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Hemirings characterized by the properties of their generalized fuzzy ideals

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We accept this dissertation as conforming to the required standard

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2010

Dedicated To

My loving Parents

and

My cute niece

Amina

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In the name of ***Allah*** who is kind and merciful.

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Abstract

The fundamental concept of a fuzzy set, introduced by L.A. Zadeh in his classic paper [16] of 1965, provides a natural framework for generalizing some of the basic notions of algebra. In [12], Rosenfeld formulated the elements of a theory of fuzzy groups. Bhakat and Das generalized Rosenfeld's fuzzy subgroups and introduced the (α,β)-fuzzy subgroups by using the notion of "belongingness" and "quasi-coincidence" of fuzzy point and fuzzy set, which was introduced by Ming et al. [9].

In this dissertation we characterized the regular, weakly regular and some other related classes of hemirings using $(\in,\in\vee q)$, $(\bar{\in},\bar{\in}\vee\bar{q})$-fuzzy ideals and fuzzy ideals with thresholds $(\alpha,\beta]$. This dissertation consists of three chapters.

Chapter 1 is of an introductory nature. In **chapter 2**, we characterize different classes of hemirings by the properties of their $(\in,\in\vee q)$-fuzzy ideals. In **chapter 3**, we characterize different classes of hemirings by the properties of their $(\bar{\in},\bar{\in}\vee\bar{q})$-fuzzy ideals and by fuzzy ideals with thresholds $(\alpha,\beta]$.

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Chapter 1

Fundamental concepts

In this introductory chapter we present a brief summary of basic concepts of semirings and review some of the background material that will be of value for our latter work. For undefined terms and notations of semirings, we refer to [6] and [7].

1.1 Basic concepts in semirings

1.1.1 Definition

A **semiring** is a nonempty set R together with two binary operations addition "+" and multiplication "$\cdot$" such that $(R, +)$ and $(R, \cdot)$ are semigroups and both algebraic structures are connected by the distributive laws:

$a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in R$.

1.1.2 Definition

An element $0 \in R$ is called a **zero** of the semiring $(R, +, \cdot)$ if $0x = x0 = 0$ and $0 + x = x + 0 = x$ for all $x \in R$.

1.1.3 Definition

An additively commutative semiring with zero is called a **hemiring**.

1.1.4 Definition

An element 1 of a semiring R is called **identity** of R if $1x = x1 = x$ for all $x \in R$.

1.1.5 Definition

A semiring R is said to be commutative if '$\cdot$' is commutative that is $x \cdot y = y \cdot x$ for all $x, y \in R$.

1.1.6 Definition

An element 'a' of a semiring R is called a **left absorbing** element if and only if $ra = a$ for all $0 \neq r \in R$. Right absorbing element is defined analogously.

An element "a" of a semiring R which is both a left and a right absorbing element is called an **absorbing element**.

1.1.7 Examples

(1) All rings are semirings.

(2) if $(L, \vee, \wedge)$ is a distributive lattice with 0 and 1, then L is a semiring with $+ = \vee$ and $\cdot = \wedge$.

(3) Let $A = [0, 1]$, the unit closed interval of real numbers. Then A is a semiring with $+ =$ max and $\cdot =$ min or with $+ =$ min and $\cdot =$ max or, even $+ =$ max and $\cdot =$ usual product of real numbers.

(4) Let $\mathbb{N}_\circ$ be the set of all whole numbers, then $\mathbb{N}_\circ$ is a semiring with ordinary addition and multiplication of numbers.

(5) Let $(A, +)$ be a commutative semigroup with zero "0". Then $(A, +, \cdot)$ is a semiring where '$\cdot$' is defined as $x \cdot y = 0$ for all $x, y \in A$.

1.1.8 Definition

A nonempty subset A of a semiring R is called a **subsemiring** of R if it is itself a semiring with respect to the induced operations of R.

1.1.9 Examples

(1) $(\mathbb{N}, +, \cdot)$ is a subsemiring of $(R^+, +, \cdot)$.

(2) Let $S = \text{sub}(B) =$ set of all subsets of an infinite set B. Then $(S, \cap, \cup)$ is a semiring. Let $A = f\ \text{sub}(B) =$ set of all finite subsets of B, then $(A, \cap, \cup)$ is a subsemiring of $(S, \cap, \cup)$.

(3) $(Q^+, +, \cdot)$ is a subsemiring of $(R^+, +, \cdot)$.

1.1.10 Examples

Let $M_{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in Z^+ \right\}$ be the semiring with identity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Let $A = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in Z^+ \right\}$, then A is a subsemiring of $M_{2\times 2}$ but its identity is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ which is different from $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. So it is not necessary that the subsemiring contains the identity of semiring R.

1.1.11 Definition

A nonempty subset A of a hemiring R is called a **subhemiring** of R if it contains zero and is closed with respect to addition and multiplication of R.

1.2 Ideals in semirings

1.2.1 Definition

A nonempty subset I of a semiring R is called a **left ideal** of R if it satisfies the following two conditions,

(1) $a + b \in I$, for all $a, b \in I$.

(2) $RI \subseteq I$, that is if $a \in I$ and $r \in R$, then $ra \in I$.

A right ideal of R is defined analogously.

1.2.2 Definition

A nonempty subset of a semiring R which is both left and right ideal of R is called an ideal of R.

1.2.3 Examples

(1) The set of all left absorbing element of a semiring R is an ideal of R.

(2) Let A be a nonempty subset of a semiring R. Let $(0 : A) = \{r \in R : ra = 0 \text{ for all } a \in A\}$. If $A \neq \{0\}$ then $(0 : A)$ is a left ideal of R, called the **left annihilator** ideal of A.

Right annihilator ideal of A can be defined similarly.

1.2.4 Theorem

Let $\{I_i : i \in \wedge\}$ be a family of right (left) ideals of a semiring $(R, +, \cdot)$, then $\bigcap_{i \in \wedge} I_i$ is an ideal of $(R, +, \cdot)$.

1.2.5 Corollary

Let $\{I_i : i \in \wedge\}$ be a family of ideals of a semiring $(R, +, \cdot)$, then $\bigcap_{i \in \wedge} I_i$ is an ideal of $(R, +, \cdot)$.

1.2.6 Definition

Let X be a non empty subset of R, the smallest left (right) ideal of R which contains X is called the left (right) ideal of R **generated by** X and is denoted by $(X)_L$ (resp. $(X)_R$).

1.2.7 Definition

If X is a finite set, then the left (right or two-sided) ideal generated by X is called the **finitely generated left** (right or two-sided) ideal.

1.2.8 Definition

If X is a singleton set, then the left (right or two-sided) ideal generated by X is called the **principal left** (right or two-sided) ideal.

1.2.9 Proposition

If the semiring $(R, +, \cdot)$ contains the multiplicative identity, then

(1) The right ideal generated by X is XR.

(2) The left ideal generated by X is RX.

(3) The two-sided ideal generated by X is RXR.

1.2.10 Definition

An ideal I of a semiring R is called **prime** if $HK \subseteq I$, implies that either $H \subseteq I$ or $K \subseteq I$ for ideals H and K of R.

1.2.11 Theorem

Let I be an ideal of a semiring R, then the following conditions are equivalent:

(1) I is prime.

(2) $\{arb : r \in R\} \subseteq I$ if and only if $a \in I$ or $b \in I$.

(3) If $a, b \in R$ such that $(a)(b) \subseteq I$ then either $a \in I$ or $b \in I$.

1.2.12 Definition

An ideal I of a semiring R is called **irreducible** if $I = H \cap K$, implies that either $H = I$ or $K = I$ for ideals H and K of R.

1.2.13 Theorem

Let "a" be an element of a semiring R and I be an ideal of R not containing a, then there exist an irreducible ideal H of R containing I and not cantaining a.

1.3 Regular and Weakly regular semirings

Regular rings were introduced by **von Neumann** in **1936,** in order to clarify certain aspects of operator algebra. Moreover **V.S. Rammamurthy** in [11], studied weakly regular rings whose anologue weakly regular semiring was discussed by **J. Ahsan** in [3] with some useful characterizetions.

1.3.1 Definition

Let R be a semiring. An element $a \in R$ is called **regular** if there exists some element $a' \in R$ such that $aa'a = a$.

1.3.2 Definition

A semiring R is called a **regular semiring** if every element of R is regular.

1.3.3 Proposition

Let R be a semiring then the following conditions are equivalent:

(1) R is regular.

(2) For every right ideal H and left ideal K of R, $HK = H \cap K$.

1.3.4 Definition [2]

A semiring R is called **right weakly regular** if for all $x \in R, x \in (xR)^2$.

A commutative right weakly regular semiring is a regular semiring.

1.3.5 Proposition

For a semiring R with identity, the following conditions are equivalent:

(1) R is right weakly regular.

(2) $H^2 = H$, for all right ideals H of R.

(3) $HJ = H \cap J$, for all right ideals H and two-sided ideal J of R.

1.4 Fully idempotent semirings

1.4.1 Definition

A semiring R is called fully idempotent if each (two-sided) ideal of R is idempotent, that is $I^2 = I$ for all ideals I of R.

1.4.2 Proposition

The following assertions for a semiring R are equivalent:

(1) R is fully idempotent.

(2) For each pair of ideals I, J of R, $I \cap J = IJ$.

(3) For each right ideal H and two-sided ideal J, $H \cap J \subseteq JH$.

(4) For each left ideal K and two-sided ideal $J, K \cap J \subseteq KJ$.

The following proposition shows that the concept of prime and irreducible ideals concide for a fully idempotent semirings.

1.4.3 Proposition

Let R be a fully idempotent semiring. Then the following conditions for an ideal I of R are equivalent:

(1) I is irreducible.

(2) I is prime.

1.4.4 Proposition

Let R be a semiring. Then the following conditions are equivalent:

(1) R is fully idempotent.

(2) Each proper ideal of R is the intersection of prime ideals of R which contain it.

1.5 Fuzzy semirings

The theory of fuzzy sets was introduced by **L.A. Zadeh** in his classic paper [16] of **1965.** Since its inception, the theory of fuzzy sets has been developed in many directions and is finding applications in a wide varity of fields. The notion of fuzzy set has been used in computer science and applied to various branches of mathematics, including algebra, topology and probability theory.

In this section we give the results of fuzzy semiring from [2] and [4].

1.5.1 Fuzzy subsets

Let X be a nonempty set. By a fuzzy subset f of X, we mean a membership function $f : X \to [0,1]$ which associates with each element in X a real number from the unit closed interval $[0,1]$, the value $f(x)$ represents the " grade of membership" of x in f.

A fuzzy subset $f : X \to [0,1]$ is called nonempty if f is not a constant map which assumes the value 0. For any subsets f, g of X, $f \leq g$ means that for all $x \in X, f(x) \leq g(x)$. The symbols $f \wedge g$ and $f \vee g$ will mean the following fuzzy subsets of X:

$$(f \wedge g)(x) = f(x) \wedge g(x)$$

$$(f \vee g)(x) = f(x) \vee g(x) \qquad \text{for all } x \in X.$$

More generally, if $\{f_i : i \in I\}$ is a family of fuzzy subsets of X, then the fuzzy subsets $\bigwedge_{i \in I} f_i$ and $\bigvee_{i \in I} f_i$ are defined by

$$\left(\bigwedge_{i \in I} f_i\right)(x) = \bigwedge_{i \in I} f_i(x) \quad \text{and}$$

$$\left(\bigvee_{i \in I} f_i\right)(x) = \bigvee_{i \in I} f_i(x) \qquad \text{for all } x \in R.$$

and will be called the intersection and the union of the family $\{f_i : i \in I\}$ of fuzzy subsets of X.

1.5.2 Definition

A fuzzy subset f of a semiring R is called a **fuzzy right ideal** of R, if the following conditions hold:

(1) $f(x+y) \geq f(x) \wedge f(y)$

(2) $f(xy) \geq f(x)$ for all $x, y \in R$.

Fuzzy left ideal is defined analogously.

By a fuzzy ideal of R, we mean a fuzzy subset of R which is both fuzzy right and fuzzy left ideal of R.

1.5.3 Definition

Suppose A is a subset of a set X. Then the function

$$C_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

is called the **characteristic** (indicator) function of A.

1.5.4 Proposition

Let A be a nonempty subset of a semiring R and C_A be the characteristic function of A. Then C_A is a fuzzy right (resp. left) ideal of R if and only if A is a right (resp. left) ideal of R.

1.5.5 Lemma

If f and g are two fuzzy (left, right or two sided) ideals of a semiring R, then $f \wedge g$ is a fuzzy (left, right or two sided) ideal of R.

1.5.6 Definition

Let f and g be two fuzzy ideals of a semiring R. The fuzzy subset $f + g$ (called the **sum of f and g**) of R is defined by,

$$(f+g)(x) = \bigvee_{x=y+z} [f(y) \wedge g(z)] \qquad \text{for all } x, y, z \in R.$$

1.5.7 Proposition

If f and g are two fuzzy (left, right or two sided) ideals of a semiring R, then $f + g$ is a fuzzy (left, right or two sided) ideal of R.

1.5.8 Definition

Let f and g be fuzzy ideals of a semiring R. The fuzzy subset fg (called the **product** of f and g) of R is defined by

$$(fg)(x) = \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \leq i \leq p} [f(y_i) \wedge g(z_i)] \right]$$

where $x, y_i, z_i \in R$ and $p \in \mathbb{N}$.

1.5.9 Proposition

If f and g are two fuzzy ideals of a semiring R, then fg is a fuzzy ideal of R.

1.5.10 Definition

A fuzzy ideal f of a semiring R is called **idempotent** if $ff = f^2 = f$.

1.5.11 Theorem

The following assertions for a semiring R are equivalent:

(1) R is fully idempotent.

(2) Each fuzzy ideal of R is idempotent.

(3) For each pair of fuzzy ideals f and g of R, $f \wedge g = fg$.

If R is assumed to be commutative (that is, $xy = yx$ for all $x, y \in R$), then the above asserations are equivalent to:

(4) R is von Neumann regular.

1.5.12 Theorem

The following assertions for a semiring R are equivalent:

(1) R is fully idempotent.

(2) The set of all fuzzy ideals of R (ordered by $\leq$) forms a distributive lattice $\mathcal{L}_R$ under the sum and intersection of fuzzy ideals with $f \wedge g = fg$, for each pair of fuzzy ideals f and g of R.

1.5.13 Definition

A fuzzy ideal h of a semiring R is called a fuzzy prime ideal of R if for any fuzzy ideals f and g of R, $fg \leq h$ implies either $f \leq h$ or $g \leq h$.

1.5.14 Definition

A fuzzy ideal h of a semiring R is called a fuzzy irreducible ideal of R if for fuzzy ideals f and g of R, $f \wedge g = h$ implies either $f = h$ or $g = h$.

1.5.15 Theorem

Let R be a fully idempotent semiring. For fuzzy ideal h of R, the following conditions are equivalent:

(1) h is a fuzzy prime ideal.

(2) h is a fuzzy irreducible ideal.

1.5.16 Lemma

Let R be a fully idempotent semiring. If f is a fuzzy ideal of R with $f(a) = \alpha$, where a is any element of R and $\alpha \in [0, 1]$, then there exists a fuzzy prime ideal h of R such that $f \leq h$ and $h(a) = \alpha$.

1.5.17 Theorem

The following assertions for a semiring R with 1_R are equivalent:

(1) R is fully idempotent.

(2) The set of all fuzzy ideals of R (ordered by $\leq$) forms a distributive lattice $\mathcal{L}_R$ under the sum and intersection of fuzzy ideals with $f \wedge g = fg$, for each pair of fuzzy ideals f and g of R.

(3) Each fuzzy ideal is the intersection of all those fuzzy prime ideals of R which contain it.

If R is assumed to be commutative (that is $xy = yx$ for all $x, y \in R$) then the above assertions are equivalent to:

(4) R is von Neumann regular.

1.5.18 Theorem

The following assertions for a semiring R with identity are equivalent:

(1) R is right weakly regular semiring.

(2) All right ideals of R are idempotent.

(3) $HK = H \cap K$ for all right ideals H and two-sided ideals K of R.

(4) All fuzzy right ideals of R are idempotent.

(5) $fg = f \wedge g$ for all fuzzy right f and all fuzzy two sided ideals g of R.

If R is assumed to be commutative (that is $xy = yx$ for all $x, y \in R$) then the above assertions are equivalent to:

(6) R is von Neumann regular.

Chapter 2

Characterizations of hemirings by the properties of their $(\in, \in \vee q)$ fuzzy ideals

The concept of fuzzy set, introduced by L.A. Zadeh in his classic paper [16], was applied by many researchers to generalize some of the basic concepts of algebra. Fuzzy semirings were first investigated in [2] and [4].

By a fuzzy subset f of R, we mean a map $f : R \rightarrow [0,1]$. A fuzzy subset f of the form

$$f(y) = \begin{cases} t \in (0,1] \text{ if } y = x \\ 0 \text{ if } y \neq x \end{cases}$$

is called a **fuzzy point** with the support x and the value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy subset f of the same set R, **Ming et al.** in [9] introduced the symbol $x_t \alpha f$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$.

For any fuzzy set f, the notation $x_t \in f$ means that $f(x) \geq t$. In the case $f(x) + t \geq 1$, we say that the fuzzy point x_t is quasicoincident with the fuzzy set f and write $x_t q f$. The symbol $x_t \in \vee q f$ means that $x_t \in f$ or $x_t q f$. Similarly, $x_t \in \wedge q f$ means $x_t \in f$ and $x_t q f$. $x_t \overline{\in} f$ and $x_t \overline{\in \vee q} f$ mean that $x_t \in f$ and $x_t \in \vee q f$ do not hold, respectively.

The case $\alpha = \in \wedge q$ must be omitted since for a fuzzy subset f of R such that $f(x) \leq 0.5$ for any $x \in R$, in the case $x_t \in \wedge q f$ we have $f(x) \geq t$ and $f(x) + t \geq 1$. Thus $1 < f(x) + t \leq f(x) + f(x) = 2f(x)$, which implies $f(x) \geq 0.5$. This means that $\{x_t : x_t \in \wedge q f\} = \emptyset$.

Throughout this chapter R will denote a **hemiring** with identity element 1.

2.1 $(\in, \in \vee q)$ −fuzzy ideals

2.1.1 Definition [5]

A fuzzy subset f of a hemiring R is said to be an $(\in, \in \vee q)$ fuzzy left (right) ideal of R if,

(1) $x_t, y_r \in f$ implies that $(x+y)_{\min\{t,r\}} \in \vee q f$.

(2) $x_t, \in f$ and $y \in R$ implies that $(yx)_t \in \vee q f$ (respectively, $(xy)_t \in \vee q f$) for all $x, y \in R$ and $t, r \in (0, 1]$.

2.1.2 Lemma [5]

Let f be a fuzzy subset of a hemiring R, $x, y \in R$ and $t, r \in (0,1]$. Then the following conditions are equivalent:

(1) $x_t, y_r \in f$ implies that $(x+y)_{\min\{t,r\}} \in \vee q f$.

(2) $f(x+y) \geq \min\{f(x), f(y), 0.5\}$.

2.1.3 Lemma [5]

Let f be a fuzzy subset of a hemiring R, $x, y \in R$ and $t, r \in (0,1]$. Then the following conditions are equivalent:

(1) $x_t, \in f$ and $y \in R$ implies that $(yx)_t \in \vee q f$ (respectively, $(xy)_t \in \vee q f$).

(2) $f(yx) \geq \min\{f(x), 0.5\}$, (respectively $f(xy) \geq \min\{f(x), 0.5\}$).

2.1.4 Definition

Let f and g be $(\in, \in \vee q)$ fuzzy ideals of a hemiring R. Then the product $f \odot_{0.5} g$ is defined as

$$(f \odot_{0.5} g)(x) = \left[\bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \leq i \leq p} [f(y_i) \wedge g(z_i)] \right] \right] \wedge 0.5 \text{ for all } x \in R.$$

2.1.5 Proposition

If f is an $(\in, \in \vee q)$ fuzzy left ideal and g is an $(\in, \in \vee q)$ fuzzy right ideal of R, then $f \odot_{0.5} g$ is an $(\in, \in \vee q)$ fuzzy ideal of R.

Proof. Let f be an $(\in, \in \vee q)$ fuzzy left ideal and g be an $(\in, \in \vee q)$ fuzzy right ideal of R. For any $x, x' \in R$,

$$(f \odot_{0.5} g)(x) \wedge (f \odot_{0.5} g)(x') \wedge 0.5$$
$$= \left(\left[\bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \le i \le p} (f(y_i) \wedge g(z_i)) \right] \right] \wedge 0.5 \right) \wedge \left(\left[\bigvee_{x'=\sum_{j=1}^{q} y'_j z'_j} \left[\bigwedge_{1 \le j \le q} (f(y'_j) \wedge g(z'_j)) \right] \right] \wedge 0.5 \right) \wedge 0.5$$
$$= \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \le i \le p} (f(y_i) \wedge g(z_i)) \right] \wedge \left[\bigvee_{x'=\sum_{j=1}^{q} y'_j z'_j} \left[\bigwedge_{1 \le j \le q} (f(y'_j) \wedge g(z'_j)) \right] \right] \wedge 0.5$$
$$= \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \bigvee_{x'=\sum_{j=1}^{q} y'_j z'_j} \left[\bigwedge_{1 \le i \le p} (f(y_i) \wedge g(z_i)) \right] \wedge \left[\bigwedge_{1 \le j \le q} (f(y'_j) \wedge g(z'_j)) \right] \wedge 0.5$$
$$\le \bigvee_{x+x'=\sum_{k=1}^{r} y''_k z''_k} \left[\bigwedge_{1 \le k \le r} (f(y''_k) \wedge g(z''_k)) \right] \wedge 0.5$$
$$= (f \odot_{0.5} g)(x + x').$$

This implies $(f \odot_{0.5} g)(x + x') \ge (f \odot_{0.5} g)(x) \wedge (f \odot_{0.5} g)(x') \wedge 0.5$.

