Hawking Radiation from Rotating and Accelerating Black Holes

By

Usman Alam Gillani

Supervised by

Dr. Khalid Saifullah

Department of Mathematics
Quaid-i-Azam University
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A Dissertation
Submitted in the Partial Fulfillment of the Requirements for the Degree of

MASTER OF PHILOSOPHY
IN
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CERTIFICATE

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We accept this dissertation as conforming to the required standard

1. Prof. Dr. Muhammad Ayub
(Chairman)

2. Dr. Khalid Saifullah
(Supervisor)

3. Dr. Ghulam Shabbir
(External Examiner)

Department of Mathematics
Quaid-i-Azam University
Islamabad, Pakistan
2011

بسم الله الرحمن الرحيم

Dedicated to

My Parents
And
Loving Sisters

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Usman Alam Gillani

Abstract

The tunneling probability and Hawking temperature of the fermion particles and the scalar particles from the rotating and accelerating black holes are discussed in this dissertation. We divide our scheme in four chapters. In Chapter 1 we provide some basic concepts of black hole physics while in Chapter 2 we briefly discuss the Einstein Field Equations and its some well known solutions like Schwarzschild black hole, the Riessner-Nordström black hole, the kerr and the Kerr-Newman black holes. Chapter 3 constitutes the Hawking radiations and the review work of two papers in which Kerner and Mann [16] discussed the tunneling probability and Hawking temperature of the fermion particles for Riessner-Nordström black hole and Kerr black hole. In the fourth chapter we extend this work to the rotating and accelerating black holes. We also calculate tunneling probability and Hawking temperature of scalar particles in this chapter. We observe that tunneling probability and Hawking temperature of fermions and scalar particles are same for the rotating and accelerating black holes. Finally we provide conclusion at the end of this chapter.

Contents

Chapter 1

Black Holes

1.1 Our Universe

Our understanding of the universe on the largest scale of space and time has increased dramatically in the recent years. This thesis does not have enough space to describe the observational details about the universe, and it does not assume the breadth of physics necessary to analyze all the processes that are important for the structure of the universe. We therefore concentrate on the role of relativistic gravity, introducing only the most basic three observational facts about our universe on the largest distance scales [1].

The universe consists of stars and gases in gravitationally bound collections of matter called galaxies, diffused radiations, dark matter of unknown character and vacuum energy. Some hundred of thousands of years after the big band the temperature dropped enough that previously free electrons combined with nuclei to make neutral, transparent matter, mostly hydrogen and helium. As the universe expanded, both matter and radiation cooled. Light emitted at that time when the temperature was approximately 3000 K has been traveling to us ever since and form a cosmic background radiation. The intervening expansion has cooled the radiation to a temperature of 2.73 K above absolute zero. A map of the temperature of this radiation on the sky is as close as we can come to the picture of the universe at the big bang.

How is the detectable matter and radiation in the universe organized on the large scales? How did this organization change over time?. To answer such questions, the location and distribution of the matter and radiations in the universe must be mapped. This is not easy.

The distances are vast, the time is long. We have a few maps of this very large place. These maps provided compelling evidence that on the largest scales the universe is isotropic (the same in one direction as in any other) and homogeneous (the same in one place as in any other).

1.2 Star

A star is a massive, luminous ball of plasma which is held together by gravity. In most cases the density of the accumulated mass is small, hence the gravitational force is described by Newton's law of gravitation [1, 2] . Such stars are classified as normal stars. For most of its life, a star shines due to thermonuclear fusion in its core releasing energy that traverses the star's interior and then radiates into the outer space. The nearest star to Earth is the Sun, which is the source of most of the energy on Earth.

Most of the things you can see in the night sky are stars. A star is a hot ball of mostly hydrogen gas; the Sun is an example of a typical ordinary star. Gravity keeps the gas from evaporating into space, and pressure due to the star's high temperature and density keep the ball from shrinking. In the core of the star, the temperature and densities are high enough to sustain nuclear fusion reactions, and the energy produced by these reactions works its way to the surface and radiates into space as heat and light. When the fuel for the fusion reactions is depleted, the structure of the star changes. The process of building up heavier elements from lighter ones by nuclear reactions, and adjusting the internal structure to balance gravity and pressure, is called stellar evolution.

Looking at a star through a telescope can tell us many of its important properties. The colour of a star tells us its temperature, and the temperature depends on some combination of the star's mass and evolutionary phase. Stars are not static objects. As a star consumes fuel in its nuclear reactions, its structure and composition changes, affecting its colour and luminosity. Thus it shows that it has different stages in its evolutionary history.

1.3 Gravitational Collapse

The life of a star is the interplay between the contracting force of gravity and the expanding forces (outward pressure) of gases heated by reactions which combine the nuclei and release energy. This process is called thermonuclear burning. A star begins its life when a cloud of interstellar gas consisting mostly of hydrogen and helium collapses gravitationally. This interstellar gas is momentarily cooler, denser, or lower in kinetic energy than its surroundings. Compressional heating increases the core temperature high enough to ignite the thermonuclear reactions, in which hydrogen is burnt to make helium, and energy is released. Then the star reaches a steady state in which the energy lost to radiation is balanced by that produced by thermonuclear burning of hydrogen. This is the present state of our Sun [1].

Eventually, a significant amount of the hydrogen in the star's core is exhausted and there remains no longer enough thermonuclear fuel to provide the energy lost to radiation. Then gravitational contraction starts. Again the compressional heating raises the core temperature unless the reactions which burn helium to make other elements ignite. The star gets brighter and its surface temperature changes. Eventually, a significant amount of the helium will be exhausted, the core will again contract further more and a new stage of thermonuclear burning will be initiated. When a star runs out of thermonuclear fuel then there are two possibilities: Either the end state is an equilibrium star, in which nonthermal source of pressure is balanced by the force of gravity, or the end state is ongoing gravitational collapse. There are several possible nonthermal sources of pressure. One of them is the pressure due to the Pauli exclusion principle. The Pauli exclusion principle does not allow two electrons (fermions) to have the same quantum state. This pressure is called the electron Fermi pressure. There are similar Fermi pressures for neutron and protons as well. There are also the nonthermal pressures arising from repulsive nuclear forces. The stars supported against the forces of gravitational collapse by the Fermi pressure of electrons are called white-dwarf stars or simply white dwarfs. The stars supported against the forces of gravitational collapse by the Fermi pressure of neutrons and by nuclear forces are called Neutron stars. These two equilibrium end states of stellar evolution are much smaller and denser than the ordinary stars. A white dwarf might have a mass of the same order as that of the Sun but with a radius of only a few thousand Kms. A neutron star of the same mass might have a radius of $10Km$ only.

When the star runs out of thermonuclear fuel and a significant amount of the fuel in the core is exhausted so that it does not support the nonthermal source of pressure to balance the gravity, then the inward gravitational force overcomes the outward pressure and the volume of the core decreases eventually so that motion of the interstellar gas molecules appears to be vibrations. In such a way, end state of the star is ongoing gravitational collapse which leads the star to a black hole.

1.4 Black hole

A black hole is a region of space in which the gravitational field is so powerful that nothing, not even light, can escape. The black hole has a one-way surface, called an event horizon, into which objects can fall, but out of which nothing can come. It is called "black" because it absorbs all the light that hits it, reflecting nothing, just like a perfect black-body in thermodynamics [2].

Although black holes are created in nature through gravitational collapse but general relativity predicts that black holes are remarkably simple objects characterized by just a few number of parameters like mass, charge and the angular momentum.

There are different conditions of mass for a star to become a black hole, proposed by different scientists. One of them is the Chandrasekhar limit. According to this, a star can become a black hole by gravitational collapse if it possesses at least 1.4 solar masses. So our Sun does not have sufficient mass to become a black hole.

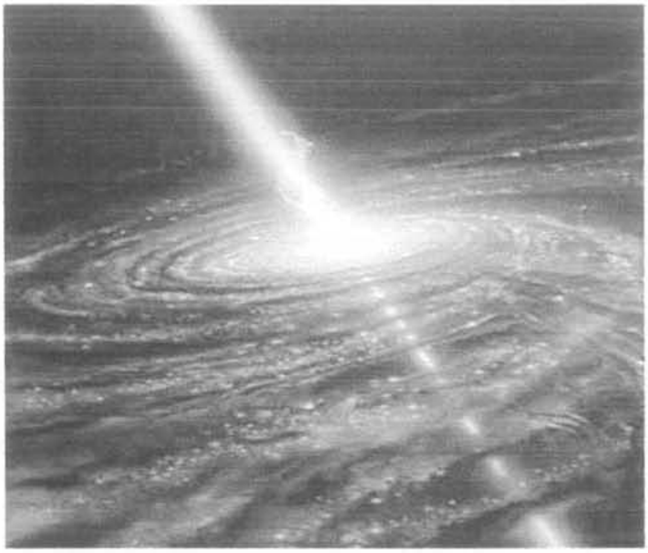

Figure 1,1 Black hole (Taken from Wikipedia)

1.4.1 Singularity

According to general relativity, under certain extreme conditions some regions of space-time develop infinitely large curvatures, thus becoming singularities where the normal laws of physics break down. Black holes, for example, should contain singularities hidden inside the event horizon.

1.4.2 Horizon

Horizons are the boundaries surrounded by a black hole through which matter, informations and light etc. can fall into the black hole and can never get back.

An event horizon is a boundary in space-time, most often an area surrounded by a black hole, beyond which events cannot affect an outside observer. Light emitted from beyond the horizon can never reach the observer, and any object that approaches the horizon from the observer's side appears to slow down and never quite pass through the horizon. In other words we can say that the event horizon is a region of no escape.

1.4.3 Ergosphere

The ergosphere is a region located outside a rotating black hole. Its name is derived from the Greek word ergon, which means "work". It received this name because it is theoretically possible to extract energy and mass from the black hole in this region [2].

The ergosphere is ellipsoidal in shape and is situated so that at the poles of a rotating black hole it touches the event horizon and stretches out to a distance that is equal to the radius of the event horizon. Within the ergosphere, space-time is dragged along in the direction of the rotation of the black hole at a speed greater than the speed of light in relation to the rest of the universe. This process is known as the lense-thirring effect or frame-dragging. Because of this dragging effect, objects within the ergosphere are not stationary with respect to the rest of the universe unless they travel faster than the speed of light, which is impossible based on the laws of physics. But in truth, particles are not moving with that speed, it is the space-time of the ergosphere that moves with a speed higher than the speed of light. Another result of this dragging of space is the existence of negative energies within the ergosphere [2].

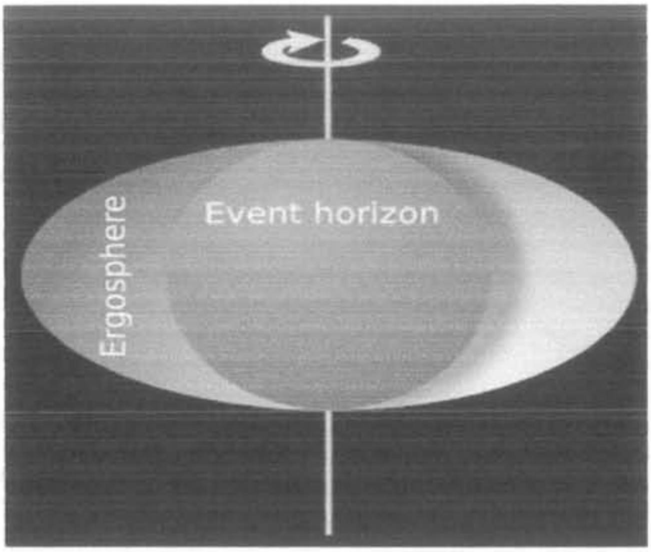

Figure 1,2 Egrosphere of the black hole (Taken from Wikipedia)

The outer limit of the ergosphere is called the stationary limit or static limit. At the stationary limit, objects moving at the speed of light are stationary with respect to the rest of the universe. This is because the space here is being dragged at exactly the speed of light relative to the rest of space. Outside this limit space is still dragged, but at a rate less than the speed of light. Since the ergosphere is outside the event horizon, it is still possible for objects to escape from the gravitational pull of the black hole. An object can gain energy by entering the black hole's rotation and then escaping from it, thus taking some of the black hole's energy with it. This process of removing energy from a rotating black hole was proposed by the mathematician Roger Penrose in 1969, and is called the Penrose process [4]. The theoretical maximum of possible energy extraction is 29% of the total energy of a rotating black hole. When this energy is removed, the black hole loses its spin and the ergosphere no longer exists.

Chapter 2

Black Holes in General Reletivity

2.1 Introduction

Einstein presented his field equations in 1915. These are the basic equations which play the central role in general relativity, relativistic astrophysics and cosmology. These equations are as follows [2]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4}T_{\mu\nu},$$

where $R_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ metric tensor, R Ricci scalar, Λ the cosmological constant, G Newtonian coupling constant, c the speed of light and $T_{\mu\nu}$ is the energy-momentum tensor. The cosmological constant was first introduced by Einstein in his field equations. He introduced this constant to study the static behavior of the universe assuming that the universe was neither expanding nor contracting. But later on, he put this to zero assuming this introduction a big blunder by him. However, many scientists have been using this constant taking small values of it. In this chapter we discuss some of the solutions of these field equations thats represent black holes.

2.2 The Schwarzschild Black Hole

The simplest case to consider, after the flat Minkowski space, was the case of the simple point gravitational source at the origin, which is clearly spherically symmetric and static. The line element of the Schwarzschild solution is given as [1–3]

$$ds^2 = (1 - \frac{r_s}{r})dt^2 - (1 - \frac{r_s}{r})^{-1}dr^2 - r^2 d\Omega^2.$$

Here

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi,$$

M is the mass of the black hole and r is the radial coordinate. It is clear that the Schwarzschild metric exhibits unusual behavior at the Schwarzscild radius $r_s = 2M$. For $r > 2M$, $g_{tt} > 0$ and $g_{rr} < 0$. This means that a world line along the t axis has $ds^2 < 0$ and so describes the spacelike curve. Whereas when the world line along the r axis has $ds^2 > 0$ it describes a timelike curve. This means that the massive particle inside the Schwarzschild radius could not remain stationary at the constant value of r. Now we consider the first term of the metric which is g_{tt}. We see that at $r = r_s$

$$g_{tt} = 1 - \frac{2M}{2M} = 0.$$

While this is well behaved mathematically, the term g_{tt} vanished means that the surface $r_s = 2M$ is a surface of infinite redshift. While nothing unusual happens to $g_{\theta\theta}$ and $g_{\phi\phi}$, we see that g_{rr} behaves very badly

$$g_{rr} = -\frac{1}{(1 - \frac{2M}{r})} \rightarrow \infty \quad \text{as } r \rightarrow 2M.$$

Its mathematical expression goes to infinity at some point, which is called a singularity. The question is whether the singularity is physically real or it is due to the bad choice of the coordinates we have made. While the surface $r_s = 2M$ has some unusual properties, the singularity is due to the choice of coordinates, and so is a coordinate singularity or in other word we can say that it is removable coordinate which can be removed by some suitable coordinates. However, we will see that the point $r_s = 0$ is due to infinite curvature and cannot be removed by a change in coordinates. This type of singularity is called essential singularity.

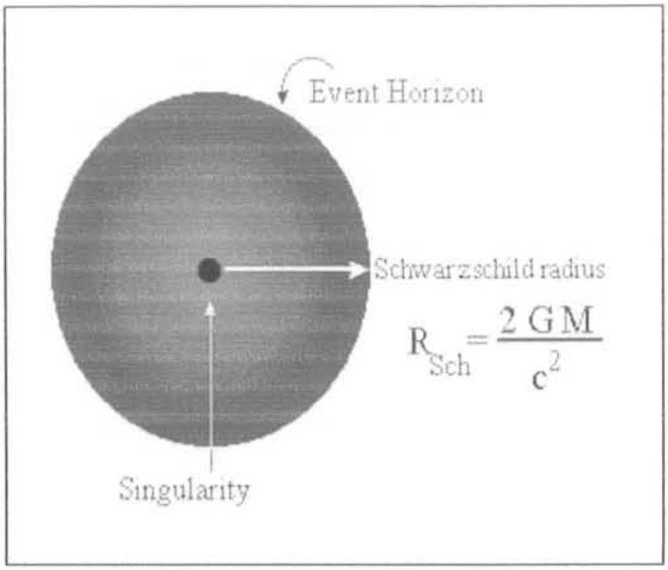

Figure 2.1 Schwarzchild radius (Taken from Wikipedia)

We discuss the following transformations in which we see how we can remove the singularity. The first attempt to get rid of the problem was made by Eddington-Finkelstein.

Eddington-Finkelstein Coordinates

First we will introduce a new coordinate r^* called the tortoise coordinate [3] given by

$$r^* = r + 2M\ln(\frac{r}{2M}), \tag{2.1}$$

along with two null coordinates

$$u = t - r^* \quad \text{and} \quad v = t + r^*. \tag{2.2}$$

From Eq. (2.1) we find

$$dr^* = \frac{dr}{1 - \frac{2M}{r}}.$$

Now we use Eq. (2.2) to write

$$\begin{aligned} dt &= dv - dr^{\star}, \\ dt &= dv - \frac{dr}{(1-\frac{2M}{r})}, \\ dt^2 &= dv^2 - 2\frac{dv\ dr}{(1-\frac{2M}{r})} + \frac{dr^2}{(1-\frac{2M}{r})^2}. \end{aligned}$$

Substitution of this result into the Schwarzschild metric gives the Eddington-Finkelstein form of the metric

$$ds^2 = (1 - \frac{2M}{r})dv^2 - 2dvdr - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

There is still a singularity at $r = 0$, the essential singularity which cannot be removed but in these new coordinates the metric has no longer a singularity at $r_s = 2M$. Let us consider the radial path of the light rays by putting $d\theta = d\phi = 0$ and $ds^2 = 0$,

$$(1 - \frac{2M}{r})dv^2 - 2dvdr = 0.$$

If we check at $r_s = 2M$, we have $\frac{dr}{dv} = 0$, that is the radial coordinate velocity of light has vanished. We intergrate to find that $r(v) =$ constant, which shows that light rays are neither outgoing nor ingoing. Rearranging the terms we have

$$\frac{dr}{dv} = \frac{2}{(1-\frac{2M}{r})}.$$

So

$$v(r) = 2(r + 2M\ln|r - 2M|) + \text{constant}.$$

This equation gives us the path that the radial light rays will follow using (v, r) coordinates. If we discussed the case at $r_s > 2M$ then if r_s increases v increases. This shows that the radial light rays are outgoing while on the other hand, if $r_s < 2M$, as r_s decreases, v increases so the light rays are ingoing [3].

