Non-orthogonal stagnation point flow of a non-Newtonian fluid

By

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTER OF PHILOSOPHY

We accept this dissertation as conforming to the required standard.

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2011

Dedicated

To

My

Ammi and Abbu

Whose affection is the reason of every success in my life.

Who've always given me perpetual love, Care, and cheers. Whose prayers have always been a source of great Inspiration for me and whose sustained hope in me led me to where I stand today.

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RASHID MEHMOOD

Preface

In recent years, considerable attention has been given to the non-Newtonian fluids regarding their importance in industrial applications. Moreover great amount of interest has been shown in the study of stagnation-point flow because of their worth in numerous engineering problems. For example, the extrusion of plastic sheets, fabrication of adhesive tapes and application of coating layers onto rigid substrates. Polymer sheets are manufactured by continuous extrusion of the polymer from a die to a windup roller. H.S. Takhar [1] studied that thin polymer sheet constitutes a continuously moving surface with a non-uniform velocity through the ambient fluid. Crane [2] investigated the steady two-dimensional flow of an incompressible fluid over a stretching sheet which moves in its own plane with a velocity varying linearly with the distance from a fixed point. Chiam [4] studied the steady two-dimensional and the axisymmetric stagnation-point flow of a viscous Newtonian incompressible fluid towards a stretching surface. Mahapatra and Gupta [5] studied the heat transfer in a stagnation-point flow towards a stretching sheet. Lok et al. [6] investigated the non-orthogonal stagnation-point flow towards a stretching sheet. Reza and Gupta [7] studied the steady two-dimensional oblique stagnation-point flow of a Newtonian fluid towards a stretching surface. Rajagopal et al. [8] studied steady flow of a second-order fluid past a stretching sheet. The temperature distribution of a steady flow of a second-order fluid was investigated by Bhattacharyya et al. [9]. Issues concerning the status of second grade fluids can be found in the paper by Dunn and Rajagopal [13]. Nazar and Amin [20] discussed the stagnation point flow of a micro polar fluid towards a stretching sheet. Rees and Bassom [24] examined the Blasius boundary-layer flow of a micro polar fluid. Free convection boundary-layer flow of a micro polar fluid from a vertical flat plate was studied by Rees and I. Pop [25].

In the present thesis study, we divide our work in three chapters. Chapter 1 contains some basic definitions and relevant Governing equations which are quite substantial for the subsequent chapters.

In chapter 2 we consider the steady two-dimensional non-orthogonal stagnation-point flow of a viscoelastic second-grade fluid towards a stretching surface with heat transfer. An analytical technique known as Homotopy analysis method is used to find the solutions of the governing non-linear ordinary differential equations.

In chapter 3 we consider the steady two-dimensional non-orthogonal stagnation-point flow of a micropolar fluid towards a stretching surface with heat transfer. An analytical technique known as Homotopy analysis method is used to find the solutions of the governing non-linear ordinary differential equations.

Contents

Chapter 1

Relevant definitions and equations

The main purpose of this chapter is to provide some relevant definitions and equations for the subsequent chapters. One can find these definitions in the books of "F.M White" and "Fox and Mc donald".

1.1 Basic definitions

"*Fluid*" is defined as a substance that deforms continuously under the action of applied shear stresses of any magnitude. The basic difference between solids and fluids is that in case of solids, the deformation generated by applied shear stresses is not continuous. "*Fluid mechanics*" is the branch of engineering which is associated with the study of fluids at rest or in motion. The branch of engineering dealing with the fluids in motion is known as "*fluid dynamics*". The branch of engineering that deals with the study of fluids at rest is known as "*fluid statics*". "*Density*" of any substance (fluid) is defined as the mass of unit volume of the substance (fluid) at a given temperature and pressure. However (in case of fluids) if the density of the fluid varies throughout the system, then the density at a point is defined as the limiting value in the following way

$$\rho = \lim_{\delta v \to 0} \left(\frac{\delta m}{\delta v} \right). \tag{1.1}$$

In above equation δm denotes the mass element, δv is the volume element enclosing the point under consideration and ρ indicates the fluid density.

" *Viscosity*" is defined as the ability of a fluid to resist the flow, or it is the internal resistance of a fluid. In a more scientific and more compact way, the ratio of shear stress to the rate of shear strain is known as viscosity. The mathematical relationship for viscosity is

$$\text{Viscosity } (\mu) = \frac{\text{shear stress}}{\text{rate of shear strain}}. \tag{1.2}$$

Depending upon certain conditions in the several cases, viscosity is also termed as absolute, kinematic or dynamic viscosity. It is an important property of a fluid which plays obvious role in experimental and mathematical analysis regarding flow. Classification of fluids is also made on the basis of viscosity. *Kinematic viscosity* is defined as the ratio of dynamic viscosity to the density of the fluid. In mathematical form one can write

$$\text{Kinematic viscosity } (v) = \frac{\mu}{\rho}. \tag{1.3}$$

"*Pressure*" is known as the magnitude of the applied force to the object (in the perpendicular direction to the surface) per unit area. Mathematically one can write

$$\text{pressure} = \frac{\text{Magnitude of applied force}}{\text{area}} = \frac{\mathbf{F}}{\mathbf{A}} \tag{1.4}$$

We know that the fluid goes under deformation when different forces act upon it. If the deformation increases continuously or indefinitely then this is known as "*flow*". The flow in which the physical properties of the fluid (i.e. velocity, pressure, density etc.) at each point of the flow field remain invariant with respect to time is named as "*steady flow*". For any fluid property ζ we then write $\frac{\partial \zeta}{\partial t} = 0$. The flow in which the fluid property changes with time is called the "*unsteady flow*". In mathematical notation we have $\frac{\partial \zeta}{\partial t} \neq 0$. Flow of constant density fluid is known as "*incompressible flow*". In general all liquids are considered to have an incompressible flow. The flow for which density varies is known as "*compressible flow*". Flow of all the gases have been treated as the compressible flows. A flow is classified as one-, two-, or three dimensional depending upon the number of space coordinates appearing in the velocity field.The imaginary line in the fluid drawn in such a way that the tangent to it at any point gives the direction of flow at that point, is called "*stream line*". Thus the stream line shows the direction of motion of a number of particles at the same time. A function, which describes the form of pattern of flow, or in other words the discharge per unit thickness is called the "*stream function*". It describes flow fields in term of either mass flow rate, for compressible fluids, or volume flow rate, for incompressible fluids. Mathematically for a steady state two dimensional flow field, we may write

$$\mathbf{V} = \nabla \times \psi, \tag{1.5}$$

where $V = (u, v, 0)$, therefore $\psi = (0, 0, \psi)$. In Cartesian coordinate system, the velocity components in terms of stream function may be defined as

$$u = \frac{\partial \psi}{\partial y},\ v = -\frac{\partial \psi}{\partial x}. \tag{1.6}$$

The stream function can be used to plot the stream lines (ψ = constant) to analyze the flow behavior graphically. Fluids of negligible viscosity are known as "*Ideal fluids*". These fluids do not offer any resistance to the shear forces and thus do not practically exist in nature. However, from engineering point of view, gasses are considered as the ideal fluids. On the other hand, fluids of finite viscosity are known as "*real fluids*". These fluids offer considerable resistance against the shear forces. Such fluids are further classified in to two sub classes namely the Newtonian and non-Newtonian fluids. The fluid for which shear stress is directly proportional to the linear rate of strain is termed as "*Newtonian fluid*". For such fluids, the graph between shear stress and deformation rate is a straight line. Mathematical expression satisfied by such fluid is given below

$$\tau_{yx} \propto \frac{du}{dy}, \tag{1.7}$$

or

$$\tau_{yx} = \mu \frac{du}{dy}, \tag{1.8}$$

where τ_{yx} is the shear stress, μ is the dynamic viscosity (a constant of proportionality) and du/dy is the rate of strain (velocity gradient perpendicular to the direction of shear) for a unidirectional and one-dimensional flow.

For a "*Newtonian fluid*", the viscosity, by definition, depends only on temperature and pressure, not on the forces acting upon it. In common terms, this means that the fluid continues to flow, regardless of the forces acting on it. If the fluid is incompressible and viscosity is constant across the fluid, the above equation governing the shear stress can be generalized in the Cartesian coordinate system as follows

$$\tau_{ij}= \mu\left(\frac{\partial u_i}{\partial y_j}+\frac{\partial u_j}{\partial y_i}\right). \tag{1.9}$$

In above expression

τ_{ij} is the shear stress on the i^{th} face of a fluid element in the j^{th} direction

u_i is the velocity in the i^{th} direction

y_j is the j^{th} direction coordinate

The most common examples of such fluids are water and gasoline. "*Non-Newtonian fluids*" are fluids in which shear stress is not directly proportional to deformation rate. For one-dimensional flows

$$\tau_{yx} = k(\frac{du}{dy})^n, \tag{1.10}$$

where the exponent n, is called the flow behavior index and the coefficient k, the consistency index. This equation reduces to Newton,s law of viscosity for $n = 1$ with $k = \mu$.To ensure that τ_{yx} has the same sign as du/dy, above equation is rewritten in the form

$$\tau_{yx} = k(\frac{du}{dy})^{n-1}\frac{du}{dy}= \eta\frac{du}{dy}. \tag{1.11}$$

The term $\eta = k(du/dy)^{n-1}$ is referred to as the apparent viscosity.

Fluids in which the apparent viscosity decreases with increasing deformation rate, this type of fluids are known as "*pseudoplastic*" or shear thinning fluids. Examples include polymer solution, colloidal suspension etc. If the apparent viscosity increases by increasing deformation rate, this type of fluids are named as "*dilatant fluids*". Suspensions of stratch and of sand are examples of dilatant fluids. A fluid that behaves as a solid until a minimum yield stress τ_{xy}, is exceeded and subsequently exhibits a linear relation between stress and rate of deformation are called "*Bingham plastic*". Mathematically

$$\tau_{yx} = \tau_y + \mu_p\left(\frac{du}{dy}\right). \tag{1.12}$$

Clay suspensions, drilling muds and toothpaste are examples of substances exhibiting this behavior. Fluids for which apparent viscosity η decrease with time under a constant applied shear stress are termed as "*Thixotropic fluids*". Paints are examples of thixotropic fluids. Fluids that show an increase in η with time are termed as "*rheopectic fluids*". Some fluids after deformation partially return to their original shape when the applied stress is released are known as "*viscoelastic fluids*". "*Stagnation point*" is a point in a field of flow about a body where the fluid particles have zero velocity with respect to the body. Such forces which act on the surface of any medium through direct contact with the surface are called"*Surface forces*". Examples of such forces include pressure and stress. Such forces which act throughout the volume of the fluid and are independent of any type of physical contact are called "*body forces*". Gravity and magnetic forces are examples of two body forces.

"*Volume flow rate*" is the volume of fluid which passes through a section of pipe or channel in unit time. It is usually represented by the symbol Q. Given an area A, and a fluid flowing through it with uniform velocity V with an angle θ away from the perpendicular to A, then the volume flow rate is

$$Q = A\mathbf{V}\cos\theta. \tag{1.13}$$

For flow perpendicular to the area A we have $\theta = 0$ and thus the volume flow rate is

$$Q = A\mathbf{V}. \tag{1.14}$$

When a fluid flows, the outer most molecules of the fluid near the solid boundary stick with the boundary and the fluid velocity at the boundary is equal to that of the solid boundary. This is known as the "*no-slip condition*". Although no-slip condition is extensively used in flows of Newtonian and non-Newtonian fluids but in most engineering applications, the no-slip condition does not always hold in reality. For example a large class of polymeric materials slip or stick-slip on the solid boundaries. To counter this situation Navier proposed a general boundary condition that incorporate the possibility of fluid slip at the solid boundary. According to Navier, the relative velocity between the fluid and the solid boundary in the x-direction (at a solid boundary) is directly proportional to the shear stress at that boundary, i.e.

$$u_f - u_w \propto \tau_{xy}, \tag{1.15}$$

or

$$u_f - u_w = \pm \frac{\beta}{\mu} \tau_{xy}, \tag{1.16}$$

where β (constant of proportionality) is the slip parameter having dimension of length, the plus and the minus signs are due to direction of the normal on the wall, u_f is the velocity of the fluid and u_w is the velocity of the wall. This is known as "*slip condition*" at solid boundary. For $\beta = 0$ we recover the case of no-slip condition. We know that the total kinetic energy of the system is known as "*heat*". Heat is one of the most common form of energy that plays a vital role in transfer of energy from one place to another due to difference in temperature (average kinetic energy of the system). "*Heat transfer*" is the process that deals with the flow of heat within the system. It is different from thermodynamics in the sense that thermodynamics only deals with the flow of heat across the boundary and it is inadequate to explain the flow of heat within the system. As all of the transfer phenomenon are triggered by some gradient, in case of heat transfer the cause is difference in temperature. Heat flows from hotter to cooler side, and it keeps on flowing unless the temperature gradient is zero (or the heat is uniformly distributed throughout the system). Following are the modes through which heat can be transferred from one place to another. The transfer of heat, when it takes place from more energetic particles to the less energetic ones due to particle to particle collusions, is known as "*conduction*". Most of the heat transfer taking place in solids is due to conduction, it also takes place in liquids and gases but not as a major mode of heat transfer. Common example of conduction is rise of temperature of one end of an iron rod, when the other end is heated by any source. Heat transfer when it takes place between a solid boundary and the fluid moving adjacent to the boundary, is termed as "*convection*". It involves the combined effects of conduction and fluid motion. Conduction is the mode of heat transfer that is responsible for the transfer of heat in fluids. Example of convection can be taken as heating up of water when it is boiled in any container.

If no external force or agent is involved in the process, or the fluid motion occurs purely due to density difference induced by the temperature difference, then the process is called "*natural or free convection*". The temperature changes in the whole control volume produces a difference in density that in turn induces body forces, these body forces are responsible for generation of flow in case of free convection. These body forces are actually generated by pressure gradients imposed on the whole fluid. Gravity is the most common source of this imposed pressure fields. The body forces in this case are in common termed as buoyancy forces. In general we can say that natural convection would not be possible without thermal expansion and gravity. "*Forced convection*" is the type of convection that involves the fluid flow due to some external agent or source e.g. due to a fan or a pump. Buoyancy forces are negligible is this case. Matter in all its forms emit, absorb and transmit "*radiations*". These radiations are in form of electromagnetic waves. The transfer of heat by this mode has the speciality that it does not require any medium of propagation, and radiations can travel through vacuum. Heat transfer by this mode is explained by modified Stephan-Boltzmann law. Specific heat The amount of heat energy required to increase the temperature of one kg of any substance by one degree, is know as "*specific heat*" of that substance. The ability to transmit or to conduct heat energy for different materials is different. It is the measure of this ability of a material to conduct heat that is known as "*thermal conductivity*". It is denoted by k. A substance with a large k is a good conductor of heat e.g. iron, whereas a material with low k is a poor conductor but a good insulator e.g. air and wood. The ratio of amount of heat conducted to the amount of heat stored per unit volume is known as "*thermal diffusivity*". "*Viscous dissipation*" is the transformation of kinetic energy to the internal energy of the fluid due to viscous effects, in other words it is the heating up of fluid.