Note that for an orbitrary expression form of $x + x'$ say $\sum_{k=1}^{r} y''_k z''_k$, it is not necessarily possible to write x as $\sum_{k=1}^{t} y''_k z''_k$ and x' as $\sum_{k=t+1}^{r} y''_k z''_k$.

Also,

$$(f \odot_{0.5} g)(x) \wedge 0.5$$
$$= \left[\bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \le i \le p} (f(y_i) \wedge g(z_i)) \right] \right] \wedge 0.5 \wedge 0.5$$
$$= \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \le i \le p} (f(y_i) \wedge (g(z_i) \wedge 0.5)) \right] \wedge 0.5$$
$$\le \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \le i \le p} (f(y_i) \wedge g(z_i a)) \right] \wedge 0.5$$
$$\le \bigvee_{xa=\sum_{j=1}^{q} y'_j z'_j} \left[\bigwedge_{1 \le j \le q} (f(y'_j) \wedge g(z'_j)) \right] \wedge 0.5$$
$$= (f \odot_{0.5} g)(xa).$$

Therefore, $(f \odot_{0.5} g)(xa) \ge (f \odot_{0.5} g)(x) \wedge 0.5$.

Hence, $f \odot_{0.5} g$ is an $(\in, \in \vee q)$ fuzzy right ideal of R.

Similarly, $f \odot_{0.5} g$ is an $(\in, \in \vee q)$ fuzzy left ideal of R. ■

2.1.6 Definition

Let f and g be $(\in, \in \vee q)$ fuzzy ideals of a hemiring R. Then the sum $f +_{0.5} g$ is defined as

$$(f +_{0.5} g)(x) = \left[\bigvee_{x=y+z} [f(y) \wedge g(z)]\right] \wedge 0.5 \quad \text{for all } x \in R.$$

2.1.7 Proposition

Let f and g be any $(\in, \in \vee q)$ fuzzy left (right) ideals of R then their sum $f +_{0.5} g$ is also an $(\in, \in \vee q)$ fuzzy left (right) ideal of R, respectively.

Proof. Let f and g be $(\in, \in \vee q)$ fuzzy left ideals of R. For any $x, x' \in R$,

$$\begin{aligned}
&(f +_{0.5} g)(x) \wedge (f +_{0.5} g)(x') \wedge 0.5 \\
&= \left(\left[\bigvee_{x=y+z} (f(y) \wedge g(z))\right] \wedge 0.5\right) \wedge \left(\left[\bigvee_{x'=y'+z'} (f(y') \wedge g(z'))\right] \wedge 0.5\right) \wedge 0.5 \\
&= \bigvee_{\substack{x=y+z \\ x'=y'+z'}} [[f(y) \wedge g(z)] \wedge [f(y') \wedge g(z')]] \wedge 0.5 \\
&= \bigvee_{\substack{x=y+z \\ x'=y'+z'}} [[f(y) \wedge f(y') \wedge 0.5] \wedge [g(z) \wedge g(z') \wedge 0.5]] \wedge 0.5 \\
&\leq \bigvee_{\substack{x=y+z \\ x'=y'+z'}} [f(y+y') \wedge g(z+z')] \wedge 0.5 \\
&\leq \bigvee_{x+x'=a+b} [f(a) \wedge g(b)] \wedge 0.5 \\
&\leq (f +_{0.5} g)(x + x').
\end{aligned}$$

This implies, $(f +_{0.5} g)(x + x') \geq (f +_{0.5} g)(x) \wedge (f +_{0.5} g)(x') \wedge 0.5.$

Again,

$$\begin{aligned}
&(f +_{0.5} g)(x) \wedge 0.5 \\
&= \left[\bigvee_{x=y+z} (f(y) \wedge g(z))\right] \wedge 0.5 \wedge 0.5 \\
&= \bigvee_{x=y+z} [[f(y) \wedge 0.5] \wedge [g(z) \wedge 0.5]] \wedge 0.5 \\
&\leq \bigvee_{x=y+z} [f(ay) \wedge g(az)] \wedge 0.5
\end{aligned}$$

(where a is any element of R)

$\leq \bigvee_{ax=y'+z'} [f(y') \wedge g(z')] \wedge 0.5$

$= (f +_{0.5} g)(ax)$.

This implies, $(f +_{0.5} g)(ax) \geq (f +_{0.5} g)(x) \wedge 0.5$.

Hence, $f +_{0.5} g$ is an $(\in, \in \vee q)$ fuzzy left (right) ideal of semiring R. ∎

2.1.8 Corollary

For $(\in, \in \vee q)$ fuzzy ideals f and g of R, $f +_{0.5} g$ is also an $(\in, \in \vee q)$ fuzzy ideal of R.

2.1.9 Definition

Let f be a fuzzy subset of a hemiring R. We define the lower part of f as follows,

$f^-(x) = f(x) \wedge 0.5$ for all $x \in R$.

2.1.10 Lemma

If f is an $(\in, \in \vee q)$ fuzzy ideal of R, then f^- is also an $(\in, \in \vee q)$ fuzzy ideal of R.

Proof. Let f be an $(\in, \in \vee q)$ fuzzy ideal of R and $x, y \in R$. Then

$$\begin{aligned} f^-(x) \wedge f^-(y) \wedge 0.5 &= [f(x) \wedge 0.5 \wedge f(y) \wedge 0.5] \wedge 0.5 \\ &= [f(x) \wedge f(y) \wedge 0.5] \wedge 0.5 \\ &\leq f(x+y) \wedge 0.5 \\ &= f^-(x+y). \end{aligned}$$

$$\text{Thus,} \quad f^-(x+y) \geq f^-(x) \wedge f^-(y) \wedge 0.5.$$

Now

$$\begin{aligned} f^{-}(x) \wedge 0.5 &= [f(x) \wedge 0.5] \wedge 0.5 \\ &\leq f(xy) \wedge 0.5 \\ &= f^{-}(xy). \\ \text{Thus, } f^{-}(xy) &\geq f^{-}(x) \wedge 0.5. \end{aligned}$$

Similarly we can show, $f^{-}(xy) \geq f^{-}(y) \wedge 0.5$.

Hence, f^{-} is an $(\in, \in \vee q)$ fuzzy ideal of hemiring R. ∎

2.1.11 Definition

Let X be a nonempty subset of a hemiring R. Then the lower part of the characteristic function is,

$$C_X^{-}(x) = \begin{cases} 0.5 & \text{if } x \in X \\ 0 & \text{if } x \notin X \end{cases}$$

2.1.12 Lemma

If X and Y are subsets of a hemiring R, then $C_X^{-} = C_Y^{-}$ if and only if $X = Y$.

Proof. Obvious. ∎

2.1.13 Lemma

If X and Y are subsets of a hemiring R, then $C_X \odot_{0.5} C_Y = C_{XY} \wedge 0.5$.

Proof. Let a be any element of hemiring R.

If $a \in XY$, then there exist $x_i \in X$, and $y_i \in Y$ such that $a = \sum_{i=1}^{p} x_i y_i$. So,

$(C_X \odot_{0.5} C_Y)(a)$

$$= \bigvee_{a=\sum_{i=1}^{p} x_i y_i} \left[\bigwedge_{1 \le i \le p} [C_X(x_i) \wedge C_Y(y_i)] \right] \wedge 0.5$$

$$= \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \le i \le p} (1 \wedge 1) \right] \wedge 0.5$$

$= 1 \wedge 0.5.$

$= 0.5.$

Since, $a \in XY$. Therefore $C_{XY}(a) = 1$.

Hence, $C_{XY}(a) \wedge 0.5 = 1 \wedge 0.5 = 0.5$.

If $a \notin XY$, then $a \neq \sum_{i=1}^{p} x_i y_i$ for $x_i \in X$ and $y_i \in Y$.

If $a = \sum_{i=1}^{p} s_i t_i$ for some $s_i, t_i \in R$, then we have

$(C_X \odot_{0.5} C_Y)(a)$

$$= \bigvee_{a=\sum_{i=1}^{p} s_i t_i} \left[\bigwedge_{1 \le i \le p} [C_X(s_i) \wedge C_Y(t_i)] \right] \wedge 0.5$$

$= 0 \wedge 0.5$

$= 0.$

$= C_{XY}(a) \wedge 0.5.$

If $a \neq \sum_{i=1}^{p} s_i t_i$ for $s_i, t_i \in R$, then

$(C_X \odot_{0.5} C_Y)(a) = 0 = C_{XY}(a) \wedge 0.5.$

Thus in any case, we have $C_X \odot_{0.5} C_Y = C_{XY} \wedge 0.5$. ∎

2.1.14 Lemma

If X and Y are subsets of a hemiring R, then $C_X \wedge_{0.5} C_Y = C_{X \cap Y} \wedge 0.5$.

Proof. Obvious. ∎

2.1.15 Lemma

The characteristic function C_L of a nonempty subset L of R is an $(\in, \in \vee q)$ fuzzy left ideal of a hemiring R if and only if L is a left ideal of R.

Proof. Let L be a left ideal of the hemiring R.

Let $x, y \in R$, if $x, y \in L$ then $x + y \in L$.

Thus $C_L(x+y) = 1 \geq C_L(x) \wedge C_L(y) \wedge 0.5$.

If x or $y \notin L$ then $C_L(x) \wedge C_L(y) \wedge 0.5 = 0 \leq C_L(x+y)$.

This implies $C_L(x+y) \geq \min\{C_L(x), C_L(y), 0.5\}$.

Again,

If $y \in L$ then $xy \in L$.

Thus, $C_L(xy) = 1 \geq C_L(y) \wedge 0.5$.

If $y \notin L$ then $C_L(y) \wedge 0.5 = 0 \leq C_L(xy)$.

This implies $C_L(xy) \geq \min\{C_L(y), 0.5\}$.

Hence, C_L is an $(\in, \in \vee q)$ fuzzy left ideal of a hemiring R.

Conversely,

Let $x, y \in L$ this implies that $C_L(x) = 1 = C_L(y)$

Since, $C_L(x+y) \geq \min\{C_L(x), C_L(y), 0.5\} = \min\{1, 1, 0.5\} = 0.5$.

Therefore, $C_L(x+y) = 1$, this implies $x + y \in L$.

For second condition, let $x \in R$ and $y \in L$ this implies that $C_L(y) = 1$.

Since, $C_L(xy) \geq \min\{C_L(y), 0.5\}$

$$= \min\{1, 0.5\} = 0.5.$$

Therefore, $C_L(xy) = 1$ this implies $xy \in L$.

Hence, L is a left ideal of hemiring R. ■

2.1.16 Definition

Let f and g be $(\in, \in \vee q)$-fuzzy ideals of a hemiring R. Then the lower part of $f \wedge_{0.5} g$ is defined as, $(f \wedge_{0.5} g)(x) = (f \wedge g)(x) \wedge 0.5$ where $x \in R$.

2.1.17 Lemma

Let f be an $(\in, \in \vee q)$-fuzzy right ideal and g be an $(\in, \in \vee q)$ fuzzy left ideal of a hemiring R, then $f \odot_{0.5} g \leq f \wedge_{0.5} g$.

Proof. Let f and g be $(\in, \in \vee q)$ fuzzy right and fuzzy left ideals of R, respectively. For any $x \in R$,

$$
\begin{aligned}
&(f \odot_{0.5} g)(x) \\
&= \left[\bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \leq i \leq p} (f(y_i) \wedge g(z_i))\right]\right] \wedge 0.5 \\
&= \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \leq i \leq p} (f(y_i) \wedge 0.5) \wedge (g(z_i) \wedge 0.5)\right] \wedge 0.5 \\
&\leq \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \leq i \leq p} f(y_i z_i) \wedge g(y_i z_i)\right] \wedge 0.5 \\
&= \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\left[\bigwedge_{1 \leq i \leq p} f(y_i z_i) \wedge 0.5\right] \wedge \left[\bigwedge_{1 \leq i \leq p} g(y_i z_i) \wedge 0.5\right]\right] \wedge 0.5 \\
&\leq \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[f\left(\sum_{i=1}^{p} y_i z_i\right) \wedge g\left(\sum_{i=1}^{p} y_i z_i\right)\right] \wedge 0.5 \\
&= \bigvee_{x=\sum_{i=1}^{p} y_i z_i} [f(x) \wedge g(x)] \wedge 0.5 \\
&= [f(x) \wedge g(x)] \wedge 0.5 \\
&= (f \wedge g)(x) \wedge 0.5 \\
&= (f \wedge_{0.5} g)(x).
\end{aligned}
$$

This implies, $f \odot_{0.5} g \leq f \wedge_{0.5} g$. ■

2.2 Regular and Weakly regular hemirings

Recall that a hemiring R is **regular** if for each $x \in R$ there exists an $a \in R$ such that $x = xax$ and R is **right weakly regular** if each $x \in R$, $x \in (xR)^2$.

2.2.1 Lemma

A hemiring R is regular if and only if $f \odot_{0.5} g = f \wedge_{0.5} g$, for every $(\in, \in \vee q)$ fuzzy right ideal f and for every $(\in, \in \vee q)$ fuzzy left ideal g of R.

Proof. Let f and g be $(\in, \in \vee q)$ fuzzy right and fuzzy left ideals of R respectively. For $x \in R$,

$(f \odot_{0.5} g)(x) \leq (f \wedge_{0.5} g)(x)$ by Lemma 2.1.17.

On the other hand, since R is regular hemiring, so for every $x \in R$, there exists $s \in R$ such that $x = xsx$. Consider,

$$\begin{aligned} (f \wedge_{0.5} g)(x) &= (f \wedge g)(x) \wedge 0.5 \\ &= f(x) \wedge g(x) \wedge 0.5 \\ &\leq f(xs) \wedge g(x) \wedge 0.5 \\ &\leq \bigvee_{x=\sum_{i=1}^{q} a_i b_i} \left[\bigwedge_{1 \leq i \leq q} [f(a_i) \wedge g(b_i)] \right] \wedge 0.5 \\ &= (f \odot_{0.5} g)(x). \end{aligned}$$

$$\text{Thus,} \quad (f \wedge_{0.5} g)(x) \leq (f \odot_{0.5} g)(x).$$

$$\text{Hence,} \quad f \odot_{0.5} g = f \wedge_{0.5} g.$$

Conversely, let A be a right ideal and B be a left ideal of R. Then by Lemma 2.1.15, C_A and C_B are $(\in, \in \vee q)$ fuzzy right ideal and left ideal of R, respectively.

By hypothesis, $C_A \odot_{0.5} C_B = C_A \wedge_{0.5} C_B$. By Lemma 2.1.13 and 2.1.14 we have, $C_A \odot_{0.5} C_B = C_{AB} \wedge 0.5$ and $C_A \wedge_{0.5} C_B = C_{A \cap B} \wedge 0.5$ respectively.

This implies, $C_{AB} \wedge 0.5 = C_{A \cap B} \wedge 0.5$.

By Lemma 2.1.12 we get, $AB = A \cap B$.

Hence, by Proposition 1.3.3, R is a regular hemiring. ■

2.2.2 Definition

An $(\in, \in \vee q)$ fuzzy ideal f of R is said to be idempotent if $f \odot_{0.5} f = f \wedge 0.5 = f^-$.

2.2.3 Theorem

The following assertions for a hemiring R are equivalent;

(1) R is right weakly regular hemiring.

(2) All $(\in, \in \vee q)$ fuzzy right ideals of R are idempotent.

(3) $f \odot_{0.5} g = f \wedge_{0.5} g$ for all $(\in, \in \vee q)$ fuzzy right f and all $(\in, \in \vee q)$ fuzzy two sided ideals g of R.

Proof. (1) $\Longrightarrow$ (2)

Let f be an $(\in, \in \vee q)$ fuzzy right ideal of R. By Lemma 2.1.17, $f \odot_{0.5} f \leq f^-$.

On the other hand, since R is right weakly regular hemiring, so $x \in (xR)^2$.

Hence, $x = \sum_{i=1}^{q} xa_ixb_i$ for some $a_i, b_i \in R$ and $q \in \mathbb{N}$. So,

$$\begin{aligned}
f(x) &= f(x) \wedge f(x) \\
f(x) \wedge 0.5 &= f(x) \wedge f(x) \wedge 0.5 \\
f^-(x) &= (f(x) \wedge 0.5) \wedge (f(x) \wedge 0.5) \wedge 0.5 \\
&\leq f(xa_i) \wedge f(xb_i) \wedge 0.5 \quad \text{for } 1 \leq i \leq q \\
f^-(x) &\leq \bigwedge_{1 \leq i \leq q} [f(xa_i) \wedge f(xb_i)] \wedge 0.5 \\
&\leq \bigvee_{x=\sum_{j=1}^{r} y_jz_j} \left[\bigwedge_{1 \leq j \leq r} [f(y_j) \wedge f(z_j)] \right] \wedge 0.5 \\
&= (f \odot_{0.5} f)(x) \\
\text{Thus, } f^-(x) &\leq (f \odot_{0.5} f)(x). \\
\text{Hence, } f \odot_{0.5} f &= f^-.
\end{aligned}$$

This shows that, all $(\in, \in \vee q)$ fuzzy right ideals of R are idempotent.

$(2) \Longrightarrow (1)$

Let $x \in R$, we claim that $x \in (xR)^2$.

Let $xR = A$ be the right ideal generated by x, and C_A be the characteristic function of A, by Lemma 2.1.15 it is an $(\in, \in \vee q)$ fuzzy right ideal of R.

By our assumption, $C_A^- = C_A \odot_{0.5} C_A$

By Lemma 2.1.13, $C_A \odot_{0.5} C_A = C_{AA} \wedge 0.5$

So we have, $C_A \wedge 0.5 = C_{AA} \wedge 0.5$.

Therefore, by Lemma 2.1.12 we have, $A = AA$.

Now since $x \in A$, this imlpies $x \in A^2$.

Therefore, $x \in (xR)^2$.

Hence, R is a right weakly regular hemiring.

$(1) \Longrightarrow (3)$

Let f be an $(\in, \in \vee q)$ fuzzy right ideal and g be an $(\in, \in \vee q)$ fuzzy two sided of R. By Lemma 2.1.17, $f \odot_{0.5} g \leq f \wedge_{0.5} g$.

For reverse inclusion, let $x \in R$, since R is right weakly regular semiring, so $x \in (xR)^2$. Hence, $x = \sum_{i=1}^{p} xa_i xb_i$ for some a_i , $b_i \in R$ and $p \in \mathbb{N}$.

Consider,

$$\begin{aligned}
(f \wedge_{0.5} g)(x) &= (f \wedge g)(x) \wedge 0.5 \\
&= f(x) \wedge g(x) \wedge 0.5 \\
&\leq f(xa_i) \wedge g(xb_i) \wedge 0.5 \quad \text{for } 1 \leq i \leq p \\
(f \wedge_{0.5} g)(x) &\leq \bigwedge_{1 \leq i \leq p} [f(xa_i) \wedge g(xb_i)] \wedge 0.5 \\
&\leq \bigvee_{x = \sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \leq i \leq p} [f(y_i) \wedge g(z_i)] \right] \wedge 0.5 \\
&= (f \odot_{0.5} g)(x).
\end{aligned}$$

This implies, $(f \wedge_{0.5} g)(x) \leq (f \odot_{0.5} g)(x)$.

Hence, $f \wedge_{0.5} g = f \odot_{0.5} g$.

$(3) \Longrightarrow (1)$

Let A and B be the right and two sided ideals of R, respectively. Then by Lemma 2.1.15, the characteristic function C_A and C_B are $(\in, \in \vee q)$ fuzzy right ideal and fuzzy two sided ideals of R, respectively.

Hence, by hypothesis,$C_A \odot_{0.5} C_B = C_A \wedge_{0.5} C_B$.

By Lemma 2.1.13 and 2.1.14, we have $C_A \odot_{0.5} C_B = C_{AB} \wedge 0.5$ and $C_A \wedge_{0.5} C_B =$

$C_{A\cap B} \wedge 0.5$ respectively.

So we have, $C_{AB} \wedge 0.5 = C_{A\cap B} \wedge 0.5$.

Therefore, by Lemma 2.1.12 $AB = A \cap B$.

Hence, by Proposition 1.3.5, R is right weakly regular hemiring. ■

2.3 Hemirings in which each $(\in, \in \vee q)$ fuzzy ideal is idempotent

In this section we study those hemirings for which each $(\in, \in \vee q)$ −fuzzy ideal is idempotent.

2.3.1 Theorem

The following assertions for a hemiring R are equivalent;

(1) R is fully idempotent.

(2) Each $(\in, \in \vee q)$ fuzzy ideal of R is idempotent.

(3) For each pair of $(\in, \in \vee q)$ fuzzy ideals f and g of R, $f \wedge_{0.5} g = f \odot_{0.5} g$.

Proof. (1) $\Longrightarrow$ (3)

Let f and g be any pair of $(\in, \in \vee q)$ fuzzy ideals of R.

Then $f \odot_{0.5} g \leq f \wedge_{0.5} g$ (by Lemma 2.1.17).