Kruskal-Szekeres Coordinates

The Kruskal-Szekeres coordinates allow us to extend the Schwarzschild geometry into the region $r_s < 2M$. Two new coordinates u and v are introduced. They are related to the Schwarzschild metric $t \longrightarrow r$ in the following way, depending on the two cases $r_s < 2M$ and $r_s > 2M$ [3]. For $r_s > 2M$

$$\begin{aligned} u &= \cosh\frac{t}{4M}\exp\frac{r}{4M}\sqrt{\frac{r}{2M}-1}, \\ v &= \sinh\frac{t}{4M}\exp\frac{r_s}{4M}\sqrt{\frac{r}{2M}-1}. \end{aligned}$$

For $r_s < 2M$

$$\begin{aligned} u &= \sinh\frac{t}{4M}\exp\frac{r}{4M}\sqrt{1-\frac{r}{2M}}, \\ u &= \cosh\frac{t}{4M}\exp\frac{r}{4M}\sqrt{1-\frac{r}{2M}}. \end{aligned}$$

The coordinate singularity at $r_s = 2M$ corresponds to $u^2 - v^2 = 0$. The real curvature singularity $r = 0$ is a hyperbola that maps to

$$u^2 - v^2 = 1.$$

Once again we can examine the path of light rays by setting $ds^2 = 0$. For the new metric we have [3]

$$ds^2 = 0 = \frac{32M^3}{r}\exp\frac{-r}{2M}(du^2 - dv^2).$$

This immediately leads to

$$(\frac{du}{dv})^2 = 1.$$

In these coordinates massive bodies move inside light cones and have slope

$$(\frac{du}{dv})^2 > 1,$$

which tells us that the velocity of light is 1 everywhere. Therefore there is no boundary of light propagation in these coordinate.

The new form of the metric is given by [3]

$$ds^2 = \frac{32M^3}{r} \exp \frac{-r}{2M}(du^2 - dv^2) + r^2(d\phi^2 + \sin^2\theta d\phi^2).$$

2.3 The Reissner–Nordström Black Hole

The Reissner–Nordström metric is a static solution to the Einstein's Field Equation (EFEs) in empty space, which correspond to the gravitational field of a charged, non-rotating, spherically symmetric body of mass M. The mathematical form is [3]

$$ds^2 = (1 - \frac{r_s}{r} + \frac{r_e}{r^2})dt^2 - (1 - \frac{r_s}{r} + \frac{r_e}{r^2})^{-1}dr^2 - r^2 d\Omega^2.$$

t is the time coordinate, r is the radial coordinate, Ω is the solid angle, r_s is the Schwarzschild radius of the massive body, which is related to its mass M by

$$r_s = 2M,$$

where r_e is a length-scale corresponding to the electric charge Q of the mass

$$r_e^2 = \frac{e^2}{4\pi\epsilon_0 c^4},$$

where $1/4\pi\epsilon_0$ is Coulomb's force constant. If the charge e goes to zero, one recovers the Schwarzschild metric. The classical Newtonian theory of gravity may then be recovered in the limit as the ratio r_s/r goes to zero. In that limit, the metric returns to the Minkowski metric for special relativity

$$ds^2 = dt^2 - dr^2 - r^2 d\Omega^2.$$

Although charged black holes with $r_e << r_s$ are similar to the Schwarzschild black hole, they have two horizons. The horizon can be calculated as [3]

$$(g^{rr})^{-1} = 1 - \frac{r_s}{r} + \frac{r_e}{r^2} = \frac{1}{r^2}(r^2 - rr_s + r_e^2) = 0.$$

This second factor $r^2 - rr_s + r_e^2$ is a quadratic in r and we find its zeroes by using the quadratic formula

$$r_\pm = \frac{1}{2}(r_s \pm \sqrt{r_s^2 - 4r_e^2}).$$

2.4 The Kerr Black Hole

Observations show that astronomical objects like the Earth, Sun or a neutron star are rotating. While the Schwarzschild solution still describes the spacetime around a slowly rotating object, to accurately describe a spinning black hole we need a solution. Such a solution is given by the Kerr metric.

The Kerr metric gives some interesting new results that are unexpected. We take an example to understand the observation. An object that is placed near a spinning black hole cannot avoid rotation along the black hole, there is no matter what kind of motion we give to the object. Put the rocket ship there, fire the most powerful engines that can be constructed so that the rocket ship will move in a direction opposite to that in which the black hole is rotating. But the engines cannot help, no matter, what we do, the rocket ship will be carried along the direction of the rotation. Such black holes have different surfaces where the metric appears to have a singularity, the size and shape of these surfaces depend on the black hole's mass and angular momentum. The outer surface encloses the ergosphere and has a shape similar to a flattened sphere. The inner surface marks the "radius of no return" or the "event horizon", objects passing through this radius can never again communicate with the world outside that radius classically.

As we know a spinning object is characterized by its angular momentum. When we describe the Kerr black hole, we give the angular momentum the label J and are usually concerned with angular momentum per unit mass. This is given by $a = J/M$, where M is the mass of the gravitational object then the unit of a is given in meters.

The mathematical form of the Kerr metric describing the geometry of space-time in the vicinity of a mass M rotating with angular momentum J is [1,4]

$$ds^2 = (1 - \frac{r_s r}{\rho^2})dt^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 -$$
$$\left((r_s + \alpha^2 + \frac{r_s r_{\alpha^2}}{\rho^2}\sin^2\theta)\right)\sin^2\theta d\phi^2 - \frac{2r_s r_\alpha \sin^2\theta}{\rho^2}dtd\phi,$$

where the coordinates r, θ, ϕ are standard spherical coordinate system, and r_s is the Schwarzschild radius $r_s = 2M$, and where the length-scales α, ρ and Δ have been introduced for brevity

$$\begin{aligned} \alpha &= \frac{J}{M}, \\ \rho^2 &= r^2 + \alpha^2\cos^2\theta, \\ \Delta &= r^2 - r_s r + \alpha^2. \end{aligned}$$

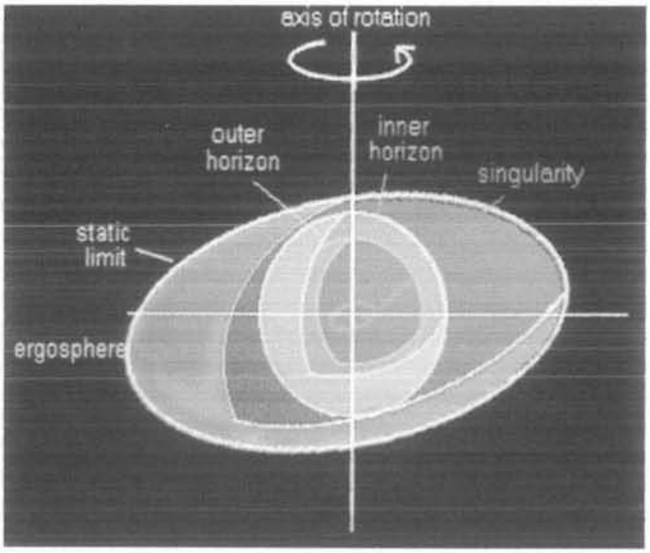

Figure 2.2 The Kerr black hole (Taken from Wikipedia)

2.5 The Kerr-Newman Black Hole

The Kerr–Newman metric [1, 4] is a solution of the Einstein-Maxwell equations in general relativity, describing the spacetime geometry in the region surrounding a charged, rotating mass. It is assumed that the cosmological constant is equals to zero. In 1965, Ezra "Ted" Newman found the axisymetric solution of EFEs for a black hole which is both rotating and electrically charged. This formula for the metric tensor $g_{\nu\mu}$ is called the Kerr-Newman metric. It is a generalization of the Kerr metric for an uncharged spinning point-mass, which had been discovered by Roy Kerr. Its metric is given by

$$\begin{aligned} ds^2 &= (1-\frac{2mr-e^2}{\Sigma})dt^2 - 2a\sin^2\theta\left(\frac{2mr-e^2}{\Sigma}\right)dtd\phi \\ &-\frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 - \left[r^2+a^2+\frac{2mr-e^2}{\Sigma}a^2\sin^2\theta\right]\sin^2\theta d\phi^2, \end{aligned}$$

where

$$\Delta = r^2 - 2mr + a^2 + e^2, \qquad \Sigma = r^2 + a^2\sin^2\theta,$$

also $a = J/M$ is the rotation parameter with angular frequency J. Event horizons are

$$\Delta = 0 \Rightarrow r_{\pm} = m \pm \sqrt{m^2 - e^2 - a^2},$$

where r_+ is called the outer horizon and r_- is called the inner horizon.

In the limit $a \longrightarrow 0$ this metric reduces to the Reissner-Nordstörm metric and in the limit $e \longrightarrow 0$ to the Kerr metric. Further the Reissner-Nordstörm metric reduces to the usual Schwarzschild metric as $e \longrightarrow 0$ and so does the Kerr metric in the limit $a \longrightarrow 0$. The Kerr and charged Kerr metrics are axially symmetric and stationary, the term involving a destroying spherical symmetry. (A metric is said to be stationary if it is time independent i.e has a time -like Killing vector but there is no space-like hypersurface globally orthogonal to it. If there is such a hypersurface then the metric is said to be static.)

2.6 The Plebański-Demiański Family of Black Holes

The Plebański-Demiański family is a collection of solution of EFEs. With the cosmological constant zero, the most general form of the Plebański-Demiański metric is given as [5]

$$\begin{aligned} ds^2 &= \frac{1}{(1-\hat{p}\hat{r})^2}[-\frac{\mathcal{Q}}{\hat{r}^2+\hat{p}^2}(d\hat{\tau}-\hat{p}^2 d\hat{\sigma})^2+\frac{\mathcal{P}}{\hat{r}^2+\hat{p}^2}(d\hat{\tau}+\hat{r}^2 d\hat{\sigma})^2 \\ &\quad +\frac{\hat{r}^2+\hat{p}^2}{\mathcal{P}}d\hat{p}^2+\frac{\hat{r}^2+\hat{p}^2}{\mathcal{Q}}d\hat{r}^2] \,, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \mathcal{P} &= \hat{k}+2\hat{n}\hat{p}-\hat{\epsilon}\hat{p}^2+2\hat{m}\hat{p}^3-\left(\hat{k}+\hat{e}^2+\hat{g}^2\right)\hat{p}^4, \\ \mathcal{Q} &= \hat{k}+\hat{e}^2+\hat{g}^2-2\hat{m}\hat{r}+\hat{\epsilon}\hat{r}^2-2\hat{n}\hat{r}^3-\hat{k}\hat{r}^4, \end{aligned}$$

and $\hat{m}$, $\hat{n}$, $\hat{e}$, $\hat{g}$, $\hat{\epsilon}$, and $\hat{k}$ are arbitrary real parameters. It is usually assumed that $\hat{m}$ and $\hat{n}$ are the mass and NUT parameters, the parameters $\hat{e}$ and $\hat{g}$ represent electric and magnetic charges. In the metric (2.3) the sources of acceleration and rotation are not clearly represented, so we introduce such parameters for which the metric is transformed in acceleration and rotation parmeters. Such parameters are

$$\hat{p}=\sqrt{\alpha\omega}p, \qquad \hat{r}=\sqrt{\frac{\alpha}{\omega}}r, \qquad \hat{\sigma}=\sqrt{\frac{\omega}{\alpha^3}}\sigma, \qquad \hat{\tau}=\sqrt{\frac{\omega}{\alpha}}\tau, \tag{2.4}$$

with the relabeling of parameters

$$\hat{m}+\iota\hat{n}=\left(\frac{\alpha}{\omega}\right)^{\frac{3}{2}}(m+\iota n), \quad \hat{e}+\iota\hat{g}=\frac{\alpha}{\omega}(e+\iota g), \quad \hat{\epsilon}=\frac{\alpha}{\omega}\epsilon, \quad \hat{k}=\alpha^2 k.$$

This introduces two additional parameters α and ω. With these changes, the metric becomes

$$\begin{aligned} ds^2 &= \frac{1}{(1-\alpha pr)^2}[-\frac{Q}{r^2+\omega^2p^2}(d\tau-\omega p^2 d\sigma)^2+\frac{P}{r^2+\omega^2p^2}(\omega d\tau+r^2 d\sigma)^2 \\ &\quad +\frac{r^2+\omega^2p^2}{P}dp^2+\frac{r^2+\omega^2p^2}{Q}dr^2] \,, \end{aligned} \tag{2.5}$$

where $P(p)$ and $Q(r)$ are quartic functions

$$P = k + 2\frac{n}{\omega}p - \epsilon p^2 + 2\alpha m p^3 - [\alpha^2(\omega^2 k + e^2 + g^2)]p^4 , \tag{2.6}$$

$$Q = (\omega^2 k + e^2 + g^2) - 2mr + \epsilon r^2 - 2\alpha\frac{n}{\omega}r^3 - \alpha^2 k r^4 . \tag{2.7}$$

and m, n, e, g, ϵ, k, α and ω are arbitrary real parameters. n is the Plebański-Demiański parameter and ω is the twist [5,6]. The component in (2.6) and (2.7) indicate the presence of a curvature singularity at $r = 0$, $p = 0$. This singularity may be considered as the source of the gravitational field. They also show the line element (2.5) is flat if $m = n = 0$ and $e = g = 0$. (The remaining parameters ϵ, k, α, ω, may be non zero in this flat limit.)

In Eq. (2.5), it is necessary that $P > 0$. Thus, the coordinate p must be restricted to a particular range between appropriate roots of P. If it is required that a singularity should appear in the boundary of the spacetime, then this range must include $p = 0$. This would require that $k > 0$. However, important non-singular solutions also exist for which the chosen range of p does not include $p = 0$. In such a case it is convenient to express the parameters ϵ, n and k which occur in the metric functions (2.6) and (2.8) in terms of new parameters a, l as [5]

$$\epsilon = \frac{\omega^2 k}{a^2 - l^2} + 4\alpha\frac{l}{\omega}m - (a^2 + 3l^2)[\frac{\alpha^2}{\omega^2}(\omega^2 k + e^2 + g^2)] , \tag{2.8}$$

$$n = \frac{\omega^2 k l}{a^2 - l^2} - \alpha\frac{a^2 - l^2}{\omega}m - (a^2 - l^2)l[\frac{\alpha^2}{\omega^2}(\omega^2 k + e^2 + g^2)] , \tag{2.9}$$

$$k = [1 + 2\alpha\frac{l}{\omega}m - 3\alpha^2\frac{l^2}{\omega^2}(e^2 + g^2)](\frac{\omega^2}{a^2 - l^2} + 3\alpha^2 l^2)^{-1} . \tag{2.10}$$

In the above, there are six physical parameters, m, e, g, a, l, α. Performing the simple transformations [5]

$$p = \frac{l}{\omega} + \frac{a}{\omega}\cos\theta , \quad \tau = t - \frac{(l + a)^2}{a}\phi , \quad \sigma = -\frac{\omega}{a}\phi ,$$

we get

$$ds^2 = \frac{1}{\bar{\Omega}^2}\{-\frac{Q}{\rho^2}[dt - (a\sin^2\theta + 4l\sin^2\frac{\theta}{2})d\phi]^2 + \frac{\rho^2}{Q}dr^2 + \frac{\rho^2}{P}d\theta^2$$

$$+\frac{P}{\rho^2}\sin^2\theta[adt - (r^2 + (a+l)^2)d\phi]^2\} , \qquad (2.11)$$

where

$$\bar{\Omega} = 1 - \frac{\alpha}{\omega}(l + a\cos\theta)r ,$$

$$\rho^2 = r^2 + (l + a\cos\theta)^2 ,$$

$$P = 1 - a_3\cos\theta - a_4\cos^2\theta ,$$

$$Q = (\omega^2 k + e^2 + g^2) - 2mr + \epsilon r^2 - 2\alpha\frac{n}{\omega}r^3 - \alpha^2 k r^4,$$

$$a_3 = 2\alpha\frac{a}{\omega}m - 4\alpha^2\frac{al}{\omega^2}(\omega^2 k + e^2 + g^2),$$

$$a_4 = -\alpha^2\frac{a^2}{\omega^2}(\omega^2 k + e^2 + g^2) .$$

It is also assumed [5] that $|a_3|$ and $|a_4|$ are sufficiently small so that P has no roots within the considered range $\theta = [0, \pi]$. When $\alpha = 0$ i.e. acceleration vanishes, the general metric reduces to the Kerr-Newman-NUT-de Sitter solution. Further if $l = 0$ then it reduces to familiar forms of the Kerr-Newman-de Sitter black hole spacetimes. If $\alpha = 0$ and the Kerr-like rotation vanishes i.e. $a = 0$ then general metric reduces to the charged NUT-de Sitter spacetime. When $\alpha = 0 = l = g$ then the Kerr-Newman metric is deduced. Further Schwarzschild metric is directly obtained if electric charge and rotation parameter vanish i.e. $e = 0 = a$. Therefore, the line element (2.11) is a very convenient metric representation of the complete class of accelerating, rotating and charged black holes of the Plebański-Demiański class. In Chapter 4 we will discuss the uncharged case of this metric, that is we will take $e = 0 = g$.