1.2 Governing Equations

"*Continuity equation*" represents the conservation of mass of the system (the transfer rate of the mass at entering and leaving the system is same).

Mathematically it can be written as for incompressible flow

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{V}) = 0, \qquad (1.17)$$

where ρ is the density of the fluid, $\mathbf{V}$ is the velocity and ∇ is the gradient operator.

For incompressible flow

$$\nabla.\mathbf{V} = 0. \qquad (1.18)$$

The equation of motion is given by

$$\rho\left[\frac{\partial}{\partial t}+(\mathbf{V}.\nabla)\right]\mathbf{V}=\rho\mathbf{b}+\nabla.\mathbf{T}, \tag{1.19}$$

where $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{b}$ is the body force, $\mathbf{V}$ is the velocity field and ρ is the fluid density.

The Eq of motion in another form is given by

$$\rho\frac{d\mathbf{V}}{dt}=\rho\mathbf{b}+\nabla.\mathbf{T} \tag{1.20}$$

where

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\mathbf{V}.\nabla \tag{1.21}$$

is the material derivative.

For Navier-Stokes equations

$$\mathbf{T}=-p\mathbf{I}+\mu\mathbf{A}_1, \tag{1.22}$$

$$\mathbf{A}_1=\nabla\mathbf{V}+(\nabla\mathbf{V})^T, \tag{1.23}$$

where p the pressure, μ the dynamic viscosity, A_1 the Rivlin Erickson tensor and T represent the transpose.

The Cauchy stress tensor in matrix form

$$\mathbf{T}=\begin{bmatrix}\sigma_{xx} & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \sigma_{yy} & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \sigma_{zz}\end{bmatrix}, \tag{1.24}$$

where σ_{xx}, σ_{yy} and σ_{yy} are the normal stresses and τ_{xy}, τ_{xz}, τ_{yx}, τ_{yz}, τ_{zx} and τ_{zy} are shear stresses.

Eq. (1.20) can be written in scalar form as the following

$$\frac{du}{dt}=\frac{1}{\rho}\frac{\partial(\sigma_{xx})}{\partial x}+\frac{1}{\rho}\frac{\partial(\tau_{xy})}{\partial y}+\frac{1}{\rho}\frac{\partial(\tau_{xz})}{\partial z}+b_x, \tag{1.25}$$

$$\frac{dv}{dt} = \frac{1}{\rho}\frac{\partial(\tau_{yx})}{\partial x} + \frac{1}{\rho}\frac{\partial(\sigma_{yy})}{\partial y} + \frac{1}{\rho}\frac{\partial(\tau_{yz})}{\partial z} + b_y, \tag{1.26}$$

$$\frac{dv}{dt} = \frac{1}{\rho}\frac{\partial(\tau_{zx})}{\partial x} + \frac{1}{\rho}\frac{\partial(\tau_{zy})}{\partial y} + \frac{1}{\rho}\frac{\partial(\sigma_{zz})}{\partial z} + b_z, \tag{1.27}$$

where b_x, b_y and b_z represents the body forces in x, y and z directions respectively.

The conservation law of energy states that the increase in the internal energy of a thermodynamical system is equal to the amount of heat energy added to the system plus (minus) the amount of energy gained (lost) by the system as a result of the work done on (by) the system by the surroundings. The general form of "*energy equation*" is

$$\rho\zeta\frac{dT}{dt} = \overline{\mathbf{T}}.(\nabla\mathbf{V}) + \nabla.(k\nabla T) \tag{1.28}$$

in which ζ is the specific heat at constant volume and k is the thermal conductivity. In case of constant thermal conductivity, Eq. (1.28) becomes

$$\rho\zeta\frac{dT}{dt} = \overline{\mathbf{T}}.(\nabla\mathbf{V}) + k\nabla^2 T). \tag{1.29}$$

1.3 Method of Solution

In topology two functions are said to be "*homotopic*" if one function can be transformed continuously in to the other. A "*homotopy*" between two continuous functions f and g from a topological space X to a topological space Y is defined to be continuous function

$$H : X * [0,1] -> Y, \tag{1.30}$$

from the product of the space X with the unit interval $[0,1]$ to Y such that for all the point x in X and

$$H(x,0) = f(x) \qquad H(x,1) = g(x) \tag{1.31}$$

The map H is called a homotopy between f and g. Any function f which is homotopic to g can be written as

$$f \simeq g. \tag{1.32}$$

We think of a homotopy as a continuous one parameter family of maps from X to Y. If we consider the parameter t as representing time, at time $t = 0$, we have the map f and as t varies the map H varies continuously so that at $t = 1$ we have the map g. For example consider two continuous functions defined on $\mathbb{R}$

$$f,\ g : \mathbb{R} \longrightarrow \mathbb{R}, \tag{1.33}$$

such that

$$f = x, \qquad g = 1 - x \tag{1.34}$$

One may develop the homotopy

$$H : X * [0,1] \longrightarrow Y. \tag{1.35}$$

In above expression $X = Y = \mathbb{R}$ and we define

$$H(x,t) = (1-t)\, f(x) + tg(x) \tag{1.36}$$

Invoking values of f and g in Eq. (1.36) we arrive at

$$H(x,t) = (1-t)\, x + t\,(1-x)\,. \tag{1.37}$$

Since $f(x)$ and $g(x)$ are continuous functions so as $H(x,t)$. Also

$$H(x,0) = x, \qquad\qquad H(x,1) = 1 - x \tag{1.38}$$

The variation of t from zero to one deforms $f(x)$ to $g(x)$. So we say that $f(x)$ and $g(x)$ are homotopic to each other.

Consider a non-linear equation governed by

$$A(u) + f(r) = 0, \tag{1.39}$$

where A is a non-linear operator,$f(r)$ is a known function and u is a unknown function. By means of "HAM" one can construct a family of equations

$$(1-p)\,\mathcal{L}\left[\widehat{v}(r,p)-u_0(r)\right]=p\hbar\{A[\widehat{v}(r,p)]-f(r)\}, \tag{1.40}$$

where $\mathcal{L}$ is the linear operator, $u_0(r)$ is an initial guess,$\hbar$ is an auxiliary parameter $p\in[0,1]$ is the embedding parameter. We expand $\widehat{v}(r,p)$in Taylor series about the embedding parameter

$$\widehat{v}(r,p)=u_0(r)+\sum_{m=1}^{\infty}u_m(r)\,p^m \tag{1.41}$$

where

$$u_m(x)=\frac{1}{m!}\frac{\partial^m\widehat{v}(r,p)}{\partial p^m}\bigg|_{p=0} \tag{1.42}$$

the convergence of the series (1.41) depends on the auxiliary parameter $\hbar$. If it is convergent at $p=1$,one has

$$u(r)=u_0(r)+\sum_{m=1}^{\infty}u_m(r) \tag{1.43}$$

Differentiating the zeroth order deformation Eq. (1.40) m-times with respect to p and then dividing them by $m!$ and finally setting $p=0$ we obtain the following mth-order deformation problem

$$\mathcal{L}\left[u_m(r)-\chi_m u_{m-1}(r)\right]=\hbar\mathcal{R}_m(r)\,, \tag{1.44}$$

in which

$$\chi_m=\begin{cases}0, & m\leq 1,\\ 1, & m>1.\end{cases} \tag{1.45}$$

$$\mathcal{R}_m(r)=\frac{1}{(m-1)!}\{\frac{d^{k-1}}{dp^{k-1}}\,A[u_0(r)+\sum_{m-1}^{\infty}u_m(r)\,p^m]\}\bigg|_{p=0} \tag{1.46}$$

There are many different ways to get the higher order deformation equations. However, according to the fundamental theorems in calculus, the terms $u_m(r)$ in the series is unique. Note that the HAM contains an auxiliary parameter $\hbar$, which provides us with a simple way to control and adjust the convergence of the series solution.

Chapter 2

Non-orthogonal stagnation-point flow towards a stretching surface in a non-Newtonian fluid with heat transfer

This chapter describe the two-dimensional oblique stagnation point flow of a second grade fluid over a stretching surface with heat transfer. The problem is formulated and then transformed into a system of non-linear ordinary differential equations with the help of suitable similarity transformations which are then solved analytically by means of homotopy analysis method (HAM). The results for velocity, temperature and skin friction coefficients are also computed, and discussed for various emerging physical parameters. The problem was solved numerically by "F. labropulu, D. Li, I.Pop" [15], and a suitable comparison is made with the numerical and HAM solutions.

2.1 Mathematical Formulation

Consider a steady two-dimensional non-orthogonal stagnation point flow of a second grade fluid towards a stretching surface. In addition heat transfer effects are considered. The flow is governed by the following equations

$$\operatorname{div} \mathbf{V}^* = 0, \tag{2.1}$$

$$\rho \frac{d\mathbf{V}^*}{dt} = \operatorname{div} \mathbf{T}^* + \rho\, \mathbf{B}^*, \tag{2.2}$$

where ρ is the fluid density and $\mathbf{T}^*$ is the Cauchy stress tensor. For second grade fluid Cauchy stress tensor is defined by

$$\mathbf{T}^* = -p^* I + \mu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2, \tag{2.3}$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are the first and second Rivlin Erickson tensors given by

$$\mathbf{A}_1 = (\operatorname{grad} \mathbf{V}^*) + (\operatorname{grad} \mathbf{V}^*)^T, \tag{2.4}$$

$$\mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1 (\operatorname{grad} \mathbf{V}^*) + (\operatorname{grad} \mathbf{V}^*)^T \mathbf{A}_1, \tag{2.5}$$

The velocity profile for present flow is taken as

$$\mathbf{V}^* = [u^*(x^*, y^*), v^*(x^*, y^*), 0], \tag{2.6}$$

For the given velocity profile we have

$$(\operatorname{grad} \mathbf{V}^*) = \begin{bmatrix} \frac{\partial u^*}{\partial x^*} & \frac{\partial u^*}{\partial y^*} & 0 \\ \frac{\partial v^*}{\partial x^*} & \frac{\partial v^*}{\partial y^*} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.7}$$

and its transpose is given by,

$$(\operatorname{grad} \mathbf{V}^*)^T = \begin{bmatrix} \frac{\partial u^*}{\partial x^*} & \frac{\partial v^*}{\partial x^*} & 0 \\ \frac{\partial u^*}{\partial y^*} & \frac{\partial v^*}{\partial y^*} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.8}$$

Making use of Eqs. (2.7) and (2.8) in Eq. (2.4), we have

$$\mathbf{A}_1 = \begin{bmatrix} 2\frac{\partial u^*}{\partial x^*} & \frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*} & 0 \\ \frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*} & 2\frac{\partial v^*}{\partial y^*} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.9}$$

for the given velocity profile

$$\mathbf{A}_2 = u^* \frac{\partial^* \mathbf{A}_1}{\partial x^*} + v^* \frac{\partial^* \mathbf{A}_1}{\partial y^*} + \mathbf{A}_1(\text{grad}\,\mathbf{V}^*) + (\text{grad}\,\mathbf{V}^*)^T \mathbf{A}_1, \tag{2.10}$$

Making use of Eqs. (2.6) to (2.9) in Eq. (2.10), we have

$$\mathbf{A}_2 = \begin{bmatrix} \begin{array}{c} 4(\frac{\partial u^*}{\partial x^*})^2 + 2\frac{\partial v^*}{\partial x^*}(\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*}) \\ +2u^*(\frac{\partial^2 u^*}{\partial x*^2}) + 2v^*(\frac{\partial^2 u^*}{\partial y*\partial x*}) \end{array} & \begin{array}{c} 2(\frac{\partial u^*}{\partial x^*})(\frac{\partial u^*}{\partial y^*}) + 2(\frac{\partial v^*}{\partial x^*})(\frac{\partial v^*}{\partial x^*}) \\ +(u^*\frac{\partial}{\partial x^*} + v^*\frac{\partial}{\partial y^*})(\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*}) \end{array} & 0 \\ \begin{array}{c} 2(\frac{\partial u^*}{\partial x^*})(\frac{\partial u^*}{\partial y^*}) + 2(\frac{\partial v^*}{\partial x^*})(\frac{\partial v^*}{\partial x^*}) \\ +(u^*\frac{\partial}{\partial x^*} + v^*\frac{\partial}{\partial y^*})(\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*}) \end{array} & \begin{array}{c} 4(\frac{\partial v^*}{\partial y^*})^2 + 2\frac{\partial u^*}{\partial y^*}(\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*}) \\ +2v^*(\frac{\partial^2 v^*}{\partial y*^2}) + 2u^*(\frac{\partial^2 v^*}{\partial x*\partial y*}) \end{array} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.11}$$

Also we have

$$\mathbf{A}_1^2 = \begin{bmatrix} 4(\frac{\partial u^*}{\partial x^*})^2 + (\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*})^2 & 0 & 0 \\ 0 & 4(\frac{\partial v^*}{\partial y^*})^2 + (\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*})^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.12}$$

Using Eqs. (2.6) to (2.12) in Eq. (2.3), we have

$$\mathbf{T}^* = \begin{bmatrix} T_{xx} & T_{xy} & 0 \\ T_{yx} & T_{yy} & 0 \\ 0 & 0 & -p^* \end{bmatrix}. \tag{2.13}$$

where

$$T_{xx} = -p^* + 2\mu \frac{\partial u^*}{\partial x^*} + \alpha_1 \begin{pmatrix} 4(\frac{\partial u^*}{\partial x^*})^2 + 2\frac{\partial v^*}{\partial x^*} \\ (\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*}) \\ +2u^*(\frac{\partial^2 u^*}{\partial x*^2}) \\ +2v^*(\frac{\partial^2 u^*}{\partial y*\partial x*}) \end{pmatrix} + \alpha_2 \left(4(\frac{\partial u^*}{\partial x^*})^2 + (\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*})^2\right)$$

$$T_{xy} = T_{yx} = \mu(\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*}) + \alpha_1 \begin{pmatrix} 2(\frac{\partial u^*}{\partial x^*})(\frac{\partial u^*}{\partial y^*}) \\ +2(\frac{\partial v^*}{\partial x^*})(\frac{\partial v^*}{\partial x^*}) \\ +(u^*\frac{\partial}{\partial x^*} + v^*\frac{\partial}{\partial y^*}) \\ (\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*}) \end{pmatrix}$$

$$T_{yy} = -p^* + 2\mu \frac{\partial v^*}{\partial y^*} + \alpha_1 \begin{pmatrix} 4(\frac{\partial v^*}{\partial y^*})^2 \\ +2\frac{\partial u^*}{\partial y^*}(\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*}) \\ +2v^*(\frac{\partial^2 v^*}{\partial y*^2}) + \\ 2u^*(\frac{\partial^2 v^*}{\partial y*\partial x*}) \end{pmatrix} + \alpha_2 \begin{pmatrix} 4(\frac{\partial v^*}{\partial y^*})^2 + \\ (\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*})^2 \end{pmatrix}$$