Since, R is fully idempotent, so $(x) = (x)^2$, thus $x = \sum_{i=1}^{p} r_i x r_i' s_i x s_i'$ for some

$r_i, r_i', s_i, s_i' \in R$. Consider

$$\begin{aligned}(f \wedge_{0.5} g)(x) &= (f \wedge g)(x) \wedge 0.5 \\ &= f(x) \wedge g(x) \wedge 0.5 \\ &\leq f(r_i x r_i') \wedge g(s_i x s_i') \wedge 0.5 \\ &\leq \bigwedge_{1 \leq i \leq p} [f(r_i x r_i') \wedge g(s_i x s_i') \wedge 0.5] \\ &\leq \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \leq i \leq p} (f(y_i) \wedge g(z_i))\right] \wedge 0.5 \\ &= (f \odot_{0.5} g)(x).\end{aligned}$$

$$\text{Therefore,} \quad f \wedge_{0.5} g \leq f \odot_{0.5} g.$$

$$\text{Hence,} \quad f \wedge_{0.5} g = f \odot_{0.5} g.$$

(3) $\Longrightarrow$ (2)

Let f and g be any pair of $(\in, \in \vee q)$ fuzzy ideals of R. We have $f \wedge_{0.5} g = f \odot_{0.5} g$. Take $g = f$.

$$\begin{aligned}f \wedge_{0.5} f &= f \odot_{0.5} f \\ (f \wedge f)(x) \wedge 0.5 &= (f \odot_{0.5} f)(x) \\ (f \wedge f)(x) \wedge 0.5 &= (f \odot_{0.5} f)(x) \\ \text{This implies,} \quad f^{-} &= f \odot_{0.5} f.\end{aligned}$$

(2) $\Longrightarrow$ (1)

Let A be an ideal of R. Then by Lemma 2.1.15, C_A is an $(\in, \in \vee q)$ fuzzy ideal of R.

Hence, $C_A \odot_{0.5} C_A = C_A^{-}$.

But, $C_A \odot_{0.5} C_A = C_{AA} \wedge 0.5$ (by Lemma 2.1.13).

Therefore, $C_{AA} \wedge 0.5 = C_A \wedge 0.5$

This implies, $AA = A$ (by Lemma 2.1.12).

Hence, R is fully idempotent hemiring. ∎

2.3.2 Theorem

The following assertions for a hemiring R are equivalent;

(1) R is fully idempotent.

(2) The set $\mathcal{L}_R = \{f^- : f$ is an $(\in, \in \vee q)$ fuzzy ideal of $R\}$, (ordered by $\leq$) form a distributive lattice under the sum and intersection of $(\in, \in \vee q)$ fuzzy ideals with $f^- \wedge_{0.5} g^- = f^- \odot_{0.5} g^-$, for each pair of $(\in, \in \vee q)$ fuzzy ideals f^- and g^- of R.

Proof. (1) $\Longrightarrow$ (2)

The set $\mathcal{L}_R$ of $(\in, \in \vee q)$ fuzzy ideals of R (ordered by $\leq$) is cleary a lattice under the sum and intersection of $(\in, \in \vee q)$ fuzzy ideals of R. Moreover, since R is a fully idempotent hemiring, it follows that $f^- \wedge_{0.5} g^- = f^- \odot_{0.5} g^-$, for each pair of $(\in, \in \vee q)$ fuzzy ideals f^- and g^- of R. We now show that $\mathcal{L}_R$ is a distributive lattice, that is, for $(\in, \in \vee q)$ fuzzy ideals f^-, g^- and h^- of R,

we have $[(f^- \wedge_{0.5} g^-) +_{0.5} h^-] = [(f^- +_{0.5} h^-) \wedge_{0.5} (g^- +_{0.5} h^-)]$. For any $x \in R$,

$[(f^- \wedge_{0.5} g^-) +_{0.5} h^-] (x)$

$= \bigvee_{x=y+z} [(f^- \wedge_{0.5} g^-) (y) \wedge h^- (z)] \wedge 0.5$

$= \bigvee_{x=y+z} [(f^- \wedge g^-) (y) \wedge 0.5 \wedge h^- (z)] \wedge 0.5$

$= \bigvee_{x=y+z} [f^- (y) \wedge g^- (y) \wedge h^- (z) \wedge h^- (z)] \wedge 0.5$

$$= \bigvee_{x=y+z} [f^-(y) \wedge h^-(z) \wedge 0.5] \wedge [g^-(y) \wedge h^-(z) \wedge 0.5] \wedge 0.5$$

$$= \left[\bigvee_{x=y+z} [f^-(y) \wedge h^-(z) \wedge 0.5]\right] \wedge \left[\bigvee_{x=y+z} [g^-(y) \wedge h^-(z) \wedge 0.5]\right] \wedge 0.5$$

$$\leq [(f^- +_{0.5} h^-)(x) \wedge (g^- +_{0.5} h^-)(x)] \wedge 0.5$$

Because, for $x = y + z$, $f^-(y) \wedge h^-(z) \wedge 0.5 \leq (f^- +_{0.5} h^-)(x)$.

Similarly, $g^-(y) \wedge h^-(z) \wedge 0.5 \leq (g^- +_{0.5} h^-)(x)$.

Thus, $[(f^- \wedge_{0.5} g^-) +_{0.5} h^-](x) \leq [(f^- +_{0.5} h^-) \wedge_{0.5} (g^- +_{0.5} h^-)](x)$ $\quad (i)$

Again,

$$[(f^- +_{0.5} h^-) \wedge_{0.5} (g^- +_{0.5} h^-)](x)$$

$$= [(f^- +_{0.5} h^-) \odot_{0.5} (g^- +_{0.5} h^-)](x)$$

$$= \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1\leq i\leq p} [(f^- +_{0.5} h^-)(y_i) \wedge (g^- +_{0.5} h^-)(z_i)]\right] \wedge 0.5$$

$$= \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1\leq i\leq p} \left[\begin{matrix} \left[\bigvee_{y_i=r_i+s_i} f^-(r_i) \wedge h^-(s_i) \wedge 0.5\right] \wedge \\ \left[\bigvee_{z_i=t_i+u_i} g^-(t_i) \wedge h^-(u_i) \wedge 0.5\right] \end{matrix}\right]\right] \wedge 0.5$$

$$= \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1\leq i\leq p} \left[\bigvee_{\substack{y_i=r_i+s_i \\ z_i=t_i+u_i}} [f^-(r_i) \wedge h^-(s_i) \wedge g^-(t_i) \wedge h^-(u_i)]\right]\right] \wedge 0.5$$

$$= \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1\leq i\leq p} \left[\bigvee_{\substack{y_i=r_i+s_i \\ z_i=t_i+u_i}} [f^-(r_i) \wedge h^-(s_i) \wedge h^-(s_i) \wedge g^-(t_i) \wedge h^-(u_i) \wedge 0.5]\right]\right] \wedge$$

0.5

$$= \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1\leq i\leq p} \left[\bigvee_{\substack{y_i=r_i+s_i \\ z_i=t_i+u_i}} \left[\begin{matrix} (f^-(r_i) \wedge 0.5) \wedge (g^-(t_i) \wedge 0.5) \wedge (h^-(s_i) \wedge 0.5) \wedge \\ (h^-(s_i) \wedge 0.5) \wedge (g^-(t_i) \wedge 0.5) \wedge (h^-(u_i) \wedge 0.5) \end{matrix}\right]\right]\right] \wedge$$

0.5

$$\leq \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1\leq i\leq p} \left[\bigvee_{\substack{y_i=r_i+s_i \\ z_i=t_i+u_i}} [f^-(r_i t_i) \wedge g^-(r_i t_i) \wedge h^-(s_i t_i) \wedge h^-(s_i u_i) \wedge h^-(r_i u_i)]\right]\right] \wedge$$

0.5

$$= \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1\leq i\leq p} \left[\bigvee_{\substack{y_i=r_i+s_i \\ z_i=t_i+u_i}} \left[\begin{matrix} (f^-(r_i t_i) \wedge g^-(r_i t_i) \wedge 0.5) \wedge \\ (h^-(s_i t_i) \wedge h^-(s_i u_i) \wedge h^-(r_i u_i) \wedge 0.5) \end{matrix}\right]\right]\right] \wedge 0.5$$

$$\leq \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1\leq i\leq p} \left[\bigvee_{\substack{y_i=r_i+s_i \\ z_i=t_i+u_i}} [(f^- \wedge_{0.5} g^-)(r_i t_i) \wedge h^-(s_i t_i + s_i u_i + r_i u_i) \wedge 0.5] \right] \right] \wedge 0.5$$

$$\leq \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1\leq i\leq p} [(f^- \wedge_{0.5} g^-) +_{0.5} h^-](y_i z_i) \right]$$

$$\leq \bigvee_{x=\sum_{i=1}^{p} y_i z_i} [(f^- \wedge_{0.5} g^-) +_{0.5} h^-](x)$$

$$= [(f^- \wedge_{0.5} g^-) +_{0.5} h^-](x).$$

Thus, $[(f^- +_{0.5} h^-) \wedge_{0.5} (g^- +_{0.5} h^-)](x) \leq [(f^- \wedge_{0.5} g^-) +_{0.5} h^-](x)$. (ii)

From (i) and (ii) we get, $[(f^- \wedge_{0.5} g^-) +_{0.5} h^-] = [(f^- +_{0.5} h^-) \wedge_{0.5} (g^- +_{0.5} h^-)]$.

$(2) \Longrightarrow (1)$

Suppose that the set $\mathcal{L}_R = \{f^-$, where f is an $(\in, \in \vee q)$ fuzzy ideal of $R\}$, (ordered by $\leq$) is a distributive lattice under the sum and intersection of $(\in, \in \vee q)$ fuzzy ideals with $f^- \wedge_{0.5} g^- = f^- \odot_{0.5} g^-$ for each pair of $(\in, \in \vee q)$ fuzzy ideals f^- and g^- of R. Then for any $(\in, \in \vee q)$ fuzzy ideal f^- of R, we have, $f^- \odot_{0.5} f^- = f^- \wedge_{0.5} f^-$.

First we have to show $f^- \odot_{0.5} f^- = f \odot_{0.5} f$.

Consider, $(f^- \odot_{0.5} f^-)(x)$

$$= \left[\bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1\leq i\leq p} (f^-(y_i) \wedge f^-(z_i)) \right] \right] \wedge 0.5$$

$$= \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1\leq i\leq p} [f(y_i) \wedge 0.5 \wedge f(z_i) \wedge 0.5] \right] \wedge 0.5$$

$$= \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1\leq i\leq p} [f(y_i) \wedge f(z_i)] \right] \wedge 0.5.$$

$$= (f \odot_{0.5} f)(x).$$

Therefore, we get $f^- \odot_{0.5} f^- = f \odot_{0.5} f$.

This implies, $f \odot_{0.5} f = f^-$.

Hence, R is fully idempotent hemiring. ■

2.3.3 Definition

An $(\in, \in \vee q)$ fuzzy ideal h of a hemiring R is called an $(\in, \in \vee q)$ fuzzy prime ideal of R if for any $(\in, \in \vee q)$ fuzzy ideal f and g of R, $f \odot_{0.5} g \leq h$, implies that $f^- \leq h^-$ or $g^- \leq h^-$.

2.3.4 Definition

An $(\in, \in \vee q)$ fuzzy ideal h of a hemiring R is called an $(\in, \in \vee q)$ fuzzy irreducible ideal of R if for any $(\in, \in \vee q)$ fuzzy ideal f and g of R, $f \wedge_{0.5} g = h$ implies that $f^- = h^-$ or $g^- = h^-$.

2.3.5 Theorem

Let R be a fully idempotent hemiring. For any $(\in, \in \vee q)$ fuzzy ideal h^- of R, the following conditions are equivalent:

(1) h^- is an $(\in, \in \vee q)$ fuzzy prime ideal.

(2) h^- is an $(\in, \in \vee q)$ fuzzy irreducible ideal.

Proof. (1) $\Longrightarrow$ (2)

Let f^- and g^- be any $(\in, \in \vee q)$ fuzzy ideals of R. Assume that h^- is an $(\in, \in \vee q)$ fuzzy prime ideal. We show that h^- ia an $(\in, \in \vee q)$ fuzzy irreducible ideal of R. Let $h^- = f^- \wedge_{0.5} g^-$. Since R is fully idempotent, so by Theorem 2.3.1, $f^- \wedge_{0.5} g^- = f^- \odot_{0.5} g^-$. Since h^- is an $(\in, \in \vee q)$ fuzzy prime ideal, this implies $f^- \leq h^-$ or $g^- \leq h^-$.

Again, since $f^- \wedge_{0.5} g^- = h^-$, this implies $h^- \leq f^-$ and $h^- \leq g^-$. It follows that,

$f^- = h^-$ or $g^- = h^-$.

Hence, h^- is an $(\in, \in \vee q)$ fuzzy irreducible ideal.

$(2) \Longrightarrow (1)$

Assume that h^- is an $(\in, \in \vee q)$ fuzzy irreducible ideal. We have to show h^- is an $(\in, \in \vee q)$ fuzzy prime ideal. Suppose that there exist $(\in, \in \vee q)$ fuzzy ideals f^- and g^- such that $f^- \odot_{0.5} g^- \leq h^-$. As, R is fully idempotent hemiring, so by Theorem 2.3.1, $f^- \wedge_{0.5} g^- = f^- \odot_{0.5} g^-$. Thus, $f^- \odot_{0.5} g^- \leq h^-$ implies $f^- \wedge_{0.5} g^- \leq h^-$. Again, since R is fully idempotent hemiring, it follows from Theorem 2.3.2, that the set of $(\in, \in \vee q)$ fuzzy ideals of R (ordered by $\leq$) is a distributive lattice with respect to the sum and intersection of $(\in, \in \vee q)$ fuzzy ideals. Hence the inequality $f^- \wedge_{0.5} g^- \leq h^-$ becomes $(f^- \wedge_{0.5} g^-) +_{0.5} h^- = h^-$, and using the distributivety of this lattice we have,

$(f^- +_{0.5} h^-) \wedge_{0.5} (g^- +_{0.5} h^-) = h^-$. Since, h^- is an $(\in, \in \vee q)$ fuzzy irreducible ideal, it follows that either $f^- +_{0.5} h^- = h^-$ or $g^- +_{0.5} h^- = h^-$. This implies, $f^- \leq h^-$or $g^- \leq h^-$.

Hence, h^- is an $(\in, \in \vee q)$ fuzzy prime ideal of hemiring. ■

2.3.6 Lemma

Let R be a fully idempotent hemiring. If f^- is an $(\in, \in \vee q)$ fuzzy ideal of R with $f^-(a) = \alpha$, where a is any element of R and $\alpha \in (0, 1]$, then there exists an $(\in, \in \vee q)$ fuzzy prime ideal h^- of R such that $f^- \leq h^-$ and $h^-(a) = \alpha$.

Proof. Let $X = \{g^-; g^-$ is an $(\in, \in \vee q)$ fuzzy ideal of R, $g^-(a) = \alpha$, and $f^- \leq g^-\}$.

Then $X \neq \emptyset$, because $f^- \in X$. Let $\mathcal{F}$ be a totally ordered subset of X, say $\mathcal{F} = \{f_i^-, i \in I\}$. We claim that $\bigvee\limits_{i\in I} f_i^-$ is an $(\in, \in \vee q)$ fuzzy ideal of R. For $x, y \in R$. Consider,

$$
\begin{aligned}
&\bigvee_{i\in I} f_i^-(x) \wedge \bigvee_{i\in I} f_i^-(y) \wedge 0.5 \\
&= \bigvee_{i\in I} f_i^-(x) \wedge \bigvee_{j\in I} f_j^-(y) \wedge 0.5 \\
&= \left[\bigvee_{i\in I} f_i^-(x)\right] \wedge \bigvee_{j\in I} \left[f_j^-(y)\right] \wedge 0.5 \\
&= \bigvee_{j} \left[\left[\bigvee_{i\in I} f_i^-(x)\right] \wedge f_j^-(y)\right] \wedge 0.5 \\
&= \bigvee_{j} \left[\bigvee_{i} \left[f_i^-(x) \wedge f_j^-(y)\right]\right] \wedge 0.5 \\
&\leq \bigvee_{j} \left[\bigvee_{i} \left[f_i^{-j}(x) \wedge f_i^{-j}(y)\right]\right] \wedge 0.5 \ \text{ where } f_i^{-j} = \max\{f_i^-, f_j^-\} : f_i^{-j} \in \{f_i^-, i \in I\} \\
&= \bigvee_{j} \left[\bigvee_{i} \left[f_i^{-j}(x) \wedge f_i^{-j}(y) \wedge 0.5\right]\right] \\
&\leq \bigvee_{j} \left[\bigvee_{i} \left[f_i^{-j}(x+y)\right]\right] \\
&= \bigvee_{i,j} \left[f_i^{-j}(x+y)\right] \\
&\leq \bigvee_{i} \left[f_i^-(x+y)\right] \\
&= \bigvee_{i} f_i^-(x+y).
\end{aligned}
$$

This imlpies $\bigvee\limits_{i} f_i^-(x+y) \geq \min\left\{\bigvee\limits_{i\in I} f_i^-(x), \bigvee\limits_{i\in I} f_i^-(y), 0.5\right\}$.

Now consider,

$$
\begin{aligned}
\left(\bigvee_{i\in I} f_i^-\right)(x) \wedge 0.5 &= \bigvee_{i\in I} \left(f_i^-(x)\right) \wedge 0.5 \\
&= \bigvee_{i} \left[f_i^-(x) \wedge 0.5\right] \\
&\leq \bigvee_{i} f_i^-(xy) \\
&= \left(\bigvee_{i} f_i^-\right)(xy).
\end{aligned}
$$

$$
\text{This implies, } \quad \left(\bigvee_{i} f_i^-\right)(xy) \geq \min\left\{\left(\bigvee_{i\in I} f_i^-\right)(x), 0.5\right\}.
$$

Thus, $\bigvee_i f_i^-$ is an $(\in, \in \vee q)$ fuzzy ideal of R. Clearly $f^- \leq \bigvee_i f_i^-$ and $\left(\bigvee_i f_i^-\right)(a) = \bigvee_i f_i^-\,(a) = \alpha$.

Therefore, $\bigvee_i f_i^-$ is the l.u.b of $\mathcal{F}$. Hence, by Zorn's Lemma, there exists an $(\in, \in \vee q)$ fuzzy ideal h^- of R which is maximal with respect to the property that $f^- \leq h^-$ and $h^-(a) = \alpha$. We now show that h^- is an $(\in, \in \vee q)$ fuzzy irreducible ideal of R.

Suppose that $h^- = k_1^- \wedge_{0.5} k_2^-$, where k_1^- and k_2^- are $(\in, \in \vee q)$ fuzzy ideals of R. This implies that $h^- \leq k_1^-$ and $h^- \leq k_2^-$. We claim that either $h^- = k_1^-$ or $h^- = k_2^-$. Suppose on contrary $h^- \neq k_1^-$ or $h^- \neq k_2^-$. Since h^- is maximal with respect to the property that $h^-(a) = \alpha$ and $h^- \lneqq k_1^-$ or $h^- \lneqq k_2^-$, it follows that, $k_1^-(a) \neq \alpha$ and $k_2^-(a) \neq \alpha$.

Hence, $\alpha = h^-(a) = \left(k_1^- \wedge_{0.5} k_2^-\right)(a) = \left(k_1^- \wedge k_2^-\right)(a) = k_1^-(a) \wedge k_2^-(a) \neq \alpha$, which is absurd.

Hence, either $h^- = k_1^-$ or $h^- = k_2^-$. This prove that h^- is an $(\in, \in \vee q)$ fuzzy irreducible ideal of R.

Hence, by Theorem 2.3.5, h^- is an $(\in, \in \vee q)$ fuzzy prime ideal of R. ∎

2.3.7 Theorem

The following assertions for a hemiring R are equivalent:

(1) R is fully idempotent.

(2) The set $\mathcal{L}_R = \{f^- : f \text{ is an } (\in, \in \vee q) \text{ fuzzy ideal of } R\}$, (ordered by $\leq$) form a distributive lattice $\mathcal{L}_R$ under the sum and intersection of $(\in, \in \vee q)$ fuzzy ideals with

$f^- \wedge_{0.5} g^- = f^- \odot_{0.5} g^-$, for each pair of $(\in, \in \vee q)$ fuzzy ideals f^- and g^- of R.

(3) Each $(\in, \in \vee q)$ fuzzy ideal is the intersection of all those $(\in, \in \vee q)$ fuzzy prime ideals of R which contain it.

Proof. (1) $\Longrightarrow$ (2) is proved in Theorem 2.3.2. Now we have to prove,

(2) $\Longrightarrow$ (3)

Let f^- be an $(\in, \in \vee q)$ fuzzy ideal of R. Let $\{g_i^-, i \in I\}$ be the family of $(\in, \in \vee q)$ fuzzy prime ideal of R which contains f^-. Obviously $f^- \leq \bigwedge_{i\in I} g_i^-$. (i)

we have to show $\bigwedge_{i\in I} g_i^- \leq f^-$.

Let "a" be any element of R, then by Lemma 2.3.6 there exists an $(\in, \in \vee q)$ fuzzy prime ideal say g_j^- such that $f^- \leq g_j^-$ and $f^-(a) = g_j^-(a)$. Thus $g_j^- \in \{g_i^-, i \in I\}$. Hence, $\bigwedge_{i\in I} g_i^- \leq g_j^-$.