2.6.1 A New Form of the Metric

If we put $\tilde{P} = P\sin^2\theta$ and also substitute for ϵ and n from Eqs. (2.8) and (2.9) in the general metric (2.11), then it reduces to the form [6]

$$\begin{aligned} ds^2 &= \frac{1}{\bar{\Omega}^2}\{-\frac{Q}{\rho^2}[dt-(a\sin^2\theta+4l\sin^2\frac{\theta}{2})d\phi]^2+\frac{\rho^2}{Q}dr^2+\frac{\rho^2}{\tilde{P}}\sin^2\theta d\theta^2 \\ &+\frac{\tilde{P}}{\rho^2}[adt-(r^2+(l+a)^2)d\phi]^2\}\ , \end{aligned} \tag{2.12}$$

where

$$\bar{\Omega} = 1-\frac{\alpha}{\omega}(l+a\cos\theta)\,r\ \ ,\ \ \rho^2 = r^2+(l+a\cos\theta)^2\ ,$$

$$\tilde{P} = \sin^2\theta(1-a_3\cos\theta-a_4\cos^2\theta)\ ,$$

$$Q = \left[(\omega^2k+e^2+g^2)(1+2\frac{\alpha l}{\omega}r)-2mr+\frac{\omega^2k}{a^2-l^2}r^2\right]\left[1+\frac{\alpha(a-l)}{\omega}r\right]\left[1-\frac{\alpha(a+l)}{\omega}r\right]\ ,$$

and

$$a_3 = 2\frac{\alpha a}{\omega}m-4\frac{\alpha^2 al}{\omega^2}(\omega^2k+e^2+g^2)\ ,$$

$$a_4 = -\frac{\alpha^2a^2}{\omega^2}(\omega^2k+e^2+g^2)\ ,$$

$$\epsilon = \frac{\omega^2k}{a^2-l^2}+4\alpha\frac{l}{\omega}m-\frac{\alpha^2}{\omega^2}(a^2+3l^2)(\omega^2k+e^2+g^2)\ ,$$

$$n = \frac{\omega^2kl}{a^2-l^2}-\alpha\frac{a^2-l^2}{\omega}m+l\frac{\alpha^2}{\omega^2}(a^2-l^2)(\omega^2k+e^2+g^2)\ ,$$

$$k = \left(1+2\alpha\frac{l}{\omega}m-3\alpha^2\frac{l^2}{\omega^2}(e^2+g^2)\right)(\frac{\omega^2}{a^2-l^2}+3\alpha^2l^2)^{-1}\ .$$

Note that here Q is in the factorized form. The above line element contains seven arbitrary parameters m, l, e, g, a, α and ω. Except ω all the remaining parameters can be varied independently and can be used to set ω to any convenient value if at least one of the parameters a or l is non-zero [6]. This can be seen that if $|l|\leq|a|$ the metric (2.12) has a curvature singularity when $\rho^2=0$, i.e. at $r=0$, $\cos\theta=-l/a$. Whereas if $|l|>|a|$, it is singularity-free. In this case, the outer and inner horizons occur at $r=r_\pm$ from the form of Q, where $r_\pm$ are the roots of the quartic equation [6, 18]

$$\frac{\omega^2 k}{a^2-l^2}r^2 - 2\{m - \frac{\alpha l}{\omega}(\omega^2 k + e^2 + g^2)\}r + (\omega^2 k + e^2 + g^2) = 0 \ .$$

There are also acceleration horizons at $\alpha r = \omega(l \pm a)^{-1}$.

If $\alpha \neq 0$, this solution represents a black hole which accelerates along the axis of symmetry in the direction $\theta = 0$. However, it is far from obvious that the complete analytical extension of this spacetime represents a pair of causally separated black holes which accelerate away from each other in opposite directions [7].

2.6.2 The Non-Accelerating Case

It can be seen that, when $\alpha = 0$, we have $\omega^2 k = a^2 - l^2$ and hence $\epsilon = 1$, $n = l$ and $\tilde{P} = \sin^2\theta$. Then Eq. (2.12) reduces to [6, 18]

$$\begin{aligned} ds^2 &= \frac{Q}{\rho^2}[dt - (a\sin^2\theta + 4l\sin^2\frac{\theta}{2})d\phi]^2 - \frac{\rho^2}{Q}dr^2 - \rho^2 d\theta^2 \\ &\quad - \frac{\sin^2\theta}{\rho^2}[adt - (r^2 + (l+a)^2)d\phi]^2 \ , \end{aligned} \tag{2.13}$$

where

$$\rho^2 = r^2 + (l + a\cos\theta)^2, \quad Q = (a^2 - l^2 + e^2 + g^2) - 2mr + r^2 \ , \tag{2.14}$$

which is exactly the Kerr-Newman-NUT solution [6] which is regular on the half-axis $\theta = 0$. This solution represents a single black hole with mass m, electric charge e, magnetic charge g, Kerr-like rotation a and NUT parameter l. If $l = 0 = g$ then the Kerr-Newman solution is obtained.

It can be seen that if $|l| \leq |a|$, then the metric (2.13) has a curvature singularity at $\rho^2 = 0$; i.e., at $r = 0$, $\cos\theta = -l/a$. However, if $|l| > |a|$, it is non-singular. Here $Q = 0$ is a coordinate singularity and gives the expression for locations of inner and outer horizons of the black hole as [6, 18]

$$r_\pm = m \pm \sqrt{m^2 + l^2 - a^2 - e^2 - g^2} \ , \tag{2.15}$$

where $m^2 \geq a^2 + e^2 + g^2 - l^2$.

Here we discuss the formation of the ergospheres in these black holes. Since we have the formula for ergosphere as

$$g_{tt} = 0 \ .$$

So from Eqs. (2.13) and (2.14)

$$r^2 - 2mr - l^2 + e^2 + g^2 + a^2 \cos^2\theta = 0 \ .$$

Its solution is [18, 19]

$$r_n(\theta) = m + \sqrt{m^2 + l^2 - e^2 - g^2 - a^2\cos^2\theta} \ ,$$

which is the relation for the ergosphere for the black hole represented by the metric (2.13). Now we are going to see its relation with the outer horizon (2.15). As we know that [19]

$$0 \le \cos^2\theta \le 1 \ ,$$

$$m^2 + l^2 - e^2 - g^2 - a^2 \le m^2 + l^2 - e^2 - g^2 - a^2\cos^2\theta \le m^2 + l^2 - e^2 - g^2 \ ,$$

$$m + \sqrt{m^2 + l^2 - e^2 - g^2 - a^2} \le m + \sqrt{m^2 + l^2 - e^2 - g^2 - a^2\cos^2\theta} \le m + \sqrt{m^2 + l^2 - e^2 - g^2} \ ,$$

$$r_+ \le r_n(\theta) \le m + \sqrt{m^2 + l^2 - e^2 - g^2} \ ,$$

$$r_+ \le r_n(\theta) \le r_a \ ,$$

where r_a is the outer horizon of the corresponding Reissner-Nordström black hole with magnetic and NUT charges g and l respectively. The above relation has a beautiful information to interpret. Since the ergosphere is dependent on θ so it will coincide the outer horizon at $\theta = 0$ and stretches out for other values of θ. However, it cannot stretch beyond the outer horizon of the corresponding Reissner-Nordström black hole with magnetic and NUT charge g and l respectively, and will coincide it at $\theta = \pi/2$.

2.6.3 Accelerating and Rotating Charged Black Holes

The Plebański-Demiański metric covers a large family of solutions which includes that of a rotating and accelerating charged black hole. Now we present a new form of the metric which is free of NUT-like behavior i.e. we take $l = 0$. If we put $l = 0,\ k = 1$, in Eq. (2.11) then $\omega = a,\ a_3 = 2\alpha m,\ a_4 = -\alpha^2(a^2 + e^2 + g^2)$ and substituting for ϵ and n, the line element (2.11) with will take the form [8, 18, 19]

$$ds^2 = \frac{1}{\bar{\Omega}^2}\{-\frac{\bar{Q}}{\rho^2}[dt - a\sin^2\theta d\phi]^2 + \frac{P}{\rho^2}\sin^2\theta[adt - (r^2 + a^2)d\phi]^2 + \frac{\rho^2}{\bar{Q}}dr^2 + \frac{\rho^2}{P}d\theta^2\}\ , \quad (2.16)$$

where

$$\bar{\Omega} = 1 - \alpha r\cos\theta\ , \quad \rho^2 = r^2 + a^2\cos^2\theta\ , \quad (2.17)$$

$$P = 1 - 2\alpha m\cos\theta + \{\alpha^2(a^2 + e^2 + g^2)\}\cos^2\theta\ ,$$

$$\bar{Q} = \{(a^2 + e^2 + g^2) - 2mr + r^2\}(1 - \alpha^2 r^2)\ . \quad (2.18)$$

The above metric contains five arbitrary parameters $m,\ e,\ g,\ \alpha$ amd a which can be varied independently and physical interpretation of which has already been described.

Here $\rho^2 = 0$ indicates the presence of a Kerr-like ring singularity at $r = 0$ and $\theta = \pi/2$. $\bar{Q} = 0$ gives the expression for the locations of the inner and outer horizons, which is identical to the inner and outer horizons of the non-accelerating Kerr-Newman black hole and is [8]

$$r_{\pm} = m \pm \sqrt{m^2 - a^2 - e^2 - g^2}\ ,$$

where $a^2 + e^2 + g^2 \leq m^2$. However, there are also acceleration horizons at $r = 1/\alpha$ and $r = 1/\alpha\cos\theta$, coming from $\bar{Q} = 0$ and $\bar{\Omega} = 0$ respectively, and are coincident with each other at $\theta = 0$

Chapter 3

Hawking Radiation From Black Holes

3.1 Introduction

In this chapter we shall briefly describe the background of the Hawking radiation and the tunnelling method. We describe the tunnelling of fermions emission from black holes. In order to do this we follow the Hamilton-Jacobi method [11, 16, 17]. In this method we apply a WKB approximation to the Dirac Equation. We shall first consider Riessner-Nordstörm black hole and confirm that the correct temperature is recovered. Then we extend this technique to the rotating black holes and find the Hawking temperature are well. From these calculations we confirm that the spin $\frac{1}{2}$ fermions particle are also emitted at the Hawking radiation. This final result, while not surprising, validates this important approach. This is one of the methods that can actually calculate the spin $\frac{1}{2}$ fermion radiation. This shows the strength of the tunnelling method as well.

A black hole is an object for which the gravity is so strong that even light cannot be passed through it. If only a classical system is considered, it would be impossible to define a temperature for the black hole because it would be impossible for anything to be in thermal equilibrium with a black hole. Hence it is proved that every thing would go into the black hole but nothing will come out.

Hawking radiation (sometimes also called Bekenstein-Hawking radiation) is a theoretical prediction from the British physicist Stephen Hawking, which explains thermal properties relating to black hole [22]. Normally, a black hole is considered to draw all matter and energy in the surrounding region into it, as a result of the intense gravitational fields. However, in 1972 the Israeli physicist Jacob Bekenstein suggested that black holes should have a well-defined entropy, and initiated the development of black hole thermodynamics, including the emission of energy [21].

3.2 Black Hole Radiation

Black hole radiation was an importance discovery because classically nothing could escape from the black hole. Basically black hole radiation depends upon the quantum gravity calculations and this emphasizes the importance of trying to find a full quantum theory of gravity. This is because a new physics should be found once a complete quantum theory of gravity is formulated and any discoveries could be as important as the black hole radiation.

The discovery of black hole radiation also opened up new mysteries such as the information loss problem. The information loss problem is about whether the black hole radiation should be purely thermal or not. If the radiation is purely that of the black hole then it should not contain any information with it and after the black hole evaporates the information of what made up the black hole will be gone forever. It is controversial whether the information actually is lost or if the radiation should have a modified emission that is not truly thermal.

3.3 Stephen Hawking on Quantum Black Holes

The quantum theory of black holes seems to lead to a new level of unpredictability in physics over and above the usual uncertainty associated with quantum mechanics. This is because black holes appear to have intrinsic entropy and to lose information from our region of the universe. Hawking says that these claims are controversial: many people working or quantum gravity, including almost all those who entered it from particle physics, would instinctively reject the idea that information about the quantum state of a system could be lost. However, they have had very little success in showing how information can get out of a black hole. He believes they will be forced to accept his suggestion that it is lost, just as they were forced to agree that black holes radiate, which went against all their preconceptions. The fact that gravity is attractive means that it will tend to draw the matter in the universe together to form objects like stars and galaxies. These can support themselves for a time against further contraction by thermal pressure, in the case of stars, or by rotation and internal motions, in the case of galaxies. However, eventually the heat or the angular momentum will be carried away and the object will begin to shrink. If the mass is less than about one and a half times that of the Sun, the contraction can be stopped by the degeneracy pressure of electrons or neutrons. The object will settle down to be a white dwarf or a neutron star, respectively. However, if the mass is greater than this limit there is nothing that can hold it up and stop it continuing to contract.

No two electrons or neutrons can occupy the same quantum state. Thus, when any collection of these particles is squeezed into a small volume, those in the highest quantum states become very energetic. The system then resists further compression, exerting an outward push called degeneracy pressure. Once it has shrunk to a certain critical size the gravitational field of its surface will be so strong that the light will be bent inward. You can see that even the outgoing light rays are bent toward each other and so are converging rather than diverging. This means that there is a closed trapped surface. Thus there must be a region of space-time from which it is not possible to escape to infinity.

3.4 Tunnelling from Black Holes

In 1974, Hawking worked out the exact theoretical model for how a black hole could emit black body radiation [22]. With the emission of Hawking radiation black hole lose their energy, shrink and eventually evaporate completely. How does this happen? When the object that is classically stable becomes quantum-mechanically unstable. The idea is that when a pair of virtual particle is created just inside the horizon, the positive energy virtual particle can tunnel out, no classical escape route exist, where it materializes as a real particle. On the other hand, from a pair created just outside the horizon the negative energy virtual particle, which is forbidden outside, can tunnel inward. In either case the negative energy particle is absorbed by a black hole, as the result the mass of the black hole decreases, while the positive energy particle escapes to infinity [20].

3.5 Hawking Temperature

For a black hole, temperature T is analogous to its surface gravity,κ, from the zeroth laws of black hole thermodynamics [19]. According to the zeroth law of thermodynamics, the temperature is constant throughout a body in thermal equilibrium, and the zeroth law of black hole thermodynamics suggests that the surface gravity for a stationary black hole is constant at the horizon. So T constant for thermal equilibrium for a normal system is analogous to surface gravity constant over the horizon of a stationary black hole.

It was Bekenstein who first claimed that these similarities were more an analogy [21]. He claimed that $Tds = \frac{\kappa}{8\pi} dA$, so that the temperature of the black hole is proportional to the surface gravity and the entropy was proportional to the area. This was later shown by Hawking [22] who calculated the temperature of the black hole

$$T_H = \frac{\kappa}{2\pi}.$$

3.6 Derivation of Hawking Radiation

There are some useful methods use for deriving Hawking temperature and calculating the black hole temperature. Recently, there has been a great interest in the method used for calculating the black hole temperature known as the tunnelling method.

The tunnelling method is a very interesting method for calculating the black hole temperature since it provides a dynamical model of the black hole radiation. The black hole tunnelling method has a lot of strength when compared to the other methods for calculating the temperature. The calculation is relatively simple. The tunnelling method can even be applied at the horizon that is not the black hole horizon, such as Rindler space-time, and calculate the temperature as well. The application to de Sitter space-time demonstrates a particular advantage of the tunnelling method. In this chapter we discuss the tunnelling method in detail and show how this method can be applied to a broad range of space-times and can be extended to model fermion emission. In the original calculation the tunnelling method was only applied to the Schwarzschild black hole [23, 24].

This method involves calculating the imaginary part of the action for the (classically forbidden) process of emission across the horizon (first considered by Kraus and Wilczek [23, 24], which in turn is related to the Boltzmann factor for emission at the Hawking Temperature. Using the WKB approximation the tunnelling probability for the classically forbidden trajectory coming from inside to outside the horizon is given as

$$\Gamma \sim \exp(-2I) \simeq \exp(-\beta E), \tag{2}$$

where β is the inverse temperature of the horizon. For calculating the temperature of the black hole, expansion of linear order is required. There are further two different approaches that are useful for calculating the imaginary part of the action for the emitted particle. At first the black hole tunnelling method was developed by Parikh and Wilczek [26] which is found in Kraus and Wilczek [23, 24] work as well. The other appraoch to black hole tunnelling method is the Hamilton-Jacobi method used by Anghaben et al, which is an extension of the complex path method.

