh wave speed without thermal effect for the said materials is shown in Table-5.3 (see [25]).

Table-5.3

Rayleigh wave speed without thermal effect

Materials	Rayleigh wave speed (m/s)
Cobalt	2811.58
Magnesium	2894.65
Zinc	2045.01

Comparing Table-5.2 and Table-5.3 we observe that one extra Rayleigh wave with some higher speed propagates under thermal effect.

5.7 Conclusions

Rayleigh wave speed in some model transversely isotropic materials, under thermal and without thermal effect, is calculated. It is observed that two Rayleigh waves propagate under thermal effect. One wave propagates with the same speed as that of the wave which propagates without thermal effect and the other one propagates with some higher speed. It is also observed that the necessary condition for Rayleigh wave propagation in transversely isotropic material remains same under thermal and without thermal effect.

Chapter 6

Concluding remarks of the thesis

This dissertation aims to contribute to some problems of Rayleigh waves in semi-infinite linear homogeneous transversely isotropic and orthotropic materials. The whole thesis may be summarized as follows:

1. It is shown that Rayleigh waves can propagate in different types of transversely isotropic and orthotropic materials with different speeds depending upon elastic constants and densities (See tables-2.2, 2.3 on page 38, 39, 40) provided the conditions of propagation i-e., $0 < \rho c^2 < \min\{c_{11}, c_{44}\}$ and $0 < \rho c^2 < \min\{c_{11}, c_{55}\}$ are satisfied for both types of materials respectively.

2. Rotational effects of both types of materials on the propagation of the waves are investigated and some practical results are shown in tables-3.1, 3.2 (on page 47, 48).

It is observed that if the frequency of rotation $\rho\dfrac{\Omega^2}{k^2}$ has the range $0 \le \rho\dfrac{\Omega^2}{k^2} \le c_{11}$, then the Rayleigh wave can propagate in both types of transversely isotropic and orthotropic materials and the speed of wave decreases slightly at a very high frequency of rotation less than or equal to c_{11}.

In table-3.1, for example, it is shown that if $0 \le \rho \frac{\Omega^2}{k^2} \le 10^{10}$, then the speed of the wave slightly slows down with the increase of rotation and the wave can propagate in all of the given transversely isotropic materials, Cobalt, Cadmium, Titanium boride, Zinc and Magnesium. If $\rho \frac{\Omega^2}{k^2} = 10^{11}$, the Raleigh wave does not propagate in Cadmium and Magnesium. If $\rho \frac{\Omega^2}{k^2} > 10^{11}$, the Rayleigh wave does not propagate in any of the given said materials.

In table-3.2, similar types of results are obtained. That is, if $0 \le \rho \frac{\Omega^2}{k^2} \le 10^{10}$, the velocity of the waves slows down with the increase of rotation of the given orthotropic materials, Iodic acid and Barium sodium niobate. If $\rho \frac{\Omega^2}{k^2} = 10^{11}$, the wave does not exist in Iodic acid. If $\rho \frac{\Omega^2}{k^2} > 10^{11}$, then the wave does not exist in both of the given said materials.

3. Rotational effects on Rayleigh waves propagation in incompressible transversely isotropic and orthotropic materials are studied. It is observed that the necessary conditions of propagation of Rayleigh waves in both of the materials i-e., $\rho c^2 < \mu_L$ and $\rho c^2 < c_{66}$ respectively are independent of rotation. The speed first begins to decrease slightly at a certain very high frequency of rotation $\frac{\Omega^2}{k^2}$, then begins to increase slightly and at last becomes constant by increasing further frequency of rotation.

Table-4.1 (on page 61) for example, shows that for the incompressible transversely isotropic material, Carbon-fibre-epoxy, the Rayleigh wave speed begins to decrease very slightly at a very high frequency of rotation of the order 10^6. This process continues till the frequency of rotation becomes of the order of 10^8. At the frequency of rotation of the order of 10^9 the speed begins to increase very slightly and attains the maximum value when the frequency of rotation becomes of the order of10^{10}. The speed does not change when the frequency of rotation becomes greater than of the order of 10^{10}.

Table-4.3 (on page63) shows the rotational effects on the Rayleigh waves speed in incompressible orthotropic material, Iodic acid. It is shown that at a high frequency of rotation of the order of 10^{3}, the speed begins to decrease very slightly. This process remains continuous till the frequency becomes of the order of 10^{7}. At the frequency of the order of 10^{8}, the speed begins to increase very slightly and attains the maximum value when the frequency becomes of the order of 10^{10}. The speed does not change when the frequency becomes greater than of the order of 10^{10}.

4. Thermal effects on Rayleigh waves speed in transversely isotropic materials are investigated. It is observed that the necessary condition for the propagation of waves i-e., $0 < \rho c^2 < \min\{c_{11}, c_{44}\}$ remain same when thermal effects are taken into account. It is also observed that two Rayleigh waves propagate under thermal effect in transversely isotropic materials. One wave propagates with the speed with which the Rayleigh wave propagates without thermal effect and other wave propagates with some higher speed. For example, see Tables-5.2, 5.3 (on page 75) for different specimen of transversely isotropic materials.

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