So, $\bigwedge_{i\in I} g_i^-(a) \leq g_j^-(a) = f^-(a)$.

This implies, $\bigwedge_{i\in I} g_i^- \leq f^-$. (ii)

From (i) and (ii), we get $\bigwedge_{i\in I} g_i^- = f^-$.

(3) $\Longrightarrow$ (1)

Let f be any $(\in, \in \vee q)$ fuzzy ideal of R then by Lemma 2.1.10, f^- is also an $(\in, \in \vee q)$ fuzzy ideal of R. Then $f^- \odot_{0.5} f^-$ is also an $(\in, \in \vee q)$ fuzzy ideal of R. Hence, according to the statement (3), $f^- \odot_{0.5} f^-$ can be written as $f^- \odot_{0.5} f^- = \bigwedge_{i\in I\,0.5} g_i^-$ where $\{g_i^-, i \in I\}$ be the family of $(\in, \in \vee q)$ fuzzy prime ideal of R which contain $f^- \odot_{0.5} f^-$.

Now $f^- \odot_{0.5} f^- \leq g_i^-$ for all $i \in I$, and since g_i^- is an $(\in, \in \vee q)$ fuzzy prime ideal So, $f^- \leq g_i^-$ for all $i \in I$.

Thus, $f^- \leq \bigwedge_{i\in I} g_i^- = f^- \odot_{0.5} f^-$ and we know that $f^- \odot_{0.5} f^- = f \odot_{0.5} f$.

This implies, $f^- \leq f \odot_{0.5} f$. $\qquad (iii)$

And by Lemma 2.1.17 we get, $f \odot_{0.5} f \leq f^-$. $\quad (iv)$

From (iii) and (iv) we have, $f \odot_{0.5} f = f^-$.

Hence, by Theorem 2.3.1 R is a fully idempotent hemiring. ■

Chapter 3

Hemirings characterized by their $(\bar{\in}, \bar{\in} \vee \bar{q})$ and fuzzy ideals with thresholds $(\alpha, \beta]$

In this chapter we characterize different classes of hemirings by the properties of their $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy ideals and also by the properties of their fuzzy ideals with thresholds $(\alpha, \beta]$.

Throughout this chapter R is a hemiring with identity 1.

3.1 $(\bar{\in}, \bar{\in} \vee \bar{q})$ −fuzzy ideals

In this section we define $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy ideals of a hemiring and prove some basic results.

3.1.1 Definition [5]

A fuzzy subset f of a hemiring R is said to be an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy left (right) ideal of R if,

(1) $(x+y)_{\min\{t,r\}} \bar{\in} f$ implies that $x_t \bar{\in} \vee \bar{q} f$ or $y_r \bar{\in} \vee \bar{q} f$.

(2) $(xy)_t \bar{\in} f$ implies that $y_t \bar{\in} \vee \bar{q} f$ (respectively $x_t \bar{\in} \vee \bar{q} f$) for all $x, y \in R$ and $t, r \in (0,1]$.

3.1.2 Theorem [14]

Let f be a fuzzy subset of a hemiring R, $x, y \in R$ and $t, r \in (0,1]$, then the following conditions are equivalent:

(1) $(x+y)_{\min\{t,r\}} \bar{\in} f$ implies that $x_t \bar{\in} \vee \bar{q} f$ or $y_r \bar{\in} \vee \bar{q} f$.

(2) $\max\{f(x+y), 0.5\} \geq \min\{f(x), f(y)\}$.

3.1.3 Theorem [14]

Let f be a fuzzy subset of a hemiring R, $x, y \in R$ and $t, r \in (0,1]$, then the following conditions are equivalent:

(1) $(xy)_t \bar{\in} f$ implies that $y_t \bar{\in} \vee \bar{q} f$ (respectively $x_t \bar{\in} \vee \bar{q} f$).

(2) $\max\{f(xy), 0.5\} \geq f(y)$ (respectively $\max\{f(xy), 0.5\} \geq f(x)$).

3.1.4 Definition

Let f and g be $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideals of a hemiring R. Then the product $f \odot^{0.5} g$ is defined as;

$$(f \odot^{0.5} g)(x) = \left[\bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \le i \le p} [f(y_i) \wedge g(z_i)] \right] \right] \vee 0.5 \quad \text{for all } x \in R.$$

3.1.5 Proposition

If f is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy left ideal and g is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy right ideal of R, then $f \odot^{0.5} g$ is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideal of R.

Proof. Let f be an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy left ideal and g be an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy right ideal of R. For any $x, x' \in R$,

$$\begin{aligned}
&(f \odot^{0.5} g)(x) \wedge (f \odot^{0.5} g)(x') \\
&= \left(\left[\bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \le i \le p} (f(y_i) \wedge g(z_i)) \right] \right] \vee 0.5 \right) \wedge \left(\left[\bigvee_{x'=\sum_{j=1}^{q} y'_j z'_j} \left[\bigwedge_{1 \le j \le q} (f(y'_j) \wedge g(z'_j)) \right] \right] \vee 0.5 \\
&= \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \le i \le p} (f(y_i) \wedge g(z_i)) \right] \wedge \left[\bigvee_{x'=\sum_{j=1}^{q} y'_j z'_j} \left[\bigwedge_{1 \le j \le q} (f(y'_j) \wedge g(z'_j)) \right] \right] \vee 0.5 \\
&= \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \bigvee_{x'=\sum_{j=1}^{q} y'_j z'_j} \left[\bigwedge_{1 \le i \le p} (f(y_i) \wedge g(z_i)) \right] \wedge \left[\bigwedge_{1 \le j \le q} (f(y'_j) \wedge g(z'_j)) \right] \vee 0.5 \\
&\le \left[\bigvee_{x+x'=\sum_{k=1}^{r} y''_k z''_k} \left[\bigwedge_{1 \le k \le r} (f(y''_k) \wedge g(z''_k)) \right] \vee 0.5 \right] \vee 0.5. \\
&= (f \odot^{0.5} g)(x + x') \vee 0.5.
\end{aligned}$$

This implies, $\max \{(f \odot^{0.5} g)(x + x'), 0.5\} \ge \min \{(f \odot^{0.5} g)(x), (f \odot^{0.5} g)(x')\}$.

Also,

$$\begin{aligned}
&(f \odot^{0.5} g)(x) \\
&= \left[\bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \le i \le p} (f(y_i) \wedge g(z_i)) \right] \right] \vee 0.5 \\
&\le \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \le i \le p} (f(y_i) \wedge (g(z_i a) \vee 0.5)) \right] \vee 0.5 \\
&\le \left[\bigvee_{xa=\sum_{j=1}^{q} y'_j z'_j} \left[\bigwedge_{1 \le j \le q} (f(y'_j) \wedge g(z'_j)) \right] \vee 0.5 \right] \vee 0.5. \\
&= (f \odot^{0.5} g)(xa) \vee 0.5.
\end{aligned}$$

Therefore, $\max\{(f \odot^{0.5} g)(xa), 0.5\} \geq (f \odot^{0.5} g)(x)$.

Hence, $f \odot^{0.5} g$ is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy right ideal of R.

Similarly, $f \odot^{0.5} g$ is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy left ideal of R. ■

3.1.6 Definition

Let f and g be $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideals of a hemiring R. Then the sum $f +^{0.5} g$ is defined as

$$(f +^{0.5} g)(x) = \left[\bigvee_{x=y+z} [f(y) \wedge g(z)]\right] \vee 0.5 \quad \text{for all } x \in R.$$

3.1.7 Proposition

Let f and g be any $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy left (right) ideals of R then their sum $f +^{0.5} g$ is also an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy left (right) ideal of R, respectively.

Proof. Let f and g be $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy left ideals of R. For any $x, x' \in R$,

$$\begin{aligned}
&(f +^{0.5} g)(x) \wedge (f +^{0.5} g)(x') \\
&= \left(\left[\bigvee_{x=y+z} (f(y) \wedge g(z))\right] \vee 0.5\right) \wedge \left(\left[\bigvee_{x'=y'+z'} (f(y') \wedge g(z'))\right] \vee 0.5\right) \\
&= \bigvee_{\substack{x=y+z \\ x'=y'+z'}} [[f(y) \wedge g(z)] \wedge [f(y') \wedge g(z')]] \vee 0.5 \\
&= \bigvee_{\substack{x=y+z \\ x'=y'+z'}} [[f(y) \wedge f(y')] \wedge [g(z) \wedge g(z')]] \vee 0.5 \\
&\leq \bigvee_{\substack{x=y+z \\ x'=y'+z'}} [(f(y+y') \vee 0.5) \wedge (g(z+z') \vee 0.5)] \vee 0.5 \\
&\leq \left[\bigvee_{x+x'=a+b} [f(a) \wedge g(b)] \vee 0.5\right] \vee 0.5. \\
&= (f +^{0.5} g)(x+x') \vee 0.5.
\end{aligned}$$

This implies, $\max\{(f +^{0.5} g)(x+x'), 0.5\} \geq \min\{(f +^{0.5} g)(x), (f +^{0.5} g)(x')\}$.

Consider,

$$\begin{aligned}
\left(f +^{0.5} g\right)(x) &= \left[\bigvee_{x=y+z} \left(f(y) \wedge g(z)\right)\right] \vee 0.5 \\
&= \bigvee_{x=y+z} \left[f(y) \wedge g(z)\right] \vee 0.5 \\
&\leq \bigvee_{x=y+z} \left[\left(f(ay) \vee 0.5\right) \wedge \left(g(az) \vee 0.5\right)\right] \vee 0.5 \\
&\quad \text{(where a is any element of R)} \\
&\leq \left[\bigvee_{ax=y'+z'} \left[f(y') \wedge g(z')\right] \vee 0.5\right] \vee 0.5. \\
&= \left(f +^{0.5} g\right)(ax) \vee 0.5.
\end{aligned}$$

This implies, $\max\left\{\left(f +^{0.5} g\right)(ax), 0.5\right\} \geq \left(f +^{0.5} g\right)(x) .0.5.$

Hence, $f +^{0.5} g$ is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy left ideal of hemiring R.

Similarly, $f +^{0.5} g$ is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy right ideal of R. ■

3.1.8 Corollary

For $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideals f and g of R, $f +^{0.5} g$ is also an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideal of R.

3.1.9 Definition

Let f be a fuzzy subset of a hemiring R. We define the upper part f^+ of f as follows,

$f^+(x) = f(x) \vee 0.5$ for all $x \in R$.

3.1.10 Lemma

If f is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideal of R, then f^+ is also an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideal of R.

Proof. Let f be an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideal of R and $x, y \in R$. Then

$$\begin{aligned} f^{+}(x) \wedge f^{+}(y) &= [f(x) \vee 0.5] \wedge [f(y) \vee 0.5] \\ &= [f(x) \wedge f(y)] \vee 0.5 \\ &\leq [f(x+y) \vee 0.5] \vee 0.5. \\ &= f^{+}(x+y) \vee 0.5. \\ \text{Thus, } \max \left\{f^{+}(x+y), 0.5\right\} &\geq \min \left\{f^{+}(x) \wedge f^{+}(y)\right\}. \end{aligned}$$

Consider,

$$\begin{aligned} f^{+}(x) &= f(x) \vee 0.5 \\ &\leq [f(xy) \vee 0.5] \vee 0.5 \\ &= f^{+}(xy) \vee 0.5. \\ \text{Thus, } \max \left\{f^{+}(xy), 0.5\right\} &\geq f^{+}(x). \end{aligned}$$

Similarly we can show, $\max \{f^{+}(xy), 0.5\} \geq f^{+}(y)$.

Hence, f^{+} is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideal of a hemiring R. ■

3.1.11 Definition

Let X be a nonempty subset of a hemiring R. Then upper part of the characteristic function is,

$$C_X^{+}(x) = \begin{cases} 1 & \text{if } x \in X \\ 0.5 & \text{if } x \notin X \end{cases}$$

3.1.12 Lemma

If X and Y are subsets of a hemiring R, then $C_X^+ = C_Y^+$ if and only if $X = Y$.

Proof. Obvious. ■

3.1.13 Lemma

If X and Y are subsets of a hemiring R, then $C_X \odot^{0.5} C_Y = C_{XY} \vee 0.5$.

Proof. Let a be any element of a hemiring R. If $a \in XY$, then there exist $x_i \in X$, and $y_i \in Y$ such that $a = \sum_{i=1}^{p} x_i y_i$. So,

$$
\begin{aligned}
&(C_X \odot^{0.5} C_Y)(a) \\
&= \left[\bigvee_{a=\sum_{i=1}^{p} x_i y_i} \left[\bigwedge_{1 \le i \le p} [C_X(x_i) \wedge C_Y(y_i)] \right] \right] \vee 0.5 \\
&= \left[\bigwedge_{1 \le i \le p} (1 \wedge 1) \right] \vee 0.5 \\
&= 1 \vee 0.5. \\
&= 1.
\end{aligned}
$$

Since, $a \in XY$. Therefore $C_{XY}(a) = 1$.

Hence, $C_{XY}(a) \vee 0.5 = 1 \vee 0.5 = 1$.

If $a \notin XY$, then $a \neq \sum_{i=1}^{p} x_i y_i$ for $x_i \in X$ and $y_i \in Y$.

If $a = \sum_{i=1}^{p} s_i t_i$ for some $s_i, t_i \in R$, then we have

$$
\begin{aligned}
&(C_X \odot^{0.5} C_Y)(a) \\
&= \left[\bigvee_{a=\sum_{i=1}^{p} s_i t_i} \left[\bigwedge_{1 \le i \le p} [C_X(s_i) \wedge C_Y(t_i)] \right] \right] \vee 0.5 \\
&= 0 \vee 0.5. \\
&= 0.5. \\
&= C_{XY}(a) \vee 0.5.
\end{aligned}
$$

Hence, $C_X \odot^{0.5} C_Y = C_{XY} \vee 0.5$. ■

3.1.14 Lemma

If X and Y are subsets of a hemiring R, then $C_X \wedge^{0.5} C_Y = C_{X \cap Y} \vee 0.5$.

Proof. Obvious. ■

3.1.15 Lemma

The characteristic function C_L of a nonempty subset L of R is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy left (right) ideal of a hemiring R if and only if L is a left (right) ideal of R.

Proof. Let L be a left ideal of the hemiring R.

Let $x, y \in R$, if $x, y \in L$ then $x + y \in L$.

This implies that $C_L(x + y) = 1$.

Thus, $C_L(x + y) \vee 0.5 \geq C_L(x) \wedge C_L(y)$.

If x or $y \notin L$ then $C_L(x) \wedge C_L(y) = 0 \leq C_L(x + y) \vee 0.5$.

This implies $\max\{C_L(x + y), 0.5\} \geq \min\{C_L(x), C_L(y), \}$.

Again,

If $y \in L$ then $xy \in L$.

This implies $C_L(xy) = 1$. Thus, $C_L(xy) \vee 0.5 \geq C_L(y)$.

If $y \notin L$ then $C_L(y) = 0 \leq C_L(xy) \vee 0.5$.

This implies $\max\{C_L(xy), 0.5\} \geq C_L(y)$.

Hence, C_L is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy left ideal of a hemiring R.

Conversely, suppose that C_L is an $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy left ideal of R. Let $x, y \in L$ this implies that $C_L(x) = 1 = C_L(y)$

Since, $\max\{C_L(x+y), 0.5\} \geq \min\{C_L(x), C_L(y),\} = \min\{1, 1\} = 1$.

This implies $\max\{C_L(x+y), 0.5\} \geq 1$.

Thus $C_L(x+y) = 1$ which imlies, $x + y \in L$.

For second condition, let $x \in R$ and $y \in L$ this implies that $C_L(y) = 1$.

Since, $\max\{C_L(xy), 0.5\} \geq C_L(y) = 1$

Therefore, $C_L(xy) \vee 0.5 \geq 1$.

This implies $C_L(xy) = 1$, hence $xy \in L$.

Thus L is a left ideal of R.

Similarly we can show the characteristic function C_L is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy right ideal of R if and only if L is a right ideal of R. ∎

3.1.16 Definition

Let f and g be $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy ideals of a hemiring R. Then the upper part of $f \wedge^{0.5} g$ is defined as, $(f \wedge^{0.5} g)(x) = (f \wedge g)(x) \vee 0.5$ where $x \in R$.

3.1.17 Lemma

Let f be an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy right ideal and g be an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy left ideal of a hemiring R, then $f \odot^{0.5} g \leq f \wedge^{0.5} g$.

Proof. Let f and g be $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy right and fuzzy left ideals of R, respectively. For any $x \in R$,

$$
\begin{aligned}
&(f \odot^{0.5} g)(x) \\
&= \left[\bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \le i \le p} (f(y_i) \wedge g(z_i)) \right] \right] \vee 0.5 \\
&= \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \le i \le p} (f(y_i) \wedge g(z_i)) \right] \vee 0.5 \\
&\le \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \le i \le p} (f(y_i z_i) \vee 0.5) \wedge (g(y_i z_i) \vee 0.5) \right] \vee 0.5 \\
&= \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\left(\bigwedge_{1 \le i \le p} f(y_i z_i) \right) \wedge \left(\bigwedge_{1 \le i \le p} g(y_i z_i) \right) \right] \vee 0.5 \\
&\le \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\left[f\left(\textstyle\sum_{i=1}^{p} y_i z_i\right) \vee 0.5 \right] \wedge \left[g\left(\textstyle\sum_{i=1}^{p} y_i z_i\right) \vee 0.5 \right] \right] \vee 0.5 \\
&= \bigvee_{x=\sum_{i=1}^{p} y_i z_i} [f(x) \wedge g(x)] \vee 0.5 \\
&= [f(x) \wedge g(x)] \vee 0.5 \\
&= (f \wedge g)(x) \vee 0.5 \\
&= (f \wedge^{0.5} g)(x).
\end{aligned}
$$

This implies, $f \odot^{0.5} g \le f \wedge^{0.5} g$. ■

3.2 Regular and Weakly regular hemirings

In this section we characterize regular and weakly regular hemirings by the properties of their $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy ideals.

3.2.1 Lemma

A hemiring R is regular if and only if $f \odot^{0.5} g = f \wedge^{0.5} g$, for every $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy right ideal f and for every $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy left ideal g of R.

Proof. Let f and g be $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy right and fuzzy left ideals of R, respectively. For any $x \in R$,

$(f \odot^{0.5} g)(x) \leq (f \wedge^{0.5} g)(x)$ by Lemma 3.1.17.

On the other hand, since R is regular hemiring, so for every $x \in R$, there exists $s \in R$ such that $x = xsx$.

Now

$$\begin{aligned}
(f \wedge^{0.5} g)(x) &= (f \wedge g)(x) \vee 0.5 \\
&= f(x) \wedge g(x) \vee 0.5 \\
&\leq f(xs) \wedge g(x) \vee 0.5 \\
&\leq \bigvee_{x=\sum_{i=1}^{q} a_i b_i} \left[\bigwedge_{1 \leq i \leq q} \left[f(a_i) \wedge g(b_i) \right] \right] \vee 0.5 \\
&= f \odot^{0.5} g(x).
\end{aligned}$$

$$\text{Thus, } (f \wedge^{0.5} g)(x) \leq (f \odot^{0.5} g)(x).$$

$$\text{Hence, } f \odot^{0.5} g = f \wedge^{0.5} g.$$

Conversely, let A be a right ideal and B be a left ideal of R. Then by lemma 3.1.15 C_A and C_B are $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy right ideal and left ideals of R, respectively.

By hypothesis, $C_A \odot^{0.5} C_B = C_A \wedge^{0.5} C_B$. By Lemma 3.1.13 and 3.1.14 we have, $C_A \odot^{0.5} C_B = C_{AB} \vee 0.5$ and $C_A \wedge^{0.5} C_B = C_{A \cap B} \vee 0.5$ respectively.

This implies, $C_{AB} \vee 0.5 = C_{A \cap B} \vee 0.5$.

Thus by Lemma 3.1.12, $AB = A \cap B$.

Hence, by Proposition 1.3.3 R is a regular hemiring. ■

3.2.2 Definition

An $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideal f of R is said to be idempotent if $f \odot^{0.5} f = f \vee 0.5 = f^{+}$.

3.2.3 Theorem

The following assertions for a hemiring R are equivalent;

(1) R is right weakly regular hemiring.

(2) All $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy right ideals of R are idempotent.

(3) $f \odot^{0.5} g = f \wedge^{0.5} g$ for all $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy right ideals f and all $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy two sided ideals g of R.

Proof. (1) $\Longrightarrow$ (2)

let f be an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy right ideal of R. By Lemma 3.1.17, $f \odot^{0.5} f \leq f^{+}$.

On the other hand, since R is right weakly regular hemiring, so $x \in (xR)^2$.

Hence, $x = \sum\limits_{i=1}^{q} xa_i xb_i$ for some a_i, $b_i \in R$ and $q \in \mathbb{N}$. So,

$$\begin{aligned}
f(x) &= f(x) \wedge f(x) \\
f(x) \vee 0.5 &= (f(x) \wedge f(x)) \vee 0.5 \\
f^{+}(x) &= f(x) \wedge f(x) \vee 0.5 \\
&\leq (f(xa_i) \vee 0.5) \wedge (f(xb_i) \vee 0.5) \vee 0.5 \qquad \text{for } 1 \leq i \leq q \\
f^{+}(x) &\leq \bigwedge_{1 \leq i \leq q} [f(xa_i) \wedge f(xb_i)] \vee 0.5 \\
&\leq \bigvee_{x = \sum_{j=1}^{r} y_j z_j} \left[\bigwedge_{1 \leq j \leq r} [f(y_j) \wedge f(z_j)] \right] \vee 0.5 \\
&= \left(f \odot^{0.5} f\right)(x).
\end{aligned}$$

$$\text{Thus, } f^{+}(x) \leq \left(f \odot^{0.5} f\right)(x).$$

$$\text{Hence, } f \odot^{0.5} f = f^{+}.$$

This shows that, all $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy right ideals of R are

idempotent.