3.7 Hawking Radiation of Dirac Particles via Tunneling from the Reissner-Nordstörm Black Hole

In this section we discuss the tunneling radiation of fermion from the Reissner Nordstörm black hole. Here the electromagnetic field would couple with the matter field and gravity field, so in this case the Dirac equation of charged particles is introduced and the pure thermal spectrum of fermions from Reissner-Nordstorm black hole is derived. The line element of the Reissner-Nordstorm black hole is given by [9]

$$ds^2 = -f(r)\,dt^2 + f^{-1}(r)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{3.1}$$

where

$$f(r) = \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right) = 0.$$

At horizon

$$f(r_\pm) = \left(1 - \frac{2M}{r_\pm} + \frac{e^2}{r_\pm^2}\right) = 0.$$

Here $r_{\pm}$ show the inner and outer horizons, M, e are the mass and charge of the black hole. The non vanishing component of the electromagnetic vector potential is given as [9]

$$A_{\mu} = \left(\frac{-e}{r}, 0, 0, 0\right).$$

3.7.1 Calculation of the Tunneling Probability and Hawking Temperature

There are two methods to find the tunneling probability. Firstly Hamilton-Jacobi method and secondly the null geodesics method. Here we discuss the Hamilton-Jacobi method. For this method we use Dirac equation and calculate the tunneling probability and Hawking temperature

$$i\hbar\gamma^{\mu}\left(\partial_{\mu} + \Omega_{\mu} - \iota q A_{\mu}\right)\psi + m\psi = 0, \tag{3.2}$$

which in expanded form becomes

$$i\hbar\gamma^{t}\left(\partial_{t} + \Omega_{t} - \iota q A_{t}\right)\psi + i\hbar\gamma^{r}\left(\partial_{r} + \Omega_{r} - \iota q A_{r}\right)\psi$$

$$+i\hbar\gamma^{\theta}\left(\partial_{\theta} + \Omega_{\theta} - \iota q A_{\theta}\right)\psi + i\hbar\gamma^{\phi}\left(\partial_{\phi} + \Omega_{\phi} - \iota q A_{\phi}\right)\psi + m\psi = 0,$$

where

$$\begin{aligned}
\Omega_{\mu} &= \frac{1}{2}\iota\Gamma_{\mu}^{\alpha\beta}\Sigma_{\alpha\beta}, \\
\Sigma_{\alpha\beta} &= \frac{1}{4}\iota\left[\gamma^{\alpha}, \gamma^{\beta}\right], \quad \text{so } \Omega_{\mu} = \frac{-1}{8}\Gamma_{\mu}^{\alpha\beta}\left[\gamma^{\alpha}, \gamma^{\beta}\right],
\end{aligned}$$

and γ^{μ} matrices satisfy $[\gamma^{\mu}, \gamma^{\nu}] = 2g^{\mu\nu} \times I$, ($I$ is the identity matrix); m, q are the mass and charge of the fermion particles, respectively. To deal with the fermion tunneling radiation, it is important to choose γ^{μ} matrices:

$$\gamma^{t} = \frac{1}{\sqrt{f(r)}}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \gamma^{r} = \sqrt{f(r)}\begin{pmatrix} 0 & \sigma^{3} \\ \sigma^{3} & 0 \end{pmatrix}, \tag{3.3}$$

$$\gamma^{\theta} = \frac{1}{r}\begin{pmatrix} 0 & \sigma^{1} \\ \sigma^{1} & 0 \end{pmatrix}, \quad \gamma^{\phi} = \frac{1}{r\sin\theta}\begin{pmatrix} 0 & \sigma^{2} \\ \sigma^{2} & 0 \end{pmatrix}. \tag{3.4}$$

Here, σ^i are the Pauli sigma matrices given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For a fermion with spin $\frac{1}{2}$, the wave function has two spin states namely spin up ($\uparrow$) and spin down ($\downarrow$) so we can take the following ansatz for this wave function

$$\begin{aligned} \psi_\uparrow &= \begin{pmatrix} A(t,r,\theta,\phi) \\ 0 \\ B(t,r,\theta,\phi) \\ 0 \end{pmatrix} \exp\left(\frac{i}{\hbar} I_\uparrow(t,r,\theta,\phi)\right), \\ \psi_\downarrow &= \begin{pmatrix} 0 \\ C(t,r,\theta,\phi) \\ 0 \\ D(t,r,\theta,\phi) \end{pmatrix} \exp\left(\frac{i}{\hbar} I_\downarrow(t,r,\theta,\phi)\right), \end{aligned} \tag{3.5}$$

where $\psi_\uparrow$ denotes the wave function of the spin up particle and $\psi_\downarrow$ is for the spin down case. Here A, B, C, D are arbitrary functions of the coordinates. We will only show the spin up case since the spin down case is similar to this other than some changes in the sign. Inserting Eq. (3.5) in Eq. (3.2), after dividing by the exponential term and multiplying by $\hbar$, and taking leading order term in $\hbar$ we obtain

$$\frac{iA(\partial_t I_\uparrow - qA_t)}{\sqrt{f(r)}} + B\sqrt{f(r)}\partial_r I_\uparrow - mA = 0, \tag{3.6}$$

$$\frac{iB(\partial_t I_\uparrow - qA_t)}{\sqrt{f(r)}} - A\sqrt{g(r)}\partial_r I_\uparrow + mB = 0, \tag{3.7}$$

$$\frac{-B}{r}\left(\partial_\theta I_\uparrow + \frac{i}{\sin\theta}\partial_\phi I_\uparrow\right) = 0, \tag{3.8}$$

$$\frac{-A}{r}\left(\partial_\theta I_\uparrow + \frac{i}{\sin\theta}\partial_\phi I_\uparrow\right) = 0. \tag{3.9}$$

Now it is difficult to solve the above equations. So we can carry out the following standard solution [9]

$$I_{\uparrow} = -Et + W(r) + J(\theta, \phi), \tag{3.10}$$

then we find

$$-\frac{iA(E+qA_t)}{\sqrt{f(r)}} + B\sqrt{f(r)}W'(r) - Am = 0, \tag{3.11}$$

$$\frac{iB(E+qA_t)}{\sqrt{f(r)}} + A\sqrt{f(r)}W'(r) - Bm = 0, \tag{3.12}$$

$$\frac{-B}{r}\left(J_\theta + \frac{i}{\sin\theta}J_\phi\right) = 0, \tag{3.13}$$

$$\frac{-A}{r}\left(J_\theta + \frac{i}{\sin\theta}J_\phi\right) = 0. \tag{3.14}$$

We neglect the equation which depends upon "θ", because these equations do not contribute to the imaginary part of the action. Eqs. (3.11) and (3.12) become

$$-\frac{iA(E+qA_t)}{\sqrt{f(r)}} + B\sqrt{f(r)}W'(r) - Am = 0, \tag{3.15}$$

$$\frac{iB(E+qA_t)}{\sqrt{f(r)}} - A\sqrt{f(r)}W'(r) - Bm = 0. \tag{3.16}$$

3.7.2 The Massless Case

In the massless case ($m = 0$) Eq. (3.15) and (3.16) become

$$-\frac{iA(E+qA_t)}{\sqrt{f(r)}} + B\sqrt{f(r)}W'(r) = 0, \tag{3.17}$$

$$\frac{iB(E+qA_t)}{\sqrt{f(r)}} - A\sqrt{f(r)}W'(r) = 0. \tag{3.18}$$

These equations give two possible solutions

$$A = -\iota B \quad \text{and} \quad W'(r) = W'_+(r) = \frac{E+qA_t}{f(r)}, \tag{3.19}$$

$$A = \iota B \quad \text{and} \quad W'(r) = W'_-(r) = -\frac{(E+qA_t)}{f(r)}, \tag{3.20}$$

where W_+ corresponds to the outgoing solution and W_- corresponds to the incoming solution. For simplification we used Taylor's series at outer horizon and neglect the higher powers to get

$$f(r) = f(r_+) + (r - r_+)f_r(r_+). \tag{3.21}$$

At the horizon

$$f(r_+) = 0.$$

Eq. (3.21) becomes

$$f(r) = (r - r_+)f_r(r_+). \tag{3.22}$$

Using Eq. (3.22) in Eq. (3.19) and (3.20) we get

$$W'_+(r) = \frac{(E + qA_t)}{(r - r_+)f'(r_+)}.$$

After intergrating around the pole (and dropping the + subscript) and putting the value of $f'(r_+)$ as well, we get

$$\begin{aligned} W(r) &= \frac{\iota\pi(E - w_0)\left(M^2 + M\sqrt{M^2 - Q^2} - \frac{1}{2}Q^2\right)}{\sqrt{M^2 - Q^2}}, \\ \operatorname{Im} W(r) &= \frac{\pi(E - w_0)\left(M^2 + M\sqrt{M^2 - Q^2} - \frac{1}{2}Q^2\right)}{\sqrt{M^2 - Q^2}}. \end{aligned}$$

Here $w_0 = qV_0 = qQ/r_+$. So the tunneling probabilities of fermion charge particles is

$$\begin{aligned} Prob[out] &\propto exp[-2\operatorname{Im} I] = exp[-2(\operatorname{Im} W_+ + \operatorname{Im}\Theta)], \\ Prob[in] &\propto exp[-2\operatorname{Im} I] = exp[-2(\operatorname{Im} W_- + \operatorname{Im}\Theta)]. \end{aligned}$$

Since $\operatorname{Im} W_+ = -\operatorname{Im} W_-$

$$\Gamma = exp[-4\operatorname{Im} W_+], \tag{3.23}$$

and the resulting tunneling probability is

$$\Gamma = exp[-\frac{4\pi(E-w_0)\left(M^2+M\sqrt{M^2-Q^2}-\frac{1}{2}Q^2\right)}{\sqrt{M^2-Q^2}}]. \tag{3.24}$$

comparing this with $\Gamma = \exp(-\beta E)$ where $\beta = 1/T$, this gives the expected Hawking temperature as [10]

$$T_H = \frac{\sqrt{M^2-Q^2}}{4\pi\left(M^2+M\sqrt{M^2-Q^2}-\frac{1}{2}Q^2\right)}. \tag{3.25}$$

3.7.3 The Massive Case

In the massive case ($m \neq 0$) solving Eqs. (3.6) and (3.7) for A and B lead to the result

$$(\frac{A}{B})^2 = \frac{-\iota(E+qA_t)+\sqrt{f(r)}m}{\iota(E+qA_t)+\sqrt{f(r)}m}.$$

Near the horizon it can be seen that

$$\lim_{r\longrightarrow r_0}(\frac{A}{B})^2 = -1,$$

the other steps are same as in the massless case. We shall obtain the same result for the Hawking temperature as in the massless case.

The spin down case is very similar to the spin up case and just the sign is different. For the massive and massless spin down the same Hawking temperature as in Eq. (3.25) is recovered.

3.8 Hawking Radiation of Dirac Particles via Tunneling from the Kerr Black Hole

The Plebański-Demiański metric covers a large family of solutions of Einstein's field equation and it also includes rotating black holes with cosmological constant $\Lambda = 0$. Among the various subfamilies identified in the metric, Kerr metric in spherical polar coordinates (t, r, θ, ϕ) is given as [11,16]

$$ds^2 = -f(r,\theta)dt^2 + \frac{dr^2}{g(r,\theta)} + \Sigma(r,\theta)d\theta^2 + K(r,\theta)d\phi^2 - 2H(r,\theta)dtd\phi, \tag{3.26}$$

where $f(r,\theta)$, $g(r,\theta)$, $\Sigma(r,\theta)$, $K(r,\theta)$, $H(r,\theta)$ are defined below

$$f(r,\theta) = \frac{1}{\Omega^2}\left(\frac{Q - a^2 P\sin^2\theta}{\rho^2}\right), \tag{3.27}$$

$$g(r,\theta) = \frac{Q\Omega^2}{\rho^2}, \tag{3.28}$$

$$\Sigma(r,\theta) = \frac{\rho^2}{P\Omega^2}, \tag{3.29}$$

$$K(r,\theta) = \left(\frac{\sin^2\theta[P(r^2+a^2)^2 - Qa^2\sin^2\theta]}{\rho^2\Omega^2}\right), \tag{3.30}$$

$$H(r,\theta) = \left(\frac{2a\sin^2\theta\left[P(r^2+a^2) - Q\right]}{\rho^2\Omega^2}\right), \tag{3.31}$$

with

$$\Omega = 1, \tag{3.32}$$

$$\rho^2 = r^2 + a^2\cos^2\theta, \tag{3.33}$$

$$P = 1, \tag{3.34}$$

$$Q = (a^2 - 2Mr + r^2). \tag{3.35}$$

The event horizon of the accelerating and rotating black hole can be calculated by putting

$$\frac{1}{g_{11}} = 0, \tag{3.36}$$

which implies that

$$r_{\pm} = M \pm \sqrt{M^2 - a^2}. \tag{3.37}$$

Here $r_{\pm}$ represent the outer horizon and inner horizon corresponding to the Kerr black hole. Now we define the function

$$F(r,\theta) = f(r,\theta) + \frac{H^2(r,\theta)}{K(r,\theta)}. \tag{3.38}$$

Putting the values of $f(r,\theta),\ K(r,\theta),\ H(r,\theta)$ in Eq. (3.38) we get

$$F(r,\theta) = \frac{QP\rho^2}{(P(r^2+a^2)^2 - Qa^2\sin^2\theta)(\Omega^2)}. \tag{3.39}$$

The angular velocity for the metric (3.26) takes the form [13]

$$\Omega_H = \frac{H(r_+,\theta)}{K(r_+,\theta)}. \tag{3.40}$$

Putting the values of $H(r_+,\theta),\ K(r_+,\theta),$ we get

$$\Omega_H = \frac{a(P(r_+^2+a^2) - Q(r_+)}{Q(r_+)a^2\sin^2\theta + P(r_+^2+a^2)^2}. \tag{3.41}$$

Here we have used $Q(r_+) = 0$ [12]

$$\Omega_H = \frac{a}{r_+^2+a^2}. \tag{3.42}$$

We shall only show the calculations for the spin up case because the final result is the same for the spin down case apart from the change in the sign.

3.8.1 Calculation of the Tunneling Probability and Hawking Temperature

The Dirac equation is introduced for the uncharged fermions particles as

$$\iota\gamma^\mu(D_\mu)\Psi + \frac{m}{\hbar}\Psi = 0. \tag{3.43}$$

Here the Greek indices $\mu = (0,1,2,3)$ and m is the mass of the fermion particles, and

$$D_\mu = \partial_\mu + \Omega_\mu, \qquad \Omega_\mu = \frac{1}{2}\iota\Gamma^{\alpha\beta}{}_\mu\Sigma_{\alpha\beta}, \qquad \Sigma_{\alpha\beta} = \frac{1}{4}\iota[\gamma^\alpha,\gamma^\beta], \tag{3.44}$$

and γ^μ matrices satisfy $[\gamma^\alpha,\gamma^\beta] = 2g^{\mu\nu}I$, (I is the identity matrix). For fermion tunneling radiation, it is important to choose γ^μ matrices. The γ^μ matrices can be taken as

$$\gamma^t = \sqrt{\frac{(P(r^2+a^2)^2 - Qa^2\sin^2\theta)(\Omega^2)}{PQ\rho^2}}\gamma^0, \qquad \gamma^r = \sqrt{\frac{Q\Omega^2}{\rho^2}}\gamma^3, \qquad \gamma^\theta = \sqrt{\frac{P\Omega^2}{\rho^2}}\gamma^1,$$

$$\gamma^{\phi} = \frac{\rho\Omega\gamma^2}{\sin\theta(\sqrt{P(r^2+a^2)^2}) - Qa^2\sin^2\theta} + \frac{a(P(r^2+a^2) - Q)\gamma^0}{\sqrt{F(r,Q)}(P(r^2+a^2)^2) - Qa^2\sin^2\theta)}. \tag{3.45}$$

Here

$$\gamma^0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}. \tag{3.46}$$

Here $\sigma^i (i = 1, 2, 3)$ are the Pauli sigma matrices given as

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -\iota \\ \iota & 0 \end{pmatrix},$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.47}$$

The spin up and spin down solutions for the Dirac equation is

$$\Psi_{\uparrow}(t,r,\theta,\phi) = \begin{pmatrix} A(t,r,\theta,\phi)\xi_{\uparrow} \\ B(t,r,\theta,\phi)\xi_{\uparrow} \end{pmatrix} exp\{\frac{\iota I_{\uparrow}(t,r,\theta,\phi)}{\hbar}, \tag{3.48}$$

$$\Psi_{\downarrow}(t,r,\theta,\phi) = \begin{pmatrix} C(t,r,\theta,\phi)\xi_{\downarrow} \\ D(t,r,\theta,\phi)\xi_{\downarrow} \end{pmatrix} exp\{\frac{\iota.I_{\downarrow}(t,r,\theta,\phi)}{\hbar}. \tag{3.49}$$

where $[\gamma^{\alpha}, \gamma^{\beta}]$ satisfies the commutative relations

$$[\gamma^{\alpha}, \gamma^{\beta}] = -[\gamma^{\beta}, \gamma^{\alpha}] \qquad if \quad \alpha \neq \beta, \quad [\gamma^{\alpha}, \gamma^{\beta}] = 0 \quad if \quad \alpha = \beta. \tag{3.50}$$

By using Eq. (3.50) all the terms cancelled out and the reduced form of Eq. (3.43) is

$$(\iota\gamma^t\partial_t + \iota\gamma^r\partial_r + \iota\gamma^{\theta}\partial_{\theta} + \iota\gamma^{\phi}\partial_{\phi})\Psi + \frac{m}{\hbar}\Psi = 0. \tag{3.51}$$

Using Eqs. (3.45) to (3.48) in Eq. (3.51). Finally we obtain four equations.

$$
\begin{aligned}
0 &= -B(\frac{1}{\sqrt{F(r,\theta)}}\partial_t I_\uparrow + \sqrt{\frac{\Omega^2 Q}{\rho^2}}\partial_r I_\uparrow \\
&+ \frac{a(P(r^2+a^2)-Q)}{\sqrt{F(r,\theta)}(P(r^2+a^2)^2 - Qa^2\sin^2\theta)}\partial_\phi I_\uparrow) + Am, \quad (3.52)\\
0 &= -B(\sqrt{\frac{\Omega^2 P}{\rho^2}}\partial_\theta I_\uparrow + \frac{\iota\rho\Omega}{\sin\theta(\sqrt{P(r^2+a^2)^2 - Qa^2\sin^2\theta}}\partial_\phi I_\uparrow), \quad (3.53)\\
0 &= +A(\frac{1}{\sqrt{F(r,\theta)}}\partial_t I_\uparrow - \sqrt{\frac{\Omega^2 Q}{\rho^2}}\partial_r I_\uparrow \\
&+ \frac{a(P(r^2+a^2)-Q)}{\sqrt{F(r,\theta)}(P(r^2+a^2)^2) - Qa^2\sin^2\theta)}\partial_\phi I_\uparrow) + Bm, \quad (3.54)\\
0 &= -A(\sqrt{\frac{\Omega^2 P}{\rho^2}}\partial_\theta I_\uparrow + \frac{\iota\rho\Omega}{\sin\theta(\sqrt{P(r^2+a^2)^2) - Qa^2\sin^2\theta}}\partial_\phi I_\uparrow). \quad (3.55)
\end{aligned}
$$

It is difficult to solve the above equations. So, we assume that [11, 16]

$$
I_\uparrow = -Et + J\phi + W(r,\theta). \quad (3.56)
$$

So the above four equations become

$$
\begin{aligned}
0 &= -B(\frac{1}{\sqrt{F(r,\theta)}}(-E) + \sqrt{\frac{\Omega^2 Q}{\rho^2}}W'(r,\theta) \\
&+ \frac{a(P(r^2+a^2)-Q)}{\sqrt{F(r,\theta)}(P(r^2+a^2)^2) - Qa^2\sin^2\theta)}J) + Am, \quad (3.57)\\
0 &= -B(\sqrt{\frac{\Omega^2 P}{\rho^2}}W_\theta(r,\theta) + \frac{\iota\rho\Omega}{\sin\theta(\sqrt{P(r^2+a^2)^2) - Qa^2\sin^2\theta}}J), \quad (3.58)\\
0 &= +A(\frac{1}{\sqrt{F(r,\theta)}}(-E) - \sqrt{\frac{\Omega^2 Q}{\rho^2}}W'(r,\theta) \\
&+ \frac{a(P(r^2+a^2)-Q)}{\sqrt{F(r,\theta)}(P(r^2+a^2)^2) - Qa^2\sin^2\theta)}J) + Bm, \quad (3.59)\\
0 &= -A(\sqrt{\frac{\Omega^2 P}{\rho^2}}W_\theta(r,\theta) + \frac{\iota\rho\Omega}{\sin\theta(\sqrt{P(r^2+a^2)^2) - Qa^2\sin^2\theta}}J). \quad (3.60)
\end{aligned}
$$

