Matrix Wiener-Hopf Analysis of Some Scattering Problems of Acoustic Waves

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بسم الله الرحمن الرحيم

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A Thesis
Submitted in the Partial Fulfillment of the
Requirements for the Degree of

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IN
MATHEMATICS**

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We accept this thesis as conforming to the required standard

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Dedicated

Most Respectfully to

The Most Respectable
The Greatest of Prophets
The Most Beloved
The Holy Prophet

Hazrat Muhammad

Sallallah-o-Alaih-e-Wassallum

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Nomenclature

$f(x)$	Function of x
$\overline{f}(\alpha)$	Fourier transform of $f(x)$
α	Fourier transform parameter
σ, τ	Real and imaginary parts of α
ω	Oscillating frequency
t	Time
c	Speed of sound
$\widehat{p}$	Acoustic pressure
M	Constant
ξ	Complex variable
$K(\xi)$	Kernel function
a	Constant
$A(\alpha)$, $B(\alpha)$, $C(\alpha)$	Functions of complex variable α
λ	± 1
π	Constant
ψ_+	Regular in upper half or in the domain q to ∞
ψ_1	Regular in the domain p to q
ψ_-	Regular in lower half or in the domain $-\infty$ to p
$L_\pm(\alpha)$, $D_\pm(\alpha)$	Split functions
$J(\alpha)$, $P(\alpha)$	Polynomials in α
τ'_-, τ'_+, A	Limits of integration

x, y, z	Cartesian coordinates
ϕ_t, ψ_t	Total velocity potential
ϕ_i, ψ_i	Incident wave
ϕ_r, ψ_r	Reflected wave
ϕ, ψ	Diffracted wave
D_1, D_2	Constants
k	Wave number
$D_-(\alpha)$, $S_-(\alpha)$	Difference and sum of two negative functions
θ_0	Angle of incidence
θ	Angle of observer
$\gamma(\alpha)$	Branch cuts
$K(\alpha)$	Branch cuts
$H_\pm(\alpha)$	Split functions
χ	Large parameter
$C_i\,(i = 1-6)$	Constants
$j(z)$, $h(z)$	Analytic functions
χ_0, ϑ	Polar form of $\chi\left(=\chi_0 e^{i\vartheta}\right)$
β	Complex specific admittance
a_1, a_2	Scalars
kd	Strip length
u, v	Real and imaginary parts of $h(z)$

I, $I_i\,(i=1-6)$	Integrals
$\widehat{J}$	Max. value of $j\,(z)$
U	Max. value of u on the path A to B
s	Large parameter
Γ	Gamma function
$\widehat{\mu}$	Scalar in Kharapkov's method
$z_0 = x_0 + iy_o$	Saddle point
r_0, θ_0	Polar coordinates
$\mathbf{H}\,(\alpha)\,,\ \mathbf{M}\,(\alpha)\,,\ \mathbf{W}\,(\alpha)$	Kernel matrices
$\mathbf{H}_{\pm}\,(\alpha)\,,\ \mathbf{W}_{\pm}\,(\alpha)$	Factors of kernel matrices
$\widetilde{\mathbf{Q}}\,(\alpha)$	Formal inverse of $\mathbf{Q}\,(\alpha)$ Matrix
$\mathbf{C}$	Commutant Matrix
l, m, n	Scalars
g, f	Polynomials
ϱ_1, ϱ_2	Eigen values
ϵ	Index
ϵ_1, ϵ_2	Functions
F	A quantity in Kharapkov's method
$\mathcal{F}$, $\widetilde{\mathcal{F}}$	Fresnel functions in line and point source solutions
$F\,(\alpha)$	A function in strip/slit problems

$\mathbf{I}$	Identity matrix
$g(x), p(x), q(x), r(x)$	Continuous functions
$y(x)$	Unknown function
δ	Dirac delta function
$M_\pi(z)$	Maliuzhinetz function
ε	A small positive quantity
$W(.)$	Wronskian
$\widetilde{\gamma}$	$\ln 2$
φ	An angle in argument of Maliuzhinetz function
$C_1(\alpha), C_2(\alpha), C(\alpha)$	Functions of α
(x_o, y_0)	Position of line source
n	Unit normal in the positive direction
$A(\alpha), B(\alpha), C(\alpha)$	Functions of complex variable α
$\Psi_\pm(\alpha), \Psi_1(\alpha), \mathbf{Q}(\alpha)$	2×1 matrices
sgn	Signum function
$q(\alpha)$ and $r(\alpha)$	Integrals in the solution of line source excitation
$\mathbf{P}(\alpha)$	Matrix
p^*, p^{**}	Constants in solutions of line and point sources
$\varkappa(\alpha)$	Analytic function
$T_\pm(\alpha), S_\pm(\alpha)$	Split functions
ζ, t_1	Real variables

Φ_t	Total velocity potential in case of point source
μ	Transform variable for z
μ_1, μ_2	Functions in the solution of point source (Appendix C)
η_1	Branch cut in the solution of point source
$\Theta = \nu$	Saddle point in case of point source excitation
k_0	Free space wave number
η	Specific impedance of the surface
$\mathcal{L}_\pm$	Integration lines in the complex plane
$\mathcal{L}$	Linear operator
σ_1	$= k\sqrt{1 - \frac{1}{\eta^2}}$
$\overline{\Lambda}_+(\alpha), \overline{\Lambda}'_+(\alpha)$	Functions of α
$\overline{\Phi}$	Transform of Φ
$\kappa(\alpha)$	A function in determinant of $\mathrm{W}(\alpha)$
$g_1(\mu), g_2(\mu), f_1(\mu)$	Functions of the variable μ in Appendix C
s_1, s_2	Functions
$H(.)$	Heaviside function
P	A function in the point source excitation
$\Omega, \widehat{\Omega}$	Functions in point source solution
$H_0^{(1)}$	Hankel function of order zero, kind one
ε_1	Lower limit of integration in point source problem
$\widehat{\tau}, \tau_{R_1}, \tau_{R_2}$	Functions in point source solution

p, q	Edges of strip/slit
$G(\alpha), \widetilde{G}(\alpha)$	Known functions of α in strip/slit geometries
$\mathbf{A}(\alpha)$	Known column vector
$\mathbf{U}_{\pm}, \mathbf{V}_{\pm}, \mathbf{R}_{\pm}, \mathbf{S}_{\pm}$	Split (matrices) functions of α in strip/slit geometries
$T(\alpha), T_1(\alpha), T_2(\alpha)$	Known functions in case of strip/slit geometries
$W_{m,n}$	Whittaker function
$\mathbf{S}_{+}^{*}, \mathbf{D}_{+}^{*}, \mathbf{F}_{+}^{*}$	Known functions in case of strip/slit geometries
$R_i\,(i = 1-6)$	Functions in strip/slit geometries
$P_i\,(i = 1-6)$	Functions in strip/slit geometries
$G_i\,(i = 1-6)$	Functions in strip/slit geometries
C_1, C_2	Known constants in strip/slit geometry
$\widetilde{C}_1, \widetilde{C}_2$	Known constants in strip/slit geometry
E_{-1}, D_{-1}, D_0, E_r	Known constants in strip/slit geometry
$k\rho$	Observer distance from the point of observation (usually origin)

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Abstract

The aim of this thesis is to address the scattering problems of acoustic waves by different geometries (e.g., half plane, junction, strip and slit) satisfying soft-hard boundary conditions which are substantial and important in the existing and prevailing diffraction theory and has been addressed since long and by many researchers [33, 34, 151 – 153] working in the area of scattering of waves. The soft-hard boundary conditions on a barrier or on a part of barrier involves tedious mathematical analysis since the applications of these conditions result into the coupled Wiener-Hopf (W-H) equations that cannot be decoupled trivially. The mathematical route of the presented problems comprises of an integral transform, the W-H technique in Jones' interpretation, the steepest descent asymptotic method and the Geometrical Theory of Diffraction (GTD).

In chapters 3 and 4 two problems of line source and point source scattering of acoustic waves by a soft-hard half plane and by the junction of transmissive and soft-hard half planes have been investigated. Both of these problems are not only physically important but mathematically difficult to work out because from a line source and point source excitations the waves are coming from a known position as compared to the case of plane wave excitation in which the waves are known to be incident from infinity and hence the line and point sources are considered to be better substitutes than a plane wave situation [54]. Both of the problems also involve the multiplicative splitting of the kernel matrices appearing in these problems. The far field solution of both of these problems is obtained using the method of steepest descent and hence extending the recent works of [33] and [70].

The plane wave situation graphs can be recovered by shifting the line source to a far off distance and the mathematical results of [33] and [70] are modified by a multiplicative factor in the case of line source excitation and agrees well with the existing evidences. Point source problems are examined using the results of line source cases.

The second problem considered is that of diffraction of a plane acoustic wave by a finite soft-hard strip. Using the Fourier integral transform and Jones' method [14, 135] the boundary value problem resulted into a matrix W-H functional equation which is solved by a procedure outlined in [14]. Several integrals in the analysis of the problem formulated for strip are approximated by using the physical concept of GTD which takes into account that the strip length is large as compared to the incident wavelength. The far-field solution is determined by using the steepest descent asymptotic method. In addition, some graphs are also plotted and discussed about the parameters of interest.

The final and last problem considered concerns about the diffraction of plane acoustic waves by a slit aperture in an infinite soft-hard expanse. The mathematical treatment of the problem for diffraction from the slit aperture is same as that of strip barrier. The slit width is assumed to be larger than the incident wavelength so that integrals in the analysis can be evaluated by using the GTD. The effects of various parameters like observer distance from the origin, the wave number parameter and slit width parameter on the separated diffracted field produced by the two edges of the slit are also noted.

The problems considered in this thesis complete to some extent the discussion for the soft-hard half plane and offer considerable scope for further research.

Contents

Chapter 1

Introduction

Acoustics is the science of sound. One of the most fascinating subject of mankind and is as old as our universe is. Although acoustic wave motion has fascinated many generations of applied mathematicians, acoustic physicists, communication engineers, numerical simulists, geophysicists and otologists etc., yet the scientific study of sound is generally attributed to Greeks. The word acoustics is derived from the Greek word **'akouein'** means **'to hear'** and **'Sauver'** appears to be the first person to apply the term acoustics to the science of sound in 1701 [1].

Historical developments in acoustics [2]

Acoustic waves have found their applications in many areas such as music, architecture, engineering, medical science (acoustic tomography), oil and gas exploration (bore-hole sounding) aerodynamics and linguistics etc. Nowadays acoustics have been established as the most modern branch of science with itself has so many branches. To name a few only e.g., Architectural Acoustics, Physical Acoustics, Engineering Acoustics, Structural Acoustics, Underwater Acoustics, Physiological and psychological Acoustics and many others.

Like many other branches of science and technology, the developments in acoustics were proceeded by empirical observation. It is not surprising because the required solutions are very complicated and problems are generally not amenable to the direct theoretical treatment. Many existing acoustical phenomena are not less than three dimensions and are transient in nature as well. Most of the scattering of acoustic waves is taking place in the medium which is neither at rest nor isotropic. Further, the boundary conditions to which wave equation is subjected are not necessarily on a regular shaped geometry and hence require complex analytic treatment in order to solve the problem under consideration. Anyone who has just taken a casual look at the equations which arise while solving a relatively simple problem of acoustic wave scattering realizes that why a scientist working in this field has frequently turned to his apparatus instead of turning towards the pencil. It is also worthwhile to mention at this stage that a subject can be understood better when something about its history is known. Here we mention some of the important and distinct experiments that took place in the history of acoustics during the various centuries.

Before the 18th century, the scientific apparatus in acoustics was of the simplest kind. Pythagoras established mathematics in the Greek culture and studied vibrating strings and musical sounds before $16th$ century. Galileo $(1564 - 1642)$ used pendulum as a demonstration instrument. Mersenne $(1588 - 1648)$, using pendulum, measured the speed of sound to be 1038 ft./sec. The first serious attempt to measure velocity of sound was made by Sir Isaac Newton. Flamsteed and Halley (1708) calculated the speed of sound to be 1142 ft./sec. Tuning fork was invented in 1711 by John Shore, a trumpeter in the service of George I of England. A variable standard of frequency was given by Stancari in 1706. A commission at French Academy of Sciences in 1738 again calculated the speed of sound to be 1094 ft./sec. at $0°C$. In the history of acoustics, no remarkable event took place between 1750 and 1800.

In the nineteenth century, tremendous progress in the field of acoustics has been achieved. Chaldni (1802) determined the wave patterns of vibrating bodies by means of sand figures, longitudinal and torsional vibrations of rods and strings and transverse vibrations of bars and plates. In 1807, Thomas Young described a graphical apparatus for accurate determination of frequency. Wheatstone in 1833 proved the existence of nodal lines.

Scott adopted the smoked surface technique for the measurement of air waves in 1858. Rudolph Koeing presented his results in the form of large collection of phonograms in London in 1862, which comprises all the applications of the method that have so far been discovered in acoustics.

An optical method to determine the strength of sound waves was first described by Biot in 1820 and developed further by Kundt in 1864 and by Mach in 1872. A second optical method was developed by Toepler and Boltzmann in 1870. Acoustic siren, which is the principal source of continuous variable sound, in its present form was first constructed by Seeback (1841), improved by Koenig in 1867.

The other great names that appeared on the important published papers on acoustics are those of Rayleigh, Stokes, Lamb, Helmholtz, Tyndall, Morse, Taylor, Sabine and Webster. Lord Rayleigh's two-volume treatise on '*The Theory of Sound*' was based mainly on the mathematical treatment of the subject of acoustics and was soon acknowledged by the scientific world as authoritative. This great treatise contains the whole physical theory and a logical order of a huge material collected from all sources. Another great mass of work on acoustics may be found in the classical volumes of '*Vibrations and Sound*' contributed by P.M. Morse. With this bird's eye view of the history of acoustics, attention is now focused on the scattering of acoustic waves.

Literature survey

In this thesis, the emphasis will be on some scattering problems of acoustic waves using matrix Wiener-Hopf approach. A brief account of scattering phenomenon, the Wiener-Hopf (W-H) technique, matrix factorization methods and geometries related to the solved problems etc. and some details of relevant literature survey are now presented.

The scattering of acoustic waves was first investigated mathematically by Lord

Rayleigh under the assumption that scatterers are small as compared to the wavelength. The solution of scattering by rigid, immovable circular cylinders and spheres, not necessarily small compared to the wavelength, was given by Morse [3].

Scattering is a physical process where some forms of radiation such as light, sound or moving particles are forced to deviate from a straight path by one or more localized non-uniformities in the medium through which they pass. In conventional applications, this also includes deviation of reflected radiation from the angle given by the law of reflection. The types of non-uniformities that can cause scattering are known as scatterers or scattering centers. The effects of such features on the path of almost any type of propagating wave or moving particle can be described in the framework of scattering theory. Scattering theory has played a central and vital role in the twentieth century mathematical physics. Scattering phenomenon has attracted, perplexed and challenged mathematicians, scientists, physicists and engineers for centuries because the problem of calculating an exact analytical solution for the problem of scattering of acoustic/electromagnetic waves by an arbitrarily shaped body is generally intractable. However, for particular geometries, e.g., half planes, junctions, wedges, strips, slits and certain other restricted classes of problems, it is possible to obtain an analytical solution of the governing boundary value problem. Broadly speaking, the scattering theory is concerned with the effect that is caused on an incident wave by the inhomogeneous medium.

Although the mathematical analysis of scattering of light by faceted objects was the focus of attention of many medieval scientists, one pertinent name is that of Ibn-ul-Haitham of Basra, who flourished in 10th century A. D. The extensive studies of the scattering of acoustic/electromagnetic waves proceeded after the investigations of half plane and wedge problems by Poincar'e [4] and Sommerfeld [5]. Poincare [4] calculated the asymptotic field for diffraction off a wedge. Sommerfeld [5] considered the diffraction of a plane wave from the half plane by employing the method of images on a Riemann surface. The technique of constructing the requisite many valued solution of the wave equation was simplified by Sommerfeld in a subsequent paper [6].

Finally, Carslaw [7] replaced the image method with a direct construction of a solution that yields simpler formulae. These had already been obtained independently by McDonald [8] by summing the Fourier series representation of the Green's function. Sommerfeld's results were also obtained by Lamb [9] by using parabolic coordinates. It was later shown by Magnus [10] that the problem of diffraction of sound waves of small amplitude can also be reduced to the solution of a singular integral equation. Later on the integral equation apparatus was also used by Levine and Schwinger [11], Miles [12] and Copson [13] to study the diffraction of waves by a plane screen.

Copson [13] solved the integral equation arose in his work with the help of W-H technique and showed that his solution was in accordance with that of Sommerfeld's [5, 6] half plane problem. Afterwards, it was shown by Levine and Schwinger [11] that all problems involving diffraction (scattering) of a plane wave by a number of semi-infinite parallel cylinders or plates can be formulated into W-H type integral equation(s) which is/are capable of exact solution(s).

W-H technique [14] was introduced around 1931 to solve certain integral equations, of the type

$$\int_0^\infty f(\xi) K(x-\xi) \, d\xi = g(x) \qquad (0 < x < \infty), \tag{1.1}$$

where K and g are given and f is to be found. In fact, Copson [13] was the first to apply this method to solve diffraction problems by formulating the problem of diffraction of sound waves by a perfectly reflecting half plane in terms of an integral equation. The elegance and analytical sophistication of this method, now called the W-H technique, impresses all who use it. Its applicability to almost all branches of engineering, mathematics, physics and applied mathematics such as diffraction of acoustic, elastic, sonar, radio and electromagnetic waves, crystal growth, fracture mechanics, flow problems, diffusion models, hydrodynamics, electrodynamics, aerodynamics, neutron transport theory, nuclear reaction theory, plasma physics and mathematical finance etc., is borne out of many thousands of papers published in these areas since its conception. The W-H technique remains an extremely important tool for modern day scientists and the areas of applications continue to broaden [15]. Lawrie and Abrahams [15] pronounced the Noble's monograph [14] to be the Bible of a W-H practitioner.

This technique is based on the application of integral transforms and the theory of analytic continuation of complex valued functions. Here we describe the mathematical maneuvers that are central to the W-H technique. In this procedure, the associated mathematical boundary value problem is transformed to the W-H functional equation involving two unknown functions of a complex variable. These unknown functions, which are Fourier transform of the solution of a partial differential equation or an integral equation, are analytic in two overlapping half planes. This provides sufficient information to conclude by means of theory of analytic continuation, that the two functions are the representations in their half planes of analyticity of a function which is analytic in the whole complex plane. In other words, these two functions represent an entire function. From the asymptotic behavior of this entire function, enough information can be obtained to determine the Fourier transform and from these, by inverse transformation, the solutions of the original equation [16].

An important step in the solution of the W-H functional equation is to decompose the kernel function. In general, it is known function of a complex variable with a number of poles characterizing the underlying physical process. Thus it is required to split the kernel function into a product or sum of two functions, one being regular in upper/right and other being regular in lower/left half of the complex plane. The procedure for decomposition is relatively simple for a scalar kernel. However, in the case of system of W-H equations, one has to work with a matrix W-H equation involving a matrix kernel. For the coupled or system of W-H equations, one needs the explicit factorization of a kernel matrix into a product of two matrices, one being regular in the upper (right) and the other being regular in the lower (left) half plane.

The factor matrices should be non-singular and of algebraic growth at infinity. To find such factors of the kernel matrix is both vital and difficult at the same time. The non-commutativity and the satisfaction of radiation conditions present further problems. There is, as yet, no general and comprehensive procedure for factorization of such matrices, although the factorization of the restricted class of matrices has been achieved. The absence of general procedure for finding the factor matrices has been a stumbling block in finding the solutions of many problems. Nevertheless, the development and improvement of matrix factorization techniques has been progressing steadily. For example, the Wiener-Hopf-Hilbert (W-H-H) method introduced by [17], [18] and [19] is a powerful tool in the case when the kernel matrix has only branch-point singularities, while the Daniele-Kharapkov method proposed independently by Daniele [20] and Kharapkov [21], is effective for the class of kernel matrices having pole-singularities and branch-point singularities. A major breakthrough for a fairly large class of matrices has been made by Jones [22] who for the first time examined the question of commutative factorization of the kernel matrices and presented a natural extension of Kharapkov's [21] work. It was shown by Asghar and Hassan [23] that Jones' [22] and Kharpkov's [21] methods may be considered as methods that cover the whole range of matrix factorization methods worked out so far. Another detailed survey of matrix factorization methods with reference to applications of these methods to different diffraction problems may also be found in a paper of Büyükaksoy and Serbest [24]. Although the target of factorizing a general 2×2 matrix is known to exist from the work of Gohberg and Krein [25] yet it has only been achieved for the restricted class of matrices. The problem of factorizing a matrix having exponentially growing elements has been addressed by Abrahams and Wickham [26]. They suggested to obtain a single scalar integral equation, the solution of which generates the required factors. It is worthwhile to mention here that use of Padé approximants for kernel factorization proposed by Abrahams [27] has successfully been applied to obtain explicit exact factorization for both scalar and matrix kernels by [28, 29].

For factorizing scalar kernels Carrier [30] in his remarkable paper made very useful suggestions in which he pointed out that if a complicated scalar kernel is replaced by another simpler scalar kernel, provided that the substituted kernel has the same singularity, the same area and the same first moment as that of the original kernel, even then the obtained results are of great importance and work equally as good as those factors which are very complicated functions of the complex variable.

Bates and Mittra [31] introduced an integral representation for the W-H factorization of a class of scalar functions in a form convenient for numerical processing. This proposed representation is particularly suitable for the radiation problems involving waveguide structures. Recently Crighton [32] has proposed that matched asymptotic expansions (MAE) may be used for the asymptotic factorization of W-H kernels.

With these some details about the phenomenon of scattering, the W-H technique and the decomposition of the kernel function, the scattering problems discussed in this thesis and the literature survey relevant to these problems is now presented.

Motivation and thesis plan

The scattering of two dimensional plane acoustic waves from a semi-infinite planar screen is a fundamental problem in the theory of acoustic scattering. Recently, Büyükaksoy [33] studied the problem of diffraction of plane waves by a soft-hard half plane. The continued interest in the problem of diffraction from a soft-hard half plane stems from the fact that it constitutes the simplest half plane problem that can be formulated as a system of coupled W-H equations that cannot be decomposed trivially [33].

Rawlins [34] was the first who solved the problem of diffraction of acoustic waves by a soft-hard half plane. Rawlins [34] pointed out that two unusual features arose in this boundary value problem and adopted an adhoc method for the solution of this boundary value problem. After the lapse of many years, Büyükaksoy [33] not only reconsidered the problem solved by Rawlins [34] but also proposed an appropriate method for the solution of the said boundary value problem. What had not been done is the consideration of a line source (cylindrical wave) and point source (spherical wave) diffraction of acoustic waves from a soft-hard half plane.

Numerous past investigations have been devoted to the study of classical problems of line source and point source scattering of acoustic, electromagnetic, shear horizontal (SH) and seismic waves by various types of half planes, cylinders and other types of objects. To name a few only, e.g., the line source diffraction of electromagnetic waves by a perfectly conducting half plane [35]. Hohmann [36] considered the cylindrical inhomogenity buried in a conductive half space with line source excitation, line source diffraction of acoustic waves by a hard half plane attached to a wake in still air as well as when the medium is convective [37], line source diffraction of acoustic waves by an absorbing half plane [38]. Boersma and Lee [39] studied the electric line source diffraction by a perfectly conducting half plane. Hongo and Nakajima [40] considered the diffraction problem of an anisotropic cylindrical wave by a cylindrical obstacle. Engheta and Papas [41] obtained the far-zone radiated fields due to a line source located at the interface of two homogenous media using an asymptotic technique.

The other significant contributions regarding the line source excitation are line source diffraction of sound waves by an absorbent semi-infinite plane such that the two faces of half plane have different impedances [42], Sanyal and Bhattacharyya [43] obtained a uniform asymptotic expansion of the Maliuzhinetz's exact solution for the plane wave and line source illuminations by a half plane with two face impedances by using Vander Waerden's method. Büyükaksoy and Uzgören [44] studied the magnetic line source diffraction by the edges of cylindrically curved surface with different face impedances. Rawlins et al [45] considered the line source diffraction by an acoustically penetrable or an electromagnetically dielectric half plane. Line source and point source diffraction by three half planes in a moving fluid has been discussed by Asghar et al [46], line source diffraction by a rectangular cylinder on an infinite impedance plane has been examined by Tayyar and Büyükaksoy [47], line source diffraction of acoustic waves by an absorbing half plane using Myre's conditions has been contributed by Ahmad [48]. Hussain [49] analyzed the problem of line source diffraction of electromagnetic waves by a perfectly conducting half plane in a homogeneous bi-isotropic medium. Recently, Ayub et al [50, 51, 52] studied the line source diffraction phenomenon by a junction, reactive step and an impedance step and more recently Ahmed and Naqvi [53] studied the response of a coated nihility circular cylinder subjected to directive electromagnetic radiation produced by a line source.

The phenomenon of point source excitation has also been continuously and rigorously investigated by several authors. Point sources are regarded as fundamental radiating devices [54] and are considered to be better substitutes for plane wave or line source incidences. The solutions of point source problems are regarded as fundamental solutions of the differential solution [55]. Some important contributions regarding the point source scattering situations can be found in the works of Vlaar [56], Ghosh [57], Wenzel [58], Chattopadhyay et al [59], Balasubramanyam [60], Asghar et al [46, $61-64$], Hayat and Asghar [65], Rawlins [66], Ahmad [67] and Ayub et al [50, 52].

Inspired by the above mentioned studies for line source and point source excitations, Chapters 3 and 4 of this thesis are devoted to investigate the problems of line source and point source diffraction of acoustic waves by a soft-hard half plane and by the junction of transmissive and soft-hard half planes.

Soft and hard boundaries are not only well known in acoustics but now these have also been artificially made to study the diffraction of dually polarized electromagnetic waves from a large number geometries, e.g., plane reflector, rationally symmetric reflector, circular cylinder and cylindrical waveguides etc. [68]. The problem of diffraction of waves from a soft-hard half plane is both mathematically difficult and physically important because it results in a matrix W-H equation. In order to complete the solution of the problem the involved matrix kernel has to be factorized. Incidentally for the case of line and point source excitations, the kernel matrix remains unchanged which has been factorized by Büyükaksoy [33]. However, for the sake of completeness the missing steps in the factorizations of the kernel matrices have been incorporated and details are reported in Appendix A of the thesis.

The introduction of line source changes the incident field and the method of solution requires a careful analysis in working out the diffracted field. The consideration of point source diffraction will help understand the acoustic differentiation and will go a step further to complete the discussion for the soft-hard half plane. The mathematical importance of the point source lies in the fact that introduction of point source introduces another variable. The difficulty arises in the solution of the integrals which occur while taking the inverse transform. The integrals are difficult to handle because of the presence of branch points and are amenable to solution via asymptotic methods. The results for the line source and point source diffraction of acoustic waves by a soft-hard plane are presented in Chapter 3 of this thesis. It is noted that the line source incidence, Büyükaksoy [33] results are modified by a multiplicative factor of the form

$$\sqrt{\frac{2\pi}{k\rho_0}}\exp\left(ik\rho_0+i\frac{\pi}{4}\right), \tag{1.2}$$

which agrees well with the results already known [35, 46]. The results obtained for line source incidence lead to the consideration of point source and diffracted field due to the point source is also presented. The mathematical route of both problems consists of Fourier integral transform, the W-H technique in Jones' interpretation [14] and the method of steepest descent [69]. The contents of Chapter 3 have been published in **Archives of Mechanics, 62** (2), (2010) 157 – 174.

In Chapter 4, firstly the analysis of Büyükaksoy et al [70] for the scattering of plane waves by the junction of transmissive (impedance) and soft-hard half planes is extended to the case of line source. Using the results of line source, the analysis is further extended to the case of point source. The kernel matrix appearing in this problem was same as that of [70] but has been reported with sufficient details in Appendix B of the thesis. Mathematically, the results of [50] differ from those of [70] by a multiplicative factor of the form Eq. (1.2). Several graphs for noting the effects of various parameters on the scattered field are also plotted. It is observed that the graphs of [70] can be recovered by shifting the line source to a large distance, which can be considered as check of correctness of presented results in [50]. The importance of present work stems from the facts that:

(a) The scattering properties of a surface are functions of both its geometrical and material properties.

(b) The edge scattering by dihedral structures whose surfaces can be modeled by the impedance [Leontovich] boundary condition has been the focus of attention of many scientists for both acoustic and electromagnetic waves [71].

(c) The junction geometry constitutes a canonical problem for scattering because it requires the use of more sophisticated analytical techniques involving Bessel/Hankel transforms. The diffraction coefficients related to discontinuities in the junction geometry are quite complicated and these have to be used in the problems of practical interest [72].

The pattern of solution of problems of Chapter 4 is same as that of Chapter 3. The contents of Chapter 4 have been published in **Journal of Mathematical Analysis and Applications, 346** (2008) 280 – 295.

Now turning to another class of canonical geometries consisting of strips and slits which are important in diffraction theory and have received a great deal of attention and appreciation. Because of their practical applications in science, engineering and communication systems, strips and slits are typical examples among a number of simple obstacles, and scattering and diffraction problems related to these geometries have been extensively investigated by many authors using a variety of numerical and analytical techniques. From both the strip and slit geometries, the phenomenon of multiple diffraction often occurs which is of great importance in diffraction theory [73]. It has been reported in [74] that the problem of diffraction by a strip with parallel edges was first solved by Fox and in his two subsequent papers, the same author studied the diffraction of pulses by a slit and by a grating. Morse and Rubenstein [75] studied the diffraction of acoustic waves by ribbons and by slits using the method of separation of variables.

Bowman et al [76] summarized and reviewed much of the work done on half planes, strips, slits and cylinders etc. Myre's [77] used symmetry-like principles to study the wave scattering by strip geometry.

Another important technique namely geometrical theory of diffraction (GTD) introduced by Keller [78] has been applied by Senior [79], Tiberio and Kouyoumjian [80] and Tiberio et al [81] to study the diffraction of electromagnetic waves by strips satisfying various types of boundary conditions.

Some of the authors, e.g., Bowman [82], Chakrabarti [83] and Wickham [84] attempted the problems corresponding to strip configuration by using the method of successive approximations. Another well-known technique which proved to be promising, to study scattering by canonical strip and slit configurations, is the W-H technique. Many scientists, e.g., Jones [35], Kobayashi [73], Noble [14], Faulkner [85], Chakrabarti [86], Asghar [87], Asghar et al [88 – 90], Asghar and Hayat [91 – 92], Hayat and Asghar [93, 94] and Ayub et al [95 – 100] successfully employed the W-H technique [14] to study the scattering of acoustic/electromagnetic waves by strips (satisfying different types of boundary conditions).

Another distinct contribution regarding the studies of diffraction by strips using W-H technique in conjunction with ray optical method and spectral iteration technique (SIT) has been developed by Serbest and Büyükaksoy [101] and applied by Büyükaksoy et al [102], Büyükaksoy and Uzgören [103], Serbest et al [104], Büyükaksoy and Uzgören [105], Erdogan et al [106], Büyükaksoy and Alkumru [107, 108] and Cinar and Büyükaksoy [109].

Recently, Imran et al [110, 111] used the Kobayashi's potential method and Castro and Kapanadze [112] used the theory of Bessel potential spaces to study the diffraction by a strip geometry.

Motivated by these studies, Chapter 5 of the present thesis is devoted to the study of diffraction of a plane acoustic wave by a soft-hard strip. The resulting functional matrix W-H equation in the problem is solved by the W-H technique [14]. Some graphs for the different parameters of interest are plotted and discussed. The key attributes of the use of Wiener-Hopf technique are:

(a) Uniform asymptotic solution obtained for the diffracted field has no restriction on incident and observation angles, contrary to GTD [73].

(b) As compared to the numerical techniques which are valid only for boundaries of finite length, the W-H method does not have such a restriction [113].

The contents of Chapter 5 have also appeared in **Optics Communications 282**, (2009) 4322 – 4328.

The last and sixth chapter of the dissertation is dedicated to the study of scattering of acoustic waves by a slit in an infinite soft-hard plane. Almost all the methods that are available for strip geometry are equally good to study the diffraction from a slit geometry with minor modifications. Keller [114] applied the GTD to study the diffraction from a slit of any shape in a thick screen. Clemmow [115] derived dual integral equations for the diffracted field by a slit. Diffraction of plane electromagnetic waves by a slit in an infinite conduction plane has been studied by Hamid et al [116] and Karp and Russek [117]. Levine [118] studied diffraction of acoustic waves by a slit in an infinite hard plane. Büyükaksoy and Topsakal [119] and Birbir and Büyükaksoy [120] studied the diffraction of electromagnetic waves by a slit in a thick impedance slit using the W-H technique in conjunction with an iterative procedure. Using W-H technique, the diffraction of waves (acoustic and electromagnetic) have been treated by Asghar et al [121, 122], Asghar and Hayat [123], Hayat et al [124, 125] and Ayub et al [126, 127].

Another important configuration consisting of a slit in an impedance plane and a parallel complementary strip, which may be used for the purpose of electromagnetic shielding and optimal coupling between incident and transmitted field, has been considered by Cinar and Büyükaksoy [128] to study the electromagnetic plane wave diffraction. Using Kobayashi's potential, method based on discontinuity properties of Weber-Schafheitlin integral, Hongo [129] and Imran et al [130] studied the diffraction of electromagnetic waves by a slit configuration. Recently, Naveed et al [131] used Maliuzhinitz function to study the diffraction of electromagnetic plane waves by a slit in an impedance screen and Ahmad and Naqvi [132] discussed the problem of electromagnetic scattering from a two dimensional perfect electromagnetic conductor (PEMC) strip and PEMC strip grating (slit) using numerical simulation and presented a semi analytical solution.

A new technique namely Sommerfeld-Maliuzhinetz integral representation has been recently developed by Bernard [133] to study the delicate problem of three part impedance plane such that the impedance of each part is different.

Keeping in view of the importance of slit geometry, the last and sixth chapter of the thesis throws light on the diffraction of plane acoustic waves by a slit in an infinite soft-hard half plane. Integrals transforms, the W-H technique and asymptotic methods are used to analyze the situation. The explicit expressions for the singly diffracted field (separated field) and doubly diffracted field (interaction of one edge upon the other) are obtained.

The contents of this chapter have been published in **Progress in Electromagnetics Research B, 11** (2009) 103 – 131.

Throughout the thesis, the following considerations have been taken into account:

- The time dependence of the form $e^{-i\omega t}$ is assumed and suppressed.
- For the problems under consideration, a matrix W-H equation is formulated and then solved to give the solution of the corresponding problem.
- Far field approximation is used so that the scattered field dominates and the effects of surface waves can be neglected.

Chapter 2

Mathematical Preliminaries

In this chapter, some definitions and mathematical preliminaries which will be used in the subsequent chapters are presented. These consist of the analytical properties of the Fourier transform [134], the W-H technique [14, 54], Jones' method [14, 135], the method of steepest descent used for the asymptotic evaluation of certain integrals appearing in different diffraction problems [69, 136 – 138], the Daniele-Kharapkov methods for the matrix factorization [20, 21, 139], the Green's function [140, 141], and the famous Maliuzhinetz half plane function [142 – 148]. The contents of this chapter can be found in many standard text books [14, 54, 69, 134, 136 – 138, 140, 141] and in some technical reports [139, 143, 148, 150]. These are presented for the sake of completion of the thesis document.

2.1 Analytic properties of the Fourier transform [134]

Consider

$$\overline{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x} dx, \tag{2.1.1}$$

where the transform variable α is complex in general. In order to discuss the properties of the function $\overline{f}(\alpha)$ we write

$$f(x) = f_+(x) + f_-(x), \tag{2.1.2}$$

where

$$f_+(x) = \begin{cases} 0 & x < 0 \\ f(x) & x > 0, \end{cases} \tag{2.1.3}$$

$$f_-(x) = \begin{cases} f(x) & x < 0 \\ 0 & x > 0. \end{cases} \tag{2.1.4}$$

Thus,

$$\overline{f}_+(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x)\, e^{i\alpha x} dx \tag{2.1.5}$$

and

$$\overline{f}_-(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} f(x)\, e^{i\alpha x} dx. \tag{2.1.6}$$

The analytic properties of $\overline{f}(\alpha)$ by establishing the properties of $\overline{f}_+(\alpha)$ and $\overline{f}_-(\alpha)$ are now discussed. Firstly, consider Eq. (2.1.5)

$$\overline{f}_+(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x)\, e^{i\alpha x} dx.$$

If the function $f_+(x)$ is of exponential order, i.e.,

$$|f_+(x)| < M\ e^{\tau_- x} \quad as \quad x \longrightarrow \infty, \tag{2.1.7}$$

then the function $\overline{f}_+(\alpha)$ is an analytic function of the complex variable $\alpha = \sigma + i\tau$ in the domain $\operatorname{Im}\alpha > \tau_-$ and in this domain $\overline{f}_+(\alpha) \longrightarrow 0$ as $|\alpha| \longrightarrow \infty$. Noticing that

$$M\ e^{\tau_- x}\ e^{i\alpha x} = M\ e^{\tau_- x}\ e^{(i\sigma - \tau)x} = M\ e^{(\tau_- - \tau)x}\ e^{i\sigma x}, \tag{2.1.8}$$

is bounded when $(\tau_- - \tau) < 0$ or equivalently $\tau > \tau_-$ or $\operatorname{Im}\alpha > \tau_-$. Now by taking the inverse Fourier transform

$$f_+(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f}_+(\alpha)\, e^{-i\alpha x} d\alpha, \tag{2.1.9}$$

where the integration will be performed over any straight line $\operatorname{Im}\alpha > \tau_-$ parallel to the real axis in the complex $\alpha - plane$. Now for the problems considered in this thesis, the strips of analyticity can be calculated by considering the following cases.

- For $\tau_- < 0$, (i.e., $f_+(x)$ decreases at infinity), the domain of analyticity of $\overline{f}_+(\alpha)$ contains the real axis and Eq. (2.1.9) can be integrated along the real axis.

- For $\tau_- > 0$, (i.e., $f_+(x)$ increases at infinity but not faster than the exponential function with the linear exponent), the domain of analyticity of $\overline{f}_+(\alpha)$ lies above the real axis of the complex $\alpha - plane$ and Eq. (2.1.9) can be integrated above the real axis.

Similarly for the function

$$f_{-}(x) = \begin{cases} f(x) & x < 0 \\ 0 & x > 0, \end{cases}$$

satisfying the exponential order condition

$$|f_{-}(x)| < M\, e^{\tau_{+} x} \qquad \text{as} \quad x \longrightarrow \infty. \tag{2.1.10}$$

Then consider Eq. (2.1.6)

$$\overline{f}_{-}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} f(x)\, e^{i\alpha x} dx,$$

is analytic for the complex variable α in the domain $\operatorname{Im}\alpha < \tau_{+}$. By the inverse Fourier transform

$$f_{-}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f}_{-}(\alpha)\, e^{-i\alpha x} d\alpha, \tag{2.1.11}$$

for $\tau_{+} > 0$, the domain of analyticity of $\overline{f}_{-}(\alpha)$ contains the real axis and for $\tau_{+} < 0$, the domain of analyticity does not contain the real axis. Hence the Eq. (2.1.11) is analytic in the domain $\tau_{-} < \operatorname{Im}\alpha < \tau_{+}$ as shown in Fig. 2.1.

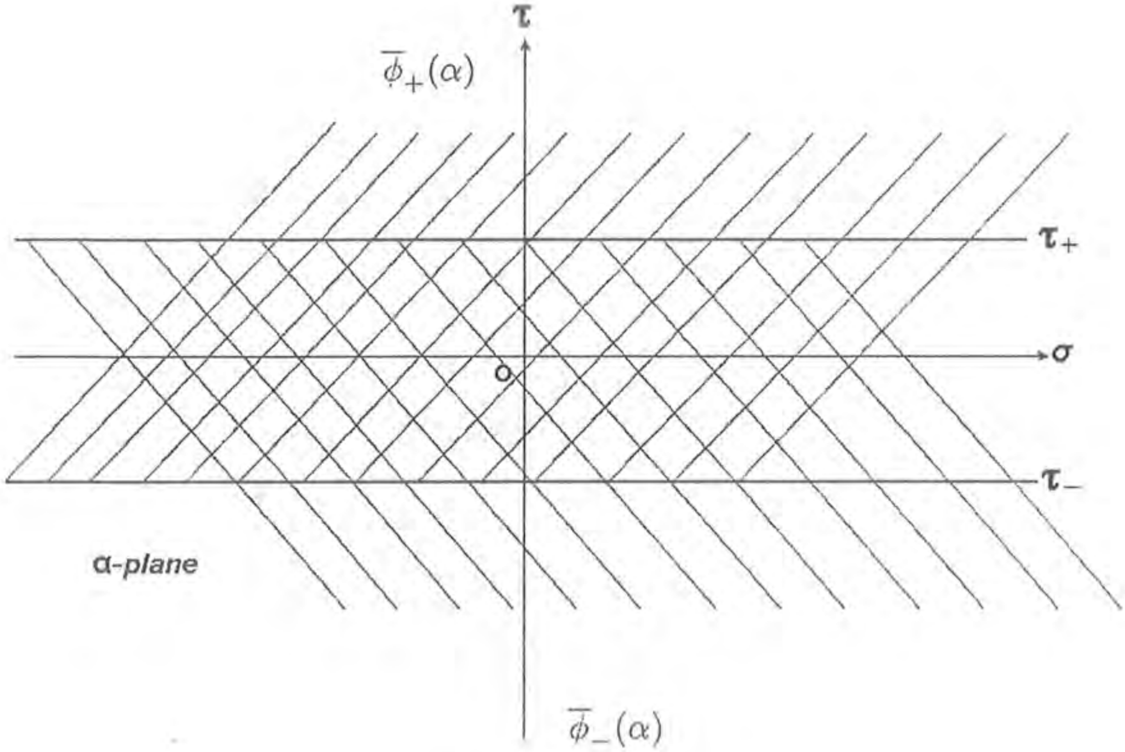

Fig. 2.1 Strip of analyticity

The functions $f(x)$ and $\overline{f}(\alpha)$ are related by the inverse Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+\iota\tau}^{\infty+\iota\tau} \overline{f}(\alpha) e^{-i\alpha x} d\alpha, \tag{2.1.12}$$

where the integration is performed along the straight line parallel to the real axis of the complex $\alpha - plane$ lying in the strip $\tau_- < \operatorname{Im}\alpha < \tau_+$. In particular, the function $\overline{f}(\alpha)$ is analytic for $\tau_+ > 0$ and $\tau_- < 0$ in the strip containing the real axis of the complex $\alpha - plane$.

2.2 The Wiener-Hopf technique [14, 54]

Historically W-H technique is the outcome of the collaboration between N. Wiener and E. Hopf which was initiated by their mutual interest in the differential equations governing the problems of radiation equilibrium of stars [15]. This collaboration also resulted in their famous paper [149] in which they established the tool by which such equations could be solved.

Considerable difficulty is usually encountered in finding solutions for the shapes not covered by the method of separation of variables. The W-H technique provides a significant extension of the range of problems that can be solved by Fourier, Laplace and Mellin integral transforms. The W-H technique was originally invented to solve an integral equation of the form

$$\int_0^\infty f(\xi) K(x-\xi)\, d\xi = g(x) \qquad (0 < x < \infty),$$

where K and g are the given functions and f is to be calculated [14]. An important modification was done by D. S. Jones [135] known as Jones' method based on the application of integral transform directly to the partial differential equation and the related boundary conditions, by-passes the initial derivation of the integral equation and directly resulted into the formation of complex W-H functional equation.

2.2.1 General scheme of Wiener-Hopf procedure

The typical problem obtained by applying Fourier transform to partial differential equations is the following. It is required to determine the functions $\psi_+(\alpha)$ and $\psi_-(\alpha)$ of a complex variable α, which are analytic respectively in the half planes $\operatorname{Im}\alpha > \tau_-$ and $\operatorname{Im}\alpha < \tau_+$ ($\tau_- < \tau_+$) and tend to zero as $|\alpha| \to \infty$ in both domains of analyticity and in the strip $\tau_- < \operatorname{Im}\alpha < \tau_+$, satisfy the functional equation [14]

$$A(\alpha)\psi_+(\alpha) + B(\alpha)\psi_-(\alpha) + C(\alpha) = 0, \tag{2.2.1.1}$$

where $A(\alpha)$, $B(\alpha)$ and $C(\alpha)$ are given functions of the complex variable α analytic in the strip $\tau_- < \operatorname{Im}\alpha < \tau_+$ and $A(\alpha)$ and $B(\alpha)$ are non-zero in the strip.

The fundamental step in the Wiener-Hopf technique for the solution of the equation is to find $L_+(\alpha)$ regular and non-zero in $\operatorname{Im}\alpha > \tau_-$ and $L_-(\alpha)$ regular and non-zero in $\operatorname{Im}\alpha < \tau_+$ such that

$$\frac{A(\alpha)}{B(\alpha)} = \frac{L_+(\alpha)}{L_-(\alpha)}. \tag{2.2.1.2}$$

Then using Eq. (2.2.1.2) in Eq. (2.2.1.1) we may write

$$L_+(\alpha)\psi_+(\alpha) + L_-(\alpha)\psi_-(\alpha) + L_-(\alpha)\frac{C(\alpha)}{B(\alpha)} = 0. \tag{2.2.1.3}$$

The last term in Eq. (2.2.1.3) may be decomposed as

$$L_-(\alpha)\frac{C(\alpha)}{B(\alpha)} = D_+(\alpha) + D_-(\alpha), \tag{2.2.1.4}$$

where the functions $D_+(\alpha)$ and $D_-(\alpha)$ are analytic in the half planes $\operatorname{Im}\alpha > \tau_-$ and

$\operatorname{Im}\alpha < \tau_+$ respectively. In the strip the following equation holds true

$$L_+(\alpha)\psi_+(\alpha) + D_+(\alpha) = -L_-(\alpha)\psi_+(\alpha) - D_-(\alpha) = J(\alpha). \tag{2.2.1.5}$$

This equation defines only in the strip $\tau_- < \operatorname{Im}\alpha < \tau_+$. The first part of Eq. (2.2.1.5) is a function analytic in the half plane $\operatorname{Im}\alpha > \tau_-$, and the second part is a function analytic in the domain $\operatorname{Im}\alpha < \tau_+$. Hence by the analytic continuation principle we can define $J(\alpha)$ over the whole $\alpha - plane$. Now suppose that

$$\left|L_+(\alpha)\psi_+(\alpha) + D_+(\alpha)\right| < |\alpha|^p \qquad \text{as } \alpha \longrightarrow \infty, \quad \operatorname{Im}\alpha > \tau_-, \tag{2.2.1.6}$$

$$\left|L_-(\alpha)\psi_-(\alpha) + D_-(\alpha)\right| < |\alpha|^q \qquad \text{as } \alpha \longrightarrow \infty, \quad \operatorname{Im}\alpha < \tau_+. \tag{2.2.1.7}$$

Then by the extended Liouville's theorem [14], which states that "If $J(\alpha)$ is an integral function such that $|J(\alpha)| < M\,|\alpha|^p$ as $\alpha \longrightarrow \infty$ where M and p are constants then $J(\alpha)$ is a polynomial of degree less than or equal to $[p]$ where $[p]$ is the integral part of p." $J(\alpha)$ is a polynomial $P(\alpha)$ of degree less than or equal to the integral part of (p, q) i.e.,

$$\psi_+(\alpha) = \frac{P(\alpha) - D_+(\alpha)}{L_+(\alpha)}, \tag{2.2.1.8}$$

$$\psi_-(\alpha) = \frac{P(\alpha) + D_-(\alpha)}{-L_-(\alpha)}. \tag{2.2.1.9}$$

These equations determine $\psi_+(\alpha)$ and $\psi_-(\alpha)$ in terms of $P(\alpha)$. Thus the use of Wiener-Hopf technique is based on the representations (2.2.1.8) and (2.2.1.9). For the factorization of certain functions in W-H procedure the following results are quite useful. These results and their proofs are now presented.

2.3 Additive decomposition theorem [14]

Statement

Let a function $\overline{f}(\alpha)$ be an analytic function in the strip $\tau_- < \operatorname{Im}\alpha < \tau_+$ and $\overline{f}(\alpha)$ tends to zero uniformly in the strip as $|\alpha| \to \infty$, then in the strip

$$\overline{f}(\alpha) = \overline{f}_+(\alpha) + \overline{f}_-(\alpha), \tag{2.3.1}$$

where $\overline{f}_+(\alpha)$ is analytic in the domain $\operatorname{Im}\alpha > \tau_-$ and $\overline{f}_-(\alpha)$ is analytic in the domain $\operatorname{Im}\alpha < \tau_+$.

Proof

Consider an arbitrary point α lying in the given strip and construct a rectangle $P_1P_2P_3P_4$ containing the point α and bounded by the straight lines $\operatorname{Im}\alpha = \tau'_-$, $\operatorname{Im}\alpha = \tau'_+$, Re $\alpha = -A$ and Re $\alpha = A$ such that $\tau_- < \tau'_- < \tau'_+ < \tau_+$ as shown in Fig. 2.2.

Then by Cauchy's integral formula

$$\overline{f}(\alpha) = \frac{1}{2\pi i}\int\limits_{-A+i\tau'_-}^{A+i\tau'_-} \frac{f(\xi)}{\xi-\alpha}d\xi + \frac{1}{2\pi i}\int\limits_{A+i\tau'_-}^{A+i\tau'_+} \frac{f(\xi)}{\xi-\alpha}d\xi$$
$$+\frac{1}{2\pi i}\int\limits_{A+i\tau'_+}^{-A+i\tau'_+} \frac{f(\xi)}{\xi-\alpha}d\xi + \frac{1}{2\pi i}\int\limits_{-A+i\tau'_+}^{-A+i\tau'_-} \frac{f(\xi)}{\xi-\alpha}d\xi. \tag{2.3.2}$$

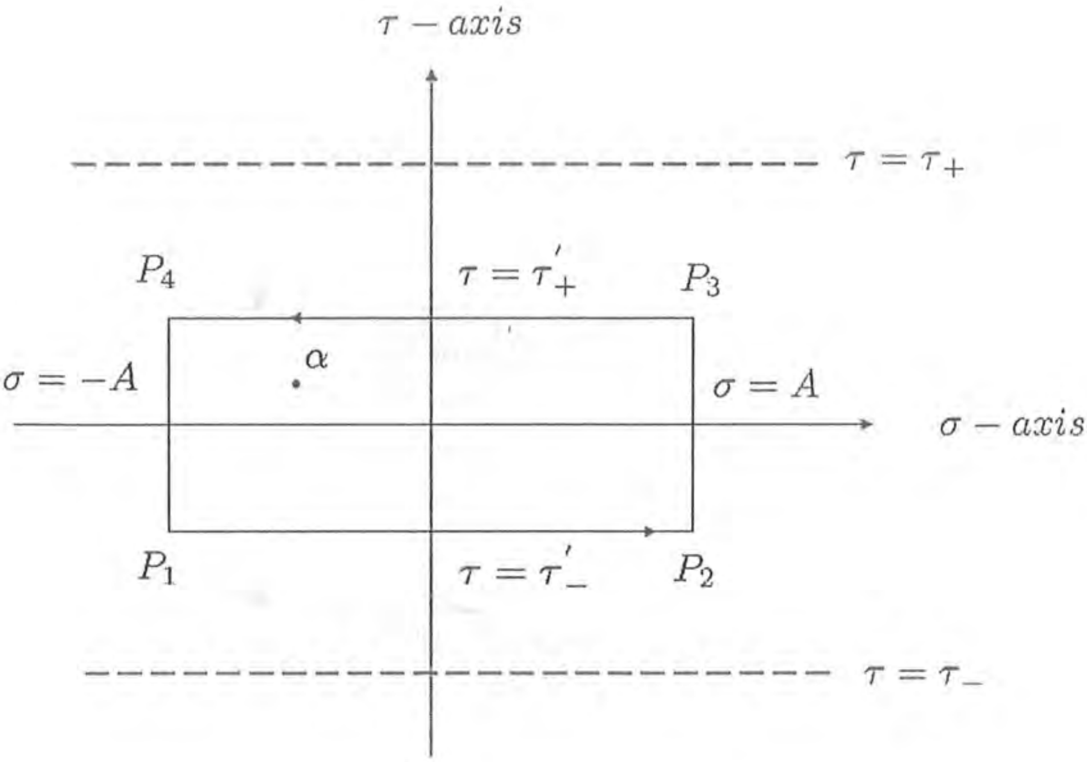

Fig. 2.2 Contour of integration

Taking the limit $A \longrightarrow \infty$, the second and fourth integrals on the right hand side of Eq. (2.3.2) will tend to zero and hence Eq. (2.3.2) reduces to

$$\overline{f}(\alpha) = \overline{f}_-(\alpha) + \overline{f}_+(\alpha), \tag{2.3.3}$$

where

$$\overline{f}_-(\alpha) = -\frac{1}{2\pi i} \int_{-\infty + i\tau'_+}^{\infty + i\tau'_+} \frac{f(\xi)}{\xi - \alpha} d\xi, \tag{2.3.4}$$

is regular in the lower $\alpha - plane$ $\tau < \tau_+$ and

$$\overline{f}_+(\alpha) = \frac{1}{2\pi i} \int_{-\infty + i\tau'_-}^{\infty + i\tau'_-} \frac{f(\xi)}{\xi - \alpha} d\xi, \tag{2.3.5}$$

is regular in the upper $\alpha-plane$ defined by $\tau > \tau_-$. These integrals represent analytic functions provided that the point α does not lie on the contour of integration. Since the point α and the straight lines $\mathrm{Im}\,\alpha = \tau'_-$ and $\mathrm{Im}\,\alpha = \tau'_+$ are arbitrary, therefore, this proves the theorem.

2.4 Multiplicative decomposition theorem [14]

Statement

Let a function $\overline{\psi}(\alpha)$ be analytic and non zero in the strip $\tau_- < \mathrm{Im}\,\alpha < \tau_+$ and $\overline{\psi}(\alpha)$ tends to zero uniformly as $|\alpha| \to \infty$ in the strip. Then in the given strip $\overline{\psi}(\alpha)$ can be factorized such that

$$\overline{\psi}(\alpha) = \overline{\psi}_+(\alpha)\,\overline{\psi}_-(\alpha), \tag{2.4.1}$$

where the functions $\overline{\psi}_-(\alpha)$ and $\overline{\psi}_+(\alpha)$ are analytic and non-zero in the half planes $\mathrm{Im}\,\alpha < \tau_+$ and $\mathrm{Im}\,\alpha > \tau_-$, respectively.

Proof

Let

$$\overline{f}(\alpha) = \log \overline{\psi}(\alpha), \tag{2.4.2}$$

which satisfies all the conditions of the additive decomposition theorem. Thus, the function $\overline{f}(\alpha)$ can be splitted as

$$\overline{f}(\alpha) = \overline{f}_-(\alpha) + \overline{f}_+(\alpha). \tag{2.4.3}$$

Substituting

$$\begin{aligned} \overline{\psi}_+(\alpha) &= \exp\left(\overline{f}_+(\alpha)\right), \\ \overline{\psi}_-(\alpha) &= \exp\left(\overline{f}_-(\alpha)\right), \end{aligned} \tag{2.4.4}$$

so that

$$\overline{f}_+(\alpha) = \log \overline{\psi}_+(\alpha) \tag{2.4.5a}$$

and

$$\overline{f}_-(\alpha) = \log \overline{\psi}_-(\alpha). \tag{2.4.5b}$$

Thus using Eqs. (2.4.5a, b) in Eq. (2.4.3), result in

$$\log \overline{\psi}(\alpha) = \log \overline{\psi}_+(\alpha) + \log \overline{\psi}_-(\alpha), \tag{2.4.6}$$

which proves the result

$$\overline{\psi}(\alpha) = \overline{\psi}_+(\alpha)\, \overline{\psi}_-(\alpha).$$

2.5 The Jones' method [14, 135]

The possible methods for the solution of Sommerfeld's half plane diffraction problem may be

- Green's function : Integral equation method
- Dual integral equation method
- Jones' method

In this thesis, the Jones' method proposed by D. S. Jones [135] is followed, which provides a straight forward procedure for solving the problems using W-H technique.

2.5.1 Problem formulation

Consider a half plane problem. Steady state waves with harmonic time dependence $e^{-i\omega t}$ exists in two dimensional (x, y) space. There is a rigid boundary along the negative real axis and the plane waves

$$\phi_i = e^{-ikx\cos\theta - iky\sin\theta} \qquad (0 < \theta < \pi), \tag{2.5.1.1}$$

are incident on the screen. The total velocity potential $\phi_t(x, y)$ can be written as

$$\phi_t = \phi_i + \phi, \tag{2.5.1.2}$$

where ϕ is the diffracted potential and satisfies

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + k^2\phi = 0, \tag{2.5.1.3}$$

where k is assumed to have positive imaginary part which is necessary for the application of W-H technique [14]. The following conditions apply:

$$\frac{\partial\phi_t}{\partial y} = 0 \quad \text{on } y = 0, \qquad -\infty < x \leq 0, \tag{2.5.1.4}$$

or equivalently

$$\frac{\partial \phi}{\partial y} = ik \sin\theta e^{-ikx\cos\theta}, \ y = 0, \qquad -\infty < x \leq 0, \tag{2.5.1.5}$$

$$\frac{\partial \phi_t}{\partial y} \text{ and hence } \frac{\partial \phi}{\partial y} \text{ are continuous on } y = 0, \qquad -\infty < x < \infty, \tag{2.5.1.6}$$

$$\phi_t \text{ and hence } \phi \text{ are continuous on } y = 0, \qquad 0 < x < \infty. \tag{2.5.1.7}$$

The different regions in which various potentials exist are shown in Fig. 2.3.