In component form Eq. (2.2) is written as

$$\rho[u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*}] = \frac{\partial}{\partial x^*}(T_{xx}) + \frac{\partial}{\partial y^*}(T_{xy}). \tag{2.14}$$

$$\rho[u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*}] = \frac{\partial}{\partial x^*}(T_{yx}) + \frac{\partial}{\partial y^*}(T_{yy}). \tag{2.15}$$

Making use of Eq. (2.13) in Eqs. (2.14) and (2.15), we obtain

$$\begin{aligned} u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} + \frac{1}{\rho}\frac{\partial p^*}{\partial x^*} \;&=\; \nu \nabla^{*2} u^* + \frac{\alpha_1}{\rho}\{\frac{\partial}{\partial x^*}[4(\frac{\partial u^*}{\partial x^*})^2 + 2\frac{\partial v^*}{\partial x^*}(\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*}) \\ &+2u^*(\frac{\partial^2 u^*}{\partial x*^2}) + 2v^*(\frac{\partial^2 u^*}{\partial y*\partial x*})] + \frac{\partial}{\partial y^*}[2(\frac{\partial u^*}{\partial x^*})(\frac{\partial u^*}{\partial y^*}) \\ &+2(\frac{\partial v^*}{\partial x^*})(\frac{\partial v^*}{\partial y^*}) + (u^*\frac{\partial}{\partial x^*} + v^*\frac{\partial}{\partial y^*})(\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*})\}] \\ &+\frac{\alpha_2}{\rho}\frac{\partial}{\partial x^*}[4(\frac{\partial u^*}{\partial x^*})^2 + (\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*})^2], \end{aligned} \tag{2.16}$$

$$
\begin{aligned}
u^*\frac{\partial v^*}{\partial x^*}+v^*\frac{\partial v^*}{\partial y^*}+\frac{1}{\rho}\frac{\partial p^*}{\partial y^*} &= \nu\nabla^{*^2}v^*+\frac{\alpha_1}{\rho}\{\frac{\partial}{\partial x^*}[2(\frac{\partial u^*}{\partial x^*})(\frac{\partial u^*}{\partial y^*}))+2(\frac{\partial v^*}{\partial x^*})(\frac{\partial v^*}{\partial y^*}) \\
&+(u^*\frac{\partial}{\partial x^*}+v^*\frac{\partial}{\partial y^*})(\frac{\partial u^*}{\partial y^*}+\frac{\partial v^*}{\partial x^*})]+\frac{\partial}{\partial y^*}[4(\frac{\partial v^*}{\partial y^*})^2 \\
&+2\frac{\partial u^*}{\partial y^*}(\frac{\partial u^*}{\partial y^*}+\frac{\partial v^*}{\partial x^*})+2v^*(\frac{\partial^2 v^*}{\partial x*^2})+2u^*(\frac{\partial^2 v^*}{\partial y*\partial x*})]\} \\
&+\frac{\alpha_2}{\rho}\frac{\partial}{\partial y^*}[4(\frac{\partial v^*}{\partial y^*})^2+(\frac{\partial u^*}{\partial y^*}+\frac{\partial v^*}{\partial x^*})^2],
\end{aligned} \tag{2.17}
$$

where $p^*(x^*,y^*)$ is the fluid pressure function, $\nu=\mu/\rho$ is the kinematic viscosity, ρ is the constant fluid density and μ is the constant coefficient of viscosity.

Law of conservation of energy is given by

$$\rho\frac{de}{dt}=\sigma^*.\mathrm{L}-\operatorname{div}\mathbf{q}+\rho r, \tag{2.18}$$

where $e=C_p\sigma^*$, $\mathbf{q}=-k\mathrm{div}\sigma^*$ is the heat flux vector, σ^* is the fluid temperature, k is the thermal conductivity, C_p is the specific heat and r is the internal heat generation. In the absence of viscous dissipation effect and the radiant heating, the energy Eq. can be written as

$$\rho C_p[u^*\frac{\partial\sigma^*}{\partial x^*}+v^*\frac{\partial\sigma^*}{\partial y^*}]=k\nabla^{*^2}\sigma^*, \tag{2.19}$$

or

$$u^*\frac{\partial\sigma^*}{\partial x^*}+v^*\frac{\partial\sigma^*}{\partial y^*}=\alpha^*\nabla^{*^2}\sigma^*, \tag{2.20}$$

Where $\alpha^*=k/\rho C_p$ is the thermal diffusivity of the fluid.

The corresponding boundary conditions are given by

$$u^*=cx^*,\ v^*=0,\ \sigma^*=\sigma_w\ \ on\ y^*=0, \tag{2.21}$$

$$u^*=ax^*+by^*,\quad \sigma^*=\sigma_\infty\ \ as\ \ y^*\to\infty. \tag{2.22}$$

Where a, b, and c are positive constants with dimensions of inverse time, σ_w is the constant temperature of the plate while the uniform temperature of the ambient fluid is σ_∞. Introducing

$$\begin{aligned} x &= x^*\sqrt{\frac{c}{\nu}},\ y = y^*\sqrt{\frac{c}{\nu}},\ u = \frac{1}{\sqrt{\nu c}}u^*,\ v = \frac{1}{\sqrt{\nu c}}v^*, \\ p &= \frac{1}{\rho\nu c}p^*,\ \ \sigma = \frac{\sigma^* - \sigma_\infty}{\sigma_w - \sigma_\infty}. \end{aligned} \quad (2.23)$$

Using Eq. (2.23) in Eqs. (2.1), (2.16), (2.17) and (2.20), we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (2.24)$$

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial x} &= \nabla^2 u + We\{\frac{\partial}{\partial x}[4(\frac{\partial u}{\partial x})^2 + 2\frac{\partial v}{\partial x}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) + 2u(\frac{\partial^2 u}{\partial x^2}) \\ &+2v(\frac{\partial^2 u}{\partial y\partial x})] + \frac{\partial}{\partial y}[2(\frac{\partial u}{\partial x})(\frac{\partial u}{\partial y}) + 2(\frac{\partial v}{\partial x})(\frac{\partial v}{\partial y}) \\ &+(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})]\} + \lambda\frac{\partial}{\partial x}[4(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})^2] \end{aligned} \quad (2.25)$$

$$\begin{aligned} u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} &= \nabla^2 v + We\{\frac{\partial}{\partial x}[2(\frac{\partial u}{\partial x})(\frac{\partial u}{\partial y}) + 2(\frac{\partial v}{\partial x})(\frac{\partial v}{\partial y}) + (u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})] \\ &+\frac{\partial}{\partial y}[4(\frac{\partial v}{\partial y})^2) + 2\frac{\partial u}{\partial y}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) + 2v(\frac{\partial^2 v}{\partial x^2}) + 2u(\frac{\partial^2 v}{\partial y\partial x})]\} \\ &+\lambda\frac{\partial}{\partial y}[4(\frac{\partial v}{\partial y})^2 + (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})^2], \end{aligned} \quad (2.26)$$

$$\Pr[u\frac{\partial \sigma}{\partial x} + v\frac{\partial \sigma}{\partial y}] = \nabla^2\sigma, \quad (2.27)$$

where $We = \alpha_1 C/\rho\nu$ is the Weissenberg number and $\lambda = \alpha_2 C/\rho\nu$, and $\Pr = \mu C_p/k$ is the prandtl number.

Introducing the stream function relation

$$u = \frac{\partial \Psi}{\partial y},\ v = -\frac{\partial \Psi}{\partial x}. \quad (2.28)$$

Substitution of Eq. (2.28) in Eqs. (2.25) to (2.27), and elimination of pressure from the resulting equations using $P_{xy} = P_{yx}$ yields

$$\frac{\partial(\Psi, \nabla^2\Psi)}{\partial(x,y)} - We\frac{\partial(\Psi, \nabla^4\Psi)}{\partial(x,y)} + \nabla^4\Psi = 0. \tag{2.29}$$

$$\Pr[\frac{\partial\Psi}{\partial y}\frac{\partial\sigma}{\partial x} - \frac{\partial\Psi}{\partial x}\frac{\partial\sigma}{\partial y}] = \nabla^2\sigma. \tag{2.30}$$

The corresponding non-dimensional boundary conditions in terms of the stream function $\Psi(x, y)$ are given by

$$\Psi = 0, \frac{\partial\Psi}{\partial y} = x, \qquad \sigma = 1 \ \ at \ \ y = 0, \tag{2.31}$$

$$\Psi = \frac{a}{c}xy + \frac{1}{2}\gamma y^2, \ \sigma = 0 \ at \ y \to \infty. \tag{2.32}$$

where $\gamma = b/c$ represents the shear in the stream.

we seek solutions of Eqs. (2.29) and (2.30) of the form

$$\Psi(x, y) = x \ f(y) + g(y), \quad \sigma = \theta(y) \ , \tag{2.33}$$

where the functions $f(y)$ and $g(y)$ are referring to as the normal and tangential component of the flow respectively and prime denotes differentiation with respect to y. substituting Eq. (2.33) in Eqs. (2.29) and (2.30), we obtain the following ordinary differential equations after one integration

$$f''' + ff'' - f'^2 - We(ff^{IV} - 2f'f''' + f''^2) + C_1 = 0, \tag{2.34}$$

$$g''' + fg'' - f'g' - We(f\ g^{IV} - f'g''' + f''g'' - f'''g') + C_2 = 0, \tag{2.35}$$

$$\theta'' + \Pr f\ \theta' = 0. \tag{2.36}$$

where prime denotes differentiation with respect to y and C_1, C_2 are constants of integration. Using Eq. (2.33) the boundary conditions (2.31) and (2.32) give

$$f(0) = 0, \ f'(0) = 0, \ f'(\infty) = \frac{a}{c}, \tag{2.37}$$

$$g(0) = 0, \ g'(0) = 0, \ g''(\infty) = \gamma, \tag{2.38}$$

$$\theta(0) = 1,\ \theta(\infty) = 0. \tag{2.39}$$

Taking the limit $y \to \infty$ in Eq. (2.34) and using the boundary condition $f^{'}(\infty) = a/c$, we get $C_1 = (a/c)^2$. An analysis of the boundary layer Eq. (2.34) implies that $f(y)$ behaves as $f(y) = (\frac{a}{c})\, y + A$ as $y \to \infty$, where $A = A(We, a/c)$ is a constant that accounts for the boundary layer displacement. Taking the limit as $y \to \infty$ in Eq. (2.35) and using the infinity boundary condition $g^{''}(\infty) = \gamma$ we get that $C_2 = -A\gamma$. Thus Eqs. (2.34) and (2.35) become

$$f^{'''} + ff^{''} - (f^{'})^2 - We(f\, f^{IV} - 2f^{'}f^{'''} + (f^{''})^2) + \frac{a^2}{c^2} = 0, \tag{2.40}$$

$$g^{'''} + fg^{''} - f'g' - We(f\, g^{IV} - f'g''' + f''g'' - f^{'''}g') - A\gamma = 0, \tag{2.41}$$

$$\theta'' + \Pr f\, \theta' = 0. \tag{2.42}$$

Introducing a new variable,

$$g'(y) = \gamma\, h(y). \tag{2.43}$$

Using Eq. (2.43) in Eqs. (2.38) and (2.41) we have

$$h^{''} + f\, h' - f^{'} h - We(f\, h''' - f'\, h'' + f''\, h' - f^{'''}h) - A = 0, \tag{2.44}$$

$$h(0) = 0, \quad h^{'}(\infty) = 1. \tag{2.45}$$

2.2 Homotopy Analysis Solution

We express $f(\eta)$, $h(\eta)$ and $\theta(\eta)$ by a set of base functions

$$\{\eta^k \exp(-n\eta) | k \geq 0,\ n \geq 0\}, \tag{2.46}$$

in the form

$$f(\eta) = a_{0,0}^0 + \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{m,n}^k \eta^k \exp(-n\eta), \tag{2.47}$$

$$\theta(\eta) = b_{0,0}^0 + \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} b_{m,n}^k \eta^k \exp(-n\eta), \tag{2.48}$$

$$h(\eta) = c_{0,0}^{0} + \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} c_{m,n}^{k}\eta^{k}\exp(-n\eta), \tag{2.49}$$

in which $a_{m,n}^{k}$, $b_{m,n}^{k}$ and $c_{m,n}^{k}$ are the coefficients.

For the HAM solutions, the initial guesses are given by

$$f_0(\eta) = (1 - \frac{a}{c}) * (1 - \exp(-\eta)) + \frac{a}{c}\eta, \tag{2.50}$$

$$\theta_0(\eta) = \exp(-\eta), \tag{2.51}$$

$$h_0(\eta) = 1 - \exp(-\eta)) + \eta. \tag{2.52}$$

The corresponding linear operators are given by

$$\pounds_f = \frac{d^3 f}{d\eta^3} + \frac{d^2 f}{d\eta^2}, \tag{2.53}$$

$$\pounds_\theta = \frac{d^2 \theta}{d\eta^2} + \frac{d\theta}{d\eta}, \tag{2.54}$$

$$\pounds_h = \frac{d^3 h}{d\eta^3} + \frac{d^2 h}{d\eta^2}, \tag{2.55}$$

which have the following property

$$\pounds_f[C_1 + C_2\eta + C_3\exp(-\eta)] = 0, \tag{2.56}$$

$$\pounds_\theta[C_4 + C_5\exp(-\eta)] = 0, \tag{2.57}$$

$$\pounds_h[C_6 + C_7\eta + C_8\exp(-\eta)] = 0, \tag{2.58}$$

where C_i $(i = 1 - 8)$ are arbitrary constants.