(2) $\Longrightarrow$ (1)

Let $x \in R$, we claim that $x \in (xR)^2$.

Let $xR = A$ be the right ideal generated by x, and C_A be the characteristic function of A, by Lemma 3.1.15 it is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy right ideal of R.

By our assumption, $C_A^+ = C_A \odot^{0.5} C_A$.

By Lemma 3.1.13 $C_A \odot^{0.5} C_A = C_{AA} \vee 0.5$.

So we have, $C_A \vee 0.5 = C_{AA} \vee 0.5$.

Therefore, by Lemma 3.1.12 we have, $A = AA$.

Now since $x \in A$, this imlpies $x \in A^2$.

Therefore, $x \in (xR)^2$.

Hence, R is a right weakly regular hemiring.

(1) $\Longrightarrow$ (3)

Let f be an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy right ideal and g be an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy two sided ideal of R.

By Lemma 3.1.17, $f \odot^{0.5} g \leq f \wedge^{0.5} g$.

For reverse inclusion, let $x \in R$, since R is right weakly regular hemiring, so

$x \in (xR)^2$. Hence, $x = \sum_{i=1}^{p} x a_i x b_i$ for some a_i , $b_i \in R$ and $p \in \mathbb{N}$. Now

$$\begin{aligned}
\left(f \wedge^{0.5} g\right)(x) &= (f \wedge g)(x) \vee 0.5 \\
&= f(x) \wedge g(x) \vee 0.5 \\
&\leq (f(xa_i) \vee 0.5) \wedge (g(xb_i) \vee 0.5) \vee 0.5 \quad \text{for } 1 \leq i \leq p \\
&\leq \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \leq i \leq p} [f(y_i) \wedge g(z_i)] \right] \vee 0.5 \\
&= \left(f \odot^{0.5} g\right)(x).
\end{aligned}$$

This implies, $\left(f \wedge^{0.5} g\right)(x) \leq \left(f \odot^{0.5} g\right)(x)$.

Hence, $f \wedge^{0.5} g = f \odot^{0.5} g$.

$(3) \Longrightarrow (1)$

Let A and B be the right and two sided ideal of R respectively. Then by Lemma 3.1.15, the characteristic function C_A and C_B are $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy right ideal and fuzzy two sided ideal of R.

Hence, by hypothesis, $C_A \odot^{0.5} C_B = C_A \wedge^{0.5} C_B$.

By Lemma 3.1.13, $C_A \odot^{0.5} C_B = C_{AB} \vee 0.5$ and $C_A \wedge^{0.5} C_B = C_{A \cap B} \vee 0.5$.

So we have, $C_{AB} \vee 0.5 = C_{A \cap B} \vee 0.5$.

Therefore, by Lemma 3.1.12 $AB = A \cap B$.

Hence, by Proposition 1.3.5, R is right weakly regular semiring. ∎

3.3 Hemirings in which each $(\bar{\in}, \bar{\in} \vee \bar{q})$ –fuzzy ideal is idempotent

In this section we study those hemirings for which each $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy ideal is idempotent.

3.3.1 Theorem

The following assertions for a hemiring R are equivalent;

(1) R is fully idempotent.

(2) Each $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideal of R is idempotent.

(3) For each pair of $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideals f and g of R, $f \wedge^{0.5} g = f \odot^{0.5} g$.

Proof. (1) $\Longrightarrow$ (3)

Let f and g be any pair of $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideals of R.

Then, $f \odot^{0.5} g \leq f \wedge^{0.5} g$ (by Lemma 3.1.17).

Since, R is fully idempotent, so $(x) = (x)^2$ we have, $x = \sum_{i=1}^{p} r_i x r_i' t_i x t_i'$. Thus,

$$\begin{aligned}
(f \wedge^{0.5} g)(x) &= (f \wedge g)(x) \vee 0.5 \\
&= f(x) \wedge g(x) \vee 0.5 \\
&\leq \bigwedge_{1 \leq i \leq p} [f(r_i x r_i') \wedge g(t_i x t_i') \vee 0.5] \\
&\leq \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \leq i \leq p} (f(y_i) \wedge g(z_i)) \right] \vee 0.5 \\
&= (f \odot^{0.5} g)(x).
\end{aligned}$$

Therefore, $(f \wedge^{0.5} g)(x) \leq (f \odot^{0.5} g)(x)$.

Hence, $f \wedge^{0.5} g = f \odot^{0.5} g$.

(3) $\Longrightarrow$ (2)

Let f and g be any pair of $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideals of R. We have $f \wedge^{0.5} g = f \odot^{0.5} g$. Take $g = f$.

Thus,

$$\begin{aligned} f \wedge^{0.5} f &= f \odot^{0.5} f \\ (f \wedge f)(x) \vee 0.5 &= (f \odot^{0.5} f)(x) \\ f(x) \vee 0.5 &= (f \odot^{0.5} f)(x) \\ \text{This implies, } f^{+} &= f \odot^{0.5} f. \end{aligned}$$

(2) $\Longrightarrow$ (1)

Let A be an ideal of R. Then by Lemma 3.1.15 C_A is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideal of R.

Hence, $C_A \odot^{0.5} C_A = C_A^{+}$

But, $C_A \odot^{0.5} C_A = C_{AA} \vee 0.5$ (by Lemma 3.1.13).

Therefore, $C_{AA} \vee 0.5 = C_A \vee 0.5$

This implies, $AA = A$ (by Lemma 3.1.12).

Hence, R is fully idempotent hemiring. ■

3.3.2 Theorem

The following assertions for a hemiring R are equivalent;

(1) R is fully idempotent.

(2) The set $\mathcal{L}_R = \{f^{+} : f \text{ is an } (\bar{\in}, \bar{\in} \vee \bar{q}) \text{ fuzzy ideal of } R\}$, (ordered by $\leq$) form a distributive lattice under the sum and intersection of $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideals with

$f^+ \wedge^{0.5} g^+ = f^+ \odot^{0.5} g^+$, for each pair of $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideals f^+ and g^+ of R.

Proof. (1) $\Longrightarrow$ (2)

The set $\mathcal{L}_R = \{f^+ : f$ is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideal$\}$ of $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideals of R (ordered by $\leq$) is cleary a lattice under the sum and intersection of $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideals. Moreover, since R is a fully idempotent hemiring, it follows that $f^+ \wedge^{0.5} g^+ = f^+ \odot^{0.5} g^+$, for each pair of $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideals f^+ and g^+ of R. We now show that $\mathcal{L}_R$ is a distributive lattice, that is, for $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideals f^+, g^+ and h^+ of R,

we have $[(f^+ \wedge^{0.5} g^+) +^{0.5} h^+] = [(f^+ +^{0.5} h^+) \wedge^{0.5} (g^+ +^{0.5} h^+)]$. For any $x \in R$,

$$
\begin{aligned}
&[(f^+ \wedge^{0.5} g^+) +^{0.5} h^+](x) \\
&= \left[\bigvee_{x=y+z} [(f^+ \wedge^{0.5} g^+)(y) \wedge h^+(z)]\right] \vee 0.5 \\
&= \bigvee_{x=y+z} [(f^+ \wedge g^+)(y) \vee 0.5 \wedge h^+(z)] \vee 0.5 \\
&= \bigvee_{x=y+z} [f^+(y) \wedge g^+(y) \wedge h^+(z) \wedge h^+(z)] \vee 0.5 \\
&= \bigvee_{x=y+z} [f^+(y) \wedge h^+(z) \vee 0.5] \wedge [g^+(y) \wedge h^+(z) \vee 0.5] \vee 0.5 \\
&= \left[\bigvee_{x=y+z} [f^+(y) \wedge h^+(z)] \vee 0.5\right] \wedge \left[\bigvee_{x=y+z} [g^+(y) \wedge h^+(z)] \vee 0.5\right] \vee 0.5. \\
&\leq [(f^+ +^{0.5} h^+)(x) \wedge (g^+ +^{0.5} h^+)(x)] \vee 0.5.
\end{aligned}
$$

Thus, $[(f^+ \wedge^{0.5} g^+) +^{0.5} h^+](x) \leq [(f^+ +^{0.5} h^+) \wedge^{0.5} (g^+ +^{0.5} h^+)](x)$. $\quad (i)$

Again,

$$
\begin{aligned}
&[(f^+ +^{0.5} h^+) \wedge^{0.5} (g^+ +^{0.5} h^+)](x) \\
&= [(f^+ +^{0.5} h^+) \odot^{0.5} (g^+ +^{0.5} h^+)](x) \\
&= \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \leq i \leq p} [(f^+ +^{0.5} h^+)(y_i) \wedge (g^+ +^{0.5} h^+)(z_i)]\right] \vee 0.5 \\
&= \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \leq i \leq p} \left[\begin{array}{c} \left[\bigvee_{y_i=r_i+s_i} (f^+(r_i) \wedge h^+(s_i)) \vee 0.5\right] \wedge \\ \left[\bigvee_{z_i=t_i+u_i} (g^+(t_i) \wedge h^+(u_i)) \vee 0.5\right] \end{array}\right]\right] \vee 0.5
\end{aligned}
$$

$$= \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1\leq i\leq p} \left[\bigvee_{\substack{y_i=r_i+s_i\\ z_i=t_i+u_i}} [f^+(r_i) \wedge h^+(s_i) \wedge g^+(t_i) \wedge h^+(u_i)] \right]\right] \vee 0.5$$

$$= \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1\leq i\leq p} \left[\bigvee_{\substack{y_i=r_i+s_i\\ z_i=t_i+u_i}} \left[\begin{array}{c} f^+(r_i) \wedge h^+(s_i) \wedge h^+(s_i) \\ \wedge g^+(t_i) \wedge h^+(u_i) \end{array} \right]\right]\right] \vee 0.5$$

$$\leq \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1\leq i\leq p} \left[\bigvee_{\substack{y_i=r_i+s_i\\ z_i=t_i+u_i}} \left[\begin{array}{c} (f^+(r_i t_i) \vee 0.5) \wedge (g^+(r_i t_i) \vee 0.5) \wedge \\ (h^+(s_i t_i) \vee 0.5) \wedge (h^+(s_i u_i) \vee 0.5) \wedge (h^-(r_i u_i) \vee 0.5) \end{array} \right]\right]\right] \vee 0.5$$

$$= \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1\leq i\leq p} \left[\bigvee_{\substack{y_i=r_i+s_i\\ z_i=t_i+u_i}} \left[\begin{array}{c} (f^+(r_i t_i) \wedge g^+(r_i t_i) \vee 0.5) \wedge \\ (h^+(s_i t_i) \wedge h^+(s_i u_i) \wedge h^+(r_i u_i)) \end{array} \right]\right]\right] \vee 0.5$$

$$\leq \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1\leq i\leq p} \left[\bigvee_{\substack{y_i=r_i+s_i\\ z_i=t_i+u_i}} \left[\begin{array}{c} (f^+ \wedge^{0.5} g^+)(r_i t_i) \wedge \\ h^+(s_i t_i + s_i u_i + r_i u_i) \end{array} \right] \vee 0.5 \right]\right] \vee 0.5$$

$$\leq \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1\leq i\leq p} [(f^+ \wedge^{0.5} g^+) +^{0.5} h^+](y_i z_i) \right]$$

$$\leq \bigvee_{x=\sum_{i=1}^{p} y_i z_i} [(f^+ \wedge^{0.5} g^+) +^{0.5} h^+](x)$$

$$= [(f^+ \wedge^{0.5} g^+) +^{0.5} h^+](x).$$

Thus, $[(f^+ +^{0.5} h^+) \wedge^{0.5} (g^+ +^{0.5} h^+)](x) \leq [(f^+ \wedge^{0.5} g^+) +^{0.5} h^+](x)$. $\quad (ii)$

From (i) and (ii) we get, $[(f^+ \wedge^{0.5} g^+) +^{0.5} h^+] = [(f^+ +^{0.5} h^+) \wedge^{0.5} (g^+ +^{0.5} h^+)]$.

$(2) \Longrightarrow (1)$

Suppose that the set $\mathcal{L}_R = \{f^+$, where f is an $(\in, \in \vee q)$ fuzzy ideal of $R\}$, (ordered by $\leq$) is a distributive lattice under the sum and intersection of $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideals with $f^+ \wedge^{0.5} g^+ = f^+ \odot^{0.5} g^+$ for each pair of $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideals f^+ and g^+ of R. Then for any $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideal f^+ of R, we have, $f^+ \odot^{0.5} f^+ = f^+ \wedge^{0.5} f^+$.

First we have to show that $f^+ \odot^{0.5} f^+ = f \odot^{0.5} f$.

Consider, $(f^+ \odot^{0.5} f^+)(x)$

$$= \left[\bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \le i \le p} \left(f^+ (y_i) \wedge f^+ (z_i) \right) \right] \right] \vee 0.5$$

$$= \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \le i \le p} \left[f (y_i) \vee 0.5 \wedge f (z_i) \vee 0.5 \right] \right] \vee 0.5$$

$$= \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \le i \le p} \left[f (y_i) \wedge f (z_i) \right] \right] \vee 0.5.$$

$$= \left(f \odot^{0.5} f \right) (x) .$$

Therefore, we get $f^+ \odot^{0.5} f^+ = f \odot^{0.5} f$.

This implies, $f \odot^{0.5} f = f^+$.

Hence, R is fully idempotent hemiring. ∎

3.3.3 Definition

An $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideal h of a hemiring R is called an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy prime ideal of R if for any $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideals f and g of R, $f \odot^{0.5} g \le h$, implies that $f^+ \le h^+$ or $g^+ \le h^+$.

3.3.4 Definition

An $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideal h of a hemiring R is called an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy irreducible ideal of R if for any $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideals f and g of R, $f \wedge^{0.5} g = h$ implies that $f^+ = h^+$ or $g^+ = h^+$.

3.3.5 Theorem

Let R be a fully idempotent hemiring. For any $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideal h^+ of R, the following conditions are equivalent:

(1) h^+ is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy prime ideal.

(2) h^+ is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy irreducible ideal.

Proof. (1) $\Longrightarrow$ (2)

Let f^+ and g^+ be any $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideals of R. Assume that h^+ is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy prime ideal. We show that h^+ ia an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy irreducible ideal of R. Let $h^+ = f \wedge^{0.5} g$. Since R is fully idempotent, so by Lemma 3.3.1 $f^+ \wedge^{0.5} g^+ = f^+ \odot^{0.5} g^+$. This implies $f^+ \odot^{0.5} g^+ = h^+$. Since h^+ is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy prime ideal, therefore $f^+ \leq h^+$ or $g^+ \leq h^+$.

Again, since $f^+ \wedge^{0.5} g^+ = h^+$, this implies $h^+ \leq f^+$ and $h^+ \leq g^+$. It follows that, $f^+ = h^+$ or $g^+ = h^+$.

Hence, h^+ is an $(\in, \in \vee q)$ fuzzy irreducible ideal of R.

(2) $\Longrightarrow$ (1)

Assume that h^+ is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy irreducible ideal. We have to show h^+ is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy prime ideal. Suppose that there exist $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideals f^+ and g^+ such that $f \odot^{0.5} g \leq h^+$. As, R is fully idempotent hemiring, so by Theorem 3.3.1 $f^+ \wedge^{0.5} g^+ = f^+ \odot^{0.5} g^+$. Thus, $f^+ \odot^{0.5} g^+ \leq h^+$ implies, $f^+ \wedge^{0.5} g^+ \leq h^+$. Again, since R is fully idempotent hemiring, it follows from Theorem 3.3.2 that the set of $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideals of R (ordered by $\leq$) is a distributive lattice with respect to the sum and intersection of $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideals. Hence the inequality $f^+ \wedge^{0.5} g^+ \leq h^+$ becomes $(f^+ \wedge^{0.5} g^+) +^{0.5} h^+ = h^+$, and using the distributivety of this lattice we have,

$(f^+ +^{0.5} h^+) \wedge^{0.5} (g^+ +^{0.5} h^+) = h^+$. Since, h^+ is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy irreducible ideal, it follows that either $f^+ +^{0.5} h^+ = h^+$ or $g^+ +^{0.5} h^+ = h^+$. This implies,

$f^+ \leq h^+$or $g^+ \leq h^+$.

Hence, h^+ is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy prime ideal of hemiring. ∎

3.3.6 Lemma

Let R be a fully idempotent hemiring. If f^+ is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideal of R with $f^+(a) = \alpha$, where a is any element of R and $\alpha \in (0, 1]$, then there exists an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy prime ideal h^+ of R such that $f^+ \leq h^+$ and $h^+(a) = \alpha$.

Proof. Let $X = \{g^+ \,; g^+$ is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideal of R, $g^+(a) = \alpha$, and $f^+ \leq g^+\}$. Then $X \neq \emptyset$, since $f^+ \in X$. Let $\mathcal{F}$ be a totally ordered subset of X, say $\mathcal{F} = \{f_i^+, i \in I\}$. We claim that $\bigvee\limits_{i \in I} f_i^+$ is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideal of R. For any $x \in R$.

$$\bigvee_{i\in I} f_i^+(x) \wedge \bigvee_{i\in I} f_i^+(y)$$
$$= \bigvee_{i\in I} f_i^+(x) \wedge \bigvee_{j\in I} f_j^+(y)$$
$$= \left[\bigvee_{i\in I} f_i^+(x)\right] \wedge \bigvee_{j\in I} \left[f_j^+(y)\right]$$
$$= \bigvee_{j} \left[\left[\bigvee_{i\in I} f_i^+(x)\right] \wedge f_j^+(y)\right]$$
$$= \bigvee_{j} \left[\bigvee_{i} \left[f_i^+(x) \wedge f_j^+(y)\right]\right]$$
$$\leq \bigvee_{j} \left[\bigvee_{i} \left[f_i^{+j}(x) \wedge f_i^{+j}(y)\right]\right] \quad \text{where } f_i^{+j} = \max\{f_i^+, f_j^+\} : f_i^{+j} \in \{f_i^+, i \in I\}$$
$$= \bigvee_{j} \left[\bigvee_{i} \left[f_i^{+j}(x) \wedge f_i^{+j}(y)\right]\right]$$
$$\leq \bigvee_{j} \left[\bigvee_{i} \left[f_i^{+j}(x+y) \vee 0.5\right]\right]$$
$$= \bigvee_{i,j} \left[f_i^{+j}(x+y) \vee 0.5\right]$$
$$\leq \bigvee_{i} \left[f_i^{+}(x+y) \vee 0.5\right]$$
$$= \bigvee_{i} f_i^{+}(x+y) \vee 0.5.$$

This imlpies, $\max\left\{\bigvee\limits_{i} f_i^+(x+y), 0.5\right\} \geq \min\left\{\bigvee\limits_{i\in I} f_i^+(x), \bigvee\limits_{i\in I} f_i^+(y)\right\}$.

Now consider,

$$\begin{aligned}\left(\bigvee_{i\in I} f_i^+\right)(x) &= \bigvee_{i\in I}\left(f_i^+(x)\right)\\ &= \bigvee_i \left[f_i^+(x)\right]\\ &\leq \bigvee_i \left[f_i^+(xy)\vee 0.5\right]\\ &= \left(\bigvee_i f_i^+\right)(xy)\vee 0.5\\ \text{Thus, } \max\left\{\left(\bigvee_i f_i^+\right)(xy)\vee 0.5\right\} &\geq \min\left(\bigvee_{i\in I} f_i^+\right)(x).\end{aligned}$$

Thus, $\bigvee_i f_i^+$ is an $(\bar{\in}, \bar{\in}\vee\bar{q})$ fuzzy ideal of R. Clearly $f^+ \leq \bigvee_i f_i^+$ and $\left(\bigvee_i f_i^+\right)(a) = \bigvee_i f_i^+(a) = \alpha$.

Therefore, $\bigvee_i f_i^+$ is l.u.b of $\mathcal{F}$. Hence, by Zorns lemma, there exists an $(\bar{\in}, \bar{\in}\vee\bar{q})$ fuzzy ideal h^+ of R which is maximal with respect to the property that $f^+ \leq h^+$ and $h^+(a) = \alpha$. We now show that h^+ is an $(\bar{\in}, \bar{\in}\vee\bar{q})$ fuzzy irreducible ideal of R.

Suppose that $h^+ = k_1^+ \wedge^{0.5} k_2^+$, where k_1^+ and k_2^+ are $(\bar{\in}, \bar{\in}\vee\bar{q})$ fuzzy ideals of R. This implies that $h^+ \leq k_1^+$ and $h^+ \leq k_2^+$. We claim that either $h^+ = k_1^+$ or $h^+ = k_2^+$. Suppose on contrary $h^+ \neq k_1^+$ or $h^+ \neq k_2^+$. Since h^+ is maximal with respect to the property that $h^+(a) = \alpha$ and $h^+ \lneqq k_1^+$ or $h^+ \lneqq k_2^+$, it follows that, $k_1^+(a) \neq \alpha$ and $k_2^+(a) \neq \alpha$.