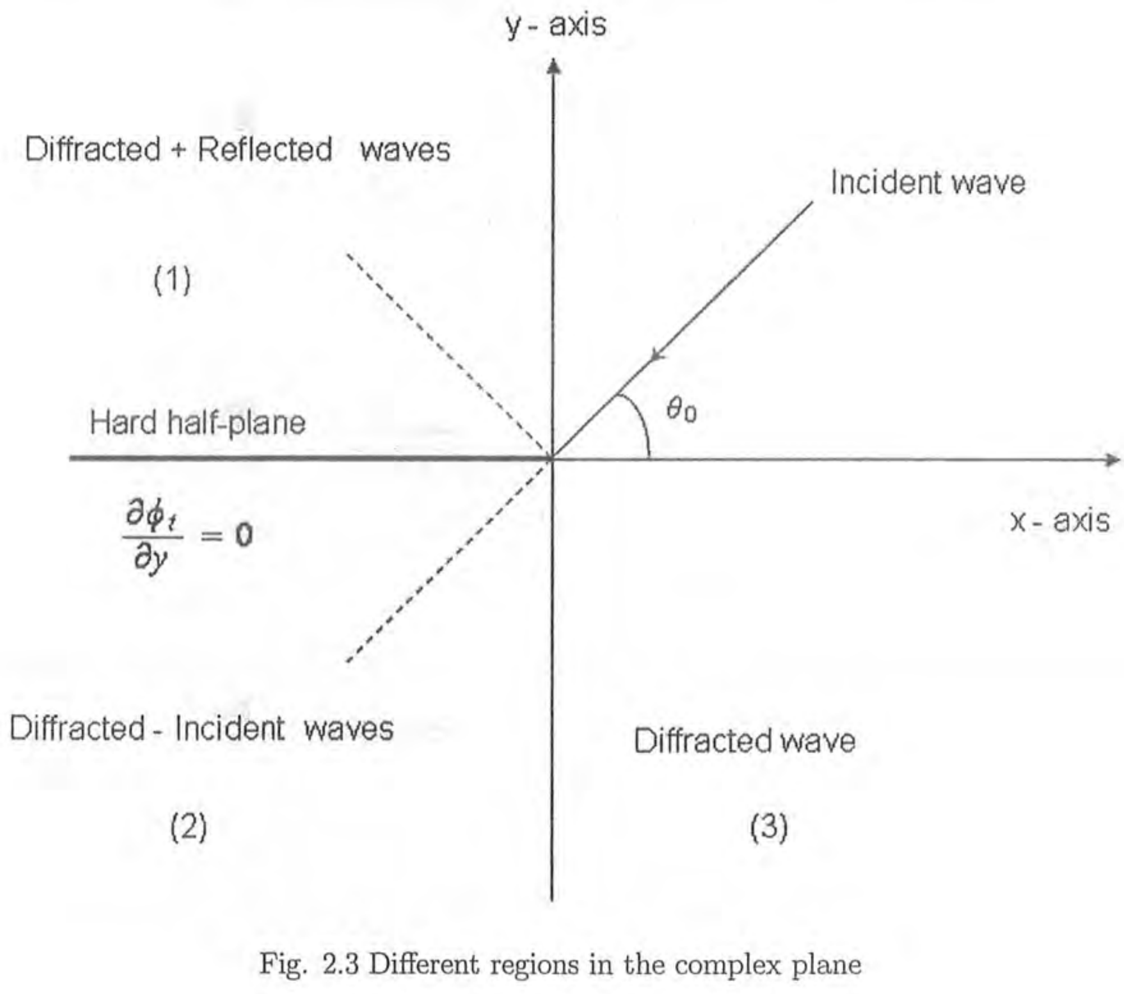

Fig. 2.3 Different regions in the complex plane

Defining

$$\overline{\phi}(\alpha, y) = \overline{\phi}_+(\alpha, y) + \overline{\phi}_-(\alpha, y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi(x, y)e^{i\alpha x}dx, \tag{2.5.1.8}$$

where

$$\overline{\phi}_-(\alpha, y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} \phi(x, y)e^{i\alpha x}dx, \tag{2.5.1.9}$$

$$\overline{\phi}_+(\alpha, y) = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} \phi(x, y)e^{i\alpha x}dx. \tag{2.5.1.10}$$

Further it is required that

$$\begin{aligned} |\overline{\phi}| &< D_1 e^{-k_2 x} \quad \text{as} \quad x \longrightarrow \infty, \\ |\overline{\phi}| &< D_1 e^{k_2 x \cos\theta} \quad \text{as } x \longrightarrow -\infty. \end{aligned} \tag{2.5.1.11}$$

So that $\overline{\phi}_+$ is analytic for $\operatorname{Im}\alpha > -k_2$ and $\overline{\phi}_-$ is analytic for $\operatorname{Im}\alpha < k_2\cos\theta$. Applying Fourier transform (2.5.1.8) on above Eq. (2.5.1.3), yields

$$\frac{d^2\overline{\phi}}{dy^2} - \gamma^2\overline{\phi} = 0, \tag{2.5.1.12}$$

where

$$\gamma^2 = \sqrt{\alpha^2 - k^2}. \tag{2.5.1.13}$$

Eq. (2.5.1.12) has solution

$$\overline{\phi}(\alpha, y) = \begin{cases} A_1(\alpha)e^{-\gamma y} + B_1(\alpha)e^{\gamma y}, & y \geq 0, \\ A_2(\alpha)e^{-\gamma y} + B_2(\alpha)e^{\gamma y}, & y \leq 0. \end{cases} \tag{2.5.1.14}$$

In Eq. (2.5.12), the real part of γ is always positive in the strip $-k_2 < \operatorname{Im}\alpha < k_2\cos\theta$ and therefore in Eq. (2.5.1.14) one must have $A_2 = B_1 = 0$. From condition (2.5.1.6)

$$\begin{aligned} \frac{d\overline{\phi}(\alpha,0^+)}{dy} &= -\gamma A_1(\alpha), \\ \frac{d\overline{\phi}(\alpha,0^-)}{dy} &= \gamma B_2(\alpha). \end{aligned} \tag{2.5.1.15}$$

Thus letting $-A_1 = B_2 = A$, in Eq. (2.5.1.14) will give

$$\overline{\phi}(\alpha,y) = \begin{cases} A(\alpha)e^{-\gamma y}, & y \geq 0, \\ -A(\alpha)e^{\gamma y}, & y \leq 0. \end{cases} \tag{2.5.1.16}$$

When a transform is discontinuous across $y = 0$, the notation can be extended as follows:

$$\overline{\phi}_-(\alpha,0^\pm) = \overline{\phi}_-(0^\pm) = \lim_{y\longrightarrow\pm0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \phi e^{i\alpha x}dx. \tag{2.5.1.17}$$

By using condition (2.5.1.7)

$$\overline{\phi}_+(0^+) = \overline{\phi}_+(0^-) = \overline{\phi}_+(0), \tag{2.5.1.18}$$

and by condition (2.5.1.6)

$$\overline{\phi}'_+(0^+) = \overline{\phi}'_+(0^-) = \overline{\phi}'_+(0), \tag{2.5.1.19}$$

and similarly

$$\overline{\phi}'_-(0^+) = \overline{\phi}'_-(0^-) = \overline{\phi}'_-(0). \tag{2.5.1.20}$$

Applying conditions (2.5.1.18 – 2.5.1.20) in Eq. (2.5.1.16) will result in the following

$$\overline{\phi}_+(0) + \overline{\phi}_-(0^+) = A\left(\alpha\right), \tag{2.5.1.21}$$

$$\overline{\phi}_{+}(0) + \overline{\phi}_{-}(0^{-}) = -A\,(\alpha)\,, \tag{2.5.1.22}$$

$$\overline{\phi}'_{+}(0) + \overline{\phi}'_{-}(0) = -\gamma A\,(\alpha)\,. \tag{2.5.1.23}$$

Now at this stage it is required that one has to deal with the equations which contain only the functions whose regions of regularity are known. Addition of Eqs. (2.5.1.21) and (2.5.1.22) will result in

$$2\overline{\phi}_{+}(0) = -\overline{\phi}_{-}(0^{+}) - \overline{\phi}_{-}(0^{-}), \tag{2.5.1.24}$$

whereas subtraction of Eqs. (2.5.1.21) and (2.5.1.22) will yield

$$\overline{\phi}_{-}(0^{+}) - \overline{\phi}_{-}(0^{-}) = 2A(\alpha). \tag{2.5.1.25}$$

Using Eq. (2.5.1.25) in Eq. (2.5.1.23) will result into

$$\overline{\phi}'_{+}(0) + \overline{\phi}'_{-}(0) = -\frac{\gamma}{2}\left(\overline{\phi}_{-}(0^{+}) - \overline{\phi}_{-}(0^{-})\right). \tag{2.5.1.26}$$

Taking Fourier transform of Eq. (2.5.1.5) will give

$$\overline{\phi}'_{-}(0) = \frac{k\sin\theta}{\sqrt{2\pi}\,(\alpha - k\cos\theta)}. \tag{2.5.1.27}$$

For convenience, let us introduce the following notations

$$\begin{aligned} \overline{\phi}_{-}(0^{+}) - \overline{\phi}_{-}(0^{-}) &= 2D_{-}, \\ \overline{\phi}_{-}(0^{+}) + \overline{\phi}_{-}(0^{-}) &= 2S_{-}. \end{aligned} \tag{2.5.1.28}$$

Therefore, Eqs. (2.5.1.24) and (2.5.1.26) become

$$\overline{\phi}_{+}(0) = -S_{-}, \tag{2.5.1.29}$$

$$\overline{\phi}'_{+}(0)+\frac{k\sin\theta}{\sqrt{2\pi}\,(\alpha-k\cos\theta)}=-\gamma D_{-}. \tag{2.5.1.30}$$

In Eqs. (2.5.1.29) and (2.5.1.30) $\overline{\phi}_{+}(0)$, $\overline{\phi}'_{+}(0)$, S_{-} and D_{-} are unknown functions and each of these equations holds in the strip of analyticity $-k_2 < \operatorname{Im}\alpha < k_2\cos\theta$ and is in standard W-H form. Substituting $\gamma=\sqrt{\alpha+k}\sqrt{\alpha-k}$ in Eq. (2.5.1.30) and arranging it in the form

$$\frac{\overline{\phi}'_{+}(0)}{\sqrt{\alpha+k}}+\frac{k\sin\theta}{\sqrt{2\pi}\sqrt{\alpha+k}\,(\alpha-k\cos\theta)}=-\sqrt{\alpha-k}D_{-}, \tag{2.5.1.31}$$

where $\sqrt{\alpha+k}$ being regular in the upper half plane $\operatorname{Im}\alpha > -k_2$ and $\sqrt{\alpha-k}$ being regular in the lower half plane $\operatorname{Im}\alpha < k_2\cos\theta$. It is noticed that first term on the left hand side of Eq. (2.5.1.31) is regular in the upper half plane and the second term is regular in the strip $-k_2 < \operatorname{Im}\alpha < k_2\cos\theta$ whereas the right hand side of this equation is regular in the lower half plane. The middle term of Eq. (2.5.1.31) can be splitted in the following manner

$$\frac{k\sin\theta}{\sqrt{2\pi}\sqrt{\alpha+k}\,(\alpha-k\cos\theta)}=\frac{k\sin\theta}{\sqrt{2\pi}\,(\alpha-k\cos\theta)}\left[\frac{1}{\sqrt{\alpha+k}}-\frac{1}{\sqrt{k+k\cos\theta}}\right]$$
$$+\frac{k\sin\theta}{\sqrt{2\pi}\,(\alpha-k\cos\theta)\sqrt{k+k\cos\theta}}, \tag{2.5.1.32}$$

or

$$\frac{k\sin\theta}{\sqrt{2\pi}\sqrt{\alpha+k}\,(\alpha-k\cos\theta)}=H_{+}(\alpha)+H_{-}(\alpha), \tag{2.5.1.33}$$

where $H_{+}(\alpha)$ is regular in $\operatorname{Im}\alpha > -k_2$ and $H_{-}(\alpha)$ is regular in $\operatorname{Im}\alpha < k_2\cos\theta$. Substituting Eq. (2.5.1.33) in Eq. (2.5.1.31) yields

$$\overline{\phi}'_{+}(0)\,(\alpha+k)^{-\frac{1}{2}}+H_{+}(\alpha)=-\,(\alpha-k)^{\frac{1}{2}}\,D_{-}-H_{-}(\alpha)=J(\alpha). \tag{2.5.1.34}$$

In the present form Eq. (2.5.1.34) defines a function $J(\alpha)$ which is regular in $\operatorname{Im}\alpha > -k_2$ and in $\operatorname{Im}\alpha < k_2\cos\theta$, so it is regular in the strip $-k_2 < \operatorname{Im}\alpha < k_2\cos\theta$, provided that $J(\alpha)$ has algebraic behavior as $\alpha \longrightarrow \infty$, hence one can use the extended form of the Liouville's theorem to determine the exact form of $J(\alpha)$. Now examining the behaviors of the functions appearing in Eq. (2.5.1.34) and by using the edge conditions [14, 150] as $\alpha \longrightarrow \infty$ will yield

$$\begin{aligned}
&\left|\overline{\phi}_-(0^+)\right| < C_1\left|\alpha\right|^{-1} \text{ as } \alpha \longrightarrow \infty \text{ in } \tau < k_2\cos\theta,\\
&\left|\overline{\phi}'_+(0)\right| < C_2\left|\alpha\right|^{-\frac{1}{2}} \text{ as } \alpha \longrightarrow \infty \text{ in } \tau > -k_2,\\
&H_-(\alpha) < C_3\left|\alpha\right|^{-1}, \text{ as } \alpha \longrightarrow \infty \text{ in } \tau < k_2\cos\theta,\\
&H_+(\alpha) < C_4\left|\alpha\right|^{-1}, \text{ as } \alpha \longrightarrow \infty \text{ in } \tau > -k_2.
\end{aligned} \tag{2.5.1.35}$$

Using these asymptotic approximates (2.5.1.35) of the various functions in Eq. (2.5.1.34), it is observed that

$$J(\alpha) < C_5\left|\alpha\right|^{-\frac{1}{2}} \text{ as } \alpha \longrightarrow \infty \text{ in } \tau < k_2\cos\theta, \tag{2.5.1.36}$$

$$J(\alpha) < C_6\left|\alpha\right|^{-1} \text{ as } \alpha \longrightarrow \infty \text{ in } \tau > -k_2.$$

This implies that $J(\alpha)$ is regular in the whole $\alpha-plane$ and tends to zero as $\alpha \longrightarrow \infty$. Hence by extended Liouville's theorem, $J(\alpha)$ must be identically equal to constant (zero). Therefore

$$\begin{aligned}
&\overline{\phi}'_+(0) = -\left(\alpha + k\right)^{\frac{1}{2}} H_+(\alpha),\\
&D_- = -\left(\alpha - k\right)^{\frac{1}{2}} H_-(\alpha).
\end{aligned} \tag{2.5.1.37}$$

Substituting $\overline{\phi}'_+(0)$ and $\overline{\phi}'_-(0)$ in Eq. (2.5.1.23), $A(\alpha)$ is found out to be

$$A(\alpha) = -\frac{k\sin\theta}{\sqrt{2\pi}\sqrt{\alpha - k}\,(\alpha - k\cos\theta)\,\sqrt{k + k\cos\theta}}. \tag{2.5.1.38}$$

Finally by taking the inverse Fourier transform and substituting the value of $A(\alpha)$ into Eq. (2.5.1.16), one has

$$\phi(x, y) = \mp\frac{1}{2\pi}\sqrt{k - k\cos\theta}\int\limits_{-\infty+ia}^{\infty+ia}\frac{e^{-i\alpha x \mp \gamma y}}{\sqrt{\alpha - k}\,(\alpha - k\cos\theta)}d\alpha. \tag{2.5.1.39}$$

The above contour integral of a certain type can be solved by using asymptotic methods. In sequel an asymptotic method for the solution of certain integrals will be outlined. Lastly the main steps while using the Jones' method can be summarized as:

(a) Extend the range of definition of the integral/partial differential equation from $-\infty$ to ∞.

(b) Apply the integral transform (Fourier, two-sided Laplace, Mellin etc).

(c) Determine the line of junction (often it is called strip of analyticity).

(d) Carry out the additive/multiplicative factorization of the kernel formed in the problem.

(e) Separate the W-H functional equation into positive and negative portions.

(f) Apply the extended Liouville's theorem to conclude the entire function $= J(\alpha)$.

(g) Determine $J(\alpha)$ from the behavior of $\overline{\phi}(\alpha)$ (which can usually be gleaned from physical consideration) at small x.

(h) Evaluate the corresponding inverse transform.

2.6 Method of steepest descent [69, 136-138]

The radiation and diffraction fields can be represented in terms of an integral representation. These integrals are often difficult and rather sometimes impossible to be evaluated in closed form (due to diverging field parameters). The integrands of these integrals in most of the cases contains a large parameter say χ and one can approximate the integral in terms of χ (due to large value contributing towards the integral) [138]. Some of the famous methods to solve such integrals are

(i) Integration by parts

(ii) Laplace method

(iii) Method of stationary phase

(iv) Method of steepest descent

(v) Numerical integration methods etc.

An appreciable number of standard text books on these methods are available e.g., $[69, 136-138]$. In the forthcoming chapters of this thesis the method of steepest descent has been widely applied to approximate several integrals appearing in different boundary value problems. The method of steepest descent was originated by Riemann and developed by Debey [137].

Considering an integral of the form

$$I(\chi) = \int_A^B j(z) e^{\chi h(z)} dz, \qquad (\chi \to \infty) \tag{2.6.1}$$

in which it is assumed that $j(z)$ and $h(z)$ are the analytic functions of the complex variable $z = x + iy$ along the path A to B in the complex $z - plane$ and $\varkappa$ to be the large and positive parameter. One can consider χ to be real, if not, writing $\chi = \chi_0 e^{i\vartheta}$ and $e^{i\vartheta}$ can be absorbed into $h(z)$. The magnitude of the integral crucially depends on the real part of $h(z)$. Writing $h(z) = u(x, y) + iv(x, y)$, ($u$ and v are real) then

$$\left|e^{\chi h(z)}\right| = e^{\chi u}. \tag{2.6.2}$$

The bound for the integral I is given by

$$|I| \leq L\widehat{J}e^{\chi U}, \tag{2.6.3}$$

where L is the length of the contour, $\widehat{J}$ is the maximum value of $|j(z)|$ on the path and U is the maximum value of u on the path from A to B. Clearly Eq. (2.6.3) might be an over estimate since a path deformation might produces a much less value of u. Hence the best bound is chosen by taking the path such that u is as small as possible.

2.6.1 Contours of $u(x, y)$

Imagine $u(x, y)$ to be the height of the surface above a reference plane $u = 0$. It will be found expedient to deform the path of integration so that it join the end points A and B and passes along the *low ground* of the surface where u is as small as possible. An overall picture of $u(x, y)$ can be obtained by considering the contours $u = constant$. It is also useful to visualize the family of contours $v = constant$ which are the lines of constant phase $\operatorname{Im} h(z) = constant$. The two families of the curves are easily seen to be mutually orthogonal as

$$\nabla u.\nabla v = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}. \tag{2.6.4}$$

By using Cauchy-Riemann equations, Eq. (2.6.4) will take the form

$$\nabla u.\nabla v = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} = 0, \tag{2.6.5}$$

i.e., ∇u is perpendicular to ∇v. The points at which $h'(z) = 0$ is called a *saddle point* or *col. Let* $z = z_0 = x_0 + iy_0$ be a saddle point. As $h'(z) = 0$ so

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} = 0. \tag{2.6.6}$$

Observe that u (and also v) cannot have a maxima or minima at a saddle point (x_0, y_0) because $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$. Also it is observed that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ implies that $u_{xx} = -u_{yy}$ which means that both of the quantities u_{xx} and u_{yy} have opposite signs. Further

$$\begin{vmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{vmatrix} = u_{xx}u_{yy} - u_{xy}^2 = -\left(u_{xx}^2 + u_{xy}^2\right) \leq 0. \tag{2.6.7}$$

The above relation also implies that $z_0 = (x_0, y_0)$ is neither a maxima nor a minima. The saddle point links the *valleys* and *ridges on the surface* $u(x, y)$, the curve $v =$ *constant* will go either up a *ridge* or down a *valley* since these are the derivatives of the greatest change. These are the points of steepest descent for which the neighborhood of the saddle point produces the most significant contribution. A sketch of the saddle points and the paths of steepest descent is shown in Fig. 2.4.

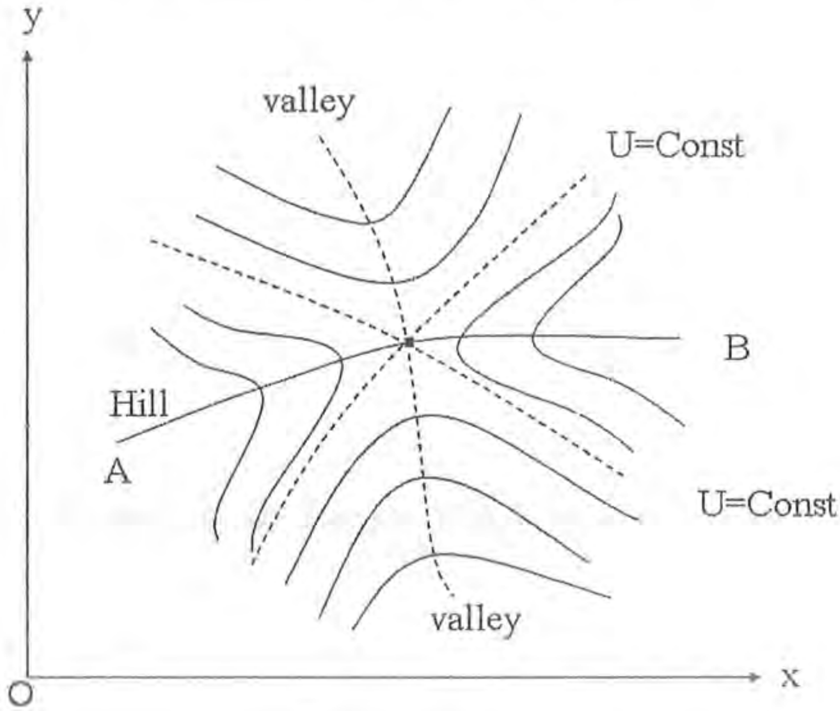

Fig. 2.4 Vallies and Ridges

At a saddle point we have $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$ that is $h'(z_0) = 0$. Further it is assumed that $h''(z_0) \neq 0$ and along the path of steepest descent one can expand $h(z)$ in a Taylor's series as

$$h(z) = h(z_0) + \frac{1}{2}(z - z_0)^2 h''(z_0) + ..., \tag{2.6.8}$$

or

$$h(z) - h(z_0) = \frac{1}{2}(z - z_0)^2 h''(z_0). \tag{2.6.9}$$

As $v = constant$ so right hand side of above equation is purely real. Thus introducing a variable t^2 such that

$$h(z) - h(z_0) = -t^2, \tag{2.6.10}$$

and it changes the complex variable z to a real variable t and describes the path from valley to valley along a route that descends most rapidly on the either side of the point z_0. Now using Eq. (2.6.10) in Eq. (2.6.1), we have

$$I(\chi) = \int_A^B j(z) e^{\chi\left(h(z_0) - t^2\right)} dz. \tag{2.6.11}$$

The above integral can be rearranged in another suitable form as

$$I(\chi) \approx e^{\chi h(z_0)} \int_A^B j(z) e^{-\chi t^2} \frac{dz}{dt} dt. \tag{2.6.12}$$

As the exponential in the above integrand decays more rapidly as compared to the function $j(z)$, therefore further simplification of above integral will result in

$$I(\chi) \approx e^{\chi h(z_0)} j(z_0) \int_{-t_1}^{t_1} e^{-\chi t^2} \frac{dz}{dt} dt, \tag{2.6.13}$$

provided that $j(z_0)$ is not singular in the vicinity of $z = z_0$. The remaining task left is to calculate the value of $\frac{dz}{dt}$. Writing Eq. (2.6.10) in the form

$$-t^2 = \frac{1}{2}(z - z_0)^2 h''(z_0) \tag{2.6.14}$$

and introducing the polar coordinates

$$z - z_0 = r_0 e^{i\theta_0}, \tag{2.6.15}$$

in Eq. (2.6.14), it can easily be concluded from Eq. (2.6.14) that

$$\arg\left[\frac{1}{2} r_0^2 e^{2i\theta_0} h''(z_0)\right] = 0. \tag{2.6.16}$$

This gives

$$r_0^2 = \frac{-t^2}{\left|\frac{1}{2} h''(z_0)\right|},$$

which simplifies to

$$r_0 = e^{\frac{i\pi}{2}} t \left| \frac{1}{2} h''(z_0) \right|^{-\frac{1}{2}}. \tag{2.6.17}$$

By using Eq. (2.6.17) in Eq. (2.6.15) and calculating $\frac{dz}{dt}$, we arrive at

$$\frac{dz}{dt} = \left| \frac{1}{2} h''(z_0) \right|^{-\frac{1}{2}} e^{\frac{i\pi}{2} + i\theta_0}. \tag{2.6.18}$$

Substitution of Eq. (2.6.18) in Eq. (2.6.13) will give

$$I(\chi) \approx e^{\chi h(z_0)} j(z_0) \frac{1}{\left|\frac{1}{2} h''(z_0)\right|^{\frac{1}{2}}} \sqrt{\frac{\pi}{\chi}} e^{\frac{i\pi}{2} + i\theta_0},$$

or

$$I(\chi) \approx e^{\chi h(z_0)} j(z_0) \sqrt{\frac{2\pi}{\chi \left| h''(z_0) \right|}} e^{i\left(\frac{\pi}{2} + \theta_0\right)}. \tag{2.6.19}$$

In order to elaborate the method of steepest descent, it has been applied to two famous examples, first to establish Stirling's formula and second to approximate the Bessel's function of first kind and order zero.

The Stirling's Formula

$$s! = \sqrt{\frac{\pi}{s}} e^{-s} s^{s+\frac{1}{2}} \qquad s \longrightarrow \infty. \tag{2.6.20}$$

Consider the integral

$$s! = \int_0^\infty e^{-\alpha} \alpha^s d\alpha \qquad s \longrightarrow \infty. \tag{2.6.21}$$

In order to solve the above integral, substitute $\alpha = sz$, therefore

$$s! = s^{s+1} \int_0^\infty e^{-sz} z^s dz, \tag{2.6.22}$$

which can also be written as

$$s! = s^{s+1}\int_{0}^{\infty} e^{(\ln z - z)s} dz. \tag{2.6.23}$$

On comparing Eq. (2.6.23) with Eq. (2.6.1) it is noted that

$$h(z) = (\ln z - z), \qquad j(z) = 1, \qquad \left|h''(z)\right|_{z=1} = 1. \tag{2.6.24}$$

Using various values from Eq. (2.6.24) into Eq. (2.6.19), the value of the integral in Eq. (2.6.23) becomes

$$s! = \sqrt{\frac{\pi}{s}} e^{-s} s^{s+\frac{1}{2}}. \tag{2.6.25}$$

The Bessel's function

Consider

$$J_0(\alpha) = \frac{1}{\pi}\int_{-1}^{1} \frac{e^{i\alpha z}}{\sqrt{1-z^2}} dz, \qquad \alpha \longrightarrow \infty, \tag{2.6.26}$$

where $J_0(\alpha)$ is called the Bessel's function of the first kind and order zero. It is noted that integration by parts fails because for the choice of first function to be $e^{i\alpha z}$ and the second function to be $(1-z^2)^{-\frac{1}{2}} dz$, leads to a non asymptotic expansion. Alternatively if first function is taken to $(1-z^2)^{-\frac{1}{2}}$ and the second function is taken to be $e^{i\alpha z} dz$ leads to a singular expansion at $z = \pm 1$. Thus to use the method of steepest descent, the contour of integration has to be deformed into a constant phase contour. Here

$$h(z) = iz = i(x+iy) = ix - y. \tag{2.6.27}$$

For $z = \pm 1$, the phase is ± 1. The contour of integration is shown in Fig. 2.5.

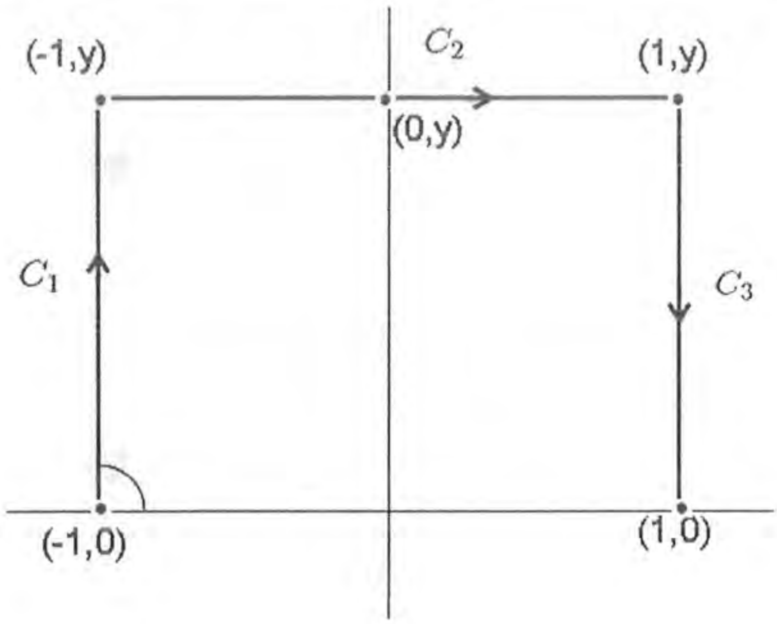

Fig. 2.5 Contour of integration of Bessel's function

Thus

$$\int_{-1}^{1} \frac{e^{i\alpha z}}{\sqrt{1-z^2}} dz = \int_{C_1+C_2+C_3} \frac{e^{i\alpha z}}{\sqrt{1-z^2}} dz. \qquad (2.6.28)$$

As $y \longrightarrow \infty$, the integral along C_2 vanishes because the integrand vanishes uniformly there. On $C_1 : z = -1 + iy$ and on $C_2 : z = 1 + iy$. Thus

$$J_0(\alpha) = \frac{1}{\pi} \int_{-1}^{-1+i\infty} \frac{e^{i\alpha z}}{\sqrt{1-z^2}} dz + \frac{1}{\pi} \int_{1+i\infty}^{1} \frac{e^{i\alpha z}}{\sqrt{1-z^2}} dz,$$

$$= \frac{ie^{-i\alpha}}{\pi} \int_{0}^{\infty} \frac{e^{-\alpha y}}{\sqrt{2iy+y^2}} d\tau + \frac{ie^{i\alpha}}{\pi} \int_{\infty}^{0} \frac{e^{-\alpha y}}{\sqrt{-2iy+y^2}} dy. \qquad (2.6.29)$$

The above integral after some mathematical manipulation results in

$$J_0(\alpha) = \frac{ie^{-i\alpha-\frac{i\pi}{4}}}{\sqrt{2}\pi}\int_0^\infty e^{-\alpha y} y^{-\frac{1}{2}}\left(1-\frac{1}{2}iy\right)^{-\frac{1}{2}} dy$$

$$-\frac{ie^{i\alpha+\frac{i\pi}{4}}}{\sqrt{2}\pi}\int_0^\infty e^{-\alpha y} y^{-\frac{1}{2}}\left(1+\frac{1}{2}iy\right)^{-\frac{1}{2}} dy. \tag{2.6.30}$$

The above integrals are the Laplace type integrals and only the immediate neighborhood of $y = 0$ contributes to their asymptotic developments for large α. For the leading order term integrals (2.6.30) can be written as

$$J_0(\alpha) \approx \frac{e^{-i\alpha+\frac{i\pi}{4}}}{\sqrt{2}\pi}\int_0^\infty e^{-\alpha y} y^{-\frac{1}{2}} dy + \frac{ie^{i\alpha-\frac{i\pi}{4}}}{\sqrt{2}\pi}\int_0^\infty e^{-\alpha y} y^{-\frac{1}{2}} dy. \tag{2.6.31}$$

Substituting $\alpha y = t$, will yield

$$\begin{aligned} J_0(\alpha) &\approx \frac{1}{\sqrt{2\alpha}\pi}\left[e^{-i\alpha+\frac{i\pi}{4}} + e^{i\alpha-\frac{i\pi}{4}}\right]\int_0^\infty e^{-t} t^{-\frac{1}{2}} dt, \\ &\approx \frac{2}{\sqrt{2\alpha}\pi}\cos\left(\alpha-\frac{\pi}{4}\right)\Gamma\left(\frac{1}{2}\right), \end{aligned} \tag{2.6.32}$$

which finally simplifies to be

$$J_0(s) \approx \sqrt{\frac{2}{\alpha\pi}}\cos\left(\alpha-\frac{\pi}{4}\right) \qquad \alpha \longrightarrow \infty. \tag{2.6.33}$$

2.7 The Daniele-Kharapkov method [20, 21, 139]

The central problem in solving the matrix W-H equation is the factorization of the kernel matrix (usually a 2×2 matrix). Let α be the complex variable and $\mathrm{G}(\alpha)$ be the 2×2 matrix whose elements are the functions of α, then the splitting of the form

$$G(\alpha) = G_+(\alpha)\ G_-(\alpha) = G_-(\alpha)\ G_+(\alpha), \tag{2.7.1}$$

is required, where $G_+(\alpha)$ is regular in the upper half plane and $G_-(\alpha)$ is regular in the lower half plane. It is very important to emphasize at this stage that the above factorization (2.7.1) must be carried out in such a way that the elements of the factor matrices have algebraic behavior as $|\alpha| \longrightarrow \infty$ in the appropriate half planes, respectively. No method is known of factoring a general 2×2 matrix. However, a number of ways exist for the matrices of a particular form. These are [139]

- The scalar method,
- The method of logarithm,
- The Wiener-Hopf-Hilbert method [17 – 19] ,
- The Daniele method [20] ,
- The Kharapkov method [21] ,
- The Jones' method [22] .

Since there are many similarities between the Daniele's method [20] and the Kharapkov's method [21] yet it can be said safely that the later contains the former method in it. Now a gist of the Kharapkov's method [21] is presented. Kharapov [21] considered a matrix of the form

$$G(\alpha) = I + \widehat{\mu} Q, \tag{2.7.2}$$

where $\widehat{\mu}$ is a scalar and Q is a polynomial matrix which can be written as

$$\mathrm{G}(\alpha) = \mathrm{I}+\frac{\widehat{\mu}}{2}\left(\mathrm{Q}+\widetilde{\mathrm{Q}}\right) + \frac{\widehat{\mu}}{2}\left(\mathrm{Q}-\widetilde{\mathrm{Q}}\right), \tag{2.7.3}$$

where tilda denotes the formal inverse, i.e., $\widetilde{\mathrm{Q}} = (\det \mathrm{Q})\, \mathrm{Q}^{-1}$ [139]. Eq. (2.7.3) can be put in another form as

$$\mathrm{G}(\alpha) = a_1\mathrm{I}+a_2\mathbf{C}, \tag{2.7.4}$$

with a_1, a_2 being scalars and

$$\mathbf{C} = \mathrm{Q}-\widetilde{\mathrm{Q}} = \begin{bmatrix} l & m \\ n & -l \end{bmatrix}, \tag{2.7.5}$$

where l, m, n are polynomials and C is called the '*commutant matrix*' by Kharapkov [21] and its formal inverse is equal to negative of it and

$$\mathrm{C}^2 = \left(l^2 + mn\right)\mathrm{I}. \tag{2.7.6}$$

Kharapkov [21] writes $l^2 + mn = g^2 f$, where g and f are polynomials and f being non-square of minimum degree. The eigen values of G are calculated to be

$$\begin{aligned} \varrho_1 &= a_1 + a_2 g\sqrt{f}, \\ \varrho_2 &= a_1 - a_2 g\sqrt{f}. \end{aligned} \tag{2.7.7}$$

The eigen values ϱ_1 and ϱ_2 are related through an '*index*' ϵ. Once the branch $\sqrt{f}$ is decided, the index ϵ is computed by the formula

$$\epsilon = \frac{1}{2}\log\frac{\varrho_1}{\varrho_2}. \tag{2.7.8}$$

Let us introduce a quantity F defined as

$$F = \frac{\epsilon}{\sqrt{f}}. \tag{2.7.9}$$

On eliminating a_1 and a_2 from Eq. (2.7.7) in view of Eq. (2.7.8), one gets

$$\begin{aligned} a_1 &= \varrho_2 \left(\frac{1+e^{2\epsilon}}{2}\right) = \varrho_2 e^{\epsilon} \cosh \epsilon, \\ a_2 &= \varrho_2 \left(\frac{e^{2\epsilon}-1}{2g\sqrt{f}}\right) = \varrho_2 e^{\epsilon} \frac{\sinh \epsilon}{g\sqrt{f}}. \end{aligned} \tag{2.7.10}$$

Since $\det \mathbf{G} = a_1^2 - a_2^2 g^2 f$. Therefore

$$\sqrt{\det \mathbf{G}} = \varrho_2 e^{\epsilon}. \tag{2.7.11}$$

By using Eq. (2.7.11) into Eq. (2.7.10), will yield

$$\begin{aligned} a_1 &= \sqrt{\det \mathbf{G}} \cosh \epsilon, \\ a_2 &= \sqrt{\det \mathbf{G}} \frac{\sinh \epsilon}{g\sqrt{f}}. \end{aligned} \tag{2.7.12}$$

By substituting Eq. (2.7.12) into Eq. (2.7.4), one gets

$$\mathbf{G}(\alpha) = \sqrt{\det \mathbf{G}} \left[\cosh \epsilon \, \mathbf{I} + \frac{\sinh \epsilon}{\mathbf{g}\sqrt{\mathbf{f}}} \, \mathbf{C}\right]. \tag{2.7.13}$$

On introducing F from Eq. (2.7.9), it takes the form

$$\mathbf{G}(\alpha) = \sqrt{\det \mathbf{G}} \left[\cosh \left(F\sqrt{f}\right) \, \mathbf{I} + \frac{\sinh \left(F\sqrt{\mathbf{f}}\right)}{\mathbf{g}\sqrt{\mathbf{f}}} \, \mathbf{C}\right]. \tag{2.7.14}$$

Matrices of this form, whose determinants and indices differ, can be multiplied simply by taking the product of their determinants and sum of their indices. This suggests how $\mathbf{G}$ can be factorized. Therefore

$$\det \mathbf{G} = (\det \mathbf{G})_+ (\det \mathbf{G})_- \tag{2.7.15}$$

and

$$\mathcal{F} = \mathcal{F}_+ + \mathcal{F}_-. \tag{2.7.16}$$

Then, G commutes, i.e.,

$$\mathbf{G} = \mathbf{G}_+\mathbf{G}_- = \mathbf{G}_-\mathbf{G}_+, \tag{2.7.17}$$

with

$$\mathbf{G}_\pm = \left(\sqrt{\det \mathbf{G}}\right)_\pm \left[\cosh \mathcal{F}_\pm\sqrt{f}\ \mathbf{I} + \frac{\sinh\left(\mathcal{F}_\pm\sqrt{\mathrm{f}}\right)}{\mathrm{g}\sqrt{\mathrm{f}}}\ \mathbf{C}\right]. \tag{2.7.18}$$

Lastly the factors given in Eq. (2.7.18) sometimes suffer from the presence of poles occurring due to zeros of g. Luckily this hurdle can be removed as follows. Noting that

$$\mathbf{C}^2 = g^2 f\ \mathbf{I} = g_+^2 g_-^2 f_+ f_-\ \mathbf{I}. \tag{2.7.19}$$

Let us write

$$\mathbf{G} = \mathbf{G}\mathbf{I},$$

which implies

$$\mathbf{G} = \left(\frac{\mathbf{G}_+\mathbf{C}}{g_+^2 f_+}\right)\left(\frac{\mathbf{C}\mathbf{G}_-}{g_-^2 f_-}\right). \tag{2.7.20}$$

Thus, a new factorization is achieved

$$\mathbf{G} = \widehat{\mathbf{G}}_+\widehat{\mathbf{G}}_- = \widehat{\mathbf{G}}_-\widehat{\mathbf{G}}_+, \tag{2.7.21}$$

where

$$\widehat{\mathbf{G}}_\pm = \frac{\left(\sqrt{\det \mathbf{G}}\right)_\pm}{g_\pm^2 f_\pm}\left[\cosh\left(\mathcal{F}_\pm\sqrt{f}\right).\mathbf{C} + \mathrm{g}\sqrt{\mathrm{f}}\sinh\left(\mathcal{F}_\pm\sqrt{\mathrm{f}}\right).\mathbf{I}\right]. \tag{2.7.22}$$

These factors do not have undesirable poles.

2.8 The Green's function [140, 141]

Let $\mathcal{L}$ be a differential operator and $g(x)$ be a continuous function, $\mathcal{L}$ and $g(x)$ are given and remaining is to find an unknown function $y(x)$ which satisfies

$$\mathcal{L}\left[y(x)\right] = g(x), \tag{2.8.1}$$

for specified boundary conditions. If the operator $\mathcal{L}$ is one-one then the $\mathcal{L}^{-1}$ also exists such that

$$y(x) = \mathcal{L}^{-1}\left[g(x)\right]. \tag{2.8.2}$$

Then Eq. (2.8.2) can also be expressed as

$$y(x) = \int \mathcal{L}^{-1}\left[g(x_0)\delta\left(x - x_0\right)dx_0\right]. \tag{2.8.3}$$

The solution of Eq. (2.8.1) when

$$g(x) = \delta\left(x - x_0\right), \tag{2.8.4}$$

is called the Green's function $G\left(x, x_0\right)$. The Green's function then satisfies

$$\mathcal{L}\left[G\left(x, x_0\right)\right] = \delta\left(x - x_0\right), \tag{2.8.5}$$

with the same boundary conditions as on the function $y(x)$. In engineering terminology the Green's function is the impulse response of the system and also known as the '*transfer function*' [140].

2.8.1 Construction of the Green's function

The Green's function will be different for the different equations. Since every second order non-homogenous differential equation can be converted into Sturm-Liouville (S-L) form. Therefore, now the construction of the Green's function for the S-L boundary value problem is presented.

$$\left[\frac{d}{dx}\left\{p(x)\frac{dy}{dx}\right\} - q(x)y\right] + \varrho r(x)y = g(x), \tag{2.8.6}$$

subject to the homogeneous boundary conditions [141]. An alternate form of Eq. (2.8.6) is

$$[\mathcal{L} + \varrho r(x)] = g(x), \tag{2.8.7}$$

where ϱ is an eigen value of the corresponding S-L system. Now if the Green's function exist for Eq. (2.8.6) then its solution can be written as

$$y(x) = \int_a^b g(x_0)G(x, x_0)\, dx_0. \tag{2.8.8}$$

For a unit impulse deriving function, Eq.(2.8.6) will take the form

$$\left[\frac{d}{dx}\left\{p(x)\frac{dG}{dx}\right\} - q(x)G\right] + \varrho r(x)G = \delta(x - x_0), \tag{2.8.9}$$

where $G(x, x_0)$ is the Green's function. At $x \neq x_0$ Eq. (2.8.9) will take the form

$$\left[\frac{d}{dx}\left\{p(x)\frac{dG}{dx}\right\} - q(x)G\right] + \varrho r(x)G = 0, \tag{2.8.10}$$

the solution of Eq. (2.8.10) can be written in the form

$$G(x, x_0) = \begin{cases} A_1 y_1(x) & a \leq x \leq x_0, \\ A_2 y_2(x) & x_0 \leq x \leq b. \end{cases} \tag{2.8.11}$$

By using the various properties of the Green's function, the constants A_1 and A_2 appearing in Eq. (2.8.11) can be determined. The continuity of $G(x, x_0)$ at $x = x_0$ implies that

$$A_1 y_1(x_0) - A_2 y_2(x_0) = 0. \tag{2.8.12}$$

To establish that the derivative of $G(x, x_0)$ is discontinuous at $x = x_0$, integration of Eq. (2.8.9) from $x = x_0 - \varepsilon$ to $x_0 + \varepsilon$ will yield

$$\lim_{\varepsilon \longrightarrow 0} \int\limits_{x_0-\varepsilon}^{x_0+\varepsilon} \left[\frac{d}{dx}\left(p(x)\frac{dG}{dx}\right) + \{-q(x) + \varrho r(x)\} G \right] dx = \lim_{\varepsilon \longrightarrow 0} \int\limits_{x_0-\varepsilon}^{x_0+\varepsilon} \delta\,(x - x_0)\, dx. \tag{2.8.13}$$

Since $q(x)$, $r(x)$ and $G(x, x_0)$ are continuous at $x = x_0$, Eq. (2.8.13) reduces to

$$\lim_{\varepsilon \longrightarrow 0} \left[p(x_0) \left\{ \frac{dG(x_0 + \varepsilon, x_0)}{dx} - \frac{dG(x_0 - \varepsilon, x_0)}{dx} \right\} \right] = 1, \tag{2.8.14}$$

which simplifies to

$$\left\{ \frac{dG(x_{0+}, x_0)}{dx} - \frac{dG(x_{0-}, x_0)}{dx} \right\} = \frac{1}{p(x_0)}, \tag{2.8.15}$$

or

$$A_2 y_2'(x_0) - A_1 y_1'(x_0) = \frac{1}{p(x_0)}, \tag{2.8.16}$$

which shows the discontinuity of the derivative of the Green's function. Solving Eqs. (2.8.12) and (2.8.16) will lead to

$$A_1 = \frac{y_2'(x_0)}{W(x_0)p(x_0)}, \qquad A_2 = \frac{y_1'(x_0)}{W(x_0)p(x_0)}. \tag{2.8.17}$$

Substituting the values of A_1 and A_2 into Eq. (2.8.11) gives the Green's function in the closed form as

$$G(x,x_0)=\begin{cases}\frac{y_2'(x_0)}{W(x_0)p(x_0)}y_1(x) & a\leq x\leq x_0,\\ \frac{y_1'(x_0)}{W(x_0)p(x_0)}y_2(x) & x_0\leq x\leq b.\end{cases} \tag{2.8.18}$$

Thus some of the properties of the Green's function can be summarized as follows:

- $G(x,x_0)$ satisfies the homogeneous differential equation except at $x=x_0$.
- $G(x,x_0)$ is symmetric with respect to x and x_0.
- $G(x,x_0)$ satisfies the homogeneous boundary conditions.
- $G(x,x_0)$ is continuous at $x=x_0$.
- The derivative of $G(x,x_0)$ is discontinuous at $x=x_0$.

2.9 The Maliuzhinetz function [142-148]

In the study of diffraction of acoustic/electromagnetic waves by an impedance half plane the central role is played by the function $M_\pi(z)$, defined as

$$M_\pi(z)=\exp\left\{-\frac{1}{8\pi}\int_0^z\frac{\pi\sin t-2\sqrt{2}\sin\frac{t}{2}+2t}{\cos t}dt\right\}, \tag{2.9.1}$$

known as Maliuzhinetz's function introduced by Maliuzhinetz [142]. In (2.9.1), z is complex and the integration is along the path not crossing any of the singularities of the integrand [143]. In [142], the author discussed some the analytic properties of the introduced function shortly. In another paper [144] Maliuzhinetz mentioned that M. P. Sacharowa tabulated the Maliuzhinetz function for numerical computation purposes but no reference to the literature was given. The two main propositions in Maliuzhinetz method are the inversion formula for the Sommerfeld integral and the nullification theorem which can be found in [145]. Bucci [143] and Bowman [82] derived some analytic properties of the Maliuzhinetz function in sufficient details. Since the apparent complication of the expression (2.9.1) is a major deterrent to its use so it is desirable to compute it in a convenient manner and hence it can easily be incorporated into the scattering code.

Volakis and Senior [146] derived two simple expressions, one for small argument and the other is for large complex argument of $M_\pi(z)$.

For small argument,

$$M_\pi(z) = 1 - az^2 + O\left(z^4\right), \tag{2.9.2}$$

where $a = \frac{1}{16}\left(1 - \sqrt{2} + \frac{2}{\pi}\right) = 0.01390$, and a small argument approximation to $M_\pi(z)$ is therefore

$$M_\pi(z) = 1 - 0.01390z^2. \tag{2.9.3}$$

If $y \gg 0$,

$$M_\pi(z) = 1.05302\left[\cos\frac{1}{4}(z - i\widetilde{\gamma})\right]^{\frac{1}{2}} \qquad y > 8. \tag{2.9.4}$$

where $\widetilde{\gamma} = \ln 2 = 0.69315$. These expressions (2.9.3) and (2.9.4) must be employed within the strip $0 \leq x \leq \frac{\pi}{2}$. For other values of x the following recurrence relations of $M_\pi(z)$ must be employed to relate $M_\pi(z)$ to its value at the corresponding point within the strip

$$M_\pi(z) = \left[M_\pi\left(\frac{\pi}{2}\right)\right]^2 \frac{\cos\left(\frac{z}{4} - \frac{\pi}{8}\right)}{M_\pi(z-\pi)}, \tag{2.9.5}$$

with $M_\pi\left(\frac{\pi}{2}\right) = 0.96562$,

$$M_\pi(-z) = M_\pi(z), \tag{2.9.6}$$

and

$$M_\pi(z^*) = M_\pi^*(z), \tag{2.9.7}$$

where the asteriks denote the complex conjugate. Hu et al [147] applied the tanh transformation for the numerical computation of the Maliuzhinetz function $M_n(z)$ and showed that their analysis is valid for all values of z and n. Another detailed account of excellent work for the development of FORTRAN subroutines for the numerical computation of Maliuzhinetz function can be found in the report of Osipov and Stein [148]. Now some of the important analytic properties of the Maliuzhinetz function are enlisted here.

2.9.1 Some properties of the Maliuzhinetz function [82]

(a) $M_\pi(z)$ is an even meromorphic function of z.

(b) Its logarithmic derivative is given to be

$$\frac{M'_\pi(z)}{M_\pi(z)} = -\frac{\sin z}{8\cos z} + \frac{\sqrt{2}\sin\left(\frac{z}{2}\right)}{4\cos z} - \frac{z}{4\pi\cos z}. \tag{2.9.7}$$

(c) Two Maliuzhinetz's functions are multiplied as follows

$$M_\pi\left(z+\frac{\pi}{2}\right) M_\pi\left(z-\frac{\pi}{2}\right) = \left[M_\pi\left(\frac{\pi}{2}\right)\right]^2 \cos\frac{z}{4}. \tag{2.9.8}$$

(d) By successive application of Eq. (2.9.8), the following expressions can be obtained

$$M_\pi(z+\pi)\,M_\pi(z-\pi)=\frac{\left[M_\pi\left(\frac{\pi}{2}\right)\right]^4}{2\left[M_\pi(z)\right]^2}\left[\cos\frac{z}{2}+\cos\frac{\pi}{4}\right], \tag{2.9.9}$$

$$M_\pi\left(z+\frac{3\pi}{2}\right)M_\pi\left(z-\frac{3\pi}{2}\right)=\frac{1}{2}\left[M_\pi\left(\frac{\pi}{2}\right)\right]^2\frac{\cos\frac{z}{2}}{\cos\frac{z}{4}}, \tag{2.9.10}$$

$$\frac{M_\pi(z+\pi)}{M_\pi(z-\pi)}=\frac{\cos\left(\frac{z}{4}+\frac{\pi}{8}\right)}{\cos\left(\frac{z}{4}-\frac{\pi}{8}\right)}, \tag{2.9.11}$$

and

$$\frac{M_\pi(z+2\pi)}{M_\pi(z-2\pi)}=\cot\left(\frac{z}{2}+\frac{\pi}{4}\right). \tag{2.9.12}$$

(e) The function $M(z)$ can be expressed in terms of the function $M_\pi(z)$ by the product

$$M(z)=M_\pi(z+\pi+\theta)\,M_\pi(z+\pi-\theta)\,M_\pi(z-\pi-\varphi)\,M_\pi(z-\pi+\varphi). \tag{2.9.13}$$

(f) By manipulating expression (2.9.13) with expression (2.9.12) one can derive two more identities of the function $M(z)$.

$$\begin{aligned}\frac{M(\pi+z)}{M(\pi-z)}&=\frac{M_\pi'(z+\theta+2\pi)\,M_\pi'(z-\theta+2\pi)}{M_\pi'(z+\theta-2\pi)\,M_\pi'(z-\theta-2\pi)}\\&=\cot\left(\frac{z}{2}+\frac{\theta}{2}+\frac{\pi}{4}\right)\cot\left(\frac{z}{2}-\frac{\theta}{2}+\frac{\pi}{4}\right)=\frac{\cos\theta-\sin z}{\cos\theta+\sin z},\end{aligned} \tag{2.9.14}$$

and similarly

$$\frac{M(-\pi-z)}{M(-\pi+z)}=\frac{\cos\varphi-\sin z}{\cos\varphi+\sin z}. \tag{2.9.15}$$

Chapter 3

Wiener-Hopf Analysis Of Diffraction Of Acoustic Waves By A Soft-Hard Half Plane

The problem of diffraction of acoustic waves by a soft-hard half plane is a very special and substantial academic problem because:

(i) Two unusual features arose in this boundary value problem [34].

(ii) It constitutes the simplest half plane problem which can be casted in term of two coupled W-H equations that cannot be decoupled trivially [33].

After Rawlins' work [34], the same problem was again addressed by Chakrabarti [151] and he claimed to add the correct edge condition in the corresponding boundary value problem and pointed out that the W-H equations which arose in the paper of Rawlins [34] did not hold in the strip of analyticity which is one of the most important feature about the W-H analysis. The same boundary value problem was later on addressed by Heins [152]. He employed a function theoretic method which is based on the combination of ideas of Wiener and Hopf, and Carleman on singular integral equations. In a subsequent paper Heins [153] presented the detailed matrix factorization of the matrix appearing in the same boundary value problem.

Keeping in view of the above mentioned contributions about the soft-hard half plane, in this chapter two problems have been studied. The first problem deals with a line source diffraction by a soft-hard half plane. The introduction of line source changes the incident field and the method of solution requires a careful analysis in working out the diffracted field. The second problem is related to the point source diffraction by a soft-hard half plane. The introduction of a point source introduces another variable and an additional Fourier transform is required to transform the problem into two dimensions. These considerations are important since the line source and the point source are better substitutes for the plane wave situation and also the point sources are regarded as fundamental radiating devices [54]. The mathematical route of the problem consists of Fourier transform, the W-H technique and the method of steepest descent. A key attribute of the W-H technique is that it is not only independent of the incidence and reflection angles [73] but also provides an insight into the physical structure of the diffracted field [14]. The mathematical results of this chapter for the line source incidence modify the results of plane wave incidence [33] by a multiplicative factor which agrees well with the results already known [35, 46]. This can be considered as a check of correctness of the results presented in this chapter. Some graphs showing the effects of parameters $k\rho$ (distance of the observer from the point of observation), $k\rho_0$ (distance of the source from the point of observation) and θ_0 (the angle of incidence) on the diffracted field ψ are also plotted.

3.1 The boundary conditions

The Dirichlet (soft or pressure release) and Neumann (hard or rigid) boundary conditions are the classical one in acoustics and these conditions are also involved in the standard form of Babinet's principle. If ϕ is the velocity potential then the boundary conditions on a soft surface is the Dirichlet one, i.e., $\phi = 0$. At the hard or rigid surface the normal component of the fluid velocity is zero and thus giving rise to the Neumann boundary condition $\frac{\partial \phi}{\partial n} = 0$ [154].

Another way to derive the soft-hard boundary conditions is to consider a surface which may yield a little under the influence of pressure, e.g., a surface having an absorbing lining on one of its face [34]. Such a surface is described by an impedance relation between pressure $\widehat{p}$ and the normal velocity fluctuation and is mathematically described by a relation of the form

$$\frac{\partial \widehat{p}}{\partial n} - ik\beta\widehat{p} = 0,$$

[155], such that $\mathrm{Re}(\beta) > 0$, where n is the normal pointing into the absorbing lining,

$k = \frac{\omega}{c}$, c is a velocity of sound and β is the complex specific admittance of the absorbing lining. For an acoustically hard surface $|\beta| \to 0$ and $|\beta| \to \infty$ corresponds to an acoustically soft surface. Rawlins [34] examplified such a surface to be a barrier made of hard board having a foam rubber sheet on one of its face.

3.2 The line source diffraction problem

Consider the diffraction of an acoustic wave due to a line source located at the position (x_o, y_o) by a soft-hard half plane located at $x > 0$, $y = 0$ so that the edges lie along the $z - axis$. Thus the field is assumed to be independent of the $z - axis$ and let θ_0 be the angle of incidence. The geometry of the problem is depicted in Figure 3.1.

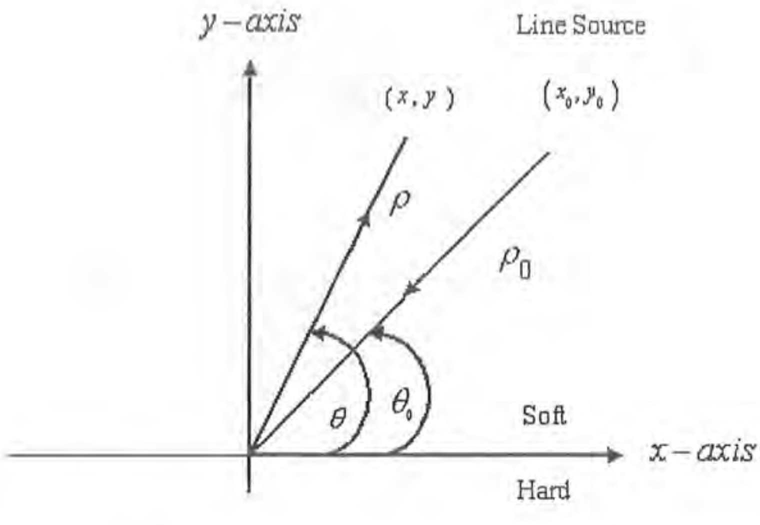

Fig. 3.1 Geometry of the half-plane problem

For harmonic acoustic vibrations of the time dependence $e^{-i\omega t}$, which are assumed and suppressed, a solution of the wave equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)\psi_t(x,y)=\delta(x-x_0)\,\delta(y-y_0), \tag{3.1}$$

is required, where ψ_t is the total velocity potential, and the boundary and continuity conditions are given as follows:

$$\psi_t\left(x,0^+\right)=0, \qquad x>0, \tag{3.2}$$

$$\frac{\partial\psi_t\left(x,0^-\right)}{\partial y}=0, \qquad x>0, \tag{3.3}$$

$$\psi_t\left(x,0^+\right)=\psi_t\left(x,0^-\right), \qquad x<0, \tag{3.4}$$

and

$$\frac{\partial\psi_t\left(x,0^+\right)}{\partial y}=\frac{\partial\psi_t\left(x,0^-\right)}{\partial y}, \qquad x<0. \tag{3.5}$$

For a unique solution of the problem, it is required that the radiation condition [14]

$$\sqrt{r}\left(\frac{\partial}{\partial r}-ik\right)\psi_t\to 0 \quad \text{as} \quad r=\left(x^2+y^2\right)^{\frac{1}{2}}\to\infty, \tag{3.6}$$

must be satisfied. Following [34, 37], for the analysis purpose it is convenient to express the total field for the line source incidence as

$$\psi_t(x,y)=\begin{cases}\psi_i(x,y)+\psi(x,y) & y>0,\\ \psi(x,y) & y<0,\end{cases} \tag{3.7}$$

where $\psi_i(x,y)$ accounts for the inhomogeneous source term and $\psi(x,y)$ represents the diffracted field. In Eq. (3.7) $\psi_i(x,y)$ is the incident field satisfying the equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)\psi_i(x,y)=\delta(x-x_0)\,\delta(y-y_0), \tag{3.8}$$

and the diffracted field $\psi(x,y)$ satisfies the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)\psi(x,y)=0. \tag{3.9}$$

For analytic convenience it is assumed that the wave number k has small imaginary part for which $k=k_r+ik_i$, where k_r and k_i are both positive and $k_i\to 0^+$ is the loss factor of the medium. The appropriate Fourier transform pair is defined as

$$\overline{\psi}(\alpha,y)=\int_{-\infty}^{\infty}\psi(x,y)\,e^{i\alpha x}dx, \tag{3.10}$$

and

$$\psi(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\psi}(\alpha,y)\,e^{-i\alpha x}d\alpha. \tag{3.11}$$

The solution of inhomogeneous Eq. (3.8) can be obtained by using the Green's function method as [140] as follows:

Applying Eq. (3.10) to Eq. (3.8) will result in

$$\left(\frac{d^2}{dx^2}+K^2\right)G(\alpha,y;x_0,y_0)=e^{i\alpha x_0}\delta(y-y_0), \tag{3.12}$$

where $K^2=(k^2-\alpha^2)$ and $G(\alpha,y,x_0,y_0)$ is the Green's function corresponding to the concentrated source located at (x_0,y_0). The homogenous solution of Eq. (3.12) can be written as

$$G(\alpha,y;x_0,y_0)=C_1(\alpha)e^{-\gamma(\alpha)y}+C_2(\alpha)e^{\gamma(\alpha)y}. \tag{3.13}$$

As the Green's function satisfies homogeneous boundary conditions, therefore, a suitable form of the radiated field can expressed as

$$G(\alpha, y; x_0, y_0) = \begin{cases} C(\alpha)e^{\gamma(\alpha)y} & -\infty < y < y_0 \\ C(\alpha)e^{-\gamma(\alpha)y} & y_0 < y < \infty, \end{cases} \tag{3.14}$$

where C is an unknown constant, C is taken to be same for both cases by using the property that the Green's function is continuous across the boundary $y = y_0$. The above result can further be simplified as

$$G(\alpha, y; x_0, y_0) = Ce^{-\gamma(\alpha)|y-y_0|}. \tag{3.15}$$

From Eq. (3.15) one gets

$$\frac{dG}{dy} = -C\gamma(\alpha)\, e^{-\gamma(\alpha)|y-y_0|} \operatorname{sgn}(y - y_0), \tag{3.16}$$

where sgn denotes the *signum* function defined as

$$\operatorname{sgn} x = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0. \end{cases} \tag{3.17}$$

To find the unknown constant appearing in Eq. (3.15), integrating Eq. (3.12) from $y_0 - \varepsilon$ to $y_0 + \varepsilon$, where ε is a small vanishing quantity, will result into

$$\lim_{\varepsilon \to 0} \left. \frac{dG}{dy} \right|_{y_0-\varepsilon}^{y_0+\varepsilon} = e^{i\alpha x_0}. \tag{3.18}$$

The constant C can be determined by utilizing another property of the Green's function that the derivative of Green's function is discontinuous at $y = y_0$. Using Eq. (3.15) into Eq. (3.18) and simplifying, C is determined to be

$$C = \frac{e^{i\alpha x_0}}{2iK(\alpha)}, \tag{3.19}$$

where $\gamma(\alpha) = -iK(\alpha)$ [14] has been used in Eq. (3.19). Using $\gamma(\alpha) = -iK(\alpha)$ and the value of C in Eq. (3.15), the Green's function (influence function) due to a line source located at (x_0, y_0) is determined to be

$$\overline{\psi}_i(\alpha, y) = \frac{1}{2iK(\alpha)} e^{i\alpha x_0 + iK(\alpha)|y-y_0|}, \tag{3.20}$$

where $K(\alpha) = \sqrt{k^2 - \alpha^2}$. Defining $K(\alpha)$, the square root function, to be that branch which reduces to $+k$ when $\alpha = 0$ and the complex plane is cut either from $\alpha = k$ to $\alpha = k\infty$ or from $\alpha = -k$ to $\alpha = -k\infty$.