2.3 Zeroth-order deformation equation

If $p \in [0,1]$ is an embedding parameter and $\hbar_f$, $\hbar_h$, $\hbar_\theta$, indicate the non zero auxiliary parameters respectively then the zeroth order deformation problems are

$$(1-p)\,\mathcal{L}_f[\hat{f}(\eta;p) - \hat{f}_0\,(\eta)] = p\hbar_f N_f[\hat{f}(\eta;p), \hat{h}(\eta;p), \hat{\theta}(\eta;p)], \tag{2.59}$$

$$(1-p)\,\mathcal{L}_h[\hat{h}(\eta;p) - \hat{h}_0\,(\eta)] = p\hbar_h N_h[\hat{f}(\eta;p), \hat{h}(\eta;p)], \tag{2.60}$$

$$(1-p)\,\mathcal{L}_\theta[\hat{\theta}(\eta;p) - \hat{\theta}_0\,(\eta)] = p\hbar_\theta N_\theta[\hat{f}(\eta;p), \hat{\theta}(\eta;p)], \tag{2.61}$$

$$\hat{f}(0;p) = 0, \hat{f}^{'}(0;p) = 1,\ \hat{h}(0;p) = 0,\ \hat{\theta}(0;p) = 1, \tag{2.62}$$

$$\hat{f}^{'}(\infty;p) = \frac{a}{c},\ \hat{h}^{'}(\infty;p) = 1,\ \hat{\theta}(\infty;p) = 0, \tag{2.63}$$

in which the non linear operators N_f, N_h, N_θ are

$$\begin{aligned} N_f[\hat{f}(p;\eta), \hat{h}(p;\eta)] &= \frac{\partial^3 \hat{f}(\eta;p)}{\partial \eta^3} + \hat{f}(\eta;p)\frac{\partial^2 \hat{f}(\eta;p)}{\partial \eta^2} - (\frac{\partial \hat{f}(\eta;p)}{\partial \eta})^2 - We(\hat{f}(\eta;p)\frac{\partial^4 \hat{f}(\eta;p)}{\partial \eta^4} \\ &\quad -2\frac{\partial \hat{f}(\eta;p)}{\partial \eta}\frac{\partial^3 \hat{f}(\eta;p)}{\partial \eta^3} + (\frac{\partial^2 \hat{f}(\eta;p)}{\partial \eta^2})^2) + (\frac{a}{c})^2, \end{aligned} \tag{2.64}$$

$$N_\theta[\hat{f}(p;\eta), \hat{\theta}(p;\eta)] = \frac{\partial^2 \hat{\theta}(\eta;p)}{\partial \eta^2} + \Pr \hat{f}(\eta;p)\frac{\partial \hat{\theta}(\eta;p)}{\partial \eta}, \tag{2.65}$$

$$\begin{aligned} N_h[\hat{f}(p;\eta), \hat{h}(p;\eta)] &= \frac{\partial^2 \hat{h}(p;\eta)}{\partial \eta^2} + \hat{f}(\eta;p)\frac{\partial \hat{h}(p;\eta)}{\partial \eta} - \frac{\partial \hat{f}(\eta;p)}{\partial \eta}\hat{h}(p;\eta) - \\ We(\hat{f}(\eta;p)\frac{\partial^3 \hat{h}(p;\eta)}{\partial \eta^3} - \frac{\partial \hat{f}(\eta;p)}{\partial \eta}\frac{\partial^2 \hat{h}(p;\eta)}{\partial \eta^2} &\quad + \frac{\partial^2 \hat{f}(\eta;p)}{\partial \eta^2}\frac{\partial \hat{h}(p;\eta)}{\partial \eta} - \frac{\partial^3 \hat{f}(\eta;p)}{\partial \eta^3}\hat{h}(p;\eta)) - \end{aligned} \tag{2.66}$$

Obviously,

$$\hat{f}(\eta;0) = f_0(\eta) \quad \hat{f}(\eta;1) = f(\eta), \tag{2.67}$$

$$\hat{\theta}(\eta;0) = \theta_0(\eta) \quad \hat{\theta}(\eta;1) = \theta(\eta), \tag{2.68}$$

$$\hat{h}(\eta;0) = h_0(\eta) \quad \hat{h}(\eta;1) = h(\eta). \tag{2.69}$$

As p goes from 0 to 1, $\hat{f}(\eta;p)$, $\hat{h}(\eta;p)$, $\hat{\theta}(\eta;p)$ vary from initial guesses $f_0(\eta)$, $h_0(\eta)$, $\theta_0(\eta)$ to final solutions $f(\eta)$, $h(\eta)$, $\theta(\eta)$ respectively. Making the assumption that the auxiliary parameters $\hbar_f$, $\hbar_h$ $\hbar_\theta$ are so properly chosen that the Taylor series of $f(\eta;p)$, $h(\eta;p)$, $\theta(\eta;p)$ expanded with respect to embedding parameters converges at $p = 1$. Thus we can write

$$\hat{f}(\eta;p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta)p^m, \tag{2.70}$$

$$\hat{h}(\eta;p) = h_0(\eta) + \sum_{m=1}^{\infty} h_m(\eta)p^m, \tag{2.71}$$

$$\hat{\theta}(\eta;p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta)p^m, \tag{2.72}$$

where

$$\begin{aligned} f_m(\eta) &= \frac{1}{m!}\frac{\partial^m f(\eta;p)}{\partial \eta^m}\bigg|_{p=0}, h_m(\eta) = \frac{1}{m!}\frac{\partial^m h(\eta;p)}{\partial \eta^m}\bigg|_{p=0}, \\ \theta_m(\eta) &= \frac{1}{m!}\frac{\partial^m \theta(\eta;p)}{\partial \eta^m}\bigg|_{p=0}. \end{aligned} \tag{2.73}$$

With the help of Eq. (2.73) Eqs.(2.70) to (2.72) can be written as

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \tag{2.74}$$

$$h(\eta) = h_0(\eta) + \sum_{m=1}^{\infty} h_m(\eta), \tag{2.75}$$

$$\theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta). \tag{2.76}$$

The mth-order deformation equations are defined as

$$\mathcal{L}_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = \hbar_f R_m^f(\eta), \tag{2.77}$$

$$\mathcal{L}_h[h_m(\eta) - \chi_m h_{m-1}(\eta)] = \hbar_h R_m^h(\eta), \tag{2.78}$$

$$\mathcal{L}_\theta[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = \hbar_\theta R_m^\theta(\eta), \tag{2.79}$$

The corresponding boundary conditions for mth deformation problems are

$$f_m(0) = f_m^{'}(0) = 1, h_m(0) = 0, \theta_m(0) = 1, \tag{2.80}$$

$$f_m^{'}(\infty) = \frac{a}{c}, h_m^{'}(\infty) = 1, \theta_m(\infty) = 0, \tag{2.81}$$

where

$$\begin{aligned} R_m^f(\eta) &= f_{m-1}^{'''} + \sum_{k=0}^{m-1} f_k f_{m-1-k}^{''} - \sum_{k=0}^{m-1} f_k^{'} f_{m-1-k}^{'} - We(\sum_{k=0}^{m-1} f_k f_{m-1-k}^{IV} - 2\sum_{k=0}^{m-1} f_k^{'} f_{m-1-k}^{'''} \\ &+ \sum_{k=0}^{m-1} f_k^{''} f_{m-1-k}^{''}) + (\frac{a}{c})^2, \end{aligned} \tag{2.82}$$

$$R_m^\theta(\eta) = \theta_{m-1}^{''} + \Pr \sum_{k=0}^{m-1} f_k \theta_{m-1-k}^{'} \tag{2.83}$$

$$\begin{aligned} R_m^h(\eta) &= h_{m-1}^{''} + \sum_{k=0}^{m-1} f_k h_{m-1-k}^{'} - \sum_{k=0}^{m-1} f_k^{'} h_{m-1-k} - We(\sum_{k=0}^{m-1} f_k h_{m-1-k}^{'''} - \sum_{k=0}^{m-1} f_k^{'} h_{m-1-k}^{''} \\ &+ \sum_{k=0}^{m-1} f_k^{''} h_{m-1-k}^{'} - \sum_{k=0}^{m-1} f_k^{'''} h_{m-1-k}) - A, \end{aligned} \tag{2.84}$$

in which

$$\chi_m = \begin{cases} 0, m \leq 1 \\ 1, m > 1. \end{cases} \tag{2.85}$$

The general solutions of Eqs.(2.82) to (2.84) can be written as

$$f_m(\eta) = f_m^*(\eta) + C_1 + C_2\eta + C_3\exp(-\eta), \tag{2.86}$$

$$\theta_m(\eta) = \theta_m^*(\eta) + C_4 + C_5\exp(-\eta), \tag{2.87}$$

$$h_m(\eta) = h_m^*(\eta) + C_6 + C_7\eta + C_8\exp(-\eta), \tag{2.88}$$

where C_i $(i = 1 - 8)$ are constants.

2.4 Convergence of the HAM Solutions

Obviously the series solutions depend upon the non-zero auxiliary parameters $\hbar_f$, $\hbar_h$, $\hbar_\theta$ which can adjust and control the convergence of the HAM solutions. In order to see the range of admissible values of $\hbar_f$, $\hbar_h$, $\hbar_\theta$ and the $\hbar-curve$ of the functions $f^{''}(0)$, $h^{'}(0)$, $\theta^{'}(0)$ and are sketched for 15-order of approximations in Fig 2.1a. It is found that the range of admissible values of $\hbar_f$, $\hbar_h$, $\hbar_\theta$ are $-1 \leq \hbar_f \leq -0.3$, $-1.1 \leq \hbar_h \leq -0.6$, $-1.3 \leq \hbar_\theta \leq -0.5$,

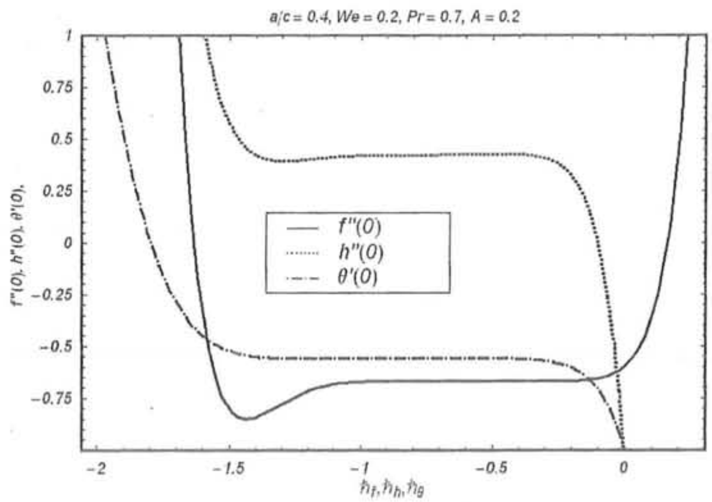

Fig: 2.1(a) $\hbar-$ Curves for f, h and θ

Table 2.1 : Convergence of HAM solution for different order of approximations

$order of approximations$	$f''(0)$	$-\theta'(0)$	$h''(0)$
1	-0.64200	0.78400	0.09411
10	-0.66888	0.55924	0.33182
15	-0.66888	0.55875	0.33194
17	-0.66888	0.55874	0.33192
19	-0.66888	0.55874	0.33192
21	-0.66888	0.55874	0.33192
23	-0.66888	0.55874	0.33192
25	-0.66888	0.55874	0.33192
27	-0.66888	0.55874	0.33192
29	-0.66888	0.55874	0.33192
31	-0.66888	0.55874	0.33192

2.5 Results and discussion

In this section the influence of emerging parameters on the velocity components f', h',temperature profile θ and stream line patterns are discussed. For this purpose Figs. (2.2) to (2.13) are plotted to see the variation of velocity a/c, Weissenberg number We, and Prandtl number Pr on the f', h' and θ. Fig. (2.2) is plotted to see the effects of a/c on velocity component f'. Fig. (2.2) describe that velocity f' is an increasing function of a/c. From Fig. (2.3) we see that velocity f' increases with an increase in We. The boundary layer thickness also increase with We. Fig. (2.4) gives the variation of a/c on h'. This Fig. shows that initially h' increases and after $y = 1$, h' decrease with an increase in a/c. The effect of We on h' is qualitatively opposite to that of a/c ($see Fig.$(2.5)).Fig. (2.6) and (2.7) are plotted to see the effects of We and Pr on θ. As expected θ is a decreasing function of Pr. Thermal boundary layer thickness also decreases with Pr see ($Fig.$2.7). Fig. (2.8) depicts that θ is also a decreasing function of a/c. Numerical values of $f''(0)$, $h'(0)$, and $-\theta'(0)$ for different values of We and a/c are shown in *Tables* 1 to 4 to predict the behavior of skin friction and local heat flux.From these tables one can see that HAM solution has an excellent agreement with numerical solution. Figs. (2.9) to (2.13) are streamline patterns for the oblique flow for various values of the parameters We, γ, and a/c. It can be clearly seen from these figures that for fixed values of We and a/c, the streamlines are oblique towards the left of the stagnation-point with increase in γ (positive). On the other hand, the stream lines are more and more oblique towards the right of the stagnation point see Figs. (2.11) and (2.13).