Using Taylor's series in Eq. (3.28) and neglecting the second order powers we get [11, 16]

$$g(r,\theta) = g(r_+,\theta) + (r - r_+)g_r(r_+,\theta). \tag{3.61}$$

At the horizon

$$g(r_+,\theta) = 0, \quad F(r_+,\theta) = 0. \tag{3.62}$$

Rest of the equation becomes

$$g(r,\theta) = (r - r_+)g_r(r_+,\theta). \tag{3.63}$$

Taking the partial derivative of Eq. (3.28) with respect to r and evaluating at the outer horizon, we get

$$g_r(r_+,\theta) = \frac{(2r_+ - 2M)}{(r_+^2 + a^2\cos^2\theta)}. \tag{3.64}$$

Using Eqs. (3.64) and (3.62) in (3.61) we get

$$g(r,\theta) = (r - r_+)(\frac{(2r_+ - 2M)}{(r_+^2 + a^2\cos^2\theta)}). \tag{3.65}$$

Using the same procedure Eq. (3.39) becomes

$$F(r,\theta) = (r - r_+)(\frac{(r_+^2 + a^2\cos^2\theta)(2r_+ - 2M)}{((r_+^2 + a^2)^2)\Omega^2}). \tag{3.66}$$

Now expanding Eqs. (3.52) to (3.55) near the black hole horizon and using Eqs. (3.65) and (3.66) we get

$$\begin{aligned}
0 &= -B(\frac{(-E)}{\sqrt{(r-r_+)F_r(r_+,\theta)}} + \sqrt{(r-r_+)g_r(r_+,\theta)}W'(r,\theta) \\
&+ \frac{a(P(r_+^2+a^2)-Q)}{\sqrt{(r-r_+)F_r(r_+,\theta)}(P(r^2+a^2)^2)-Qa^2\sin^2\theta)}J) + Am, && (3.67) \\
0 &= -B(\sqrt{\frac{\Omega^2 P}{\rho^2}}W_\theta(r,\theta) + \frac{\iota\rho\Omega}{\sin\theta(\sqrt{P(r^2+a^2)^2)-Qa^2\sin^2\theta}}J), && (3.68) \\
0 &= +A(\frac{(-E)}{\sqrt{(r-r_+)F_r(r_+,\theta)}} - \sqrt{(r-r_+)g_r(r_+,\theta)}W'(r,\theta) \\
&+ \frac{a(P(r_+^2+a^2)-Q)}{\sqrt{(r-r_+)F_r(r_+,\theta)}(P(r^2+a^2)^2)-Qa^2\sin^2\theta)}J) + Bm, && (3.69) \\
0 &= -A(\sqrt{\frac{\Omega^2 P}{\rho^2}}W_\theta(r,\theta) + \frac{\iota\rho\Omega}{\sin\theta(\sqrt{P(r^2+a^2)^2)-Qa^2\sin^2\theta}}J). && (3.70)
\end{aligned}$$

We neglect the equation which depends upon "θ", because these equation do not contribution to the imaginary part of the action. Using Eq. (3.41) in Eqs. (3.52) and (3.54) we get

$$\begin{aligned}
0 &= -B(\frac{-E+\Omega_H J}{\sqrt{(r-r_+)F_r(r_+,\theta)}} + \sqrt{(r-r_+)g_r(r_+,\theta)}W'(r,\theta)) \\
&+ Am, && (3.71) \\
0 &= +A(\frac{-E+\Omega_H J}{\sqrt{(r-r_+)F_r(r_+,\theta)}} - \sqrt{(r-r_+)g_r(r_+,\theta)}W'(r,\theta)) \\
&+ Bm. && (3.72)
\end{aligned}$$

At the horizon we can further separate $W(r,\theta)$ as

$$W(r,\theta) = W(r) + \Theta(\theta) \tag{3.73}$$

The Massless Case

If $m = 0$, there exist two possible solutions

$$B = 0, \quad \text{or} \quad W'(r) = W'_+(r) = \frac{(E-\Omega_H J)}{\sqrt{(r-r_+)F_r(r_+,\theta)}\sqrt{(r-r_+)g_r(r_+,\theta)}}.$$

Eq. (3.72) becomes

$$A=0, \quad \text{or} \quad W'(r)=W'_-(r)=\frac{-(E-\Omega_H J)}{\sqrt{(r-r_+)F_r(r_+,\theta)}\sqrt{(r-r_+)g_r(r_+,\theta)}}.$$

Putting the values of $F_r(r_+,\theta)$ and $g_r(r_+,\theta)$ in the above equations we get

$$W'_+(r) = \frac{(E-\Omega_H J)(r_+^2+a^2)}{(r-r_+)(2r_+-2M)}, \tag{3.74}$$

$$W'_-(r) = \frac{-(E-\Omega_H J)(r_+^2+a^2)}{(r-r_+)(2r_+-2M)}. \tag{3.75}$$

Here the prime denotes the derivative with respect to r and $+/-$ corresponds to outgoing/incoming solutions. For finding the value of $W(r)$ we integrate the above result. Here $r=r_+$ is the simple pole. Integrating around the pole we get

$$W_+(r)=\pi\iota\frac{(E-\Omega_H J)(r_+^2+a^2)}{(2r_+-2M)}.$$

Dropping the $+$ subscript we obtain

$$W(r) = \pi\iota\frac{(E-\Omega_H J)(r_+^2+a^2)}{(2r_+-2M)},$$

$$\operatorname{Im} W = \frac{\pi}{2}\frac{(E-\Omega_H J)(r_+^2+a^2)}{(r_+-M)}. \tag{3.76}$$

So the tunneling probabilities of fermion charged particles is

$$Prob[out] \propto exp[-2\operatorname{Im} I]=exp[-2(\operatorname{Im} W_+ + \operatorname{Im}\Theta)], \tag{3.77}$$

$$Prob[in] \propto exp[-2\operatorname{Im} I]=exp[-2(\operatorname{Im} W_- + \operatorname{Im}\Theta)]. \tag{3.78}$$

Since $\operatorname{Im} W_+ = -\operatorname{Im} W_-$ so

$$\Gamma=exp[-4\operatorname{Im} W_+]. \tag{3.79}$$

The resulting tunneling probability is

$$\Gamma=exp[-2\pi\frac{(E-\Omega_H J)(r_+^2+a^2)}{(r_+-M)}]. \tag{3.80}$$

Comparing this with $\Gamma = \exp(-\beta E)$ where $\beta = 1/T$, this gives the expected Hawking temperature as which gives

$$T_H = \frac{(1+\sqrt{M^2-a^2})}{2\pi(2M^2+2M\sqrt{M^2-a^2})}. \tag{3.81}$$

The Massive Case

In the massive case Eqs. (3.71) and (3.72) no longer decouple and analysis of the tunneling is more subtle. We shall begin by eliminating the function $W'(r,\theta)$ from these two equation and will find an equation relating A and B in terms of the known quantities. Multiplying Eq. (3.72) by B and Eq. (3.71) by A and subtracting yields

$$\frac{A}{B} = \frac{-(E-J\Omega_H) \pm \sqrt{(E-J\Omega_H)^2 + m^2 F_r(r_+,\theta)(r-r_+)}}{m\sqrt{F_r(r_+,\theta)(r-r_+)}}, \tag{3.82}$$

where

$$\lim_{r\to r_+}\left(\frac{A}{B}\right) = \lim_{r\to r_+}\left(\frac{-(E-J\Omega_H) \pm \sqrt{(E-J\Omega_H)^2 + m^2 F_r(r_+,\theta)(r-r_+)}}{m\sqrt{F_r(r_+,\theta)(r-r_+)}}\right). \tag{3.83}$$

$$\lim_{r\to r_+}\left(\frac{A}{B}\right) = \left\{ \begin{array}{c} 0 \\ -\infty \end{array} \right. ,$$

for the upper/lower sign respectively.

Consequently at the horizon either $A/B \to 0$ or $A/B \to -\infty$, i.e. either $A \to 0$ or $B \to 0$. For $A \to 0$ we find the value of m from Eq. (3.72),

$$m = \frac{A}{B}\left(\frac{E-J\Omega_H}{\sqrt{F_r(r_+,\theta)(r-r_+)}} + \sqrt{g_r(r_+,\theta)(r-r_+)}W'(r)\right).$$

Putting in Eq. (3.71) and simplifying we get

$$W_r(r,\theta) = W'_+(r) = \frac{(E-J\Omega_H)}{\sqrt{F_r(r_+,\theta)g_r(r_+,\theta)}(r-r_+)}(1+A^2/B^2)/(1-A^2/B^2).$$

Integrating with respect to r we get

$$W_+(r) = \int \frac{(E-J\Omega_H)}{\sqrt{F_r(r_+,\theta)g_r(r_+,\theta)}(r-r_+)}(1+A^2/B^2)/(1-A^2/B^2)dr.$$

Here $r = r_+$ is the simple pole. Integrating around the pole we get

$$W_+(r) = \frac{\pi i(E - J\Omega_H)}{\sqrt{F_r(r_+, \theta) g_r(r_+, \theta)}}.$$

Using the values of functions $F_r(r_+, \theta)$ and $g_r(r_+, \theta)$ and simplifying we obtain

$$W_+(r) = \pi i \frac{(E - \Omega_H J)(r_+^2 + a^2)}{(2r_+ - 2M)}.$$

Dropping the $+$ subscript we obtain

$$\begin{aligned} W(r) &= \pi \iota \frac{(E - \Omega_H J)(r_+^2 + a^2)}{(2r_+ - 2M)}, \\ \operatorname{Im} W &= \frac{\pi}{2} \frac{(E - \Omega_H J)(r_+^2 + a^2)}{(r_+ - M)}. \end{aligned} \tag{3.84}$$

For $B \to 0$, we can simply rewrite the expression in term of B/A to get

$$W_r(r, \theta) = W'_-(r) = \frac{-(E - J\Omega_H)}{\sqrt{F_r(r_+, \theta) g_r(r_+, \theta)}(r - r_+)}(1 + B^2/A^2)/(1 - B^2/A^2),$$

$$W_-(r) = \pi \iota \frac{-(E - \Omega_H J)(r_+^2 + a^2)}{(2r_+ - 2M)}. \tag{3.85}$$

Here the final result is the same as in the massless case.

The spin down case is very similar to the spin up case and just the sign is different. The equations are of the same form as in the spin up case. For the massive and massless spin down case the Hawking Temperature (3.81) is recovered.

Chapter 4

Hawking Radiation from Accelerating and Rotating Black Holes

4.1 Introduction

The Plebanski-Demianski metric [14] covers a large family of solutions of EFEs and it also includes accelerating and rotating black holes with cosmological constant $\Lambda = 0$. In spherical polar coordinates (t, r, θ, ϕ) this metric can be written as [15]

$$ds^2 = \frac{-1}{\Omega^2}\{\frac{Q}{\rho^2}[dt - a\sin^2\theta d\phi]^2 - \frac{\rho^2}{Q}dr^2 - \frac{\rho^2}{P}d\theta^2 - \frac{P\sin^2\theta}{\rho^2}[adt - (r^2 + a^2)d\phi]^2\}, \qquad (4.1)$$

which in expanded form becomes

$$\begin{aligned} ds^2 &= \frac{-1}{\Omega^2}\{\frac{Q}{\rho^2}[dt^2 + a^2\sin^4\theta d\phi^2 - 2a\sin^2\theta dtd\theta] - \frac{\rho^2}{Q}dr^2 - \frac{\rho^2}{P}d\theta^2 \\ &- \frac{P\sin^2\theta}{\rho^2}[a^2dt^2 + (r^2 + a^2)^2d\phi^2 - 2a(r^2 + a^2)dtd\phi]\}. \end{aligned} \qquad (4.2)$$

Another convenient form of this metric is

$$
\begin{aligned}
ds^2 &= \frac{1}{\Omega^2}\{-\{\frac{Q}{\rho^2} - \frac{a^2 P\sin^2\theta}{\rho^2}\}dt^2 + \frac{\rho^2}{Q}dr^2 + \frac{\rho^2}{P}d\theta^2 \\
&+(\frac{P(r^2+a^2)^2\sin^2\theta}{\rho^2} - \frac{Qa^2\sin^4\theta}{\rho^2})\}d\phi^2 - \frac{2a\sin^2\theta(P(r^2+a^2)-Q)dtd\phi}{\rho^2\Omega^2}, \quad (4.3)
\end{aligned}
$$

where

$$
\begin{aligned}
\Omega &= 1-\alpha r\cos\theta, &(4.4)\\
\rho^2 &= r^2+a^2\cos^2\theta, &(4.5)\\
P &= 1-2\alpha M\cos\theta+\alpha^2a^2\cos^2\theta, &(4.6)\\
Q &= (a^2-2Mr+r^2)(1-\alpha^2r^2). &(4.7)
\end{aligned}
$$

Here M, a and α are the arbitrary parameters. M is the mass of the black hole, a is rotation and α is the acceleration of the black hole. Now using the notation of [16], the above metric defined by Eq. (4.1) can be written as

$$
ds^2 = -f(r,\theta)dt^2 + \frac{dr^2}{g(r,\theta)} + \Sigma(r,\theta)d\theta^2 + K(r,\theta)d\phi^2 - 2H(r,\theta)dtd\phi, \quad (4.8)
$$

where $f(r,\theta)$, $g(r,\theta)$, $\Sigma(r,\theta)$, $K(r,\theta)$ and $H(r,\theta)$ are defined below

$$
\begin{aligned}
f(r,\theta) &= \frac{1}{\Omega^2}(\frac{Q-a^2P\sin^2\theta}{\rho^2}), &(4.9)\\
g(r,\theta) &= \frac{Q\Omega^2}{\rho^2}, &(4.10)\\
\Sigma(r,\theta) &= \frac{\rho^2}{P\Omega^2}, &(4.11)\\
K(r,\theta) &= \left(\frac{\sin^2\theta\left[P(r^2+a^2)^2-Qa^2\sin^2\theta\right]}{\rho^2\Omega^2}\right), &(4.12)\\
H(r,\theta) &= \left(\frac{2a\sin^2\theta\left[P(r^2+a^2)-Q\right]}{\rho^2\Omega^2}\right). &(4.13)
\end{aligned}
$$

The event horizons can be calculated by putting

$$\frac{1}{g_{11}} = 0, \tag{4.14}$$

which implies that

$$g(r,\theta) = \frac{Q\Omega^2}{\rho^2} = 0. \tag{4.15}$$

Thus we get

$$\Omega^2 = 0, \qquad Q = 0. \tag{4.16}$$

Putting Eq. (4.16) in Eqs. (4.4) and (4.7) this becomes

$$\begin{aligned} 0 &= (1-\alpha^2 r^2), \qquad (1-\alpha r\cos\theta)^2 = 0, \\ 0 &= (a^2 - 2Mr + r^2). \end{aligned} \tag{4.17}$$

Finally, we obtain

$$r_{\pm} = \pm\frac{1}{\alpha}, \quad r = \frac{1}{\alpha\cos\theta}, \quad \text{and} \quad r_{\pm} = M \pm \sqrt{M^2 - a^2}, \tag{4.18}$$

where $r_{\pm}$ represent the outer and inner horizons corresponding to the Kerr-Newman black holes. Here the other two horizons are acceleration horizons. Now we define the function as shown in [16]

$$F(r,\theta) = f(r,\theta) + \frac{H^2(r,\theta)}{K(r,\theta)}. \tag{4.19}$$

Using the values of $f(r,\theta)$, $K(r,\theta)$ and $H(r,\theta)$ from Eqs. (4.9), (4.12) and (4.13) and after simplification we get

$$F(r,\theta) = \frac{PQ\rho^2}{\left[P(r^2+a^2)^2 - Qa^2\sin^2\theta\right]\Omega^2}. \tag{4.20}$$

The angular velocity, for the metric (4.1) takes the form [16]

$$\Omega_H = \frac{H(r_+,\theta)}{K(r_+,\theta)}. \tag{4.21}$$

Using the values of $K(r_+,\theta)$ and $H(r_+,\theta)$ from Eqs. (4.12) and (4.13) we get

$$\Omega_H = \frac{a(P(r_+^2+a^2) - Q(r_+)}{Q(r_+)a^2\sin^2\theta + P(r_+^2+a^2)^2}. \tag{4.22}$$

If we use $Q(r_+) = 0$,this takes the form

$$\Omega_H = \frac{a}{r_+^2+a^2}. \tag{4.23}$$

We shall only show the calculations for the spin up case. The calculations for the spin down case are similar, apart from the change in the sign.