For the diffracted field the solution of homogeneous Eq. (3.9) satisfying the radiation condition can formally be written as

$$\overline{\psi}(\alpha, y) = \begin{cases} A_1(\alpha) e^{iK(\alpha)y} & y > 0 \\ A_2(\alpha) e^{-iK(\alpha)y} & y < 0, \end{cases} \tag{3.21}$$

where $A_1(\alpha)$ and $A_2(\alpha)$ are the unknown coefficients to be determined.

Taking the Fourier transform of the boundary and continuity conditions $(3.2 - 3.5)$ will yield

$$\overline{\psi}_+(\alpha, 0^+) = -\int_0^\infty \psi_i(x, 0^+) e^{i\alpha x} dx, \tag{3.22}$$

$$\frac{\partial \overline{\psi}_+(\alpha, 0^-)}{\partial y} = -\int_0^\infty \frac{\partial \psi_i(x, 0^-)}{\partial y} e^{i\alpha x} dx, \tag{3.23}$$

$$\overline{\psi}_-(\alpha, 0^+) = \overline{\psi}_-(\alpha, 0^-), \tag{3.24}$$

and

$$\frac{\partial \overline{\psi}_-(\alpha, 0^+)}{\partial y} = \frac{\partial \overline{\psi}_-(\alpha, 0^-)}{\partial y}. \tag{3.25}$$

For a unique solution of the problem, the edge conditions require that ψ_t and its normal derivative must be bounded near $x = 0$ and these must be of the following orders [33]

$$\psi_t(x,0) = -1 + O\left(x^{\frac{1}{4}}\right), \qquad x \to 0, \tag{3.26}$$

$$\frac{\partial \psi_t(x,0)}{\partial y} = O\left(x^{-\frac{3}{4}}\right), \qquad x \to 0. \tag{3.27}$$

The substitution of solution (3.21) into boundary and continuity conditions (3.22 – 3.25) will yield

$$A_1(\alpha) = -\int_0^\infty \psi_i\left(x, 0^+\right) e^{i\alpha x} dx + \overline{\psi}_{-1}(\alpha), \tag{3.28}$$

$$K(\alpha)A_2(\alpha) = -i\int_0^\infty \frac{\partial \psi_i\left(x, 0^-\right)}{\partial y} e^{i\alpha x} dx + \overline{\psi}_{-2}(\alpha), \tag{3.29}$$

$$A_1(\alpha) - A_2(\alpha) = \overline{\psi}_{+1}(\alpha), \tag{3.30}$$

and

$$A_1(\alpha) + A_2(\alpha) = \frac{\overline{\psi}_{+2}(\alpha)}{K(\alpha)}, \tag{3.31}$$

where $\overline{\psi}_{-\,1,2}(\alpha)$ and $\overline{\psi}_{+\,1,2}(\alpha)$ are defined as follows:

$$\overline{\psi}_{-1}(\alpha) = \int_{-\infty}^{0} \psi(x, 0^+) e^{i\alpha x} dx, \tag{3.32}$$

$$\overline{\psi}_{-2}(\alpha) = i\int_{-\infty}^{0} \frac{\partial \psi(x, 0^-)}{\partial y} e^{i\alpha x} dx, \tag{3.33}$$

$$\overline{\psi}_{+1}(\alpha) = \int_0^\infty \left[\psi(x, 0^+) - \psi(x, 0^-)\right] e^{i\alpha x} dx, \tag{3.34}$$

and

$$\overline{\psi}_{+2}(\alpha)(\alpha) = -i\int_{0}^{\infty}\left[\frac{\partial\psi(x,0^{+})}{\partial y} - \frac{\partial\psi(x,0^{-})}{\partial y}\right]e^{i\alpha x}dx. \tag{3.35}$$

Due to the analytical properties of the Fourier integrals [134], $\overline{\psi}_{-\,1,2}(\alpha)$ and $\overline{\psi}_{+\,1,2}(\alpha)$ are regular functions of α in the half planes $\operatorname{Im}\alpha > \operatorname{Im}k\cos\theta_0$ and $\operatorname{Im}\alpha < \operatorname{Im}k$, respectively. By using the edge conditions in $(3.26-3.27)$, it can be shown that when $|\alpha| \to \infty$ in the respective regions of regularity then

$$\overline{\psi}_{-1}(\alpha) = -\frac{1}{i\alpha} + O\left(\alpha^{-\frac{5}{4}}\right), \tag{3.36}$$

$$\overline{\psi}_{+1}(\alpha) = O\left(\alpha^{-\frac{5}{4}}\right), \tag{3.37}$$

$$\overline{\psi}_{\pm 2}(\alpha) = O\left(\alpha^{-\frac{1}{4}}\right). \tag{3.38}$$

The elimination of $A_1(\alpha)$ and $A_2(\alpha)$ between $(3.28-3.31)$ leads to the following matrix Wiener-Hopf equation

$$\mathbf{H}(\alpha)\Psi_{+}(\alpha) = 2\Psi_{-}(\alpha) - 2\begin{bmatrix} q(\alpha) \\ r(\alpha) \end{bmatrix}, \tag{3.39}$$

valid in the strip of analyticity $\operatorname{Im}k\cos\theta_0 < \operatorname{Im}\alpha < \operatorname{Im}k$, as shown in Fig. 3.2.

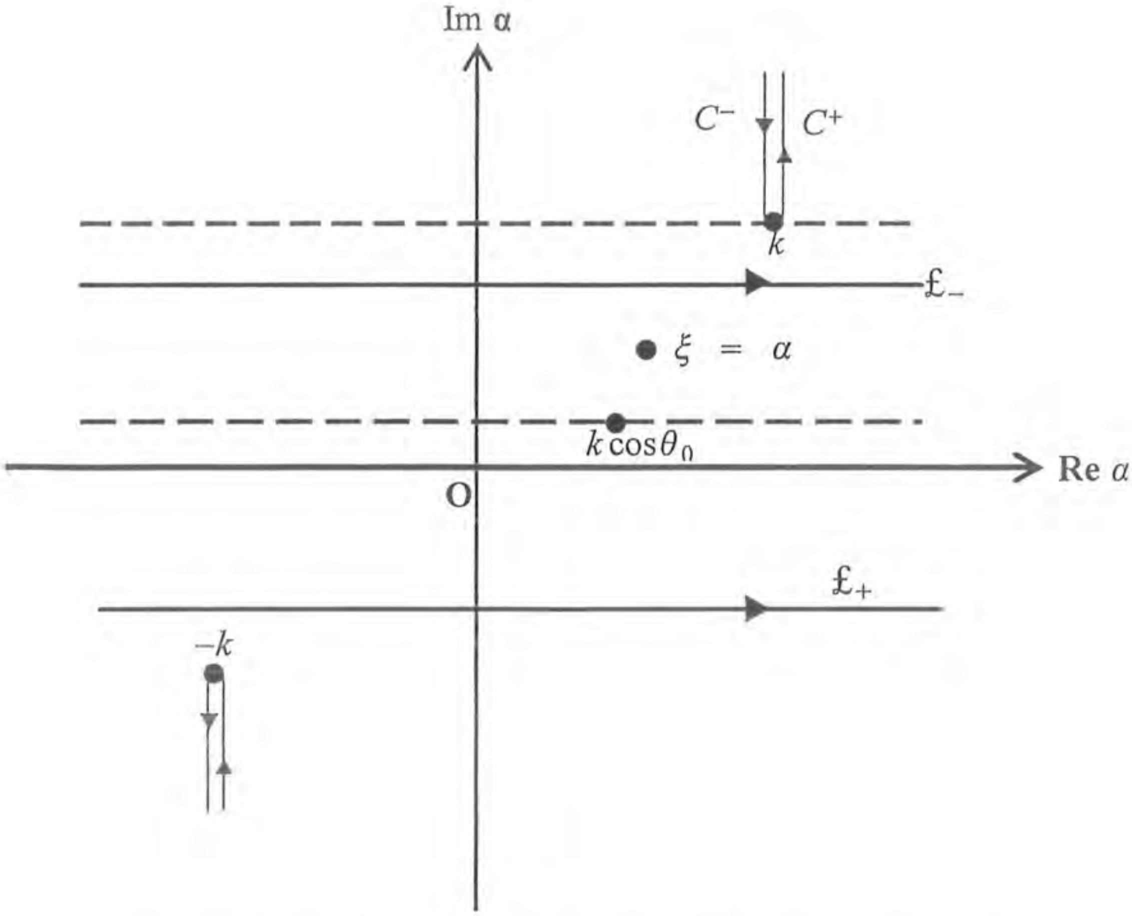

Fig. 3.2 Branch cuts and integration lines in the complex plane

The bold letters are used to denote the matrices and $\mathbf{H}(\alpha)$, $\Psi_{\pm}(\alpha)$, $q(\alpha)$ and $r(\alpha)$ are given as follows:

$$\mathbf{H}(\alpha) = \begin{bmatrix} 1 & 1/K(\alpha) \\ -K(\alpha) & 1 \end{bmatrix}, \tag{3.40}$$

$$\Psi_{\pm}(\alpha) = \begin{bmatrix} \overline{\psi}_{\pm 1}(\alpha) \\ \overline{\psi}_{\pm 2}(\alpha) \end{bmatrix}, \tag{3.41}$$

$$q(\alpha) = \int_0^\infty \psi_i\left(x, 0^+\right) e^{i\alpha x} dx, \tag{3.42}$$

and

$$r(\alpha) = i \int_0^\infty \frac{\partial \psi_i\left(x, 0^-\right)}{\partial y} e^{i\alpha x} dx. \tag{3.43}$$

3.3 Solution of the matrix W-H equation

Incidentally, the kernel matrix $\mathbf{H}(\alpha)$, which can be written as

$$\mathbf{H}(\alpha) = \mathbf{I} + \frac{1}{K(\alpha)} \begin{bmatrix} 0 & 1 \\ -\left(k^2 - \alpha^2\right) & 0 \end{bmatrix}, \tag{3.44}$$

is the same as in [33], where $\mathbf{I}$ is the unit matrix. Although the matrix $\mathbf{H}(\alpha)$ has been factorized by Büyükaksoy [33] by using the Daniele-Kharapkov methods [20, 21] yet for the sake of completeness and readers convenience, the complete factorization details have been given in Appendix A. Some of the important results are given below

$$\mathbf{H}_{+}(\alpha) = 2^{1/4} \begin{bmatrix} \cosh \varkappa(\alpha) & \sinh \varkappa(\alpha) / \gamma(\alpha) \\ \gamma(\alpha) \sinh \varkappa(\alpha) & \cosh \varkappa(\alpha) \end{bmatrix}, \tag{3.45}$$

with

$$\mathbf{H}_{-}(\alpha) = \mathbf{H}_{+}(-\alpha), \tag{3.46}$$

where

$$\varkappa(\alpha) = -\frac{i}{4} \arccos \frac{\alpha}{k}, \ \varkappa(-\alpha) = -\frac{i}{4} \left[\pi - \arccos \frac{\alpha}{k}\right], \tag{3.47}$$

and

$$\gamma(\alpha) = \sqrt{\alpha^2 - k^2}. \tag{3.48}$$

Also we note that as $|\alpha| \to \infty$, the orders of the elements of $\mathbf{H}_{+}(\alpha)$ can be calculated as follows. From Eq. (3.47)

$$\varkappa(\alpha) = -\frac{i}{4}\arccos\frac{\alpha}{k} = \frac{1}{4}\ln\left[\frac{\alpha+\sqrt{\alpha^2-k^2}}{k}\right], \tag{3.49}$$

or in another convenient form

$$e^{\varkappa(\alpha)} = \left[\frac{\alpha+\sqrt{\alpha^2-k^2}}{k}\right]^{\frac{1}{4}}. \tag{3.50}$$

As $|\alpha| \to \infty$

$$e^{\varkappa(\alpha)} \sim \left(\frac{2}{k}\right)^{\frac{1}{4}} O(\alpha)^{\frac{1}{4}} \tag{3.51}$$

and

$$e^{-\varkappa(\alpha)} \sim \left(\frac{2}{k}\right)^{-\frac{1}{4}} O(\alpha)^{-\frac{1}{4}}. \tag{3.52}$$

Hence

$$cosh\varkappa(\alpha) \sim (2)^{-\frac{3}{4}} k^{-\frac{1}{4}} O(\alpha)^{\frac{1}{4}} \tag{3.53}$$

and

$$2^{\frac{1}{4}} cosh\varkappa(\alpha) \sim (4k)^{-\frac{1}{4}} O(\alpha)^{\frac{1}{4}}. \tag{3.54}$$

Keeping in view of the Eqs. (3.49 – 3.52) the order estimates of the other elements of the matrix $\mathbf{H}_{+}(\alpha)$ can be easily determined and these are found to be

$$2^{\frac{1}{4}} sinh\varkappa(\alpha)/\gamma(\alpha) \sim (4k)^{-\frac{1}{4}} O(\alpha)^{-\frac{3}{4}}, \tag{3.55}$$

and

$$2^{\frac{1}{4}} sinh\varkappa(\alpha)\gamma(\alpha) \sim (4k)^{-\frac{1}{4}} O(\alpha)^{\frac{5}{4}}. \tag{3.56}$$

Also keeping in view $\mathbf{H}_+(-\alpha) = \mathbf{H}_-(+\alpha)$ we found the orders to be

$$\mathbf{H}_\pm(\alpha) \sim O(4k)^{-1/4} \begin{bmatrix} (\pm\alpha)^{1/4} & (\pm\alpha)^{-3/4} \\ (\pm\alpha)^{5/4} & (\pm\alpha)^{1/4} \end{bmatrix}. \tag{3.57}$$

Using the factorization of the kernel matrix, Eq. (3.39) can be rearranged as

$$\mathbf{H}_+\Psi_+(\alpha) = 2\left[\mathbf{H}_-(\alpha)\right]^{-1}\Psi_-(\alpha) - 2\left[\mathbf{H}_-(\alpha)\right]^{-1} \begin{bmatrix} q(\alpha) \\ r(\alpha) \end{bmatrix}. \tag{3.58}$$

Eq. (3.58) is the matrix Wiener-Hopf equation. To make it regular in the upper and lower half planes it is required to split the term

$$\left[\mathbf{H}_-(\alpha)\right]^{-1} \begin{bmatrix} q(\alpha) \\ r(\alpha) \end{bmatrix}.$$

To achieve this end, the additive decomposition theorem [14] is applied which results in

$$\left[\mathbf{H}_-(\alpha)\right]^{-1} \begin{bmatrix} q(\alpha) \\ r(\alpha) \end{bmatrix} = \begin{bmatrix} T(\alpha) \\ S(\alpha) \end{bmatrix} = \begin{bmatrix} T_+ + T_- \\ S_+ + S_- \end{bmatrix} = \begin{bmatrix} T_+ \\ S_+ \end{bmatrix} + \begin{bmatrix} T_- \\ S_- \end{bmatrix}, \tag{3.59}$$

where

$$\begin{aligned} T_\pm(\alpha) &= \pm\frac{1}{2\pi i}\int\limits_{-\infty+ic}^{\infty+ic} \frac{T(\xi)}{(\xi-\alpha)}d\xi \\ &= \pm\frac{1}{2\pi i}\int\limits_{-\infty+ic}^{\infty+ic} \frac{2^{-\frac{1}{4}}\left[cosh\varkappa(-\xi)\,q(\xi) - r(\xi)\,sinh\varkappa(-\xi)/\gamma(-\xi)\right]}{(\xi-\alpha)}d\xi, \end{aligned} \tag{3.60}$$

and

$$
\begin{aligned}
S_{\pm}(\alpha) &= \pm\frac{1}{2\pi i}\int\limits_{-\infty+id}^{\infty+id}\frac{S(\xi)}{(\xi-\alpha)}d\xi \\
&= \pm\frac{1}{2\pi i}\int\limits_{-\infty+id}^{\infty+id}\frac{2^{-\frac{1}{4}}\left[-q(\xi)\gamma(-\xi)\, sinh\varkappa(-\xi)+r(\xi)\, cosh\varkappa(-\xi)\right]}{(\xi-\alpha)}d\xi.
\end{aligned}
\tag{3.61}
$$

Hence Eq. (3.58) can finally be arranged as follows:

$$
\mathbf{H}_{+}\Psi_{+}(\alpha)+2\begin{bmatrix} T_{+}(\alpha) \\ S_{+}(\alpha)\end{bmatrix} = 2\left[\mathbf{H}_{-}(\alpha)\right]^{-1}\Psi_{-}(\alpha)-2\begin{bmatrix} T_{-}(\alpha) \\ S_{-}(\alpha)\end{bmatrix}. \tag{3.62}
$$

The left hand side of Eq. (3.62) is regular in the upper half plane $\operatorname{Im}\alpha > \operatorname{Im}k\cos\theta_0$ and the right hand side is regular in the lower half plane $\operatorname{Im}\alpha < \operatorname{Im}k$ and hence by the analytic continuation principle they define an entire matrix valued function $\mathbf{P}^{*}(\alpha)$. By taking into account the order relations (3.36 – 3.38), Eqs. (3.46) and (3.57), it can be concluded from the extended Liouville's theorem that $\mathbf{P}^{*}$ is a constant matrix of the form

$$
\mathbf{P}^{*} = p^{*}\begin{bmatrix} 0 \\ 2 \end{bmatrix}. \tag{3.63}
$$

Thus the solution of Eq. (3.62) becomes

$$
\left[\mathbf{H}_{-}(\alpha)\right]^{-1}\Psi_{-}(\alpha)-\begin{bmatrix} T_{-}(\alpha) \\ S_{-}(\alpha)\end{bmatrix} = \begin{bmatrix} 0 \\ p^{*}\end{bmatrix}, \tag{3.64}
$$

where the unknown constant p^* can be determined as follows. Simplification of Eq. (3.64) will give

$$\Psi_-(\alpha) = \mathbf{H}_-(\alpha) \begin{bmatrix} T_-(\alpha) \\ p^* + S_-(\alpha) \end{bmatrix}. \tag{3.65}$$

By considering the order relation in Eq. (3.57), the unknown constant p^* can be specified with the help of Eq. (3.65) as follows,

$$\begin{bmatrix} \overline{\psi}_{-1}(\alpha) \\ \overline{\psi}_{-2}(\alpha) \end{bmatrix} \approx (4k)^{-\frac{1}{4}} \left[p^* - \widetilde{T}_-\right] \begin{bmatrix} (-\alpha)^{-\frac{3}{4}} \\ (-\alpha)^{-\frac{1}{4}} \end{bmatrix} + O \begin{bmatrix} (-\alpha)^{-\frac{5}{4}} \\ (-\alpha)^{-\frac{1}{4}} \end{bmatrix} \tag{3.66}$$

with

$$\widetilde{T}_- = \lim_{\alpha \to \infty} \alpha T_-. \tag{3.67}$$

The correct behaviors of $\overline{\psi}_{-1}(\alpha)$ and $\overline{\psi}_{-2}(\alpha)$ are recovered if

$$p^* - \widetilde{T}_- = 0. \tag{3.68}$$

Hence the expressions for $\overline{\psi}_{-1}(\alpha)$ and $\overline{\psi}_{-2}(\alpha)$, the elements of $\Psi_-(\alpha)$ defined in Eq. (3.41), are given as

$$\overline{\psi}_{-1}(\alpha) = \left[cosh\varkappa(-\alpha) \left\{ -\frac{1}{2\pi i} \int_{-\infty+ia}^{\infty+ia} \frac{[cosh\varkappa(-\xi)\, q(\xi) - r(\xi)\, sinh\varkappa(-\xi)/\gamma(-\xi)]}{(\xi-\alpha)} d\xi \right\} \right.$$

$$\left. + \frac{sinh\varkappa(-\alpha)}{\gamma(-\alpha)} \left\{ 2^{\frac{1}{4}} p^* - \frac{1}{2\pi i} \int_{-\infty+ia}^{\infty+ia} \frac{[-q(\xi)\,\gamma(-\xi)\, sinh\varkappa(-\xi) + r(\xi)\, cosh\varkappa(-\xi)]}{(\xi-\alpha)} d\xi \right\} \right], \tag{3.69}$$

and

$$\overline{\psi}_{-2}(\alpha)=\left[\gamma(-\alpha)\,sinh\varkappa(-\alpha)\left\{-\frac{1}{2\pi i}\int\limits_{-\infty+ia}^{\infty+ia}\frac{[cosh\varkappa(-\xi)\,q(\xi)-r(\xi)\,sinh\varkappa(-\xi)\,/\gamma(-\xi)]}{(\xi-\alpha)}d\xi\right\}\right.$$

$$\left.+cosh\varkappa(-\alpha)\left\{2^{\frac{1}{4}}p^{*}-\frac{1}{2\pi i}\int\limits_{-\infty+ia}^{\infty+ia}\frac{[-q(\xi)\,\gamma(-\xi)\,sinh\varkappa(-\xi)+r(\xi)\,cosh\varkappa(-\xi)]}{(\xi-\alpha)}d\xi\right\}\right]. \tag{3.70}$$

3.4 Far field solution

Now by substituting Eq. (3.69) into Eq. (3.28) and then the result in Eq. (3.21) and taking the inverse Fourier transform, the diffracted far field for $y>0$ is given to be

$$\psi(x,y)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left[cosh\varkappa(-\alpha)\left\{-\frac{1}{2\pi i}\int\limits_{-\infty+ia}^{\infty+ia}\frac{[cosh\varkappa(-\xi)\,q(\xi)-r(\xi)\,sinh\varkappa(-\xi)\,/\gamma(-\xi)]}{(\xi-\alpha)}d\xi\right\}\right.$$

$$\left.+\frac{sinh\varkappa(-\alpha)}{\gamma(-\alpha)}\left\{2^{\frac{1}{4}}p^{*}-\frac{1}{2\pi i}\int\limits_{-\infty+ia}^{\infty+ia}\frac{[-q(\xi)\,\gamma(-\xi)\,sinh\varkappa(-\xi)+r(\xi)\,cosh\varkappa(-\xi)]}{(\xi-\alpha)}d\xi\right\}\right]$$

$$\times e^{iK(\alpha)y-i\alpha x}d\alpha-\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left\{\int\limits_{0}^{\infty}\psi_i\left(x,0^{+}\right)e^{i\alpha x}dx\right\}e^{iK(\alpha)y-i\alpha x}d\alpha. \tag{3.71}$$

To determine the far field behavior of the diffracted field the following substitutions can be introduced

$$x = \rho\cos\theta,\quad y=\rho\sin\theta\qquad(0<\theta<\pi), \tag{3.72}$$

$$x_0 = \rho_0\cos\theta_0,\ y_0=\rho_0\sin\theta_0\ \ (\pi<\theta_0<0), \tag{3.73}$$

and the transformation

$$\alpha = -k\cos(\theta + i\zeta), \tag{3.74}$$

where ζ given in Eq. (3.74) is real. The contour of integration over α in Eq. (3.71) goes into the branch of hyperbola around $-ik$ if $\frac{\pi}{2} < \theta < \pi$. It is further observed that in deforming the contour into a hyperbola the pole $\alpha = \xi$ may be crossed. If one makes the transformation $\xi = k\cos(\theta_o + it_1)$ the contour over ξ also goes into a hyperbola. The two hyperbolae will not cross each other if $\theta < \theta_0$. However, if the inequality is reversed there will be a contribution from the pole which cancels the incident wave in the shadow region. The explicit expression for the unknown constant p^* is determined with the help of Eqs. (3.60), (3.67) and (3.68) and found to be

$$p^* = \frac{1}{2\pi i}\int\limits_{-\infty+ic}^{\infty+ic} 2^{-\frac{1}{4}}\left[cosh\varkappa(-\xi)\, q(\xi) - r(\xi)\, sinh\varkappa(-\xi)/\gamma(-\xi)\right] d\xi. \tag{3.75}$$

By using Eqs. (3.42), (3.43), (3.73) and the transformation $\xi = k\cos(\theta_o + it_1)$ in Eq. (3.75), p^* is determined to be

$$p^* = \frac{2^{-\frac{1}{4}}}{2\pi i}\int\limits_{-\infty+ic}^{\infty+ic} \left[cosh\varkappa(-k\cos(\theta_o + it_1)) - isinh\varkappa(-k\cos(\theta_o + it_1))\right] e^{ik\rho_0\cosh t_1} dt_1. \tag{3.76}$$

A saddle point for the integral appearing in the last Eq. (3.76) occurs at $t_1 = 0$. By using the method of steepest descent [69], the explicit expression of p^* is given by

$$p^* = \frac{2^{\frac{1}{4}}\sin\frac{\theta_o}{4}}{2\pi i}\frac{e^{ik\rho_0 + \frac{i\pi}{4}}}{\sqrt{2\pi k\rho_o}}. \tag{3.77}$$

By using Eqs. (3.72 – 3.74) and (3.77) in Eq. (3.71) and the method of steepest descent [69], we obtain

$$\psi(\rho,\theta) \approx \frac{i}{4\pi^2}\left[\sin\frac{\theta_o}{4}\frac{sinh\varkappa\,(k\cos\theta)}{\gamma\,(k\cos\theta)} - \left\{cosh\varkappa\,(k\cos\theta)\sin\frac{\theta_o}{4}\right.\right.$$

$$\left.\left.+\frac{sinh\varkappa\,(k\cos\theta)}{\gamma\,(k\cos\theta)}\cos\frac{\theta_0}{4}k\sin\theta_0\right\}\frac{1}{k\,(\cos\theta+\cos\theta_0)}\frac{\sqrt{2}e^{ik(\rho+\rho_0)}}{\sqrt{\rho\rho_0}}\sin\theta.\right. \tag{3.78}$$

Substituting

$$\begin{aligned} sinh\varkappa\,(k\cos\theta) &= -i\sin\frac{\theta}{4}, \\ cosh\varkappa\,(k\cos\theta) &= \cos\frac{\theta}{4}, \\ \gamma\,(k\cos\theta) &= -ik\sin\theta \end{aligned} \tag{3.79}$$

in Eq. (3.78) the field due to a line source at a large distance from the plate for $y > 0$ is given as

$$\psi(\rho,\theta) \approx \frac{i}{4\pi^2}\left\{\sin\frac{\theta_o}{4}\sin\frac{\theta}{4} - \frac{\cos\frac{\theta}{4}\sin\frac{\theta_o}{4}\sin\theta + \sin\frac{\theta}{4}\cos\frac{\theta_0}{4}\sin\theta_0}{\cos\theta+\cos\theta_0}\right\}\frac{\sqrt{2}e^{ik(\rho+\rho_0)}}{k\sqrt{\rho\rho_0}}, \tag{3.80}$$

which after some trigonometric simplification reduces to

$$\psi(\rho,\theta) \approx -\frac{e^{\frac{i\pi}{2}}}{\sqrt{2}\pi^2}\left(\frac{\sin\frac{\theta_o}{4}\sin\frac{\theta}{4}}{\cos\theta+\cos\theta_0}\right)\left(1+\cos\frac{\theta}{2}+\cos\frac{\theta_0}{2}\right)\frac{e^{ik(\rho+\rho_0)}}{k\sqrt{\rho\rho_0}}. \tag{3.81}$$

The contour of steepest descent is shown in Fig. 3.3.

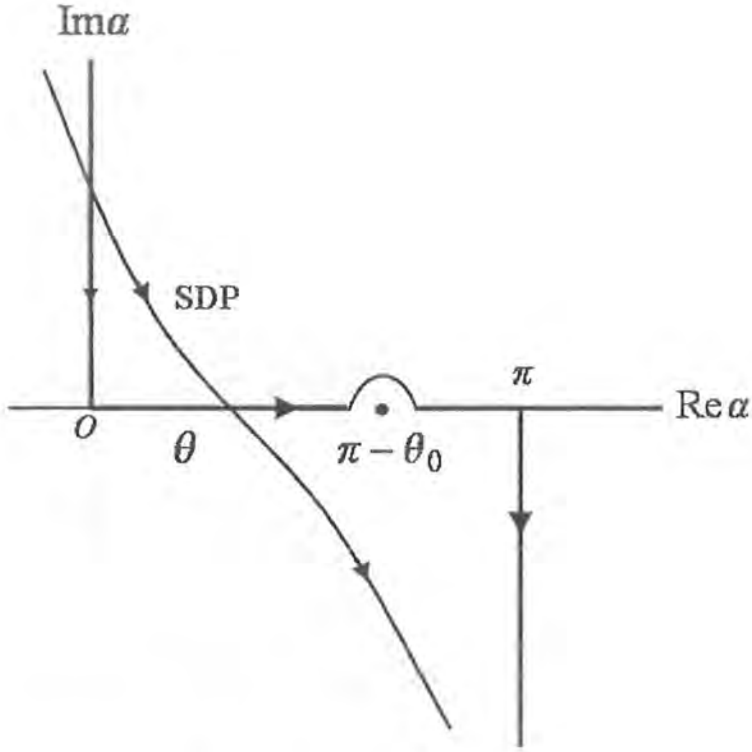

Fig. 3.3 The path of steepest descent

3.5 The point source diffraction problem

For the case of point source scattering, suppose that a point source is occupying the position (x_0, y_0, z_0). Thus for harmonic time variations $e^{-i\omega t}$, the solution of the equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+k^2\right)\Phi_t(x,y,z)=\delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0), \qquad (3.82)$$

subject to the following boundary conditions,

for $x>0$

$$\Phi_t\left(x,0^+,z\right)=0, \qquad -\infty<z<\infty, \qquad (3.83)$$

$$\frac{\partial \Phi_t\left(x,0^-,z\right)}{\partial y}=0, \qquad -\infty<z<\infty, \tag{3.84}$$

and for $x<0$

$$\Phi_t\left(x,0^+,z\right)=\Phi_t\left(x,0^-,z\right), \qquad -\infty<z<\infty, \tag{3.85}$$

$$\frac{\partial \Phi_t\left(x,0^+,z\right)}{\partial y}=\frac{\partial \Phi_t\left(x,0^-,z\right)}{\partial y}, \qquad -\infty<z<\infty, \tag{3.86}$$

is required, where Φ_t is the total acoustic field defined as

$$\Phi_t(x,y,z)=\Phi_0(x,y,z)+\Phi(x,y,z), \tag{3.87}$$

where Φ is the scattered field and Φ_0 represents the effect due to a point source.

Let us define the Fourier transform and the inverse Fourier transform with respect to the variable z as follows

$$\overline{\Phi}\left(x,y,\mu\right)=\int_{-\infty}^{\infty}\Phi\left(x,y,z\right)e^{ik\mu z}dz, \tag{3.88}$$

$$\Phi\left(x,y,z\right)=\frac{k}{2\pi}\int_{-\infty}^{\infty}\overline{\Phi}\left(x,y,\mu\right)e^{-ik\mu z}d\mu. \tag{3.89}$$

Taking Fourier transform of the Eqs. $(3.82-3.86)$, the problem with boundary conditions in the transformed domain μ takes the following form

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\eta_1^2\right)\overline{\Phi}_t=a\delta\left(x-x_0\right)\delta\left(y-y_0\right), \tag{3.90}$$

with $\eta_1=\sqrt{1-\mu^2}$, and $a=e^{ik\mu z_0}$.

The transformed boundary conditions take the form

$$\overline{\Phi}_t\left(x,0^+,\mu\right)=0 \qquad x>0, \tag{3.91}$$

$$\frac{\partial\overline{\Phi}_t(x,0^-,\mu)}{\partial y}=0 \qquad x>0, \tag{3.92}$$

$$\overline{\Phi}_t(x,0^+,\mu)=\overline{\Phi}_t(x,0^-,\mu) \qquad x<0, \tag{3.93}$$

$$\frac{\partial\overline{\Phi}_t(x,0^+,\mu)}{\partial y}=\frac{\partial\overline{\Phi}_t(x,0^-,\mu)}{\partial y} \qquad x<0. \tag{3.94}$$

Thus, it is seen that the problem (3.90) together with the boundary conditions (3.91–3.94) in the transformed domain μ is the same as in the case of two dimensions formulated in the section 2 except that $k^2\eta_1^2$ replaces k^2[46, 60, 61].

3.6 Solution of the problem

As mentioned before, the mathematical problem (3.90) together with the boundary conditions (3.91 − 3.94) in the transformed domain μ is the same as in the case of two dimensions formulated in the section 3.2 except that $k^2\eta_1^2$ replaces k^2. Thus, using the solution obtained in section 3.2, the diffracted field due to a point source is given to be:

For $y>0$, we have

$$\overline{\Phi}(\rho,\theta,\mu)\approx-\frac{e^{\frac{i\pi}{2}}}{\pi^2}\left(\frac{\sin\frac{\theta_0}{4}\sin\frac{\theta}{4}}{\cos\theta+\cos\theta_0}\right)\left(1+\cos\frac{\theta}{2}+\cos\frac{\theta_0}{2}\right)\frac{e^{ik\eta_1(\rho+\rho_0)+ik\mu z_0}}{k\eta_1\sqrt{2\rho\rho_0}}. \tag{3.95}$$

The scattered field in the spatial domain can now be obtained by taking the inverse Fourier transform of Eq. (3.95). Thus, for $y>0$

$$\Phi(\rho,\theta,z)\approx\frac{k}{2\pi}\int_{-\infty}^{\infty}\left\{-\frac{e^{\frac{i\pi}{2}}}{\pi^2}\left(\frac{\sin\frac{\theta_0}{4}\sin\frac{\theta}{4}}{\cos\theta+\cos\theta_0}\right)\left(1+\cos\frac{\theta}{2}+\cos\frac{\theta_0}{2}\right)\right\}\frac{e^{ik\eta_1(\rho+\rho_0)-ik\mu(z-z_0)}}{k\eta_1\sqrt{2\rho\rho_0}}d\mu. \tag{3.96}$$

In order to solve the problem completely, the following substitutions [60, 61] in the Eq. (3.96)

$$\mu = \cos\Theta,\ \eta_1 = \sqrt{1-\mu^2} = \sin\Theta, \tag{3.97}$$

$$\rho+\rho_0 = R_1 \sin\nu,\ z-z_0 = R_1\cos\nu, \tag{3.98}$$

$$R_1 = \sqrt{(z-z_0)^2+(\rho+\rho_0)^2}, \tag{3.99}$$

are introduced. Using the method of steepest descent [69], the integral appearing in Eq. (3.96) can be evaluated asymptotically for large kR_1. The contour of integration is taken such that it passes through the point of steepest descent $\Theta = \nu$. Therefore, for $kR_1 \gg 1$, omitting the details of calculations, the final form of field for $y > 0$ is given as follows.

For $y > 0$,

$$\Phi(\rho,\theta,z) \approx \frac{1}{2\pi^2}\left[\left(\frac{\sin\frac{\theta_o}{4}\sin\frac{\theta}{4}}{\cos\theta+\cos\theta_0}\right)\left(1+\cos\frac{\theta}{2}+\cos\frac{\theta_0}{2}\right)\right]\frac{e^{-ikR_1+i\frac{\pi}{4}}}{\sqrt{\pi k R_1 \rho\rho_0}}. \tag{3.100}$$

3.7 Graphical results and discussions

In this section, some graphical results showing the effect of some dimensionless parameters such as the observer distance from the origin $k\rho$, source distance from the origin $k\rho_0$ and the observation angle θ_0 on the diffracted field ψ produced by the line source are presented.

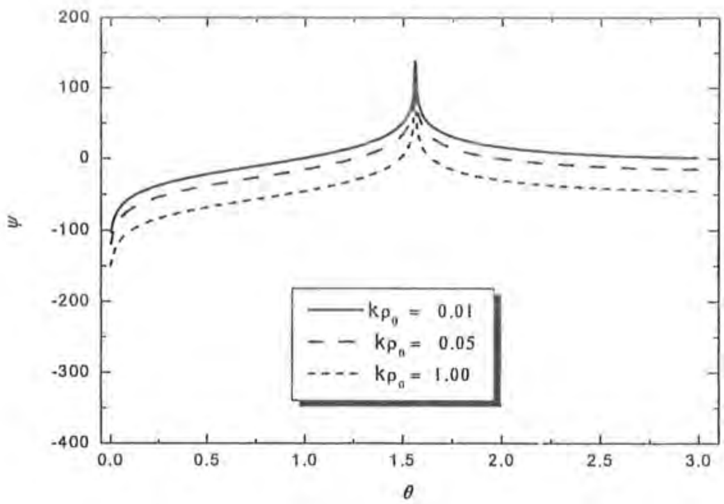

Fig. 3.4 Plots of ψ Vs θ for $k\rho = 1$ and $\theta_0 = \frac{\pi}{2}$.

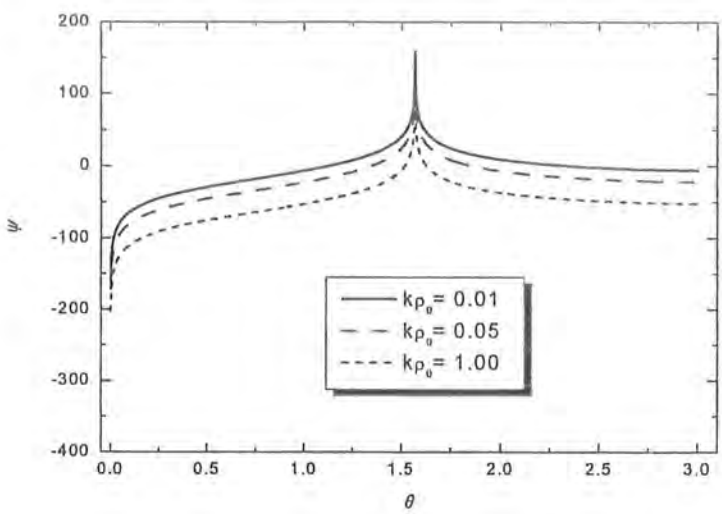

Fig. 3.5 Plots of ψ Vs θ for $k\rho = 2$ and $\theta_0 = \frac{\pi}{2}$.

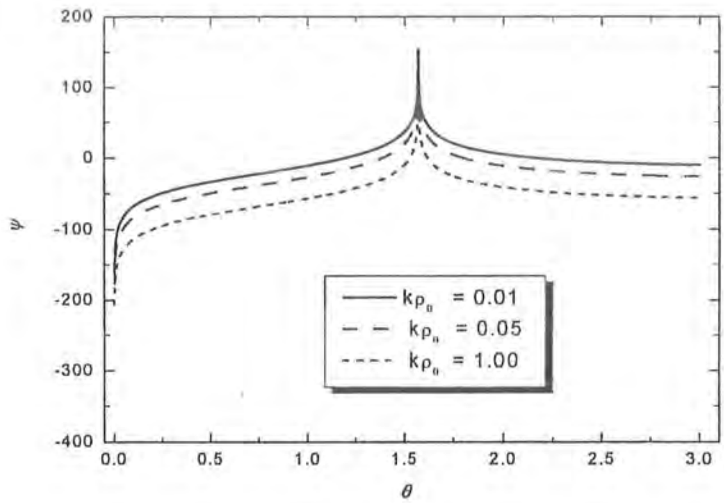

Fig. 3.6 Plots of ψ Vs θ for $k\rho = 3$ and $\theta_0 = \frac{\pi}{2}$.

- Figures (3.4 − 3.6) show the variation of the parameter $k\rho_0$ by fixing $\theta_0 = \pi/2$ and $k\rho = 1, 2, 3$ respectively. It is observed that by increasing the parameter $k\rho_0$ the magnitude of the diffracted field decreases.

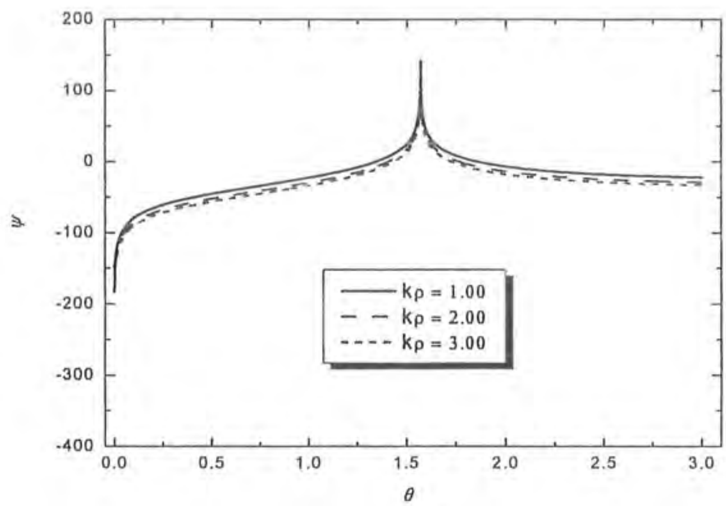

Fig. 3.7 Plots of ψ Vs θ for $k\rho_0 = 0.01$ and $\theta_0 = \frac{\pi}{2}$.

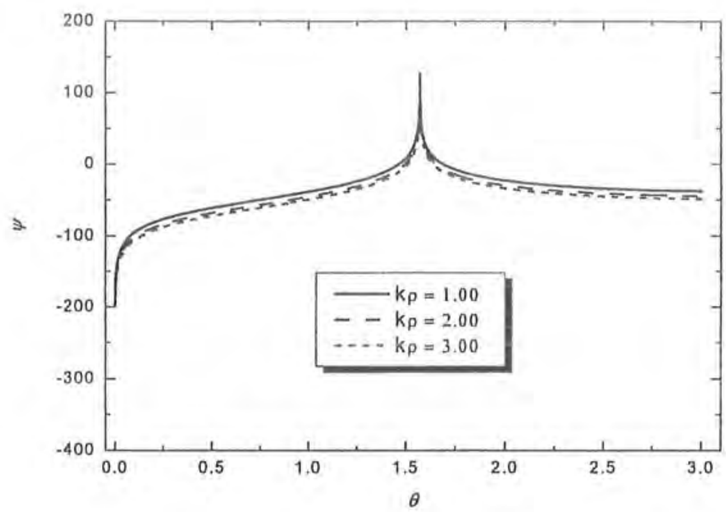

Fig. 3.8 Plots of ψ Vs θ for $k\rho_0 = 0.05$ and $\theta_0 = \frac{\pi}{2}$.

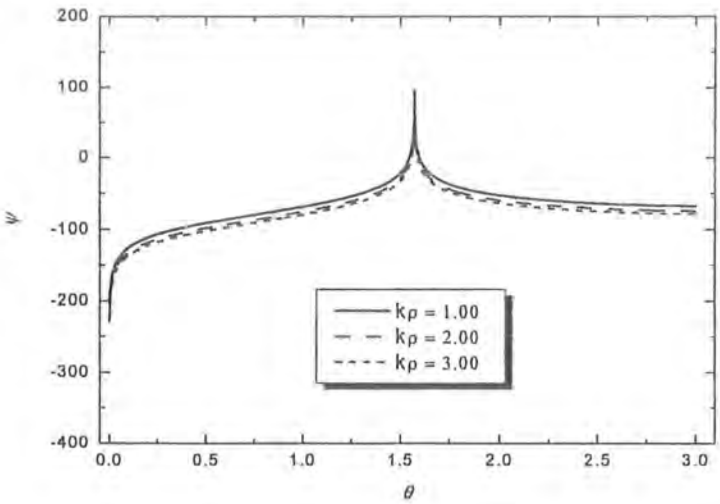

Fig. 3.9 Plots of ψ Vs θ for $k\rho_0 = 1$ and $\theta_0 = \frac{\pi}{2}$.

- Figures $(3.7 - 3.9)$ are plotted to note the variation of the parameter $k\rho$ on the diffracted field. It is observed that the diffracted field decreases by increasing the parameter $k\rho$ and fixing the other parameters to be $\theta_0 = \pi/2$ and $k\rho_0 = 0.01, 0.05, 1$ respectively, but the field lines are almost very close in these figures as compared to the Figures $(3.2 - 3.4)$.

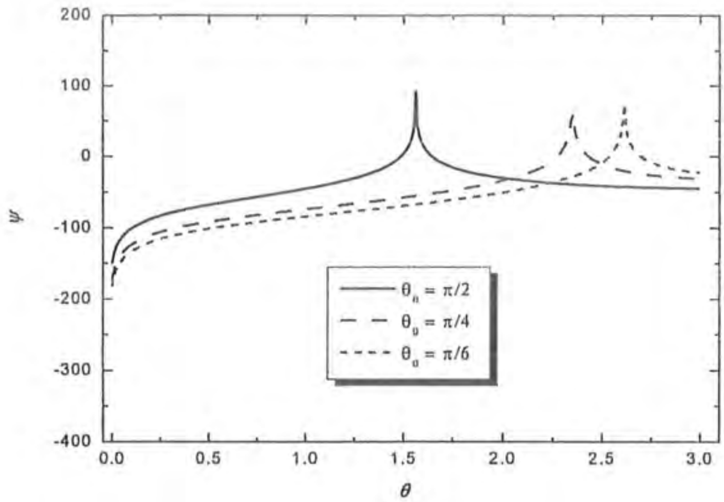

Fig. 3.10 Plots of ψ Vs θ for $k\rho_0 = 1$ and $k\rho = 1$.

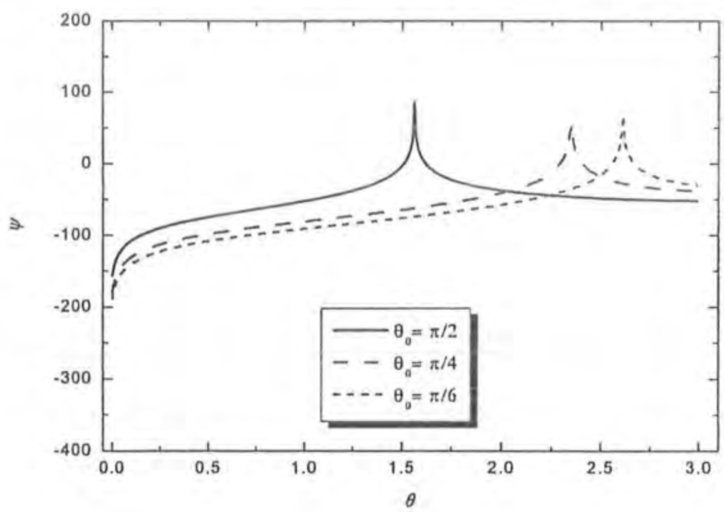

Fig. 3.11 Plots of ψ Vs θ for $k\rho_0 = 1$ and $k\rho = 2$.

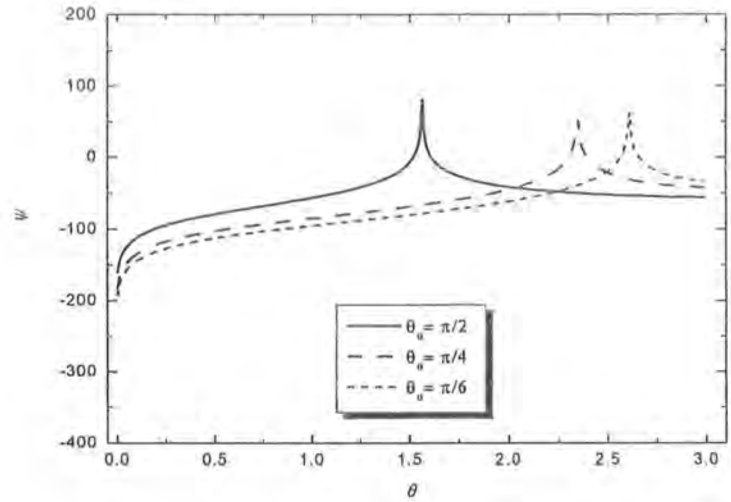

Fig. 3.12 Plots of ψ Vs θ for $k\rho_0 = 1$ and $k\rho = 3$.

- Figures $(3.10 - 3.12)$ depict the variation of the observation angle θ_0 on the diffracted field. It is observed that the highest curve corresponds to the normal incidence and the magnitude of the diffracted field decreases and the peaks shift toward right as the angle of incidence decreases which is as expected.

3.8 Concluding remarks

In this chapter, the line source and the point source scattering of acoustic waves by the soft/hard half plane are studied. By means of Fourier transform technique, the boundary value problem is reduced to the matrix Wiener-Hopf equation whose solution is obtained by considering the Wiener-Hopf factorization of the kernel matrix. It is observed that the results obtained in this chapter for the case of line source incidence differ from [33] by a multiplicative factor. The result of point source excitation is also obtained. Some graphs, showing the effect of sundry parameters, for the case of line source situation are also plotted and discussed.

- The line source results modify the plane wave situation [33] results by a multiplicative factor of

$$\sqrt{\frac{2\pi}{k\rho_0}}e^{ik\rho_0+i\pi/4}$$

  which agrees well with the already known results [35, 46].

- Point source results are based on line source results.

- The explicit analytical expressions for the diffracted field produced by the line source and the point source in the half space $y < 0$ can also be calculated in a similar manner.

Chapter 4

Line Source And Point Source Scattering Of Acoustic Waves By The Junction Of Transmissive And Soft-Hard Half Planes

The aim of this chapter is to study the line source and point source scattering of acoustic waves by the junction of partially transmissive (Senior's resistive-type) [] and soft-hard half planes. The junction configuration has been the focus of attention of many researchers both in acoustics and electromagnetics. To name a few only, e.g., the problem of diffraction by the junction of impedance half planes was first treated by Maliuzhinetz as a special case of wedge diffraction. Rojas employed

Wiener-Hopf technique to develop diffraction coefficients for impedance junction illuminated at skew incidence. Senior used dual integral equation approach to derive diffraction coefficients for resistive/conductive sheet junctions [72]. Büyükaksoy et al [70] studied the scattering of plane waves by the junction of transmissive and soft-hard half planes. Recently Ahmad [67] studied the diffraction of a spherical acoustic wave by the coupling of pressure release and absorbing half planes. Much more work related to the junction configuration can be found in the book of Senior and Volakis [72]. It has been mentioned by Rojas [71] that scattering properties of a surface are functions of both of its geometrical and material properties. Diffraction by a junction configuration is an important topic in diffraction theory and it constitutes a canonical boundary value problem for diffraction because of abrupt changes in the material properties, besides this it is also relevant to many engineering applications.

4.1 Approximate boundary conditions

The boundary conditions on a partially transmissive half plane are the first order impedance (Leontovich) conditions relating field and its normal derivative and sometimes also referred as standard impedance boundary conditions [72]. For detailed historical discussion on impedance boundary conditions the reader may be referred to [156]. These conditions were first introduced by Rytov [157] and subsequently were used to model radio waves propagation along the surface of the earth and near conducting obstacles [158]. Mathematically the first order impedance conditions are given by [72]

$$\frac{\partial \phi}{\partial n} - ik_0 \frac{z_0^*}{\eta} \phi = 0,$$

where ϕ is the velocity potential, η is the specific impedance of the surface, z_0^* is the intrinsic impedance of the surrounding medium and k_0 is the free space wave number. It is also worthwhile to mention here that a surface which may yield a little under the influence of pressure is also characterized by the condition (4.1) (see [154]). Approximate boundary conditions are also used for computational purposes, e.g., absorbing boundary conditions [155], Myre's impedance conditions [159] etc. have been employed by many researchers, e.g., Rawlins [38], Asghar [61], Asghar and Hayat [63, 64] and Ahmed [48] etc. Recently Ayub et al [51, 52] used such first order impedance conditions to study magnetic line source, and line source and point source scattering of electromagnetic waves by an impedance and reactive steps. Another detailed account of higher order impedance and absorbing boundary conditions can be found in [160, 161]. Keeping in view of the all above mentioned studies about the geometry of junction and impedance boundary conditions, this chapter is dedicated to study the scattering of cylindrical and spherical acoustic waves by a junction of transmissive and soft-hard half planes.

4.2 The line source scattering problem

Consider the problem of scattering of an acoustic wave from a line source located at (x_0, y_0) by the junction of the soft-hard half plane located at $y = 0$, $x > 0$, and the penetrable half plane located at $y = 0$, $x < 0$, respectively so that their edges lie along the $z - axis$. Thus it can be said that the field is independent of the $z - axis$.

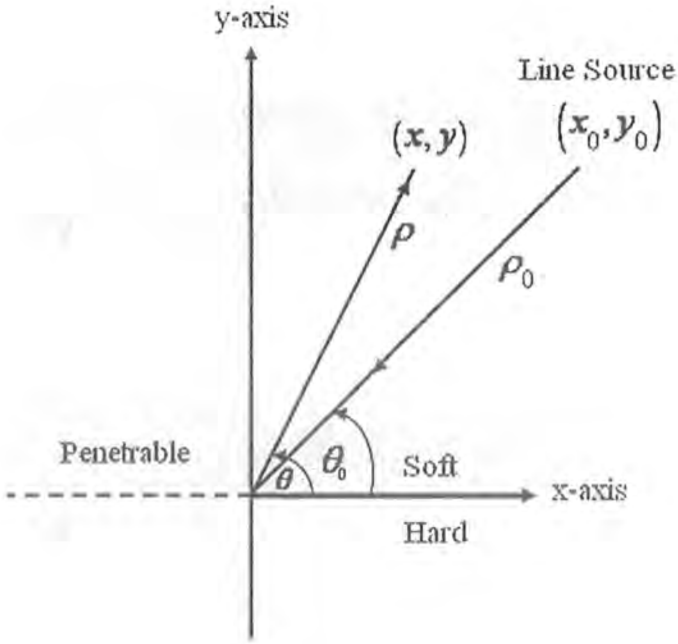

Fig. 4.1 Geometry of the junction problem

The geometry of the problem is shown in Figure 4.1. For the harmonic acoustic vibrations of time dependence, the solution of the following equation is required

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right) \psi_t(x, y) = \delta(x - x_0)\, \delta(y - y_0), \tag{4.1}$$

where ψ_t is the total velocity potential, and the boundary conditions at the soft and hard surfaces are

$$\psi_t\left(x,0^+\right)=0, \qquad x>0, \tag{4.2}$$

$$\frac{\partial\psi_t\left(x,0^-\right)}{\partial y}=0, \qquad x>0, \tag{4.3}$$

and at the partially transmissive surface are [154]

$$\frac{\partial\psi_t\left(x,0^+\right)}{\partial y}+\frac{ik}{\eta}\psi_t\left(x,0^+\right)=0, \qquad x<0, \tag{4.4}$$

$$\frac{\partial\psi_t\left(x,0^-\right)}{\partial y}-\frac{ik}{\eta}\psi_t\left(x,0^-\right)=0, \qquad x<0, \tag{4.5}$$

$$\psi_t\left(x,0^+\right)-\psi_t\left(x,0^-\right)=0, \qquad x<0. \tag{4.6}$$

In above relations η is the normal specific impedance of the material relative to the impedance of the surrounding medium, k is the wave number, and a time factor $e^{-i\omega t}$ is assumed and suppressed. The boundary conditions in $(4.4-4.6)$ represent the situation in which the pressure on both sides of the sheet is equal and producing the jump discontinuity in the normal component of the fluid velocity across it. These are the valid conditions from the mathematical view point and are acoustic counter part of an electrically resistive sheet in which ψ is then the tangential component of the electric field [154].

It is assumed that the wave number k has positive imaginary part. The lossless case can be obtained by making $\operatorname{Im} k \to 0$ in the final expressions. For the analysis purpose it is convenient to express the total field as follows [37, 38]

$$\psi_t(x,y) = \psi_0(x,y) + \psi(x,y), \tag{4.7}$$

where $\psi_0(x,y)$ is regarded as the unperturbed field that would exist if the whole plane $y = 0^+$ were a soft boundary. Hence the complementary part $\psi(x,y)$ represents the diffracted field. In Eq. (4.7), we have

$$\psi_0(x,y) = \begin{cases} \psi_i(x,y) + \psi_r(x,y) & \text{for } y > 0 \\ 0 & \text{for } y < 0, \end{cases} \tag{4.8}$$

where ψ_i is the incident field satisfying the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)\psi_i(x,y) = \delta(x - x_0)\,\delta(y - y_0), \tag{4.9}$$

and ψ_r is the corresponding reflected field. The scattered field $\psi(x,y)$ satisfies the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)\psi(x,y) = 0. \tag{4.10}$$

For analytic convenience, it is assumed that k has small imaginary part for which $k = k_r + ik_i$, where k_r and k_i are both positive. It is appropriate to define the following Fourier transform pair as follows

$$\overline{\psi}(\alpha,y) = \int_{-\infty}^{\infty} \psi(x,y)\, e^{i\alpha x} dx, \tag{4.11}$$

and

$$\psi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\psi}(\alpha,y) e^{-i\alpha x} d\alpha. \tag{4.12}$$

Using Eq. (4.11), Eq. (4.10) can be written as

$$\frac{d^2\overline{\psi}}{dy^2} + K^2\overline{\psi} = 0, \tag{4.13}$$

where $K(\alpha) = \sqrt{k^2 - \alpha^2}$. The square root function is defined in the complex $\alpha-plane$ cut such that $K(0) = k$. The solution of Eq. (4.13) satisfying the radiation conditions can be written as

$$\overline{\psi}(\alpha, y) = \begin{cases} A(\alpha)e^{iK(\alpha)y} & y > 0 \\ B(\alpha)e^{-iK(\alpha)y} & y < 0, \end{cases} \tag{4.14}$$

where $A(\alpha)$ and $B(\alpha)$ are the unknown coefficients to be determined.