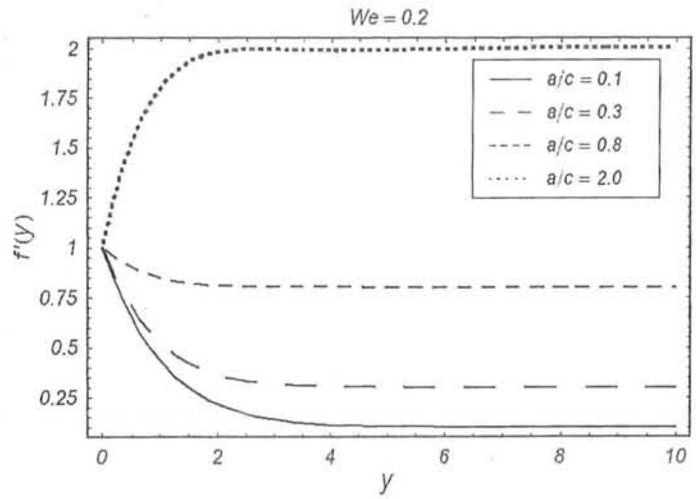

Fig: 2.2 Variation of $f'(y)$ for various values of a/c, when $We = 0.2$

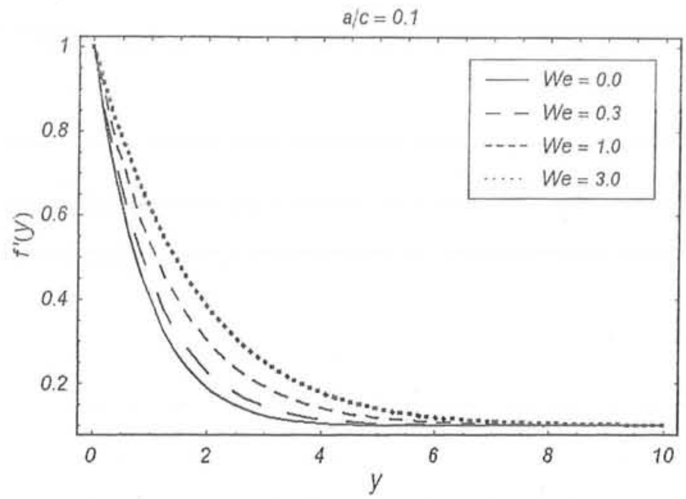

Fig: 2.3 Variation of $f^{'}(y)$ for various values of We, when $a/c = 0.1$

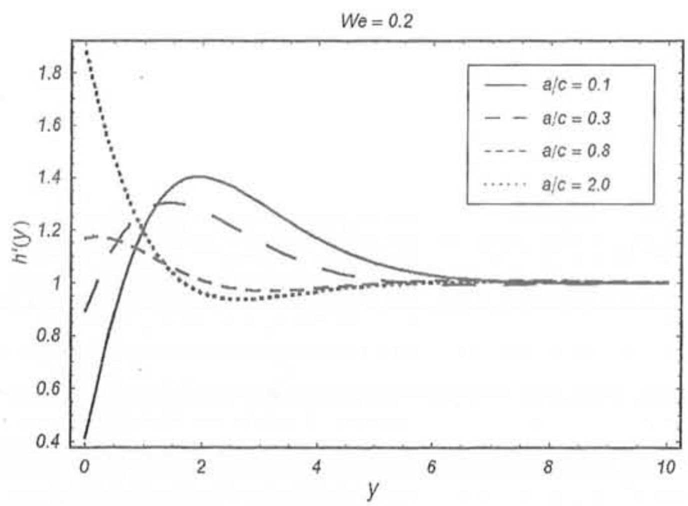

Fig: 2.4 Variation of $h^{'}(y)$ for various values of a/c, when $We = 0.2$

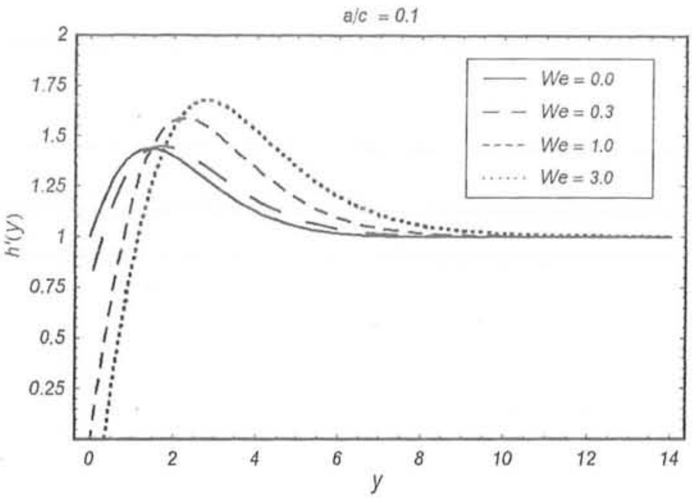

Fig: 2.5 Variation of $h'(y)$ for various values of We, when

$a/c = 0.1$

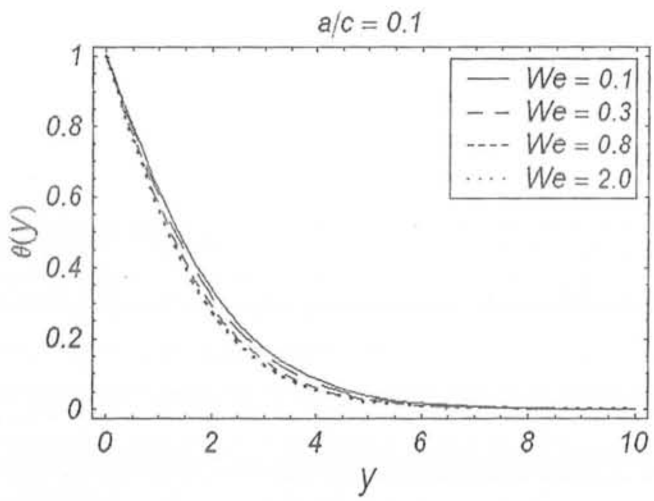

Fig: 2.6 Variation of θ(y) for various values of We, when

$\Pr = 0.5$ and $a/c = 0.1$

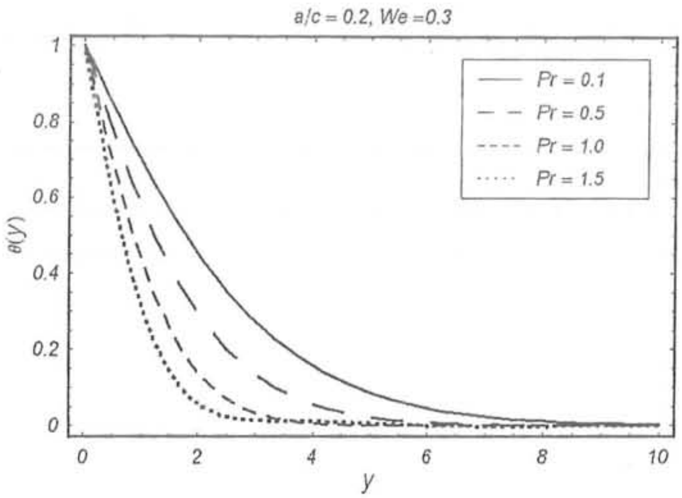

Fig: 2.7 Variation of θ(y) for various values of Pr, when $We = 0.3$ and $a/c = 0.2$

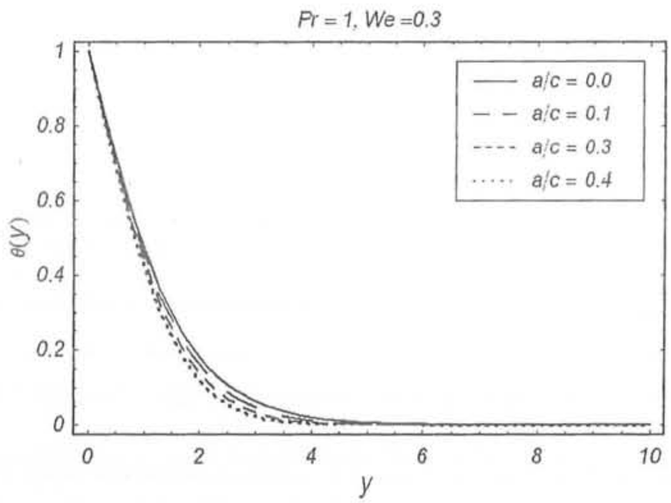

Fig: 2.8 Variation of θ(y) for various values of a/c, when $We = 0.3$ and $\Pr = 1$

$TABLE\ 1\ (f''\ (0))$

$parameters$	HAM [NUM]	HAM [NUM]	HAM [NUM]	HAM [NUM]
$\frac{a}{c}\downarrow, We \rightarrow$	0	0.2	0.5	1
0.1	-0.96939 [-0.96938]	-0.87660 [-0.87659]	-0.77542 [-0.77541]	-0.66327[-0.66328]
0.3	-0.84942 [-0.84942]	-0.75115 [-0.75114]	-0.64992 [-0.64992]	-0.54442[-0.54434]
0.8	-0.29938 [-0.29938]	-0.24972 [-0.24971]	-0.20647 [-0.20647]	-0.16689[-0.16689]
1	0 [0]	0 [0]	0 [0]	0 [0]
2	2.0087 [2.0175]	1.4889 [1.4890]	1.1518 [1.1518]	0.8925[0.8925]
3	4.7291 [4.7292]	3.2130 [3.2132]	2.4055 [2.4056]	1.8305[1.8307]

Table 1 : Comparison of HAM and numerical solutions [15] of $f''(0)$ for various values of We and a/c

$TABLE$ 2 ($h'(0)$)

$parameters$	HAM [NUM]	HAM [NUM]	HAM [NUM]	HAM [NUM]
$\frac{a}{c}\downarrow, We \rightarrow$	0	0.2	0.5	1
0.1	0.26275 [0.26278]	0.36382 [0.36384]	0.48622 [0.48624]	0.60812 [0.60812]
0.3	0.60571 [0.60573]	0.67905 [0.67908]	0.75114 [0.75116]	0.82137 [0.82139]
0.8	0.93430 [0.93430]	0.95291 [0.95292]	0.97101 [0.97102]	0.97785 [0.97787]
1	1 [1]	1 [1]	1 [1]	1 [1]

Table 2: Comparison of HAM and numerical solutions [15] of $h'(0)$ for various values of We and a/c

$TABLE$ 3 ($-\theta'(0)$), $Pr = 1$

$parameters$	HAM [NUM]	HAM [NUM]	HAM [NUM]	HAM [NUM]
$\frac{a}{c}\downarrow, We \rightarrow$	0	0.2	0.5	1
0.1	0.60276 [0.60281]	0.61942 [0.61941]	0.63866 [0.63866]	0.66051 [0.66052]
0.3	0.64728 [0.64732]	0.66190 [0.66189]	0.67751 [0.67751]	0.69456 [0.69456]
0.8	0.75710 [0.75709]	0.76197 [0.76193]	0.76651 [0.76650]	0.77109 [0.77109]
1	0.79790 [0.79788]	0.79790 [0.79788]	0.79790 [0.79788]	0.79790 [0.79788]
2	0.97876 [0.97872]	0.95059 [0.95031]	0.92810 [0.92878]	0.90943 [0.90940]

Table 3 : Comparison of HAM and numerical solutions [15] of $(-\theta'(0))$ for various values of We and a/c, when Pr =1

$TABLE$ 4 $(-\theta'(0))$, $Pr = 1.5$

parameters	HAM [NUM]	HAM [NUM]	HAM [NUM]	HAM [NUM]
$\frac{a}{c}\downarrow, We \rightarrow$	0	0.2	0.5	1
0.1	0.77680 [0.77681]	0.79531 [0.79529]	0.81580 [0.81564]	0.83890 [0.83842]
0.3	0.81910 [0.81911]	0.83545 [0.83545]	0.85263 [0.85262]	0.87103 [0.87104]
0.8	0.93274 [0.93306]	0.93835 [0.93869]	0.94356 [0.94390]	0.94847 [0.94904]
1	0.97689 [0.97720]	0.97683 [0.97720]	0.97683 [0.97720]	0.97683 [0.97720]
2	1.1719 [1.1780]	1.1399 [1.1443]	1.1059 [1.1191]	1.0969 [1.09697]

Table 4: Comparison of HAM and numerical solutions [15] of $(-\theta'(0))$ for various values of We and a/c when Pr =1.5

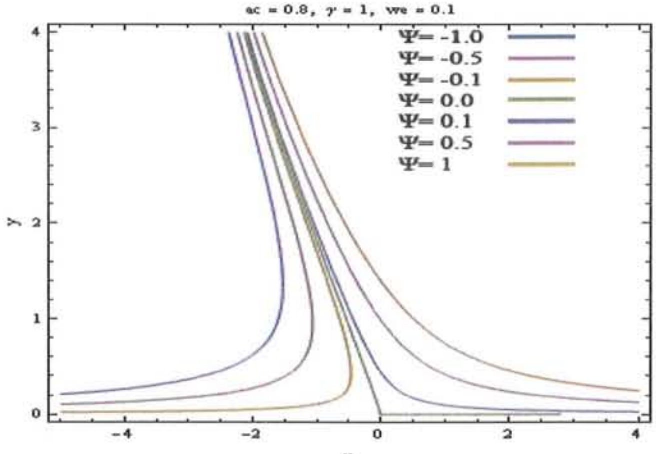

Fig: 2.9 Streamline pattern for $\gamma = 1, We = 0.1$

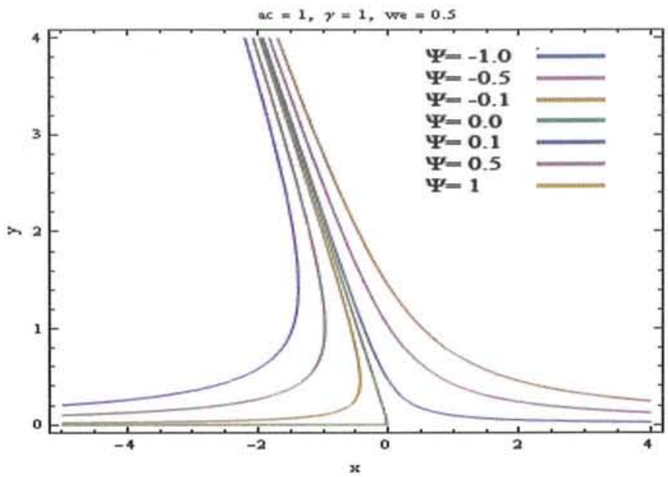

Fig: 2.10 Streamline pattern for $\gamma = 1, We = 0.5$

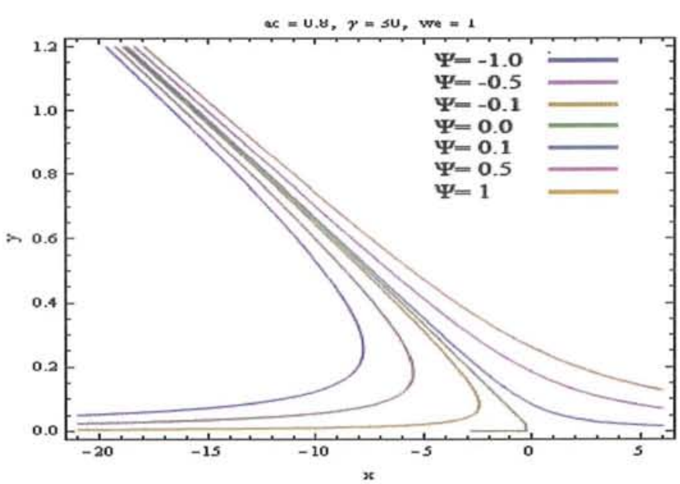

Fig: 2.11 Streamline pattern for $\gamma = 30, We = 1$

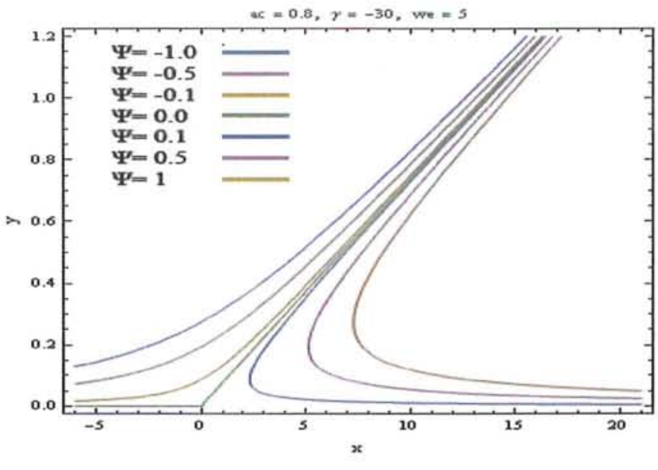

Fig: 2.12 Streamline pattern for $\gamma = -30, We = 5$

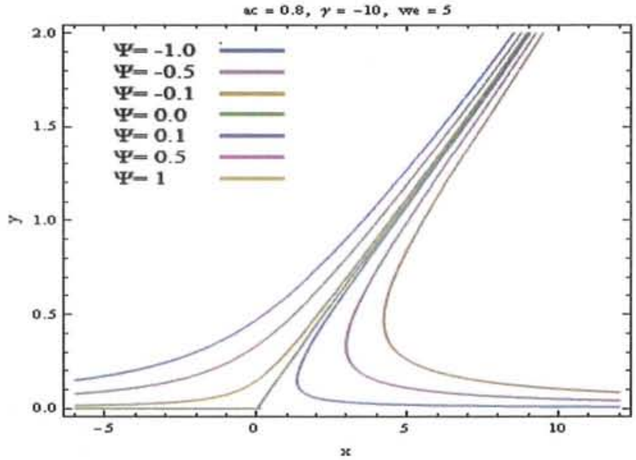

Fig: 2.13 Streamline pattern for $\gamma = -10, We = 5$

Chapter 3

Non-orthogonal stagnation-point flow of a micropolar fluid towards a stretching surface with heat transfer

This chapter emphasizes on the heat transfer analysis for stagnation point flow of a micropolar fluid, over a stretching surface. The governing equations of motion for two dimensional flow are modelled and then simplified with the help of suitable similarity transformations.The reduced nonlinear coupled equations are then solved analytically with the help of homotopy analysis method (HAM).The effects of several flow parameters are examined on the velocity, temperature and micro-rotation profile.The stream lines for the problem are also made.