4.1.1 Calculation of the Tunneling Probability and Hawking Temperature

The Dirac equation [9,17] for the uncharged fermion particles is

$$\iota\gamma^\mu(D_\mu)\Psi + \frac{m}{\hbar}\Psi = 0, \tag{4.24}$$

The quantities γ's are defined as

$$\begin{aligned}
\gamma^t &= \sqrt{\frac{(P(r^2+a^2)^2 - Qa^2\sin^2\theta)(\Omega^2)}{PQ\rho^2}}\gamma^0, \qquad \gamma^r = \sqrt{\frac{Q\Omega^2}{\rho^2}}\gamma^3, \qquad \gamma^\theta = \sqrt{\frac{P\Omega^2}{\rho^2}}\gamma^1, \\
\gamma^\phi &= \frac{\rho\Omega\gamma^2}{\sin\theta\sqrt{P(r^2+a^2)^2) - Qa^2\sin^2\theta}} + \frac{a(P(r^2+a^2) - Q)\gamma^0}{\sqrt{F(r,Q)}(P(r^2+a^2)^2) - Qa^2\sin^2\theta)},
\end{aligned} \tag{4.25}$$

where the matrices and the Pauli sigma matrices is defined in Eqs. (3.46), (3.47). The solutions of the spin up and spin down particles respectively can be assumed to be [17].

$$\Psi_\uparrow(t,r,\theta,\phi) = \begin{pmatrix} A(t,r,\theta,\phi)\xi_\uparrow \\ B(t,r,\theta,\phi)\xi_\uparrow \end{pmatrix} exp\left[\frac{\iota}{\hbar}I_\uparrow(t,r,\theta,\phi)\right], \tag{4.26}$$

and

$$\Psi_\downarrow(t,r,\theta,\phi) = \begin{pmatrix} C(t,r,\theta,\phi)\xi_\downarrow \\ D(t,r,\theta,\phi)\xi_\downarrow \end{pmatrix} exp\left[\frac{\iota}{\hbar}I_\uparrow(t,r,\theta,\phi)\right], \tag{4.27}$$

where $I_{\uparrow/\downarrow}$ denote the action of the emitted spin up and spin down particles, respectively. We shall only show the spin up case since the spin down case is similar expect for some change in the sign.

$$D_\mu = \partial_\mu + \frac{1}{8}\iota^2 \Gamma_\mu^{\alpha\beta}[\Upsilon^\alpha, \Upsilon^\beta], \tag{4.28}$$

where $[\gamma^\alpha, \gamma^\beta]$ satisfies the commutative relations

$$[\gamma^\alpha, \gamma^\beta] = -[\gamma^\beta, \gamma^\alpha], \qquad if \quad \alpha \neq \beta, \quad [\gamma^\alpha, \gamma^\beta] = 0, \quad if \quad \alpha = \beta. \tag{4.29}$$

Giving variation to α and β this becomes

$$\begin{aligned} D_\mu \;=\; & \partial_\mu + \frac{1}{8}\iota^2[\Gamma_\mu^{00}[\gamma^0,\gamma^0] + \Gamma_\mu^{01}[\gamma^0,\gamma^1] + \Gamma_\mu^{02}[\gamma^0,\gamma^2] + \Gamma_\mu^{03}[\gamma^0,\gamma^{03}] + \Gamma_\mu^{10}[\gamma^1,\gamma^0] \\ & + \Gamma_\mu^{11}[\gamma^1,\gamma^1] + \Gamma_\mu^{12}[\gamma^1,\gamma^2] + \Gamma_\mu^{13}[\gamma^1,\gamma^3] + \Gamma_\mu^{20}[\gamma^2,\gamma^0] + \Gamma_\mu^{21}[\gamma^2,\gamma^1] \\ & + \Gamma_\mu^{22}[\gamma^2,\gamma^2] + \Gamma_\mu^{23}[\gamma^2,\gamma^3] + \Gamma_\mu^{30}[\gamma^3,\gamma^0] + \Gamma_\mu^{31}[\gamma^3,\gamma^1] + \Gamma_\mu^{32}[\gamma^3,\gamma^2] + \Gamma_\mu^{33}[\gamma^3,\gamma^3]]. \end{aligned} \tag{4.30}$$

By using Eq. (4.29) all the terms in Eq. (4.30) cancelled out except ∂_μ. Thus the reduced form of Eq. (4.24) is

$$(\iota\gamma^t\partial_t + \iota\gamma^r\partial_r + \iota\gamma^\theta\partial_\theta + \iota\gamma^\phi\partial_\phi)\Psi + \frac{m}{\hbar}\Psi = 0. \tag{4.31}$$

Now consider the first term of Eq. (4.31)

$$(\iota\gamma^t\partial_t)\Psi = \iota\frac{1}{F(r,\theta)}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\partial_t\begin{pmatrix} A(t,r,\theta,\phi)\xi_\uparrow \\ B(t,r,\theta,\phi)\xi_\uparrow \end{pmatrix} exp[\frac{\iota}{\hbar}I].$$

Taking derivative of the matrix with respect to t we get

$$(\iota\gamma^t\partial_t)\Psi = \frac{\iota}{F(r,\theta)}\begin{pmatrix} (B_t + B\frac{\iota}{\hbar}\partial_t I_\uparrow)I\xi_\uparrow \\ -(A_t + A\frac{\iota}{\hbar}\partial_t I_\uparrow)I\xi_\uparrow \end{pmatrix} exp[\frac{\iota}{\hbar}I].$$

Now

$$I\xi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

therefore the above equation becomes

$$(\iota\gamma^t\partial_t)\Psi = \frac{\iota}{F(r,\theta)}\begin{pmatrix} (B_t + B\frac{\iota}{\hbar}\partial_t I_\uparrow) \\ 0 \\ -(A_t + A\frac{\iota}{\hbar}\partial_t I_\uparrow) \\ 0 \end{pmatrix} exp[\frac{\iota}{\hbar}I]. \tag{4.32}$$

Now consider the second term of Eq. (4.31)

$$(\iota\gamma^r\partial_r)\Psi = \iota\sqrt{\frac{\Omega^2 Q}{\rho^2}}\begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}\partial_r\begin{pmatrix} A(t,r,\theta,\phi)\xi_\uparrow \\ B(t,r,\theta,\phi)\xi_\uparrow \end{pmatrix} exp[\frac{\iota}{\hbar}I].$$

Taking derivative of the matrix with respect to r we obtain

$$(\iota\gamma^r\partial_r)\Psi = \iota\sqrt{\frac{\Omega^2 Q}{\rho^2}}\begin{pmatrix} (B_r + B\frac{\iota}{\hbar}\partial_r I_\uparrow)\sigma^3\xi_\uparrow \\ (A_r + A\frac{\iota}{\hbar}\partial_r I_\uparrow)\sigma^3\xi_\uparrow \end{pmatrix} exp[\frac{\iota}{\hbar}I].$$

Now

$$\sigma^3\xi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

therefore the above equation becomes

$$(\iota\gamma^r\partial_r)\Psi = \iota\sqrt{\frac{\Omega^2 Q}{\rho^2}}\begin{pmatrix} B_r + B\frac{\iota}{\hbar}\partial_r I_\uparrow \\ 0 \\ A_r + A\frac{\iota}{\hbar}\partial_r I_\uparrow \\ 0 \end{pmatrix} exp[\frac{\iota}{\hbar}I]. \tag{4.33}$$

The third term of Eq. (4.31) is

$$(\iota\gamma^\theta\partial_\theta)\Psi = \iota\sqrt{\frac{\Omega^2 P}{\rho^2}}\begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}\partial_\theta\begin{pmatrix} A(t,r,\theta,\phi)\xi_\uparrow \\ B(t,r,\theta,\phi)\xi_\uparrow \end{pmatrix} exp[\frac{\iota}{\hbar}]I.$$

Taking derivative of the matrix with respect to θ we get

$$(\iota\gamma^{\theta}\partial_{\theta})\Psi = \iota\sqrt{\frac{\Omega^2 P}{\rho^2}}\begin{pmatrix} (B_{\theta} + B\frac{\iota}{\hbar}\partial_{\theta}I_{\uparrow})\sigma^1\xi_{\uparrow} \\ (A_{\theta} + A\frac{\iota}{\hbar}\partial_{\theta}I_{\uparrow})\sigma^1\xi_{\uparrow} \end{pmatrix} exp[\frac{\iota}{\hbar}I].$$

Since

$$\sigma^1\xi_{\uparrow} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

thus the above equation takes the form

$$(\iota\gamma^{\theta}\partial_{\theta})\Psi = \iota\sqrt{\frac{\Omega^2 P}{\rho^2}}\begin{pmatrix} 0 \\ B_{\theta} + B\frac{\iota}{\hbar}\partial_{\theta}I_{\uparrow} \\ 0 \\ A_{\theta} + A\frac{\iota}{\hbar}\partial_{\theta}I_{\uparrow} \end{pmatrix} exp[\frac{\iota}{\hbar}I]. \tag{4.34}$$

The last term of Eq. (4.31) is

$$\begin{aligned}(\iota\gamma^{\phi}\partial_{\phi})\Psi &= (\frac{\iota\rho\Omega\gamma^2}{\sin\theta\sqrt{P(r^2+a^2)^2 - Qa^2\sin^2\theta)}} \\ &+ \frac{a(P(r^2+a^2)-Q)\gamma^0}{\sqrt{F(r,Q)}(P(r^2+a^2)^2 - Qa^2\sin^2\theta))})\partial\phi\begin{pmatrix} A(t,r,\theta,\phi)\xi_{\uparrow} \\ B(t,r,\theta,\phi)\xi_{\uparrow} \end{pmatrix} exp[\frac{\iota}{\hbar}I].\end{aligned}$$

Taking derivative with respect to ϕ we obtain

$$\begin{aligned}(\iota\gamma^{\phi}\partial_{\phi})\Psi &= \frac{\iota\rho\Omega}{\sin\theta\sqrt{P(r^2+a^2)^2 - Qa^2\sin^2\theta)}}\begin{pmatrix} (B_{\phi} + B\frac{\iota}{\hbar}\partial_{\phi}I_{\uparrow})\sigma^2\xi_{\uparrow} \\ (B_{\phi} + B\frac{\iota}{\hbar}\partial_{\phi}I_{\uparrow})\sigma^2\xi_{\uparrow} \end{pmatrix} exp[\frac{\iota}{\hbar}I] \\ &+ \frac{a[P(r^2+a^2)-Q]}{\sqrt{F(r,Q)}[P(r^2+a^2)^2 - Qa^2\sin^2\theta)]})\begin{pmatrix} (B_{\phi} + B\frac{\iota}{\hbar}\partial_{\phi}I_{\uparrow})I\xi_{\uparrow} \\ -(A_{\phi} + B\frac{\iota}{\hbar}\partial_{\phi}I)_{\uparrow}I\xi_{\uparrow} \end{pmatrix} exp[\frac{\iota}{\hbar}I].\end{aligned}$$

As

$$\sigma^1\xi_{\uparrow} = \begin{pmatrix} 0 \\ \iota \end{pmatrix}, \text{ and } I\xi_{\uparrow} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

the above equation becomes

$$(\iota\gamma^{\phi}\partial_{\phi})\Psi = \frac{\iota\rho\Omega}{\sin\theta(\sqrt{P(r^2+a^2)^2-Qa^2\sin^2\theta)}}\begin{pmatrix} 0 \\ \iota(B_{\phi}+B\frac{\iota}{\hbar}\partial_{\phi}I_{\uparrow}) \\ 0 \\ \iota(A_{\phi}+A\frac{\iota}{\hbar}\partial_{\phi}I_{\uparrow}) \end{pmatrix} exp[\frac{\iota}{\hbar}I]$$
$$+ \frac{a(P(r^2+a^2)-Q)}{\sqrt{F(r,Q)}(P(r^2+a^2)^2-Qa^2\sin^2\theta))}) \begin{pmatrix} B_{\phi}+B\frac{\iota}{\hbar}\partial_{\phi}I_{\uparrow} \\ 0 \\ -(A_{\phi}+A\frac{\iota}{\hbar}\partial_{\phi}I_{\uparrow}) \\ 0 \end{pmatrix} exp[\frac{\iota}{\hbar}]. \tag{4.35}$$

Using Eqs. (4.32) to (4.35) in (4.31) we obtain the following four equations.

$$0 = -B(\frac{1}{\sqrt{F(r,\theta)}}\partial_t I_{\uparrow} + \sqrt{\frac{\Omega^2 Q}{\rho^2}}\partial_r I_{\uparrow} + \frac{a(P(r^2+a^2)-Q)}{\sqrt{F(r,\theta)}(P(r^2+a^2)^2)-Qa^2\sin^2\theta)}\partial_{\phi}I_{\uparrow}) + Am, \tag{4.36}$$

$$0 = -B(\sqrt{\frac{\Omega^2 P}{\rho^2}}\partial_{\theta}I_{\uparrow} + \frac{\iota\rho\Omega}{\sin\theta(\sqrt{P(r^2+a^2)^2)}-Qa^2\sin^2\theta}\partial_{\phi}I_{\uparrow}), \tag{4.37}$$

$$0 = +A(\frac{1}{\sqrt{F(r,\theta)}}\partial_t I_{\uparrow} - \sqrt{\frac{\Omega^2 Q}{\rho^2}}\partial_r I_{\uparrow} + \frac{a(P(r^2+a^2)-Q)}{\sqrt{F(r,\theta)}(P(r^2+a^2)^2)-Qa^2\sin^2\theta)}\partial_{\phi}I_{\uparrow}) + Bm, \tag{4.38}$$

$$0 = -A(\sqrt{\frac{\Omega^2 P}{\rho^2}}\partial_{\theta}I_{\uparrow} + \frac{\iota\rho\Omega}{\sin\theta(\sqrt{P(r^2+a^2)^2)}-Qa^2\sin^2\theta}\partial_{\phi}I_{\uparrow}). \tag{4.39}$$

We apply the following anstaz [17] for solving the above system of equations

$$I_{\uparrow} = -Et + J\phi + W(r,\theta). \tag{4.40}$$

The above four equations become

$$
\begin{aligned}
0 &= -B(\frac{1}{\sqrt{F(r,\theta)}}(-E)+\sqrt{\frac{\Omega^2 Q}{\rho^2}}W'(r,\theta) \\
&\quad +\frac{a(P(r^2+a^2)-Q)}{\sqrt{F(r,\theta)}(P(r^2+a^2)^2-Qa^2\sin^2\theta))}J)+Am, && (4.41)\\
0 &= -B(\sqrt{\frac{\Omega^2 P}{\rho^2}}W_\theta(r,\theta)+\frac{\iota\rho\Omega}{\sin\theta(\sqrt{P(r^2+a^2)^2-Qa^2\sin^2\theta)}}J), && (4.42)\\
0 &= +A(\frac{1}{\sqrt{F(r,\theta)}}(-E)-\sqrt{\frac{\Omega^2 Q}{\rho^2}}W'(r,\theta) \\
&\quad +\frac{a(P(r^2+a^2)-Q)}{\sqrt{F(r,\theta)}(P(r^2+a^2)^2-Qa^2\sin^2\theta))}J)+Bm, && (4.43)\\
0 &= -A(\sqrt{\frac{\Omega^2 P}{\rho^2}}W_\theta(r,\theta)+\frac{\iota\rho\Omega}{\sin\theta(\sqrt{P(r^2+a^2)^2-Qa^2\sin^2\theta)}}J). && (4.44)
\end{aligned}
$$

Using Taylor's theorem in Eq. (4.10) and neglecting square and higher powers we get [11,16,17]

$$g(r,\theta)=g(r_+,\theta)+(r-r_+)g_r(r_+,\theta). \tag{4.45}$$

At the horizon

$$g(r_+,\theta)=0, \quad F(r_+,\theta)=0.$$

Thus Eq. (4.49) becomes

$$g(r,\theta)=(r-r_+)g_r(r_+,\theta). \tag{4.46}$$

Taking partial derivative of Eq. (4.10) with respect to r and evaluating at the horizon, we get

$$g_r(r_+,\theta)=\frac{(1-\alpha r_+\cos\theta)^2(2r_+-2M)(1-\alpha^2 r_+^2)}{(r_+^2+a^2\cos^2\theta)}. \tag{4.47}$$

Using Eq. (4.47) in Eq. (4.46) we get

$$g(r,\theta)=(r-r_+)(\frac{(1-\alpha r_+\cos\theta)^2(2r_+-2M)(1-\alpha^2 r_+^2)}{(r_+^2+a^2\cos^2\theta)}). \tag{4.48}$$

Using the same procedure Eq. (4.20) becomes

$$F(r,\theta) = (r - r_+)(\frac{(r_+^2 + a^2\cos^2\theta)(2r_+ - 2M)(1 - \alpha^2 r_+^2)}{((r_+^2 + a^2)^2)\Omega^2}). \tag{4.49}$$

Now expanding Eqs. (4.41) to (4.44) near the black hole horizon and using Eqs. (4.48) and (4.49) we get

$$\begin{aligned} 0 &= -B(\frac{(-E)}{\sqrt{(r-r_+)F_r(r_+,\theta)}} + \sqrt{(r-r_+)g_r(r_+,\theta)}W'(r,\theta) \\ &\quad + \frac{a(P(r_+^2+a^2)-Q)}{\sqrt{(r-r_+)F_r(r_+,\theta)}(P(r^2+a^2)^2 - Qa^2\sin^2\theta))}J) + Am, \end{aligned} \tag{4.50}$$

$$0 = -B(\sqrt{\frac{\Omega^2 P}{\rho^2}}W_\theta(r,\theta) + \frac{\iota\rho\Omega}{\sin\theta\sqrt{P(r^2+a^2)^2 - Qa^2\sin^2\theta)}}J), \tag{4.51}$$

$$\begin{aligned} 0 &= +A(\frac{(-E)}{\sqrt{(r-r_+)F_r(r_+,\theta)}} - \sqrt{\frac{\Omega^2 Q}{\rho^2}}W'(r,\theta), \\ &\quad + \frac{a(P(r_+^2+a^2)-Q)}{\sqrt{(r-r_+)F_r(r_+,\theta)}(P(r^2+a^2)^2 - Qa^2\sin^2\theta))}J) + Bm, \end{aligned} \tag{4.52}$$

$$0 = -A(\sqrt{\frac{\Omega^2 P}{\rho^2}}W_\theta(r,\theta) + \frac{\iota\rho\Omega}{\sin\theta\sqrt{P(r^2+a^2)^2 - Qa^2\sin^2\theta)}}J). \tag{4.53}$$

We neglect the equation which depends upon "θ". Although these equation could provided a contribution to the imaginary part of the action, but its total contribution to the tunneling rate are cancelled out. Using Eq. (4.22) in Eqs. (4.50) and (4.52) we get

$$\begin{aligned} 0 &= -B(\frac{-E+\Omega_H J}{\sqrt{(r-r_+)F_r(r_+,\theta)}} + \sqrt{(r-r_+)g_r(r_+,\theta)}W'(r,\theta)) \\ &\quad + Am, \end{aligned} \tag{4.54}$$

$$\begin{aligned} 0 &= +A(\frac{-E+\Omega_H J}{\sqrt{(r-r_+)F_r(r_+,\theta)}} - \sqrt{(r-r_+)g_r(r_+,\theta)}W'(r,\theta)) \\ &\quad + Bm. \end{aligned} \tag{4.55}$$

At the horizon we can further separate $W(r,\theta)$ by using Eq. (3.73) and divide our solution into two parts, the massless and the massive case.