Using Eq. (4.11), from Eq. (4.9) the incident field and the corresponding reflected field can be calculated by using the Green's function method as given in Chapter 3 and [140] as follows:

$$\overline{\psi}_i(\alpha, y) = \frac{1}{2iK}e^{i\alpha x_0 + iK(\alpha)|y-y_0|}, \tag{4.15}$$

and

$$\overline{\psi}_r(\alpha, y) = -\frac{1}{2iK}e^{i\alpha x_0 + iK(\alpha)|y+y_0|}. \tag{4.16}$$

Taking Fourier transform of the boundary conditions (4.2 – 4.6), will give

$$\overline{\psi}_+(\alpha, 0^+) = -\int_0^\infty \psi_0(x, 0^+)\, e^{i\alpha x}dx, \tag{4.17}$$

$$\frac{\partial\overline{\psi}_+(\alpha, 0^-)}{\partial y} = 0, \tag{4.18}$$

$$\frac{\partial\overline{\psi}_-(\alpha, 0^+)}{\partial y} + \frac{ik}{\eta}\overline{\psi}_-(\alpha, 0^+) = -\int_{-\infty}^0 \left[\frac{\partial\overline{\psi}_0(x, 0^+)}{\partial y} + \frac{ik}{\eta}\overline{\psi}_0(x, 0^+)\right] e^{i\alpha x}dx, \tag{4.19}$$

$$\frac{\partial\overline{\psi}_-(\alpha,0^-)}{\partial y}-\frac{ik}{\eta}\overline{\psi}_-(\alpha,0^-)=0, \tag{4.20}$$

$$\overline{\psi}_-(\alpha,0^+)-\overline{\psi}_-(\alpha,0^-)=-\int_{-\infty}^{0}\psi_0(x,0^+)\,e^{i\alpha x}dx. \tag{4.21}$$

In order to obtain the unique solution it is necessary to take into account the following edge conditions

$$\psi(x,0)=O\left(x^{\frac{1}{4}}\right)\qquad \text{as } x\to 0, \tag{4.22}$$

$$\frac{\partial\psi(x,0)}{\partial y}=O\left(x^{-\frac{3}{4}}\right)\qquad \text{as } x\to 0. \tag{4.23}$$

The substitution of Eq. (4.14) into boundary conditions $(4.17-4.21)$ will yield the following integral equations

$$A(\alpha)=\overline{\psi}_-(\alpha,0^+)-\int_{0}^{\infty}\psi_0(x,0^+)\,e^{i\alpha x}dx, \tag{4.24}$$

$$B(\alpha)=\frac{\overline{\psi}'_-(\alpha,0^-)}{K(\alpha)}, \tag{4.25}$$

$$A(\alpha)-B(\alpha)=\overline{\Lambda}_+(\alpha)-\int_{-\infty}^{0}\psi_0(x,0^+)\,e^{i\alpha x}dx, \tag{4.26}$$

$$\begin{aligned}\left[\frac{2k}{\eta}+K(\alpha)\right]A(\alpha)+K(\alpha)B(\alpha) &= \overline{\Lambda}'_+(\alpha)-\frac{2k}{\eta}\left[\int_{-\infty}^{\infty}\psi_0(x,0^+)\,e^{i\alpha x}dx\right]\\ &\quad -\frac{1}{i}\int_{-\infty}^{0}\psi'_0(x,0)\,e^{i\alpha x}dx,\end{aligned} \tag{4.27}$$

where prime denotes the differentiation with respect to y and $\overline{\psi}_-(\alpha,0^+)$, $\overline{\psi}'_-(\alpha,0^-)$, $\overline{\Lambda}_+(\alpha)$ and $\overline{\Lambda}'_+(\alpha)$ are defined by

$$\overline{\psi}_-(\alpha,0^+)=\int_{-\infty}^{0}\psi(x,0^+)\,e^{i\alpha x}dx, \tag{4.28}$$

$$\overline{\psi}'_{-}(\alpha, 0^{-}) = i \int_{-\infty}^{0} \frac{\partial \psi(x, 0^{-})}{\partial y} e^{i\alpha x} dx, \tag{4.29}$$

$$\overline{\Lambda}_{+}(\alpha) = \int_{0}^{\infty} \left[\psi(x, 0^{+}) - \psi(x, 0^{-}) \right] e^{i\alpha x} dx, \tag{4.30}$$

$$\overline{\Lambda}'_{+}(\alpha) = \int_{0}^{\infty} \left[\frac{\partial \psi(x, 0^{+})}{\partial y} - \frac{\partial \psi(x, 0^{-})}{\partial y} \right] e^{i\alpha x} dx. \tag{4.31}$$

Due to the analytic properties of the Fourier integrals [134] $\overline{\psi}_{-}(\alpha, 0^{+})$, $\overline{\psi}'_{-}(\alpha, 0^{-})$, $\overline{\Lambda}_{+}(\alpha)$ and $\overline{\Lambda}'_{+}(\alpha)$ are regular functions of α in the half planes $\mathrm{Im}\,\alpha < \mathrm{Im}\,k\cos\theta_0$ and $\mathrm{Im}\,\alpha > \mathrm{Im}\,(-k)$, respectively. By using the edge conditions (4.22 – 4.23) it can be easily shown that when $|\alpha| \to \infty$ in the respective regions of regularity one obtains:

$$\overline{\psi}_{-}(\alpha, 0^{+}) = O\left(\alpha^{-\frac{5}{4}}\right), \tag{4.32}$$

and

$$\overline{\psi}'_{-}(\alpha, 0^{-}) = O\left(\alpha^{-\frac{1}{4}}\right). \tag{4.33}$$

The elimination of $A(\alpha)$ and $B(\alpha)$ among Eqs. (4.24 – 4.27) leads to the following matrix W-H equation valid in the strip $\mathrm{Im}\,(-k) < \mathrm{Im}\,\alpha < \mathrm{Im}\,k\cos\theta_0$

$$\begin{bmatrix} 1 & -\frac{1}{K(\alpha)} \\ \frac{2k}{\eta} + K(\alpha) & 1 \end{bmatrix} \begin{bmatrix} \overline{\psi}_{-}(\alpha, 0^{+}) \\ \overline{\psi}'_{-}(\alpha, 0^{-}) \end{bmatrix} = \begin{bmatrix} \overline{\Lambda}_{+}(\alpha) \\ \overline{\Lambda}'_{+}(\alpha) \end{bmatrix} + \begin{bmatrix} q \\ r \end{bmatrix}, \tag{4.34}$$

as shown in Fig. 4.2

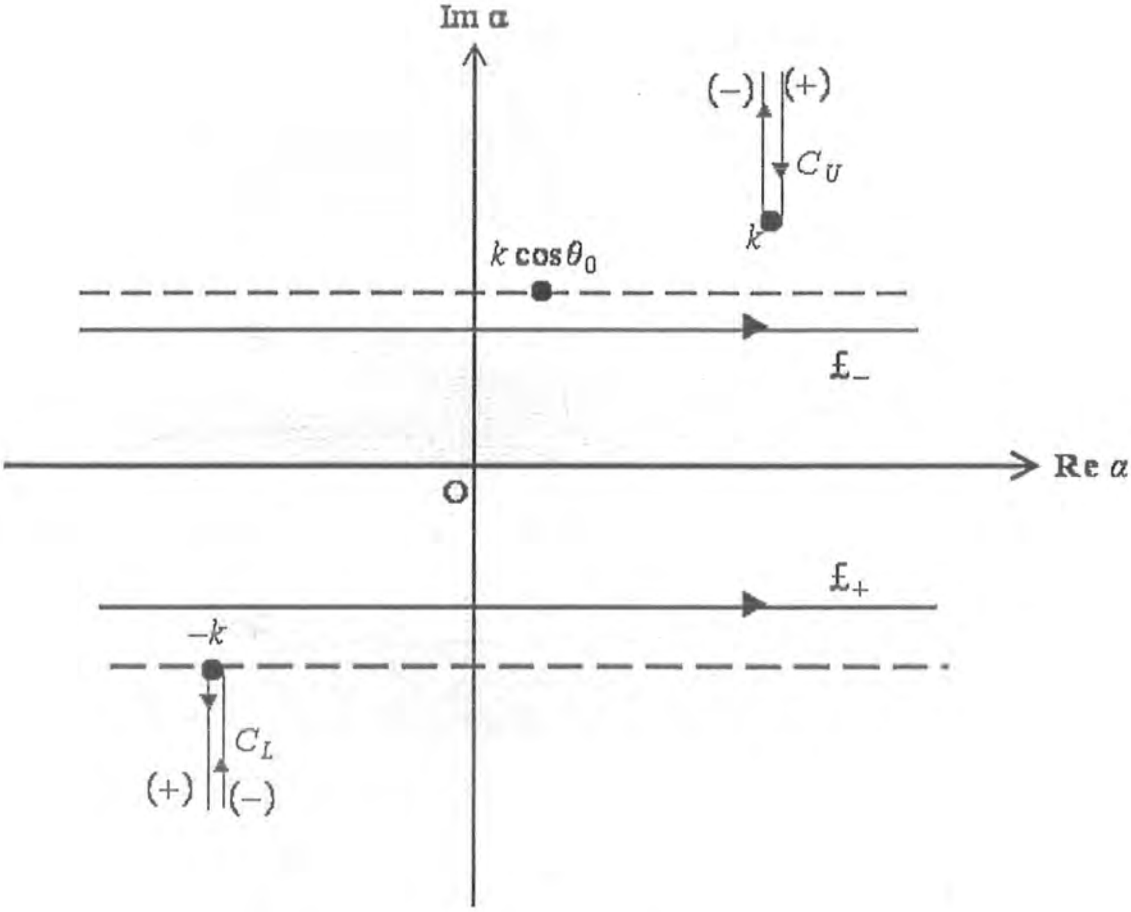

Fig. 4.2 Strip of analyticity for the problem

and the various quantities appearing in Eq. (4.34) are defined as

$$q = -\int_{-\infty}^{0} \psi_0\left(x, 0^+\right) e^{i\alpha x} dx + \int_{0}^{\infty} \psi_0\left(x, 0^+\right) e^{i\alpha x} dx, \tag{4.35}$$

$$r = -\frac{2k}{\eta}\left[\int_{-\infty}^{\infty} \psi_0\left(x, 0^+\right) e^{i\alpha x} dx\right] - \frac{1}{i}\int_{-\infty}^{0} \psi_0'\left(x, 0\right) e^{i\alpha x} dx$$

$$+\left(\frac{2k}{\eta} + K\left(\alpha\right)\right)\int_{-\infty}^{0} \psi_0\left(x, 0^+\right) e^{i\alpha x} dx, \tag{4.36}$$

with

$$\mathrm{M}(\alpha)=\begin{bmatrix} 1 & -\frac{1}{K(\alpha)} \\ \frac{2k}{\eta}+K(\alpha) & 1 \end{bmatrix}. \tag{4.37}$$

In order to obtain the explicit solution of Eq. (4.34), it is required to factorize the kernel matrix $\mathrm{M}(\alpha)$ as the product of two non-singular matrices say $\mathrm{M}_{+}(\alpha)$ and $\mathrm{M}_{-}(\alpha)$ whose entries are the regular functions of α in the upper and lower half planes, respectively. The kernel matrix $\mathbf{M}(\alpha)$ is factorized by [70] using the Daniele-Kharapkov methods [20, 21, 139]. Further details can be found in [70]. The Daniele-Kharapkov methods suggest pre-multiplication of the matrix given in Eq. (4.37) by the following constant matrix

$$\mathrm{C}=\begin{bmatrix} 1 & 0 \\ -\frac{2k}{\eta} & 1 \end{bmatrix}, \tag{4.38}$$

and then writing it in the form suitable for the application of Daniele-Kharapkov methods. Thus

$$\mathrm{W}(\alpha)=\mathrm{CM}(\alpha)=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+\frac{1}{K(\alpha)}\begin{bmatrix} 0 & -1 \\ (k^2-\alpha^2) & \frac{2k}{\eta} \end{bmatrix}=\begin{bmatrix} F_+ & G_+ \\ H_+ & J_+ \end{bmatrix}\begin{bmatrix} F_- & G_- \\ H_- & J_- \end{bmatrix}. \tag{4.39}$$

The matrix $\mathrm{W}(\alpha)$ is a special form which can be factorized through the Daniele-

Kharapkov methods. Omitting the details [70] the final expression for $\mathrm{W}_+(\alpha)$ is

$$\mathrm{W}_+(\alpha) = \left(\frac{2}{\eta}\right)^{\frac{1}{4}} \frac{1}{\sqrt{\kappa_+(\alpha)}} \begin{bmatrix} \cosh \varkappa(\alpha) - \frac{k}{\eta}\frac{\sinh \varkappa(\alpha)}{\sqrt{\alpha^2-\sigma_1^2}} & -\frac{\sinh \varkappa(\alpha)}{\sqrt{\alpha^2-\sigma_1^2}} \\ (k^2-\alpha^2)\frac{\sinh \varkappa(\alpha)}{\sqrt{\alpha^2-\sigma_1^2}} & \cosh \varkappa(\alpha) + \frac{k}{\eta}\frac{\sinh \varkappa(\alpha)}{\sqrt{\alpha^2-\sigma_1^2}} \end{bmatrix}$$

$$= \begin{bmatrix} F_+ & G_+ \\ H_+ & J_+ \end{bmatrix}, \tag{4.40}$$

so that

$$\mathrm{W}_-(\alpha) = \mathrm{W}_+(-\alpha), \tag{4.41}$$

$$\varkappa(\alpha) = \frac{1}{4}\ln\left\{\frac{\left(\sigma_1^2 + k\alpha - \sqrt{\alpha^2-\sigma_1^2}\sqrt{k^2-\sigma_1^2}\right)\left(\alpha+\sqrt{k^2-\alpha^2}\right)}{\sigma_1^2(k+\alpha)}\right\}, \tag{4.42}$$

and

$$\sigma_1 = k\sqrt{1-\frac{1}{\eta^2}}. \tag{4.43}$$

In Eq. (4.40), $\kappa_+(\alpha)$ and $\kappa_-(\alpha) = \kappa_+(-\alpha)$ are the split functions regular and free of zeros in the upper and lower half planes, respectively, resulting from the factorization of

$$\kappa(\alpha) = \frac{K(\alpha)}{k+\eta K(\alpha)}, \tag{4.44}$$

as

$$\kappa(\alpha) = \kappa_-(\alpha)\,\kappa_+(\alpha). \tag{4.45}$$

Noticing that $\kappa_+(\alpha)$ and $\kappa_-(\alpha)$ can be expressed in terms of Maliuzhinetz function [162] as follows:

$$\kappa_{-}(k\cos\theta) = 2^{\frac{3}{2}}\sqrt{\frac{2}{\eta}}\sin\frac{\theta}{2}\left\{\frac{M_\pi\left(\frac{3\pi}{2}-\theta-\varphi\right)M_\pi\left(\frac{\pi}{2}-\theta+\varphi\right)}{M_\pi^2\left(\frac{\pi}{2}\right)}\right\}$$

$$\times\left\{\left[1+\sqrt{2}\cos\left(\frac{\left(\frac{\pi}{2}-\theta+\varphi\right)}{2}\right)\right]\left[1+\sqrt{2}\cos\left(\frac{\left(\frac{3\pi}{2}-\theta-\varphi\right)}{2}\right)\right]\right\}^{-1}, \tag{4.46}$$

with

$$\sin\varphi = \frac{1}{\eta}, \tag{4.47}$$

and

$$M_\pi(z) = \exp\left\{-\frac{1}{8\pi}\int_0^z \frac{\pi\sin u - 2\sqrt{2}\sin\frac{u}{2} + 2u}{\cos u}du\right\}, \tag{4.48}$$

and as $|\alpha| \to \infty$ in the upper half plane, one obtains

$$\mathbf{W}_+(\alpha) \approx \frac{1}{\sqrt{2}}\left\{\frac{k-\sqrt{k^2-\sigma_1^2}}{\sigma_1^2}\right\}^{\frac{1}{4}}\begin{bmatrix}\alpha^{\frac{1}{4}} & \alpha^{-\frac{3}{4}} \\ \alpha^{\frac{5}{4}} & \alpha^{\frac{1}{4}}\end{bmatrix}. \tag{4.49}$$

With this factorization of the kernel matrix, Eq. (4.34) can be rearranged as

$$\mathbf{W}_+(\alpha)\,\mathbf{W}_-(\alpha)\begin{bmatrix}\overline{\psi}_-(\alpha,0^+) \\ \overline{\psi}'_-(\alpha,0^-)\end{bmatrix} = \mathbf{C}\begin{bmatrix}\overline{\Lambda}_+(\alpha) \\ \overline{\Lambda}'_+(\alpha)\end{bmatrix} + \mathbf{C}\begin{bmatrix}q \\ r\end{bmatrix}, \tag{4.50}$$

or

$$\mathbf{W}_-(\alpha)\begin{bmatrix}\overline{\psi}_-(\alpha,0^+) \\ \overline{\psi}'_-(\alpha,0^-)\end{bmatrix} = \mathbf{W}_+^{-1}(\alpha)\,\mathbf{C}\begin{bmatrix}\overline{\Lambda}_+(\alpha) \\ \overline{\Lambda}'_+(\alpha)\end{bmatrix} + \mathbf{W}_+^{-1}(\alpha)\,\mathbf{C}\begin{bmatrix}q \\ r\end{bmatrix}. \tag{4.51}$$

Eq. (4.51) is the matrix Wiener-Hopf equation. To make it regular in the upper and lower half planes one has to split the term

$$\mathbf{W}_+^{-1}(\alpha)\,\mathbf{C}\begin{bmatrix}q \\ r\end{bmatrix}. \tag{4.52}$$

This can be achieved by using the additive decomposition theorem [14]. This term can be decomposed as follows

$$\mathbf{W}_+^{-1}(\alpha)\,\mathbf{C}\begin{bmatrix} q \\ r \end{bmatrix} = \begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} T_+ + T_- \\ S_+ + S_- \end{bmatrix}. \tag{4.53}$$

Using Eqs. (4.38) and (4.40) will give

$$\mathbf{W}_+^{-1}(\alpha)\,\mathbf{C}\begin{bmatrix} q \\ r \end{bmatrix} = \begin{bmatrix} J_+ q - G_+\left(-\frac{2k}{\eta} + r\right) \\ -H_+ q + F_+\left(-\frac{2k}{\eta} + r\right) \end{bmatrix} = \begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} T_+ + T_- \\ S_+ + S_- \end{bmatrix}, \tag{4.54}$$

where

$$\begin{aligned} T_\pm(\alpha) &= \pm\frac{1}{2\pi i}\int\limits_{-\infty+ic}^{\infty+ic}\frac{T(\xi)}{(\xi-\alpha)}d\xi \\ &= \pm\frac{1}{2\pi i}\int\limits_{-\infty+ic}^{\infty+ic}\left[\frac{J_+(\xi)\,q(\xi) - G_+(\xi)\left(-\frac{2k}{\eta}+r(\xi)\right)}{(\xi-\alpha)}\right]d\xi, \end{aligned} \tag{4.55}$$

$$\begin{aligned} S_\pm(\alpha) &= \pm\frac{1}{2\pi i}\int\limits_{-\infty+id}^{\infty+id}\frac{S(\xi)}{(\xi-\alpha)}d\xi \\ &= \pm\frac{1}{2\pi i}\int\limits_{-\infty+id}^{\infty+id}\left[\frac{-H_+(\xi)\,q(\xi) + F_+(\xi)\left(-\frac{2k}{\eta}+r(\xi)\right)}{(\xi-\alpha)}\right]d\xi. \end{aligned} \tag{4.56}$$

Using Eqs. (4.35), (4.36) and (4.40) in Eqs. (4.55) and (4.56), the explicit expressions for $T_-(\alpha)$ and $S_-(\alpha)$ are given as follows

$$T_-(\alpha) = \frac{1}{2\pi i}\int\limits_{-\infty+ia}^{\infty+ia}\left[\frac{\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sqrt{\kappa_+(\xi)}\exp\left(i\xi x_0 + iK(\xi)\,|y_0|\right)\frac{\sinh\varkappa(\xi)}{\sqrt{\xi^2-\sigma_1^2}}}{(\xi-\alpha)}\right]d\xi \tag{4.57}$$

and

$$S_-(\alpha) = \frac{1}{2\pi i}\int\limits_{-\infty+ia}^{\infty+ia}\left[\frac{\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sqrt{\kappa_+(\xi)}\exp\left(i\xi x_0 + iK(\xi)\,|y_0|\right)\left\{\cosh\varkappa(\xi) - \frac{k}{\eta}\frac{\sinh\varkappa(\xi)}{\sqrt{\xi^2-\sigma_1^2}}\right\}}{(\xi-\alpha)}\right]d\xi. \tag{4.58}$$

Using Eq. (4.53) in Eq. (4.51) and separating into positive and negative portions, one arrives at

$$\mathbf{W}_-(\alpha)\begin{bmatrix}\overline{\psi}_-(\alpha,0^+)\\ \overline{\psi}'_-(\alpha,0^-)\end{bmatrix}-\begin{bmatrix}T_-\\ S_-\end{bmatrix}=\mathbf{W}_+^{-1}(\alpha)\,\mathbf{C}\begin{bmatrix}\overline{\Lambda}_+(\alpha)\\ \overline{\Lambda}'_+(\alpha)\end{bmatrix}+\begin{bmatrix}T_+\\ S_+\end{bmatrix}. \quad (4.59)$$

The left hand side of Eq. (4.59) is regular in the lower half plane $\operatorname{Im}\alpha < \operatorname{Im} k\cos\theta_0$ and the right hand side is regular in the upper half plane $\operatorname{Im}\alpha > \operatorname{Im}(-k)$. Hence by analytic continuation principle both sides define an entire matrix-valued function $\mathbf{P}(\alpha)$. To find the exact value of $\mathbf{P}(\alpha)$, the order relations in Eqs. (4.32), (4.33), (4.41) and (4.49) can be taken into account which help to conclude from the extended Liouville's theorem that the $\mathbf{P}(\alpha)$ is a constant matrix of the form

$$\mathbf{P}(\alpha)=\begin{bmatrix}0\\ p^*\end{bmatrix}, \quad (4.60)$$

where p^* can be evaluated as follows.

From Eq. (4.59)

$$\begin{bmatrix}\overline{\psi}_-(\alpha,0^+)\\ \overline{\psi}'_-(\alpha,0^-)\end{bmatrix}=\begin{bmatrix}J_- & -G_-\\ -H_- & F_-\end{bmatrix}\begin{bmatrix}T_-\\ S_-+p^*\end{bmatrix}. \quad (4.61)$$

The above equation can further be simplified to get

$$\begin{bmatrix}\overline{\psi}_-(\alpha,0^+)\\ \overline{\psi}'_-(\alpha,0^-)\end{bmatrix}=\begin{bmatrix}J_-T_- - G_-(S_-+p^*)\\ -H_-T_- + F_-(S_-+p^*)\end{bmatrix}. \quad (4.62)$$

The unknown constant p^* can be specified by taking into account the order relations in Eqs. (4.32) and (4.33). By using (4.41) one obtains

$$\begin{bmatrix} \overline{\psi}_-(\alpha,0^+) \\ \overline{\psi}'_-(\alpha,0^-) \end{bmatrix} \approx \sqrt{2}\left\{\frac{k-\sqrt{k^2-\sigma_1^2}}{\sigma_1^2}\right\}^{-\frac{1}{4}} \left[p^* - \widetilde{T}_-\right] \begin{bmatrix} (-\alpha)^{-\frac{3}{4}} \\ (-\alpha)^{-\frac{1}{4}} \end{bmatrix} + O\begin{bmatrix} (-\alpha)^{-\frac{5}{4}} \\ (-\alpha)^{-\frac{1}{4}} \end{bmatrix}, \tag{4.63}$$

with

$$\widetilde{T}_- = \lim_{\alpha\to\infty} \alpha T_-. \tag{4.64}$$

The correct behaviors of $\overline{\psi}_-(\alpha,0^+)$ and $\overline{\psi}'_-(\alpha,0^-)$ are recovered if

$$p^* = \widetilde{T}_-. \tag{4.65}$$

Hence the explicit expressions for $\overline{\psi}_-(\alpha,0^+)$ and $\overline{\psi}'_-(\alpha,0^-)$ are given as

$$\begin{aligned}\overline{\psi}_-(\alpha,0^+) &= \left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sqrt{\kappa_-(\alpha)}\left[\left\{\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{1}{\xi-\alpha}\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sqrt{\kappa_+(\xi)}\exp i\xi x_0 + iK(\xi)\left|y_0\right|\right.\right.\\ &\times\left(\frac{\sinh\varkappa(\xi)}{\sqrt{\xi^2-\sigma_1^2}}\right)d\xi\Bigg\}\left\{\cosh\varkappa(-\alpha)+\frac{k}{\eta}\frac{\sinh\varkappa(-\alpha)}{\sqrt{\alpha^2-\sigma_1^2}}\right\}+\left\{\frac{\sinh\varkappa(-\alpha)}{\sqrt{\alpha^2-\sigma_1^2}}\right\}\\ &\times\left\{\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\left(\frac{1}{\xi-\alpha}\right)\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sqrt{\kappa_+(\xi)}\exp\left(i\xi x_0+iK(\xi)\left|y_0\right|\right)\right.\\ &\times\left.\left.\left(\cosh\varkappa(\xi)-\frac{k}{\eta}\frac{\sinh\varkappa(\xi)}{\sqrt{\xi^2-\sigma_1^2}}\right)d\xi+p^*\right\}\right],\end{aligned} \tag{4.66}$$

and

$$\overline{\psi}'_{-}(\alpha,0^{-})=\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sqrt{\kappa_{-}(\alpha)}\left[\left\{\frac{1}{2\pi i}\int\limits_{-\infty+ia}^{\infty+ia}\frac{1}{\xi-\alpha}\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sqrt{\kappa_{+}(\xi)}\exp i\xi x_0+iK(\xi)|y_0|\right.\right.$$
$$\times\left(\frac{\sinh\varkappa(\xi)}{\sqrt{\xi^2-\sigma_1^2}}\right)d\xi\bigg\}\left\{-\left(k^2-\alpha^2\right)\frac{\sinh\varkappa(-\alpha)}{\sqrt{\alpha^2-\sigma_1^2}}\right\}+\left\{\cosh\varkappa(-\alpha)-\frac{k}{\eta}\frac{\sinh\varkappa(-\alpha)}{\sqrt{\alpha^2-\sigma_1^2}}\right\}$$
$$\times\left\{\frac{1}{2\pi i}\int\limits_{-\infty+ia}^{\infty+ia}\left(\frac{1}{\xi-\alpha}\right)\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sqrt{\kappa_{+}(\xi)}\exp\left(i\xi x_0+iK(\xi)|y_0|\right)\right.$$
$$\left.\left.\times\left(\cosh\varkappa(\xi)-\frac{k}{\eta}\frac{\sinh\varkappa(\xi)}{\sqrt{\xi^2-\sigma_1^2}}\right)d\xi+p^*\right\}\right]. \tag{4.67}$$

4.3 Far field solution

Now by substituting Eqs. (4.66) and (4.67) into Eqs. (4.24) and (4.25) and then resulting equations in Eq. (4.14) and taking the inverse Fourier transform will obtain the scattered far field for $y>0$ as,

$$\psi(x,y)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sqrt{\kappa_{-}(\alpha)}\left[\left\{\frac{1}{2\pi i}\int\limits_{-\infty+ia}^{\infty+ia}\frac{1}{\xi-\alpha}\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sqrt{\kappa_{+}(\xi)}\exp i\xi x_0+iK(\xi)|y_0|\right.\right.$$
$$\times\left(\frac{\sinh\varkappa(\xi)}{\sqrt{\xi^2-\sigma_1^2}}\right)d\xi\bigg\}\left\{\cosh\varkappa(-\alpha)+\frac{k}{\eta}\frac{\sinh\varkappa(-\alpha)}{\sqrt{\alpha^2-\sigma_1^2}}\right\}+\left\{\frac{\sinh\varkappa(-\alpha)}{\sqrt{\alpha^2-\sigma_1^2}}\right\}$$
$$\times\left\{\frac{1}{2\pi i}\int\limits_{-\infty+ia}^{\infty+ia}\left(\frac{1}{\xi-\alpha}\right)\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sqrt{\kappa_{+}(\xi)}\exp\left(i\xi x_0+iK(\xi)|y_0|\right)\right.$$
$$\left.\left.\times\left(\cosh\varkappa(\xi)-\frac{k}{\eta}\frac{\sinh\varkappa(\xi)}{\sqrt{\xi^2-\sigma_1^2}}\right)d\xi+p^*\right\}\right]\times e^{iK(\alpha)y-i\alpha x}d\alpha, \tag{4.68}$$

and for $y < 0$

$$\psi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sqrt{\kappa_-(\alpha)}\left[\left\{\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{1}{\xi-\alpha}\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sqrt{\kappa_+(\xi)}\exp i\xi x_0 + iK(\xi)|y_0|\right.\right.$$

$$\left.\times\left(\frac{\sinh\varkappa(\xi)}{\sqrt{\xi^2-\sigma_1^2}}\right)d\xi\right\}\left\{-\left(k^2-\alpha^2\right)\frac{\sinh\varkappa(-\alpha)}{\sqrt{\alpha^2-\sigma_1^2}}\right\}+\left\{\cosh\varkappa(-\alpha)-\frac{k}{\eta}\frac{\sinh\varkappa(-\alpha)}{\sqrt{\alpha^2-\sigma_1^2}}\right\}$$

$$\times\left\{\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\left(\frac{1}{\xi-\alpha}\right)\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sqrt{\kappa_+(\xi)}\exp\left(i\xi x_0+iK(\xi)|y_0|\right)\right.$$

$$\left.\left.\times\left(\cosh\varkappa(\xi)-\frac{k}{\eta}\frac{\sinh\varkappa(\xi)}{\sqrt{\xi^2-\sigma_1^2}}\right)d\xi+p^*\right\}\right]e^{-iK(\alpha)y-i\alpha x}d\alpha. \tag{4.69}$$

To determine the far field behavior of the scattered field the following substitutions

$$x=\rho\cos\theta, \qquad y=\rho\sin\theta \qquad (0<\theta<\pi), \tag{4.70}$$

$$x_0=\rho_0\cos\theta_0, \quad y_0=\rho_0\sin\theta_0. \qquad (\pi<\theta_0<0), \tag{4.71}$$

and the transformation

$$\alpha=-k\cos(\theta+i\zeta), \tag{4.72}$$

are introduced into Eqs. (4.68, 4.69). The explicit expression for the constant p^* is determined from Eqs. (4.57), (4.64) and (4.65) which give it as:

$$p^*=\frac{1}{2\pi i}\sqrt{\frac{2\pi}{k\rho_0}}\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\frac{k}{\pi}\sin\theta_0\frac{\sqrt{\kappa_+(k\cos\theta_0)}}{\sqrt{k^2\cos^2\theta-\sigma_1^2}}\sinh\varkappa(k\cos\theta_0)\exp\left(ik\rho_0+i\frac{\pi}{4}\right), \tag{4.73}$$

where t_1, given in Eq. (4.72) is real. The contour of integration over α in Eqs. (4.68) and (4.69) goes into the branch of hyperbola around $-ik$ if $\frac{\pi}{2}<\theta<\pi$. It is further observed that in deforming the contour into a hyperbola the pole $\alpha=\xi$ may be crossed. If another transformation $\xi=k\cos(\theta_0+it_1)$ is also introduced, the contour over ξ also goes into a hyperbola. The two hyperbolae will not cross each other if $\theta < \theta_0$. However, if the inequality is reversed there will be a contribution from the pole which, in fact, cancels the incident wave in the shadow region. Omitting the details of calculations, using the method of steepest descent, where the path of steepest descent is shown in Fig. 4.3,

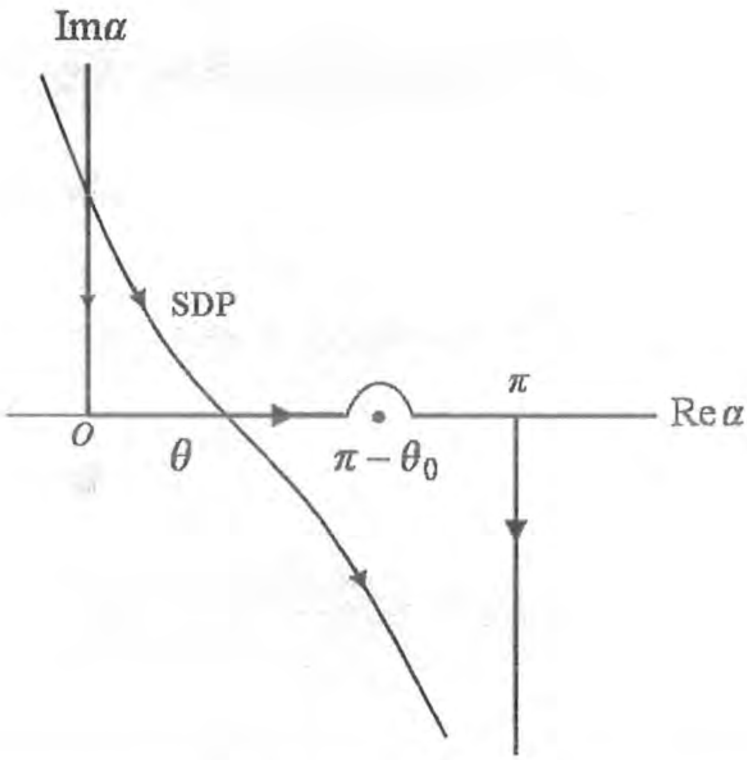

Fig. 4.3 The steepest descent path

the field due to a line source at a large distance from the edge is given for both cases $y > 0$ and $y < 0$, respectively.

For $y > 0$,

$$
\begin{aligned}
&\psi(\rho,\theta) \approx \frac{e^{\frac{-i\pi}{2}}}{2\pi}\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sqrt{\kappa_+(k\cos\theta)}\frac{\exp(ik\rho+ik\rho_0)}{\sqrt{\rho\rho_0}} \\
&\times\left[\left\{\cosh\varkappa(k\cos\theta)+\frac{k}{\eta}\frac{\sinh\varkappa(k\cos\theta)}{\sqrt{k^2\cos^2\theta-\sigma_1^2}}\right\}\left\{i\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sin\frac{\theta}{2}\sin\frac{\theta_0}{2}\right.\right. \\
&\left.\frac{\sqrt{\kappa_+(k\cos\theta_0)}}{\sqrt{k^2\cos^2\theta_0-\sigma_1^2}}\sinh\varkappa(k\cos\theta_0)\right\}+\frac{\sinh\varkappa(k\cos\theta)}{\sqrt{k^2\cos^2\theta-\sigma_1^2}} \\
&\times\left\{\left(i\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sin\frac{\theta}{2}\sin\frac{\theta_0}{2}\sqrt{\kappa_+(k\cos\theta_0)}\right)\right. \\
&\left.\left.\times\left(\cosh\varkappa(k\cos\theta_0)-\frac{k}{\eta}\frac{\sinh\varkappa(k\cos\theta_0)}{\sqrt{k^2\cos^2\theta_0-\sigma_1^2}}\right)+\sin\frac{\theta}{2}\sin\frac{\theta_0}{2}(\cos\theta_0+\cos\theta_0)\,p\right\}\right] \\
&\times\left[\mathcal{F}\left(\sqrt{2k\rho}\cos\frac{\theta-\theta_0}{2}\right)+\mathcal{F}\left(\sqrt{2k\rho}\cos\frac{\theta+\theta_0}{2}\right)\right]. \qquad (4.74)
\end{aligned}
$$

For $y < 0$, the far field is given as follows

$$
\begin{aligned}
&\psi(\rho,\theta) \approx \frac{e^{\frac{-i\pi}{2}}}{2\pi}\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sqrt{\kappa_+(k\cos\theta)}\frac{\exp(ik\rho+ik\rho_0)}{k\sqrt{\rho\rho_0}} \\
&\times\left[\left\{-k^2\sin^2\theta\frac{\sinh\varkappa(k\cos\theta)}{\sqrt{k^2\cos^2\theta-\sigma_1^2}}\right\}\times\left\{i\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\frac{\sin\frac{\theta_0}{2}}{2\cos\frac{\theta}{2}}\right.\right. \\
&\left.\times\frac{\sqrt{\kappa_+(k\cos\theta_0)}}{\sqrt{k^2\cos^2\theta_0-\sigma_1^2}}\sinh\varkappa(k\cos\theta_0)\right\}+\left\{\cosh\varkappa(k\cos\theta)-\frac{k}{\eta}\frac{\sinh\varkappa(k\cos\theta)}{\sqrt{k^2\cos^2\theta-\sigma_1^2}}\right\} \\
&\left\{\left(i\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\frac{\sin\frac{\theta_0}{2}}{2\cos\frac{\theta}{2}}\sqrt{\kappa_+(k\cos\theta_0)}\right)\times\right. \\
&\left.\left.\times\left(\cosh\varkappa(k\cos\theta_0)-\frac{k}{\eta}\frac{\sinh\varkappa(k\cos\theta_0)}{\sqrt{k^2\cos^2\theta_0-\sigma_1^2}}\right)+\frac{\sin\frac{\theta_0}{2}}{2\cos\frac{\theta}{2}}(\cos\theta_0+\cos\theta_0)\,p\right\}\right] \\
&\times\left[\mathcal{F}\left(\sqrt{2k\rho}\cos\frac{\theta-\theta_0}{2}\right)+\mathcal{F}\left(\sqrt{2k\rho}\cos\frac{\theta+\theta_0}{2}\right)\right]. \qquad (4.75)
\end{aligned}
$$

It is also worthwhile to mention that the unknown constant p^* goes into p as determined by [70] at this stage and $\mathcal{F}(z)$ stands for the Fresnel function as defined in [37, 38].

$$\mathcal{F}(z) = e^{-iz^2} \int_{z}^{\infty} e^{it^2} dt. \tag{4.76}$$

4.4 Computational results

In this section, some graphical results showing the effects of parameters resistivity η and the distance of the line source ρ_0 on the scattering phenomenon will be presented.

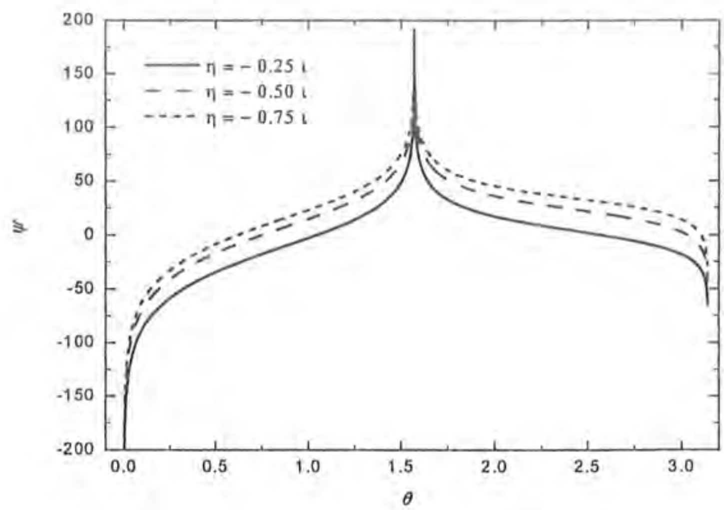

Fig. 4.4 Amplitude of the diffracted field ψ Vs θ for $\rho = 1$, $k = 1$, $\theta_0 = \frac{\pi}{2}$ and $\rho_0 = 0.001$.

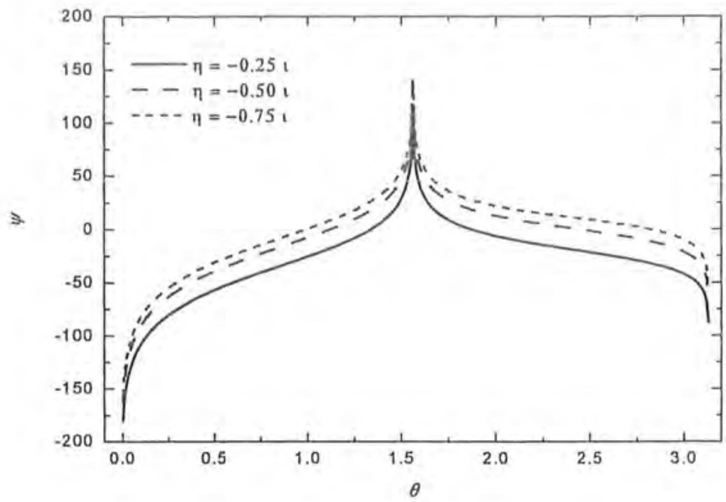

Fig. 4.5 Amplitude of the diffracted field ψ Vs θ for $\rho = 1$, $k = 1$, $\theta_0 = \frac{\pi}{2}$ and $\rho_0 = 0.01$.

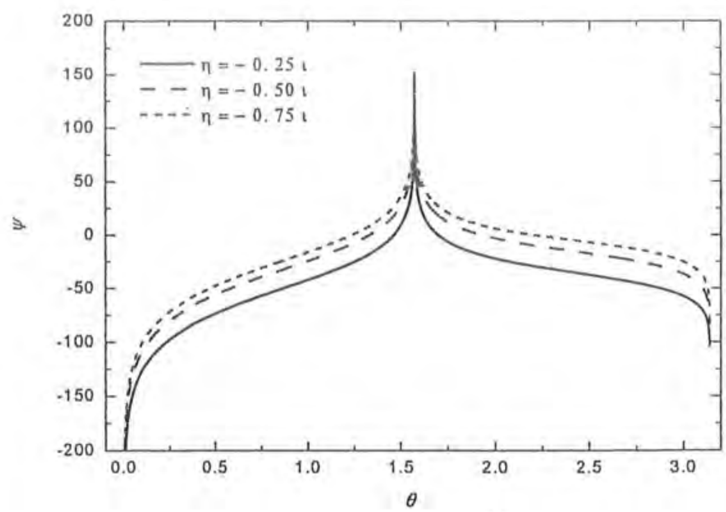

Fig. 4.6 Amplitude of the diffracted field ψ Vs θ for $\rho = 1$, $k = 1$, $\theta_0 = \frac{\pi}{2}$ and $\rho_0 = 0.05$.

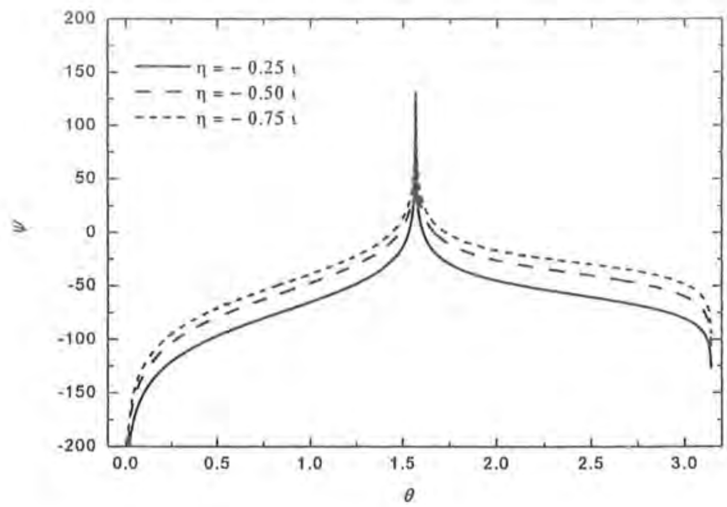

Fig. 4.7 Amplitude of the diffracted field ψ Vs θ for $\rho = 1$, $k = 1$, $\theta_0 = \frac{\pi}{2}$ and $\rho_0 = 0.5$.

- Figures (4.4 − 4.7) are plotted to note the effect of parameter η, when it is imaginary, on the amplitude of the diffracted field ψ (for $y > 0$) plotted against the observation angle θ by fixing the parameters to be $\rho_0 = 0.001$ (Fig. 4.4), $\rho_0 = 0.01$ (Fig. 4.5), $\rho_0 = 0.05$ (Fig. 4.6), $\rho_0 = 0.5$ (Fig. 4.7) and the other parameters $\theta_0 = \pi/2$, $\rho = 1$ and $k = 1$. It is observed that by decreasing the parameter η and fixing all the other parameters the amplitude of the diffracted field increases for the case $y > 0$.

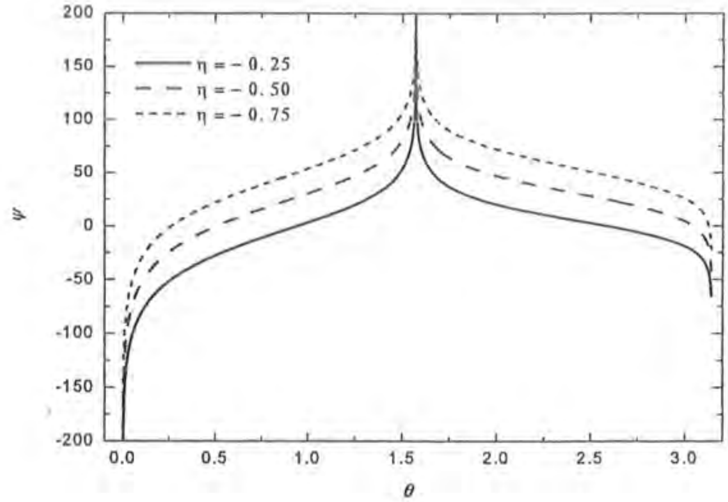

Fig. 4.8 Amplitude of the diffracted field ψ Vs θ for $\rho = 1$, $k = 1$, $\theta_0 = \frac{\pi}{2}$ and $\rho_0 = 0.001$

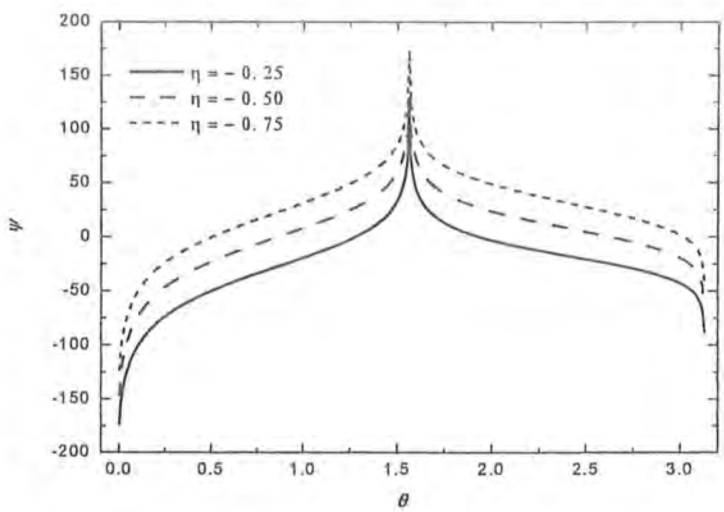

Fig. 4.9 Amplitude of the diffracted field ψ Vs θ for $\rho = 1$, $k = 1$, $\theta_0 = \frac{\pi}{2}$ and $\rho_0 = 0.01$.

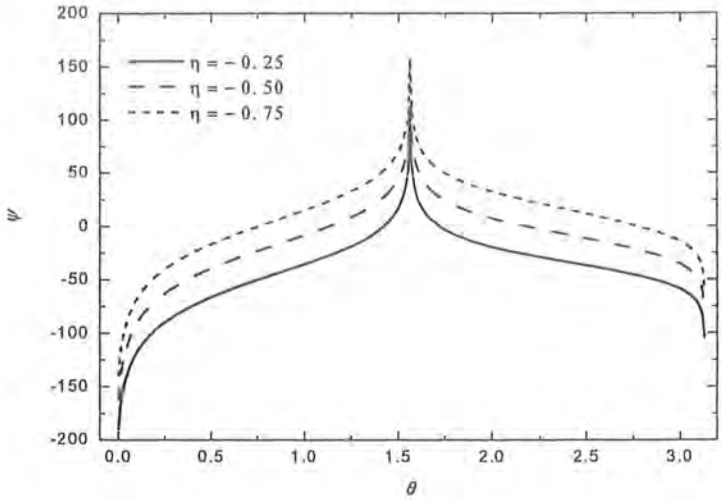

Fig. 4.10 Amplitude of the diffracted field ψ Vs θ for $\rho = 1$, $k = 1$, $\theta_0 = \frac{\pi}{2}$ and $\rho_0 = 0.05$

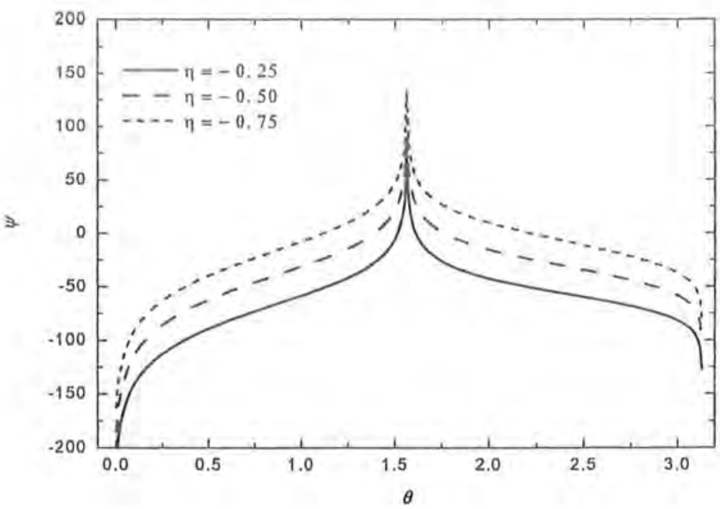

Fig. 4.11 Amplitude of the diffracted field ψ Vs θ for $\rho = 1$, $k = 1$, $\theta_0 = \frac{\pi}{2}$ and $\rho_0 = 0.5$.

- Figures $(4.8 - 4.11)$ depict the effect of parameter η, when it is real, on the amplitude of the diffracted field ψ (for $y > 0$) plotted against the observation angle θ by fixing the parameters to be $\rho_0 = 0.001$ (Fig. 4.8), $\rho_0 = 0.01$ (Fig. 4.9), $\rho_0 = 0.05$ (Fig. 4.10), $\rho_0 = 0.5$ (Fig. 4.11) and the other parameters $\theta_0 = \pi/2$, $\rho = 1$ and $k = 1$. It is noted that by decreasing the parameter η and fixing the all other parameters the amplitude of the diffracted field increases for the case $y > 0$.

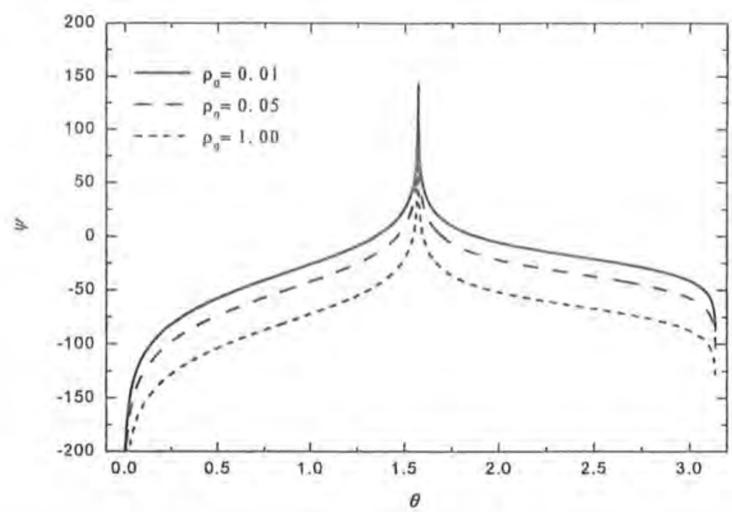

Fig. 4.12 Amplitude of the diffracted field ψ Vs θ for $\rho = 1$, $k = 1$, $\theta_0 = \frac{\pi}{2}$ and $\eta = -0.2$

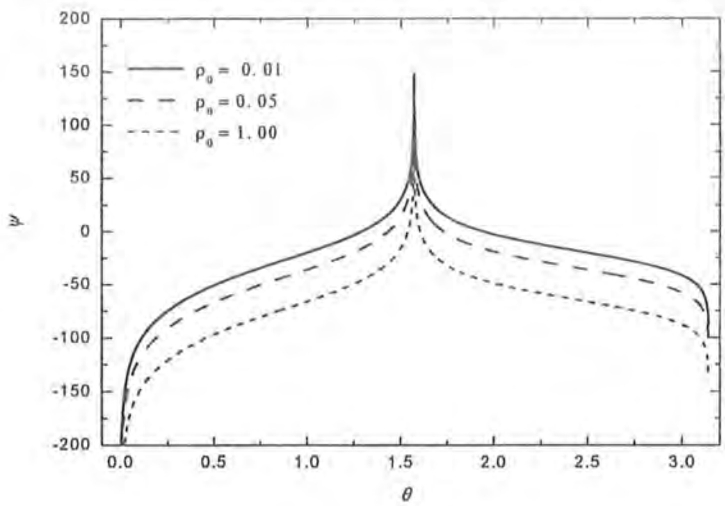

Fig. 4.13 Amplitude of the diffracted field ψ Vs θ for $\rho = 1$, $k = 1$, $\theta_0 = \frac{\pi}{2}$ and $\eta = -0.2$

- Figures (4.12) and (4.13) are plotted to study the effect of parameter ρ_0, on the amplitude of the diffracted field ψ (for $y > 0$) plotted against the observation angle θ by fixing the parameters to be $\eta = -0.25i$ (Fig. 4.12) and $\eta = -0.25$ (Fig. 4.13), and the other parameters are $\theta_0 = \pi/2$, $\rho = 1$ and $k = 1$. It is seen that by increasing the parameter ρ_0 and fixing all the other parameters the amplitude of the diffracted field decreases for the case $y > 0$.

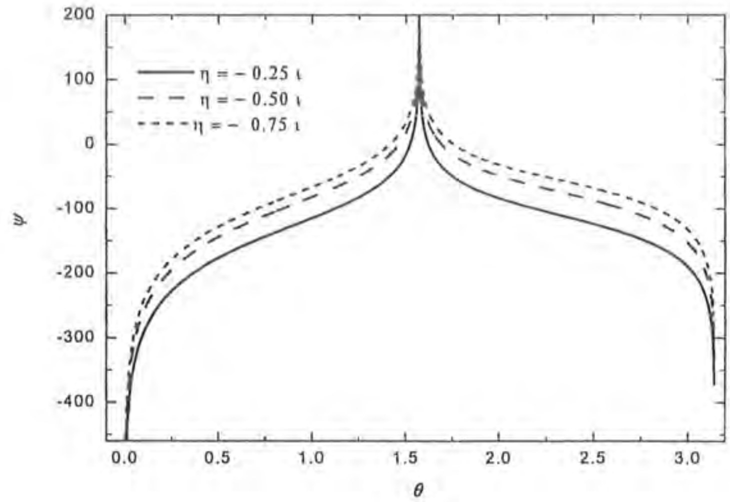

Fig. 4.14 Amplitude of the diffracted field ψ Vs θ for $\rho = 1$, $k = 1$, $\theta_0 = \frac{\pi}{2}$ and $\rho_0 = 0.00$

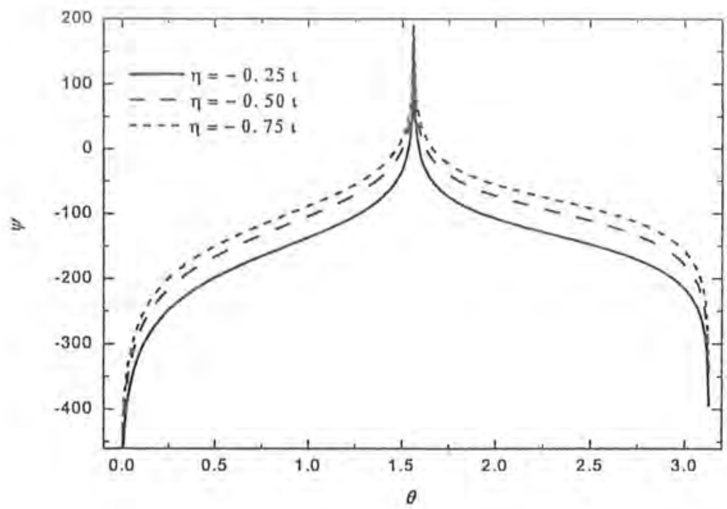

Fig. 4.15 Amplitude of the diffracted field ψ Vs θ for $\rho = 1$, $k = 1$, $\theta_0 = \frac{\pi}{2}$ and $\rho_0 = 0.01$

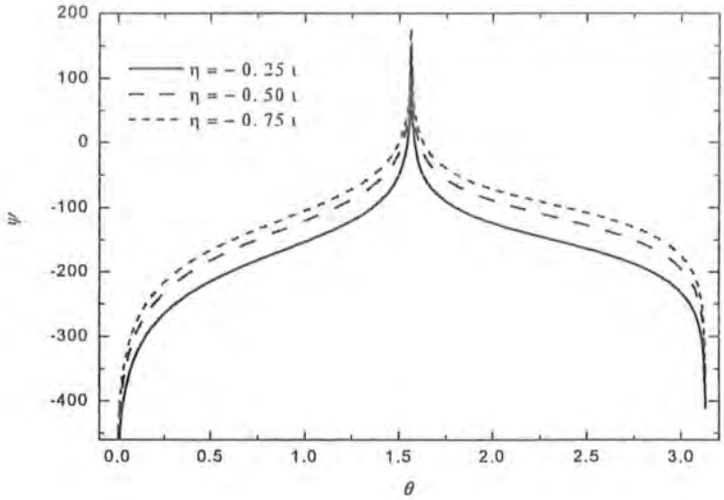

Fig. 4.16 Amplitude of the diffracted field ψ Vs θ for $\rho = 1$, $k = 1$, $\theta_0 = \frac{\pi}{2}$ and $\rho_0 = 0.05$

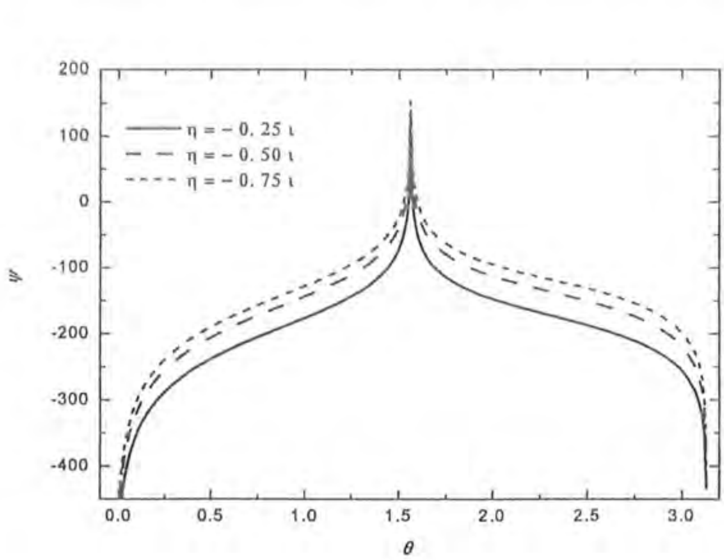

Fig. 4.17 Amplitude of the diffracted field ψ Vs θ for $\rho = 1$, $k = 1$, $\theta_0 = \frac{\pi}{2}$ and $\rho_0 = 0.5$.

- Figures (4.14 − 4.17) are sketched to note the effect of parameter η, when it is imaginary, on the amplitude of the diffracted field ψ (for $y < 0$) plotted against the observation angle θ by fixing the parameters to be $\rho_0 = 0.001$ (Fig. 4.14), $\rho_0 = 0.01$ (Fig. 4.15), $\rho_0 = 0.05$ (Fig. 4.16), $\rho_0 = 0.5$ (Fig. 4.17) and the other parameters $\theta_0 = \pi/2$, $\rho = 1$ and $k = 1$. It is observed that by decreasing the parameter η and fixing the all other parameters the amplitude of the diffracted field increases for the case $y < 0$.