Hence, $\alpha = h^+(a) = \left(k_1^+ \wedge^{0.5} k_2^+\right)(a) = \left(k_1^+ \wedge k_2^+\right)(a) = k_1^+(a) \wedge k_2^+(a) \neq \alpha$, which is absurd.

Hence, either $h^+ = k_1^+$ or $h^+ = k_2^+$. This prove that h^+ is an $(\bar{\in}, \bar{\in}\vee\bar{q})$ fuzzy irreducible ideal of R.

Hence, by Theorem 3.3.5, h^+ is an $(\bar{\in}, \bar{\in}\vee\bar{q})$ fuzzy prime ideal of R. ■

3.3.7 Theorem

The following assertions for a hemiring R are equivalent:

(1) R is fully idempotent.

(2) The set $\mathcal{L}_R = \{f^+ : f \text{ is an } (\bar{\in}, \bar{\in} \vee \bar{q}) \text{ fuzzy ideal of } R\}$, (ordered by $\leq$) form a distributive lattice under the sum and intersection of $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideals with $f^+ \wedge^{0.5} g^+ = f^+ \odot^{0.5} g^+$, for each pair of $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideals f^+ and g^+ of R.

(3) Each $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideal is the intersection of all those $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy prime ideals of R which contain it.

Proof. (1) $\Longrightarrow$ (2) is proved in Theorem 3.3.2. Now we have to prove,

(2) $\Longrightarrow$ (3)

Let f^+ be an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideal of R. Let $\{g_i^+, i \in I\}$ be the family of $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy prime ideal of R which contains f^+. Obviously $f^+ \leq \bigwedge_{i \in I} g_i^+$. $\quad (i)$

we have to show $\bigwedge_{i \in I} g_i^+ \leq f^+$.

Let "a" be any element of R, then by Lemma 3.3.6 there exists an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy prime ideal say g_j^+ such that $f^+ \leq g_j^+$ and $f^+(a) = g_j^+(a)$. Thus $g_j^+ \in \{g_i^+, i \in I\}$. Obviously $\bigwedge_{i \in I} g_i^+ \leq g_j^+$.

So, $\bigwedge_{i \in I} g_i^+(a) \leq g_j^+(a) = f^+(a)$.

This implies, $\bigwedge_{i \in I} g_i^+ \leq f^+$. $\quad (ii)$

From (i) and (ii), we get $\bigwedge_{i \in I} g_i^+ = f^+$.

(3) $\Longrightarrow$ (1)

Let f be any $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideal of R so f^+ is also. Then by Proposition 3.1.5 $f^+ \odot^{0.5} f^+$ is also an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy ideal of R. Hence, according to the statement

(3), $f^{+} \odot^{0.5} f^{+}$ can be written as $f^{+} \odot^{0.5} f^{+} = \bigwedge_{i\in I} g_i^{+}$ where $\{g_i^{+}, i \in I\}$ be the family of $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy prime ideals of R which contain $f^{+} \odot^{0.5} f^{+}$.

Now $f^{+} \odot^{0.5} f^{+} \leq g_i^{+}$ for all $i \in I$, and since g_i^{+} is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy prime ideal so, $f^{+} \leq g_i^{+}$ for all $i \in I$.

Thus, $f^{+} \leq \bigwedge_{i\in I} g_i^{+} = f^{+} \odot^{0.5} f^{+}$ and we know that $f^{+} \odot^{0.5} f^{+} = f \odot^{0.5} f$.

This implies, $f^{+} \leq f \odot^{0.5} f$. (iii)

And by Lemma 3.1.17 we get, $f \odot^{0.5} f \leq f^{+}$. (iv)

From (iii) and (iv) we have, $f \odot^{0.5} f = f^{+}$.

Hence, R is a fully idempotent hemiring. ■

3.4 Fuzzy ideals with thresholds $(\alpha, \beta]$ in hemirings

In this section and onward we study fuzzy ideals with thresholds $(\alpha, \beta]$ of a hemiring R.

3.4.1 Definition

Let $\alpha, \beta \in (0, 1]$ and $\alpha < \beta$, then a fuzzy subset f of R is called a fuzzy left (right) ideal with thresholds $(\alpha, \beta]$ of R if it satisfies the following conditions:

(1) $\max \{f(x+y), \alpha\} \geq \min \{f(x), f(y), \beta\}$

(2) $\max\{f(yx), \alpha\} \geq \min \{f(x), \beta\}$

$\max \{f(xy), \alpha\} \geq \min \{f(x), \beta\}$, respectively.

3.4.2 Definition

Let f and g be fuzzy ideals with thresholds $(\alpha, \beta]$ of a hemiring R. Then the fuzzy subset $f \circ_\alpha^\beta g$ is defined as

$$(f \circ_\alpha^\beta g)(x) = \left[\bigvee_{x=\sum_{i=1}^p y_i z_i} \left[\bigwedge_{1\le i\le p} [f(y_i) \wedge g(z_i)] \right] \wedge \beta \right] \vee \alpha \quad \text{for all } x \in R.$$

3.4.3 Proposition

If f and g are fuzzy left and right ideals with thresholds $(\alpha, \beta]$ of R respectively, then $f \circ_\alpha^\beta g$ is a fuzzy ideal with thresholds $(\alpha, \beta]$ of R.

Proof. Let f and g be fuzzy left and right ideals with thresholds $(\alpha, \beta]$ of R, respectively. For any $x, x' \in R$,

$$(f \circ_\alpha^\beta g)(x) \wedge (f \circ_\alpha^\beta g)(x') \wedge \beta$$
$$= \left(\left[\bigvee_{x=\sum_{i=1}^p y_i z_i} \left[\bigwedge_{1\le i\le p} (f(y_i) \wedge g(z_i)) \right] \wedge \beta \right] \vee \alpha \right) \wedge \left(\left[\bigvee_{x'=\sum_{j=1}^q y'_j z'_j} \left[\bigwedge_{1\le j\le q} (f(y'_j) \wedge g(z'_j)) \right] \wedge \beta \right. \right.$$
$$\beta$$
$$= \bigvee_{x=\sum_{i=1}^p y_i z_i} \left[\bigwedge_{1\le i\le p} (f(y_i) \wedge g(z_i)) \right] \wedge \left[\bigvee_{x'=\sum_{j=1}^q y'_j z'_j} \left[\bigwedge_{1\le j\le q} (f(y'_j) \wedge g(z'_j)) \right] \right] \wedge \beta \vee \alpha$$
$$= \bigvee_{x=\sum_{i=1}^p y_i z_i} \bigvee_{x'=\sum_{j=1}^q y'_j z'_j} \left[\bigwedge_{1\le i\le p} (f(y_i) \wedge g(z_i)) \right] \wedge \left[\bigwedge_{1\le j\le q} (f(y'_j) \wedge g(z'_j)) \right] \wedge \beta \vee \alpha$$
$$\le \left[\left[\bigvee_{x+x'=\sum_{k=1}^r y''_k z''_k} \left[\bigwedge_{1\le k\le r} (f(y''_k) \wedge g(z''_k)) \right] \wedge \beta \right] \vee \alpha \right] \vee \alpha.$$
$$= (f \circ_\alpha^\beta g)(x + x') \vee \alpha.$$

This implies, $\max\{(f \circ_\alpha^\beta g)(x + x'), \alpha\} \ge \min\{(f \circ_\alpha^\beta g)(x), (f \circ_\alpha^\beta g)(x'), \beta\}$.

Also,

$$(f \circ_\alpha^\beta g)(x) \wedge \beta$$
$$= \left[\bigvee_{x=\sum_{i=1}^p y_i z_i} \left[\bigwedge_{1\le i\le p} (f(y_i) \wedge g(z_i)) \right] \wedge \beta \right] \vee \alpha \wedge \beta$$

$$= \left[\bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \le i \le p} [f(y_i) \wedge (g(z_i) \wedge \beta)] \right] \wedge \beta \right] \vee \alpha$$
$$\le \left[\bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \le i \le p} (f(y_i) \wedge (g(z_i a) \vee \alpha)) \right] \wedge \beta \right] \vee \alpha$$
$$= \left[\bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \le i \le p} [f(y_i) \wedge g(z_i a)] \right] \wedge \beta \right] \vee \alpha$$
$$\le \left[\bigvee_{xa=\sum_{j=1}^{q} y'_j z'_j} \left[\bigwedge_{1 \le j \le q} (f(y'_j) \wedge g(z'_j)) \right] \wedge \beta \right] \vee \alpha \vee \alpha.$$
$$= \left(f \circ_\alpha^\beta g\right)(xa) \vee \alpha.$$

Therefore, $\max\left\{\left(f \circ_\alpha^\beta g\right)(xa), \alpha\right\} \ge \min\left\{\left(f \circ_\alpha^\beta g\right)(x), \beta\right\}$.

Hence, $f \circ_\alpha^\beta g$ is a fuzzy right ideal with thresholds $(\alpha, \beta]$ of R.

Similarly, we can show $f \circ_\alpha^\beta g$ is a fuzzy left ideal with thresholds $(\alpha, \beta]$ of R. ■

3.4.4 Definition

Let f and g be fuzzy ideals with thresholds $(\alpha, \beta]$ of a hemiring R. The fuzzy subset $f +_\alpha^\beta g$ of R is defined as

$$\left(f +_\alpha^\beta g\right)(x) = \left[\bigvee_{x=y+z} [f(y) \wedge g(z)] \wedge \beta \right] \vee \alpha \qquad \text{for all } x \in R.$$

3.4.5 Proposition

Let f and g be any fuzzy left (right) ideals with thresholds $(\alpha, \beta]$ of R, then the sum $f +_\alpha^\beta g$ is also a fuzzy left (right) ideal with thresholds $(\alpha, \beta]$ of R, respectively.

Proof. Let f and g be fuzzy left ideals with thresholds $(\alpha, \beta]$ of R. For any $x, x' \in R$,

$$\left(f +_\alpha^\beta g\right)(x) \wedge \left(f +_\alpha^\beta g\right)(x') \wedge \beta$$
$$= \left(\left[\bigvee_{x=y+z} (f(y) \wedge g(z)) \wedge \beta \right] \vee \alpha \right) \wedge \left(\left[\bigvee_{x'=y'+z'} (f(y') \wedge g(z')) \wedge \beta \right] \vee \alpha \right) \wedge \beta$$

$$= \left[\bigvee_{\substack{x=y+z \\ x'=y'+z'}} [[f(y) \wedge g(z)] \wedge [f(y') \wedge g(z')]] \wedge \beta \right] \vee \alpha$$
$$= \bigvee_{\substack{x=y+z \\ x'=y'+z'}} [[f(y) \wedge f(y') \wedge \beta] \wedge [g(z) \wedge g(z') \wedge \beta]] \vee \alpha$$
$$\leq \left[\bigvee_{\substack{x=y+z \\ x'=y'+z'}} [(f(y+y') \vee \alpha) \wedge (g(z+z') \vee \alpha)] \right] \vee \alpha$$
$$= \left[\bigvee_{x+x'=(y+y')+(z+z')} [f(y+y') \wedge g(z+z')] \wedge \beta \right] \vee \alpha$$
$$\leq \left[\left[\bigvee_{x+x'=a+b} [f(a) \wedge g(b)] \wedge \beta \right] \vee \alpha \right] \vee \alpha.$$
$$= \left(f +_{\alpha}^{\beta} g\right)(x+x') \vee \alpha.$$

This implies, $\max\left\{\left(f +_{\alpha}^{\beta} g\right)(x+x'), \alpha\right\} \geq \min\left\{\left(f +_{\alpha}^{\beta} g\right)(x), \left(f +_{\alpha}^{\beta} g\right)(x'), \beta\right\}$.

Again,

$$\left(f +_{\alpha}^{\beta} g\right)(x) \wedge \beta$$
$$= \left[\bigvee_{x=y+z} (f(y) \wedge g(z)) \wedge \beta \right] \vee \alpha \wedge \beta$$
$$= \left[\bigvee_{x=y+z} [(f(y) \wedge \beta) \wedge (g(z) \wedge \beta)] \wedge \beta \right] \vee \alpha$$
$$\leq \left[\bigvee_{x=y+z} [(f(ay) \vee \alpha) \wedge (g(az) \vee \alpha)] \wedge \beta \right] \vee \alpha$$

(where a is any element of R)

$$= \left[\bigvee_{ax=ay+az} [f(ay) \wedge g(az)] \wedge \beta \right] \vee \alpha$$
$$\leq \left[\bigvee_{ax=y'+z'} [f(y') \wedge g(z')] \wedge \beta \right] \vee \alpha \vee \alpha.$$
$$= \left(f +_{\alpha}^{\beta} g\right)(ax) \vee \alpha.$$

This implies, $\max\left\{\left(f +_{\alpha}^{\beta} g\right)(ax), \alpha\right\} \geq \min\left\{\left(f +_{\alpha}^{\beta} g\right)(x), \beta\right\}$.

Hence, $f +_{\alpha}^{\beta} g$ is a fuzzy left (right) ideal with thresholds $(\alpha, \beta]$ of R. ■

3.4.6 Corollary

For fuzzy ideals f and g with thresholds $(\alpha, \beta]$ of R, $f +_{\alpha}^{\beta} g$ is a fuzzy ideal with thresholds $(\alpha, \beta]$ of R.

3.4.7 Definition

Let f be a fuzzy subset of a hemiring R and $\alpha, \beta \in (0, 1]$ such that $\alpha < \beta$. We define the fuzzy subset f_{α}^{β} of R as follows, $f_{\alpha}^{\beta}(x) = (f(x) \wedge \beta) \vee \alpha$ for $x \in R$.

3.4.8 Lemma

If f is a fuzzy ideal with thresholds $(\alpha, \beta]$ of a hemiring R, then $f_{\alpha}^{\beta} = (f \wedge \beta) \vee \alpha$ is also a fuzzy ideal with thresholds $(\alpha, \beta]$ of hemiring R.

Proof. Let f be a fuzzy right ideal with thresholds $(\alpha, \beta]$ of a hemiring R. For any $x, y \in R$,consider,

$$\begin{aligned}
\max\left\{f_{\alpha}^{\beta}(x+y), \alpha\right\} &= \max\left\{(f(x+y) \wedge \beta) \vee \alpha, \alpha\right\} \\
&= (f(x+y) \wedge \beta) \vee \alpha \\
&= (f(x+y) \vee \alpha) \wedge (\beta \vee \alpha) \\
&= [(f(x+y) \vee \alpha) \vee \alpha] \wedge \beta \\
&\geq [(f(x) \wedge f(y) \wedge \beta) \vee \alpha] \wedge \beta \\
&= [(f(x) \wedge \beta) \vee \alpha] \wedge [(f(y) \wedge \beta) \vee \alpha] \wedge \beta \\
&= f_{\alpha}^{\beta}(x) \wedge f_{\alpha}^{\beta}(y) \wedge \beta.
\end{aligned}$$

This implies, $\max\left\{f_{\alpha}^{\beta}(x+y), \alpha\right\} \geq \min\left\{f_{\alpha}^{\beta}(x), f_{\alpha}^{\beta}(y), \beta\right\}$.

Now

$$\begin{aligned} \max\ \left\{f_\alpha^\beta(xy), \alpha\right\} &= \max\ \{(f(xy) \wedge \beta) \vee \alpha, \alpha\} \\ &= (f(xy) \wedge \beta) \vee \alpha \\ &= (f(xy) \vee \alpha) \wedge (\beta \vee \alpha) \\ &= [(f(xy) \vee \alpha) \vee \alpha] \wedge \beta \\ &\geq [(f(x) \wedge \beta) \vee \alpha] \wedge \beta \\ &= f_\alpha^\beta(x) \wedge \beta. \\ \text{This implies, } \max\ \left\{f_\alpha^\beta(xy), \alpha\right\} &\geq \min\ \left\{f_\alpha^\beta(x), \beta\right\}. \end{aligned}$$

Similarly we can show, $\max\ \left\{f_\alpha^\beta(xy), \alpha\right\} \geq \min\ \left\{f_\alpha^\beta(y), \beta\right\}$.

Hence, f_α^β is a fuzzy ideal with thresholds $(\alpha, \beta]$ of a hemiring R. ■

3.4.9 Definition

Let X be a nonempty subset of a hemiring R and C_X be the characteristic function of X, then $(C_X)_\alpha^\beta(x)$ is defined as

$$(C_X)_\alpha^\beta(x) = \begin{cases} \beta & \text{if } x \in X \\ \alpha & \text{if } x \notin X \end{cases}$$

3.4.10 Lemma

If X and Y are subsets of a hemiring R, then $(C_X \wedge \beta) \vee \alpha = (C_Y \wedge \beta) \vee \alpha$ if and only if $X = Y$.

Proof. Obvious. ■

3.4.11 Lemma

If X and Y are subsets of a hemiring R , then $C_X \circ_\alpha^\beta C_Y = (C_{XY} \wedge \beta) \vee \alpha$.

Proof. Let a be any element of hemiring R. If $a \in XY$, then there exist $x_i \in X$, and $y_i \in Y$ such that $a = \sum_{i=1}^{p} x_i y_i$. So,

$$\begin{aligned}
&\left(C_X \circ_\alpha^\beta C_Y\right)(a) \\
&= \left[\bigvee_{a=\sum_{i=1}^{p} x_i y_i} \left[\bigwedge_{1\le i\le p} \left[C_X(x_i) \wedge C_Y(y_i)\right]\right] \wedge \beta\right] \vee \alpha \\
&\ge \left[\bigwedge_{1\le i\le p} (1 \wedge 1) \wedge \beta\right] \vee \alpha \\
&= \beta \vee \alpha. \\
&= \beta.
\end{aligned}$$

Since, $a \in XY$. Therefore $C_{XY}(a) = 1$.

Hence, $(C_{XY}(a) \wedge \beta) \vee \alpha = (1 \wedge \beta) \vee \alpha = \beta \vee \alpha = \beta$.

Therefore, $\left(C_X \circ_\alpha^\beta C_Y\right)(a) = (C_{XY}(a) \wedge \beta) \vee \alpha$.

If $a \notin XY$, then $a \neq \sum_{i=1}^{p} x_i y_i$ for $x_i \in X$ and $y_i \in Y$.

If $a = \sum_{i=1}^{p} s_i t_i$ for some $s_i, t_i \in R$, then we have

$$\begin{aligned}
&\left(C_X \circ_\alpha^\beta C_Y\right)(a) \\
&= \left[\bigvee_{a=\sum_{i=1}^{p} s_i t_i} \left[\bigwedge_{1\le i\le p} \left[C_X(s_i) \wedge C_Y(t_i)\right] \wedge \beta\right]\right] \vee \alpha \\
&= (0 \wedge \beta) \vee \alpha. \\
&= 0 \\
&= (C_{XY}(a) \wedge \beta) \vee \alpha.
\end{aligned}$$

If $a \neq \sum_{i=1}^{p} s_i t_i$ for $s_i, t_i \in R$, then

$\left(C_X \circ_\alpha^\beta C_Y\right)(a) = 0 = (C_{XY}(a) \wedge \beta) \vee \alpha$.

Hence, in any case, we have $C_X \circ_\alpha^\beta C_Y = (C_{XY} \wedge \beta) \vee \alpha$. ∎

3.4.12 Lemma

If X and Y are subsets of a hemiring R , then $C_X \wedge_\alpha^\beta C_Y = (C_{X\cap Y} \wedge \beta) \vee \alpha$.

Proof. Obvious. ■

3.4.13 Lemma

The characteristic function C_L of a nonempty subset L of R is a fuzzy left (right) ideal with thresholds $(\alpha, \beta]$ of R if and only if L is a left (right) ideal of R.

Proof. Let L be a left ideal of the hemiring R.

Let $x, y \in R$, if $x, y \in L$ then $x + y \in L$.

This implies that $C_L(x+y) = 1$.

Thus, $C_L(x+y) \vee \alpha \geq C_L(x) \wedge C_L(y) \wedge \beta$.

If x or $y \notin L$ then $C_L(x) \wedge C_L(y) \wedge \beta = 0 \leq C_L(x+y) \vee \alpha$.

This implies, $\max\{C_L(x+y), \alpha\} \geq \min\{C_L(x), C_L(y), \beta\}$.

Again,

If $y \in L$ then $xy \in L$. This implies, $C_L(xy) = 1$.

Thus $C_L(xy) \vee \alpha \geq C_L(y) \wedge \beta$.

If $y \notin L$ then $C_L(y) \wedge \beta = 0 \leq C_L(xy) \vee \alpha$.

This implies, $\max\{C_L(xy), \alpha\} \geq \min\{C_L(y), \beta\}$.

Hence, C_L is a fuzzy left ideal with thresholds $(\alpha, \beta]$ of a hemiring R.

Conversely, let $x, y \in L$ this implies that $C_L(x) = 1 = C_L(y)$

Since, $\max\{C_L(x+y), \alpha\} \geq \min\{C_L(x), C_L(y), \beta\}$

$$= \min\{1, 1, \beta\} = \beta.$$

Therefore, $C_L(x+y)=1$, this implies $x+y\in L$.

For second condition, let $x\in R$ and $y\in L$ this implies that $C_L(y)=1$.

Since, $\max\{C_L(xy),\alpha\}\geq\min\{C_L(y),\beta\}$

$$=\min\{1,\beta\}=\beta.$$

Thus $C_L(xy)=1$. This implies $xy\in L$.