4.1.2 The Massless Case

In the massless case we put $m = 0$ in Eqs. (4.54) - (4.55), then there exist two possible solutions

$$B = 0, \quad W'(r) = W'_+(r) = \frac{(E - \Omega_H J)}{\sqrt{(r - r_+)F_r(r_+, \theta)}\sqrt{(r - r_+)g_r(r_+, \theta)}}.$$

Eq. (4.55) become

$$A = 0, \quad W'(r) = W'_-(r) = \frac{-(E - \Omega_H J)}{\sqrt{(r - r_+)F_r(r_+, \theta)}\sqrt{(r - r_+)g_r(r_+, \theta)}}.$$

Putting the values of $F_r(r_+, \theta)$ and $g_r(r_+, \theta)$ in above equations we get

$$W'_+(r) = \frac{(E - \Omega_H J)(r_+^2 + a^2)}{(r - r_+)(2r_+ - 2M)(1 - \alpha^2 r_+^2)}, \tag{4.56}$$

$$W'_-(r) = \frac{-(E - \Omega_H J)(r_+^2 + a^2)}{(r - r_+)(2r_+ - 2M)(1 - \alpha^2 r_+^2)}. \tag{4.57}$$

Here the prime denotes the derivative with respect to r and $+/-$ corresponds to outgoing/incoming solution. For finding the value of $W(r)$ we integrate the above result

$$W_+(r) = \int \frac{(E - \Omega_H J)(r_+^2 + a^2)}{(r - r_+)(2r_+ - 2M)(1 - \alpha^2 r_+^2)}. \tag{4.58}$$

Here $r = r_+$ is the simple pole. Integrating around the pole we get

$$W_+(r) = \pi\iota \frac{(E - \Omega_H J)(r_+^2 + a^2)}{(2r_+ - 2M)(1 - \alpha^2 r_+^2)}.$$

Dropping the + subscript we obtain

$$W(r) = \pi\iota \frac{(E - \Omega_H J)(r_+^2 + a^2)}{(2r_+ - 2M)(1 - \alpha^2 r_+^2)},$$

$$\operatorname{Im} W = \frac{\pi}{2} \frac{(E - \Omega_H J)(r_+^2 + a^2)}{(r_+ - M)(1 - \alpha^2 r_+^2)}. \tag{4.59}$$

So the tunneling probabilities of fermion charge particles are

$$Prob[out] \propto \exp[-2\,\text{Im}\, I] = exp[-2(\text{Im}\, W_+ + \text{Im}\, \Theta)], \tag{4.60}$$

$$Prob[in] \propto exp[-2\,\text{Im}\, I] = exp[-2(\text{Im}\, W_- + \text{Im}\, \Theta)]. \tag{4.61}$$

Since $\text{Im}\, W_+ = -\,\text{Im}\, W_-$

$$\Gamma = exp[-4\,\text{Im}\, W_+]. \tag{4.62}$$

The resulting tunneling probability is

$$\Gamma = exp[-2\pi \frac{(E - \Omega_H J)(r_+^2 + a^2)}{(r_+ - M)(1 - \alpha^2 r_+^2)}]. \tag{4.63}$$

Comparing this with $\Gamma = exp(-\beta E)$ where $\beta = 1/T_H$ we get

$$T_H = \frac{(r_+ - M)(1 - \alpha^2 r_+^2)}{2\pi(r_+^2 + a^2)}, \tag{4.64}$$

which is the Hawking temperature for the accelerating and rotating black hole at the outer horizon.

4.1.3 The Massive Case

In the massive case we shall eliminate the function $W'(r,\theta)$ from Eqs. (4.54) and (4.55). Multiplying Eq. (4.54) by A and Eq. (4.55) by B and subtracting yields

$$A^2 m - B^2 m + 2\frac{AB(E - J\Omega_H)}{\sqrt{F_r(r_+,\theta)(r - r_+)}} = 0. \tag{4.65}$$

Multiplying the whole equation by $\sqrt{F_r(r_+,\theta)(r - r_+)}$ and dividing by B^2 we get

$$m \sqrt{F_r(r_+,\theta)(r - r_+)}(A/B)^2 + 2(E - J\Omega_H)(A/B) - m\sqrt{F_r(r_+,\theta)(r - r_+)} = 0 \tag{4.66}$$

$$A/B = \frac{-(E - J\Omega_H) \pm \sqrt{(E - J\Omega_H)^2 + m^2 F_r(r_+,\theta)(r - r_+)}}{m\sqrt{F_r(r_+,\theta)(r - r_+)}}, \tag{4.67}$$

where

$$\lim_{r\to r_+}(A/B) = \lim_{r\to r_+}(\frac{-(E-J\Omega_H) \pm \sqrt{(E-J\Omega_H)^2 + m^2 F_r(r_+,\theta)(r-r_+)}}{m\sqrt{F_r(r_+,\theta)(r-r_+)}}). \tag{4.68}$$

Now

$$\lim_{r\to r_+} A/B = \begin{cases} 0 \\ -\infty \end{cases},$$

for the upper/lower sign respectively.

Consequently at the horizon either $A/B \to 0$ or $A/B \to -\infty$, i.e. either $A \to 0$ or $B \to 0$. For $A \to 0$ we find the value of m from Eq. (4.55),

$$m = -A/B(\frac{-E + J\Omega_H}{\sqrt{F_r(r_+,\theta)(r-r_+)}} - \sqrt{g_r(r_+,\theta)(r-r_+)}W'(r)).$$

Puting in Eq. (4.54) and simplifying we get

$$W_r(r,\theta) = W'_+(r) = \frac{(E-J\Omega_H)}{\sqrt{F_r(r_+,\theta)g_r(r_+,\theta)}(r-r_+)}(1+A^2/B^2)/(1-A^2/B^2).$$

Integrating with respect to r we have

$$W_+(r) = \int \frac{(E-J\Omega_H)}{\sqrt{F_r(r_+,\theta)g_r(r_+,\theta)}(r-r_+)}(1+A^2/B^2)/(1-A^2/B^2)dr.$$

Here $r = r_+$ is the simple pole. Integrating around the pole we get

$$W_+(r) = \frac{\pi i(E-J\Omega_H)}{\sqrt{F_r(r_+,\theta)g_r(r_+,\theta)}}.$$

Using the values of functions $F_r(r_+,\theta)$ and $g_r(r_+,\theta)$ and simplifying we get

$$W_+(r) = \pi i\frac{(E-\Omega_H J)(r_+^2 + a^2)}{(2r_+ - 2M)(1-\alpha^2 r_+^2)}.$$

Dropping the + subscript we obtain

$$W(r) = \pi \iota \frac{(E - \Omega_H J)(r_+^2 + a^2)}{(2r_+ - 2M)(1 - \alpha^2 r_+^2)},$$
$$\operatorname{Im} W = \frac{\pi}{2} \frac{(E - \Omega_H J)(r_+^2 + a^2)}{(r_+ - M)(1 - \alpha^2 r_+^2)}. \tag{4.69}$$

For $B \to 0$ we can simply rewrite the expression in term of B/A to get

$$W_r(r,\theta) = W'_-(r) = \frac{-(E - J\Omega_H)}{\sqrt{F_r(r_+,\theta) g_r(r_+,\theta)}(r - r_+)}(1 + B^2/A^2)/(1 + B^2/A^2),$$

$$W_-(r) = \pi \iota \frac{-(E - \Omega_H J)(r_+^2 + a^2)}{(2r_+ - 2M)(1 - \alpha^2 r_+^2)}. \tag{4.70}$$

The final result is the same as in the massless case.

4.2 The Acceleration Horizon

For the acceleration horizon, the function $F_r(r,\theta)$ and $g_r(r,\theta)$ will be

$$F_r(r_+,\theta) = \frac{(r_+^2 + a^2 \cos^2\theta)(a^2 - 2Mr_+ + r_+^2)(-2r_+\alpha^2)}{((r_+^2 + a^2)^2)(1 - \alpha r_+ \cos\theta)^2} \tag{4.71}$$
$$g_r(r_+,\theta) = \frac{(1 - \alpha r_+ \cos\theta)^2(\alpha^2 a^2 - 2M\alpha + 1)(-2r_+\alpha^2)}{(r_+^2 + a^2 \cos^2\theta)}. \tag{4.72}$$

Eqs. (4.54) and (4.55) take the form

$$0 = -B\Big(\frac{-E + \Omega_H J}{\sqrt{(r - r_+)F_r(r_+,\theta)}} + \sqrt{(r - r_+)g_r(r_+,\theta)}W'(r,\theta)\Big)$$
$$+ Am, \tag{4.73}$$
$$0 = +A\Big(\frac{-E + \Omega_H J}{\sqrt{(r - r_+)F_r(r_+,\theta)}} - \sqrt{(r - r_+)g_r(r_+,\theta)}W'(r,\theta)\Big)$$
$$+ Bm. \tag{4.74}$$

4.2.1 The Massless Case

In the massless case we put $m = 0$ in Eqs. (4.73) - (4.74), then there exist two possible solutions

$$B = 0, \quad W'(r) = W'_+(r) = \frac{(E - \Omega_H J)}{\sqrt{(r - r_+)F_r(r_+, \theta)}\sqrt{(r - r_+)g_r(r_+, \theta)}}.$$

Similarly from Eq. (4.74) we get

$$A = 0, \quad W'(r) = W'_-(r) = \frac{-(E - \Omega_H J)}{\sqrt{(r - r_+)F_r(r_+, \theta)}\sqrt{(r - r_+)g_r(r_+, \theta)}}.$$

Putting the values of functions $F_r(r_+, \theta)$ and $g_r(r_+, \theta)$ in the above equations we get

$$W'_-(r) = \frac{-(E - \Omega_H J)(1 + \alpha^2 a^2)}{2\alpha(r - r_+)(\alpha^2 a^2 - 2M\alpha + 1)}, \tag{4.75}$$

$$W'_+(r) = \frac{(E - \Omega_H J)(1 + \alpha^2 a^2)}{2\alpha(r - r_+)(\alpha^2 a^2 - 2M\alpha + 1)}. \tag{4.76}$$

Here the prime denotes the derivative with respect to r and $+/-$ corresponds to outdoing/incoming solution. For finding the value of $W(r,)$ we integrate the above result

$$W_+(r) = \int \frac{(E - \Omega_H J)(1 + \alpha^2 a^2)}{2\alpha(r - r_+)(\alpha^2 a^2 - 2M\alpha + 1)}, \tag{4.77}$$

Here $r = r_+$ is the simple pole. Integrating around the pole we get

$$W_+(r) = \pi\iota \frac{(E - \Omega_H J)(1 + \alpha^2 a^2)}{2\alpha(\alpha^2 a^2 - 2M\alpha + 1)}. \tag{4.78}$$

Dropping the + subscript we obtain

$$\begin{aligned} W(r) &= \frac{\pi\iota}{2}\frac{(E - \Omega_H J)(1 + \alpha^2 a^2)}{\alpha(\alpha^2 a^2 - 2M\alpha + 1)}, \\ \operatorname{Im} W &= \frac{\pi}{2}\frac{(E - \Omega_H J)(1 + \alpha^2 a^2)}{\alpha(\alpha^2 a^2 - 2M\alpha + 1)}. \end{aligned} \tag{4.79}$$

Similarly we get

$$W_-(r) = -\frac{\pi}{2}\frac{(E - \Omega_H J)(1 + \alpha^2 a^2)}{\alpha(\alpha^2 a^2 - 2M\alpha + 1)}.$$

The resulting tunneling probability is

$$\Gamma = exp[-4\,\mathrm{Im}\,W_+]. \tag{4.80}$$

$$\Gamma = exp[-2\pi\frac{(E-\Omega_H J)(1+\alpha^2a^2)}{\alpha\,(\alpha^2a^2-2M\alpha+1)}]. \tag{4.81}$$

Comparing this with $\Gamma = exp(-\beta E)$ where $\beta = 1/T_H$ we get

$$T_H = \frac{\alpha\,(\alpha^2a^2-2M\alpha+1)}{2\pi(1+\alpha^2a^2)}.$$

4.2.2 The Massive Case

In the massive case we shall eliminate the function $W'(r,\theta)$ from Eqs. (4.73) and (4.74). Multiplying Eq. (4.73) by A and Eq. (4.74) by B and subtracting yields we get

$$A^2m - B^2m - 2\frac{AB(-E+J\Omega_H)}{\sqrt{F_r(r_+,\theta)(r-r_+)}} = 0.$$

Multiplying the whole equation by $\sqrt{F_r(r_+,\theta)(r-r_+)}$ and dividing by B^2

$$M \quad \sqrt{F_r(r_+,\theta)(r-r_+)}(\frac{A}{B})^2 + 2(E-J\Omega_H)(\frac{A}{B}) - m\sqrt{F_r(r_+,\theta)(r-r_+)} = 0$$

$$\frac{A}{B} = \frac{-(E-J\Omega_H)\pm\sqrt{(E-J\Omega_H)^2+m^2F_r(r_+,\theta)(r-r_+)}}{m\sqrt{F_r(r_+,\theta)(r-r_+)}}, \tag{4.82}$$

where

$$\lim_{r\to r_+}(\frac{A}{B}) = \lim_{r\to r_+}(\frac{-(E-J\Omega_H)\pm\sqrt{(E-J\Omega_H)^2+m^2F_r(r_+,\theta)(r-r_+)}}{m\sqrt{F_r(r_+,\theta)(r-r_+)}}),$$

$$\lim_{r\to r_+}(\frac{A}{B}) = \begin{cases} 0 \\ -\infty \end{cases},$$

for the upper/lower sign respectively.

Consequently at the horizon either $A/B \to 0$ or $A/B \to -\infty$, i.e. either $A \to 0$ or $B \to 0$. For $A \to 0$ we find the value of m from Eq. (4.74),

$$m = \frac{-A}{B}(\frac{-E+J\Omega_H}{\sqrt{F_r(r_+,\theta)(r-r_+)}} - \sqrt{g_r(r_+,\theta)(r-r_+)}W'(r)).$$

Puting in Eq. (4.73) and simplifying we get

$$W_r(r,\theta) = W'_+(r) = \frac{(E - J\Omega_H)}{\sqrt{F_r(r_+,\theta)g_r(r_+,\theta)}(r-r_+)}(1 + A^2/B^2)/(1 - A^2/B^2). \tag{4.83}$$

Here $r = r_+$ is the simple pole. Integrating around the pole we obtain

$$W_+(r) = \frac{\pi i(E - J\Omega_H)}{\sqrt{F_r(r_+,\theta)g_r(r_+,\theta)}}.$$

Using the values of $F_r(r_+,\theta)$ and $g_r(r_+,\theta)$ and simplifying we get

$$W_+(r) = \frac{\pi\iota}{2}\frac{(E - \Omega_H J)(1 + \alpha^2 a^2)}{\alpha\,(\alpha^2 a^2 - 2M\alpha + 1)}.$$

Dropping the + subscript we obtain

$$\begin{aligned} W(r) &= \frac{\pi\iota}{2}\frac{(E - \Omega_H J)(1 + \alpha^2 a^2)}{\alpha(r - r_+)\,(\alpha^2 a^2 - 2M\alpha + 1)}, \\ \operatorname{Im} W &= \frac{\pi}{2}\frac{(E - \Omega_H J)(1 + \alpha^2 a^2)}{\alpha\,(\alpha^2 a^2 - 2M\alpha + 1)}. \end{aligned} \tag{4.84}$$

For $B \to 0$ we can simply rewrite the expression in term of B/A to get

$$W_r(r,\theta) = W'_-(r) = \frac{-(E - J\Omega_H)}{\sqrt{F_r(r_+,\theta)g_r(r_+,\theta)}(r-r_+)}(1 + B^2/A^2)/(1 + B^2/A^2),$$

$$W_-(r) = -\frac{\pi\iota}{2}\frac{(E - \Omega_H J)(1 + \alpha^2 a^2)}{\alpha\,(\alpha^2 a^2 - 2M\alpha + 1)}.$$

The final result is the same as in the massless case. So the resulting tunneling probability is

$$\Gamma = exp[-4\operatorname{Im} W_+],$$

$$\Gamma = exp[-2\pi\frac{(E - \Omega_H J)(1 + \alpha^2 a^2)}{\alpha\,(\alpha^2 a^2 - 2M\alpha + 1)}], \tag{4.85}$$

Comparing this with $\Gamma = exp(-\beta E)$ where $\beta = 1/T_H$ we get

$$T_H = \frac{\alpha\left(\alpha^2 a^2 - 2M\alpha + 1\right)}{2\pi(1 + \alpha^2 a^2)}.$$

4.3 Calculation of the Action

We use the seperation of variables

$$W(r,\theta) = R(r) + \Theta(\theta),$$

in Eqs. (4.54) and (4.55) to get

$$\begin{aligned} 0 &= -B\left(\frac{-E+\Omega_H J}{\sqrt{(r-r_+)F_r(r_+,\theta)}} + \sqrt{(r-r_+)g_r(r_+,\theta)}R'(r)\right) \\ &+ Am, \qquad (4.86) \\ 0 &= +A\left(\frac{-E+\Omega_H J}{\sqrt{(r-r_+)F_r(r_+,\theta)}} - \sqrt{(r-r_+)g_r(r_+,\theta)}R'(r)\right) \\ &+ Bm. \qquad (4.87) \end{aligned}$$

If $m = 0$, from Eq. (4.86) we get two possible solutions

$$B = 0, \quad R'(r) = R'_+(r) = \frac{(E - \Omega_H J)}{\sqrt{(r-r_+)F_r(r_+,\theta)}\sqrt{(r-r_+)g_r(r_+,\theta)}}.$$

Similarly from Eq. (4.87) we get

$$A = 0, \quad R'(r) = R'_-(r) = \frac{-(E - \Omega_H J)}{\sqrt{(r-r_+)F_r(r_+,\theta)}\sqrt{(r-r_+)g_r(r_+,\theta)}}.$$

Putting the values of functions $F_r(r_+,\theta)$ and $g_r(r_+,\theta)$ in the above equation we get

$$R'_-(r) = \frac{-(E-\Omega_H J)(r_+^2 + a^2)}{(r-r_+)(2r_+ - 2M)(1-\alpha^2 r_+^2)}, \qquad (4.88)$$

$$R'_+(r) = \frac{(E-\Omega_H J)(r_+^2 + a^2)}{(r-r_+)(2r_+ - 2M)(1-\alpha^2 r_+^2)}, \qquad (4.89)$$

where the prime denotes the derivative with respect to r and $+/-$ corresponds to outgoing/incoming solution. For finding the value of $R(r)$ we integrating the Eqs. (4.88) and (4.89) we get

$$R_+(r) = \frac{(E-\Omega_H J)(r_+^2 + a^2)}{(2r_+ - 2M)(1-\alpha^2 r_+^2)} ln(r - r_+). \qquad (4.90)$$

$$R_-(r) = \frac{-(E-\Omega_H J)(r_+^2+a^2)}{(2r_+-2M)(1-\alpha^2 r_+^2)} ln(r-r_+). \quad (4.91)$$

Now we come to the massive case.

$$R'(r) = \frac{mA}{B\sqrt{\partial_r g\,(r_+,\theta)\,(r-r_+)}} - \frac{(-E+\Omega_H J)}{\sqrt{\partial_r F\,(r_+,\theta)\,\partial_r g\,(r_+,\theta)}\,(r-r_+)},$$

where A and B are functions of (r,θ,ϕ). Integrating with respect to r we get

$$R\,(r) = R_+\,(r) = \int \frac{mA}{B\sqrt{\partial_r g\,(r_+,\theta)\,(r-r_+)}}dr - \frac{(-E+\Omega_H J)}{\sqrt{\partial_r F\,(r_+,\theta)\,\partial_r g\,(r_+,\theta)}}\ln\,(r-r_+)\,. \quad (4.92)$$

which corresponds to the outgoing particles. Similarly from Eq. (4.87) we get

$$R\,(r) = R_-\,(r) = \int \frac{mB}{A\sqrt{\partial_r g\,(r_+,\theta)\,(r-r_+)}}dr + \frac{(-E+\Omega_H J)}{\sqrt{\partial_r F\,(r_+,\theta)\,\partial_r g\,(r_+,\theta)}}\ln\,(r-r_+)\,. \quad (4.93)$$

The rest of Eqs. (4.51) and (4.53) become

$$0 = -B(\sqrt{\frac{\Omega^2 P}{\rho^2}}\Theta_\theta(r,\theta) + \frac{\iota\rho\Omega}{\sin\theta(\sqrt{P(r_+^2+a^2)^2 - Qa^2\sin^2\theta}}J), \quad (4.94)$$

$$0 = -A(\sqrt{\frac{\Omega^2 P}{\rho^2}}\Theta_\theta(r,\theta) + \frac{\iota\rho\Omega}{\sin\theta(\sqrt{P(r_+^2+a^2)^2 - Qa^2\sin^2\theta}}J). \quad (4.95)$$

We can see that Eqs. (4.94) and (4.95) are similar. So we get the same equation from both the equations regardless of the values of A or B.