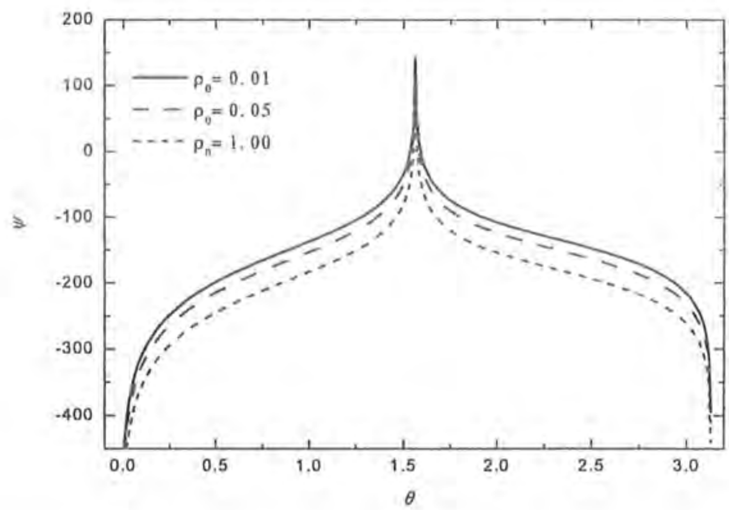

Fig. 4.18 Amplitude of the diffracted field ψ Vs θ for $\rho = 1$, $k = 1$, $\theta_0 = \frac{\pi}{2}$ and $\eta = -0.2$

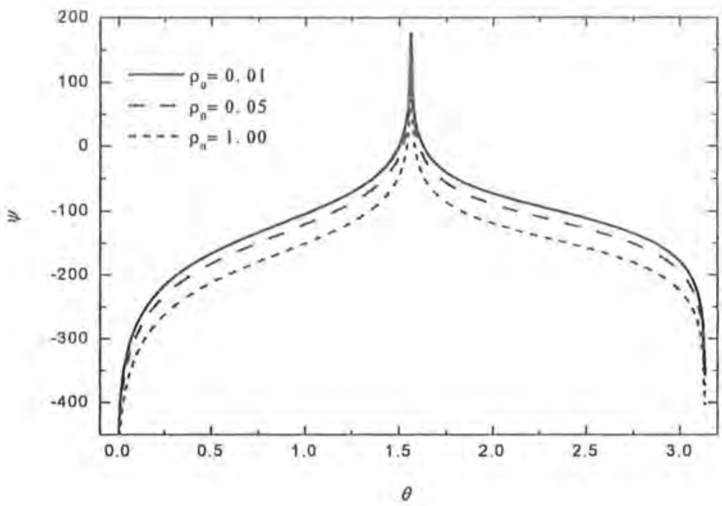

Fig. 4.19 Amplitude of the diffracted field ψ Vs θ for $\rho = 1$, $k = 1$, $\theta_0 = \frac{\pi}{2}$ and $\eta = -0.5i$

- Figures (4.18) and (4.19) are plotted to study the effect of parameter ρ_0, on the amplitude of the diffracted field ψ (for $y < 0$) plotted against the observation angle θ by fixing the parameters to be $\eta = -0.25i$ (Fig. 4.18) and $\eta = -0.25$ (Fig. 4.19), and the other parameters are $\theta_0 = \pi/2$, $\rho = 1$ and $k = 1$. It is seen that by increasing the parameter ρ_0 and fixing all the other parameters the amplitude of the diffracted field decreases for the case $y < 0$.

4.5 The point source scattering problem

For the case of point source scattering, suppose that a point source is occupying the position (x_0, y_0, z_0). Thus the appropriate equation representing the point source incidence is

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+k^2\right)\Phi_t\left(x,y,z\right)=\delta\left(x-x_0\right)\delta\left(y-y_0\right)\delta\left(z-z_0\right), \tag{4.77}$$

subject to the following boundary conditions, for $x>0$

$$\Phi_t\left(x,0^+,z\right)=0, \tag{4.78}$$

$$\frac{\partial\Phi_t\left(x,0^-,z\right)}{\partial y}=0, \tag{4.79}$$

and for $x<0$

$$\frac{\partial\Phi_t\left(x,0^+,z\right)}{\partial y}+\frac{ik}{\eta}\Phi_t\left(x,0^+,z\right)=0, \tag{4.80}$$

$$\frac{\partial\Phi_t\left(x,0^-,z\right)}{\partial y}-\frac{ik}{\eta}\Phi_t\left(x,0^-,z\right)=0, \tag{4.81}$$

$$\Phi_t\left(x,0^+,z\right)-\Phi_t\left(x,0^-,z\right)=0, \tag{4.82}$$

where Φ_t is the total acoustic field, defined as

$$\Phi_t(x,y,z)=\Phi_0(x,y,z)+\Phi(x,y,z), \tag{4.83}$$

where Φ is the scattered field and Φ_0 represents the effect due to a point source.

Let us define the Fourier transform and the inverse Fourier transform with respect to the variable z as follows

$$\overline{\Phi}\left(x,y,\mu\right)=\int_{-\infty}^{\infty}\Phi\left(x,y,z\right)e^{ik\mu z}dz, \tag{4.84}$$

$$\Phi(x,y,z) = \frac{k}{2\pi}\int_{-\infty}^{\infty}\overline{\Phi}(x,y,\mu)\, e^{-ik\mu z} d\mu. \qquad (4.85)$$

Taking Fourier transform of the Eqs. (4.77) to (4.82), the problem with boundary conditions in the transformed domain μ takes the following form

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\eta_1^2\right)\overline{\Phi}_t = a\delta(x - x_0)\,\delta(y - y_0), \qquad (4.86)$$

with $\eta_1 = \sqrt{1-\mu^2}$, and $a = e^{ik\mu z_0}$.

The transformed boundary conditions take the form

$$\overline{\Phi}_t\left(x, 0^+, \mu\right) = 0, \qquad (4.87)$$

$$\frac{\partial\overline{\Phi}_t\left(x, 0^-, \mu\right)}{\partial y} = 0, \qquad (4.88)$$

$$\frac{\partial\overline{\Phi}_t\left(x, 0^+, \mu\right)}{\partial y} + \frac{ik}{\eta}\overline{\Phi}_t\left(x, 0^+, \mu\right) = 0, \qquad (4.89)$$

$$\frac{\partial\overline{\Phi}_t\left(x, 0^-, \mu\right)}{\partial y} - \frac{ik}{\eta}\overline{\Phi}_t\left(x, 0^-, \mu\right) = 0, \qquad (4.90)$$

$$\overline{\Phi}_t\left(x, 0^+, \mu\right) - \overline{\Phi}_t\left(x, 0^-, \mu\right) = 0. \qquad (4.91)$$

Thus the problem (4.86) together with the boundary conditions $(4.87 - 4.91)$ in the transformed domain μ is the same as in the case of two dimensions formulated in the Section 4.2 except that $k^2\eta_1^2$ replaces k^2.

4.6 Solution of the problem

As mentioned before, the mathematical problem (4.86) together with the boundary conditions (4.87 – 4.91) in the transformed domain μ is the same as in the case of two dimensions formulated in the Section 4.2 except that $k^2\gamma^2$ replaces k^2 [46, 60, 61]. Thus making use of the equations (4.74) and (4.75), the scattered field due to a point source can be calculated as follows:

For $y > 0$

$$\begin{aligned}
\overline{\Phi}(\rho,\theta,\mu) \approx & \left(\frac{e^{\frac{-i\pi}{2}}}{2\pi}\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sqrt{\kappa_{+}(k\eta_1\cos\theta)}\right) \\
\times & \left[\left\{\cosh\varkappa(k\eta_1\cos\theta)+\frac{k\eta_1}{\eta}\frac{\sinh\varkappa(k\eta_1\cos\theta)}{\sqrt{k^2\eta_1^2\cos^2\theta-\sigma_1^2}}\right\}\right. \\
\times & \left\{i\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sin\frac{\theta}{2}\sin\frac{\theta_0}{2}\frac{\sqrt{\kappa_{+}(k\eta_1\cos\theta_0)}}{\sqrt{k^2\eta_1^2\cos^2\theta_0-\sigma_1^2}}\sinh\varkappa(k\eta_1\cos\theta_0)\right\} \\
+ & \frac{\sinh\varkappa(k\eta_1\cos\theta)}{\sqrt{k^2\eta_1^2\cos^2\theta-\sigma_1^2}}\left\{\left(i\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sin\frac{\theta}{2}\sin\frac{\theta_0}{2}\sqrt{\kappa_{+}(k\eta_1\cos\theta_0)}\right)\right. \\
\times & \left.\left.\left(\begin{array}{c}\cosh\varkappa(k\eta_1\cos\theta_0) \\ -\frac{k\eta_1}{\eta}\frac{\sinh\varkappa(k\eta_1\cos\theta_0)}{\sqrt{k^2\eta_1^2\cos^2\theta_0-\sigma_1^2}}\end{array}\right)+\sin\frac{\theta}{2}\sin\frac{\theta_0}{2}(\cos\theta+\cos\theta_0)p^{**}\right\}\right] \\
\times & \left[\widetilde{\mathcal{F}}\left(\sqrt{2k\eta_1\rho}\cos\frac{\theta-\theta_0}{2}\right)+\widetilde{\mathcal{F}}\left(\sqrt{2k\eta_1\rho}\cos\frac{\theta+\theta_0}{2}\right)\right] \\
\times & \frac{\exp[ik\eta_1(\rho+\rho_0)+ik\mu z_0]}{\sqrt{\rho\rho_0}}.
\end{aligned} \tag{4.92}$$

For $y < 0$

$$\begin{aligned}
&\overline{\Phi}(\rho,\theta,\mu) \approx \left(\frac{e^{\frac{-i\pi}{2}}}{2\pi}\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sqrt{\kappa_{+}(k\eta_1\cos\theta)}\right) \\
&\times\left[\left\{-k^2\eta_1^2\sin^2\theta\frac{\sinh\varkappa(k\eta_1\cos\theta)}{\sqrt{k^2\eta_1^2\cos^2\theta-\sigma_1^2}}\right\}\times\left\{i\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\frac{\sin\frac{\theta_0}{2}}{2\cos\frac{\theta}{2}}\right.\right. \\
&\left.\times\frac{\sqrt{\kappa_{+}(k\eta_1\cos\theta_0)}}{\sqrt{k^2\eta_1^2\cos^2\theta_0-\sigma_1^2}}\sinh\varkappa(k\eta_1\cos\theta_0)\right\}+\left(\cosh\varkappa(k\eta_1\cos\theta)-\frac{k\eta_1}{\eta}\frac{\sinh\varkappa(k\eta_1\cos\theta)}{\sqrt{k^2\eta_1^2\cos^2\theta-\sigma_1^2}}\right) \\
&\times\left\{\left(i\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\frac{\sin\frac{\theta_0}{2}}{2\cos\frac{\theta}{2}}\frac{\sqrt{\kappa_{+}(k\eta_1\cos\theta_0)}}{\sqrt{k^2\eta_1^2\cos^2\theta_0-\sigma_1^2}}\right)\times\left(\begin{array}{c}\sqrt{k^2\eta_1^2\cos^2\theta_0-\sigma_1^2}\cosh\varkappa(k\eta_1\cos\theta_0) \\ -\frac{k\gamma}{\eta}\sinh\varkappa(k\eta_1\cos\theta_0)\end{array}\right)\right. \\
&\left.\left.+\frac{\sin\frac{\theta_0}{2}}{2\cos\frac{\theta}{2}}(\cos\theta+\cos\theta_0)\,p^{**}\right\}\right] \\
&\times\left[\widetilde{\mathcal{F}}\left(\sqrt{2k\eta_1\rho}\cos\frac{\theta-\theta_0}{2}\right)+\widetilde{\mathcal{F}}\left(\sqrt{2k\eta_1\rho}\cos\frac{\theta+\theta_0}{2}\right)\right] \\
&\times\frac{\exp\left[ik\eta_1(\rho+\rho_0)+ik\mu z_0\right]}{k\eta_1\sqrt{\rho\rho_0}}.
\end{aligned} \tag{4.93}$$

The scattered field in the spatial domain can now be obtained by taking the inverse Fourier transform of equations (4.92) and (4.93).

Thus, for $y > 0$,

$$\begin{aligned}
\Phi(\rho,\theta,z) &\approx \frac{k}{2\pi}\int_{-\infty}^{\infty}\left(\frac{e^{\frac{-i\pi}{2}}}{2\pi}\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sqrt{\kappa_+(k\eta_1\cos\theta)}\right)\\
&\times\left[\left\{\cosh\varkappa(k\eta_1\cos\theta)+\frac{k\eta_1}{\eta}\frac{\sinh\varkappa(k\eta_1\cos\theta)}{\sqrt{k^2\eta_1^2\cos^2\theta-\sigma_1^2}}\right\}\left\{i\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sin\frac{\theta}{2}\sin\frac{\theta_0}{2}\right.\right.\\
&\left.\times\frac{\sqrt{\kappa_+(k\eta_1\cos\theta_0)}}{\sqrt{k^2\eta_1^2\cos^2\theta_0-\sigma_1^2}}\sinh\varkappa(k\eta_1\cos\theta_0)\right\}+\frac{\sinh\varkappa(k\eta_1\cos\theta)}{\sqrt{k^2\eta_1^2\cos^2\theta-\sigma_1^2}}\\
&\times\left\{\left(i\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sin\frac{\theta}{2}\sin\frac{\theta_0}{2}\sqrt{\kappa_+(k\eta_1\cos\theta_0)}\right)\right.\\
&\left.\left.\times\left(\cosh\varkappa(k\eta_1\cos\theta_0)-\frac{k\eta_1}{\eta}\frac{\sinh\varkappa(k\eta_1\cos\theta_0)}{\sqrt{k^2\eta_1^2\cos^2\theta_0-\sigma_1^2}}\right)+\sin\frac{\theta}{2}\sin\frac{\theta_0}{2}(\cos\theta+\cos\theta_0)p^{**}\right\}\right]\\
&\times\left[\widetilde{\mathcal{F}}\left(\sqrt{2k\eta_1\rho}\cos\frac{\theta-\theta_0}{2}\right)+\widetilde{\mathcal{F}}\left(\sqrt{2k\eta_1\rho}\cos\frac{\theta+\theta_0}{2}\right)\right]\\
&\times\frac{\exp\left[ik\eta_1(\rho+\rho_0)+ik\mu z_0-ik\mu z\right]}{k\eta_1\sqrt{\rho\rho_0}}d\mu, \qquad (4.94)
\end{aligned}$$

and for $y < 0$

$$
\begin{aligned}
&\Phi(\rho,\theta,\mu) \approx \frac{k}{2\pi}\int_{-\infty}^{\infty}\left(\frac{e^{\frac{-i\pi}{2}}}{2\pi}\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sqrt{\kappa_{+}(k\eta_1\cos\theta)}\right)\\
&\times\left[\left\{-k^2\eta_1^2\sin^2\theta\frac{\sinh\varkappa(k\eta_1\cos\theta)}{\sqrt{k^2\eta_1^2\cos^2\theta-\sigma_1^2}}\right\}\left\{i\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\frac{\sin\frac{\theta_0}{2}}{2\cos\frac{\theta}{2}}\right.\right.\\
&\left.\times\frac{\sqrt{\kappa_{+}(k\eta_1\cos\theta_0)}}{\sqrt{k^2\eta_1^2\cos^2\theta_0-\sigma_1^2}}\sinh\varkappa(k\eta_1\cos\theta_0)\right\}+\left(\cosh\varkappa(k\eta_1\cos\theta)-\frac{k\eta_1}{\eta}\frac{\sinh\varkappa(k\eta_1\cos\theta)}{\sqrt{k^2\eta_1^2\cos^2\theta-\sigma_1^2}}\right)\\
&\times\left\{i\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\frac{\sin\frac{\theta_0}{2}}{2\cos\frac{\theta}{2}}\frac{\sqrt{\kappa_{+}(k\eta_1\cos\theta_0)}}{\sqrt{k^2\eta_1^2\cos^2\theta_0-\sigma_1^2}}\begin{pmatrix}\sqrt{k^2\eta_1^2\cos^2\theta_0-\sigma_1^2}\cosh\varkappa(k\eta_1\cos\theta_0)\\-\frac{k\eta_1}{\eta}\sinh\varkappa(k\eta_1\cos\theta_0)\end{pmatrix}\right.\\
&\left.\left.+\frac{\sin\frac{\theta_0}{2}}{2\cos\frac{\theta}{2}}(\cos\theta+\cos\theta_0)\,p^{**}\right\}\right]\\
&\times\left[\widetilde{\mathcal{F}}\left(\sqrt{2k\eta_1\rho}\cos\frac{\theta-\theta_0}{2}\right)+\widetilde{\mathcal{F}}\left(\sqrt{2k\eta_1\rho}\cos\frac{\theta+\theta_0}{2}\right)\right]\\
&\times\frac{\exp\left[ik\eta_1(\rho+\rho_0)+ik\mu z_0-ik\mu z\right]}{k\gamma\sqrt{\rho\rho_0}}d\mu. \qquad (4.95)
\end{aligned}
$$

The integrals appearing in expressions (4.94) and (4.95) can be evaluated asymptotically by the method of steepest descent [69] (see Appendix C), and the far field for $y > 0$ and $y < 0$ are finally given as follows:

For $y > 0$

$$
\begin{aligned}
&\Phi(\rho,\theta,z) \approx \exp[-i\frac{\pi}{4}+ikR_1]\\
&[g_1(s_1)\in_1\sqrt{\frac{a_{1+}\rho+\rho_0}{R_1(R_{11}+R_1)}}\widetilde{\mathcal{F}}(\tau_{R_1})+g_1(s_2)\in_2\sqrt{\frac{a_{2+}\rho+\rho_0}{R_1(R_{12}+R_1)}}\widetilde{\mathcal{F}}(\tau_{R_2})]\\
&+\sqrt{\frac{a_1}{2\pi}}\exp[-i\frac{\pi}{4}+ikR_{11}]g_1(\frac{a_1}{R_{11}})\frac{1}{R_{11}}H(-\in_1)+\sqrt{\frac{a_2}{2\pi}}\exp[-i\frac{\pi}{4}+ikR_{12}]\\
&\times g_1(\frac{a_2}{R_2})\frac{1}{R_{12}}H(-\in_2)]. \qquad (4.96)
\end{aligned}
$$

For $y < 0$

$$\Phi(\rho,\theta,z) \approx \exp[-i\frac{\pi}{4} + ikR_1]$$
$$[g_2(s_1) \in_1 \sqrt{\frac{a_{1+}\rho + \rho_0}{R_1(R_{11}+R_1)}}\widetilde{\mathcal{F}}(\tau_{R_1}) + g_2(s_2) \in_2 \sqrt{\frac{a_{2+}\rho + \rho_0}{R_1(R_{12}+R_1)}}\widetilde{\mathcal{F}}(\tau_{R_2})]$$
$$+\sqrt{\frac{a_1}{2\pi}}\exp[-i\frac{\pi}{4} + ikR_{11}]g_2(\frac{a_1}{R_{11}})\frac{1}{R_{11}}H(-\in_1) + \sqrt{\frac{a_2}{2\pi}}\exp[-i\frac{\pi}{4} + ikR_{12}]$$
$$\times g_2(\frac{a_2}{R_2})\frac{1}{R_{12}}H(-\in_2)], \tag{4.97}$$

where

$$s_1 = \frac{\sqrt{\tau_{R_1}^2(2kR_{11} + \tau_{R_1}^2) + k^2a_1^2}}{\tau_{R_1}^2 + kR_{11}} \text{ and } s_2 = \frac{\sqrt{\tau_{R_2}^2(2kR_{12} + \tau_{R2}^2) + k^2a_2^2}}{\tau_{R_1}^2 + kR_{12}}. \tag{4.98}$$

In expressions (4.96) and (4.97) $H(.)$ is the usual Heaviside function and the quantities $\mu_1, f_1(\mu), g_1(\mu), a_1, R_{11}, \tau_{R_1}, \in_1$ and R_1 have already been explained in the Appendix C. It is important to remark here that the other quantities for e.g., $R_{12}, \tau_{R_2}, \in_2$ etc. may be seen from [60, 61].

The unknown constant p^{**} for the case of point source scattering is given as follows

$$p^{**} = \left(\frac{\eta}{2}\right)^{\frac{1}{4}}\frac{k\eta_1}{\pi i}\frac{\sqrt{\kappa_+(k\eta_1\cos\theta_0)}}{\sqrt{k^2\eta_1^2\cos^2\theta - \sigma_1^2}}\sinh\varkappa(k\eta_1\cos\theta_0). \tag{4.99}$$

4.7 Concluding remarks

In this chapter, the line source and the point source scattering of acoustic waves by the junction of partially transmissive and soft-hard half planes are studied. The first boundary value problem of line source excitation is reduced to a matrix W-H equation by using the Fourier transform technique. Then solution of the problem requires the W-H factorization of the kernel matrix involved in the equation. This factorization is performed by Büyükaksoy et al [70] which can be used for the analysis presented in this chapter. The problem is then solved completely. It is observed that:

- For the case of line source incidence our results obtained in this chapter differs from [70] by a multiplicative factor which agrees well with the already existing literature [35, 46].
- If a line source is removed to a far-off distance (at infinity) the graphical results of plane wave situation [70] can be recovered.
- Mathematically point source problem is strongly based on the results obtained for the line source situation and it is also well known in the existing literature e.g., see [46, 60, 61].
- Various graphs of interest showing the effects of different parameters on the amplitude of the scattered field produced by the line source are also plotted and discussed.

Chapter 5

Diffraction Of A Plane Wave By A Soft-Hard Strip

An attempt is made in this chapter to investigate the diffraction of a plane acoustic wave by a finite soft-hard barrier. Scattering/Diffraction by strips/slits is an important and classical topic both in electromagnetic and acoustic wave theory. As mentioned earlier that in order to study diffraction patterns (single or multiple) from a strip a large number of analytical, numerical or a combination of both analytical and numerical methods such as separation of variables [75], geometrical theory of diffraction (GTD) [78, 163], Kobayashi's potential method [110, 111], spectral iteration technique (SIT) [101], method of successive approximations [83, 84], and the W-H technique [87 – 100] have been successfully applied. Some recent developments in the literature are also based on Bessel's potential spaces [112] and Maliuzhinetz-

Sommerfeld integral representation [133].

The problem of a plane acoustic wave diffraction presented in this chapter resulted into a typical matrix W-H functional equation of the form

$$e^{i\alpha q}\Psi_{+}(\alpha) - \mathbf{H}(\alpha)\,\Psi_{1}(\alpha) + e^{i\alpha p}\Psi_{-}(\alpha) = \frac{G(\alpha)}{i\sqrt{2\pi}}\mathbf{A}(\alpha),$$

where $\Psi_{+}(\alpha)$ and $\Psi_{-}(\alpha)$ are the unknown functions being regular in the upper half plane $\tau > k_2 \cos\theta_0$ and the lower half plane $\tau < k_2$, $\Psi_1(\alpha)$ is an entire function, $\mathbf{H}(\alpha)$ is the kernel matrix, $G(\alpha)$ is a known function and $\mathbf{A}(\alpha)$ is a known column vector respectively and the bold letters are used to indicate matrices and will be defined in the sequel.

By using the Fourier transform technique the related boundary value problem is reduced to a matrix W-H equation which in turn requires the factorization of the kernel matrix involved as a product of two non-singular matrices having entries which are regular and of algebraic growth in certain overlapping halves of the complex plane. Luckily for the problem under consideration the kernel matrix remains the same as that of [33] and has been factorized by [33]. However for the sake of completeness we have given the complete factorization details in the Appendix A of this thesis. By applying the W-H technique [14] in the Jones' interpretation [135] the problem under consideration is completely solved. Finally, by calculating the undetermined coefficient $A(\alpha)$, expressed in terms of functions whose regions of regularity are known, the inverse Fourier transform is calculated using the method of steepest descent [69, 136 − 138], and the scattered field is presented. It is also shown that the scattered field is sum of two fields, i.e., separated field $\psi_{sep}(x,y)$ (field radiated from each edge p and q) and the interacted field $\psi_{int}(x,y)$ (interaction of one edge upon the other edge).

It is also important to mention here that the problem presented in this chapter is solved under the physical assumption that the strip width is large as compared to the incident wavelength and hence a high frequency approximate solution of the problem can be obtained by using GTD [163]. Several integrals $(I_1 - I_6)$ in the presented analysis have been evaluated under this assumption.

Some practical applications of the strip geometry may be found with reference to noise reduction by barriers [99], decontamination chambers of hospital in which textiles and operation apparatuses are routinely sterilized using highly toxic organic gases [90], microwave filters, reflectors and design of frequency selective surfaces. Uniform high frequency expressions for the field scattered from strip are also obtained for plane, cylindrical and spherical wave illuminations by using the promising W-H technique and can be found in the works [95, 96, 100], [87, 97, 99] and [89, 94, 98].

5.1 Mathematical formulation of the problem

Considering the diffraction of a plane acoustic wave incident on the finite soft-hard plane $S = \{x \in (p,q)\,,\, y = 0,\, z \in (-\infty,\infty)\}$. The geometry of the problem is depicted in Figure 5.1.

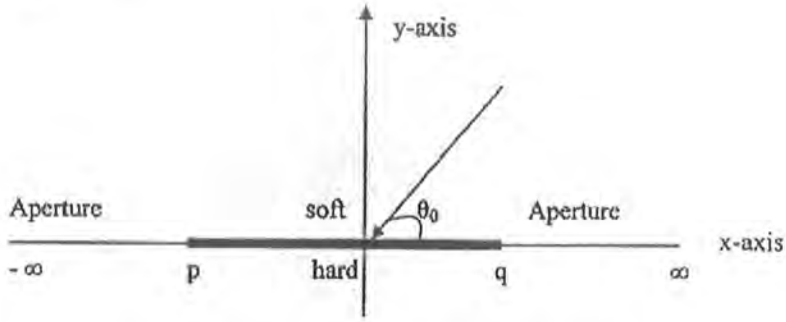

Fig. 5.1 Geometry of the strip problem

The plane is assumed to be of infinitesimal thickness and soft (pressure release) at the top and hard (rigid) at the bottom. A time factor $e^{-i\omega t}$ is assumed and suppressed throughout. The wave equation satisfied by the total velocity potential ψ_t is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right) \psi_t (x, y) = 0, \tag{5.1}$$

subject to the boundary and continuity conditions:

$$\psi_t (x, 0^+) = 0, \qquad \text{on} \qquad p < x < q, \tag{5.2}$$

$$\frac{\partial \psi_t (x, 0-)}{\partial y} = 0, \qquad \text{on} \qquad p < x < q, \tag{5.3}$$

and

$$\psi_t (x, 0^+) - \psi_t (x, 0^-) = 0, \qquad \text{on} \qquad \left\{ \begin{array}{c} -\infty < x < p \\ q < x < \infty \end{array} \right\}, \tag{5.4}$$

$$\frac{\partial \psi_t (x, 0+)}{\partial y} - \frac{\partial \psi_t (x, 0-)}{\partial y} = 0, \qquad \text{on} \qquad \left\{ \begin{array}{c} -\infty < x < p \\ q < x < \infty \end{array} \right\}. \tag{5.5}$$

For the unique solution of the problem it is required that ψ_t and its normal derivative must be bounded and these must be of the following orders [33, 34]

$$\psi_t(x,0) = \begin{cases} -1 + O(x-p)^{\frac{1}{4}} & \text{as } x \rightarrow p, \\ -1 + O(x-q)^{\frac{1}{4}} & \text{as } x \rightarrow q, \end{cases} \tag{5.6}$$

and

$$\frac{\partial \psi_t(x,0)}{\partial y} = \begin{cases} O(x-p)^{-\frac{3}{4}} & \text{as } x \rightarrow p, \\ O(x-q)^{-\frac{3}{4}} & \text{as } x \rightarrow q. \end{cases} \tag{5.7}$$

Let a plane acoustic wave

$$\psi_i = e^{-ik(x\cos\theta_0 + y\sin\theta_0)}, \tag{5.8}$$

be incident upon the soft-hard half finite plate occupying the position $p < x < q$, $y = 0$. In Eq. (5.8), θ_0 is the angle of incidence and for the analytic convenience it is assumed that the wave number k has positive imaginary part.

For the analysis purpose it is convenient to express the total field ψ_t as [14]

$$\psi_t(x,y) = \psi_i(x,y) + \psi(x,y), \tag{5.9}$$

where ψ is the diffracted field and ψ_i is the incident field given by Eq. (5.8). Thus, scattered field satisfies the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)\psi(x,y) = 0, \tag{5.10}$$

subject to the boundary conditions

$$\psi\left(x,0^+\right) = -e^{-ikx\cos\theta_0} \quad \text{on} \quad p < x < q, \tag{5.11}$$

$$\frac{\partial \psi\left(x,0^{-}\right)}{\partial y}=ik\sin\theta_{0}e^{-ikx\cos\theta_{0}}\quad \text{on}\quad p<x<q, \tag{5.12}$$

and the continuity conditions

$$\psi\left(x,0^{+}\right)=\psi\left(x,0^{-}\right)\qquad \text{on } -\infty<x<p,\ q<x<\infty, \tag{5.13}$$

and

$$\frac{\partial \psi\left(x,0^{+}\right)}{\partial y}=\frac{\partial \psi\left(x,0-\right)}{\partial y}\qquad \text{on } -\infty<x<p,\ q<x<\infty. \tag{5.14}$$

The Fourier transform pair is defined as follows

$$\begin{aligned}\overline{\psi}\left(\alpha,y\right) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi\left(x,y\right)e^{i\alpha x}dx,\\ &= e^{i\alpha p}\overline{\psi}_{-}\left(\alpha,y\right)+\overline{\psi}_{1}\left(\alpha,y\right)+e^{i\alpha q}\overline{\psi}_{+}\left(\alpha,y\right),\end{aligned} \tag{5.15}$$

and

$$\psi\left(x,y\right)=\int_{-\infty}^{\infty}\overline{\psi}\left(\alpha,y\right)e^{-i\alpha x}d\alpha, \tag{5.16}$$

where

$$\begin{aligned}\overline{\psi}_{-}\left(\alpha,y\right) &= \frac{1}{2\pi}\int_{-\infty}^{p}\psi\left(x,y\right)e^{i\alpha(x-p)}dx,\\ \overline{\psi}_{1}\left(\alpha,y\right) &= \frac{1}{2\pi}\int_{p}^{q}\psi\left(x,y\right)e^{i\alpha x}dx,\\ \overline{\psi}_{+}\left(\alpha,y\right) &= \frac{1}{2\pi}\int_{q}^{\infty}\psi\left(x,y\right)e^{i\alpha(x-q)}dx.\end{aligned} \tag{5.17}$$

The function $\overline{\psi}_{-}\left(\alpha,y\right)$ is regular in the lower half plane $\operatorname{Im}\alpha<\operatorname{Im}k$, $\overline{\psi}_{+}\left(\alpha,y\right)$ is regular in the upper half plane $\operatorname{Im}\alpha>\operatorname{Im}k\cos\theta_{0}$ and $\overline{\psi}_{1}\left(\alpha,y\right)$ is an analytic function and therefore regular in the common region $\operatorname{Im}k\cos\theta_{0}<\operatorname{Im}\alpha<\operatorname{Im}k$.

The Fourier transform of the Eq. (5.10) will yield

$$\frac{d^2\overline{\psi}(\alpha,y)}{dy^2}+K^2\overline{\psi}(\alpha,y)=0, \tag{5.18}$$

where $K(\alpha)=\sqrt{k^2-\alpha^2}$.

Defining $K(\alpha)$, the square root function, to be that branch which reduces to $+k$ when $\alpha=0$ and when the complex $\alpha-$plane is cut either from $\alpha=k$ to $\alpha=k\infty$ or from $\alpha=-k$ to $\alpha=-k\infty$ as shown in Figure 3.2.

The solution of Eq. (5.18), representing the outgoing waves at infinity, can formally be written as

$$\overline{\psi}(\alpha,y)=\begin{cases} A(\alpha)e^{iK(\alpha)y} & y>0, \\ B(\alpha)e^{-iK(\alpha)y} & y<0, \end{cases} \tag{5.19}$$

where $A(\alpha)$ and $B(\alpha)$ are the unknown coefficients which are to be determined. The Fourier transform of the boundary conditions $(5.11-5.14)$ yield

$$\overline{\psi}_1\left(\alpha,0^+\right)=-\frac{1}{i\sqrt{2\pi}}G(\alpha), \tag{5.20}$$

$$\frac{\partial\overline{\psi}_1\left(\alpha,0^-\right)}{\partial y}=\frac{k\sin\theta_0}{\sqrt{2\pi}}G(\alpha), \tag{5.21}$$

and

$$\overline{\psi}_\pm\left(\alpha,0^+\right)=\overline{\psi}_\pm\left(\alpha,0^+\right), \tag{5.22}$$

$$\frac{\partial\overline{\psi}_\pm\left(\alpha,0^+\right)}{\partial y}=\frac{\partial\overline{\psi}_\pm\left(\alpha,0^-\right)}{\partial y}, \tag{5.23}$$

where

$$G(\alpha)=\frac{e^{i(\alpha-k\cos\theta_0)q}-e^{i(\alpha-k\cos\theta_0)p}}{\sqrt{2\pi}\left(\alpha-k\cos\theta_0\right)}. \tag{5.24}$$

Using Eqs. (5.20 − 5.23) in Eq. (5.19), will give

$$A(\alpha) = e^{i\alpha p}\overline{\psi}_{-1}(\alpha) - \frac{1}{i\sqrt{2\pi}}G(\alpha) + e^{i\alpha q}\overline{\psi}_{+1}(\alpha), \tag{5.25}$$

$$K(\alpha)B(\alpha) = e^{i\alpha p}\overline{\psi}_{-2}(\alpha) + \frac{ik\sin\theta_0}{\sqrt{2\pi}}G(\alpha) + e^{i\alpha q}\overline{\psi}_{+2}(\alpha), \tag{5.26}$$

$$A(\alpha) - B(\alpha) = 2\overline{\psi}_1(\alpha), \tag{5.27}$$

$$A(\alpha) + B(\alpha) = 2\frac{\overline{\psi}_2(\alpha)}{K(\alpha)}, \tag{5.28}$$

where

$$\overline{\psi}_{-1}(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{p} \psi(x, 0^+)\, e^{i\alpha(x-p)}dx, \tag{5.29}$$

$$\overline{\psi}_{+1}(\alpha) = \frac{1}{2\pi}\int_{q}^{\infty} \psi(x, 0^+)\, e^{i\alpha(x-q)}dx, \tag{5.30}$$

$$\overline{\psi}_{-2}(\alpha) = \frac{i}{2\pi}\int_{-\infty}^{p} \frac{\partial\psi(x, 0^-)}{\partial y}e^{i\alpha(x-p)}dx, \tag{5.31}$$

$$\overline{\psi}_{+2}(\alpha) = \frac{i}{2\pi}\int_{q}^{\infty} \frac{\partial\psi(x, 0^-)}{\partial y}e^{i\alpha(x-q)}dx, \tag{5.32}$$

$$\overline{\psi}_1(\alpha) = \frac{1}{4\pi}\int_{p}^{q} \left[\psi(x, 0^+) - \psi(x, 0^-)\right] e^{i\alpha x}dx, \tag{5.33}$$

and

$$\overline{\psi}_2(\alpha) = \frac{1}{4\pi i K(\alpha)}\int_{p}^{q} \left[\frac{\partial\psi(x, 0^+)}{\partial y} - \frac{\partial\psi(x, 0^-)}{\partial y}\right] e^{i\alpha x}dx. \tag{5.34}$$

The elimination of the coefficients $A(\alpha)$ and $B(\alpha)$ among the Eqs. (5.25 − 5.28) will lead to the following matrix Wiener-Hopf equation valid in the strip of analyticity

$\operatorname{Im} k \cos\theta_0 < \operatorname{Im}\alpha < \operatorname{Im} k$,

$$e^{i\alpha q}\begin{bmatrix} \overline{\psi}_{+1}(\alpha) \\ \overline{\psi}_{+2}(\alpha) \end{bmatrix} - \begin{bmatrix} 1 & \frac{1}{K(\alpha)} \\ -K(\alpha) & 1 \end{bmatrix}\begin{bmatrix} \overline{\psi}_{1}(\alpha) \\ \overline{\psi}_{2}(\alpha) \end{bmatrix} + e^{i\alpha p}\begin{bmatrix} \overline{\psi}_{-1}(\alpha) \\ \overline{\psi}_{-2}(\alpha) \end{bmatrix}$$

$$= \frac{1}{i\sqrt{2\pi}}G(\alpha)\begin{bmatrix} 1 \\ k\sin\theta_0 \end{bmatrix}. \tag{5.35}$$

In compact form Eq. (5.35) can further be arranged as

$$e^{i\alpha q}\Psi_{+}(\alpha) - \mathbf{H}(\alpha)\,\Psi_1(\alpha) + e^{i\alpha p}\Psi_{-}(\alpha) = \frac{G(\alpha)}{i\sqrt{2\pi}}\mathbf{A}(\alpha), \tag{5.36}$$

where bold letters are used to denote the matrices. Also Eq. (5.36) is an equation of the form (5.60) available in [14]. In Eq. (5.36) $\mathbf{H}(\alpha)$ is the kernel matrix and in order to solve it, one has to factorize the matrix $\mathbf{H}(\alpha)$ as the product of two non-singular factor matrices such that one factor matrix being regular in the lower half plane and the other factor matrix being regular in the upper half plane with the additional requirements that both the factor matrices as well as their inverses contain elements of algebraic growth at infinity and both of these factor matrices should commute with each other. The factorization of $\mathbf{H}(\alpha)$ satisfying the conditions mentioned above, has been done in [33] by using the Daniele-Kharapkov methods [20, 21] and the result is as follows:

$$\mathbf{H}_{+}(\alpha) = 2^{\frac{1}{4}}\begin{bmatrix} \cosh\chi(\alpha) & \sinh\chi(\alpha)/\gamma(\alpha) \\ \gamma(\alpha)\sinh\chi(\alpha) & \cosh\chi(\alpha) \end{bmatrix}, \tag{5.37}$$

with

$$\mathbf{H}_{-}(\alpha) = \mathbf{H}_{+}(-\alpha), \tag{5.38}$$

where

$$\chi(\alpha) = -\frac{i}{4}\arccos\frac{\alpha}{k}, \qquad \chi(-\alpha) = -\frac{i}{4}\left[\pi - \arccos\frac{\alpha}{k}\right], \tag{5.39}$$

and

$$\gamma(\alpha) = \sqrt{\alpha^2 - k^2}. \tag{5.40}$$

Also as $|\alpha| \to \infty$, it is noted that

$$\mathbf{H}_{\pm}(\alpha) \sim (4k)^{-\frac{1}{4}} \begin{bmatrix} (\pm\alpha)^{\frac{1}{4}} & (\pm\alpha)^{-\frac{3}{4}} \\ (\pm\alpha)^{\frac{5}{4}} & (\pm\alpha)^{\frac{1}{4}} \end{bmatrix}. \tag{5.41}$$

Since the factorization of the matrix $\mathbf{H}(\alpha)$ has been accomplished, therefore Eq. (5.36) can be rearranged as:

$$e^{i\alpha q}\Psi_+(\alpha) - \mathbf{H}_+(\alpha)\,\mathbf{H}_-(\alpha)\,\Psi_1(\alpha) + e^{i\alpha p}\Psi_-(\alpha) = \frac{G(\alpha)}{i\sqrt{2\pi}}\mathrm{A}(\alpha), \tag{5.42}$$

or

$$e^{i\alpha q}\Psi_+(\alpha) - \mathbf{H}_-(\alpha)\,\mathbf{H}_+(\alpha)\,\Psi_1(\alpha) + e^{i\alpha p}\Psi_-(\alpha) = \frac{G(\alpha)}{i\sqrt{2\pi}}\mathrm{A}(\alpha). \tag{5.43}$$

5.2 Solution of the matrix W-H equation

Pre-multiplying Eq. (5.42) by $e^{-i\alpha q}\left[\mathbf{H}_+(\alpha)\right]^{-1}$, substituting the value of $G(\alpha)$ from Eq. (5.24) and simplifying will result into

$$\begin{aligned} &\left[\mathbf{H}_+(\alpha)\right]^{-1}\Psi_+(\alpha) - e^{-i\alpha q}\mathbf{H}_-(\alpha)\,\Psi_1(\alpha) + e^{i\alpha(p-q)}\left[\mathbf{H}_+(\alpha)\right]^{-1}\Psi_-(\alpha) \\ = \; &\frac{e^{-ik\cos\theta_0 q}}{2\pi(\alpha - k\cos\theta_0)}\left[\mathbf{H}_+(\alpha)\right]^{-1}\mathrm{A} - \frac{e^{i\alpha(p-q)-ik\cos\theta_0 p}}{2\pi(\alpha - k\cos\theta_0)}\left[\mathbf{H}_+(\alpha)\right]^{-1}\mathrm{A}. \end{aligned} \tag{5.44}$$

According to the procedure defined in [14] different terms occurring in Eq. (5.44) can be decomposed as follows,

$$e^{i\alpha(p-q)}\left[\mathbf{H}_{+}(\alpha)\right]^{-1}\Psi_{-}(\alpha)=\mathbf{U}_{+}(\alpha)+\mathbf{U}_{-}(\alpha), \tag{5.45}$$

$$\frac{e^{i\alpha(p-q)-ik\cos\theta_0 p}}{2\pi\left(\alpha-k\cos\theta_0\right)}\left[\mathbf{H}_{+}(\alpha)\right]^{-1}\mathbf{A}=\mathbf{V}_{+}(\alpha)+\mathbf{V}_{-}(\alpha). \tag{5.46}$$

The pole contribution of the first term on right hand side of Eq. (5.44) can be expressed as

$$\frac{e^{-ik\cos\theta_0 q}}{2\pi\left(\alpha-k\cos\theta_0\right)}\left[\left\{\mathbf{H}_{+}(\alpha)\right\}^{-1}-\left\{\mathbf{H}_{+}(k\cos\theta_0)\right\}^{-1}+\left\{\mathbf{H}_{+}(k\cos\theta_0)\right\}^{-1}\right]\mathbf{A}. \tag{5.47}$$

Using Eqs. (5.45 – 5.47) in Eq. (5.44) and separating it into positive and negative terms, will yield

$$\begin{aligned}
&\left[\mathbf{H}_{+}(\alpha)\right]^{-1}\Psi_{+}(\alpha)+\mathbf{U}_{+}(\alpha)-\frac{e^{-ik\cos\theta_0 q}}{2\pi\left(\alpha-k\cos\theta_0\right)}\\
&\times\left[\left\{\mathbf{H}_{+}(\alpha)\right\}^{-1}-\left\{\mathbf{H}_{+}(k\cos\theta_0)\right\}^{-1}\right]\mathbf{A}+\mathbf{V}_{+}(\alpha)\\
=\;& e^{-i\alpha q}\mathbf{H}_{-}(\alpha)\Psi_1(\alpha)-\mathbf{U}_{-}(\alpha)-\mathbf{V}_{-}(\alpha)+\frac{e^{-ik\cos\theta_0 q}}{2\pi\left(\alpha-k\cos\theta_0\right)}\left\{\mathbf{H}_{+}(k\cos\theta_0)\right\}^{-1}\mathbf{A}.
\end{aligned} \tag{5.48}$$

Now pre-multiplying Eq. (5.44) by $e^{-i\alpha p}\left[\mathbf{H}_{-}(\alpha)\right]^{-1}$, substituting the value of $G(\alpha)$ from Eq. (5.24) and simplifying will give

$$\begin{aligned}
&e^{i\alpha(q-p)}\left[\mathbf{H}_{-}(\alpha)\right]^{-1}\Psi_{+}(\alpha)-e^{-i\alpha p}\mathbf{H}_{+}(\alpha)\Psi_1(\alpha)+\left[\mathbf{H}_{-}(\alpha)\right]^{-1}\Psi_{-}(\alpha)\\
=\;&\frac{e^{i\alpha(q-p)-ik\cos\theta_0 q}}{2\pi\left(\alpha-k\cos\theta_0\right)}\left[\mathbf{H}_{-}(\alpha)\right]^{-1}\mathbf{A}-\frac{e^{-ik\cos\theta_0 p}}{2\pi\left(\alpha-k\cos\theta_0\right)}\left[\mathbf{H}_{-}(\alpha)\right]^{-1}\mathbf{A}.
\end{aligned} \tag{5.49}$$

Decomposing different terms in Eq. (5.49) by following [14], will obtain

$$e^{i\alpha(q-p)}\left[\mathbf{H}_-(\alpha)\right]^{-1}\Psi_+(\alpha)=\mathbf{R}_+(\alpha)+\mathbf{R}_-(\alpha), \tag{5.50}$$

$$\frac{e^{i\alpha(q-p)-ik\cos\theta_0 q}}{2\pi(\alpha-k\cos\theta_0)}\left[\mathbf{H}_-(\alpha)\right]^{-1}\mathbf{A}=\mathbf{S}_+(\alpha)+\mathbf{S}_-(\alpha). \tag{5.51}$$

Using Eqs. (5.50, 5.51) in Eq. (5.49) and separating it into positive and negative portions will show that

$$\begin{aligned}&\mathbf{R}_-(\alpha)+\left[\mathbf{H}_-(\alpha)\right]^{-1}\Psi_-(\alpha)-\mathbf{S}_-(\alpha)+\frac{e^{-ik\cos\theta_0 p}}{2\pi(\alpha-k\cos\theta_0)}\left[\mathbf{H}_-(\alpha)\right]^{-1}\mathbf{A}\\ =\;& e^{-i\alpha p}\mathbf{H}_+(\alpha)\,\Psi_1(\alpha)-\mathbf{R}_+(\alpha)+\mathbf{S}_+(\alpha).\end{aligned} \tag{5.52}$$

The left hand side of Eq. (5.48) and right hand side of Eq. (5.52) are regular in $\operatorname{Im}\alpha > \operatorname{Im} k\cos\theta_0$ and $\operatorname{Im}\alpha < \operatorname{Im} k$. Hence using the extended form of the Liouville's theorem each side of Eqs. (5.48) and (5.52) is equal to zero, i.e.,

$$\begin{aligned}&\left[\mathbf{H}_+(\alpha)\right]^{-1}\Psi_+(\alpha)+\mathbf{U}_+(\alpha)-\frac{e^{-ik\cos\theta_0 q}}{2\pi(\alpha-k\cos\theta_0)}\\ &\times\left[\{\mathbf{H}_+(\alpha)\}^{-1}-\{\mathbf{H}_+(k\cos\theta_0)\}^{-1}\right]\mathbf{A}+\mathbf{V}_+(\alpha)=0\end{aligned} \tag{5.53}$$

and

$$\mathbf{R}_-(\alpha)+\left[\mathbf{H}_-(\alpha)\right]^{-1}\Psi_-(\alpha)-\mathbf{S}_-(\alpha)+\frac{e^{-ik\cos\theta_0 p}}{2\pi(\alpha-k\cos\theta_0)}\left[\mathbf{H}_-(\alpha)\right]^{-1}\mathbf{A}=0. \tag{5.54}$$

The explicit expressions for $\mathbf{U}_+(\alpha)$, $\mathbf{V}_+(\alpha)$, $\mathbf{R}_-(\alpha)$ and $\mathbf{S}_-(\alpha)$ are given as follows:

$$\mathbf{U}_+(\alpha)=\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{e^{i\xi(p-q)}\left[\mathbf{H}_+(\xi)\right]^{-1}\Psi_-(\xi)}{\xi-\alpha}d\xi, \tag{5.55}$$

$$\mathbf{V}_+(\alpha)=\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{e^{i\xi(p-q)-ik\cos\theta_0 p}\left[\mathbf{H}_+(\xi)\right]^{-1}\mathbf{A}}{2\pi(\xi-\alpha)(\xi-k\cos\theta_0)}d\xi, \tag{5.56}$$

$$\mathrm{R}_{-}(\alpha) = -\frac{1}{2\pi i}\int\limits_{-\infty+id}^{\infty+id} \frac{e^{i\xi(q-p)}\left[\mathrm{H}_{-}(\xi)\right]^{-1}\Psi_{+}(\xi)}{\xi-\alpha}d\xi \tag{5.57}$$

and

$$\mathrm{S}_{-}(\alpha) = -\frac{1}{2\pi i}\int\limits_{-\infty+id}^{\infty+id} \frac{e^{i\xi(q-p)-ik\cos\theta_0 q}\left[\mathrm{H}_{-}(\xi)\right]^{-1}\mathrm{A}}{2\pi(\xi-\alpha)(\xi-k\cos\theta_0)}d\xi, \tag{5.58}$$

where $-\operatorname{Im}\alpha < c < \operatorname{Im}k\cos\theta_0$ and $-\operatorname{Im}\alpha < d < \operatorname{Im}k\cos\theta_0$, also $\operatorname{Im}\alpha > c$ in Eqs. (5.55, 5.56) and $\operatorname{Im}\alpha < d$ in Eqs. (5.57, 5.58) as given in [14]. Using Eqs. (5.55, 5.56) in Eq. (5.53) and Eqs. (5.57, 5.58) in Eq. (5.54) and simplifying these equations will give

$$\begin{aligned}\left[\mathrm{H}_{+}(\alpha)\right]^{-1}\Psi_{+}^{*}(\alpha) + \frac{e^{-ik\cos\theta_0 q}\left[\mathrm{H}_{+}(k\cos\theta_0)\right]^{-1}\mathrm{A}}{2\pi(\alpha-k\cos\theta_0)}\\ +\frac{1}{2\pi i}\int\limits_{-\infty+ic}^{\infty+ic}\frac{e^{i\xi(p-q)}\left[\mathrm{H}_{+}(\xi)\right]^{-1}\Psi_{-}(\xi)}{(\xi-\alpha)}d\xi = 0\end{aligned} \tag{5.59}$$

and

$$\left[\mathrm{H}_{-}(\alpha)\right]^{-1}\Psi_{-}(\alpha) - \frac{1}{2\pi i}\int\limits_{-\infty+id}^{\infty+id}\frac{e^{i\xi(q-p)}\left[\mathrm{H}_{-}(\xi)\right]^{-1}\Psi_{+}^{*}(\alpha)}{(\xi-\alpha)}d\xi = 0, \tag{5.60}$$

where

$$\Psi_{+}^{*}(\alpha) = \Psi_{+}(\alpha) - \frac{e^{-ik\cos\theta_0 q}\mathrm{A}}{2\pi(\alpha-k\cos\theta_0)} \tag{5.61}$$

and

$$\Psi_{-}(\alpha) = \Psi_{-}(\alpha) + \frac{e^{-ik\cos\theta_0 p}\mathrm{A}}{2\pi(\alpha-k\cos\theta_0)}. \tag{5.62}$$

From the assumption that $0 < \theta_0 < \frac{\pi}{2}$, one can choose a such that $-k_2\cos\theta_0 < a < k_2\cos\theta_0$ and $d = -c = a$ [14]. In Eq. (5.59) replacing ξ by $-\xi$ and in Eq. (5.60) α by $-\alpha$ and also noting that $\mathrm{H}_{-}(-\alpha) = \mathrm{H}_{+}(\alpha)$ will give

$$[\mathbf{H}_+(\alpha)]^{-1}\,\boldsymbol{\Psi}_+^*(\alpha)+\frac{e^{-ik\cos\theta_0 q}\,[\mathbf{H}_+(k\cos\theta_0)]^{-1}\,\mathbf{A}}{2\pi(\alpha-k\cos\theta_0)}$$
$$-\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{e^{i\xi(q-p)}\,[\mathbf{H}_-(\xi)]^{-1}\,\boldsymbol{\Psi}_-(-\xi)}{(\xi+\alpha)}d\xi=0 \tag{5.63}$$

and

$$[\mathbf{H}_+(\alpha)]^{-1}\,\boldsymbol{\Psi}_-(-\alpha)-\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{e^{i\xi(q-p)}\,[\mathbf{H}_-(\xi)]^{-1}\,\boldsymbol{\Psi}_+^*(\alpha)}{(\xi+\alpha)}d\xi=0. \tag{5.64}$$

Addition and subtraction of Eqs. (5.63) and (5.64), will result into

$$[\mathbf{H}_+(\alpha)]^{-1}\,\mathbf{S}_+^*(\alpha)+\frac{e^{-ik\cos\theta_0 q}\,[\mathbf{H}_+(k\cos\theta_0)]^{-1}\,\mathbf{A}}{2\pi(\alpha-k\cos\theta_0)}$$
$$-\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{e^{i\xi(q-p)}\,[\mathbf{H}_-(\xi)]^{-1}\,\mathbf{S}_+^*(\xi)}{(\xi+\alpha)}d\xi=0 \tag{5.65}$$

and

$$[\mathbf{H}_+(\alpha)]^{-1}\,\mathbf{D}_+^*(\alpha)+\frac{e^{-ik\cos\theta_0 q}\,[\mathbf{H}_+(k\cos\theta_0)]^{-1}\,\mathbf{A}}{2\pi(\alpha-k\cos\theta_0)}$$
$$+\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{e^{i\xi(q-p)}\,[\mathbf{H}_-(\xi)]^{-1}\,\mathbf{D}_+^*(\xi)}{(\xi+\alpha)}d\xi=0, \tag{5.66}$$

where

$$\mathbf{S}_+^*(\alpha)=\boldsymbol{\Psi}_+^*(\alpha)+\boldsymbol{\Psi}_-(-\alpha), \tag{5.67}$$

$$\mathbf{D}_+^*(\alpha)=\boldsymbol{\Psi}_+^*(\alpha)-\boldsymbol{\Psi}_-(-\alpha). \tag{5.68}$$

The Eqs. (5.65) and (5.66) are of the same type and an approximate solution can be obtained by a method due to Jones [164]. Setting

$$\mathbf{S}_+^*(\alpha)=\mathbf{D}_+^*(\alpha)=\mathbf{F}_+^*(\alpha), \tag{5.69}$$

the Eqs. (5.65 – 5.66) will take the form

$$[\mathbf{H}_+(\alpha)]^{-1}\mathbf{F}_+^*(\alpha)+\frac{\lambda}{2\pi i}\int\limits_{-\infty+ia}^{\infty+ia}\frac{e^{i\xi(q-p)}\left[\mathbf{H}_-(\xi)\right]^{-1}\mathbf{F}_+^*(\xi)}{(\xi+\alpha)}d\xi$$

$$=-\frac{e^{-ik\cos\theta_0 q}\left[\mathbf{H}_+(k\cos\theta_0)\right]^{-1}\mathbf{A}(\alpha)}{2\pi(\alpha-k\cos\theta_0)}, \tag{5.70}$$

where

$$\mathbf{F}_+^*(\alpha)=\mathbf{F}_+(\alpha)-\frac{\mathbf{A}e^{-ik\cos\theta_0 q}}{2\pi(\alpha-k\cos\theta_0)}+\frac{\lambda\mathbf{A}e^{-ik\cos\theta_0 p}}{2\pi(\alpha+k\cos\theta_0)}, \tag{5.71}$$

$$\mathbf{F}_+(\alpha)=\mathbf{\Psi}_+(\alpha)-\lambda\mathbf{\Psi}_-(-\alpha), \tag{5.72}$$

and $\lambda=\pm1$.