Hence, L is a left ideal of R.

Similarly we can show, the characteristic function C_L is a fuzzy right ideal with thresholds $(\alpha,\beta]$ of R if and only if L is a right ideal of R. ∎

3.4.14 Definition

Let f and g be fuzzy ideals with thresholds $(\alpha,\beta]$ of a hemiring R, then $f\wedge_\alpha^\beta g$ is defind as, $\left(f\wedge_\alpha^\beta g\right)(x)=((f\wedge g)(x)\wedge\beta)\vee\alpha$, where $x\in R$.

3.4.15 Lemma

Let f be a fuzzy right ideal with thresholds $(\alpha,\beta]$ and g be a fuzzy left ideal with thresholds $(\alpha,\beta]$ of a hemiring R, then $f\circ_\alpha^\beta g\leq f\wedge_\alpha^\beta g$.

Proof. Let f and g be fuzzy right and fuzzy left ideals with thresholds $(\alpha,\beta]$ of R, respectively. For any $x\in R$,

$$\begin{aligned}&\left(f\circ_\alpha^\beta g\right)(x)\\ &=\left[\bigvee_{x=\sum_{i=1}^p y_iz_i}\left[\bigwedge_{1\leq i\leq p}\left(f(y_i)\wedge g(z_i)\right)\right]\wedge\beta\right]\vee\alpha\\ &=\left[\bigvee_{x=\sum_{i=1}^p y_iz_i}\left[\bigwedge_{1\leq i\leq p}\left(\left(f(y_i)\wedge\beta\right)\wedge\left(g(z_i)\wedge\beta\right)\right)\right]\wedge\beta\right]\vee\alpha\end{aligned}$$

$$\leq \left[\bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1\leq i\leq p} (f(y_i z_i) \vee \alpha) \wedge (g(y_i z_i) \vee \alpha) \right] \wedge \beta \right] \vee \alpha$$

$$= \left[\bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\left(\bigwedge_{1\leq i\leq p} f(y_i z_i) \wedge \beta \right) \wedge \left(\bigwedge_{1\leq i\leq p} g(y_i z_i) \wedge \beta \right) \right] \wedge \beta \right] \vee \alpha$$

$$\leq \left[\bigvee_{x=\sum_{i=1}^{p} y_i z_i} [[f(\textstyle\sum_{i=1}^{p} y_i z_i) \vee \alpha] \wedge [g(\textstyle\sum_{i=1}^{p} y_i z_i) \vee \alpha]] \wedge \beta \right] \vee \alpha$$

$$= \left[\bigvee_{x=\sum_{i=1}^{p} y_i z_i} [f(x) \wedge g(x)] \wedge \beta \right] \vee \alpha$$

$$= [[f(x) \wedge g(x)] \wedge \beta] \vee \alpha$$

$$= ((f \wedge g)(x) \wedge \beta) \vee \alpha.$$

$$= \left(f \wedge_{\alpha}^{\beta} g\right)(x).$$

This implies, $f \circ_{\alpha}^{\beta} g \leq f \wedge_{\alpha}^{\beta} g$. ■

3.5 Regular and Weakly regular hemirings

In this section we characterize regular and right weakly regular hemirings by the properties of their fuzzy ideals with thresholds $(\alpha, \beta]$.

3.5.1 Lemma

A hemiring R is regular if and only if $f \circ_{\alpha}^{\beta} g = f \wedge_{\alpha}^{\beta} g$, for every fuzzy right ideal f and for every fuzzy left ideal g with thresholds $(\alpha, \beta]$ of R.

Proof. Let f and g be any fuzzy right and fuzzy left ideals with thresholds $(\alpha, \beta]$ of R, respectively. For any $x \in R$,

$\left(f \circ_{\alpha}^{\beta} g\right)(x) \leq \left(f \wedge_{\alpha}^{\beta} g\right)(x)$. (by Lemma 3.4.15)

On the other hand, since R is regular hemiring, so for every $x \in R$, there exists $s \in R$ such that $x = xsx$.

Now

$$\begin{aligned}
\left(f \wedge_{\alpha}^{\beta} g\right)(x) &= ((f \wedge g)(x) \wedge \beta) \vee \alpha \\
&= (f(x) \wedge g(x) \wedge \beta) \vee \alpha \\
&\leq [f(xs) \wedge g(x) \wedge \beta] \vee \alpha \\
&\leq \bigvee_{x=\sum_{i=1}^{q} a_i b_i} \left[\bigwedge_{1 \leq i \leq q} [f(a_i) \wedge g(b_i)] \wedge \beta\right] \vee \alpha \\
&= f \circ_{\alpha}^{\beta} g(x).
\end{aligned}$$

$$\text{This implies, } \left(f \wedge_{\alpha}^{\beta} g\right)(x) \leq \left(f \circ_{\alpha}^{\beta} g\right)(x).$$

$$\text{Hence, } \quad , f \circ_{\alpha}^{\beta} g = f \wedge_{\alpha}^{\beta} g.$$

Conversely, let A be a right ideal and B be a left ideal of R. Then by Lemma 3.4.13 C_A and C_B are fuzzy right and left ideals with thresholds $(\alpha, \beta]$ of R, respectively.

By hypothesis, $C_A \circ_{\alpha}^{\beta} C_B = C_A \wedge_{\alpha}^{\beta} C_B$. By Lemma 3.4.11 and 3.4.12 we have, $C_A \circ_{\alpha}^{\beta} C_B = (C_{AB} \wedge \beta) \vee \alpha$ and $C_A \wedge_{\alpha}^{\beta} C_B = (C_{A \cap B} \wedge \beta) \vee \alpha$ respectively.

This implies, $(C_{AB} \wedge \beta) \vee \alpha = (C_{A \cap B} \wedge \beta) \vee \alpha$.

By Lemma 3.4.10 we get, $AB = A \cap B$.

Hence, by Proposition 1.3.3 R is a regular hemiring. ■

3.5.2 Definition

A fuzzy ideal f with thresholds $(\alpha, \beta]$ of R is said to be idempotent if $f \circ_{\alpha}^{\beta} f = f_{\alpha}^{\beta}$ where $f_{\alpha}^{\beta} = (f \wedge \beta) \vee \alpha$.

3.5.3 Theorem

The following assertions for a hemiring R are equivalent;

(1) R is right weakly regular hemiring.

(2) All fuzzy right ideals with thresholds $(\alpha,\beta]$ of R are idempotent.

(3) $f \circ_{\alpha}^{\beta} g = f \wedge_{\alpha}^{\beta} g$ for all fuzzy right ideals f and all fuzzy two sided ideals g with thresholds $(\alpha,\beta]$ of R.

Proof. (1) $\Longrightarrow$ (2)

Let f be a fuzzy right ideal with thresholds $(\alpha,\beta]$ of R. By Lemma 3.4.15, $f \circ_{\alpha}^{\beta} f \leq f_{\alpha}^{\beta}$. On the other hand, since R is right weakly regular hemiring, so $x \in (xR)^2$.

Hence, $x = \sum_{i=1}^{q} xa_i xb_i$ for some a_i, $b_i \in R$ and $q \in \mathbb{N}$. So,

$$\begin{aligned}
f(x) &= f(x) \wedge f(x) \\
(f(x) \wedge \beta) \vee \alpha &= (f(x) \wedge f(x) \wedge \beta) \vee \alpha \\
f_{\alpha}^{\beta}(x) &= ((f(x) \wedge \beta) \wedge (f(x) \wedge \beta) \wedge \beta) \vee \alpha \\
&\leq ((f(xa_i) \vee \alpha) \wedge (f(xb_i) \vee \alpha) \wedge \beta) \vee \alpha \quad \text{for } 1 \leq i \leq q \\
f_{\alpha}^{\beta}(x) &\leq \left[\bigwedge_{1\leq i\leq q} [f(xa_i) \wedge f(xb_i)] \wedge \beta\right] \vee \alpha \\
&\leq \bigvee_{x=\sum_{j=1}^{r} y_j z_j} \left[\bigwedge_{1\leq j\leq r} [f(y_j) \wedge f(z_j)] \wedge \beta\right] \vee \alpha \\
&= \left(f \circ_{\alpha}^{\beta} f\right)(x).
\end{aligned}$$

$$\text{This implies,} \quad f_{\alpha}^{\beta}(x) \leq \left(f \circ_{\alpha}^{\beta} f\right)(x).$$

$$\text{Hence,} \quad f \circ_{\alpha}^{\beta} f = f_{\alpha}^{\beta}.$$

This shows that all fuzzy right ideals with thresholds $(\alpha,\beta]$ of R are idempotent.

(2) $\Longrightarrow$ (1)

Let $x \in R$, we claim that $x \in (xR)^2$.

Let $xR = A$ be the right ideal generated by x, and C_A be the characteristic function of A. By Lemma 3.4.13 C_A is a fuzzy right ideal with thresholds $(\alpha, \beta]$ of R.

By our assumption, $(C_A \wedge \beta) \vee \alpha = C_A \circ_\alpha^\beta C_A$.

By Lemma 3.4.11 $C_A \circ_\alpha^\beta C_A = (C_{AA} \wedge \beta) \vee \alpha$.

So we have, $(C_A \wedge \beta) \vee \alpha = (C_{AA} \wedge \beta) \vee \alpha$.

Therefore, by Lemma 3.4.10 we have, $A = AA$.

Now since $x \in A$, this imlpies $x \in A^2$.

Therefore, $x \in (xR)^2$.

Hence, R is a right weakly regular hemiring.

(1) $\Longrightarrow$ (3)

Let f be a fuzzy right ideal with thresholds $(\alpha, \beta]$ and g be a fuzzy two sided ideal with thresholds $(\alpha, \beta]$ of R.

By Lemma 3.4.15, $f \circ_\alpha^\beta g \leq f \wedge_\alpha^\beta g$.

For reverse inclusion, let $x \in R$, since R is right weakly regular hemiring, so $x \in (xR)^2$. Hence, $x = \sum_{i=1}^{p} xa_i xb_i$ for some a_i , $b_i \in R$ and $p \in \mathbb{N}$.

Now

$$
\begin{aligned}
\left(f \wedge_{\alpha}^{\beta} g\right)(x) &= ((f \wedge g)(x) \wedge \beta) \vee \alpha \\
&= (f(x) \wedge g(x) \wedge \beta) \vee \alpha \\
&= [(f(x) \wedge \beta) \wedge (g(x) \wedge \beta) \wedge \beta] \vee \alpha \\
&\leq [(f(xa_i) \vee \alpha) \wedge (g(xb_i) \vee \alpha) \wedge \beta] \vee \alpha \quad \text{where } 1 \leq i \leq p \\
\left(f \wedge_{\alpha}^{\beta} g\right)(x) &\leq \left[\bigwedge_{1 \leq i \leq p} [f(xa_i) \wedge g(xb_i)] \wedge \beta\right] \vee \alpha \\
&\leq \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \leq i \leq p} [f(y_i) \wedge g(z_i)] \wedge \beta\right] \vee \alpha \\
&= \left(f \circ_{\alpha}^{\beta} g\right)(x).
\end{aligned}
$$

This implies, $\left(f \wedge_{\alpha}^{\beta} g\right)(x) \leq \left(f \circ_{\alpha}^{\beta} g\right)(x)$.

Hence, $f \wedge_{\alpha}^{\beta} g = f \circ_{\alpha}^{\beta} g$.

$(3) \Longrightarrow (1)$

Let A and B be the right and two sided ideal of R, respectively.Then by Lemma 3.4.13, the characteristic function C_A and C_B are fuzzy right and fuzzy two sided ideals with threshold $(\alpha, \beta]$ of R.

Hence, by hypothesis,$C_A \circ_{\alpha}^{\beta} C_B = C_A \wedge_{\alpha}^{\beta} C_B$.

By Lemma 3.4.11, $C_A \circ_{\alpha}^{\beta} C_B = (C_{AB} \wedge \beta) \vee \alpha$ and $C_A \wedge_{\alpha}^{\beta} C_B = (C_{A\cap B} \wedge \beta) \vee \alpha$.

So we have, $(C_{AB} \wedge \beta) \vee \alpha = (C_{A\cap B} \wedge \beta) \vee \alpha$.

Therefore, by Lemma 3.4.10 $AB = A \cap B$.

Hence, by Proposition 1.3.5, R is right weakly regular semiring. ■

3.6 Hemirings in which each fuzzy ideal with thresholds $(\alpha, \beta]$ is idempotent

3.6.1 Theorem

The following assertions for a hemiring R are equivalent;

(1) R is fully idempotent.

(2) Each fuzzy ideal with thresholds $(\alpha, \beta]$ of R is idempotent.

(3) For each pair of fuzzy ideals f, g of R with thresholds $(\alpha, \beta]$, $f \wedge_\alpha^\beta g = f \circ_\alpha^\beta g$.

Proof. $(1) \Longrightarrow (3)$

Let f and g be any pair of fuzzy ideals with thresholds $(\alpha, \beta]$ of R. Then $f \circ_\alpha^\beta g \leq f \wedge_\alpha^\beta g$ (by Lemma 3.4.15). Since R is fully idempotent, so $(x) = (x)^2$. This means $x = \sum_{i=1}^{p} r_i x s_i x t_i$.

$$
\begin{aligned}
\left(f \wedge_\alpha^\beta g\right)(x) &= ((f \wedge g)(x) \wedge \beta) \vee \alpha \\
&= [(f(x) \wedge g(x)) \wedge \beta] \vee \alpha \\
&\leq [(f(r_i x s_i) \wedge g(x t_i)) \wedge \beta] \vee \alpha \\
&\leq \bigwedge_{1 \leq i \leq p} [[(f(r_i x s_i) \wedge g(x t_i)) \wedge \beta] \vee \alpha] \\
&\leq \left[\bigvee_{x = \sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1 \leq i \leq p} (f(y_i) \wedge g(z_i))\right] \wedge \beta\right] \vee \alpha \\
&= \left(f \circ_\alpha^\beta g\right)(x).
\end{aligned}
$$

$$\text{Therefore,} \quad f \wedge_\alpha^\beta g \leq f \circ_\alpha^\beta g.$$

$$\text{Hence,} \quad f \wedge_\alpha^\beta g = f \circ_\alpha^\beta g.$$

$(3) \Longrightarrow (2)$

Let f and g be any pair of fuzzy ideals with thresholds $(\alpha, \beta]$ of R. We have $f \wedge_{\alpha}^{\beta} g = f \circ_{\alpha}^{\beta} g$. Take $g = f$. Thus, $f \wedge_{\alpha}^{\beta} f = f \circ_{\alpha}^{\beta} f$, where $f \wedge_{\alpha}^{\beta} f = f_{\alpha}^{\beta}$. This implies that $f_{\alpha}^{\beta} = f \circ_{\alpha}^{\beta} f$. Hence, each fuzzy ideal with thresholds $(\alpha, \beta]$ of R is idempotent.

$(2) \Longrightarrow (1)$

Let A be an ideal of R. Then by Lemma 3.4.13, C_A is a fuzzy ideal with thresholds $(\alpha, \beta]$ of R.

Hence, $C_A \circ_{\alpha}^{\beta} C_A = (C_A \wedge \beta) \vee \alpha$

But, $C_A \circ_{\alpha}^{\beta} C_A = (C_{AA} \wedge \beta) \vee \alpha$ (by Lemma 3.4.11).

Therefore, $(C_{AA} \wedge \beta) \vee \alpha = (C_A \wedge \beta) \vee \alpha$.

This implies, $AA = A$ (by Lemma 3.4.10).

Hence, R is fully idempotent semiring. ■

3.6.2 Theorem

The following assertions for a hemiring R are equivalent;

(1) R is fully idempotent.

(2) The set $\mathcal{L}_R = \{f_{\alpha}^{\beta} : f$ is a fuzzy ideal with thresholds $(\alpha, \beta]$ of $R\}$, (ordered by $\leq$) form a distributive lattice under the sum and intersection of fuzzy ideals f_{α}^{β} and g_{α}^{β} with $f_{\alpha}^{\beta} \wedge_{\alpha}^{\beta} g_{\alpha}^{\beta} = f_{\alpha}^{\beta} \circ_{\alpha}^{\beta} g_{\alpha}^{\beta}$.

Proof. $(1) \Longrightarrow (2)$

The set $\mathcal{L}_R = \left\{f_{\alpha}^{\beta} : f \text{ is a fuzzy ideal with threshold } (\alpha, \beta] \text{ of } R\right\}$, (ordered by $\leq$) is a lattice under the sum and intersection of fuzzy ideals with thresholds $(\alpha, \beta]$. Moreover, since R is a fully idempotent semiring, it follows that $f \wedge_{\alpha}^{\beta} g = f \circ_{\alpha}^{\beta} g$, for each pair of fuzzy ideals f_α^β and g_α^β of R. We now show that $\mathcal{L}_R$ is a distributive lattice, that is, for fuzzy ideals $f_\alpha^\beta, g_\alpha^\beta$ and h_α^β of R,

we have $\left[\left(f \wedge_\alpha^\beta g\right) +_\alpha^\beta h\right] = \left[\left(f +_\alpha^\beta h\right) \wedge_\alpha^\beta \left(g +_\alpha^\beta h\right)\right]$. For any $x \in R$,

$$\begin{aligned}
&\left[\left(f \wedge_\alpha^\beta g\right) +_\alpha^\beta h\right](x) \\
&= \left[\bigvee_{x=y+z} \left[\left(f \wedge_\alpha^\beta g\right)(y) \wedge h(z)\right] \wedge \beta\right] \vee \alpha \\
&= \left[\bigvee_{x=y+z} \left[\left(\left(f \wedge g\right)(y) \wedge \beta\right) \vee \alpha \wedge h(z)\right] \wedge \beta\right] \vee \alpha \\
&= \left[\bigvee_{x=y+z} \left[f(y) \wedge g(y) \wedge h(z) \wedge h(z)\right] \wedge \beta\right] \vee \alpha \\
&= \bigvee_{x=y+z} \left[\left[\left(f(y) \wedge h(z) \wedge \beta\right) \vee \alpha\right] \wedge \left[\left(g(y) \wedge h(z) \wedge \beta\right) \vee \alpha\right] \wedge \beta\right] \vee \alpha \\
&\leq \left[\left[\left(f +_\alpha^\beta h\right)(x) \wedge \left(g +_\alpha^\beta h\right)(x)\right] \wedge \beta\right] \vee \alpha.
\end{aligned}$$

Thus, $\left[\left(f \wedge_\alpha^\beta g\right) +_\alpha^\beta h\right](x) \leq \left[\left(f +_\alpha^\beta h\right) \wedge_\alpha^\beta \left(g +_\alpha^\beta h\right)\right](x)$. $\quad (i)$

Again,

$$\begin{aligned}
&\left[\left(f +_\alpha^\beta h\right) \wedge_\alpha^\beta \left(g +_\alpha^\beta h\right)\right](x) \\
&= \left[\left(f +_\alpha^\beta h\right) \circ_\alpha^\beta \left(g +_\alpha^\beta h\right)\right](x) \\
&= \left[\bigvee_{x=\sum_{i=1}^p y_i z_i} \left[\bigwedge_{1\leq i\leq p} \left[\left(f +_\alpha^\beta h\right)(y_i) \wedge \left(g +_\alpha^\beta h\right)(z_i)\right]\right] \wedge \beta\right] \vee \alpha \\
&= \left[\bigvee_{x=\sum_{i=1}^p y_i z_i} \left[\bigwedge_{1\leq i\leq p} \left[\begin{array}{c} \left[\left(\bigvee_{y_i=r_i+s_i} f(r_i) \wedge h(s_i) \wedge \beta\right) \vee \alpha\right] \wedge \\ \left[\left(\bigvee_{z_i=t_i+u_i} g(t_i) \wedge h(u_i) \wedge \beta\right) \vee \alpha\right] \end{array}\right]\right] \wedge \beta\right] \vee \alpha \\
&= \left[\bigvee_{x=\sum_{i=1}^p y_i z_i} \left[\bigwedge_{1\leq i\leq p} \left[\bigvee_{\substack{y_i=r_i+s_i \\ z_i=t_i+u_i}} \left[f(r_i) \wedge h(s_i) \wedge g(t_i) \wedge h(u_i)\right]\right]\right] \wedge \beta\right] \vee \alpha \\
&= \left[\bigvee_{x=\sum_{i=1}^p y_i z_i} \left[\bigwedge_{1\leq i\leq p} \left[\bigvee_{\substack{y_i=r_i+s_i \\ z_i=t_i+u_i}} \left[f(r_i) \wedge h(s_i) \wedge h(s_i) \wedge g(t_i) \wedge h(u_i)\right]\right]\right] \wedge \beta\right] \vee \alpha \\
&= \left[\bigvee_{x=\sum_{i=1}^p y_i z_i} \left[\bigwedge_{1\leq i\leq p} \left[\bigvee_{\substack{y_i=r_i+s_i \\ z_i=t_i+u_i}} \left[\begin{array}{c} \left(f(r_i) \wedge \beta\right) \wedge \left(g(t_i) \wedge \beta\right) \wedge \left(h(s_i) \wedge \beta\right) \wedge \\ \left(h(s_i) \wedge \beta\right) \wedge \wedge \left(h(u_i) \wedge \beta\right) \end{array}\right]\right]\right] \wedge \beta\right] \vee \\
&\alpha
\end{aligned}$$

$$\leq \left[\bigvee_{x=\sum_{i=1}^{p} y_i z_i}\left[\bigwedge_{1\leq i\leq p}\left[\bigvee_{\substack{y_i=r_i+s_i\\ z_i=t_i+u_i}}\left[\begin{array}{c}(f(r_it_i)\vee\alpha)\wedge(g(r_it_i)\vee\alpha)\\ \wedge(h(s_it_i)\vee\alpha)\wedge(h(s_iu_i)\vee\alpha)\wedge(h(r_iu_i)\vee\alpha)\end{array}\right]\right]\right]\wedge\beta\right]\vee$$

α

$$= \left[\bigvee_{x=\sum_{i=1}^{p} y_i z_i}\left[\bigwedge_{1\leq i\leq p}\left[\bigvee_{\substack{y_i=r_i+s_i\\ z_i=t_i+u_i}}\left[\begin{array}{c}((f(r_it_i)\wedge g(r_it_i)\wedge\beta)\vee\alpha)\wedge\\ (h(s_it_i)\wedge h(s_iu_i)\wedge h(r_iu_i)\wedge\beta)\vee\alpha\end{array}\right]\right]\right]\wedge\beta\right]\vee$$

α

$$\leq \left[\bigvee_{x=\sum_{i=1}^{p} y_i z_i}\left[\bigwedge_{1\leq i\leq p}\left[\bigvee_{\substack{y_i=r_i+s_i\\ z_i=t_i+u_i}}\left[(f\wedge_\alpha^\beta g)(r_it_i)\wedge h(s_it_i+s_iu_i+r_iu_i)\right]\right]\right]\wedge\beta\right]\vee\alpha$$

$$\leq \bigvee_{x=\sum_{i=1}^{p} y_i z_i}\left[\bigwedge_{1\leq i\leq p}\left[(f\wedge_\alpha^\beta g)+_\alpha^\beta h\right](y_iz_i)\right]$$

$$\leq \bigvee_{x=\sum_{i=1}^{p} y_i z_i}\left[(f\wedge_\alpha^\beta g)+_\alpha^\beta h\right](x)$$

$$= \left[(f\wedge_\alpha^\beta g)+_\alpha^\beta h\right](x).$$

Thus, $\left[(f+_\alpha^\beta h)\wedge_\alpha^\beta(g+_\alpha^\beta h)\right](x)\leq\left[(f\wedge_\alpha^\beta g)+_\alpha^\beta h\right](x)$. *(ii)*

From (i) and (ii) we get, $\left[(f\wedge_\alpha^\beta g)+_\alpha^\beta h\right]=\left[(f+_\alpha^\beta h)\wedge_\alpha^\beta(g+_\alpha^\beta h)\right]$.