3.1 Mathematical Formulation

Consider a steady two-dimensional non-orthogonal stagnation point flow towards a stretching surface of a micro polar fluid. In addition heat transfer effects are considered. The flow is governed by the following equations

$$\operatorname{div}\mathbf{V}^* = 0, \tag{3.1}$$

$$\rho\frac{\mathrm{D}\mathbf{V}^*}{\mathrm{Dt}} = \operatorname{div}\mathbf{T}^* + k\nabla^* \times \mathbf{N}, \tag{3.2}$$

$$\rho j\frac{\mathrm{DN}^*}{\mathrm{Dt}} = \gamma\nabla^*(\nabla^*.\mathrm{N}^*) - \gamma\nabla^* \times (\nabla^* \times \mathrm{N}^*) + k\nabla^* \times \mathrm{V}^* - 2k\mathrm{N}^*, \tag{3.3}$$

where

$$\mathrm{V}^* = [u^*(x^*, y^*), v^*(x^*, y^*), 0], \tag{3.4}$$

$$\mathrm{N}^* = [0, 0, N^*(x^*, y^*)]. \tag{3.5}$$

For the given velocity profile

$$\nabla^* \times \mathrm{V}^* = \left(0, 0, \frac{\partial v^*(x^*, y^*)}{\partial x^*} - \frac{\partial u^*(x^*, y^*)}{\partial y^*}\right), \tag{3.6}$$

$$\nabla^* \times \mathrm{N}^* = \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x^*} & \frac{\partial}{\partial y^*} & \frac{\partial}{\partial z^*} \\ 0 & 0 & N^*(x^*, y^*) \end{pmatrix}, \tag{3.7}$$

or

$$\nabla^* \times \mathrm{N}^* = \left(\frac{\partial \mathrm{N}^*}{\partial y^*}, -\frac{\partial \mathrm{N}^*}{\partial x^*}, 0\right), \tag{3.8}$$

and

$$\nabla^* \times (\nabla^* \times \mathrm{N}^*) = \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x^*} & \frac{\partial}{\partial y^*} & \frac{\partial}{\partial z^*} \\ \frac{\partial \mathrm{N}^*}{\partial y^*} & -\frac{\partial \mathrm{N}^*}{\partial x^*} & 0 \end{pmatrix}, \tag{3.9}$$

$$\nabla^* \times (\nabla^* \times \mathrm{N}^*) = \left(0, 0, -\nabla^{*^2}\mathrm{N}^*\right), \tag{3.10}$$

also

$$\nabla^*.\mathrm{N}^* = [\frac{\partial}{\partial x^*}, \frac{\partial}{\partial y^*}, \frac{\partial}{\partial z^*}].[0, 0, N^*(x^*, y^*)], \tag{3.11}$$

or

$$\nabla^*.\mathrm{N}^* = 0, \tag{3.12}$$

where $\mathbf{D}/\mathbf{Dt}$ is the material derivative, V^* and N^* represent the velocity and micro rotation vectors, $p^*(x^*, y^*)$ is the fluid pressure function, $\nu = \mu/\rho$ is the kinematic viscosity, ρ and $j = \nu/\mathrm{c}$ are the density and gyrartion parameters of the fluid, $\gamma = (\mu + k/2)\, j$ [24] and k are the spin gradient viscosity and the vortex viscosity respectively.

Law of conservation of energy is given by

$$\rho\frac{de}{dt} = \sigma^{*}.\mathbf{L} - \operatorname{div}\mathbf{q}+\rho r, \tag{3.13}$$

where $e = C_p\sigma^{*}$, $\mathbf{q} = -k\operatorname{div}\sigma^{*}$ is the heat flux vector, σ^{*} is the fluid temperature, k is the thermal conductivity, C_p is the specific heat and r is the internal heat generation. In the absence of viscous dissipation effect and the radiant heating, the energy equation can be written as

$$\rho C_p[u^{*}\frac{\partial\sigma^{*}}{\partial x^{*}} + v^{*}\frac{\partial\sigma^{*}}{\partial y^{*}}]=k\nabla^{*^2}\sigma^{*}, \tag{3.14}$$

or

$$u^{*}\frac{\partial\sigma^{*}}{\partial x^{*}} + v^{*}\frac{\partial\sigma^{*}}{\partial y^{*}} = \alpha^{*}\nabla^{*^2}\sigma^{*}, \tag{3.15}$$

where $\alpha^{*} = k/\rho C_p$ is the thermal diffusivity of the fluid. Using Eqs. (3.4) to (3.12) in Eqs. (3.1) to (3.3), we get

$$\frac{\partial u^{*}}{\partial x^{*}} + \frac{\partial v^{*}}{\partial y^{*}} = 0, \tag{3.16}$$

$$\begin{aligned} u^{*}\frac{\partial u^{*}}{\partial x^{*}} + v^{*}\frac{\partial u^{*}}{\partial y^{*}} + \frac{1}{\rho}\frac{\partial p^{*}}{\partial x^{*}} &= (\nu+\frac{k}{\rho})\nabla^{*^2}u^{*} + \frac{\alpha_1}{\rho}\{\frac{\partial}{\partial x^{*}}[4(\frac{\partial u^{*}}{\partial x^{*}})^2 \\ &+2\frac{\partial v^{*}}{\partial x^{*}}(\frac{\partial u^{*}}{\partial y^{*}} + \frac{\partial v^{*}}{\partial x^{*}}) + 2u^{*}(\frac{\partial^2 u^{*}}{\partial x*^2}) + 2v^{*} \\ &(\frac{\partial^2 u^{*}}{\partial y*\partial x*})] + \frac{\partial}{\partial y^{*}}[2(\frac{\partial u^{*}}{\partial x^{*}})(\frac{\partial u^{*}}{\partial y^{*}}) \\ &+2(\frac{\partial v^{*}}{\partial x^{*}})(\frac{\partial v^{*}}{\partial y^{*}}) + (u^{*}\frac{\partial}{\partial x^{*}} + v^{*}\frac{\partial}{\partial y^{*}})(\frac{\partial u^{*}}{\partial y^{*}} + \frac{\partial v^{*}}{\partial x^{*}})]\} \\ &+\frac{\alpha_2}{\rho}\frac{\partial}{\partial x^{*}}[4(\frac{\partial u^{*}}{\partial x^{*}})^2 + (\frac{\partial u^{*}}{\partial y^{*}} + \frac{\partial v^{*}}{\partial x^{*}})^2] + \frac{k}{\rho}\frac{\partial N^{*}}{\partial y^{*}}, \end{aligned} \tag{3.17}$$

$$\begin{aligned} u^{*}\frac{\partial v^{*}}{\partial x^{*}} + v^{*}\frac{\partial v^{*}}{\partial y^{*}} + \frac{1}{\rho}\frac{\partial p^{*}}{\partial y^{*}} &= (\nu+\frac{k}{\rho})\nabla^{*^2}v^{*} + \frac{\alpha_1}{\rho}\{\frac{\partial}{\partial x^{*}}[2(\frac{\partial u^{*}}{\partial x^{*}})(\frac{\partial u^{*}}{\partial y^{*}}) + 2(\frac{\partial v^{*}}{\partial x^{*}})(\frac{\partial v^{*}}{\partial y^{*}}) \\ &+(u^{*}\frac{\partial}{\partial x^{*}} + v^{*}\frac{\partial}{\partial y^{*}})(\frac{\partial u^{*}}{\partial y^{*}} + \frac{\partial v^{*}}{\partial x^{*}})] + \frac{\partial}{\partial y^{*}}[4(\frac{\partial v^{*}}{\partial y^{*}})^2 \\ &+2\frac{\partial u^{*}}{\partial y^{*}}(\frac{\partial u^{*}}{\partial y^{*}} + \frac{\partial v^{*}}{\partial x^{*}}) + 2v^{*}(\frac{\partial^2 v^{*}}{\partial x*^2}) + 2u^{*}(\frac{\partial^2 v^{*}}{\partial y*\partial x*})]\} \\ &+\frac{\alpha_2}{\rho}\frac{\partial}{\partial y^{*}}[4(\frac{\partial v^{*}}{\partial y^{*}})^2 + (\frac{\partial u^{*}}{\partial y^{*}} + \frac{\partial v^{*}}{\partial x^{*}})^2] - \frac{k}{\rho}\frac{\partial N^{*}}{\partial x^{*}}, \end{aligned} \tag{3.18}$$

$$u^*\frac{\partial N^*}{\partial x^*}+v^*\frac{\partial N^*}{\partial y^*}=\frac{\gamma}{\rho j}\nabla^{*2}N^*-\frac{k}{\rho j}[2N^*+\frac{\partial u^*}{\partial y^*}-\frac{\partial v^*}{\partial x^*}],$$

$$[u^*\frac{\partial \sigma^*}{\partial x^*}+v^*\frac{\partial \sigma^*}{\partial y^*}]=\alpha^*\nabla^{*2}\sigma^*. \quad (\ldots$$

The corresponding boundary conditions are given by

$$u^*=cx^*,\ v^*=0,\ \sigma^*=\sigma_w,\ N^*(x^*,y^*)=-m_0\frac{\partial u^*}{\partial y^*}\ on\ y^*=0, \quad (\ldots$$

$$u^*=ax^*+by^*,\ \sigma^*=\sigma_\infty,\ N^*(x^*,y^*)\rightarrow\ constant\ as\ y^*\rightarrow\infty, \quad (\ldots$$

where a, b, and c are positive constants with dimensions of inverse time, σ_w is the cons temperature of the plate while the uniform temperature of the ambient fluid is σ_∞. Introdu

$$x = x^*\sqrt{\frac{c}{\nu}},\ y=y^*\sqrt{\frac{c}{\nu}},\ u=\frac{1}{\sqrt{\nu c}}u^*,\ v=\frac{1}{\sqrt{\nu c}}v^*,$$
$$p = \frac{1}{\rho\nu c}p^*,\ \sigma=\frac{\sigma^*-\sigma_\infty}{\sigma_w-\sigma_\infty},\ N=\frac{1}{c}N^*. \quad (3\ldots$$

Eqs. (3.16) to (3.20) take the form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \quad (3\ldots$$

$$\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial x} &= (1+K)\nabla^2 u+We\{\frac{\partial}{\partial x}[4(\frac{\partial u}{\partial x})^2+2\frac{\partial v}{\partial x}(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}) \\ &+2u(\frac{\partial^2 u}{\partial x^2})+2v(\frac{\partial^2 u}{\partial y\partial x})]+\frac{\partial}{\partial y}[2(\frac{\partial u}{\partial x})(\frac{\partial u}{\partial y}) \\ &+2(\frac{\partial v}{\partial x})(\frac{\partial v}{\partial y})+(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y})(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})]\} \\ &+\lambda\frac{\partial}{\partial x}[4(\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})^2]+K\frac{\partial N}{\partial y}, \end{aligned} \quad (3.\ldots$$

$$
\begin{aligned}
u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial y} &= (1+K)\nabla^2 v+We\{\frac{\partial}{\partial x}[2(\frac{\partial u}{\partial x})(\frac{\partial u}{\partial y})+2(\frac{\partial v}{\partial x})(\frac{\partial v}{\partial y})\\
&+(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y})(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})]+\frac{\partial}{\partial y}[4(\frac{\partial v}{\partial y})^2\\
&+2\frac{\partial u}{\partial y}(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})+2v(\frac{\partial^2 v}{\partial x^2})+2u(\frac{\partial^2 v}{\partial y\partial x})]\}\\
&+\lambda\frac{\partial}{\partial y}[4(\frac{\partial v}{\partial y})^2+(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})^2]-K\frac{\partial N}{\partial x}, \qquad (3.26)
\end{aligned}
$$

$$
u\frac{\partial N}{\partial x}+v\frac{\partial N}{\partial y}=(1+\frac{K}{2})\nabla^2 N-K[2N+\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}], \qquad (3.27)
$$

$$
\Pr[u\frac{\partial \sigma}{\partial x}+v\frac{\partial \sigma}{\partial y}]=\nabla^2\sigma, \qquad (3.28)
$$

where $We=\alpha_1 C/\rho\nu$ is the Weissenberg number, $\lambda=\alpha_2 C/\rho\nu$, $K=k/\mu$ is the material parameter and $\Pr=\mu C_p/k$ is the prandtl number.

Introducing the stream function relations

$$
u=\frac{\partial \Psi}{\partial y}, v=-\frac{\partial \Psi}{\partial x}. \qquad (3.29)
$$

Substitution of Eq. (3.29) into Eqs. (3.25) to (3.28) and elimination of pressure from the resulting equations using $P_{xy}=P_{yx}$ yield

$$
(1+K)\nabla^4\Psi-We\frac{\partial(\Psi,\nabla^4\Psi)}{\partial(x,y)}+\frac{\partial(\Psi,\nabla^2\Psi)}{\partial(x,y)}+K\nabla^2 N=0, \qquad (3.30)
$$

$$
\frac{\partial \Psi}{\partial y}\frac{\partial N}{\partial x}-\frac{\partial \Psi}{\partial x}\frac{\partial N}{\partial y}=(1+\frac{K}{2})\nabla^2 N-K[2N+\nabla^2\Psi(x,y)], \qquad (3.31)
$$

$$
\Pr[\frac{\partial \Psi}{\partial y}\frac{\partial \sigma}{\partial x}-\frac{\partial \Psi}{\partial x}\frac{\partial \sigma}{\partial y}]=\nabla^2\sigma, \qquad (3.32)
$$

The corresponding non-dimensional boundary conditions in terms of the stream function $\Psi(x,y)$ are given by

$$
\Psi=0, \frac{\partial \Psi}{\partial y}=x,\ N=-m_0\nabla^2\Psi,\ \sigma=1\ at\ y=0, \qquad (3.33)
$$

$$
\Psi=\frac{a}{c}xy+\frac{1}{2}\gamma_1 y^2,\ N(x,y)\rightarrow\ constant,\ \sigma=0\ at\ y\rightarrow\infty, \qquad (3.34)
$$

where $\gamma_1=b/c$ represents the shear in the stream.