Thus $B=0$ and

$$(\sqrt{\frac{\Omega^2 P}{\rho^2}}\Theta_\theta(r,\theta) + \frac{\iota\rho\Omega}{\sin\theta(\sqrt{P(r_+^2+a^2)^2 - Qa^2\sin^2\theta}}J) = 0$$

$$\dot{\Theta}(\theta) = -\frac{\iota\rho^2 J}{\sin\theta\sqrt{P}\sqrt{P(r_+^2+a^2)^2 - Qa^2\sin^2\theta}}.$$

At horizon $Q(r_+) = 0$

$$\dot{\Theta}(\theta) = -\frac{\iota\rho^2(r_+,\theta)J}{(r_+^2 + a^2)P\sin\theta}. \tag{4.96}$$

Putting the values of $\rho^2(r_+,\theta)$ and P

$$\dot{\Theta}(\theta) = -\frac{\iota J(r_+^2 + a^2\cos^2\theta)}{(r_+^2 + a^2)\sin\theta[1 - 2\alpha M\cos\theta + \alpha^2a^2)\cos^2\theta]}.$$

Integrating the above equation we get

$$\Theta(\theta) = -\frac{\iota J}{(r_+^2 + a^2)}\int\frac{(r_+^2 + a^2\cos^2\theta)d\theta}{\sin\theta[1 - 2\alpha M\cos\theta + \alpha^2a^2\cos^2\theta]},$$

$$I = \int\frac{(r_+^2 + a^2\cos^2\theta)d\theta}{\sin\theta[1 - 2\alpha M\cos\theta + \alpha^2a^2\cos^2\theta]}.$$

We put $z = \cos\theta$ and use the partial fraction method to solve the integral. Finally we get

$$I_2 = -\int\left[\frac{L_1}{1-z}dz + \frac{L_2}{1+z} + \frac{L_3z + L_4}{1 - 2\alpha Mz + \alpha^2a^2z^2}\right], \tag{4.97}$$

where values of L_1, L_2, L_3, L_4 are given below

$$L_1 = \frac{-iJ}{2[1 - 2\alpha M + \alpha^2a^2]}, \tag{4.98}$$

$$L_2 = -\frac{+iJ}{2[1 + 2\alpha M + \alpha^2a^2]}, \tag{4.99}$$

$$L_3 = \frac{-2\alpha^3a^2MiJ}{[1 - 2\alpha M + \alpha^2a^2][1 + 2\alpha M + \alpha^2a^2]}, \tag{4.100}$$

$$L_4 = -\frac{2iJ\alpha^2a^2[1 + \alpha^2a^2] - 8iJ\alpha^2M^2}{2[1 - 2\alpha M + \alpha^2a^2][1 + 2\alpha M + \alpha^2a^2]}, \tag{4.101}$$

After simplification we obtain

$$\begin{aligned}\Theta &= -\frac{+iJa^2}{2\left(r_+^2+a^2\right)\alpha\sqrt{M^2-a^2}}\ln\left(\frac{a^2\alpha\cos\theta-M-\sqrt{M^2-a^2}}{a^2\alpha\cos\theta-M+\sqrt{M^2-a^2}}\right)\\&+\frac{-iJ}{2\left[1-2\alpha M+\alpha^2a^2\right]}\ln\left(1-\cos\theta\right)+\frac{+iJ}{2\left[1+2\alpha M+\alpha^2a^2\right]}\ln\left(1+\cos\theta\right)\\&+\frac{\alpha MiJ}{\left[1-2\alpha M+\alpha^2a^2\right]\left[1+2\alpha M+\alpha^2a^2\right]}\times\ln\left[1-2\alpha M\cos\theta+\alpha^2a^2\cos^2\theta\right]\\&-\frac{a^2\left[-2iJ\alpha^2-2iJ\alpha^4a^2\right]+4iJ\alpha^2M^2}{4\alpha\sqrt{M^2-a^2}\left[\left(1+\alpha^2a^2\right)^2-4\alpha^2M^2\right]}\\&\times\ln\left[\frac{\alpha a^2\cos\theta-M-\sqrt{M^2-a^2}}{\alpha a^2\cos\theta-M+\sqrt{M^2-a^2}}\right].\end{aligned}\tag{4.102}$$

4.4 Quantum Tunneling of Scalar Particles

4.4.1 Tunneling Probability at the Rotation Horizon

In this section we shall find the quantum tunneling of scalar particles from the accelerating and rotating black hole with metric given by Eq. (4.1)

$$ds^2=-f\left(r,\theta\right)dt^2+\frac{dr^2}{g\left(r,\theta\right)}+\Sigma\left(r,\theta\right)d\theta^2+K\left(r,\theta\right)d\phi^2-2H\left(r,\theta\right)dtd\phi,\tag{4.103}$$

where values of $f\left(r,\theta\right)$, $g\left(r,\theta\right)$, $\Sigma\left(r,\theta\right)$, $K\left(r,\theta\right)$ and $H\left(r,\theta\right)$ are given in Eqs. (4.9) to (4.13). The Klein-Gordon equation will be solved for this purpose which is given as

$$g^{\mu\nu}\partial_\mu\partial_\nu\phi-\frac{m^2}{\hbar^2}\phi=0.\tag{4.104}$$

The wave function $\phi(t,r,\theta,\phi)$ is defined as

$$\phi(t,r,\theta,\phi)=\exp(\frac{\iota}{\hbar}I(t,r,x^a)+I_1(t,r,x^a)+O(\hbar).\tag{4.105}$$

Using Eq. (4.105) in Eq. (4.104) we get

$$g^{\mu\nu}\left(\partial_\mu I\right)\left(\partial_\nu I\right)+m^2=0,\tag{4.106}$$

where m is the mass of scalar particles, $g^{\mu\nu}$ is the inverse of metric, and I is the action. Expanding Eq. (4.106) and simplifying we get

$$-\frac{(\partial_t I)^2}{F(r,\theta)}+g(r,\theta)(\partial_r I)^2-\frac{2H(r,\theta)}{F(r,\theta)K(r,\theta)}(\partial_t I)(\partial_\phi I)+\frac{f(r,\theta)}{F(r,\theta)K(r,\theta)}(\partial_\phi I)^2+\frac{(\partial_\theta I)^2}{\rho^2(r,\theta)}+m^2=0, \tag{4.107}$$

where value of $F(r,\theta)$ is given by Eq. (4.20). We shall chose the following ansatz for the calculation of tunneling probability

$$I=-Et+W(r)+J\phi. \tag{4.108}$$

Using Eq. (4.108) in Eq. (4.107) we get

$$-\frac{E^2}{F(r,\theta)}+g(r,\theta)W'^2(r)+\frac{2H(r,\theta)}{F(r,\theta)K(r,\theta)}(E)(J)+\frac{f(r,\theta)}{F(r,\theta)K(r,\theta)}J^2+m^2=0. \tag{4.109}$$

After some calculation this takes the form

$$-\frac{1}{F(r,\theta)}\left[E-\frac{H(r,\theta)}{K(r,\theta)}J\right]^2+\left[\frac{H^2(r,\theta)}{F(r,\theta)K(r,\theta)}+\frac{f(r,\theta)}{F(r,\theta)}\right]\frac{J^2}{K(r,\theta)}+g(r,\theta)W'^2(r)+m^2=0. \tag{4.110}$$

Here we have added and subtracted $\frac{H^2(r,\theta)}{F(r,\theta)K^2(r,\theta)}(J-qA_\phi)^2$ to make first term a completing square. Simplifying, so Eq. (4.110) becomes

$$-\frac{1}{F(r,\theta)}\left[E-\frac{H(r,\theta)}{K(r,\theta)}J\right]^2+\frac{J^2}{K(r,\theta)}+g(r,\theta)W'^2(r)+m^2=0. \tag{4.111}$$

Near the horizon $r=r_+$ expanding Eq. (4.111) similar as in the case of Dirac particles we get

$$\begin{aligned}0 &= -\frac{(1-\alpha r_+\cos\theta)^2\left(r_+^2+a^2\right)^2}{2\left(r_+^2+a^2\cos^2\theta\right)(r_+-M)\left(1-\alpha^2r_+^2\right)(r-r_+)}(E-\Omega_H J)^2+\frac{J^2}{K(r_+,\theta)}\\ &\quad+\frac{2(1-\alpha r_+\cos\theta)^2(r_+-M)\left(1-\alpha^2r_+^2\right)(r-r_+)}{\left(r_+^2+a^2\cos^2\theta\right)}W'^2(r)+m^2.\end{aligned} \tag{4.112}$$

Solving this equation for $W(r)$ we get

$$W^2(r) = \frac{\left(r_+^2+a^2\right)^2}{4\left(r_+-M\right)^2\left(1-\alpha^2 r_+^2\right)^2\left(r-r_+\right)^2}\left(E-\Omega_H J\right)^2 - \left(\frac{J^2}{K\left(r_+,\theta\right)}+m^2\right)\times\frac{\left(r_+^2+a^2\cos^2\theta\right)}{2\left(1-\alpha r_+\cos\theta\right)^2\left(r_+-M\right)\left(1-\alpha^2 r_+^2\right)\left(r-r_+\right)}, \tag{4.113}$$

or

$$W_\pm(r) = \pm\int\frac{\left(r_+^2+a^2\right)}{2\left(r_+-M\right)\left(1-\alpha^2 r_+^2\right)\left(r-r_+\right)}dr \times\sqrt{\left(E-\Omega_H J\right)^2-\frac{2\left(r_+^2+a^2\cos^2\theta\right)\left(1-\alpha^2 r_+^2\right)\left(r_+-M\right)\left(r-r_+\right)}{\left(1-\alpha r_+\cos\theta\right)^2}\left(\frac{J^2}{K\left(r_+,\theta\right)}+m^2\right)}. \tag{4.114}$$

Here $r=r_+$ is the singularity so using residue theory integrating Eq. (4.114) we get

$$W_\pm(r) = \pm\frac{\pi i\left(E-\Omega_H J\right)\left(r_+^2+a^2\right)}{2\left(r_+-M\right)\left(1-\alpha^2 r_+^2\right)}, \tag{4.115}$$

or

$$\operatorname{Im} W_\pm(r) = \pm\frac{\pi\left(E-\Omega_H J\right)\left(r_+^2+a^2\right)}{2\left(r_+-M\right)\left(1-\alpha^2 r_+^2\right)}. \tag{4.116}$$

So tunneling of outgoing scalar particles is

$$\Gamma = \exp\left[-4\operatorname{Im} W_+\right].$$

Using value of $\operatorname{Im} W_+$ in the above equation we get

$$\Gamma = \exp\left[-2\pi\frac{\left(E-\Omega_H J\right)\left(r_+^2+a^2\right)}{\left(r_+-M\right)\left(1-\alpha^2 r_+^2\right)}\right]. \tag{4.117}$$

Note that the tunneling probability of scalar particles Eq. (4.117) is same as in the case of Dirac particles. Thus we recover the Hawking temperature at the outer horizon.

4.4.2 Tunneling Probability at Acceleration Horizon

In this section we shall find the tunneling probability at the acceleration horizon $r_+ = \frac{1}{\alpha}$. The calculations for the probability at the acceleration horizon proceeds in the same way as in the case of outer horizon $r = r_+$. At the acceleration horizon $r_+ = \frac{1}{\alpha}$,and Eq. (4.112) takes the form

$$\begin{aligned} 0 &= \frac{(1-\cos\theta)^2 \left(r_+^2 + a^2\right)^2}{2\alpha \left(r_+^2 + a^2\cos^2\theta\right)\left[r_+^2 - 2Mr_+^2 + a^2\right](r - r_+)} (E - \Omega_H J)^2 \\ &\quad - \frac{2\alpha (1-\cos\theta)^2 \left[r_+^2 - 2Mr_+ + a^2\right](r - r_+)}{\left(r_+^2 + a^2\cos^2\theta\right)} W^2(r) + \frac{J^2}{K(r_+,\theta)} + m^2. \quad (4.118) \end{aligned}$$

Solving this equation for $W(r)$ we get

$$\begin{aligned} W^2(r) &= \frac{\left(r_+^2 + a^2\right)^2}{4\alpha^2 \left[r_+^2 - 2Mr_+ + a^2\right]^2 (r - r_+)^2} (E - \Omega_H J)^2 \\ &\quad + \frac{\left(r_+^2 + a^2\cos^2\theta\right)}{2\alpha (1-\cos\theta)^2 \left[r_+^2 - 2Mr_+ + a^2\right](r - r_+)} \left(\frac{J^2}{K(r_+,\theta)} + m^2\right), \end{aligned}$$

or

$$\begin{aligned} W_\pm(r) &= \pm \int \frac{\left(r_+^2 + a^2\right)}{2\alpha \left[r_+^2 - 2Mr_+ + a^2\right](r - r_+)} dr \\ &\quad \times \sqrt{(E - \Omega_H J)^2 + \frac{2\alpha \left(r_+^2 + a^2\cos^2\theta\right)\left[r_+^2 - 2Mr_+ + a^2\right](r - r_+)}{\left(r_+^2 + a^2\right)^2 (1-\cos\theta)^2} \left(\frac{J^2}{K(r_+,\theta)} + m^2\right)}. \end{aligned}$$

Here $r = r_+$ is the singularity. So integrating the above equation using residue the theory we get

$$W_\pm(r) = \pm \frac{\pi i \left(r_+^2 + a^2\right)}{2\alpha \left[r_+^2 - 2Mr_+ + a^2\right]} (E - \Omega_H J).$$

or

$$\operatorname{Im} W_\pm(r) = \pm \frac{\pi \left(r_+^2 + a^2\right)}{2\alpha \left[r_+^2 - 2Mr_+ + a^2\right]} (E - \Omega_H J).$$

Now the tunneling probability of outgoing scalar particles is found by the following farmula

$$\Gamma = \exp\left[-4 \operatorname{Im} W_+\right].$$

Using value of $\operatorname{Im} W_+$ in the above equation we get the resulting tunneling probability of scalar particles as

$$\Gamma = \exp\left[-\frac{2\pi\left(r_+^2 + a^2\right)}{\alpha\left[r_+^2 - 2Mr_+ + a^2\right]}\left(E - \Omega_H J\right)\right].$$

Note that the tunneling probility of scalar particles is the same as that of the Dirac particles at the acceleration horizon. Thus we recover the Hawking temperature at the acceleration horizon.

4.5 Conclusion

In this dissertation we have studied the tunneling probability and Hawking temperature of fermion and scalar particles from different black holes. In Chapter 1, we have briefly discussed some basics of the black holes physics. In Chapters 2 we have worked out to explain the EFEs and its some well known solutions including the Schwarzschild black hole, the Riessner-Nordstrom black hole, the kerr and the Kerr-Newman black holes. Also we have explained the correspondence of Plebanski-Demianski metric to accelerating and rotating black holes in this chapter.

Hawking radiations which are the quantum mechianlly aspects of black holes have been discussed in Chapter 3. According to Stephen Hawking when a pair of particle is created just outside or inside the horizon the negative energy particle is absorbed by the black hole and the positive energy particle escaped to infinity appearing as Hawking radiation. Parikh and Wilczek observed that outgoing particles create barrier. Later on by using Hamilton-Jacobi method Kerner and Mann discussed the tunneling probability of fermion particles.

Here in Chapter 3, we have reviewed two papers. In the first paper we have calculated the tunneling probability of fermions from the Riessner-Nordström black hole using Hamilton-Jacobi method. The solution of Dirac equation leads to four complicated equations. First we have made these equation simple with the WKP approximation method and then more simpler with ansatz solution which have been given by Kerner and Mann. Then we have calculated the tunneling probility of fermions for both massive and massless cases and have concluded that the tunnelling probility of fermions is same for both the cases. Also Hawking temperature has been worked out in this chapter. Similarly in the second paper we have calculated the tunneling probability and Hawking temperature from the Kerr black hole.

In the last chapter we have extended this approach to find the tunneling probability of the fermion particles from the rotating and accelerating black holes. We have followed the method of Kerner and Mann and observed an extra term in the tunnelling probability and Hawking temperature in Eqs. (4.68) and (4.70) due to acceleration of black holes. When this extra term appraoches to zero we get the tunnelling probability and Hawking temperature observed by Kerner and Mann. Further we have worked out to investigate the tunnelling probability and Hawking temperature of the scalar particles for the same black holes. For the scalar particle we have used the Klein-Gordon equation and noticed that tunneling probability of fermions and the scalar particles are same in both the massless and the massive cases which indicates that both the particles emit from the black hole at the same rate.

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