$(2)\Longrightarrow(1)$

Suppose that the set $\mathcal{L}_R=\{f_\alpha^\beta$, where f is a fuzzy ideal with thresholds $(\alpha,\beta]$ of $R\}$, (ordered by $\leq$) is a distributive lattice under the sum and intersection of fuzzy ideals with $f_\alpha^\beta\wedge_\alpha^\beta g_\alpha^\beta=f_\alpha^\beta\circ_\alpha^\beta g_\alpha^\beta$, for each pair of fuzzy ideals f_α^β and g_α^β of R. Then for any fuzzy ideal f_α^β of R, we have, $f_\alpha^\beta\circ_\alpha^\beta f_\alpha^\beta=f_\alpha^\beta\wedge_\alpha^\beta f_\alpha^\beta$.

First we have to show that $f_\alpha^\beta\circ_\alpha^\beta f_\alpha^\beta=f\circ_\alpha^\beta f$.

Consider, $\left(f_\alpha^\beta\circ_\alpha^\beta f_\alpha^\beta\right)(x)$

$$=\left[\bigvee_{x=\sum_{i=1}^{p} y_i z_i}\left[\bigwedge_{1\leq i\leq p}\left(f_\alpha^\beta(y_i)\wedge f_\alpha^\beta(z_i)\right)\right]\wedge\beta\right]\vee\alpha$$

$$=\left[\bigvee_{x=\sum_{i=1}^{p} y_i z_i}\left[\bigwedge_{1\leq i\leq p}\left[((f(y_i)\wedge\beta)\vee\alpha\wedge(f(z_i)\wedge\beta)\vee\alpha\right]\right]\wedge\beta\right]\vee\alpha$$

$$=\left[\bigvee_{x=\sum_{i=1}^{p} y_i z_i}\left[\bigwedge_{1\leq i\leq p}\left[f(y_i)\wedge f(z_i)\right]\right]\wedge\beta\right]\vee\alpha.$$

$= \left(f \circ_{\alpha}^{\beta} f\right)(x)$.

Therefore, we get $f_{\alpha}^{\beta} \circ_{\alpha}^{\beta} f_{\alpha}^{\beta} = f \circ_{\alpha}^{\beta} f$.

This implies, $f \circ_{\alpha}^{\beta} f = f_{\alpha}^{\beta}$.

Hence, R is fully idempotent semiring. ■

3.6.3 Definition

A fuzzy ideal h with thresholds $(\alpha, \beta]$ of a hemiring R is called a fuzzy prime ideal with thresholds $(\alpha, \beta]$ of R if for any fuzzy ideals f and g with thresholds $(\alpha, \beta]$ of R, $f \circ_{\alpha}^{\beta} g \leq h$ implies that $f_{\alpha}^{\beta} \leq h_{\alpha}^{\beta}$ or $g_{\alpha}^{\beta} \leq h_{\alpha}^{\beta}$.

3.6.4 Definition

A fuzzy ideal h with thresholds $(\alpha, \beta]$ of a hemiring R is called a fuzzy irreducible ideal with thresholds $(\alpha, \beta]$ if for fuzzy ideals f and g with thresholds $(\alpha, \beta]$ of R, $f \wedge_{\alpha}^{\beta} g = h$ implies that $f_{\alpha}^{\beta} = h_{\alpha}^{\beta}$ or $g_{\alpha}^{\beta} = h_{\alpha}^{\beta}$.

3.6.5 Theorem

Let R be a fully idempotent hemiring. For fuzzy ideal h with thresholds $(\alpha, \beta]$ of R, the following conditions are equivalent;

(1) h is fuzzy prime ideal with thresholds $(\alpha, \beta]$ of R.

(2) h is fuzzy irreducible ideal with thresholds $(\alpha, \beta]$ of R.

Proof. $(1) \Longrightarrow (2)$

Let f and g be any fuzzy ideals with thresholds $(\alpha, \beta]$ of R. Assume that h is a fuzzy prime ideal with thresholds $(\alpha, \beta]$. We show that h is a fuzzy irreducible ideal with thresholds $(\alpha, \beta]$ of R. Let $h = f \wedge_\alpha^\beta g$. Since R is fully idempotent, so by Lemma 3.6.1 $f \wedge_\alpha^\beta g = f \circ_\alpha^\beta g$. Since h is a fuzzy prime ideal with thresholds $(\alpha, \beta]$, this implies $f_\alpha^\beta \leq h_\alpha^\beta$ or $g_\alpha^\beta \leq h_\alpha^\beta$.

Again, since $f \wedge_\alpha^\beta g = h$, this implies $h \leq f$ and $h \leq g$. This implies $h_\alpha^\beta \leq f_\alpha^\beta$ and $h_\alpha^\beta \leq g_\alpha^\beta$. It follows that, $f_\alpha^\beta = h_\alpha^\beta$ or $g_\alpha^\beta = h_\alpha^\beta$.

Hence, h is a fuzzy irreducible ideal with thresholds $(\alpha, \beta]$ of R.

$(2) \Longrightarrow (1)$

Assume that h is a fuzzy irreducible ideal with thresholds $(\alpha, \beta]$ of R. We have to show h is a fuzzy prime ideal with thresholds $(\alpha, \beta]$ of R. Suppose that there exist fuzzy ideals f and g with threshold $(\alpha, \beta]$ such that $f \circ_\alpha^\beta g \leq h$. As, R is fully idempotent hemiring, so by Theorem 3.6.1 $f \wedge_\alpha^\beta g = f \circ_\alpha^\beta g$. This implies, $f \wedge_\alpha^\beta g \leq h$. Again, since R is fully idempotent hemiring, it follows from Theorem 3.6.2 that the set of fuzzy ideals with thresholds $(\alpha, \beta]$ of R (ordered by $\leq$) is a distributive lattice with respect to the sum and intersection of fuzzy ideals with thresholds $(\alpha, \beta]$. Hence the inequality $f \wedge_\alpha^\beta g \leq h$ becomes $\left(f \wedge_\alpha^\beta g\right) +_\alpha^\beta h = h_\alpha^\beta$, and using the distributivety of this lattice we have,

$\left(f +_\alpha^\beta h\right) \wedge_\alpha^\beta \left(g +_\alpha^\beta h\right) = h_\alpha^\beta$. Since, h is a fuzzy irreducible ideal with thresholds $(\alpha, \beta]$ so h_α^β is also, It follows that either $f +_\alpha^\beta h = h_\alpha^\beta$ or $g +_\alpha^\beta h = h_\alpha^\beta$. This implies $f_\alpha^\beta \leq h_\alpha^\beta$ or $g_\alpha^\beta \leq h_\alpha^\beta$.

Hence, h is a fuzzy prime ideal with thresholds $(\alpha, \beta]$ of hemiring R. ■

3.6.6 Lemma

Let R be a fully idempotent hemiring. If f_α^β is a fuzzy ideal with thresholds $(\alpha, \beta]$ of R with $f_\alpha^\beta(a) = \gamma$, where a is any element of R and $\gamma \in (0, 1]$, then there exists a fuzzy prime ideal h_α^β with thresholds $(\alpha, \beta]$ such that $f_\alpha^\beta \leq h_\alpha^\beta$ and $h_\alpha^\beta(a) = \gamma$.

Proof. Let $X = \{g_\alpha^\beta \,; g_\alpha^\beta$ is a fuzzy ideal with thresholds $(\alpha, \beta]$ of R, $g_\alpha^\beta(a) = \gamma$, and $f_\alpha^\beta \leq g_\alpha^\beta\}$. Then $X \neq \emptyset$, since $f_\alpha^\beta \in X$. Let $\mathcal{F}$ be a totally ordered subset of X, say $\mathcal{F} = \left\{f_{i\alpha}^\beta, i \in I\right\}$. We claim that $\bigvee\limits_{i\in I} f_{i\alpha}^\beta$ is a fuzzy ideal with thresholds $(\alpha, \beta]$ of R. For any $x \in R$.

$$
\begin{aligned}
& \bigvee_{i\in I} f_{i\alpha}^\beta(x) \wedge \bigvee_{i\in I} f_{i\alpha}^\beta(y) \wedge \beta \\
&= \bigvee_{i\in I} f_{i\alpha}^\beta(x) \wedge \bigvee_{j\in I} f_{j\alpha}^\beta(y) \wedge \beta \\
&= \left[\bigvee_{i\in I} f_{i\alpha}^\beta(x)\right] \wedge \bigvee_{j\in I}\left[f_{j\alpha}^\beta(y)\right] \wedge \beta \\
&= \bigvee_{j}\left[\left[\bigvee_{i\in I} f_{i\alpha}^\beta(x)\right] \wedge f_{j\alpha}^\beta(y)\right] \wedge \beta \\
&= \bigvee_{j}\left[\bigvee_{i}\left[f_{i\alpha}^\beta(x) \wedge f_{j\alpha}^\beta(y)\right]\right] \wedge \beta \\
&\leq \bigvee_{j}\left[\bigvee_{i}\left[f_{i\alpha}^{j\beta}(x) \wedge f_{i\alpha}^{j\beta}(y)\right]\right] \wedge \beta \quad ;\text{where } f_i^{-j} = \max\left\{f_{i\alpha}^\beta, f_{j\alpha}^\beta\right\} : f_{i\alpha}^{j\beta} \in \left\{f_{i\alpha}^\beta, i \in I\right\} \\
&= \bigvee_{j}\left[\bigvee_{i}\left[f_{i\alpha}^{j\beta}(x) \wedge f_{i\alpha}^{j\beta}(y) \wedge \beta\right]\right] \\
&\leq \bigvee_{j}\left[\bigvee_{i}\left[f_{i\alpha}^{j\beta}(x+y) \vee \alpha\right]\right] \\
&= \bigvee_{i,j}\left[f_{i\alpha}^{j\beta}(x+y) \vee \alpha\right] \\
&\leq \bigvee_{i}\left[f_{i\alpha}^{\beta}(x+y) \vee \alpha\right] \\
&= \bigvee_{i} f_{i\alpha}^{\beta}(x+y) \vee \alpha.
\end{aligned}
$$

This imlpies $\max\left\{\bigvee\limits_{i} f_{i\alpha}^\beta(x+y), \alpha\right\} \geq \min\left\{\bigvee\limits_{i\in I} f_{i\alpha}^\beta(x), \bigvee\limits_{i\in I} f_{i\alpha}^\beta(y), \beta\right\}$.

Now consider,

$$\begin{aligned}\left(\bigvee_{i\in I} f_{i\alpha}^{\beta}\right)(x)\wedge\beta &= \bigvee_{i\in I}\left(f_{i\alpha}^{\beta}(x)\right)\wedge\beta\\ &= \bigvee_{i}\left[f_{i\alpha}^{\beta}(x)\wedge\beta\right]\\ &\leq \bigvee_{i} f_{i\alpha}^{\beta}(xy)\vee\alpha\\ &= \left(\bigvee_{i} f_{i\alpha}^{\beta}\right)(xy)\vee\alpha.\\ \text{This implies, } \max\left\{\left(\bigvee_{i} f_{i\alpha}^{\beta}\right)(xy),\alpha\right\} &\geq \min\left\{\left(\bigvee_{i\in I} f_{i\alpha}^{\beta}\right)(x),\beta\right\}.\end{aligned}$$

Thus, $\bigvee_{i} f_{i\alpha}^{\beta}$ is a fuzzy ideal with thresholds $(\alpha,\beta]$ of R. Clearly $f_{\alpha}^{\beta}\leq\bigvee_{i} f_{i\alpha}^{\beta}$ and $\left(\bigvee_{i} f_{i\alpha}^{\beta}\right)(a)=\bigvee_{i} f_{i\alpha}^{\beta}(a)=\gamma$.

Therefore, $\bigvee_{i} f_{i\alpha}^{\beta}$ is l.u.b of $\mathcal{F}$. Hence by Zorns Lemma, there exists a fuzzy ideal h_{α}^{β} with thresholds $(\alpha,\beta]$ of R which is maximal with respect to the property that $f_{\alpha}^{\beta}\leq h_{\alpha}^{\beta}$ and $h_{\alpha}^{\beta}(a)=\gamma$. We now show that h_{α}^{β} is a fuzzy irreducible ideal with thresholds $(\alpha,\beta]$ of R.

Suppose that $h_{\alpha}^{\beta}=k_{1\alpha}^{\beta}\wedge_{\alpha}^{\beta}k_{2\alpha}^{\beta}$, where $k_{1\alpha}^{\beta}$ and $k_{2\alpha}^{\beta}$ are fuzzy ideals with thresholds $(\alpha,\beta]$ of R. This implies that $h_{\alpha}^{\beta}\leq k_{1\alpha}^{\beta}$ and $h_{\alpha}^{\beta}\leq k_{2\alpha}^{\beta}$. We claim that either $h_{\alpha}^{\beta}=k_{1\alpha}^{\beta}$ or $h_{\alpha}^{\beta}=k_{2\alpha}^{\beta}$. Suppose on contrary $h_{\alpha}^{\beta}\neq k_{1\alpha}^{\beta}$ or $h_{\alpha}^{\beta}\neq k_{2\alpha}^{\beta}$. Since h_{α}^{β} is maximal with respect to the property that $h_{\alpha}^{\beta}(a)=\gamma$ and $h_{\alpha}^{\beta}\lneqq k_{1\alpha}^{\beta}$ or $h_{\alpha}^{\beta}\lneqq k_{2\alpha}^{\beta}$, it follows that, $k_{1\alpha}^{\beta}(a)\neq\gamma$ and $k_{2\alpha}^{\beta}(a)\neq\gamma$.

Hence, $\gamma=h_{\alpha}^{\beta}(a)=\left(k_{1\alpha}^{\beta}\wedge_{\alpha}^{\beta}k_{2\alpha}^{\beta}\right)(a)=\left(k_{1\alpha}^{\beta}\wedge k_{2\alpha}^{\beta}\right)(a)=k_{1\alpha}^{\beta}(a)\wedge k_{2\alpha}^{\beta}(a)\neq\gamma$, which is absurd.

Hence, either $h_{\alpha}^{\beta}=k_{\alpha}^{\beta}$ and $h_{\alpha}^{\beta}=k_{2\alpha}^{\beta}$. This prove that h_{α}^{β} is a fuzzy irreducible ideal of R.

Hence, by Theorem 3.6.5, h_{α}^{β} is a fuzzy prime ideal with threshold of R. ■

3.6.7 Theorem

The following assertions for a hemiring R are equivalent:

(1) R is fully idempotent.

(2) The set $\mathcal{L}_R = \{f_\alpha^\beta : f$ is a fuzzy ideal with thresholds $(\alpha, \beta]$ of $R\}$, (ordered by $\leq$) form a distributive lattice under the sum and intersection of fuzzy ideals f_α^β and g_α^β with thresholds $(\alpha, \beta]$ such as $f_\alpha^\beta \wedge_\alpha^\beta g_\alpha^\beta = f_\alpha^\beta \circ_\alpha^\beta g_\alpha^\beta$.

(3) Each fuzzy ideal with thresholds $(\alpha, \beta]$ is the intersection of all those fuzzy prime ideals with thresholds $(\alpha, \beta]$ of R which contain it.

Proof. (1) $\Longrightarrow$ (2) is proved in Theorem 3.6.2. Now we have to prove,

(2) $\Longrightarrow$ (3)

Let f_α^β be a fuzzy ideal with thresholds $(\alpha, \beta]$ of R. Let $\left\{g_{i\alpha}^\beta, i \in I\right\}$ be the family of fuzzy prime ideal with thresholds $(\alpha, \beta]$ of R which contains f_α^β. Obviously $f_\alpha^\beta \leq \bigwedge_{i \in I} g_{i\alpha}^\beta$. $\qquad (i)$

we have to show $\bigwedge_{i \in I} g_{i\alpha}^\beta \leq f_\alpha^\beta$.

Let "a" be any element of R, then by Lemma 3.6.6 there exists a fuzzy prime ideal with thresholds $(\alpha, \beta]$ say $g_{j\alpha}^\beta$ such that $f_\alpha^\beta \leq g_{j\alpha}^\beta$ and $f_\alpha^\beta(a) = g_{j\alpha}^\beta(a)$. Thus $g_{j\alpha}^\beta \in \left\{g_{i\alpha}^\beta, i \in I\right\}$. Hence, $\bigwedge_{i \in I} g_{i\alpha}^\beta \leq g_{j\alpha}^\beta$.

So, $\bigwedge_{i \in I} g_{i\alpha}^\beta(a) \leq g_{j\alpha}^\beta(a) = f_\alpha^\beta(a)$.

This implies, $\bigwedge_{i \in I} g_{i\alpha}^\beta \leq f_\alpha^\beta$. $\qquad (ii)$

From (i) and (ii), we get $\bigwedge_{i \in I} g_{i\alpha}^\beta = f_\alpha^\beta$.

(3) $\Longrightarrow$ (1)

Let f be any fuzzy ideal with thresholds $(\alpha, \beta]$ of R so f_α^β is also. Then $f_\alpha^\beta \circ_\alpha^\beta f_\alpha^\beta$

is also a fuzzy ideal with thresholds $(\alpha, \beta]$ of R. Hence, according to the statement (3), $f_{\alpha}^{\beta} \circ_{\alpha}^{\beta} f_{\alpha}^{\beta}$ can be written as $f_{\alpha}^{\beta} \circ_{\alpha}^{\beta} f_{\alpha}^{\beta} = \bigwedge_{i \in I} g_{i\alpha}^{\beta}$ where $\left\{g_{i\alpha}^{\beta}, i \in I\right\}$ be the family of fuzzy prime ideal with thresholds $(\alpha, \beta]$ of R which contain $f_{\alpha}^{\beta} \circ_{\alpha}^{\beta} f_{\alpha}^{\beta}$.

Now $f_{\alpha}^{\beta} \circ_{\alpha}^{\beta} f_{\alpha}^{\beta} \leq g_{i\alpha}^{\beta}$ for all $i \in I$, and since $g_{i\alpha}^{\beta}$ is a fuzzy prime ideal with thresholds $(\alpha, \beta]$. So, $f_{\alpha}^{\beta} \leq g_{i\alpha}^{\beta}$ for all $i \in I$.

Thus, $f_{\alpha}^{\beta} \leq \bigwedge_{i \in I} g_{i\alpha}^{\beta} = f_{\alpha}^{\beta} \circ_{\alpha}^{\beta} f_{\alpha}^{\beta}$, and we know that $f_{\alpha}^{\beta} \circ_{\alpha}^{\beta} f_{\alpha}^{\beta} = f \circ_{\alpha}^{\beta} f$.

This implies, $f_{\alpha}^{\beta} \leq f \circ_{\alpha}^{\beta} f$. $\qquad (iii)$

And by Lemma 3.4.15 we get, $f \circ_{\alpha}^{\beta} f \leq f_{\alpha}^{\beta}$. $\qquad (iv)$

From (iii) and (iv) we have, $f \circ_{\alpha}^{\beta} f = f_{\alpha}^{\beta}$.

Hence, R is a fully idempotent hemiring. ∎

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