We seek solutions of Eqs. (3.30) to (3.32) of the form

$$\Psi(x,y) = x\ f(y) + g(y),\ N(x,y) = x\ J(y) + S(y),\ \sigma = \theta(y), \tag{3.35}$$

where the functions $f(y)$ and $g(y)$ are referring to as the normal and tangential component of the flow, while $J(y)$ and $S(y)$ are the normal and tangential components of the micro rotation profile respectively and prime denotes differentiation with respect to y. Substituting Eq. (3.35) in Eqs. (3.30) to (3.32), we obtain the following ordinary differential equations after one integration for f , g with the corresponding boundary conditions

$$(1+K)f''' + ff'' - f'^2 - We(f\ f^{IV} - 2f'f''' + (f'')^2) + KJ' + C_1 = 0, \tag{3.36}$$

$$(1+K)g''' + fg'' - f'g' - We(f\ g^{IV} - f'g''' + f''g'' - f'''g') + KS' + C_2 = 0, \tag{3.37}$$

$$(1+\frac{K}{2})J'' - f'J + fJ' - K(f'' + 2J) = 0, \tag{3.38}$$

$$(1+\frac{K}{2})S'' - g'J + fS' - K(g'' + 2S) = 0, \tag{3.39}$$

$$\theta'' + \Pr f\ \theta' = 0, \tag{3.40}$$

$$f(0) = 0,\ f'(0) = 0,\ f'(\infty) = \frac{a}{c},\ J(0) = -m_0 f''(0),\ S(0) = -m_0 g''(0), \tag{3.41}$$

$$g(0) = 0,\ g'(0) = 0,\ g''(\infty) = \gamma_1,\ J(\infty) = 0,\ S(\infty) = -\frac{\gamma_1}{2}, \tag{3.42}$$

$$\theta(0) = 1, \theta(\infty) = 0, \tag{3.43}$$

where prime denotes differentiation with respect to y and C_1, C_2 are constants of integration.

Taking the limit $y \to \infty$ in Eq. (3.36) and using the boundary condition $f'(\infty) = a/c$, we get $C_1 = a^2/c^2$. An analysis of the boundary layer Eq. (3.36) implies that $f(y)$ behaves as $f(y) = (a/c)\ y + A$ as $y \to \infty$, where $A = A(We, a/c)$ is a constant that accounts for the boundary layer displacement. Taking the limit as $y \to \infty$ in Eq. (3.37) and using the infinity boundary condition $g''(\infty) = \gamma_1$ we get that $C_2 = -A\gamma_1$. Thus Eqs. (3.36) to (3.40) become

$$(1+K)f''' + ff'' - (f')^2 - We(f\ f^{IV} - 2f'f''' + (f'')^2) + KJ' + \frac{a^2}{c^2} = 0, \tag{3.44}$$

$$(1+K)g^{'''}+fg^{''}-f'g^{'}-We(f\,g^{IV}-f^{'}g^{'''}+f''g''-f^{'''}g')+KS^{'}-A\gamma_1=0, \tag{3.45}$$

$$(1+\frac{K}{2})J^{''}-f'J+fJ'-K(f^{''}+2J)=0, \tag{3.46}$$

$$(1+\frac{K}{2})S^{''}-g'J+fS'-K(g^{''}+2S)=0, \tag{3.47}$$

$$\theta^{''}+\Pr f\,\theta^{'}=0. \tag{3.48}$$

Introducing

$$g^{'}(y)=\gamma_1 h(y),\; S(y)=\gamma_1 s(y). \tag{3.49}$$

Using Eq.(3.49) in Eqs. (3.45) and (3.47), we have

$$(1+K)h^{''}+fh^{'}-f^{'}h-We(fh^{'''}-f^{'}h^{''}+f^{''}h^{'}-f^{'''}h)+Ks^{'}-A=0, \tag{3.50}$$

$$(1+\frac{K}{2})s^{''}-hJ+fs'-K(h^{'}+2s)=0, \tag{3.51}$$

with the boundary conditions

$$h(0)=0,\; h^{'}(\infty)=1, \tag{3.52}$$

$$s(0)=-m_0\,h'(0),\; s(\infty)=-\frac{1}{2}. \tag{3.53}$$

The Eqs. (3.44) to (3.48) are coupled nonlinear differential equations, to find the analytic solutions we use Homotopy analysis method (HAM),which is described in next section.

3.2 Homotopy analysis solution

We express $f(\eta)$, $h(\eta), \theta(\eta)$, $J(\eta)$, and $s(\eta)$ by a set of base functions

$$\{\eta^k \exp(-n\eta)|k\geq 0,\; n\geq 0\}, \tag{3.54}$$

in the forms

$$f(\eta)=a_{0,0}^0+\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_{m,n}^k\eta^k\exp(-n\eta), \tag{3.55}$$

$$h(\eta)=c_{0,0}^0+\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}c_{m,n}^k\eta^k\exp(-n\eta), \tag{3.56}$$

$$\theta(\eta) = b_{0,0}^0 + \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} b_{m,n}^k \eta^k \exp(-n\eta), \tag{3.57}$$

$$J(\eta) = d_{0,0}^0 + \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} d_{m,n}^k \eta^k \exp(-n\eta), \tag{3.58}$$

$$s(\eta) = e_{0,0}^0 + \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} e_{m,n}^k \eta^k \exp(-n\eta), \tag{3.59}$$

in which $a_{m,n}^k$, $b_{m,n}^k$, $c_{m,n}^k$ $d_{m,n}^k$, and $e_{m,n}^k$ are the coefficients. For the HAM solutions, the initial guesses $f_0(\eta)$, $h_0(\eta)$, $\theta_0(\eta)$, $J_0(\eta)$ and $s_0(\eta)$ are given by

$$f_0(\eta) = (1 - \frac{a}{c}) * (1 - \exp(-\eta)) + \frac{a}{c}\eta, \tag{3.60}$$

$$h_0(\eta) = 1 - \exp(-\eta) + \eta, \tag{3.61}$$

$$\theta_0(\eta) = \exp(-\eta), \tag{3.62}$$

$$J_0(\eta) = -m_0 f_0''(0), \tag{3.63}$$

$$s_0(\eta) = -m_0 h_0'(0), \tag{3.64}$$

with the auxiliary linear operators of the form

$$\pounds_f = \frac{d^3 f}{d\eta^3} + \frac{d^2 f}{d\eta^2}, \tag{3.65}$$

$$\pounds_h = \frac{d^3 h}{d\eta^3} + \frac{d^2 h}{d\eta^2}, \tag{3.66}$$

$$\pounds_\theta = \frac{d^2 \theta}{d\eta^2} + \frac{d\theta}{d\eta}, \tag{3.67}$$

$$\pounds_J = \frac{d^2 J}{d\eta^2} + \frac{dJ}{d\eta}, \tag{3.68}$$

$$\pounds s = \frac{d^2 s}{d\eta^2} + \frac{ds}{d\eta}, \tag{3.69}$$

and the following properties

$$\pounds_f[C_1 + C_2\eta + C_3\exp(-\eta)] = 0, \tag{3.70}$$

$$\pounds_h[C_4 + C_5\eta + C_6\exp(-\eta)] = 0, \tag{3.71}$$

$$\pounds_\theta[C_7 + C_8\exp(-\eta)] = 0, \tag{3.72}$$

$$\pounds_J[C_9 + C_{10}\exp(-\eta)] = 0, \tag{3.73}$$

$$\pounds s[C_{11} + C_{12}\exp(-\eta)] = 0, \tag{3.74}$$

Where C_i $(i = 1 - 12)$ are arbitrary constants.

3.3 Zero order deformation equation

If $p \in [0, 1]$ is an embedding parameter and h_f, h_h, h_θ, h_J and h_s indicate the non zero parameters respectively, then the zeroth order deformation of problem are

$$(1-p)\pounds_f[\hat{f}(\eta;p) - \hat{f}_0(\eta)] = ph_f N_f\ [\hat{f}(p;\eta),\ \hat{h}(p;\eta),\ \hat{\theta}(p;\eta),\ \hat{J}(\eta;p),\ \hat{s}(p;\eta)], \tag{3.75}$$

$$(1-p)\pounds_h[\hat{h}(\eta;p) - \hat{h}_0(\eta)] = ph_h N_h\ [\hat{f}(p;\eta),\ \hat{h}(p;\eta),\ \hat{\theta}(p;\eta),\ \hat{J}(\eta;p),\ \hat{s}(p;\eta)], \tag{3.76}$$

$$(1-p)\pounds_\theta[\hat{\theta}(\eta;p) - \hat{\theta}_0(\eta)] = ph_\theta N_\theta\ [\hat{f}(p;\eta),\ \hat{h}(p;\eta),\ \hat{\theta}(p;\eta),\ \hat{J}(\eta;p),\ \hat{s}(p;\eta)], \tag{3.77}$$

$$(1-p)\pounds_J[\hat{J}(\eta;p) - \hat{J}_0(\eta)] = ph_J N_J\ [\hat{f}(p;\eta),\ \hat{h}(p;\eta),\ \hat{\theta}(p;\eta),\ \hat{J}(\eta;p),\ \hat{s}(p;\eta)], \tag{3.78}$$

$$(1-p)\pounds s[\hat{s}(\eta;p) - \hat{s}_0(\eta)] = ph_s N_s\ [\hat{f}(p;\eta),\ \hat{h}(p;\eta),\ \hat{\theta}(p;\eta),\ \hat{J}(\eta;p),\ \hat{s}(p;\eta)], \tag{3.79}$$

with following conditions

$$\hat{f}(0;p) = 0,\ \hat{f}'(0;p) = 1,\ \hat{\theta}(0;p) = 0,\ \hat{h}(0;p) = 0,\ \hat{J}(0;p) = -m_0 f_0''(0),\ \hat{s}(0;p) = -m_0 h_0'(0), \tag{3.80}$$

$$\hat{f}'(\infty;p) = \frac{a}{c}, \hat{\theta}(\infty;p) = 1,\ \hat{h}'(\infty;p) = 1, \hat{J}(\infty;p) = 0,\ \hat{s}(\infty;p) = -\frac{1}{2}. \tag{3.81}$$

in which non linear operators N_f, N_h, N_θ, N_J and N_s are

$$\begin{aligned} N_f[\hat{f}(p;\eta),\hat{h}(p;\eta)] &= (1+K)\frac{\partial^3\hat{f}(\eta;p)}{\partial\eta^3}+\hat{f}(\eta;p)\frac{\partial^2\hat{f}(\eta;p)}{\partial\eta^2}-(\frac{\partial\hat{f}(\eta;p)}{\partial\eta})^2 \\ &-We(\hat{f}(\eta;p)\frac{\partial^4\hat{f}(\eta;p)}{\partial\eta^4}-2\frac{\partial\hat{f}(\eta;p)}{\partial\eta}\frac{\partial^3\hat{f}(\eta;p)}{\partial\eta^3} \\ &+(\frac{\partial^2\hat{f}(\eta;p)}{\partial\eta^2})^2)+K\frac{\partial\hat{J}(\eta;p)}{\partial\eta}+(\frac{a}{c})^2, \end{aligned} \quad (3.82)$$

$$\begin{aligned} N_h[\hat{f}(p;\eta),\hat{h}(p;\eta)] &= (1+K)\frac{\partial^2\hat{h}(p;\eta)}{\partial\eta^2}+\hat{f}(\eta;p)\frac{\partial\hat{h}(p;\eta)}{\partial\eta}-\frac{\partial\hat{f}(\eta;p)}{\partial\eta}\hat{h}(p;\eta) \\ &-We(\hat{f}(\eta;p)\frac{\partial^3\hat{h}(p;\eta)}{\partial\eta^3}-\frac{\partial\hat{f}(\eta;p)}{\partial\eta}\frac{\partial^2\hat{h}(p;\eta)}{\partial\eta^2}+\frac{\partial^2\hat{f}(\eta;p)}{\partial\eta^2}\frac{\partial\hat{h}(p;\eta)}{\partial\eta} \\ &-\frac{\partial^3\hat{f}(\eta;p)}{\partial\eta^3}\hat{h}(p;\eta))+K\frac{\partial\hat{s}(p;\eta)}{\partial\eta}-A, \end{aligned} \quad (3.83)$$

$$N_\theta[\hat{f}(p;\eta),\hat{\theta}(p;\eta)]=\frac{\partial^2\hat{\theta}(\eta;p)}{\partial\eta^2}+\Pr\hat{f}(\eta;p)\frac{\partial\hat{\theta}(\eta;p)}{\partial\eta}, \quad (3.84)$$

$$\begin{aligned} N_J[\hat{f}(p;\eta),\hat{\theta}(p;\eta),\hat{h}(p;\eta),\hat{J}(p;\eta),\hat{s}(p;\eta)] &= (1+\frac{K}{2})\frac{\partial^2\hat{J}(\eta;p)}{\partial\eta^2}-\frac{\partial\hat{f}(p;\eta)}{\partial\eta}\hat{J}(p;\eta)+\hat{f}(p;\eta) \\ &\frac{\partial\hat{J}(p;\eta)}{\partial\eta}-K(\frac{\partial^2\hat{f}(\eta;p)}{\partial\eta^2}+2\hat{J}(p;\eta)), \end{aligned} \quad (3.85)$$

$$\begin{aligned} N_s[\hat{f}(p;\eta),\hat{\theta}(p;\eta),\hat{h}(p;\eta),\hat{J}(p;\eta),\hat{s}(p;\eta)] &= (1+\frac{K}{2})\frac{\partial^2\hat{s}(\eta;p)}{\partial\eta^2}-\hat{h}(p;\eta)\hat{J}(p;\eta)+\hat{f}(p;\eta) \\ &\frac{\partial\hat{s}(p;\eta)}{\partial\eta}-K(\frac{\partial\hat{h}(\eta;p)}{\partial\eta}+2\hat{s}(p;\eta)), \end{aligned} \quad (3.86)$$