A more elaborative form of Eq. (5.70) is as follows

$$\begin{bmatrix}\cosh\varkappa(\alpha)F_+^{1*}(\alpha)-\sinh\varkappa(\alpha)F_+^{2*}(\alpha)/\gamma(\alpha)\\ -\gamma(\alpha)\sinh\varkappa(\alpha)F_+^{1*}(\alpha)+\cosh\varkappa(\alpha)F_+^{2*}(\alpha)\end{bmatrix}$$
$$+\frac{\lambda}{2\pi i}\int\limits_{-\infty+ia}^{\infty+ia}\frac{e^{i\xi(q-p)}}{(\xi+\alpha)}\begin{bmatrix}\cosh\varkappa(-\xi)F_+^{1*}(\xi)-\sinh\varkappa(-\xi)F_+^{2*}(\xi)/\gamma(\xi)\\ -\gamma(-\xi)\sinh\varkappa(-\xi)F_+^{1*}(\xi)+\cosh\varkappa(-\xi)F_+^{2*}(\xi)\end{bmatrix}d\xi$$
$$+\frac{e^{-ik\cos\theta_0 q}}{2\pi(\alpha-k\cos\theta_0)}\begin{bmatrix}A_1\cosh\varkappa(k\cos\theta_0)-A_2\sinh\varkappa(k\cos\theta_0)/\gamma(k\cos\theta_0)\\ -A_1\gamma(k\cos\theta_0)\sinh\varkappa(k\cos\theta_0)+A_2\cosh\varkappa(k\cos\theta_0)\end{bmatrix}=0. \tag{5.73}$$

The Eq. (5.71) in matrix form can be written as:

$$\begin{bmatrix}F_+^{1*}(\alpha)\\ F_+^{2*}(\alpha)\end{bmatrix}=\begin{bmatrix}F_+^{1}(\alpha)\\ F_+^{1}(\alpha)\end{bmatrix}-\frac{e^{-ik\cos\theta_0 q}}{2\pi(\alpha-k\cos\theta_0)}\begin{bmatrix}A_1\\ A_2\end{bmatrix}+\frac{\lambda e^{-ik\cos\theta_0 p}}{2\pi(\alpha+k\cos\theta_0)}\begin{bmatrix}A_1\\ A_2\end{bmatrix}. \tag{5.74}$$

Considering the first row of Eq. (5.73) and using the values of $F_+^{1*}(\alpha)$ and $F_+^{2*}(\alpha)$ in it, will give

$$\begin{aligned}
&\cosh\varkappa(\alpha)\left[F_+^1(\alpha) - \frac{e^{-ik\cos\theta_0 q}}{2\pi(\alpha - k\cos\theta_0)}A_1 + \frac{\lambda e^{-ik\cos\theta_0 p}}{2\pi(\alpha + k\cos\theta_0)}A_1\right] \\
&-\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}\left[F_+^2(\alpha) - \frac{e^{-ik\cos\theta_0 q}}{2\pi(\alpha - k\cos\theta_0)}A_2 + \frac{\lambda e^{-ik\cos\theta_0 p}}{2\pi(\alpha + k\cos\theta_0)}A_2\right] \\
&+\frac{\lambda}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{e^{i\xi(q-p)}}{(\xi+\alpha)}\left[\cosh\varkappa(-\xi)\left\{F_+^1(\xi) - \frac{e^{-ik\cos\theta_0 q}}{2\pi(\xi - k\cos\theta_0)}A_1 + \frac{\lambda e^{-ik\cos\theta_0 p}}{2\pi(\xi + k\cos\theta_0)}A_1\right\}\right. \\
&\left.-\sinh\varkappa(-\xi)/\gamma(\xi)\left\{F_+^2(\xi) - \frac{e^{-ik\cos\theta_0 q}}{2\pi(\xi - k\cos\theta_0)}A_2 + \frac{\lambda e^{-ik\cos\theta_0 p}}{2\pi(\xi + k\cos\theta_0)}A_2\right\}\right]d\xi \\
&+\frac{e^{-ik\cos\theta_0 q}}{2\pi(\alpha - k\cos\theta_0)}\left[A_1\cosh\varkappa(k\cos\theta_0) - A_2\sinh\varkappa(k\cos\theta_0)/\gamma(k\cos\theta_0)\right] = 0. \quad (5.75)
\end{aligned}$$

Writing $\gamma(\xi) = \gamma_+(\xi)\gamma_-(\xi) = \sqrt{\xi+k}\sqrt{\xi-k}$ and considering the integral arising in Eq. (5.75), one has

$$I = I_1 - \frac{e^{-ik\cos\theta_0 q}A_1}{2\pi}I_2 + \frac{\lambda e^{-ik\cos\theta_0 p}A_1}{2\pi}I_3 - I_4 + \frac{e^{-ik\cos\theta_0 q}A_2}{2\pi}I_5 + \frac{e^{-ik\cos\theta_0 p}A_2}{2\pi}I_6, \quad (5.76)$$

where

$$I_1 = \int_{-\infty+ia}^{\infty+ia}\frac{e^{i\xi(q-p)}\cosh\varkappa(-\xi)F_+^1(\xi)}{(\xi+\alpha)}d\xi, \quad (5.77)$$

$$I_2 = \int_{-\infty+ia}^{\infty+ia}\frac{e^{i\xi(q-p)}\cosh\varkappa(-\xi)}{(\xi+\alpha)(\xi - k\cos\theta_0)}d\xi, \quad (5.78)$$

$$I_3 = \int_{-\infty+ia}^{\infty+ia}\frac{e^{i\xi(q-p)}\cosh\varkappa(-\xi)}{(\xi+\alpha)(\xi + k\cos\theta_0)}d\xi, \quad (5.79)$$

$$I_4 = \int_{-\infty+ia}^{\infty+ia}\frac{e^{i\xi(q-p)}F_+^2(\xi)\sinh\varkappa(-\xi)/\sqrt{\xi+k}}{(\xi+\alpha)\sqrt{\xi-k}}d\xi, \quad (5.80)$$

$$I_5 = \int_{-\infty+ia}^{\infty+ia} \frac{e^{i\xi(q-p)} \sinh \varkappa(-\xi) / \sqrt{\xi+k}}{(\xi - k\cos\theta_0)(\xi+\alpha)\sqrt{\xi-k}} d\xi \tag{5.81}$$

and

$$I_6 = \int_{-\infty+ia}^{\infty+ia} \frac{e^{i\xi(q-p)} \sinh \varkappa(-\xi) / \sqrt{\xi+k}}{(\xi + k\cos\theta_0)(\xi+\alpha)\sqrt{\xi-k}} d\xi. \tag{5.82}$$

Integrals (5.77 – 5.82) are solved by a method described in [14] and are substituted in Eq. (5.75) to get

$$\begin{aligned}
&\cosh \varkappa(\alpha)\,[F_+^1(\alpha) - \frac{e^{-ik\cos\theta_0 q}}{2\pi(\alpha - k\cos\theta_0)} A_1 + \frac{\lambda e^{-ik\cos\theta_0 p}}{2\pi(\alpha + k\cos\theta_0)} A_1] \\
&-\frac{\sinh \varkappa(\alpha)}{\gamma(\alpha)}[F_+^2(\alpha) - \frac{e^{-ik\cos\theta_0 q}}{2\pi(\alpha - k\cos\theta_0)} A_2 + \frac{\lambda e^{-ik\cos\theta_0 p}}{2\pi(\alpha + k\cos\theta_0)} A_2] = \\
&-\lambda T(\alpha) F_+^1(k) + \lambda \frac{e^{-ik\cos\theta_0 q}}{2\pi} A_1 \{ \frac{e^{ikl\cos\theta_0}}{(\alpha + k\cos\theta_0)} \cosh \varkappa(-k\cos\theta_0) + R_2(\alpha)\} \\
&-A_1 \frac{e^{-ik\cos\theta_0 p}}{2\pi} R_1(\alpha) + \lambda T_1(\alpha) F_+^2(k) - \lambda \frac{e^{-ik\cos\theta_0 q}}{2\pi} A_2 \\
&\times \{ \frac{e^{ikl\cos\theta_0} \sinh \varkappa(-k\cos\theta_0)/\gamma_+(k\cos\theta_0)}{(\alpha + k\cos\theta_0)\gamma_-(k\cos\theta_0)} + R_4(\alpha)\} + A_2 \frac{e^{-ik\cos\theta_0 p}}{2\pi} R_3(\alpha) \\
&-\frac{e^{-ik\cos\theta_0 q}}{2\pi(\alpha - k\cos\theta_0)} [A_1 \cosh \varkappa(k\cos\theta_0) - A_2 \sinh \varkappa(k\cos\theta_0)/\gamma(k\cos\theta_0)].
\end{aligned}$$

(5.83)

Similarly considering the second row of Eq. (5.73), solving the integrals appearing in it and simplifying will yield

$$
\begin{aligned}
&-\gamma(\alpha)\sinh\varkappa(\alpha)\,[F_+^1(\alpha)-\frac{e^{-ik\cos\theta_0 q}}{2\pi(\alpha-k\cos\theta_0)}A_1+\frac{\lambda e^{-ik\cos\theta_0 p}}{2\pi(\alpha+k\cos\theta_0)}A_1]\\
&+\cosh\varkappa(\alpha)\,[F_+^2(\alpha)-\frac{e^{-ik\cos\theta_0 q}}{2\pi(\alpha-k\cos\theta_0)}A_2+\frac{\lambda e^{-ik\cos\theta_0 p}}{2\pi(\alpha+k\cos\theta_0)}A_2]\\
&=-\lambda T_2(\alpha)F_+^1(k)-\lambda\frac{e^{-ik\cos\theta_0 q}}{2\pi}A_1\{\frac{e^{ikl\cos\theta_0}\sinh\varkappa(-k\cos\theta_0)\,\gamma(-k\cos\theta_0)}{(\alpha+k\cos\theta_0)}+R_6(\alpha)\}\\
&+A_1\frac{e^{-ik\cos\theta_0 p}}{2\pi}R_5(\alpha)-\lambda T(\alpha)F_+^2(k)+\lambda\frac{e^{-ik\cos\theta_0 q}}{2\pi}A_2\\
&\times\{\frac{e^{ikl\cos\theta_0}\cosh\varkappa(-k\cos\theta_0)}{(\alpha+k\cos\theta_0)}+R_2(\alpha)\}-A_2\frac{e^{-ik\cos\theta_0 p}}{2\pi}R_1(\alpha)\\
&+\frac{e^{-ik\cos\theta_0 q}}{2\pi(\alpha-k\cos\theta_0)}\left[A_1\sinh\varkappa(k\cos\theta_0)\,\gamma(k\cos\theta_0)-A_2\cosh\varkappa(k\cos\theta_0)\right],
\end{aligned}
\tag{5.84}
$$

where $l=q-p$,

$$
T_1(\alpha)=\frac{1}{2\pi i}\frac{E_{-1}W_{-1}\{-i(k+\alpha)\,l\}\sinh\varkappa(-\alpha)}{\sqrt{\alpha+k}}, \tag{5.85}
$$

$$
R_{3,4}(\alpha)=\frac{D_{-1}\left[W_{-1}\{-i(k\pm k\cos\theta_0)\,l\}-W_{-1}\{-i(k+\alpha)\,l\}\right]}{2\pi i(\alpha\mp k\cos\theta_0)}, \tag{5.86}
$$

$$
D_{-1}=\frac{E_{-1}\sinh\varkappa(-k)}{\gamma(k)}. \tag{5.87}
$$

Now at this stage full simplification details of the Eq. (5.83) are presented and hence Eq. (5.84) can be simplified exactly on these same lines. Therefore, Eq. (5.83) can be simplified to achieve

$$
\begin{aligned}
&\cosh\varkappa(\alpha)\,F_+^1(\alpha)-\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}F_+^2(\alpha)=-\lambda T(\alpha)F_+^1(k)+\lambda T_1(\alpha)F_+^2(k)\\
&+A_1\frac{e^{-ik\cos\theta_0 q}}{2\pi}P_1(\alpha)-\lambda A_1\frac{e^{-ik\cos\theta_0 p}}{2\pi}P_2(\alpha)-A_2\frac{e^{-ik\cos\theta_0 q}}{2\pi}P_3(\alpha)+\lambda A_2\frac{e^{-ik\cos\theta_0 p}}{2\pi}P_4(\alpha)\\
&+\lambda A_1\frac{e^{-ik\cos\theta_0 q}}{2\pi}R_2(\alpha)-A_1\frac{e^{-ik\cos\theta_0 p}}{2\pi}R_1(\alpha)-\lambda A_2\frac{e^{-ik\cos\theta_0 q}}{2\pi}R_4(\alpha)+A_2\frac{e^{-ik\cos\theta_0 p}}{2\pi}R_3(\alpha),
\end{aligned}
\tag{5.88}
$$

where

$$P_{3,4}(\alpha) = \frac{\sinh\varkappa(\alpha)/\gamma(\alpha) - \sinh\varkappa(\pm k\cos\theta_0)/\gamma(\pm k\cos\theta_0)}{\alpha \mp k\cos\theta_0}. \tag{5.89}$$

Further simplification of Eq. (5.88) will yield

$$\begin{aligned}
&\cosh\varkappa(\alpha)\,F_+^1(\alpha) - \frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}F_+^2(\alpha) = -\lambda T(\alpha)F_+^1(k) + \lambda T_1(\alpha)F_+^2(k) \\
&+\frac{A_1}{2\pi}\{e^{-ik\cos\theta_0 q}P_1(\alpha) - e^{-ik\cos\theta_0 p}R_1(\alpha)\} - \frac{\lambda A_1}{2\pi}\{e^{-ik\cos\theta_0 p}P_2(\alpha) - e^{-ik\cos\theta_0 q}R_2(\alpha)\} \\
&-\frac{A_2}{2\pi}\{e^{-ik\cos\theta_0 q}P_3(\alpha) - e^{-ik\cos\theta_0 p}R_3(\alpha)\} + \frac{\lambda A_2}{2\pi}\{e^{-ik\cos\theta_0 p}P_4(\alpha) - e^{-ik\cos\theta_0 q}R_4(\alpha)\}.
\end{aligned} \tag{5.90}$$

Letting

$$G_3(\alpha) = e^{-ik\cos\theta_0 q}P_3(\alpha) - e^{-ik\cos\theta_0 p}R_3(\alpha), \tag{5.91}$$

$$G_4(\alpha) = e^{-ik\cos\theta_0 p}P_4(\alpha) - e^{-ik\cos\theta_0 q}R_4(\alpha), \tag{5.92}$$

in Eq. (5.90), the solution of the first W-H equation, obtained by considering the first row of matrices in Eq. (5.73), is given as follows:

$$\begin{aligned}
&\cosh\varkappa(\alpha)\,F_+^1(\alpha) - \frac{\sinh\varkappa(\alpha)\,F_+^2(\alpha)}{\gamma(\alpha)} = \\
&-\lambda T(\alpha)\cosh\varkappa(-k)\,F_+^1(k) + \lambda\frac{T_1(\alpha)\sinh\varkappa(-k)\,F_+^2(k)}{\sqrt{2k}} \\
&+\frac{A_1}{2\pi}\left[G_1(\alpha) - \lambda G_2(\alpha)\right] - \frac{A_2}{2\pi}\left[G_3(\alpha) - \lambda G_4(\alpha)\right].
\end{aligned} \tag{5.93}$$

The second W-H equation corresponds to the second row of the matrix Eq. (5.73) and its solution can be obtained in a similar manner as for the first row of the Eq.

(5.73). Omitting the details, the solution of the second W-H equation is given by

$$
\begin{aligned}
&-\gamma(\alpha)\sinh\varkappa(\alpha)\,F_{+}^{1}(\alpha)+\cosh\varkappa(\alpha)\,F_{+}^{2}(\alpha)= \\
&\lambda T_2(\alpha)\sqrt{2k}\sinh\varkappa(-k)\,F_{+}^{1}(k)-\lambda T(\alpha)\cosh\varkappa(-k)\,F_{+}^{2}(k) \\
&+\frac{A_2}{2\pi}\left[G_1(\alpha)-\lambda G_2(\alpha)\right]-\frac{A_1}{2\pi}\left[G_5(\alpha)-\lambda G_6(\alpha)\right],
\end{aligned}
\tag{5.94}
$$

where the various quantities used in the above equation are as follows

$$
\begin{aligned}
&T_2(\alpha)=\frac{1}{2\pi i}E_0W_0\{-i(k+\alpha)l\}\sqrt{\alpha+k}\sinh\varkappa(-\alpha), \\
&R_{5,6}(\alpha)=\frac{D_0\left[W_0\{-i(k\pm k\cos\theta_0)l\}-W_0\{-i(k+\alpha)l\}\right]}{2\pi i(\alpha\mp k\cos\theta_0)}, \\
&P_{5,6}(\alpha)=\frac{\gamma(\alpha)\sinh\varkappa(\alpha)-\gamma(\pm k\cos\theta_0)\sinh\varkappa(\pm k\cos\theta_0)}{\alpha\mp k\cos\theta_0}, \\
&G_5(\alpha)=e^{-ik\cos\theta_0 q}P_5(\alpha)-e^{-ik\cos\theta_0 p}R_5(\alpha), \\
&G_6(\alpha)=e^{-ik\cos\theta_0 p}P_6(\alpha)-e^{-ik\cos\theta_0 q}R_6(\alpha), \\
&D_0=E_0\sqrt{2k}\sinh\varkappa(-k), \\
&E_r=2e^{\frac{i\pi}{4}+ikl}l^{-r-\frac{1}{2}}i^{r}h_r,
\end{aligned}
\tag{5.95}
$$

and following are the common factors in both of the Eqs. (5.83) and (5.84), and are defined below

$$
\begin{aligned}
&T(\alpha)=\frac{1}{2\pi i}E_{-\frac{1}{2}}W_{-\frac{1}{2}}\{-i(k+\alpha)l\}\cosh\varkappa(-\alpha), \\
&R_{1,2}(\alpha)=\frac{\cosh\varkappa(-\alpha)\,E_{-\frac{1}{2}}\left[W_{-\frac{1}{2}}\{-i(k\pm k\cos\theta_0)l\}-W_{-\frac{1}{2}}\{-i(k+\alpha)l\}\right]}{2\pi i(\alpha\mp k\cos\theta_0)}, \\
&P_{1,2}(\alpha)=\frac{\cosh\varkappa(\alpha)-\cosh\varkappa(\pm k\cos\theta_0)}{\alpha\mp k\cos\theta_0}, \\
&G_1(\alpha)=e^{-ik\cos\theta_0 q}P_1(\alpha)-e^{-ik\cos\theta_0 p}R_1(\alpha), \\
&G_2(\alpha)=e^{-ik\cos\theta_0 p}P_2(\alpha)-e^{-ik\cos\theta_0 q}R_2(\alpha).
\end{aligned}
\tag{5.96}
$$

In Eqs. (5.85 − 5.87), (5.95) and (5.96) we have

$$\begin{aligned} W_{n-\frac{1}{2}}(z) &= \int_0^\infty \frac{u^n e^{-u}}{u+z} du = \\ &= \Gamma(n+1)\, e^{\frac{1}{2}z} z^{\frac{1}{2}n-\frac{1}{2}} W_{-\frac{1}{2}(n+1),\frac{1}{2}n}(z), \end{aligned} \tag{5.97}$$

where $z = -i(k+\alpha)l$ and $n = -\frac{1}{2}, 0, \frac{1}{2}$. $W_{m,n}$ is known as a Whittaker function [164, 165]. It is worthwhile to mention here that since the strip width is considered to be large as compared to the incident wavelength so the integrals appearing in the analysis have been approximated asymptotically in terms of Whittaker functions whereas for the small strip width as compared to the incident wavelength Whittaker functions can be replaced by the Fresnel Integrals [166]. By putting $\alpha = k$ in Eqs. (5.93) and (5.94) and solving these simultaneously, the values of the functions $F_+^1(k)$ and $F_+^2(k)$ are found to be

$$\begin{aligned} F_+^1(k) = \frac{A_1}{2\pi} &\left\{ \begin{array}{c} (\cosh\varkappa(k) + \lambda T(k)\cosh\varkappa(-k))\,(G_1(k) - \lambda G_2(k)) \\ -\left(\sinh\varkappa(k)/\gamma(k) + \lambda T_1(k)\sinh\varkappa(-k)/\sqrt{2k}\right)(G_5(k) - \lambda G_6(k)) \end{array} \right\} \\ +\frac{A_2}{2\pi} &\left\{ \begin{array}{c} \left(\sinh\varkappa(k)/\gamma(k) + \lambda T_1(k)\sinh\varkappa(-k)/\sqrt{2k}\right)(G_1(k) - \lambda G_2(k)) \\ -(\cosh\varkappa(k) + \lambda T(k)\cosh\varkappa(-k))\,(G_3(k) - \lambda G_4(k)) \end{array} \right\} \\ \times &\left\{ \frac{1}{\begin{array}{c} 1 + T^2(k)\cosh^2\varkappa(-k) + T_1(k)\,T_2(k)\sinh^2\varkappa(-k) \\ +2T(k)\,\lambda\cosh\varkappa(k)\cosh\varkappa(-k) + + \left(\frac{\lambda T_2(k)\sqrt{2k}}{\gamma(k)} - \frac{\lambda T_1(k)\gamma(k)}{\sqrt{2k}}\right)\sinh\varkappa(k)\sinh\varkappa(-k) \end{array}} \right\} \end{aligned}$$

(5.98)

and

$$F_+^2(k) = \frac{1}{\sinh\varkappa(k)/\gamma(k) + \lambda T_1(k)\sinh\varkappa(-k)/\sqrt{2k}}$$
$$\times\left\{(\cosh\varkappa(k) + \lambda T(k)\cosh\varkappa(-k))\,F_+^1(k)\right.$$
$$\left.+\frac{A_2}{2\pi}(G_3(k) - \lambda G_4(k)) - \frac{A_1}{2\pi}(G_1(k) - \lambda G_2(k))\right\}. \tag{5.99}$$

Now as

$$\mathbf{F}_+(\alpha) = \begin{bmatrix} F_+^1(\alpha) \\ F_+^2(\alpha) \end{bmatrix} = \begin{bmatrix} \overline{\psi}_{+1}(\alpha) \\ \overline{\psi}_{+2}(\alpha) \end{bmatrix} - \lambda \begin{bmatrix} \overline{\psi}_{-1}(\alpha) \\ \overline{\psi}_{-2}(\alpha) \end{bmatrix}. \tag{5.100}$$

Eq. (5.100) is considered for the cases of $\lambda = 1$ and $\lambda = -1$ and when the values of $F_+^1(\alpha)$ and $F_+^2(\alpha)$ are substituted in Eqs. (5.93) and (5.94) the results are as follows:

For $\lambda = 1$

$$\cosh\varkappa(\alpha)\left[\overline{\psi}_{+1}(\alpha) - \overline{\psi}_{-1}(\alpha)\right] - \frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}\left[\overline{\psi}_{+2}(\alpha) - \overline{\psi}_{-2}(\alpha)\right] =$$
$$-T(\alpha)\cosh\varkappa(-k)\,F_+^1(k)\big|_{\lambda=1} + \frac{T_1(\alpha)\sinh\varkappa(-k)}{\sqrt{2k}}\,F_+^2(k)\big|_{\lambda=1}$$
$$+\frac{A_1}{2\pi}\left[G_1(\alpha) - G_2(\alpha)\right] - \frac{A_2}{2\pi}\left[G_3(\alpha) - G_4(\alpha)\right] \tag{5.101}$$

and

$$-\gamma(\alpha)\sinh\varkappa(\alpha)\left[\overline{\psi}_{+1}(\alpha) - \overline{\psi}_{-1}(\alpha)\right] + \cosh\varkappa(\alpha)\left[\overline{\psi}_{+2}(\alpha) - \overline{\psi}_{-2}(\alpha)\right]$$
$$= -T(\alpha)\cosh\varkappa(-k)\,F_+^2(k)\big|_{\lambda=1} + \sqrt{2k}T_2(\alpha)\sinh\varkappa(-k)\,F_+^1(k)\big|_{\lambda=1}$$
$$+\frac{A_2}{2\pi}\left[G_1(\alpha) - G_2(\alpha)\right] - \frac{A_1}{2\pi}\left[G_5(\alpha) - G_6(\alpha)\right], \tag{5.102}$$

and for $\lambda = -1$

$$\begin{aligned}
&\cosh \varkappa(\alpha)\left[\overline{\psi}_{+1}(\alpha)+\overline{\psi}_{-1}(\alpha)\right]-\frac{\sinh \varkappa(\alpha)}{\gamma(\alpha)}\left[\overline{\psi}_{+2}(\alpha)+\overline{\psi}_{-2}(\alpha)\right]= \\
&+T(\alpha)\cosh \varkappa(-k)\left.F_{+}^{1}(k)\right|_{\lambda=-1}-\frac{T_1(\alpha)\sinh \varkappa(-k)}{\sqrt{2k}}\left.F_{+}^{2}(k)\right|_{\lambda=-1} \\
&+\frac{A_1}{2\pi}\left[G_1(\alpha)+G_2(\alpha)\right]-\frac{A_2}{2\pi}\left[G_3(\alpha)+G_4(\alpha)\right], \qquad (5.103)
\end{aligned}$$

and

$$\begin{aligned}
&-\gamma(\alpha)\sinh \varkappa(\alpha)\left[\overline{\psi}_{+1}(\alpha)+\overline{\psi}_{-1}(\alpha)\right]+\cosh \varkappa(\alpha)\left[\overline{\psi}_{+2}(\alpha)+\overline{\psi}_{-2}(\alpha)\right] \\
=\ & T(\alpha)\cosh \varkappa(-k)\left.F_{+}^{2}(k)\right|_{\lambda=-1}-T_2(\alpha)\sqrt{2k}\sinh \varkappa(-k)\left.F_{+}^{1}(k)\right|_{\lambda=-1} \\
&+\frac{A_2}{2\pi}\left[G_1(\alpha)+G_2(\alpha)\right]-\frac{A_1}{2\pi}\left[G_5(\alpha)+G_6(\alpha)\right]. \qquad (5.104)
\end{aligned}$$

Adding Eqs. (5.101) and (5.103), will result

$$\begin{aligned}
&\cosh \varkappa(\alpha)\overline{\psi}_{+1}(\alpha)-\frac{\sinh \varkappa(\alpha)}{\gamma(\alpha)}\overline{\psi}_{+2}(\alpha)=\frac{A_1}{2\pi}G_1(\alpha)-\frac{A_2}{2\pi}G_3(\alpha) \\
&-\frac{T(\alpha)\cosh \varkappa(-k)}{2}C_1+\frac{T_1(\alpha)\sinh \varkappa(-k)}{2\sqrt{2k}}C_2 \qquad (5.105)
\end{aligned}$$

and adding Eqs. (5.102) and (5.104) will yield

$$\begin{aligned}
&-\gamma(\alpha)\sinh \varkappa(\alpha)\overline{\psi}_{+1}(\alpha)+\cosh \varkappa(\alpha)\overline{\psi}_{+2}(\alpha)=\frac{A_2}{2\pi}G_1(\alpha)-\frac{A_1}{2\pi}G_5(\alpha) \\
&-\frac{T(\alpha)\cosh \varkappa(-k)}{2}C_2+\frac{T_2(\alpha)\sqrt{2k}\sinh \varkappa(-k)}{2}C_1, \qquad (5.106)
\end{aligned}$$

where

$$\begin{aligned}
C_1 &= \left.F_{+}^{1}(k)\right|_{\lambda=1}-\left.F_{+}^{1}(k)\right|_{\lambda=-1}, \\
C_2 &= \left.F_{+}^{2}(k)\right|_{\lambda=1}-\left.F_{+}^{2}(k)\right|_{\lambda=-1}. \qquad (5.107)
\end{aligned}$$

Eliminating $\overline{\psi}_{+2}(\alpha)$ from Eqs. (5.105) and (5.106) will yield

$$\overline{\psi}_{+1}(\alpha) = \left(\frac{A_1}{2\pi}\cosh\varkappa(\alpha) + \frac{A_2}{2\pi}\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}\right)G_1(\alpha) - \frac{A_2}{2\pi}G_3(\alpha)\cosh\varkappa(\alpha)$$
$$-\frac{A_1}{2\pi}G_5(\alpha)\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)} - \frac{T(\alpha)\cosh\varkappa(-k)}{2}\left(C_1\cosh\varkappa(\alpha) + C_2\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}\right)$$
$$+\frac{T_1(\alpha)\sinh\varkappa(-k)\cosh\varkappa(\alpha)C_2}{2\sqrt{2k}} + \frac{T_2(\alpha)\sqrt{2k}\sinh\varkappa(-k)\sinh\varkappa(\alpha)/\gamma(\alpha)C_1}{2}.$$

(5.108)

Now in order to calculate the function $\overline{\psi}_{-1}(\alpha)$, it is required to replace $G_1(\alpha)$ by $G_2(\alpha)$ (*and* $G_2(\alpha)$ *by* $G_1(\alpha)$), $G_3(\alpha)$ by $G_4(\alpha)$ (*and* $G_4(\alpha)$ *by* $G_3(\alpha)$) and $G_5(\alpha)$ by $G_6(\alpha)$ (*and* $G_6(\alpha)$ *by* $G_5(\alpha)$) and also changing α to $-\alpha$ in the Eq. (5.108), will result into

$$\overline{\psi}_{-1}(\alpha) = \left(\frac{A_1}{2\pi}\cosh\varkappa(-\alpha) + \frac{A_2}{2\pi}\frac{\sinh\varkappa(-\alpha)}{\gamma(-\alpha)}\right)G_2(-\alpha) - \frac{A_2}{2\pi}G_4(-\alpha)\cosh\varkappa(-\alpha)$$
$$-\frac{A_1}{2\pi}G_6(-\alpha)\frac{\sinh\varkappa(-\alpha)}{\gamma(-\alpha)} - \frac{T(-\alpha)\cosh\varkappa(-k)}{2}\left(\widetilde{C}_1\cosh\varkappa(-\alpha) + \widetilde{C}_2\frac{\sinh\varkappa(-\alpha)}{\gamma(-\alpha)}\right)$$
$$+\frac{T_1(-\alpha)\sinh\varkappa(-k)\cosh\varkappa(-\alpha)\widetilde{C}_2}{2\sqrt{2k}} + \frac{T_2(-\alpha)\sqrt{2k}\sinh\varkappa(-k)\sinh\varkappa(-\alpha)/\gamma(-\alpha)\widetilde{C}_1}{2},$$

(5.109)

where $\widetilde{C}_1$ and $\widetilde{C}_2$ are given as follows:

$$\begin{aligned}\widetilde{C}_1 &= \widetilde{F}^1_+(k)\Big|_{\lambda=1} - \widetilde{F}^1_+(k)\Big|_{\lambda=-1}, \\ \widetilde{C}_2 &= \widetilde{F}^2_+(k)\Big|_{\lambda=1} - \widetilde{F}^2_+(k)\Big|_{\lambda=-1},\end{aligned} \qquad (5.110)$$

where $\widetilde{F}^1_+(k)$ and $\widetilde{F}^2_+(k)$ denote the functions in which G_1 by G_2 and G_2 by G_1, G_3 by G_4 and G_4 by G_3 and G_5 by G_6 and G_6 by G_5 have also been interchanged and then evaluated for $\lambda = 1$ and $\lambda = -1$ respectively. Substituting Eqs. (5.24), (5.108) and (5.109) into Eq. (5.25) and simplifying, the unknown coefficient $A(\alpha)$ is determined to be:

$$
\begin{aligned}
A(\alpha) = & \left[\left\{ \begin{array}{c} -\frac{A_1}{2\pi}\cosh\varkappa(\alpha)\cosh\varkappa(k\cos\theta_0) - \frac{A_2}{2\pi}\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}\cosh\varkappa(k\cos\theta_0) \\ +\frac{A_2}{2\pi}\cosh\varkappa(\alpha)\frac{\sinh\varkappa(k\cos\theta_0)}{\gamma(k\cos\theta_0)} + \frac{A_1}{2\pi}\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}\gamma(k\cos\theta_0)\sinh\varkappa(k\cos\theta_0) \end{array} \right\} \right. \\
& \times \frac{e^{i(\alpha - k\cos\theta_0)q}}{\alpha - k\cos\theta_0} \\
& + \left\{ \begin{array}{l} -\frac{A_1}{2\pi}\cosh\varkappa(\alpha)R_1(\alpha) - \frac{A_2}{2\pi}\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}R_1(\alpha) \\ +\frac{A_2}{2\pi}\cosh\varkappa(\alpha)R_3(\alpha) + \frac{A_1}{2\pi}\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}R_5(\alpha) \end{array} \right\} e^{i\alpha q - ik\cos\theta_0 p} \\
& + \left\{ \begin{array}{c} -\frac{T(\alpha)\cosh\varkappa(-k)}{2}\left(C_1\cosh\varkappa(\alpha) + C_2\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}\right) \\ +\frac{T_1(\alpha)\sinh\varkappa(-k)\cosh\varkappa(\alpha)C_2}{2\sqrt{2k}} + \frac{T_2(\alpha)\sqrt{2k}\sinh\varkappa(-k)\sinh\varkappa(\alpha)/\gamma(\alpha)C_1}{2} \end{array} \right\} e^{i\alpha q} \\
& + \left\{ \begin{array}{l} -\frac{A_1}{2\pi}\cosh\varkappa(-\alpha)R_2(-\alpha) - \frac{A_2}{2\pi}\frac{\sinh\varkappa(-\alpha)}{\gamma(-\alpha)}R_2(-\alpha) \\ +\frac{A_2}{2\pi}\cosh\varkappa(-\alpha)R_4(-\alpha) + \frac{A_1}{2\pi}\frac{\sinh\varkappa(-\alpha)}{\gamma(-\alpha)}R_6(-\alpha) \end{array} \right\} e^{i\alpha p - ik\cos\theta_0 q} \\
& + \left\{ \begin{array}{c} -\frac{T(-\alpha)\cosh\varkappa(-k)}{2}\left(\widetilde{C}_1\cosh\varkappa(-\alpha) + \widetilde{C}_2\frac{\sinh\varkappa(-\alpha)}{\gamma(-\alpha)}\right) \\ +\frac{T_1(-\alpha)\sinh\varkappa(-k)\cosh\varkappa(-\alpha)\widetilde{C}_2}{2\sqrt{2k}} + \frac{T_2(-\alpha)\sqrt{2k}\sinh\varkappa(-k)\sinh\varkappa(-\alpha)/\gamma(-\alpha)\widetilde{C}_1}{2} \end{array} \right\} e^{i\alpha p} \\
& + \left\{ \begin{array}{c} \frac{A_1}{2\pi}\cosh\varkappa(-\alpha)\cosh\varkappa(-k\cos\theta_0) + \frac{A_2}{2\pi}\frac{\sinh\varkappa(-\alpha)}{\gamma(-\alpha)}\cosh\varkappa(-k\cos\theta_0) \\ -\frac{A_2}{2\pi}\cosh\varkappa(-\alpha)\frac{\sinh\varkappa(-k\cos\theta_0)}{\gamma(-k\cos\theta_0)} - \frac{A_1}{2\pi}\frac{\sinh\varkappa(-\alpha)}{\gamma(-\alpha)}\gamma(-k\cos\theta_0)\sinh\varkappa(-k\cos\theta_0) \end{array} \right\} \\
& \left. \times \frac{e^{i(\alpha - k\cos\theta_0)p}}{\alpha - k\cos\theta_0} \right]. \qquad (5.111)
\end{aligned}
$$

Since $A(\alpha)$ has been determined, the scattered field $\psi(x, y)$ can now be determined by substituting $A(\alpha)$ into Eq. (5.19) and taking the inverse Fourier transform as follows:

$$\psi(x,y) = \int_{-\infty}^{\infty} A(\alpha) e^{iK(\alpha)y - i\alpha x} d\alpha, \tag{5.112}$$

where $A(\alpha)$ is defined in Eq. (5.111). The scattered field $\psi(x,y)$ can be split up into two components as follows:

$$\psi(x,y) = \psi_{sep}(x,y) + \psi_{int}(x,y), \tag{5.113}$$

where

$$\begin{aligned}
\psi_{sep}(x,y) = \int_{-\infty}^{\infty} \Bigg[\Bigg\{ &-\frac{A_1}{2\pi} \cosh \varkappa(\alpha) \cosh \varkappa(k\cos\theta_0) - \frac{A_2}{2\pi} \frac{\sinh \varkappa(\alpha)}{\gamma(\alpha)} \cosh \varkappa(k\cos\theta_0) \\
&+ \frac{A_2}{2\pi} \cosh \varkappa(\alpha) \frac{\sinh \varkappa(k\cos\theta_0)}{\gamma(k\cos\theta_0)} + \frac{A_1}{2\pi} \frac{\sinh \varkappa(\alpha)}{\gamma(\alpha)} \gamma(k\cos\theta_0) \sinh \varkappa(k\cos\theta_0) \Bigg\} \\
&\times \frac{e^{i(\alpha - k\cos\theta_0)q}}{\alpha - k\cos\theta_0} + \Bigg\{ \frac{A_1}{2\pi} \cosh \varkappa(-\alpha) \cosh \varkappa(-k\cos\theta_0) \\
&+ \frac{A_2}{2\pi} \frac{\sinh \varkappa(-\alpha)}{\gamma(-\alpha)} \cosh \varkappa(-k\cos\theta_0) - \frac{A_2}{2\pi} \cosh \varkappa(-\alpha) \frac{\sinh \varkappa(-k\cos\theta_0)}{\gamma(-k\cos\theta_0)} \\
&- \frac{A_1}{2\pi} \frac{\sinh \varkappa(-\alpha)}{\gamma(-\alpha)} \gamma(-k\cos\theta_0) \sinh \varkappa(-k\cos\theta_0) \Bigg\} \frac{e^{i(\alpha - k\cos\theta_0)p}}{\alpha - k\cos\theta_0} \Bigg] e^{iK(\alpha)y - i\alpha x} d\alpha
\end{aligned} \tag{5.114}$$

and

$$\psi_{int}(x,y) = \int\limits_{-\infty}^{\infty} \left[\left\{ \begin{array}{c} -\frac{A_1}{2\pi}\cosh\varkappa(\alpha)\,R_1(\alpha) - \frac{A_2}{2\pi}\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}R_1(\alpha) + \\ \frac{A_2}{2\pi}\cosh\varkappa(\alpha)\,R_3(\alpha) + \frac{A_1}{2\pi}\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}R_5(\alpha) \end{array} \right\} e^{i\alpha q - ik\cos\theta_0 p} + \right.$$
$$\left\{ \begin{array}{c} -\frac{T(\alpha)\cosh\varkappa(-k)}{2}\left(C_1\cosh\varkappa(\alpha) + C_2\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}\right) + \\ \frac{T_1(\alpha)\sinh\varkappa(-k)\cosh\varkappa(\alpha)C_2}{2\sqrt{2k}} + \frac{T_2(\alpha)\sqrt{2k}\sinh\varkappa(-k)\sinh\varkappa(\alpha)/\gamma(\alpha)C_1}{2} \end{array} \right\} e^{i\alpha q} +$$
$$\left\{ \begin{array}{c} -\frac{A_1}{2\pi}\cosh\varkappa(-\alpha)\,R_2(-\alpha) - \frac{A_2}{2\pi}\frac{\sinh\varkappa(-\alpha)}{\gamma(-\alpha)}R_2(-\alpha) + \\ \frac{A_2}{2\pi}\cosh\varkappa(-\alpha)\,R_4(-\alpha) + \frac{A_1}{2\pi}\frac{\sinh\varkappa(-\alpha)}{\gamma(-\alpha)}R_6(-\alpha) \end{array} \right\} e^{i\alpha p - ik\cos\theta_0 q}$$
$$\left. \left\{ \begin{array}{c} -\frac{T(-\alpha)\cosh\varkappa(-k)}{2}\left(\widetilde{C}_1\cosh\varkappa(-\alpha) + \widetilde{C}_2\frac{\sinh\varkappa(-\alpha)}{\gamma(-\alpha)}\right) + \\ \frac{T_1(-\alpha)\sinh\varkappa(-k)\cosh\varkappa(-\alpha)\widetilde{C}_2}{2\sqrt{2k}} + \frac{T_2(-\alpha)\sqrt{2k}\sinh\varkappa(-k)\sinh\varkappa(-\alpha)/\gamma(-\alpha)\widetilde{C}_1}{2} \end{array} \right\} e^{i\alpha p} \right] e^{iK(\alpha)y - i\alpha x} d\alpha, \tag{5.115}$$

where $\psi_{sep}(x,y)$ gives the diffracted field produced by the edges at $x = p$ and at $x = q$ respectively and $\psi_{int}(x,y)$ gives the interaction of one edge upon the other edge.

5.3 Determination of the diffracted field

The calculations carried out for the three part boundary value problem formulated in terms of matrix W-H equations are quite laborious and delicate at the same time, so the far field is reported only for the case of $y > 0$. The far field for the case of $y < 0$ can be calculated in a similar manner.

In order to solve the integral appearing in Eq. (5.112) the following substitutions

$$x = \rho\cos\theta,\ y = \rho\sin\theta \text{ and } \alpha = -k\cos(\theta + it_1), \tag{5.116}$$

have been introduced into Eq. (5.112). When the length of soft-hard strip is large the main contribution to the integral in Eq. (5.112) comes from the saddle point $t_1 = 0$ with $A(-k(\cos\theta + it_1))$ is slowly varying around $t_1 = 0$ and also the quantity $(\cos\theta_0 + \cos\theta)$ is different from zero as $k\rho \to \infty$ and thus using the method of steepest descent [69], the field at the large distance from a soft-hard finite plate is given as

$$\psi(x, y) \simeq \sqrt{\frac{2\pi}{k\rho}} i \sin\theta \; A(-k\cos\theta) e^{ik\rho + i\frac{\pi}{4}}, \tag{5.117}$$

where $A(-k\cos\theta)$ can be evaluated from Eq. (5.111). The effect of surface waves can also be neglected since the far field approximation has been used in determining the diffracted field [130, 138].

5.4 Graphical results

In this section, some graphs showing the effects of sundry dimensionless parameters such as $k\rho$ and kd, where $k\rho$ is the observer distance from origin and kd is the strip width on the diffracted field produced by the two edges of the finite soft-hard plate are presented.

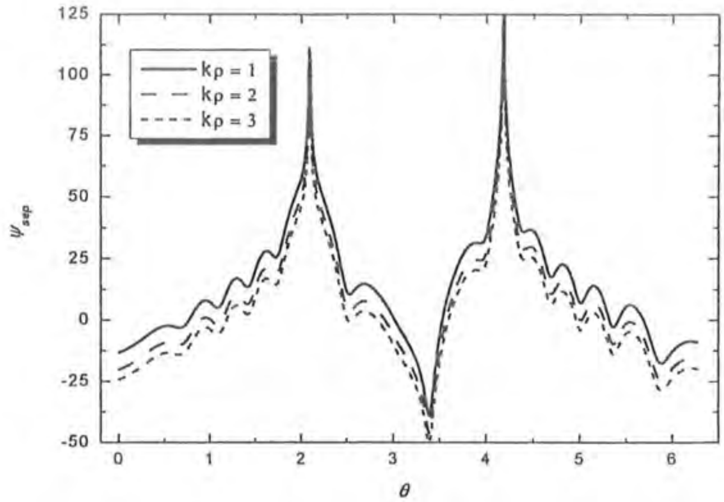

Fig. 5.2 Variation of sep. field ψ_{sep} Vs θ at $\theta_0 = \frac{\pi}{2}$ and $kd = 10$.

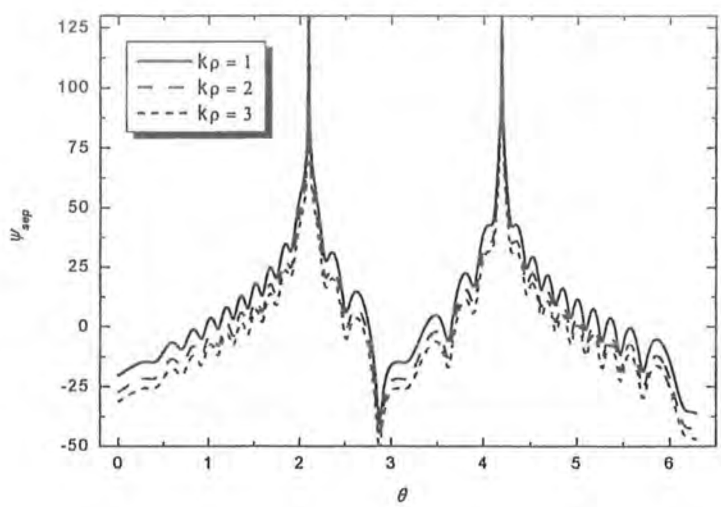

Fig. 5.3 Variation of sep. field ψ_{sep} Vs θ at $\theta_0 = \frac{\pi}{2}$ and $kd = 20$.

- Figs. 5.2 and 5.3 show the variation of separated field ψ_{sep} with observation angle θ at $\theta_0 = \pi/3$ and $kd = 10$, 20 for $k\rho = 1$, 2 and 3 respectively. It can be seen that increasing the parameter $k\rho$ causes more oscillations and the overall amplitude of the separated field ψ_{sep} decreases.

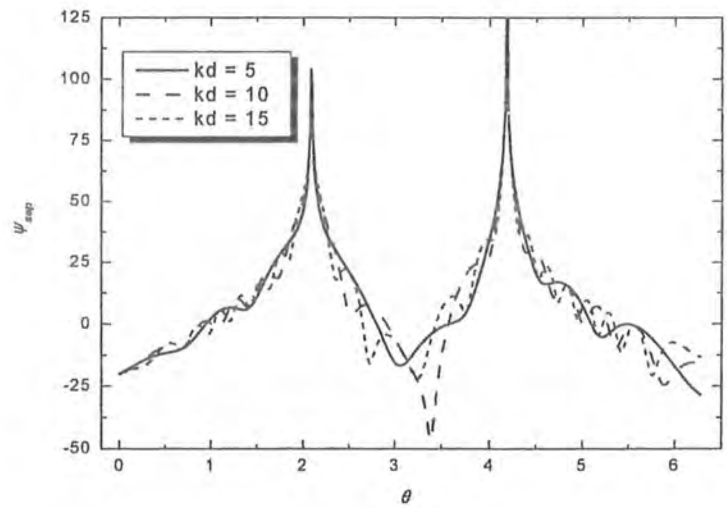

Fig. 5.4 Variation of sep. field ψ_{sep} Vs θ at $\theta_0 = \frac{\pi}{2}$ and $k\rho = 2$.

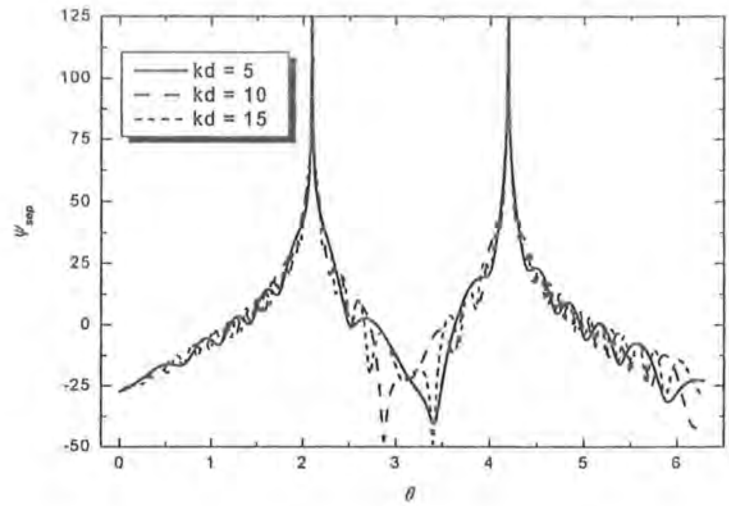

Fig. 5.5 Variation of sep. field ψ_{sep} Vs θ at $\theta_0 = \frac{\pi}{2}$ and $k\rho = 4$.

- Figs. 5.4 and 5.5 depict the separated far-field patterns ψ_{sep} with observation angle θ at $\theta_0 = \pi/3$ and $k\rho = 2, 4$ for $kd = 5$, 10 and 15 respectively. It is observed that increasing strip width parameter kd the diffraction patterns are approximately overlapped throughout most of the graph and the middle lobe in Fig. 5.5 is more steep than the middle lobe in Fig. 5.4.

5.5 Concluding remarks

In this chapter, the diffraction of a plane acoustic wave by a soft hard strip is investigated rigorously with the help of integral transform, W-H technique and the method of steepest descent. The main findings are summarized as below:

- The coupled W-H equations lead to a matrix W-H equation which is solved after using the factorization of the kernel matrix. The kernel matrix has been factorized by using the Daniele-Kharapkov methods [20, 21].

- The two edges of the soft-hard strip give rise to two diffracted fields (one from each edge), i.e., the separated field and the interaction of one edge upon the other edge, i.e., the interaction field.

- The diffracted field is presented for the far-field situation and some graphs showing the effects of various parameters on the separated diffracted field are presented and discussed.

- The soft-hard strip gives better attenuation results for the separated and interactive diffracted fields as compared to a completely rigid strip as Rawlins [34] also observed and pointed out this fact that the half plane soft on one side and hard on the other side gives better attenuation results than a completely rigid semi-infinite plane for singly diffracted fields.

- It is believed that results presented in this chapter have not been reported so far based on the W-H technique and avoids the relatively cumbersome apparatus of integral equations. This route of solution of diffraction problem is more rigorous and involves tedious mathematical calculations.

- Further the consideration of soft-hard strip will help understand acoustic diffraction and will go a step further to complete the discussion for the soft-hard half plane.

Chapter 6

Diffraction Of Plane Waves By A Slit In An Infinite Soft-Hard Plane

Diffraction of plane acoustic/electromagnetic waves by a slit configuration has received wide attention because of its importance in microwave and optical instrumentation. Also guiding structures containing thick slits or slots e.g., microwave passive filters, coupling structures have interesting reflection and transmission properties [166]. The investigations pursued in the present chapter are based on an integral transform, the W-H technique and an asymptotic method. As mentioned in chapters 1 and 5 that a large number of analytical, numerical and combination of both analytical and numerical methods are available for the solution of diffraction problems corresponding to strip/slit geometry. To name a few only e.g., Bowman et al. [76] reviewed and summarized much of the work done on slit geometry. Clemmow [115] addressed the problem of diffraction of H-polarized plane wave by a wide slit and a normally incident E-polarized plane wave by a narrow slit by using the method of plane wave spectrum representation and Achenbach [167] investigated diffraction of a plane horizontally polarized shear wave and a plane longitudinal wave by a semi-infinite slit by employing integral transforms with the W-H technique and the Cagniard de-Hoop method.

The problem of diffraction of acoustic waves by a slit in an infinite soft-hard plane has been analyzed by the elegant analytical W-H technique which impresses all those who use it and have applications in almost all branches of modern sciences [15]. To the best of authors' knowledge the problem of diffraction of acoustic waves by a slit in an infinite soft-hard plane has not been discussed previously, so it seems to be first and worthwhile attempt to address the above said boundary value problem. By using the Fourier transform technique the problem is reduced to a matrix W-H functional equation. Noble [14] and Jones [35] addressed the problems of diffraction of waves by a slit using the W-H method. Their approach has been followed very closely. This boundary value problem resulted in a peculiar W-H functional equation of the form

$$e^{i\alpha q}\Psi_{+}(\alpha) + \mathbf{H}(\alpha)\,\mathbf{Q}(\alpha) + e^{i\alpha p}\Psi_{-}(\alpha) = G(\alpha)\mathbf{A},$$

where the various quantities appearing in the above equation will be defined as the analysis is pursued further and when the slit width is larger than the incident wave length then the physical concept of the geometrical theory of diffraction (GTD) [78, 163] can be used to obtain a high frequency approximate solution of the corresponding boundary value problem. Also in working out solution of the problem of diffraction of plane acoustic waves from a slit aperture in an infinite soft-hard expanse, several integrals in the analysis have been approximated under the assumption of GTD. The approach followed in the present chapter has been used by many researchers in the literature, e.g., Birbir and Büyükaksoy [120], Kashyap and Hamid [166], Asghar et al [121, 122], Hayat et al [124, 125], Ayub et al [126, 127] and more recently by Cinar and Büyükaksoy [128].

The diffracted field obtained is shown to be the sum of the wave-fields produced by the two edges of the slit (separated field) and by the interaction of one edge upon the other edge (interaction field). Several graphs illustrating the effects of various parameters on the separated diffracted field are also plotted and discussed.

6.1 Mathematical formulation of the problem

Let (x, y, z) define the cartesian coordinate system with respect to the origin O. Consider the diffraction of a plane acoustic wave by a slit occupying the position $\{p \leq x \leq q,\ y = 0,\ z \in (-\infty, \infty)\}$. The positions of the soft-hard planes located on both sides of the slit are given by $\{-\infty < x \leq p,\ y = 0,\ z \in (-\infty, \infty)\}$ and $\{q \leq x < \infty,\ y = 0,\ z \in (-\infty, \infty)\}$, respectively and these are assumed to have vanishing thicknesses. A time factor of the type $e^{-i\omega t}$ is assumed and suppressed throughout the calculations. The geometry of the problem is shown in Figure 6.1.

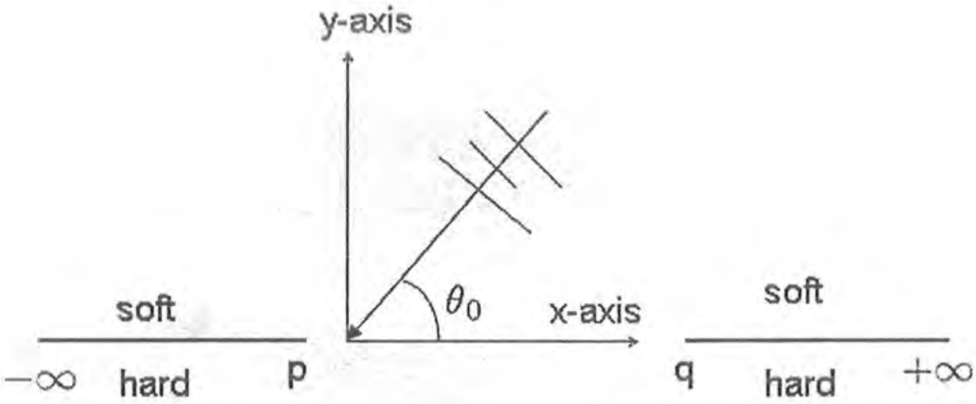

Fig. 6.1 Geometry of the slit problem

For harmonic acoustic vibrations of time dependence $e^{-i\omega t}$, the following Helmholtz's equation has to be solved

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right) \psi_t(x, y) = 0, \tag{6.1}$$

where ψ_t is the total velocity potential and the boundary and continuity conditions are given by

$$\psi_t\left(x, 0^+\right) = 0, \quad \text{on} \quad \begin{cases} -\infty < x \leq p \\ q \leq x < \infty, \end{cases} \tag{6.2}$$

$$\frac{\partial \psi_t\left(x, 0^-\right)}{\partial y} = 0, \text{ on} \quad \begin{cases} -\infty < x \leq p \\ q \leq x < \infty, \end{cases} \tag{6.3}$$

and

$$\psi_t\left(x, 0^+\right) = \psi_t\left(x, 0^-\right), \quad \text{on} \quad p < x < q, \tag{6.4}$$

$$\frac{\partial \psi_t\left(x, 0^+\right)}{\partial y} = \frac{\partial \psi_t\left(x, 0^-\right)}{\partial y}, \quad \text{on} \quad p < x < q. \tag{6.5}$$

In Eqs. $(6.2-6.5)$, the quantity $0^{\pm}$ refers to the situation that $y \rightarrow 0$ through positive or negative $y-axis$.

Let a plane acoustic wave

$$\psi_i = e^{-ik(x\cos\theta_0 + y\sin\theta_0)}, \tag{6.6}$$

be incident upon the slit occupying the position $p \leq x \leq q$, $y=0$. In Eq. (6.6), θ_0 is the angle of incidence and for the analytic convenience it is assumed that the wave number k has positive imaginary part. For the analysis purpose it is convenient to express the total field ψ_t as

$$\psi_t = \begin{cases} \psi_i + \psi_r + \psi & y > 0 \\ \psi & y < 0, \end{cases} \tag{6.7}$$

where ψ is the diffracted field and ψ_r is the reflected field from the soft surface and is given by

$$\psi_r = -e^{-ik(x\cos\theta_0 - y\sin\theta_0)},$$

[14]. For a unique solution of the problem, the edge conditions require that ψ_t and its normal derivative must be bounded and satisfy [33, 34]

$$\psi_t(x,0) = \begin{cases} -1 + O(x-p)^{\frac{1}{4}} & \text{as } x \rightarrow p^-, \\ -1 + O(x-q)^{\frac{1}{4}} & \text{as } x \rightarrow q^+, \end{cases} \tag{6.8}$$

$$\frac{\partial \psi_t(x,0)}{\partial y} = \begin{cases} O(x-p)^{-\frac{3}{4}} & \text{as } x \rightarrow p^-, \\ O(x-q)^{-\frac{3}{4}} & \text{as } x \rightarrow q^+, \end{cases} \tag{6.9}$$

where a negative sign indicates a limit taken from left and a positive sign indicates that a limit taken from right of the points p and q on the $x-axis$ [121, 125]. Thus, the scattered field satisfies the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)\psi(x,y)=0, \tag{6.10}$$

subject to the boundary conditions

$$\psi\left(x,0^+\right)=0 \quad \text{on} \quad \begin{cases} -\infty<x<p \\ q<x<\infty, \end{cases} \tag{6.11}$$

and

$$\frac{\partial\psi\left(x,0^-\right)}{\partial y}=0 \quad \text{on} \quad \begin{cases} -\infty<x<p \\ q<x<\infty, \end{cases} \tag{6.12}$$

and the continuity conditions

$$\psi\left(x,0^+\right)-\psi\left(x,0^-\right)=0 \quad \text{on} \quad p\leq x\leq q \tag{6.13}$$

and

$$\frac{\partial\psi\left(x,0^+\right)}{\partial y}-\frac{\partial\psi\left(x,0^-\right)}{\partial y}=2ik\sin\theta_0 e^{-ikx\cos\theta_0} \quad \text{on } p\leq x\leq q. \tag{6.14}$$

The Fourier transform pair is defined as follows

$$\begin{aligned}\overline{\psi}(\alpha,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(x,y)\,e^{i\alpha x}dx, \\ &= e^{i\alpha p}\overline{\psi}_-(\alpha,y)+Q(\alpha,y)+e^{i\alpha q}\overline{\psi}_+(\alpha,y),\end{aligned} \tag{6.15}$$

and its inverse as

$$\psi(x,y)=\int_{-\infty}^{\infty}\overline{\psi}(\alpha,y)\,e^{-i\alpha x}d\alpha, \tag{6.16}$$

where

$$\overline{\psi}_-(\alpha, y) = \frac{1}{2\pi}\int_{-\infty}^{p} \psi(x, y)\, e^{i\alpha(x-p)} dx,$$

$$Q(\alpha, y) = \frac{1}{2\pi}\int_{p}^{q} \psi(x, y)\, e^{i\alpha x} dx,$$

$$\overline{\psi}_+(\alpha, y) = \frac{1}{2\pi}\int_{q}^{\infty} \psi(x, y)\, e^{i\alpha(x-q)} dx. \tag{6.17}$$

The function $\overline{\psi}_-(\alpha, y)$ is regular in the lower half plane $\mathrm{Im}\,\alpha < \mathrm{Im}\,k$, $\overline{\psi}_+(\alpha, y)$ is regular in the upper half plane $\mathrm{Im}\,\alpha > \mathrm{Im}\,k\cos\theta_0$ and $Q(\alpha, y)$ is an analytic function and therefore regular in the common region $\mathrm{Im}\,k\cos\theta_0 < \mathrm{Im}\,\alpha < \mathrm{Im}\,k$ and the strip of analyticity is same as shown in Fig. 3.2.

The Fourier transform of the Eq. (6.10) will give

$$\frac{d^2\overline{\psi}(\alpha, y)}{dy^2} + K^2\overline{\psi}(\alpha, y) = 0, \tag{6.18}$$

where $K(\alpha) = \sqrt{k^2 - \alpha^2}$.

Defining $K(\alpha)$, the square root function to be that branch which reduces to $+k$ when $\alpha = 0$ and when the complex $\alpha - plane$ is cut either from $\alpha = k$ to $\alpha = k\infty$ or from $\alpha = -k$ to $\alpha = -k\infty$. The solution of Eq. (6.18), representing the outgoing waves at infinity, can formally be written as

$$\overline{\psi}(\alpha, y) = \begin{cases} A(\alpha)e^{iK(\alpha)y} & y > 0 \\ B(\alpha)e^{-iK(\alpha)y} & y < 0, \end{cases} \tag{6.19}$$

where $A(\alpha)$ and $B(\alpha)$ are the unknown coefficients which are to be determined. The

Fourier transform of the boundary conditions (6.11 − 6.14) yields

$$\overline{\psi}_{-1}\left(\alpha,0^{+}\right)=0, \tag{6.20}$$

$$\overline{\psi}_{+1}\left(\alpha,0^{+}\right)=0, \tag{6.21}$$

$$\overline{\psi}_{-2}\left(\alpha,0^{-}\right)=0, \tag{6.22}$$

$$\overline{\psi}_{+2}\left(\alpha,0^{-}\right)=0, \tag{6.23}$$

$$Q_{1}\left(\alpha,0^{+}\right)-Q_{1}\left(\alpha,0^{-}\right)=0, \tag{6.24}$$

$$Q_{2}\left(\alpha,0^{+}\right)-Q_{2}\left(\alpha,0^{-}\right)=k\sin\theta_{0}\widetilde{G}(\alpha), \tag{6.25}$$

where

$$\overline{\psi}_{-1}\left(\alpha,0^{-}\right)=\frac{1}{2\pi}\int_{-\infty}^{p}\psi\left(x,0^{-}\right)e^{i\alpha(x-p)}dx, \tag{6.26}$$

$$\overline{\psi}_{+1}\left(\alpha,0^{-}\right)=\frac{1}{2\pi}\int_{q}^{\infty}\psi\left(x,0^{-}\right)e^{i\alpha(x-q)}dx, \tag{6.27}$$

$$\overline{\psi}_{-2}\left(\alpha,0^{+}\right)=\frac{1}{2\pi i}\int_{-\infty}^{p}\frac{\partial\psi\left(x,0^{+}\right)}{\partial y}e^{i\alpha(x-p)}dx, \tag{6.28}$$

$$\overline{\psi}_{+2}\left(\alpha,0^{+}\right)=\frac{1}{2\pi i}\int_{q}^{\infty}\frac{\partial\psi\left(x,0^{+}\right)}{\partial y}e^{i\alpha(x-q)}dx, \tag{6.29}$$

$$Q_{1}\left(\alpha,0^{+}\right)=\frac{1}{2\pi}\int_{p}^{q}\psi\left(x,0^{+}\right)e^{i\alpha x}dx, \tag{6.30}$$

$$Q_{2}\left(\alpha,0^{-}\right)=\frac{1}{2\pi i}\int_{p}^{q}\frac{\partial\psi\left(x,0^{-}\right)}{\partial y}e^{i\alpha x}dx, \tag{6.31}$$

and

$$\widetilde{G}(\alpha)=\frac{e^{i(\alpha-k\cos\theta_{0})q}-e^{i(\alpha-k\cos\theta_{0})p}}{\pi\left(\alpha-k\cos\theta_{0}\right)}. \tag{6.32}$$

Using Eqs. (6.20 – 6.25) in Eq. (6.19), will give

$$A(\alpha) = Q_1\left(\alpha, 0^+\right), \tag{6.33}$$

$$B(\alpha) = -\frac{Q_2\left(\alpha, 0^-\right)}{K(\alpha)}, \tag{6.34}$$

$$A(\alpha) - B(\alpha) = -e^{i\alpha p}\overline{\psi}_{-1}\left(\alpha, 0^-\right) - e^{i\alpha q}\overline{\psi}_{+1}\left(\alpha, 0^-\right), \tag{6.35}$$

$$-K(\alpha)\left[A(\alpha) + B(\alpha)\right] = -e^{i\alpha p}\overline{\psi}_{-2}\left(\alpha, 0^+\right) - e^{i\alpha q}\overline{\psi}_{+2}\left(\alpha, 0^+\right) + ik\sin\theta_0\widetilde{G}(\alpha). \tag{6.36}$$

The elimination of the coefficients $A(\alpha)$ and $B(\alpha)$ among the Eqs. (6.33 – 6.36) will lead to the following matrix Wiener-Hopf equation valid in the strip of analyticity $\operatorname{Im} k\cos\theta_0 < \operatorname{Im}\alpha < \operatorname{Im} k$,

$$e^{i\alpha q}\begin{bmatrix} \overline{\psi}_{+1}(\alpha) \\ \overline{\psi}_{+2}(\alpha) \end{bmatrix} + \begin{bmatrix} 1 & \frac{1}{K(\alpha)} \\ -K(\alpha) & 1 \end{bmatrix}\begin{bmatrix} Q_1(\alpha) \\ Q_2(\alpha) \end{bmatrix} + e^{i\alpha p}\begin{bmatrix} \overline{\psi}_{-1}(\alpha) \\ \overline{\psi}_{-2}(\alpha) \end{bmatrix} = \widetilde{G}(\alpha)\begin{bmatrix} 0 \\ ik\sin\theta_0 \end{bmatrix}. \tag{6.37}$$

In compact form Eq. (6.37) can further be arranged as

$$e^{i\alpha q}\Psi_+(\alpha) + \mathbf{H}(\alpha)\,\mathbf{Q}(\alpha) + e^{i\alpha p}\Psi_-(\alpha) = \widetilde{G}(\alpha)\mathbf{A}, \tag{6.38}$$

where bold letters are used to denote the matrices. Eq. (6.38) is an equation analogous to the Eq. (5.60) available in [14]. In Eq. (6.38), $\mathbf{H}(\alpha)$ is the kernel matrix and in order to solve it, we have to factorize the matrix $\mathbf{H}(\alpha)$ as the product of two non-singular factor matrices such that one factor matrix being regular in the lower half plane and the other factor matrix being regular in the upper half plane with the additional requirements that both the factor matrices as well as their inverses contain elements of algebraic growth at infinity and both of these factor matrices should commute with each other. The factorization of $\mathbf{H}(\alpha)$, satisfying these conditions, has been done in [33] by using the Daniele-Kharapkov methods [20, 21] and the result is as follows:

$$\mathbf{H}_{+}(\alpha) = 2^{\frac{1}{4}} \begin{bmatrix} \cosh\chi(\alpha) & \sinh\chi(\alpha)/\gamma(\alpha) \\ \gamma(\alpha)\sinh\chi(\alpha) & \cosh\chi(\alpha) \end{bmatrix}, \tag{6.39}$$

with

$$\mathbf{H}_{-}(\alpha) = \mathbf{H}_{+}(-\alpha), \tag{6.40}$$

where

$$\chi(\alpha) = -\frac{i}{4}\arccos\frac{\alpha}{k}, \qquad \chi(-\alpha) = -\frac{i}{4}\left[\pi - \arccos\frac{\alpha}{k}\right] \tag{6.41}$$

and

$$\gamma(\alpha) = \sqrt{\alpha^2 - k^2}. \tag{6.42}$$

Also as $|\alpha| \to \infty$, it is noted that

$$\mathbf{H}_{\pm}(\alpha) \sim (4k)^{-\frac{1}{4}} \begin{bmatrix} (\pm\alpha)^{\frac{1}{4}} & (\pm\alpha)^{-\frac{3}{4}} \\ (\pm\alpha)^{\frac{5}{4}} & (\pm\alpha)^{\frac{1}{4}} \end{bmatrix}. \tag{6.43}$$

After accomplishing the factorization of the matrix $\mathbf{H}(\alpha)$, Eq. (6.38) can be re-arranged as

$$e^{i\alpha q}\Psi_{+}(\alpha) + \mathbf{H}_{+}(\alpha)\,\mathbf{H}_{-}(\alpha)\,\mathbf{Q}(\alpha) + e^{i\alpha p}\Psi_{-}(\alpha) = \widetilde{G}(\alpha)\mathbf{A}. \tag{6.44}$$

Pre-multiplying Eq. (6.44) by $e^{-i\alpha q}\left[\mathbf{H}_{+}(\alpha)\right]^{-1}$, substituting the value of $\widetilde{G}(\alpha)$ from

Eq. (6.32) and simplifying will give

$$\begin{aligned}&[\mathbf{H}_+(\alpha)]^{-1}\Psi_+(\alpha)+e^{-i\alpha q}\mathbf{H}_-(\alpha)\,\mathbf{Q}(\alpha)+e^{i\alpha(p-q)}[\mathbf{H}_+(\alpha)]^{-1}\Psi_-(\alpha)\\ =&\ \frac{e^{-ik\cos\theta_0 q}}{\pi(\alpha-k\cos\theta_0)}[\mathbf{H}_+(\alpha)]^{-1}\mathbf{A}-\frac{e^{i\alpha(p-q)-ik\cos\theta_0 p}}{\pi(\alpha-k\cos\theta_0)}[\mathbf{H}_+(\alpha)]^{-1}\mathbf{A}.\end{aligned} \tag{6.45}$$

According to the procedure defined in [14] different terms occurring in Eq. (6.45) can be decomposed as follows,

$$e^{i\alpha(p-q)}[\mathbf{H}_+(\alpha)]^{-1}\Psi_-(\alpha)=\mathbf{U}_+(\alpha)+\mathbf{U}_-(\alpha), \tag{6.46}$$

$$\frac{e^{i\alpha(p-q)-ik\cos\theta_0 p}}{\pi(\alpha-k\cos\theta_0)}[\mathbf{H}_+(\alpha)]^{-1}\mathbf{A}=\mathbf{V}_+(\alpha)+\mathbf{V}_-(\alpha). \tag{6.47}$$

The pole contribution of the first term on right hand side of Eq. (6.45) can be expressed as

$$\frac{e^{-ik\cos\theta_0 q}}{\pi(\alpha-k\cos\theta_0)}\left[\{\mathbf{H}_+(\alpha)\}^{-1}-\{\mathbf{H}_+(k\cos\theta_0)\}^{-1}+\{\mathbf{H}_+(k\cos\theta_0)\}^{-1}\right]\mathbf{A}. \tag{6.48}$$

Using Eqs. (6.46 – 6.48) in Eq. (6.45) and separating it into positive and negative terms, one obtains

$$\begin{aligned}&[\mathbf{H}_+(\alpha)]^{-1}\Psi_+(\alpha)+\mathbf{U}_+(\alpha)-\frac{e^{-ik\cos\theta_0 q}}{\pi(\alpha-k\cos\theta_0)}\\ &\times\left[\{\mathbf{H}_+(\alpha)\}^{-1}-\{\mathbf{H}_+(k\cos\theta_0)\}^{-1}\right]\mathbf{A}+\mathbf{V}_+(\alpha)\\ =&\ -e^{-i\alpha q}\mathbf{H}_-(\alpha)\,\mathbf{Q}(\alpha)-\mathbf{U}_-(\alpha)-\mathbf{V}_-(\alpha)+\frac{e^{-ik\cos\theta_0 q}}{\pi(\alpha-k\cos\theta_0)}\{\mathbf{H}_+(k\cos\theta_0)\}^{-1}\mathbf{A},\end{aligned} \tag{6.49}$$

where

$$\mathbf{U}_\pm(\alpha)=\pm\frac{1}{2\pi i}\int\limits_{-\infty+ic}^{\infty+ic}\frac{e^{i\xi(p-q)}[\mathbf{H}_+(\xi)]^{-1}\Psi_-(\xi)}{\xi-\alpha}d\xi, \tag{6.50}$$

and

$$\mathbf{V}_{\pm}(\alpha) = \pm\frac{1}{2\pi i}\int\limits_{-\infty+ic}^{\infty+ic} \frac{e^{i\xi(p-q)-ik\cos\theta_0 p}\left[\mathbf{H}_{+}(\xi)\right]^{-1}\mathbf{A}}{\pi(\xi-\alpha)(\xi-k\cos\theta_0)}d\xi. \tag{6.51}$$

Now pre-multiplying Eq. (6.44) by $e^{-i\alpha p}\left[\mathbf{H}_{-}(\alpha)\right]^{-1}$, substituting the value of $\widetilde{G}(\alpha)$ from Eq. (6.32) and simplifying will give:

$$\begin{aligned} & e^{i\alpha(q-p)}\left[\mathbf{H}_{-}(\alpha)\right]^{-1}\Psi_{+}(\alpha) + e^{-i\alpha p}\mathbf{H}_{+}(\alpha)\,\mathbf{Q}(\alpha) + \left[\mathbf{H}_{-}(\alpha)\right]^{-1}\Psi_{-}(\alpha) \\ = \; & \frac{e^{i\alpha(q-p)-ik\cos\theta_0 q}}{\pi(\alpha-k\cos\theta_0)}\left[\mathbf{H}_{-}(\alpha)\right]^{-1}\mathbf{A} - \frac{e^{-ik\cos\theta_0 p}}{\pi(\alpha-k\cos\theta_0)}\left[\mathbf{H}_{-}(\alpha)\right]^{-1}\mathbf{A}. \end{aligned} \tag{6.52}$$

Decomposing different terms in Eq. (6.52) by following [14], as:

$$e^{i\alpha(q-p)}\left[\mathbf{H}_{-}(\alpha)\right]^{-1}\Psi_{+}(\alpha) = \mathbf{R}_{+}(\alpha) + \mathbf{R}_{-}(\alpha), \tag{6.53}$$

$$\frac{e^{i\alpha(q-p)-ik\cos\theta_0 q}}{\pi(\alpha-k\cos\theta_0)}\left[\mathbf{H}_{-}(\alpha)\right]^{-1}\mathbf{A} = \mathbf{S}_{+}(\alpha) + \mathbf{S}_{-}(\alpha), \tag{6.54}$$

so that

$$\mathbf{R}_{\pm}(\alpha) = \pm\frac{1}{2\pi i}\int\limits_{-\infty+id}^{\infty+id} \frac{e^{i\xi(q-p)}\left[\mathbf{H}_{-}(\xi)\right]^{-1}\Psi_{+}(\xi)}{\xi-\alpha}d\xi, \tag{6.55}$$

and

$$\mathbf{S}_{\pm}(\alpha) = \pm\frac{1}{2\pi i}\int\limits_{-\infty+id}^{\infty+id} \frac{e^{i\xi(q-p)-ik\cos\theta_0 q}\left[\mathbf{H}_{-}(\xi)\right]^{-1}\mathbf{A}}{\pi(\xi-\alpha)(\xi-k\cos\theta_0)}d\xi, \tag{6.56}$$

where $-\operatorname{Im}\alpha < c < \operatorname{Im} k\cos\theta_0$ and $-\operatorname{Im}\alpha < d < \operatorname{Im} k\cos\theta_0$, also $\operatorname{Im}\alpha > c$ in Eqs. (6.50) and (6.51) and $\operatorname{Im}\alpha < d$ in Eqs. (6.55) and (6.56) as given in [14].

Using Eqs. (6.55) and (6.56) in Eq. (6.52) and separating it into positive and negative portions will lead to

$$\begin{aligned} & \mathbf{R}_{-}(\alpha) + \left[\mathbf{H}_{-}(\alpha)\right]^{-1}\Psi_{-}(\alpha) - \mathbf{S}_{-}(\alpha) + \frac{e^{-ik\cos\theta_0 p}}{\pi(\alpha-k\cos\theta_0)}\left[\mathbf{H}_{-}(\alpha)\right]^{-1}\mathbf{A} \\ = \; & -e^{-i\alpha p}\mathbf{H}_{+}(\alpha)\,\mathbf{Q}(\alpha) - \mathbf{R}_{+}(\alpha) + \mathbf{S}_{+}(\alpha). \end{aligned} \tag{6.57}$$

The left hand side of Eq. (6.49) and right hand side of Eq. (6.57) are regular in $\operatorname{Im}\alpha > \operatorname{Im} k\cos\theta_0$ and right hand side of Eq. (6.49) and left hand side of Eq. (6.57) are regular in $\operatorname{Im}\alpha < \operatorname{Im} k$. Hence using the extended form of the Liouville's theorem each side of Eqs. (6.49) and (6.57) is equal to zero, i.e.,

$$\begin{aligned}&[\mathbf{H}_+(\alpha)]^{-1}\Psi_+(\alpha)+\mathbf{U}_+(\alpha)-\frac{e^{-ik\cos\theta_0 q}}{\pi(\alpha-k\cos\theta_0)}\\&\left[\{\mathbf{H}_+(\alpha)\}^{-1}-\{\mathbf{H}_+(k\cos\theta_0)\}^{-1}\right]\mathbf{A}+\mathbf{V}_+(\alpha)=0,\end{aligned}\tag{6.58}$$

and

$$\mathbf{R}_-(\alpha)+[\mathbf{H}_-(\alpha)]^{-1}\Psi_-(\alpha)-\mathbf{S}_-(\alpha)+\frac{e^{-ik\cos\theta_0 p}}{\pi(\alpha-k\cos\theta_0)}[\mathbf{H}_-(\alpha)]^{-1}\mathbf{A}=0.\tag{6.59}$$

Using Eqs. (6.50) and (6.51) in Eq. (6.58), and Eqs. (6.55) and (6.56) in Eq. (6.59) and simplifying will give

$$\begin{aligned}&[\mathbf{H}_+(\alpha)]^{-1}\Psi_+^*(\alpha)+\frac{e^{-ik\cos\theta_0 q}[\mathbf{H}_+(k\cos\theta_0)]^{-1}\mathbf{A}}{\pi(\alpha-k\cos\theta_0)}\\&+\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{e^{i\xi(p-q)}[\mathbf{H}_+(\xi)]^{-1}\Psi_-(\xi)}{(\xi-\alpha)}d\xi=0\end{aligned}\tag{6.60}$$

and

$$[\mathbf{H}_-(\alpha)]^{-1}\Psi_-(\alpha)-\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id}\frac{e^{i\xi(q-p)}[\mathbf{H}_-(\xi)]^{-1}\Psi_+^*(\alpha)}{(\xi-\alpha)}d\xi=0,\tag{6.61}$$

where

$$\begin{aligned}\Psi_+^*(\alpha)&=\Psi_+(\alpha)-\frac{e^{-ik\cos\theta_0 q}\mathbf{A}}{\pi(\alpha-k\cos\theta_0)},&(6.62)\\\Psi_-(\alpha)&=\Psi_-(\alpha)+\frac{e^{-ik\cos\theta_0 p}\mathbf{A}}{\pi(\alpha-k\cos\theta_0)}.&(6.63)\end{aligned}$$

From the assumption that $0 < \theta_0 < \frac{\pi}{2}$, a can be chosen such that $-k_2 \cos\theta_0 < a < k_2 \cos\theta_0$ and $d = -c = a$, [14]. In Eq.(6.60) replacing ξ by $-\xi$ and in Eq. (6.61) α by $-\alpha$ and also noting that $\mathbf{H}_-(-\alpha) = \mathbf{H}_+(\alpha)$ will yield

$$\left[\mathbf{H}_+(\alpha)\right]^{-1} \Psi_+^*(\alpha) + \frac{e^{-ik\cos\theta_0 q}\left[\mathbf{H}_+(k\cos\theta_0)\right]^{-1}\mathbf{A}}{\pi(\alpha - k\cos\theta_0)} - \frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia} \frac{e^{i\xi(q-p)}\left[\mathbf{H}_-(\xi)\right]^{-1}\Psi_-(-\xi)}{(\xi+\alpha)} d\xi = 0 \tag{6.64}$$

and

$$\left[\mathbf{H}_+(\alpha)\right]^{-1} \Psi_-(-\alpha) - \frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia} \frac{e^{i\xi(q-p)}\left[\mathbf{H}_-(\xi)\right]^{-1}\Psi_+^*(\alpha)}{(\xi+\alpha)} d\xi = 0. \tag{6.65}$$

Addition and subtraction of Eqs. (6.64) and (6.65), will result into

$$\left[\mathbf{H}_+(\alpha)\right]^{-1} \mathbf{S}_+^*(\alpha) + \frac{e^{-ik\cos\theta_0 q}\left[\mathbf{H}_+(k\cos\theta_0)\right]^{-1}\mathbf{A}}{\pi(\alpha - k\cos\theta_0)} - \frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia} \frac{e^{i\xi(q-p)}\left[\mathbf{H}_-(\xi)\right]^{-1}\mathbf{S}_+^*(\xi)}{(\xi+\alpha)} d\xi = 0 \tag{6.66}$$

and

$$\left[\mathbf{H}_+(\alpha)\right]^{-1} \mathbf{D}_+^*(\alpha) + \frac{e^{-ik\cos\theta_0 q}\left[\mathbf{H}_+(k\cos\theta_0)\right]^{-1}\mathbf{A}}{\pi(\alpha - k\cos\theta_0)} + \frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia} \frac{e^{i\xi(q-p)}\left[\mathbf{H}_-(\xi)\right]^{-1}\mathbf{D}_+^*(\xi)}{(\xi+\alpha)} d\xi = 0, \tag{6.67}$$

where

$$\mathbf{S}_+^*(\alpha) = \Psi_+^*(\alpha) + \Psi_-(-\alpha), \tag{6.68}$$

$$\mathbf{D}_+^*(\alpha) = \Psi_+^*(\alpha) - \Psi_-(-\alpha). \tag{6.69}$$

The Eqs. (6.66) and (6.67) are of the same type and an approximate solution can be obtained by a method due to Jones [164]. Setting

$$\mathbf{S}_{+}^{*}(\alpha)=\mathbf{D}_{+}^{*}(\alpha)=\mathbf{F}_{+}^{*}(\alpha), \tag{6.70}$$

the Eqs. (6.66) and (6.67) will take the form

$$\begin{aligned}&\left[\mathbf{H}_{+}(\alpha)\right]^{-1}\mathbf{F}_{+}^{*}(\alpha)+\frac{\lambda}{2\pi i}\int\limits_{-\infty+ia}^{\infty+ia}\frac{e^{i\xi(q-p)}\left[\mathbf{H}_{-}(\xi)\right]^{-1}\mathbf{F}_{+}^{*}(\xi)}{(\xi+\alpha)}d\xi\\ =\;&-\frac{e^{-ik\cos\theta_0 q}\left[\mathbf{H}_{+}(k\cos\theta_0)\right]^{-1}\mathbf{A}}{\pi(\alpha-k\cos\theta_0)},\end{aligned} \tag{6.71}$$

where

$$\mathbf{F}_{+}^{*}(\alpha)=\mathbf{F}_{+}(\alpha)-\frac{e^{-ik\cos\theta_0 q}\mathbf{A}}{\pi(\alpha-k\cos\theta_0)}+\frac{\lambda e^{-ik\cos\theta_0 p}\mathbf{A}}{\pi(\alpha+k\cos\theta_0)}, \tag{6.72}$$

$$\mathbf{F}_{+}(\alpha)=\Psi_{+}(\alpha)-\lambda\Psi_{-}(-\alpha), \tag{6.73}$$

and $\lambda=\pm 1$.

A more elaborative form of Eq. (6.71) is as follows:

$$\begin{aligned}&\begin{bmatrix}\cosh\varkappa(\alpha)F_{+}^{1*}(\alpha)-\sinh\varkappa(\alpha)F_{+}^{2*}(\alpha)/\gamma(\alpha)\\ -\gamma(\alpha)\sinh\varkappa(\alpha)F_{+}^{1*}(\alpha)+\cosh\varkappa(\alpha)F_{+}^{2*}(\alpha)\end{bmatrix}\\ &+\frac{\lambda}{2\pi i}\int\limits_{-\infty+ia}^{\infty+ia}\frac{e^{i\xi(q-p)}}{(\xi+\alpha)}\begin{bmatrix}\cosh\varkappa(-\xi)F_{+}^{1*}(\xi)-\sinh\varkappa(-\xi)F_{+}^{2*}(\xi)/\gamma(-\xi)\\ -\gamma(-\xi)\sinh\varkappa(-\xi)F_{+}^{1*}(\xi)+\cosh\varkappa(-\xi)F_{+}^{2*}(\xi)\end{bmatrix}d\xi\\ &+\frac{e^{-ik\cos\theta_0 q}}{\pi(\alpha-k\cos\theta_0)}\begin{bmatrix}A_1\cosh\varkappa(k\cos\theta_0)-A_2\sinh\varkappa(k\cos\theta_0)/\gamma(k\cos\theta_0)\\ -A_1\gamma(k\cos\theta_0)\sinh\varkappa(k\cos\theta_0)+A_2\cosh\varkappa(k\cos\theta_0)\end{bmatrix}=0.\end{aligned} \tag{6.74}$$

Eq. (6.72) in matrix form can be written as:

$$\begin{bmatrix} F_+^{1*}(\alpha) \\ F_+^{2*}(\alpha) \end{bmatrix} = \begin{bmatrix} F_+^1(\alpha) \\ F_+^1(\alpha) \end{bmatrix} - \frac{e^{-ik\cos\theta_0 q}}{\pi(\alpha - k\cos\theta_0)} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + \frac{\lambda e^{-ik\cos\theta_0 p}}{\pi(\alpha + k\cos\theta_0)} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}. \tag{6.75}$$

Considering the first row of Eq. (6.74) and using the values of $F_+^{1*}(\alpha)$ and $F_+^{2*}(\alpha)$ in it, will obtain

$$\begin{aligned}
&\cosh\varkappa(\alpha)\left[F_+^1(\alpha) - \frac{e^{-ik\cos\theta_0 q}}{\pi(\alpha - k\cos\theta_0)}A_1 + \frac{\lambda e^{-ik\cos\theta_0 p}}{\pi(\alpha + k\cos\theta_0)}A_1\right] \\
&-\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}\left[F_+^2(\alpha) - \frac{e^{-ik\cos\theta_0 q}}{\pi(\alpha - k\cos\theta_0)}A_2 + \frac{\lambda e^{-ik\cos\theta_0 p}}{\pi(\alpha + k\cos\theta_0)}A_2\right] \\
&+\frac{\lambda}{2\pi i}\int_{-\infty+ia}^{\infty+ia} \frac{e^{i\xi(q-p)}}{(\xi+\alpha)}\cosh\varkappa(-\xi)\left\{F_+^1(\xi) - \frac{e^{-ik\cos\theta_0 q}}{\pi(\xi - k\cos\theta_0)}A_1 + \frac{\lambda e^{-ik\cos\theta_0 p}}{\pi(\xi + k\cos\theta_0)}A_1\right\} \\
&-\sinh\varkappa(-\xi)/\gamma(\xi)\left\{F_+^2(\xi) - \frac{e^{-ik\cos\theta_0 q}}{\pi(\xi - k\cos\theta_0)}A_2 + \frac{\lambda e^{-ik\cos\theta_0 p}}{\pi(\xi + k\cos\theta_0)}A_2\right\}d\xi \\
&+\frac{e^{-ik\cos\theta_0 q}}{\pi(\alpha - k\cos\theta_0)}\left[A_1\cosh\varkappa(k\cos\theta_0) - A_2\sinh\varkappa(k\cos\theta_0)/\gamma(k\cos\theta_0)\right] = 0\,.
\end{aligned} \tag{6.76}$$

Writing $\gamma(\xi) = \gamma_+(\xi)\gamma_-(\xi) = \sqrt{\xi+k}\sqrt{\xi-k}$ and considering the integrals arising in Eq. (6.76), one arrives at

$$\begin{aligned}
I &= I_1 - \frac{e^{-ik\cos\theta_0 q}A_1}{\pi}I_2 + \frac{\lambda e^{-ik\cos\theta_0 p}A_1}{\pi}I_3 - I_4 \\
&\quad + \frac{e^{-ik\cos\theta_0 q}A_2}{\pi}I_5 + \frac{e^{-ik\cos\theta_0 p}A_2}{\pi}I_6,
\end{aligned} \tag{6.77}$$

where

$$I_1 = \int_{-\infty+ia}^{\infty+ia} \frac{e^{i\xi(q-p)}\cosh\varkappa(-\xi)\,F_+^1(\xi)}{(\xi+\alpha)}d\xi, \tag{6.78a}$$

$$I_2 = \int_{-\infty+ia}^{\infty+ia} \frac{e^{i\xi(q-p)} \cosh \varkappa(-\xi)}{(\xi+\alpha)(\xi - k\cos\theta_0)} d\xi, \tag{6.78b}$$

$$I_3 = \int_{-\infty+ia}^{\infty+ia} \frac{e^{i\xi(q-p)} \cosh \varkappa(-\xi)}{(\xi+\alpha)(\xi + k\cos\theta_0)} d\xi, \tag{6.78c}$$

$$I_4 = \int_{-\infty+ia}^{\infty+ia} \frac{e^{i\xi(q-p)} F_+^2(\xi) \sinh \varkappa(-\xi)/\sqrt{\xi+k}}{(\xi+\alpha)\sqrt{\xi-k}} d\xi, \tag{6.78d}$$

$$I_5 = \int_{-\infty+ia}^{\infty+ia} \frac{e^{i\xi(q-p)} \sinh \varkappa(-\xi)/\sqrt{\xi+k}}{(\xi - k\cos\theta_0)(\xi+\alpha)\sqrt{\xi-k}} d\xi, \tag{6.78e}$$

$$I_6 = \int_{-\infty+ia}^{\infty+ia} \frac{e^{i\xi(q-p)} \sinh \varkappa(-\xi)/\sqrt{\xi+k}}{(\xi + k\cos\theta_0)(\xi+\alpha)\sqrt{\xi-k}} d\xi. \tag{6.78f}$$

Integrals (6.78 $a-f$) are solved by a method described in [14] and are substituted in Eq. (6.76) to get

$$\begin{aligned}
&\cosh \varkappa(\alpha) \left[F_+^1(\alpha) - \frac{e^{-ik\cos\theta_0 q}}{\pi(\alpha - k\cos\theta_0)} A_1 + \frac{\lambda e^{-ik\cos\theta_0 p}}{\pi(\alpha + k\cos\theta_0)} A_1 \right] \\
&-\frac{\sinh \varkappa(\alpha)}{\gamma(\alpha)} \left[F_+^2(\alpha) - \frac{e^{-ik\cos\theta_0 q}}{\pi(\alpha - k\cos\theta_0)} A_2 + \frac{\lambda e^{-ik\cos\theta_0 p}}{\pi(\alpha + k\cos\theta_0)} A_2 \right] = \\
&-\lambda T(\alpha) F_+^1(k) + \lambda \frac{e^{-ik\cos\theta_0 q}}{\pi} A_1 \left\{ \frac{e^{ikl\cos\theta_0}}{(\alpha + k\cos\theta_0)} \cosh \varkappa(-k\cos\theta_0) + R_2(\alpha) \right\} \\
&-A_1 \frac{e^{-ik\cos\theta_0 p}}{\pi} R_1(\alpha) + \lambda T_1(\alpha) F_+^2(k) \\
&-\lambda \frac{e^{-ik\cos\theta_0 q}}{\pi} A_2 \left\{ \frac{e^{ikl\cos\theta_0} \sinh \varkappa(-k\cos\theta_0)/\gamma_+(k\cos\theta_0)}{(\alpha + k\cos\theta_0)\gamma_-(k\cos\theta_0)} + R_4(\alpha) \right\} \\
&-A_2 \frac{e^{-ik\cos\theta_0 p}}{\pi} R_3(\alpha) - \frac{e^{-ik\cos\theta_0 q}}{\pi(\alpha - k\cos\theta_0)} \left[A_1 \cosh \varkappa(k\cos\theta_0) - A_2 \frac{\sinh \varkappa(k\cos\theta_0)}{\gamma(k\cos\theta_0)} \right],
\end{aligned} \tag{6.79}$$

where $l = q - p$ and

$$\begin{aligned}
T(\alpha) &= \frac{1}{2\pi i} E_{-\frac{1}{2}} W_{-\frac{1}{2}} \{-i\,(k+\alpha)\,l\}, \\
T_1(\alpha) &= \frac{1}{2\pi i} E_{-1} W_{-1} \{-i\,(k+\alpha)\,l\}, \\
R_{1,2}(\alpha) &= \frac{\cosh \varkappa\,(-k)\, E_{-\frac{1}{2}} \left[W_{-\frac{1}{2}} \{-i\,(k \pm k\cos\theta_0)\,l\} - W_{-\frac{1}{2}} \{-i\,(k+\alpha)\,l\} \right]}{2\pi i\,(\alpha \mp k\cos\theta_0)}, \\
R_{3,4}(\alpha) &= \frac{E_{-1} \left[W_{-1} \{-i\,(k \pm k\cos\theta_0)\,l\} - W_{-1} \{-i\,(k+\alpha)\,l\} \right] \sinh \varkappa\,(-k) \,/\sqrt{2k}}{2\pi i\,(\alpha \mp k\cos\theta_0)}.
\end{aligned} \tag{6.80}$$

Eq. (6.79) can further be simplified according to the procedure described in [14] and the result is

$$\begin{aligned}
&\cosh \varkappa\,(\alpha)\, F_+^1(\alpha) - \frac{\sinh \varkappa\,(\alpha)}{\gamma\,(\alpha)} F_+^2(\alpha) = -\lambda T(\alpha) F_+^1(k) + \lambda T_1(\alpha) F_+^2(k) \\
&+ A_1 \frac{e^{-ik\cos\theta_0 q}}{\pi} P_1(\alpha) - \lambda A_1 \frac{e^{-ik\cos\theta_0 p}}{\pi} P_2(\alpha) - A_2 \frac{e^{-ik\cos\theta_0 q}}{\pi} P_3(\alpha) + \lambda A_2 \frac{e^{-ik\cos\theta_0 p}}{\pi} P_4(\alpha) \\
&+ \lambda A_1 \frac{e^{-ik\cos\theta_0 q}}{\pi} R_2(\alpha) - A_1 \frac{e^{-ik\cos\theta_0 p}}{\pi} R_1(\alpha) - \lambda A_2 \frac{e^{-ik\cos\theta_0 q}}{\pi} R_4(\alpha) + A_2 \frac{e^{-ik\cos\theta_0 p}}{\pi} R_3(\alpha),
\end{aligned} \tag{6.81}$$

where

$$\begin{aligned}
P_1(\alpha) &= \frac{1}{(\alpha - k\cos\theta_0)} \left[\cosh \varkappa\,(\alpha) - \cosh \varkappa\,(k\cos\theta_0) \right], \\
P_2(\alpha) &= \frac{1}{(\alpha + k\cos\theta_0)} \left[\cosh \varkappa\,(\alpha) - \cosh \varkappa\,(-k\cos\theta_0) \right], \\
P_3(\alpha) &= \frac{1}{(\alpha - k\cos\theta_0)} \left[\frac{\sinh \varkappa\,(\alpha)}{\gamma\,(\alpha)} - \frac{\sinh \varkappa\,(k\cos\theta_0)}{\gamma\,(k\cos\theta_0)} \right], \\
P_4(\alpha) &= \frac{1}{\gamma_-(-k\cos\theta_0)\,(\alpha + k\cos\theta_0)} \left[\frac{\sinh \varkappa\,(\alpha)}{\gamma\,(\alpha)} - \frac{\sinh \varkappa\,(-k\cos\theta_0)}{\gamma_+(k\cos\theta_0)} \right].
\end{aligned} \tag{6.82}$$

Further simplification of Eq. (6.81) will lead to

$$\begin{aligned}
& \cosh \varkappa(\alpha) F_+^1(\alpha) - \frac{\sinh \varkappa(\alpha)}{\gamma(\alpha)} F_+^2(\alpha) \\
= \; & -\lambda T(\alpha) F_+^1(k) \cosh \varkappa(-k) + \lambda T_1(\alpha) F_+^2(k) \frac{\sinh \varkappa(-k)}{\sqrt{2k}} \\
& + \frac{A_1}{\pi}\{e^{-ik\cos\theta_{0q}} P_1(\alpha) - e^{-ik\cos\theta_{0p}} R_1(\alpha)\} - \frac{\lambda A_1}{\pi}\{e^{-ik\cos\theta_{0p}} P_2(\alpha) - e^{-ik\cos\theta_{0q}} R_2(\alpha)\} \\
& - \frac{A_2}{\pi}\{e^{-ik\cos\theta_{0q}} P_3(\alpha) - e^{-ik\cos\theta_{0p}} R_3(\alpha)\} + \frac{\lambda A_2}{\pi}\{e^{-ik\cos\theta_{0p}} P_4(\alpha) - e^{-ik\cos\theta_{0q}} R_4(\alpha)\}.
\end{aligned} \tag{6.83}$$

Letting

$$\begin{aligned}
G_1(\alpha) &= e^{-ik\cos\theta_{0q}} P_1(\alpha) - e^{-ik\cos\theta_{0p}} R_1(\alpha), \\
G_2(\alpha) &= e^{-ik\cos\theta_{0p}} P_2(\alpha) - e^{-ik\cos\theta_{0q}} R_2(\alpha), \\
G_3(\alpha) &= e^{-ik\cos\theta_{0q}} P_3(\alpha) - e^{-ik\cos\theta_{0p}} R_3(\alpha), \\
G_4(\alpha) &= e^{-ik\cos\theta_{0p}} P_4(\alpha) - e^{-ik\cos\theta_{0q}} R_4(\alpha),
\end{aligned} \tag{6.84}$$

in Eq. (6.83), the solution of the first W-H equation, obtained by considering the first row of matrices in Eq. (6.74), is given as follows:

$$\begin{aligned}
& \cosh \varkappa(\alpha) F_+^1(\alpha) - \frac{\sinh \varkappa(\alpha) F_+^2(\alpha)}{\gamma(\alpha)} \\
= \; & -\lambda T(\alpha) \cosh \varkappa(-k) F_+^1(k) + \lambda \frac{T_1(\alpha) \sinh \varkappa(-k) F_+^2(k)}{\sqrt{2k}} \\
& + \frac{A_1}{\pi}[G_1(\alpha) - \lambda G_2(\alpha)] - \frac{A_2}{\pi}[G_3(\alpha) - \lambda G_4(\alpha)].
\end{aligned} \tag{6.85}$$

The second W-H equation corresponds to the second row of the matrix Eq. (6.74) and its solution can be obtained in a similar manner as for the first row of Eq. (6.74).

Omitting all the similar steps and quantities arose in the solution, will lead to the following form

$$
\begin{aligned}
&-\gamma(\alpha)\sinh\varkappa(\alpha)\,F_{+}^{1}(\alpha)+\cosh\varkappa(\alpha)\,F_{+}^{2}(\alpha)\\
&=\lambda T_{2}(\alpha)\sqrt{2k}\sinh\varkappa(-k)\,F_{+}^{1}(k)-\lambda T(\alpha)\cosh\varkappa(-k)\,F_{+}^{2}(k)\\
&+\frac{A_{2}}{\pi}\left[G_{1}(\alpha)-\lambda G_{2}(\alpha)\right]-\frac{A_{1}}{\pi}\left[G_{5}(\alpha)-\lambda G_{6}(\alpha)\right], \qquad (6.86)
\end{aligned}
$$

where

$$
\begin{aligned}
&T_{2}(\alpha)=\frac{1}{2\pi i}E_{0}W_{0}\{-i(k+\alpha)\,l\},\\
&G_{5}(\alpha)=e^{-ik\cos\theta_{0}q}P_{5}(\alpha)-e^{-ik\cos\theta_{0}p}R_{5}(\alpha),\\
&G_{6}(\alpha)=e^{-ik\cos\theta_{0}p}P_{6}(\alpha)-e^{-ik\cos\theta_{0}q}R_{6}(\alpha),\\
&P_{5}(\alpha)=\frac{\gamma(\alpha)\sinh\varkappa(\alpha)-\gamma(k\cos\theta_{0})\sinh\varkappa(k\cos\theta_{0})}{\alpha-k\cos\theta_{0}},\\
&P_{6}(\alpha)=\frac{\gamma(\alpha)\sinh\varkappa(\alpha)-\gamma(-k\cos\theta_{0})\sinh\varkappa(-k\cos\theta_{0})}{\alpha+k\cos\theta_{0}},\\
&R_{5,6}(\alpha)=\frac{D_{0}\left[W_{0}\{-i(k\pm k\cos\theta_{0})\,l\}-W_{0}\{-i(k+\alpha)\,l\}\right]}{2\pi i(\alpha\mp k\cos\theta_{0})},\\
&D_{0}=E_{0}\sqrt{2k}\sinh\varkappa(-k). \qquad (6.87)
\end{aligned}
$$

In Eqs. (6.80) and (6.87),

$$
\begin{aligned}
W_{n-\frac{1}{2}}(z) &= \int_{0}^{\infty}\frac{u^{n}e^{-u}}{u+z}du\\
&= \Gamma(n+1)\,e^{\frac{1}{2}z}z^{\frac{1}{2}n-\frac{1}{2}}W_{-\frac{1}{2}(n+1),\frac{1}{2}n}(z), \qquad (6.88)
\end{aligned}
$$

where $z=-i(k+\alpha)\,l$ and $n=-\frac{1}{2},0,\frac{1}{2}$. $W_{m,n}$ is known as a Whittaker function [164, 165]. The values of the functions $F_{+}^{1}(k)$ and $F_{+}^{2}(k)$ can be calculated by putting

$\alpha = k$ in Eqs. (6.85) and (6.86) and solving these equations simultaneously. Now as

$$\mathbf{F}_{+}(\alpha) = \begin{bmatrix} F_{+}^{1}(\alpha) \\ F_{+}^{2}(\alpha) \end{bmatrix} = \begin{bmatrix} \overline{\psi}_{+1}(\alpha) \\ \overline{\psi}_{+2}(\alpha) \end{bmatrix} - \lambda \begin{bmatrix} \overline{\psi}_{-1}(\alpha) \\ \overline{\psi}_{-2}(\alpha) \end{bmatrix}, \tag{6.89}$$

Eq. (6.89) is considered for the cases $\lambda = 1$ and $\lambda = -1$ and when the values of $F_{+}^{1}(\alpha)$ and $F_{+}^{2}(\alpha)$ are substituted in Eqs. (6.85) and (6.86) the results are as follows:

For $\lambda = 1$

$$\begin{aligned} & \cosh\varkappa(\alpha)\left[\overline{\psi}_{+1}(\alpha) - \overline{\psi}_{-1}(\alpha)\right] - \frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}\left[\overline{\psi}_{+2}(\alpha) - \overline{\psi}_{-2}(\alpha)\right] \\ = \; & -T(\alpha)\cosh\varkappa(-k)\left.F_{+}^{1}(k)\right|_{\lambda=1} + \frac{T_1(\alpha)\sinh\varkappa(-k)}{\sqrt{2k}}\left.F_{+}^{2}(k)\right|_{\lambda=1} \\ & + \frac{A_1}{\pi}\left[G_1(\alpha) - G_2(\alpha)\right] - \frac{A_2}{\pi}\left[G_3(\alpha) - G_4(\alpha)\right], \end{aligned} \tag{6.90}$$

and

$$\begin{aligned} & -\gamma(\alpha)\sinh\varkappa(\alpha)\left[\overline{\psi}_{+1}(\alpha) - \overline{\psi}_{-1}(\alpha)\right] + \cosh\varkappa(\alpha)\left[\overline{\psi}_{+2}(\alpha) - \overline{\psi}_{-2}(\alpha)\right] \\ = \; & -T(\alpha)\cosh\varkappa(-k)\left.F_{+}^{2}(k)\right|_{\lambda=1} + \sqrt{2k}T_2(\alpha)\sinh\varkappa(-k)\left.F_{+}^{1}(k)\right|_{\lambda=1} \\ & + \frac{A_2}{\pi}\left[G_1(\alpha) - G_2(\alpha)\right] - \frac{A_1}{\pi}\left[G_5(\alpha) - G_6(\alpha)\right], \end{aligned} \tag{6.91}$$

and for $\lambda = -1$

$$\begin{aligned} & \cosh\varkappa(\alpha)\left[\overline{\psi}_{+1}(\alpha) + \overline{\psi}_{-1}(\alpha)\right] - \frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}\left[\overline{\psi}_{+2}(\alpha) + \overline{\psi}_{-2}(\alpha)\right] \\ = \; & +T(\alpha)\cosh\varkappa(-k)\left.F_{+}^{1}(k)\right|_{\lambda=-1} - \frac{T_1(\alpha)\sinh\varkappa(-k)}{\sqrt{2k}}\left.F_{+}^{2}(k)\right|_{\lambda=-1} \\ & + \frac{A_1}{\pi}\left[G_1(\alpha) + G_2(\alpha)\right] - \frac{A_2}{\pi}\left[G_3(\alpha) + G_4(\alpha)\right], \end{aligned} \tag{6.92}$$

and

$$
\begin{aligned}
& -\gamma(\alpha)\sinh\varkappa(\alpha)\left[\overline{\psi}_{+1}(\alpha)+\overline{\psi}_{-1}(\alpha)\right]+\cosh\varkappa(\alpha)\left[\overline{\psi}_{+2}(\alpha)+\overline{\psi}_{-2}(\alpha)\right] \\
= \; & T(\alpha)\cosh\varkappa(-k)\,F_{+}^{2}(k)\big|_{\lambda=-1}-T_{2}(\alpha)\sqrt{2k}\sinh\varkappa(-k)\,F_{+}^{1}(k)\big|_{\lambda=-1} \\
& +\frac{A_{2}}{\pi}\left[G_{1}(\alpha)+G_{2}(\alpha)\right]-\frac{A_{1}}{\pi}\left[G_{5}(\alpha)+G_{6}(\alpha)\right]. \qquad (6.93)
\end{aligned}
$$

Addition of Eqs. (6.90) and (6.92), will give

$$
\begin{aligned}
& \cosh\varkappa(\alpha)\,\overline{\psi}_{+1}(\alpha)-\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}\overline{\psi}_{+2}(\alpha)=\frac{A_{1}}{\pi}G_{1}(\alpha)-\frac{A_{2}}{\pi}G_{3}(\alpha) \\
& -\frac{T(\alpha)\cosh\varkappa(-k)}{2}C_{1}+\frac{T_{1}(\alpha)\sinh\varkappa(-k)}{2\sqrt{2k}}C_{2}, \qquad (6.94)
\end{aligned}
$$

and Eqs. (6.91) and (6.93) will yield

$$
\begin{aligned}
& -\gamma(\alpha)\sinh\varkappa(\alpha)\,\overline{\psi}_{+1}(\alpha)+\cosh\varkappa(\alpha)\,\overline{\psi}_{+2}(\alpha)=\frac{A_{2}}{\pi}G_{1}(\alpha)-\frac{A_{1}}{\pi}G_{5}(\alpha) \\
& -\frac{T(\alpha)\cosh\varkappa(-k)}{2}C_{2}+\frac{T_{2}(\alpha)\sqrt{2k}\sinh\varkappa(-k)}{2}C_{1}, \qquad (6.95)
\end{aligned}
$$

where

$$
\begin{aligned}
C_{1} &= F_{+}^{1}(k)\big|_{\lambda=1}-F_{+}^{1}(k)\big|_{\lambda=-1}, \\
C_{2} &= F_{+}^{2}(k)\big|_{\lambda=1}-F_{+}^{2}(k)\big|_{\lambda=-1}. \qquad (6.96)
\end{aligned}
$$

Eliminating $\overline{\psi}_{+2}(\alpha)$ from Eqs. (6.94) and (6.95) will lead to

$$\overline{\psi}_{+1}(\alpha) = \left(\frac{A_1}{\pi}\cosh\varkappa(\alpha) + \frac{A_2}{\pi}\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}\right)G_1(\alpha) - \frac{A_2}{\pi}G_3(\alpha)\cosh\varkappa(\alpha)$$
$$-\frac{A_1}{\pi}G_5(\alpha)\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)} - \frac{T(\alpha)\cosh\varkappa(-k)}{2}\left(C_1\cosh\varkappa(\alpha) + C_2\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}\right)$$
$$+\frac{T_1(\alpha)\sinh\varkappa(-k)\cosh\varkappa(\alpha)C_2}{2\sqrt{2k}} + \frac{T_2(\alpha)\sqrt{2k}\sinh\varkappa(-k)\sinh\varkappa(\alpha)/\gamma(\alpha)C_1}{2}$$

(6.97)

and eliminating $\overline{\psi}_{+1}(\alpha)$ between Eqs. (6.94) and (6.95) will yield

$$\overline{\psi}_{+2}(\alpha) = \left(\frac{A_1}{\pi}\gamma(\alpha)\sinh\varkappa(\alpha) + \frac{A_2}{\pi}\cosh\varkappa(\alpha)\right)G_1(\alpha) - \frac{A_2}{\pi}G_3(\alpha)\gamma(\alpha)\sinh\varkappa(\alpha)$$
$$-\frac{A_1}{\pi}G_5(\alpha)\cosh\varkappa(\alpha) - \frac{T(\alpha)\cosh\varkappa(-k)}{2}\left(C_1\gamma(\alpha)\sinh\varkappa(\alpha) + C_2\cosh\varkappa(\alpha)\right)$$
$$+\frac{T_1(\alpha)\sinh\varkappa(-k)\gamma(\alpha)\sinh\varkappa(\alpha)C_2}{2\sqrt{2k}} + \frac{T_2(\alpha)\sqrt{2k}\sinh\varkappa(-k)\cosh\varkappa(\alpha)C_1}{2}.$$

(6.98)

Now in order to calculate the function $\overline{\psi}_{-1}(\alpha)$ and $\overline{\psi}_{-2}(\alpha)$, replacing $G_1(\alpha)$ by $G_2(\alpha)$ and $G_2(\alpha)$ by $G_1(\alpha)$, $G_3(\alpha)$ by $G_4(\alpha)$ and $G_4(\alpha)$ by $G_3(\alpha)$ and $G_5(\alpha)$ by $G_6(\alpha)$ and $G_6(\alpha)$ by $G_5(\alpha)$ and also changing α to $-\alpha$ in Eqs. (6.97)and (6.98), respectively will give $\overline{\psi}_{-1}(\alpha)$ as:

$$\overline{\psi}_{-1}(\alpha) = \left(\frac{A_1}{\pi}\cosh\varkappa(-\alpha) + \frac{A_2}{\pi}\frac{\sinh\varkappa(-\alpha)}{\gamma(-\alpha)}\right)G_2(-\alpha) - \frac{A_2}{\pi}G_4(-\alpha)\cosh\varkappa(-\alpha)$$
$$-\frac{A_1}{\pi}G_6(-\alpha)\frac{\sinh\varkappa(-\alpha)}{\gamma(-\alpha)} - \frac{T(-\alpha)\cosh\varkappa(-k)}{2}\left(\widetilde{C}_1\cosh\varkappa(-\alpha) + \widetilde{C}_2\frac{\sinh\varkappa(-\alpha)}{\gamma(-\alpha)}\right)$$
$$+\frac{T_1(-\alpha)\sinh\varkappa(-k)\cosh\varkappa(-\alpha)\widetilde{C}_2}{2\sqrt{2k}} + \frac{T_2(-\alpha)\sqrt{2k}\sinh\varkappa(-k)\sinh\varkappa(-\alpha)/\gamma(-\alpha)\widetilde{C}_1}{2}$$

(6.99)

and

$$\overline{\psi}_{-2}(\alpha) = \left(\frac{A_1}{\pi}\gamma(-\alpha)\sinh\varkappa(-\alpha) + \frac{A_2}{\pi}\cosh\varkappa(-\alpha)\right)G_2(-\alpha) - \frac{A_2}{\pi}G_4(-\alpha)\gamma(-\alpha)$$
$$\times\sinh\varkappa(-\alpha) - \frac{A_1}{\pi}G_6(-\alpha)\cosh\varkappa(-\alpha) - \frac{T(-\alpha)\cosh\varkappa(-k)}{2}$$
$$\times\left(\widetilde{C}_1\gamma(-\alpha)\sinh\varkappa(-\alpha) + \widetilde{C}_2\cosh\varkappa(-\alpha)\right) + \frac{T_1(-\alpha)\sinh\varkappa(-k)\gamma(-\alpha)\sinh\varkappa(-\alpha)\widetilde{C}_2}{2\sqrt{2k}}$$
$$+\frac{T_2(-\alpha)\sqrt{2k}\sinh\varkappa(-k)\cosh\varkappa(-\alpha)\widetilde{C}_1}{2}, \tag{6.100}$$

where $\widetilde{C}_1$ and $\widetilde{C}_2$ are given by

$$\widetilde{C}_1 = \widetilde{F}^1_+(k)\Big|_{\lambda=1} - \widetilde{F}^1_+(k)\Big|_{\lambda=-1},$$
$$\widetilde{C}_2 = \widetilde{F}^2_+(k)\Big|_{\lambda=1} - \widetilde{F}^2_+(k)\Big|_{\lambda=-1}, \tag{6.101}$$

and $\widetilde{F}^1_+(k)$ and $\widetilde{F}^2_+(k)$ denote the functions in which G_1 by G_2 and G_2 by G_1, G_3 by G_4 and G_4 by G_3 and G_5 by G_6 and G_6 by G_5 have also been interchanged and then evaluated for $\lambda = 1$ and $\lambda = -1$ respectively. Since the functions $\overline{\psi}_{\pm1}(\alpha)$ and $\overline{\psi}_{\pm2}(\alpha)$ have been calculated, therefore now manipulating Eqs. (6.35) and (6.36), the unknown coefficient $A(\alpha)$ is determined to be

$$A(\alpha) = \frac{1}{2K(\alpha)}\left[e^{i\alpha p}\overline{\psi}_{-2}(\alpha) - ik\sin\theta_0 G(\alpha) + e^{i\alpha q}\overline{\psi}_{+2}(\alpha)\right] - \frac{e^{i\alpha p}\overline{\psi}_{-1}(\alpha)}{2} - \frac{e^{i\alpha q}\overline{\psi}_{+1}(\alpha)}{2}. \tag{6.102}$$

Substituting the values of $\overline{\psi}_{\pm1}(\alpha)$ and $\overline{\psi}_{\pm2}(\alpha)$ in Eq. (6.102) and simplifying one has

$$A(\alpha)=\Bigg[\Bigg\{\frac{1}{2\pi K(\alpha)}\frac{ik\sin\theta_0\cosh\varkappa\left(-\alpha\right)\cosh\varkappa\left(-k\cos\theta_0\right)}{\alpha-k\cos\theta_0}e^{i(\alpha-k\cos\theta_0)p}$$

$$-ik\sin\theta_0 R_2\left(-\alpha\right)\cosh\varkappa\left(-\alpha\right)e^{i\alpha p-k\cos\theta_0 q}$$

$$-\frac{ik\sin\theta_0\sinh\varkappa\left(-\alpha\right)\gamma\left(-\alpha\right)\sinh\varkappa\left(-k\cos\theta_0\right)}{\left(\alpha-k\cos\theta_0\right)\gamma\left(-k\cos\theta_0\right)}e^{i(\alpha-k\cos\theta_0)p}$$

$$+ik\sin\theta_0 R_4\left(-\alpha\right)\gamma\left(-\alpha\right)\sinh\varkappa\left(-\alpha\right)e^{i\alpha p-k\cos\theta_0 q}$$

$$+\left(\frac{T\left(-\alpha\right)\cosh\varkappa\left(-k\right)}{2}\left(-\gamma\left(-\alpha\right)\sinh\varkappa\left(-\alpha\right)\widetilde{C}_1+\cosh\varkappa\left(-\alpha\right)\widetilde{C}_2\right)\right.$$

$$+\frac{T_1\left(-\alpha\right)\gamma\left(-\alpha\right)\sinh\varkappa\left(-\alpha\right)\sinh\varkappa\left(-k\right)\widetilde{C}_2}{2\sqrt{2k}}$$

$$\left.+\frac{T_2\left(-\alpha\right)\sinh\varkappa\left(-k\right)\cosh\varkappa\left(-\alpha\right)\sqrt{2k}\widetilde{C}_1}{2}\right)e^{i\alpha p}\Bigg\}$$

$$+\frac{1}{2\pi K(\alpha)}\Bigg\{\frac{-ik\sin\theta_0\cosh\varkappa\left(\alpha\right)\cosh\varkappa\left(k\cos\theta_0\right)}{\alpha-k\cos\theta_0}e^{i(\alpha-k\cos\theta_0)q}$$

$$-ik\sin\theta_0 R_1\left(\alpha\right)\cosh\varkappa\left(\alpha\right)e^{i\alpha q-ik\cos\theta_0 p}$$

$$+\frac{ik\sin\theta_0\sinh\varkappa\left(\alpha\right)\gamma\left(\alpha\right)\sinh\varkappa\left(k\cos\theta_0\right)}{\left(\alpha-k\cos\theta_0\right)\gamma\left(k\cos\theta_0\right)}e^{i(\alpha-k\cos\theta_0)q}$$

$$+ik\sin\theta_0 R_3\left(\alpha\right)\gamma\left(\alpha\right)\sinh\varkappa\left(\alpha\right)e^{i\alpha q-k\cos\theta_{0p}q}$$

$$+\left(\frac{T\left(\alpha\right)\cosh\varkappa\left(-k\right)}{2}\left(-\gamma\left(\alpha\right)\sinh\varkappa\left(\alpha\right)C_1-\cosh\varkappa\left(\alpha\right)C_2\right)\right.$$

$$+\frac{T_1\left(\alpha\right)\gamma\left(\alpha\right)\sinh\varkappa\left(\alpha\right)\sinh\varkappa\left(-k\right)C_2}{2\sqrt{2k}}$$

$$\left.+\frac{T_2\left(\alpha\right)\sinh\varkappa\left(-k\right)\cosh\varkappa\left(\alpha\right)\sqrt{2k}C_1}{2}\right)e^{i\alpha q}\Bigg\}$$

$$+\frac{1}{2\pi}\Bigg\{\frac{-ik\sin\theta_0\sinh\varkappa\left(-\alpha\right)\cosh\varkappa\left(-k\cos\theta_0\right)}{\gamma\left(-\alpha\right)\left(\alpha-k\cos\theta_0\right)}e^{i(\alpha-k\cos\theta_0)p}$$

$$+\frac{ik\sin\theta_0\cosh\varkappa\left(-\alpha\right)\sinh\varkappa\left(-k\cos\theta_0\right)}{\left(\alpha-k\cos\theta_0\right)\gamma\left(-k\cos\theta_0\right)}e^{i(\alpha-k\cos\theta_0)p}$$

$$+\frac{ik\sin\theta_0 R_2\left(-\alpha\right)\sinh\varkappa\left(-\alpha\right)e^{i\alpha p-k\cos\theta_0 q}}{\gamma\left(-\alpha\right)}$$

$$
\begin{aligned}
&-ik\sin\theta_0 R_4(-\alpha)\cosh\varkappa(-\alpha)\,e^{i\alpha p-k\cos\theta_0 q}\\
&+\left(\frac{-T(-\alpha)\cosh\varkappa(-k)}{2}\left(\cosh\varkappa(-\alpha)\,\widetilde{C}_1+\frac{\sinh\varkappa(-\alpha)}{\gamma(-\alpha)}\widetilde{C}_2\right)\right.\\
&+\frac{T_1(-\alpha)\sinh\varkappa(-k)\cosh\varkappa(-\alpha)\,\widetilde{C}_2}{2\sqrt{2k}}\\
&\left.+\frac{T_2(-\alpha)\sinh\varkappa(-k)\sinh\varkappa(-\alpha)\sqrt{2k}\widetilde{C}_1}{2}\right)e^{i\alpha p}\Bigg\}\\
&+\frac{1}{2\pi}\left\{\frac{ik\sin\theta_0\sinh\varkappa(\alpha)\cosh\varkappa(k\cos\theta_0)}{\gamma(\alpha)(\alpha-k\cos\theta_0)}e^{i(\alpha-k\cos\theta_0)q}\right.\\
&-\frac{ik\sin\theta_0\cosh\varkappa(\alpha)\sinh\varkappa(k\cos\theta_0)}{(\alpha-k\cos\theta_0)\gamma(-k\cos\theta_0)}e^{i(\alpha-k\cos\theta_0)q}\\
&-ik\sin\theta_0 R_3(\alpha)\cosh\varkappa(\alpha)\,e^{i\alpha q-k\cos\theta_0 p}\\
&+\frac{ik\sin\theta_0 R_1(\alpha)\sinh\varkappa(\alpha)\,e^{i\alpha q-k\cos\theta_0 p}}{\gamma(\alpha)}\\
&-\left(\frac{-T(-\alpha)\cosh\varkappa(-k)}{2}\left(\cosh\varkappa(\alpha)\,C_1+\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}C_2\right)\right.\\
&+\frac{T_1(\alpha)\sinh\varkappa(-k)\cosh\varkappa(\alpha)\,C_2}{2\sqrt{2k}}\\
&\left.+\frac{T_2(\alpha)\sinh\varkappa(-k)\sinh\varkappa(\alpha)\sqrt{2k}C_1}{2\gamma(\alpha)}\right)e^{i\alpha q}\Bigg\}\Bigg]. \qquad (6.103)
\end{aligned}
$$

Since $A(\alpha)$ has been determined, the scattered field $\psi(x,y)$ can now be determined by substituting $A(\alpha)$ into Eq. (6.19) and taking the inverse Fourier transform as:

$$
\psi(x,y)=\int_{-\infty}^{\infty}A(\alpha)e^{iK(\alpha)y-i\alpha x}d\alpha, \qquad (6.104)
$$

where $A(\alpha)$ is defined in Eq. (6.103). The scattered field $\psi(x,y)$ can be split up into two components as follows:

$$
\psi(x,y)=\psi_{sep}(x,y)+\psi_{int}(x,y), \qquad (6.105)
$$

where

$$\psi_{sep}(x,y) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi K(\alpha)} \{ (ik \sin\theta_0 \cosh \varkappa(-\alpha) \cosh \varkappa(-k\cos\theta_0) \right.$$
$$- \frac{ik \sin\theta_0 \sinh \varkappa(-\alpha)\,\gamma(-\alpha) \sinh \varkappa(-k\cos\theta_0)}{\gamma(-k\cos\theta_0)} \Bigg)$$
$$+ \frac{1}{2\pi} \left(\frac{-ik \sin\theta_0 \sinh \varkappa(-\alpha) \cosh \varkappa(-k\cos\theta_0)}{\gamma(-\alpha)} \right.$$
$$\left. + \frac{ik \sin\theta_0 \cosh \varkappa(-\alpha) \sinh \varkappa(-k\cos\theta_0)}{\gamma(-k\cos\theta_0)} \right) \Bigg\} \frac{e^{i(\alpha - k\cos\theta_0)p}}{\alpha - k\cos\theta_0}$$
$$+ \frac{1}{2\pi K(\alpha)} \{ (-ik \sin\theta_0 \cosh \varkappa(\alpha) \cosh \varkappa(k\cos\theta_0)$$
$$+ \frac{ik \sin\theta_0 \sinh \varkappa(\alpha)\,\gamma(\alpha) \sinh \varkappa(k\cos\theta_0)}{\gamma(k\cos\theta_0)} \Bigg)$$
$$+ \frac{1}{2\pi} \left(\frac{ik \sin\theta_0 \sinh \varkappa(\alpha) \cosh \varkappa(k\cos\theta_0)}{\gamma(\alpha)} \right.$$
$$\left. \left. - \frac{ik \sin\theta_0 \cosh \varkappa(\alpha) \sinh \varkappa(k\cos\theta_0)}{\gamma(k\cos\theta_0)} \right) \Bigg\} \frac{e^{i(\alpha - k\cos\theta_0)q}}{\alpha - k\cos\theta_0} \right]$$
$$\times e^{iK(\alpha)y - i\alpha x} d\alpha \tag{6.106}$$

and

$$\psi_{int}(x,y) = \int\limits_{-\infty}^{\infty} \frac{1}{2\pi K(\alpha)} [\{(-ik\sin\theta_0 \cosh\varkappa(\alpha) R_1(\alpha)$$

$$+ik\sin\theta_0 \sinh\varkappa(\alpha)\gamma(\alpha) R_3(\alpha))e^{i\alpha q - ik\cos\theta_0 p}$$

$$+\left(\frac{T(\alpha)\cosh\varkappa(-k)}{2}(-\gamma(\alpha)\sinh\varkappa(\alpha) C_1 - \cosh\varkappa(-\alpha) C_2)\right.$$

$$+\frac{T_1(\alpha)\gamma(\alpha)\sinh\varkappa(\alpha)\sinh\varkappa(-k) C_2}{2\sqrt{2k}}$$

$$\left.+\frac{T_2(\alpha)\sqrt{2k}\sinh\varkappa(-k)\cosh\varkappa(\alpha) C_1}{2}\right) e^{i\alpha q}\Bigg\}$$

$$+\frac{1}{2\pi K(\alpha)}\{(-ik\sin\theta_0 \cosh\varkappa(-\alpha) R_2(-\alpha)$$

$$+ik\sin\theta_0 \sinh\varkappa(-\alpha)\gamma(-\alpha) R_4(-\alpha) e^{i\alpha p - ik\cos\theta_0 q}$$

$$+\left(\frac{T(-\alpha)\cosh\varkappa(-k)}{2}\left(-\gamma(-\alpha)\sinh\varkappa(-\alpha)\widetilde{C}_1 - \cosh\varkappa(-\alpha)\widetilde{C}_2\right)\right.$$

$$+\frac{T_1(-\alpha)\gamma(-\alpha)\sinh\varkappa(-\alpha)\sinh\varkappa(-k)\widetilde{C}_2}{2\sqrt{2k}}$$

$$\left.+\frac{T_2(-\alpha)\sqrt{2k}\sinh\varkappa(-k)\cosh\varkappa(-\alpha)\widetilde{C}_1}{2}\right) e^{i\alpha p}\Bigg\}$$

$$+\frac{1}{2\pi}\left\{\left(-ik\sin\theta_0 \cosh\varkappa(\alpha) R_3(\alpha) + ik\sin\theta_0 \frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)} R_1(\alpha)\right)\right.$$

$$\times e^{i\alpha q - ik\cos\theta_0 p} - \left(\frac{-T(\alpha)\cosh\varkappa(-k)}{2}\left(C_1\cosh\varkappa(\alpha) + C_2\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}\right)\right.$$

$$+\frac{T_1(\alpha)\sinh\varkappa(-k)\cosh\varkappa(\alpha) C_2}{2\sqrt{2k}}$$

$$\left.+\frac{T_2(\alpha)\sqrt{2k}\sinh\varkappa(\alpha)\sinh\varkappa(-k) C_1}{2}\right) e^{i\alpha q}\Bigg\}$$

$$+\frac{1}{2\pi}\left\{\left(\frac{ik\sin\theta_0 \sinh\varkappa(-\alpha) R_2(-\alpha)}{\gamma(-\alpha)} - ik\sin\theta_0 \cosh\varkappa(-\alpha) R_4(-\alpha)\right)\right.$$

$$\times e^{i\alpha p - ik\cos\theta_0 q}$$

$$-\left(\frac{T(-\alpha)\cosh\varkappa(-k)}{2}\left(-\widetilde{C}_1\cosh\varkappa(-\alpha) - \widetilde{C}_2\frac{\sinh\varkappa(-\alpha)}{\gamma(-\alpha)}\right)\right.$$

$$+\frac{T_1(-\alpha)\sinh\varkappa(-k)\cosh\varkappa(-\alpha)\widetilde{C}_2}{2\sqrt{2k}}$$

$$\left.+\frac{T_2(\alpha)\sqrt{2k}\sinh\varkappa(-\alpha)\sinh\varkappa(-k)\widetilde{C}_1}{2}\right) e^{i\alpha p}\Bigg\} e^{iK(\alpha)y - i\alpha x} d\alpha, \qquad (6.107)$$

where $\psi_{sep}(x, y)$ gives the diffracted field produced by the edges at $x = p$ and at $x = q$ respectively and $\psi_{int}(x, y)$ gives the interaction of one edge upon the other edge.

6.2 Determination of the far-field

The calculations carried out for the three part boundary value problem formulated in terms of matrix W-H equations are quite laborious and delicate at the same time, so the reported far field will correspond to the case of $y > 0$ only, (i.e. we determine the unknown coefficient $A(\alpha)$ only), the far field for the case of $y < 0$ can be calculated in a similar manner. Therefore, in order to solve the integral appearing in Eq. (6.104) the following substitutions

$$x = \rho\cos\theta,\ y = \rho\sin\theta \text{ and } \alpha = -k\cos(\theta + it_1), \tag{6.108}$$

have been introduced in Eq. (6.104), omitting the computational details and using the method of steepest descent, the field at the large distance from a slit in an infinite soft-hard plane is given as

$$\psi(x, y) \simeq \sqrt{\frac{2\pi}{k\rho}}i\sin\theta\ A(-k\cos\theta)e^{ik\rho+i\frac{\pi}{4}}, \tag{6.109}$$

where $A(-k\cos\theta)$ can be evaluated from Eq. (6.103).

6.3 Graphical results

In this section, some graphs showing the effects of various parameters on the diffracted field produced by the two edges of the slit in an infinite soft-hard plane are presented.

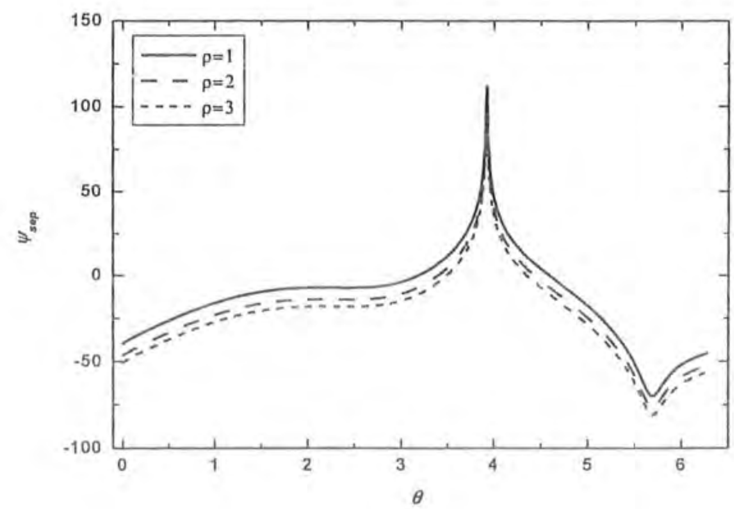

Fig. 6.2 Variation of the ψ_{sep} with θ at $\theta_0 = \frac{\pi}{4}$, $k = 1$ and $l = 1$.

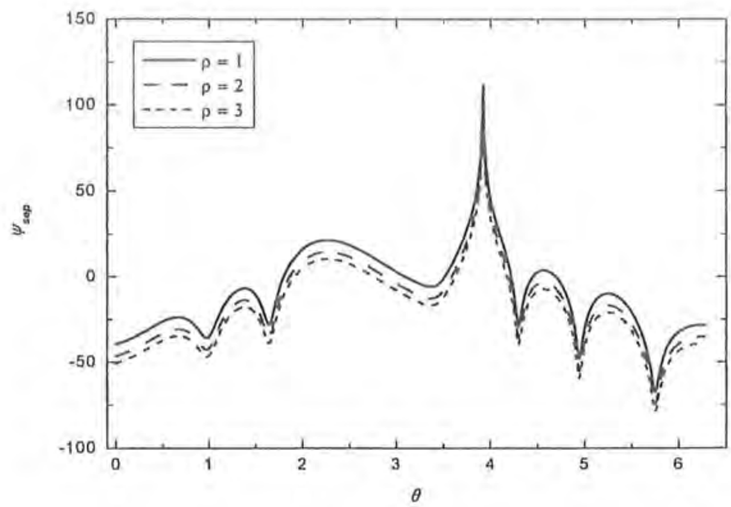

Fig. 6.3 Variation of the ψ_{sep} with θ at $\theta_0 = \frac{\pi}{4}$, $k = 1$ and $l = 5$.

- Figs. 6.2 and 6.3 show the variation of separated field ψ_{sep} with observation angle θ at $\theta_0 = \pi/4$, $k = 1$ and $\rho = 1, 2, 3$ for $l = 1$ and 5, respectively. It is observed that by increasing the parameter ρ the overall amplitude of the separated field decreases.

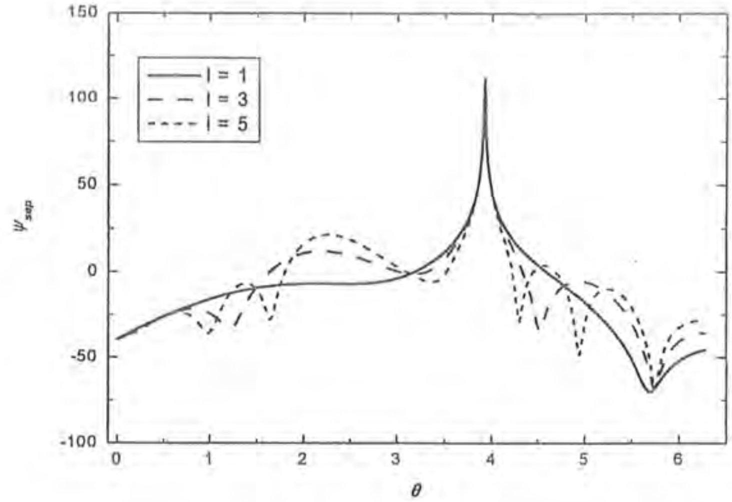

Fig. 6.4 Variation of the ψ_{sep} with θ at $\theta_0 = \frac{\pi}{4}$, $k = 1$ and $\rho = 1$.

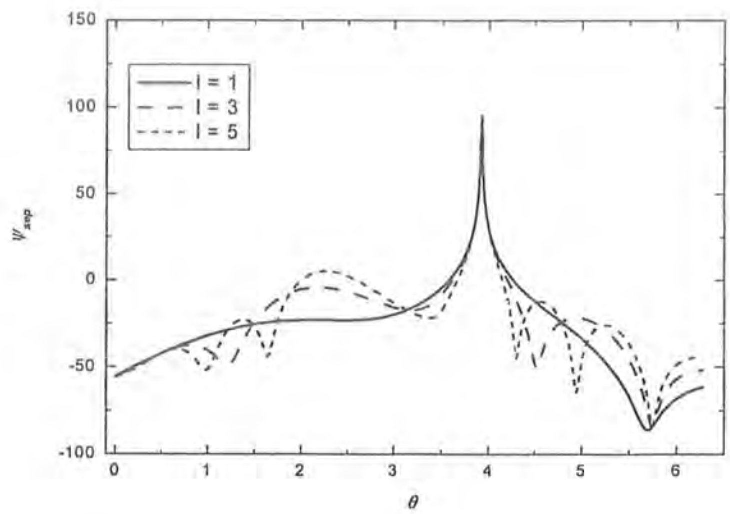

Fig. 6.5 Variation of the ψ_{sep} with θ at $\theta_0 = \frac{\pi}{4}$, $k = 1$ and $\rho = 5$.

- The effect of slit width parameter l is observed through the figures 6.4 and 6.5 in which $\theta_0 = \pi/4, k = 1$ and $l = 1, 3, 5$ for $\rho = 1$ and 5. It is noted that by keeping the other parameters fixed and increasing the parameter l causes more oscillations in the separated field and its amplitude decreases.

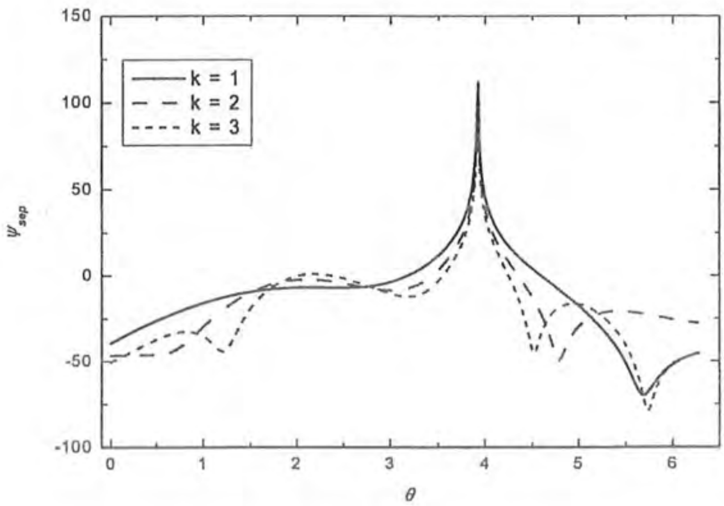

Fig. 6.6 Variation of the ψ_{sep} with θ at $\theta_0 = \frac{\pi}{4}$, $\rho = 1$ and $l = 1$.

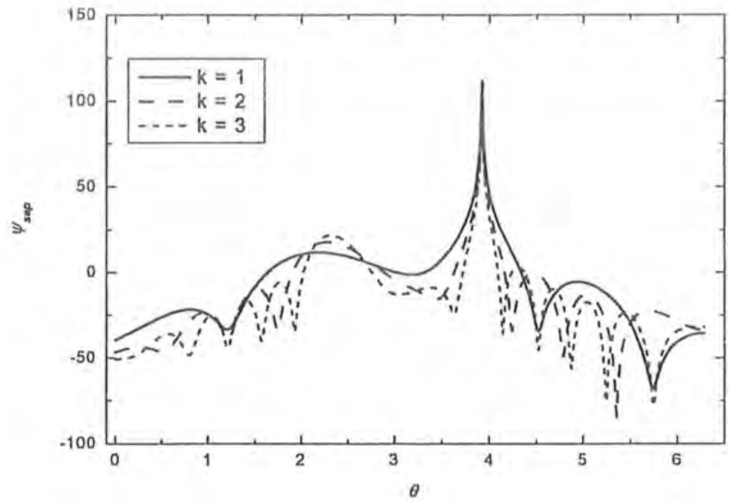

Fig. 6.7 Variation of the ψ_{sep} with θ at $\theta_0 = \frac{\pi}{4}$, $\rho = 1$ and $l = 5$.

- Finally in order to see the effects of wave number parameter k Figures 5.6 and 5.7 are plotted for $\theta_0 = \pi/4$, $\rho = 1$ and $k = 1, 2, 3$ for $l = 1$ and 5. These graphs depict that increasing the parameter k results in increasing oscillations in the separated field and the amplitude of the separated field decreases.

6.4 Concluding remarks

In this chapter, the diffraction of a plane acoustic wave by a slit in an infinite soft hard plane has been investigated rigorously with the help of Fourier integral transform, W-H technique and the method of steepest descent. The salient features of the presented analysis can be summarized as follows:

- The mathematical importance of the work on a slit in an infinite soft-hard expanse lies in the fact that in order to determine the unknown coefficient $A(\alpha)$ we have to determine four unknown functions $\overline{\psi}_{\pm1}(\alpha)$ and $\overline{\psi}_{\pm2}(\alpha)$, unlike for the strip geometry where we have to determine two unknown functions $\overline{\psi}_{\pm1}(\alpha)$.
- It is emphasized that the boundary value problem under consideration is a very special and substantial problem in the existing diffraction theory since it involves tedious mathematical calculations, it resulted into a matrix W-H equation which are usually considered difficult to handle and as pointed out by Rawlins [34] that two unusual features arose in the solution of the problem of a soft-hard half plane which is a comparatively easy problem as compared to the problem presented in this chapter (which is a three part boundary value problem).
- The two edges of the slit give rise to two diffracted fields (one from each edge) and the interaction of one edge upon the other edge.
- The consideration of a slit in an infinite soft-hard plane will help understand

acoustic diffraction and will go a step further to complete the discussion for the soft-hard half plane.

- The diffracted field is presented for the far-field situation and some graphs showing the effects of various parameters on the separated field are also plotted and discussed.

- To the author's knowledge the problem presented in this chapter has not been solved previously.

Chapter 7

Conclusions

The aim of this thesis is the study of scattering of acoustic waves by a barrier (or on a part of barrier) which satisfies soft-hard boundary conditions on it (or on a part of it). The soft-hard boundary conditions result into a canonical and substantial problem of the scattering theory. The imposition of soft-hard boundary conditions yield coupled W-H equations which cannot be decoupled trivially and one has to resort on matrix W-H approach in order to obtain the analytical solution of these equations. Since kernel factorization is one of the major steps in the successful application of the W-H technique which becomes more difficult in case of matrix kernel arose due to the application of soft-hard boundary conditions. For the problems considered in this thesis the kernel remained the same as in the previous works of Büyükaksoy [33] and Büyükaksoy et al [70] but for the sake of completeness and convenience, sufficient details have been incorporated in the factorization of kernel matrices and have been presented in appendices A and B of the thesis.

Some of the important conclusions which can be drawn from the chapters 3 – 6 are as follows:

(1) The attempted problems involving soft-hard boundary conditions have not been solved previously, constitutes a new contribution and in this way help and complete the discussion for the soft-hard half plane to some extent.

(2) The attempted problems have been presented both mathematically and graphically and the checks of correctness have also been applied where ever possible.

(3) The solutions of attempted problems required complex analytic treatment based on the application of W-H technique and involve the use of generalized functions, e.g., Dirac delta function, Fresnel function, Green's function, Hankel function, Heaviside unit step function, Maliuzhinetz function, Signum function, Whittaker function, etc. which requires a lot of mathematical skill and competence.

(4) For all the presented problems a far-zone solution is obtained and a good comparison is observed between the obtained and already known results.

(5) Several graphs for noting the effects of various parameters of interest are plotted and discussed.

Now a gist of chapter wise discussion is as follows:

Chapters 3 and 4

(i) The solutions of line source problems modified the results of plane wave situation by a multiplicative factor of the form (1.2) which is a well known and established fact in the literature [35, 46].

(ii) The solutions of point source excitations are based on line source excitations by following [46, 60, 61].

Chapters 5 and 6

(i) The strip/slit problems are solved under the physical assumption of the GTD i.e., the strip/slit length is large as compared to the incident wavelength and hence the integrals appearing in the analysis are approximated in terms of Whittaker functions.

(ii) The separated (singly diffracted field) and interacted (multiply diffracted fields) fields radiated by the strip/slit edges are calculated under the far zone approximation.

Future prospects in chapters 3 and 4

(i) The line source and point source excitations can further be extended to the cases of line and point impulses that is to include the effect of $\delta(t)$ also.

(ii) Chapters 3 and 4 can also be studied for the case of Gaussian pulse case.

(iii) In chapter 4, partially transmissive half plane can be replaced by partially conductive or modified absorbent half plane which results into other problems of practical interest and application.

Future prospects in chapters 5 and 6

(i) Soft-hard strip and slit in an infinite soft-hard expanse can be further extended to the case of line and point sources, line and point impulses and Gaussian impulse.

(ii) Presently strip/slit problems are considered for the fixed edges. One or both of these edges can be considered random and hence the said problems become more general having random edges.

(iii) The author is also planning to solve the strip/slit problems by a newly introduced technique SIT (Spectral Interaction Technique) and then to observe comparison between SIT and Noble's approach.

Chapter 8

Appendices

Appendix A

In this appendix sufficient details of the factorization of matrix $\mathbf{H}(\alpha)$

$$\mathbf{H}(\alpha) = \begin{bmatrix} 1 & \frac{1}{K(\alpha)} \\ -K(\alpha) & 1 \end{bmatrix}, \tag{A.1}$$

appeared in [33] are presented. In order to apply the Daniele-Kharapkov methods [20, 21] it is necessary to write the matrix $\mathbf{H}(\alpha)$ in the form

$$\mathbf{H}(\alpha) = \mathbf{I} + \hat{\mu}\mathbf{Q}, \tag{A.2}$$

where **I**, $\hat{\mu}$and **Q** have been defined in Section 2.7. Writing

$$\mathbf{H}(\alpha) = \begin{bmatrix} 1 & \frac{1}{K(\alpha)} \\ -K(\alpha) & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+\frac{1}{K(\alpha)} \\ -K(\alpha)+0 & 1+0 \end{bmatrix}. \tag{A.3}$$

Equation $(A.3)$ can be simplified to

$$\mathbf{H}(\alpha) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{K(\alpha)} \begin{bmatrix} 0 & 1 \\ -(k^2-\alpha^2) & 0 \end{bmatrix}. \tag{A.4}$$

Comparing Eqs. $(A.2)$ and $(A.4)$ implies $\hat{\mu} = \frac{1}{K(\alpha)}$ and

$$\mathbf{Q} = \begin{bmatrix} 0 & 1 \\ -(k^2-\alpha^2) & 0 \end{bmatrix}.$$

The matrix $\mathbf{H}(\alpha)$ belong to the class of matrices which can be represented in the form

$$\mathbf{H}(\alpha) = a_1(\alpha) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + a_2(\alpha) \begin{bmatrix} l_1(\alpha) & m(\alpha) \\ n(\alpha) & l_2(\alpha) \end{bmatrix}, \tag{A.5}$$

where $a_{1,2}(\alpha)$ are scalar functions and $l_{1,2}(\alpha)$, $m(\alpha)$ and $n(\alpha)$ are polynomials and can be factorized by the Kharapkov's method [21] in the form

$$\mathbf{H}_\pm(\alpha) = \sqrt{h_\pm(\alpha)} \left\{ \cosh\sqrt{f}F_\pm(\alpha) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{g\sqrt{f}} \sinh\sqrt{f}F_\pm(\alpha) \begin{bmatrix} l(\alpha) & m(\alpha) \\ n(\alpha) & -l(\alpha) \end{bmatrix} \right\}, \tag{A.6}$$

with

$$l(\alpha) = \frac{1}{2}\left[l_1(\alpha) - l_2(\alpha)\right], \tag{A.7}$$

and

$$l^2(\alpha) + m(\alpha)n(\alpha) = g^2(\alpha)f(\alpha). \tag{A.8}$$

Here $f(\alpha)$ and $g(\alpha)$ are polynomials and set of zeros of $f(\alpha)$ is formed by the zeros of odd multiplicity of the polynomial (l^2+mn) each taken once and $h_{\pm(\alpha)}$ and $F_{\pm}(\alpha)$ are the functions formed by multiplicative and additive split of

$$h(\alpha) = \det \mathbf{H}(\alpha) \tag{A.9}$$

and

$$F(\alpha) = \frac{1}{2\sqrt{f}} \ln\left[\frac{a_1 + a_2 g\sqrt{f}}{a_1 - a_2 g\sqrt{f}}\right]. \tag{A.10}$$

In the forms

$$h(\alpha) = h_+(\alpha)\, h_-(\alpha), \tag{A.11}$$

and

$$F(\alpha) = F_+(\alpha) + F_-(\alpha). \tag{A.12}$$

Comparing Eqs. $(A.4)$ and $(A.5)$ we have

$$a_1(\alpha) = 1,\ a_2(\alpha) = \frac{1}{K(\alpha)},\ l_1(\alpha) = 0,\ m(\alpha) = 1,\ n(\alpha) = -(k^2-\alpha^2),\ l_2(\alpha) = 0. \tag{A.13}$$

Now

$$l(\alpha) = \frac{1}{2}\left[l_1(\alpha) - l_2(\alpha)\right] = \frac{1}{2}\left[0-0\right] = 0, \tag{A.14}$$

also $(l^2+mn) = 0^2 + (1)\left(-(k^2-\alpha^2)\right) = g^2(\alpha) f(\alpha)$. This implies that

$$f(\alpha) = \alpha^2 - k^2,\ g(\alpha) = 1 \tag{A.15}$$

also

$$\det \mathbf{H}(\alpha) = \begin{vmatrix} 1 & \frac{1}{K(\alpha)} \\ -K(\alpha) & 1 \end{vmatrix} = 1 + 1 = 2 \neq 0. \tag{A.16}$$

Now consider $(A.10)$ and substitute various values from $(A.13)$ and $(A.15)$ into $(A.10)$ and simplifying will yield

$$F(\alpha) = \frac{1}{2\sqrt{\alpha^2 - k^2}} \ln \left[\frac{1 + \frac{1}{K(\alpha)}\sqrt{\alpha^2 - k^2}}{1 - \frac{1}{K(\alpha)}\sqrt{\alpha^2 - k^2}} \right],$$

which simplifies to

$$F(\alpha) = \frac{1}{2\sqrt{\alpha^2 - k^2}} \ln \left[\frac{K(\alpha)}{\sqrt{\alpha^2 - k^2}} \right]. \tag{A.17}$$

Now for additive split of $(A.17)$ we consider

$$F_{\pm}(\alpha) = \pm\frac{1}{2\pi i}\int_{\pounds_{\pm}} \frac{F(\tau_1)}{\tau_1 - \alpha}\, d\tau_1, \tag{A.18}$$

where

$$F_{-}(\alpha) = -\frac{1}{4\pi i}\int_{\pounds_{-}} \frac{1}{\sqrt{\tau_1^2 - k^2}} \ln \left\{ \frac{K(\tau_1)}{\sqrt{\tau_1^2 - k^2}} \right\} \frac{d\tau_1}{\tau_1 - \alpha}. \tag{A.19}$$

The path of integration $\pounds_{-}$ can be deformed into a contour C_{upper} enclosing the branch point at $\alpha = k$ (see Fig 3.2). The contributions from a large semicircle C_{R_0} in the upper half plane and from a small circle $C_{\widehat{\rho}_0}$ around the branch point at $\alpha = k$ tend to zero while the radii C_{R_0} and $C_{\widehat{\rho}_0}$ tend to infinity and zero, respectively. Since the integral does not have any pole singularities in the upper half plane, the integral on $\pounds_{-}$ can be written as follows:

$$F_{-}(\alpha) = -\frac{1}{4\pi i}\int_{C_{upper}} \frac{1}{\sqrt{\tau_1 - k^2}} \ln \left\{ \frac{K(\tau_1)}{\sqrt{\tau_1^2 - k^2}} \right\} \frac{d\tau_1}{\tau_1 - \alpha}. \tag{A.20}$$

After some mathematical manipulation Eq. $(A.20)$ can be simplified to

$$F_{-}(\alpha) = -\frac{1}{4\pi i}\int_{C_{upper}} \left[\frac{1}{\sqrt{\tau_1^2 - k^2}} \left\{ \ln\sqrt{\tau_1 - k} + \ln\left(\frac{i\sqrt{\tau_1 + k}}{\sqrt{\tau_1^2 - k^2}} \right) \right\} \right] \frac{d\tau_1}{\tau_1 - \alpha}. \tag{A.21}$$

The second term in expression $(A.21)$ when evaluated on both sides of the branch cut $C^{(+)}_{upper}$ and $C^{(-)}_{upper}$ vanishes and hence expression $(A.21)$ is further simplified to

$$F_{-}(\alpha) = -\frac{1}{8\pi i}\int_{C_{upper}} \left[\frac{1}{\sqrt{\tau_1^2 - k^2}} \{\ln|\tau_1 - k| + i \arg(\tau_1 - k)\}\right] \frac{d\tau_1}{\tau_1 - \alpha}. \quad (A.22)$$

Expression $(A.22)$ when evaluated along both sides of the branch cut $C^{(+)}_{upper}$ and $C^{(-)}_{upper}$ resulted into the fact that first term within {} bracket vanishes and therefore it implies

$$F_{-}(\alpha) = -\frac{1}{8\pi}\int_{k}^{\infty} \left[\frac{1}{\sqrt{\tau_1^2 - k^2}} \arg(\tau_1 - k)\right] \frac{d\tau_1}{\tau_1 - \alpha}. \quad (A.23)$$

Now substituting $\tau_1 - \alpha = x$ in $(A.23)$, the integral will take the form

$$F_{-}(\alpha) = -\frac{1}{4}\int_{k-\alpha}^{\infty} \frac{dx}{x\sqrt{(x+\alpha)^2 - k^2}}. \quad (A.24)$$

The above integral in $(A.24)$ can be easily solved and the result is as follows

$$F_{-}(\alpha) = -\frac{1}{4\sqrt{\alpha^2 - k^2}} \ln\left[\frac{\alpha + \frac{\sqrt{\alpha^2-k^2}}{k}}{-1}\right], \quad (A.25)$$

which can be further simplified to

$$F_{-}(\alpha) = \frac{1}{4\sqrt{\alpha^2 - k^2}}\left[iarccos\left(\frac{\alpha}{k}\right) - i\pi\right]. \quad (A.26)$$

Using $F_{-}(-\alpha) = F_{+}(\alpha)$ so

$$F_{+}(\alpha) = \frac{1}{4\sqrt{\alpha^2 - k^2}}\left[iarccos\left(\frac{-\alpha}{k}\right) - i\pi\right]. \quad (A.27)$$

Using the property $arccos(-x) = \pi - arccos(x)$ in expression $(A.27)$ will result in,

$$F_{+}(\alpha) = -\frac{i}{4\sqrt{\alpha^2 - k^2}} \cos^{-1}\left(\frac{\alpha}{k}\right). \quad (A.28)$$

Now as $\varkappa(\alpha) = \sqrt{f} F_{+}(\alpha)$. Therefore

$$\varkappa(\alpha) = -\frac{i}{4} \cos^{-1}\left(\frac{\alpha}{k}\right), \tag{A.29}$$

and

$$\varkappa(-\alpha) = -\frac{i}{4}\left[\pi - \cos^{-1}\left(\frac{\alpha}{k}\right)\right]. \tag{A.30}$$

The explicit factor $\mathbf{H}_{+}(\alpha)$ can be obtained by substituting the values of $h_{+}(\alpha)$, $l(\alpha)$, $m(\alpha)$, $n(\alpha)$ and $\varkappa(\alpha)$ into $(A.6)$ which gives

$$\mathbf{H}_{+}(\alpha) = 2^{\frac{1}{4}} \left\{ \cosh \varkappa(\alpha) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{\sqrt{\alpha^2 - k^2}} \sinh \varkappa(\alpha) \begin{bmatrix} 0 & 1 \\ -(k^2 - \alpha^2) & 0 \end{bmatrix} \right\}. \tag{A.31}$$

Expression $(A.31)$ is finally simplified to

$$\mathbf{H}_{+}(\alpha) = 2^{\frac{1}{4}} \begin{bmatrix} \cosh \varkappa(\alpha) & \sinh \varkappa(\alpha) / \gamma(\alpha) \\ \gamma(\alpha) \sinh \varkappa(\alpha) & \cosh \varkappa(\alpha) \end{bmatrix}, \tag{A.32}$$

which is the required factor.

Appendix B

This appendix is devoted to present sufficient details of the matrix $\mathbf{W}(\alpha)$ which arose in [70],

$$\mathbf{W}(\alpha) = \begin{bmatrix} 1 & -\frac{1}{K(\alpha)} \\ K(\alpha) & \frac{2k}{\eta K(\alpha)} + 1 \end{bmatrix}. \tag{B.1}$$

In order to apply the Daniele-Kharapkov methods [20, 21] it is necessary to write the matrix $\mathbf{W}(\alpha)$ in the form

$$\mathbf{W}(\alpha) = I + \hat{\mu} Q. \tag{B.2}$$

Eq. $(B.1)$ can be written in the form

$$\mathbf{W}(\alpha) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{K(\alpha)} \begin{bmatrix} 0 & -1 \\ (k^2 - \alpha^2) & \frac{2k}{\eta} \end{bmatrix}. \tag{B.3}$$

Comparing Eqs. $(B.2)$ and $(B.3)$ implies $\hat{\mu} = \frac{1}{K(\alpha)}$ and $Q = \begin{bmatrix} 0 & -1 \\ (k^2 - \alpha^2) & \frac{2k}{\eta} \end{bmatrix}$.

The matrix $\mathbf{W}(\alpha)$ belongs to the class of matrices which can be represented in the form

$$\mathbf{W}(\alpha) = a_1(\alpha) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + a_2(\alpha) \begin{bmatrix} l_1(\alpha) & m(\alpha) \\ n(\alpha) & l_2(\alpha) \end{bmatrix}. \tag{B.4}$$

Where $a_{1,2}(\alpha)$ are scalar functions and $l_{1,2}(\alpha)$, $m(\alpha)$ and $n(\alpha)$ are polynomials and can be factorized by the Kharapkov's method [21] in the form

$$\mathrm{W}_{\pm}(\alpha) = \sqrt{w_{\pm}(\alpha)}\left\{\cosh\sqrt{f}F_{\pm}(\alpha)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \frac{1}{g\sqrt{f}}\sinh\sqrt{f}F_{\pm}(\alpha)\begin{bmatrix}l(\alpha) & m(\alpha)\\ n(\alpha) & -l(\alpha)\end{bmatrix}\right\}, \tag{B.5}$$

with

$$l(\alpha) = \frac{1}{2}\left[l_1(\alpha) - l_2(\alpha)\right], \tag{B.6}$$

and

$$l^2(\alpha) + m(\alpha)n(\alpha) = g^2(\alpha)f(\alpha). \tag{B.7}$$

Here $f(\alpha)$ and $g(\alpha)$ are polynomials and set of zeros of $f(\alpha)$ is formed by the zeros of odd multiplicity of the polynomial $(l^2 + mn)$ each taken once and $w_{\pm(\alpha)}$ and $F_{\pm}(\alpha)$ are the functions formed by multiplicative and additive split of

$$w(\alpha) = \det \mathrm{W}(\alpha) \tag{B.8}$$

$$F(\alpha) = \frac{1}{2\sqrt{f}}\ln\left[\frac{a_1 + a_2 g\sqrt{f}}{a_1 - a_2 g\sqrt{f}}\right], \tag{B.9}$$

$$w(\alpha) = w_{+}(\alpha)w_{-}(\alpha), \tag{B.10}$$

$$F(\alpha) = F_{+}(\alpha) + F_{-}(\alpha). \tag{B.11}$$

Comparing Eqs. $(B.3)$ and $(B.4)$ will give

$$a_1(\alpha) = 1,\; a_2(\alpha) = \frac{1}{K(\alpha)}\; l_1(\alpha) = 0,\; m(\alpha) = -1,\; n(\alpha) = (k^2 - \alpha^2),\; l_2(\alpha) = \frac{2k}{\eta}. \tag{B.12}$$

Now

$$l(\alpha) = -\frac{k}{\eta}, \tag{B.13}$$

also $(l^2 + mn) = \frac{k^2}{\eta^2} - (k^2 - \alpha^2) = g^2(\alpha) f(\alpha)$ or $\alpha^2 - \sigma_1^2 = g^2(\alpha) f(\alpha)$.

This implies

$$f(\alpha) = \alpha^2 - \sigma_1^2,\ g(\alpha) = 1. \tag{B.14}$$

Also

$$\det W(\alpha) = \begin{vmatrix} 1 & -\frac{1}{K(\alpha)} \\ K(\alpha) & \frac{2k}{\eta K(\alpha)} + 1 \end{vmatrix} = \frac{2}{\eta\kappa(\alpha)}, \tag{B.15}$$

where $\kappa(\alpha) = \frac{K(\alpha)}{k+\eta K(\alpha)}$. Now consider $(B.9)$ and substitute various values from $(B.12)$ and $(B.14)$ into $(B.9)$ will yield

$$F(\alpha) = \frac{1}{2\sqrt{\alpha^2 - \sigma_1^2}} \ln\left[\frac{\sqrt{k^2 - \alpha^2}}{-\frac{k}{\eta} + \sqrt{\alpha^2 - \sigma_1^2}}\right]. \tag{B.16}$$

For additive split of $(B.16)$, consider

$$F_{\pm}(\alpha) = \pm\frac{1}{2\pi i}\int \frac{F(\tau_1)}{\tau_1 - \alpha}\, d\tau_1, \tag{B.17}$$

where

$$F_{-}(\alpha) = -\frac{1}{2\pi i}\int \frac{F(\tau_1)}{\tau_1 - \alpha}\, d\tau_1$$

$$F_{-}(\alpha) = -\frac{1}{2\pi i}\int_{\pounds_-} \frac{1}{2\sqrt{\tau_1^2 - \sigma_1^2}} \ln\left\{\frac{\sqrt{k^2 - \tau_1^2}}{-\frac{k}{\eta} + \sqrt{\tau_1^2 - \sigma_1^2}}\right\} \frac{d\tau_1}{\tau_1 - \alpha}. \tag{B.18}$$

The path of integration $\pounds_-$ can be deformed into a contour C_{upper} enclosing the branch point at $\alpha = k$ (Fig 3.2). The contributions from a large semicircle C_{R_0} in the upper half plane and from a small circle $C_{\widehat{\rho}_0}$ around the branch point at $\alpha = k$ tend to zero while the radii C_{R_0} and C_{ρ_0} tend to infinity and zero, respectively. Since the integral does not have any pole singularities in the upper half plane, the integral on $\mathcal{L}_-$ can be written as follows:

$$F_-(\alpha) = -\frac{1}{4\pi i}\left[\int_{C_{upper}} \frac{1}{\sqrt{\tau_1 - \sigma_1^2}} \ln\sqrt{\tau_1 - k} + \frac{1}{\sqrt{\tau_1^2 - \sigma_1^2}} \ln\frac{i\sqrt{\tau_1 + k}}{\sqrt{\tau_1^2 - \sigma_1^2 - \frac{k}{\eta}}}\right]\frac{d\tau_1}{\tau_1 - \alpha}. \tag{B.19}$$

The second term in expression $(B.19)$ when evaluated on both sides of the branch cut $C^{(+)}_{upper}$ and $C^{(-)}_{upper}$ vanishes and hence expression $(B.19)$ is further simplified to

$$F_-(\alpha) = -\frac{1}{8\pi i}\int_{C_{upper}}\left[\frac{1}{\sqrt{\tau_1^2 - \sigma_1^2}}\left\{\ln\sqrt{\tau_1 - k} + i\arg(\tau_1 - k)\right\}\right]\frac{d\tau_1}{\tau_1 - \alpha}. \tag{B.20}$$

After some mathematical manipulation in Eq $(B.20)$, it can be simplified to

$$F_-(\alpha) = -\frac{1}{8\pi}\int_{\tau_1 = k} \frac{1}{\sqrt{\tau_1^2 - \sigma_1^2}} \arg(\tau_1 - k)\frac{d\tau_1}{\tau_1 - \alpha}. \tag{B.21}$$

Now substituting $\tau_1 - \alpha = x$ in $(B.21)$, the integral will take the form

$$F_-(\alpha) = -\frac{1}{4}\int_{k-\alpha}^{\infty} \frac{dx}{x\sqrt{(x+\alpha)^2 - \sigma_1^2}} \tag{B.22}$$

The above integral in $(B.22)$ can be easily solved and the result is as follows

$$F_-(\alpha) = \frac{1}{4\sqrt{\alpha^2 - \sigma_1^2}} \ln\left[\frac{\alpha + \left(\sqrt{\alpha^2 - k^2}\right)(k - \alpha)}{(\alpha k - a^2) + \sqrt{k^2 - a^2}\sqrt{\alpha^2 - a^2}}\right], \tag{B.23}$$

as $F_-(-\alpha) = F_+(\alpha)$ so

$$F_+(\alpha) = \frac{1}{4\sqrt{\alpha^2 - \sigma_1^2}} \ln\left[\frac{\alpha + \left(\sqrt{\alpha^2 - \sigma_1^2} - \alpha\right)(k + \alpha)}{-\alpha k - \sigma_1^2 + \sqrt{k^2 - \sigma_1^2}\sqrt{\alpha^2 - \sigma_1^2}}\right]. \tag{B.24}$$

Defining

$$\varkappa(\alpha) = \sqrt{f(\alpha)}F_+(\alpha), \tag{B.25}$$

and the explicit factor $\mathrm{W}_+(\alpha)$ can be obtained by substituting the values of $w_+(\alpha)$, $l(\alpha)$, $m(\alpha)$, $n(\alpha)$ and $\varkappa(\alpha)$ into $(B.5)$ which gives

$$\mathrm{W}_+(\alpha) = \left(\frac{2}{\eta}\right)^{\frac{1}{n}} \frac{1}{\sqrt{\kappa_+(\alpha)}} \left\{ \cosh \varkappa(\alpha) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{\sqrt{\alpha^2 - \sigma_1^2}} \sinh \varkappa(\alpha) \begin{bmatrix} \frac{-k}{n} & -1 \\ k^2 - \alpha^2 & \frac{k}{n} \end{bmatrix} \right\}. \tag{B.26}$$

Simplification of $(B.26)$ will finally result into

$$\mathrm{W}_+(\alpha) = \left(\frac{2}{\eta}\right)^{\frac{1}{n}} \frac{1}{\sqrt{\kappa_+(\alpha)}} \begin{bmatrix} \cosh(\varkappa(\alpha)) - \frac{k}{n}\frac{\sinh(\varkappa(\alpha))}{\sqrt{\alpha^2-\sigma_1^2}} & \frac{-\sinh(\varkappa(\alpha))}{\sqrt{\alpha^2-\sigma_1^2}} \\ (k^2-\alpha^2)\frac{\sinh(\varkappa(\alpha))}{\sqrt{\alpha^2-\sigma_1^2}} & \cosh(\varkappa(\alpha)) + \frac{k}{n}\frac{\sinh(\varkappa(\alpha))}{\sqrt{\alpha^2-\sigma_1^2}} \end{bmatrix}, \tag{B.27}$$

with

$$\mathrm{W}_-(\alpha) = \mathrm{W}_+(-\alpha). \tag{B.28}$$

Appendix C

In this appendix, the following integral has been evaluated.

$$I_1 \approx \frac{k}{2\pi}\int_{-\infty}^{\infty}\left(\frac{e^{\frac{-i\pi}{2}}}{2\pi}\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sqrt{\kappa_+\left(k\eta_1\cos\theta\right)}\right)$$
$$\left[\left\{\cosh\varkappa\left(k\eta_1\cos\theta\right)+\frac{k\eta_1}{\eta}\frac{\sinh\varkappa\left(k\eta_1\cos\theta\right)}{\sqrt{k^2\eta_1^2\cos^2\theta-\sigma_1^2}}\right\}\left\{i\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sin\frac{\theta}{2}\sin\frac{\theta_0}{2}\right.\right.$$
$$\left.\times\frac{\sqrt{\kappa_+\left(k\eta_1\cos\theta_0\right)}}{\sqrt{k^2\eta_1^2\cos^2\theta_0-\sigma_1^2}}\sinh\varkappa\left(k\eta_1\cos\theta_0\right)\right\}+\frac{\sinh\varkappa\left(k\eta_1\cos\theta\right)}{\sqrt{k^2\eta_1^2\cos^2\theta-\sigma_1^2}}$$
$$\times\left\{\left(i\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sin\frac{\theta}{2}\sin\frac{\theta_0}{2}\sqrt{\kappa_+\left(k\eta_1\cos\theta_0\right)}\right)\right.$$
$$\times\left(\cosh\varkappa\left(k\eta_1\cos\theta_0\right)-\frac{k\eta_1}{\eta}\frac{\sinh\varkappa\left(k\eta_1\cos\theta_0\right)}{\sqrt{k^2\eta_1^2\cos^2\theta_0-\sigma_1^2}}\right)$$
$$\left.\left.+\sin\frac{\theta}{2}\sin\frac{\theta_0}{2}(\cos\theta+\cos\theta_0)p^{**}\right\}\right]$$
$$\times\left[\widetilde{\mathcal{F}}\left(\sqrt{2k\eta_1\rho}\cos\frac{\theta-\theta_0}{2}\right)\right]$$
$$\times\frac{\exp[ik\eta_1\rho+ik\eta_1\rho_0+ik\mu z_0-ik\mu z]}{\sqrt{\rho\rho_0}}d\mu. \tag{C.1}$$

Substitute

$$\mu_1=\sqrt{2\rho}\cos\frac{\theta-\theta_0}{2},$$

and

$$f_1(\mu)=\left(\frac{e^{\frac{-i\pi}{2}}}{2\pi}\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sqrt{\kappa_+(k\eta_1\cos\theta)}\right)$$

$$\left[\left\{\cosh\varkappa(k\eta_1\cos\theta)+\frac{k\eta_1}{\eta}\frac{\sinh\varkappa(k\eta_1\cos\theta)}{\sqrt{k^2\eta_1^2\cos^2\theta-\sigma_1^2}}\right\}\left\{i\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sin\frac{\theta}{2}\sin\frac{\theta_0}{2}\right.\right.$$

$$\left.\times\frac{\sqrt{\kappa_+(k\eta_1\cos\theta_0)}}{\sqrt{k^2\eta_1^2\cos^2\theta_0-\sigma_1^2}}\sinh\varkappa(k\eta_1\cos\theta_0)\right\}+\frac{\sinh\varkappa(k\eta_1\cos\theta)}{\sqrt{k^2\eta_1^2\cos^2\theta-\sigma_1^2}}$$

$$\times\left\{\left(i\left(\frac{\eta}{2}\right)^{\frac{1}{4}}\sin\frac{\theta}{2}\sin\frac{\theta_0}{2}\sqrt{\kappa_+(k\eta_1\cos\theta_0)}\right)\right.$$

$$\times\left(\cosh\varkappa(k\eta_1\cos\theta_0)-\frac{k\eta_1}{\eta}\frac{\sinh\varkappa(k\eta_1\cos\theta_0)}{\sqrt{k^2\eta_1^2\cos^2\theta_0-\sigma_1^2}}\right)$$

$$\left.\left.+\sin\frac{\theta}{2}\sin\frac{\theta_0}{2}(\cos\theta+\cos\theta_0)p^{**}\right\}\right].$$

The Eq. $(C.1)$ will take the form

$$I_1=\frac{k}{2\pi}\int_{-\infty}^{\infty}f_1(\mu)\widetilde{\mathcal{F}}\left(\mu_1\left(k\sqrt{1-\mu^2}\right)^{\frac{1}{2}}\right)\frac{\exp[ik\eta_1(\rho+\rho_0)-ik\mu(z-z_0)]}{\sqrt{\rho\rho_0}}d\mu. \tag{C.2}$$

Making use of the result

$$\int_z^{\infty}e^{i\widetilde{\lambda}t^2}dt_1=e^{i\widetilde{\lambda}z^2}\frac{\mathcal{F}\left(\widetilde{\lambda}^{\frac{1}{2}}z\right)}{\widetilde{\lambda}^{\frac{1}{2}}}, \tag{C.3}$$

expression $(C.2)$ will take the form [60, 61]

$$I_1=\frac{k}{2\pi}\int_{\mu_1}^{\infty}\int_{-\infty}^{\infty}\frac{f_1(\mu)\left(k\sqrt{1-\mu^2}\right)^{\frac{1}{2}}}{\sqrt{\rho\rho_0}}e^{-ik[\mu(z-z_0)-\sqrt{1-\mu^2}\left(\rho+\rho_0+t_1^2-\mu^2\right)]}d\mu dt_1. \tag{C.4}$$

Let $g_1(\mu)=\frac{f_1(\mu)\left(k\sqrt{1-\mu^2}\right)^{\frac{1}{2}}}{\sqrt{\rho\rho_0}}$ and consider the integral

$$I_1'=\int_{-\infty}^{\infty}g_1(\mu)\,e^{-ik[\mu(z-z_0)-\sqrt{1-\mu^2}\left(\rho+\rho_0^2+t_1^2-\mu^2\right)]}d\mu. \tag{C.5}$$

In order to solve the integral (C.5) the following substitutions have to be used

$$\mu = \cos\Theta, \qquad \eta_1 = \sqrt{1-\mu^2} = \sin\Theta \tag{C.6a}$$

and

$$z - z_0 = R_1 \cos\nu, \qquad P = R_1 \sin\nu. \tag{C.6b}$$

I_1' takes the form

$$I_1' = \int_{-\infty}^{\infty} g_1(\Theta)\, e^{-ikR_1 \cos(\Theta+\nu)} (-\sin\Theta)\, d\Theta. \tag{C.7}$$

By applying the method of steepest descent, the integral I_1' can be solved. For this, deforming the contour of integration so as to pass through the point of steepest descent $\Theta = -\nu$, so that the major part of integrand is given by integration over the part of deformed contour near $-\nu$ with $g_1(\Theta)$ is slowly varying around it. Therefore,

$$\begin{aligned} I_1' &\approx \pi g_1(-\nu) \sin\nu H_0^{(1)}(kR_1), \\ &\approx \pi g_1(\Omega)\, H_0^{(1)}[k\{(z-z_0)^2 + P^2\}^{\frac{1}{2}}]\Omega, \end{aligned} \tag{C.8}$$

where $\Omega = \frac{P}{[(z-z_0)^2+P^2]^{\frac{1}{2}}}$.

Using (C.8), expression (C.4) will take the form

$$\begin{aligned} I_1 = \frac{k}{2}\int_{\mu_1}^{\infty} g_1(\Omega)\, H_0^{(1)}[k\{(z-z_0)^2 + (\rho+\rho_0+t_1^2-\mu_1^2)^2\}^{\frac{1}{2}}] \\ \times \frac{(\rho+\rho_0+t_1^2-\mu_1^2)}{\{(z-z_0)^2+(\rho+\rho_0+t_1^2-\mu_1^2)^2\}^{\frac{1}{2}}} dt_1. \end{aligned} \tag{C.9}$$

Introducing the following substitutions

$$t_1^2 = -a_1 + \sqrt{a_1^2 + R_{11}^2 \sinh^2 u}, \; a_1 = \rho+\rho_0-\mu_1^2 \text{ and } R_{11}^2 = (z-z_0)^2 + a_1^2, \tag{C.10}$$

in (C.9), will yield

$$I_1 = \frac{k}{4}\int_{\varepsilon_1}^{\infty} [g_1(\widetilde{\Omega}) H_0^{(1)}(kR_{11}\cosh u) \left(\sqrt{a_1^2 + R_{11}^2\sinh^2 u} + a_1\right)^{\frac{1}{2}}] du, \tag{C.11}$$

where

$$\widetilde{\Omega} = \frac{\sqrt{a_1^2 + R_{11}^2 \sinh^2 u}}{R_{11} \cosh u} \qquad \text{and} \qquad \varepsilon_{1=} \sinh^{-1}\{\frac{\mu_1\sqrt{\mu_1^2 + 2a_1}}{R_{11}}\}.$$

The integral in expression $(C.11)$ can be solved asymptotically for $kR_{11}\cosh u \gg 1$. Therefore the Hankel function $H_0^{(1)}$ can be replaced by the first term of its asymptotic expansion to give

$$I_1 = \frac{k}{4}\int_{\varepsilon_1}^{\infty} [g_1(\widetilde{\Omega})\left\{\sqrt{\frac{2}{\pi k R_{11}\cosh u}}\right\}^{\frac{1}{2}} e^{i\left(kR_{11}\cosh u - \frac{\pi}{4}\right)}\left(\sqrt{a_1^2 + R_{11}^2\sinh^2 u} + a_1\right)^{\frac{1}{2}}]du.$$

(C.12)

By substituting $\widehat{\tau} = \sqrt{2kR_{11}}\sinh u$ in the integral appearing in expression $(C.12)$, it will take the form

$$I_1 = \sqrt{\frac{2k}{\pi}}\frac{e^{-i\frac{\pi}{4}+ikR_{11}}}{2}\int_{\tau_{R_1}}^{\infty} g_1(\widehat{\tau})\, e^{-i\tau^2} d\widehat{\tau}, \tag{C.13}$$

where

$$g_1(\widehat{\tau}) = \left\{\frac{\sqrt{\widehat{\tau}^2\left(\widehat{\tau}^2 + 2kR_{11}\right) + k^2 a_1^2} + ka_1}{\left(\widehat{\tau}^2 + kR_{11}\right)\left(\widehat{\tau}^2 + 2kR_{11}\right)}\right\}^{\frac{1}{2}} g_1(\widehat{\Omega}), \tag{C.14}$$

$$\widehat{\Omega} = \frac{\sqrt{\widehat{\tau}^2\left(\widehat{\tau}^2 + 2kR_{11}\right) + k^2 a_1^2}}{\left(\widehat{\tau}^2 + kR_{11}\right)} \tag{C.15}$$

and

$$\tau_{R_1} = \sqrt{k\left(R_1 - R_{11}\right)} \text{ and } \in_1 = sgn\left(\tau_{R_1}\right). \tag{C.16}$$

An asymptotic expansion of I_1 then follows by putting $\widehat{\tau}$ equal to its lower limit value in the non exponential part of the integrand plus the contribution from $\widehat{\tau} = 0$ if zero lies in the interval of integration. Hence in our case, for I_1 it is given in Eq. (4.94).

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Wiener–Hopf analysis of diffraction of acoustic waves by a soft/hard half-plane

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IN THIS PAPER, firstly, the far field due to a line source scattering of acoustic waves by a soft/hard half-plane is investigated. It is observed that if the line source is shifted to a large distance, the results differ from those of [16] by a multiplicative factor. Subsequently, the scattering due to a point source is also examined using the results of line source excitation. Both the problems are solved using the Wiener–Hopf technique and the steepest descent method. Some graphs showing the effects of various parameters on the diffracted field produced by the line source incidence are also plotted.

Key words: diffraction, Wiener–Hopf technique, line source diffraction, point source diffraction, steepest descent method.

1. Introduction

NUMEROUS FORMER INVESTIGATIONS have been devoted to the study of classical problems of line source and point source diffractions of electromagnetic and acoustic waves by various types of half-planes. To name a few only, e.g. the line source diffraction of electromagnetic waves by a perfectly conducting half-plane was investigated by JONES [1]. Later on, JONES [2] considered the problem of line source diffraction of acoustic waves by a hard half-plane attached to a wake in still air as well as when the medium is convective. RAWLINS [3] studied the diffraction of cylinderical waves from a line source by an absorbing half-plane in the presence of subsonic flow. AHMAD [4] considered the line source diffraction of acoustic waves by an absorbing half-plane using Myre's condition. HUSSAIN [5] analyzed the line source diffraction of electromagnetic waves by a perfectly conducting half-plane in a bi-isotropic medium. Recently AYUB *et al.* [6] studied the magnetic line source difffraction of electromagnetic waves by an impedance step.

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Line source and point source scattering of acoustic waves by the junction of transmissive and soft–hard half planes

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ABSTRACT

Firstly, the analysis of [A. Büyükaksoy, G. Cinar, A.H. Serbest, Scattering of plane waves by the junction of transmissive and soft–hard half planes, ZAMP 55 (2004) 483–499] for the scattering of plane waves by the junction of transmissive and soft–hard half planes is extended to the case of a line source. The introduction of the line source changes the incident field and the method of solution requires a careful analysis in calculating the scattered field. The graphical results are presented using MATHEMATICA. We observe that the graphs of the plane wave situation [A. Büyükaksoy, G. Cinar, A.H. Serbest, Scattering of plane waves by the junction of transmissive and soft–hard half planes, ZAMP 55 (2004) 483–499] can be recovered by shifting the line source to a large distance. Subsequently, the problem is further extended to the case of scattering due to a point source using the results obtained for a line source excitation. The introduction of a point source (three dimensions) involves another variable which then requires the calculation of an additional integral appearing in the inverse transform.

1. Introduction

The Wiener–Hopf (WH) technique provides a significant extension of the large class of problems that can be solved by Fourier, Laplace and Mellin integral transform [1]. The WH technique provides us an approach for considering the diffraction of waves by a single half plane [1]. However there are problems in dealing with other configurations which are first attacked by using matrix version of WH equations. A comprehensive procedure for tackling the matrix version of these equations is not yet available because it is not normally easy to split the matrix into the appropriate half planes. The noncommutativity of the factor matrices and the requirement of the radiation conditions also present further problems. Nevertheless the development and improvement of this technique is progressing steadily [2]. For example the Wiener–Hopf Hilbert method introduced by Hurd [3], Rawlins [4] and Rawlins and Williams [5] is a powerful tool in the case when kernel matrix has only branch point singularities, while the Daniele–Kharapkov method proposed by Daniele [6] and Kharapkov [7] is effective for the class of matrices having only pole singularities and branch-cut singularities besides pole singularities [8–12].

Diffraction from a two part surface is an important topic in diffraction theory and constitute a canonical problem for diffraction due to abrupt changes in material properties. Recently, Büyükaksoy et al. [13] considered the scattering of plane waves by a two part surface. They developed a high frequency solution for the diffraction of plane waves by the junction of two half planes. One half plane is characterized by partially transmissive boundary conditions and the other is soft at the

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Diffraction of a plane wave by a soft–hard strip

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ABSTRACT

In this paper we have studied the problem of diffraction of a plane wave by a finite soft–hard strip. By using the Fourier transform the boundary value problem is reduced to a matrix Wiener–Hopf equation. Using the matrix factorization of the kernel matrix, the problem is solved for two coupled equations using the Wiener–Hopf technique and the method of steepest descent. It is observed that the diffracted field is the sum of the fields produced by the two edges of the strip and an interaction field. Some graphs showing the effects of various parameters on the field produced by two edges of the strip are also plotted.

1. Introduction

Scattering of waves by strips/slits is an important topic both in acoustics and electromagnetics. It has attracted the attention of many researchers [1–15]. A variety of analytical and numerical techniques have been used to study scattering from strips/slits. To name a few only e.g., Morse and Rubenstein [1] studied the diffraction of acoustic waves by a strip using method of separation of variables. Bowman et al. [2] summarized and reviewed much of the work done on strip. Another important detail of works, using the Wiener–Hopf (WH) technique, consists of Jones [3], Kobayashi [4], Noble [5], Faulkner [6], Cinar and Büyükaksoy [7], Serbest and Büyükaksoy [8], Büyükaksoy and Alkumru [9], Asghar [10], Asghar et al. [11] and Ayub et al. [12,13]. Recently Ahmad and Naqvi [14] and Imran et al. [15] studied the electromagnetic scattering from a two dimensional perfect electromagnetic conductor (PEMC) strip and PEMC strip grating and by an infinitely long conducting strip on dielectric slab by using numerical simulation and Kobayashi potential method. When the strip length is large as compared to the incident wavelength a high frequency approximate solution can be obtained by using the concept of the geometrical theory of diffraction GTD [16]. (Also we have asymptotically evaluated the integrals I_1 to I_6 in the Appendix under the assumption that strip length is large [5, pp. 201] with respect to the wavelength.).

In this paper we have studied the diffraction of a plane wave by a soft–hard strip. The continued interest in the problem is due to the fact that it constitutes the simplest half plane problem which can be formulated as a system of coupled WH equations that cannot be decoupled trivially. Rawlins [17] took the lead in the discussion of diffraction of a plane acoustic waves by a semi-infinite barrier satisfying the soft (pressure release) boundary condition on its upper surface while the hard (rigid) boundary condition on its lower surface. The author [18] reconsidered the problem solved by [17] and factorized the kernel matrix appearing in the problem by Daniele–Kharapkov methods [19,20] to give the solution of the matrix WH problem.

The WH technique [5] proves to be a powerful tool to tackle, not only, the problems of diffraction by a single half plane but it may further be extended to the case of parallel half planes. In the present work, we examine a more general problem of plane-wave diffraction by a finite soft–hard strip. By using the Fourier transform technique, three-part boundary value problem is reduced to a matrix WH equation. The solution of this matrix WH problem requires the factorization of the kernel matrix appearing in the problem. This matrix factorization has been done by [18]. With the matrix factorization known, we then follow Noble's approach [5] closely to calculate the diffracted field produced by the finite soft–hard strip. It is observed that the two edges of the strip give rise to two diffracted fields (one from each edge) and the interaction of one edge upon the other edge. Finally the diffracted field is calculated using the method of steepest descent. Some graphs, showing the effect of different parameters on the diffracted field produced by the two edges of the soft–hard strip, are also plotted and discussed.

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DIFFRACTION OF PLANE WAVES BY A SLIT IN AN INFINITE SOFT-HARD PLANE

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Abstract—We have studied the problem of diffraction of plane waves by a finite slit in an infinitely long soft-hard plane. Analysis is based on the Fourier transform, the Wiener-Hopf technique and the method of steepest descent. The boundary value problem is reduced to a matrix Wiener-Hopf equation which is solved by using the factorization of the kernel matrix. The diffracted field, calculated in the far-field approximation, is shown to be the sum of the fields (separated and interaction fields) produced by the two edges of the slit. Some graphs showing the effects of various parameters on the diffracted field produced by two edges of the slit are also plotted.

1. INTRODUCTION

The problem of plane wave diffraction by a half plane which is soft at the top and hard at the bottom was first solved by Rawlins [1] who adopted an ad-hoc method for the solution of this problem. Later on Büyükaksoy [2] reconsidered the problem solved by [1] and proposed an appropriate method for the factorization of the kernel matrix appearing in it. The continued interest in the problem is due to the fact that it constitutes the simplest half plane problem which can be formulated as a system of coupled Wiener-Hopf (WH) equations that cannot be decoupled trivially [2].

In this paper we have studied the problem of diffraction of plane waves by a slit in an infinite soft-hard plane. From the existing literature it is evident that numerous past investigations have been devoted to the study of diffraction of acoustic/electromagnetic waves

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