Obviously

$$\hat{f}(\eta;0)=f_0(\eta),\ \hat{f}(\eta;1)=f(\eta), \quad (3.87)$$

$$\hat{h}(\eta;0)=h_0(\eta),\ \hat{h}(\eta;1)=h(\eta), \quad (3.88)$$

$$\hat{\theta}(\eta;0)=\theta_0(\eta),\ \hat{\theta}(\eta;1)=\theta(\eta), \quad (3.89)$$

$$\hat{J}(\eta;0)=J_0(\eta),\ \hat{J}(\eta;1)=J(\eta), \quad (3.90)$$

$$\hat{s}(\eta;0) = s_0(\eta),\ \hat{s}(\eta;1) = s(\eta), \tag{3.91}$$

As p changes from 0 to 1, $f(\eta;p)$, $h(\eta;p)$, $\theta(\eta;p)$, $J(\eta;p)$ and $s(\eta;p)$ vary from the initial guesses $f_0(\eta)$, $h_0(\eta)$, $\theta_0(\eta)$, $J_0(\eta)$,and $s_0(\eta)$, to final solutions $f(\eta)$, $h(\eta)$, $\theta(\eta)$, $J(\eta)$, and $s(\eta)$ respectively. Making the assumption that the auxiliary parameters h_f , h_h, h_θ, h_J, and h_s are chosen so properly that the Taylor series of $f(\eta;p), h(\eta;p)$, $\theta(\eta;p)$, $J(\eta;p)$,and $s(\eta;p)$ expanded with respect to embedding parameters converges at $p=1$.Thus we can write as

$$\hat{f}(\eta;p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta)p^m, \tag{3.92}$$

$$\hat{h}(\eta;p) = h_0(\eta) + \sum_{m=1}^{\infty} h_m(\eta)p^m, \tag{3.93}$$

$$\hat{\theta}(\eta;p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta)p^m, \tag{3.94}$$

$$\hat{J}(\eta;p) = J_0(\eta) + \sum_{m=1}^{\infty} J_m(\eta)p^m, \tag{3.95}$$

$$\hat{s}(\eta;p) = s_0(\eta) + \sum_{m=1}^{\infty} s_m(\eta)p^m, \tag{3.96}$$

where

$$f_m(\eta) = \frac{1}{m!}\frac{\partial^m f(\eta;p)}{\partial\eta^m}|_{p=0}, \tag{3.97}$$

$$h_m(\eta) = \frac{1}{m!}\frac{\partial^m h(\eta;p)}{\partial\eta^m}|_{p=0}, \tag{3.98}$$

$$\theta_m(\eta) = \frac{1}{m!}\frac{\partial^m \theta(\eta;p)}{\partial\eta^m}|_{p=0}, \tag{3.99}$$

$$J_m(\eta) = \frac{1}{m!}\frac{\partial^m J(\eta;p)}{\partial\eta^m}|_{p=0}, \tag{3.100}$$

$$s_m(\eta) = \frac{1}{m!}\frac{\partial^m s(\eta;p)}{\partial\eta^m}|_{p=0}, \tag{3.101}$$

By substituting the Eqs. (3.97) to (3.101) in Eqs.(3.92) to (3.96) , we get

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \tag{3.102}$$

$$h(\eta) = h_0(\eta) + \sum_{m=1}^{\infty} h_m(\eta), \tag{3.103}$$

$$\theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta), \tag{3.104}$$

$$J(\eta) = J_0(\eta) + \sum_{m=1}^{\infty} J_m(\eta), \tag{3.105}$$

$$s(\eta) = s_0(\eta) + \sum_{m=1}^{\infty} s_m(\eta), \tag{3.106}$$

mth order deformation Eqs. are written as

$$\pounds_f[f_m(\eta) - X_m f_{m-1}(\eta)] = h_f R_m^f(\eta), \tag{3.107}$$

$$\pounds_h[h_m(\eta) - X_m h_{m-1}(\eta)] = h_h R_m^h(\eta), \tag{3.108}$$

$$\pounds_\theta[\theta_m(\eta) - X_m \theta_{m-1}(\eta)] = h_\theta R_m^\theta(\eta), \tag{3.109}$$

$$\pounds_J[J_m(\eta) - X_m J_{m-1}(\eta)] = h_J R_m^J(\eta), \tag{3.110}$$

$$\pounds_s[s_m(\eta) - X_m s_{m-1}(\eta)] = h_s R_m^s(\eta), \tag{3.111}$$

The Corresponding boundary conditions for mth deformation problems are

$$f_m(0) = 0,\ f_m^{'}(0) = 1,\ \theta_m(0) = 0,\ h_m(0) = 0,\ J_m(0) = -m_0 f_m^{''}(0),\ s_m(0) = -m_0 h_m^{'}(0), \tag{3.112}$$

$$f_m^{'}(\infty) = \frac{a}{c},\ \theta_m(\infty) = 1,\ h_m^{'}(\infty) = 1,\ J_m(\infty) = 0,\ s_m(\infty) = -\frac{1}{2}, \tag{3.113}$$

where

$$\begin{aligned} R_m^f(\eta) \quad = \quad & (1+K) f_{m-1}^{'''} + \sum_{k=0}^{m-1} f_k f_{m-1-k}^{''} - \sum_{k=0}^{m-1} f_k^{'} f_{m-1-k}^{'} - We(\sum_{k=0}^{m-1} f_k f_{m-1-k}^{IV} - 2\sum_{k=0}^{m-1} f_k^{'} f_{m-1-k}^{'''} \\ & + \sum_{k=0}^{m-1} f_k^{''} f_{m-1-k}^{''}) + K J_{m-1}^{'} + (\frac{a}{c})^2, \end{aligned} \tag{3.114}$$

$$R_m^h(\eta) = (1+K)h''_{m-1} + \sum_{k=0}^{m-1} f_k h'_{m-1-k} - \sum_{k=0}^{m-1} f'_k h_{m-1-k} - We(\sum_{k=0}^{m-1} f_k h'''_{m-1-k} - \sum_{k=0}^{m-1} f'_k h''_{m-1-k} + \sum_{k=0}^{m-1} f''_k h'_{m-1-k} - \sum_{k=0}^{m-1} f'''_k h_{m-1-k}) + Ks'_{m-1} - A, \quad (3.115)$$

$$R_m^\theta(\eta) = \theta''_{m-1} + \Pr \sum_{k=0}^{m-1} f_k \theta'_{m-1-k}, \quad (3.116)$$

$$R_m^J(\eta) = (1+\frac{K}{2})J''_{m-1} - \sum_{k=0}^{m-1} J_k f'_{m-1-k} + \sum_{k=0}^{m-1} f_k J'_{m-1-k} - K(2J_{m-1} + f''_{m-1}), \quad (3.117)$$

$$R_m^s(\eta) = (1+\frac{K}{2})s''_{m-1} - \sum_{k=0}^{m-1} J_k h_{m-1-k} + \sum_{k=0}^{m-1} f_k s'_{m-1-k} - K(2s_{m-1} + h'_{m-1}), \quad (3.118)$$

in which

$$\chi_m = \begin{cases} 0, m \leq 1 \\ 1, m > 1. \end{cases} \quad (3.119)$$

The general solutions of Eqs. (3.107) to (3.111) can be written as

$$f_m(\eta) = f_m^*(\eta) + C_1 + C_2\eta + C_3 \exp(-\eta), \quad (3.120)$$

$$h_m(\eta) = h_m^*(\eta) + C_4 + C_5\eta + C_6 \exp(-\eta), \quad (3.121)$$

$$\theta_m(\eta) = \theta_m^*(\eta) + C_7 + C_8 \exp(-\eta), \quad (3.122)$$

$$J_m(\eta) = J_m^*(\eta) + C_9 + C_{10} \exp(-\eta), \quad (3.123)$$

$$s_m(\eta) = s_m^*(\eta) + C_{11} + C_{12} \exp(-\eta), \quad (3.124)$$

Where C_i $(i = 1 - 12)$ are constants.

3.4 Convergence of the HAM solution

Convergence of series solutions depend upon the non-zero auxiliary parameters $\hbar_f$, $\hbar_h$, $\hbar_\theta$,$\hbar_J$, and $\hbar_s$ which can adjust and control the convergence of the HAM solutions. In order to see the range of admissible values of $\hbar_f$, $\hbar_h$, $\hbar_\theta$, $\hbar_J$, and $\hbar_s$ and the $\hbar - curve$ of the functions $f^{''}(0)$, $h''(0)$, $\theta'(0)$, $J'(0)$, and $s'(0)$ are sketched for 10th-order of approximations in Fig. (3.1) below. It is found that the range of admissible values of $\hbar_f$, $\hbar_h$, $\hbar_\theta$, $\hbar_J$ and $\hbar_s$ are $-1 \leq \hbar_f \leq -0.3$, $-1.1 \leq \hbar_h \leq -0.4$, $-1.3 \leq \hbar_\theta \leq -0.5$, $-1.1 \leq \hbar_J \leq -0.3$, $-0.9 \leq \hbar_s \leq -0.4$,

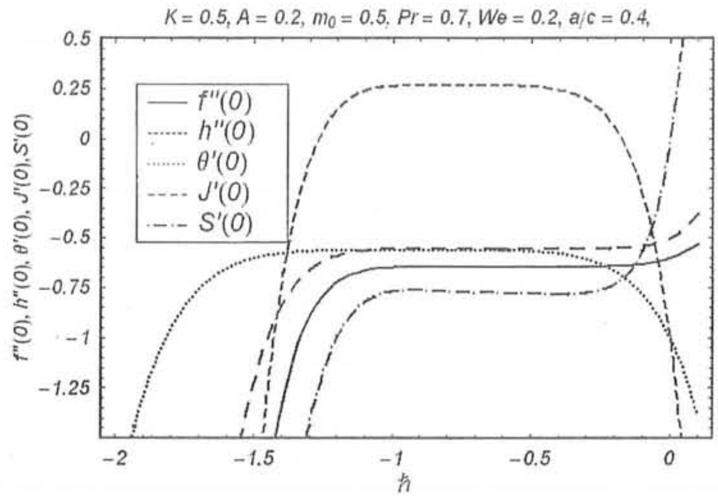

Fig:3.1 $\hbar$- Curves for f, h, θ, J, s

3.5 Results and discussion

This section highlights the variations of arising physical parameters on the fluid flow. Figs. (3.2) to (3.16). are plotted for the velocity field, temperature field, micro-rotation and the stream lines.Fig. (3.2) and (3.3) are plotted to show the effects of material parameter K on velocity f' when $m_0 = 0$ (strong concentration)and $m_0 = 1/2$ (weak concentration) respectively. we notice from these Figures that as the value of K increases, values of f' near the wall are increasing. consequently the velocity gradient at the wall increases as K increases. Figs. (3.4) and (3.5) gives the variation of K on h' for $m_0 = 0$ and $m_0 = 1/2$ respectively. These Figures shows that initially h' decreases and after $y = 1$, h' increase with an increase in K. The variation of K on the temperature is discussed in Figs. (3.6) and (3.7). It is found that θ is a decreasing function of K for $m_0 = 0$, where as it is an increasing function of K when $m_0 = 1/2$. The effect of K on the micro- rotation profile J is shown in Fig. (3.8). This Fig. suggests that the micro-rotation

J increases as K increases for the case $m_0 = 0$, the peak value of micro-rotation occurs near the wall then decrease monotonically to zero as y increases. However for $m_0 = 1/2$ ($Fig.(3.9)$), the micro-rotation J decrease continuously from its maximum value at the wall to zero far from the wall. Figs. (3.10) and (3.11) are sketched for micro-rotation profile $-s$. Fig. (3.10) express that the magnitude of the micro-rotation component s increases as y increases and reaches its maximum value that is 1/2 far away from the wall. where as for $m_0 = 1/2$ ($Fig.(3.11)$), the micro-rotation s decrease continuously from its maximum value at the wall to zero far from the wall.The stream line patterns for the oblique flows are shown in Figs. (3.12) to (3.16). The stream line $\Psi = 0$ meets the wall $y = 0$, at $x = x_0$, where x_0 is the point of stagnation and zero skin friction. It can be seen from these Figures that the stagnation point is at the left of the origin for all values of $K \neq 0$ (micropolar fluid), and the magnitude of x_0 increases as K increases. The shifting of x_0 depends upon the magnitude of K and m_0.

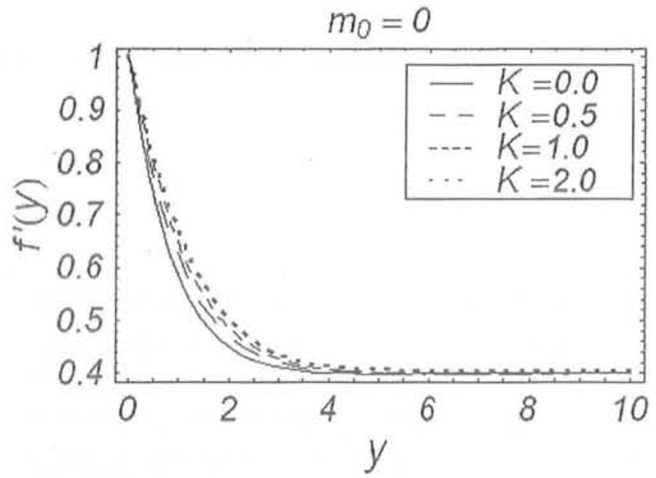

Fig: 3.2 Velocity profile f' for various values of K when $m_0 = 0$

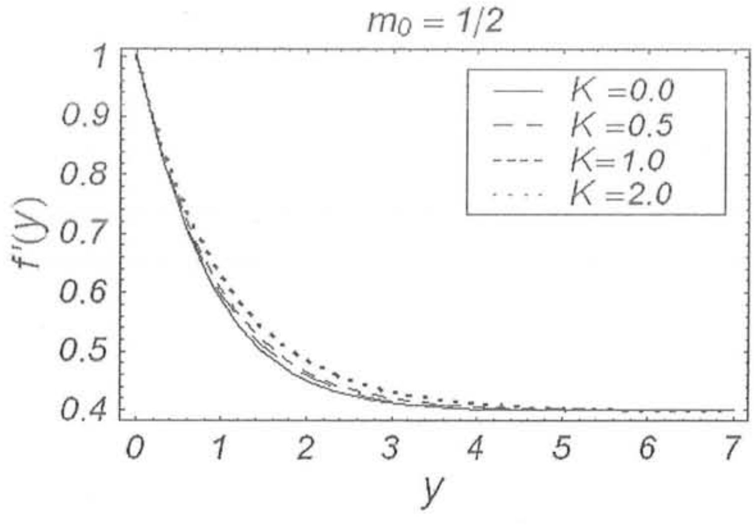

Fig: 3.3 Velocity profile f' for various values of K when

$m_0 = 0.5$

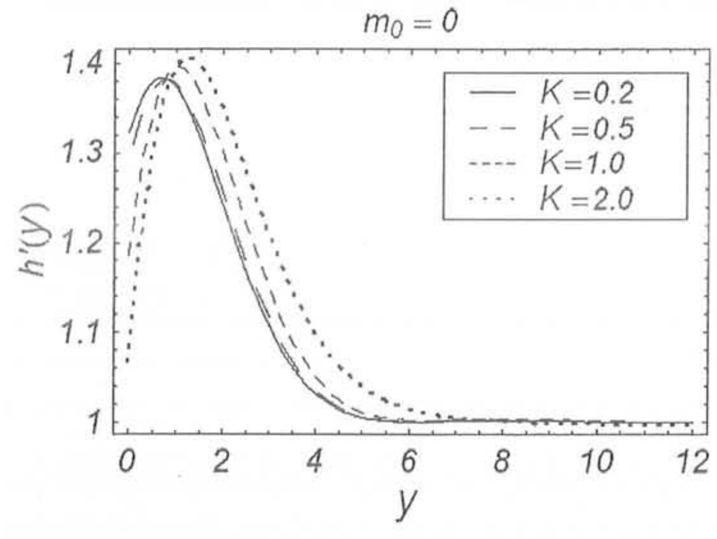

Fig: 3.4Velocity profile h' for various values of K when $m_0 = 0$

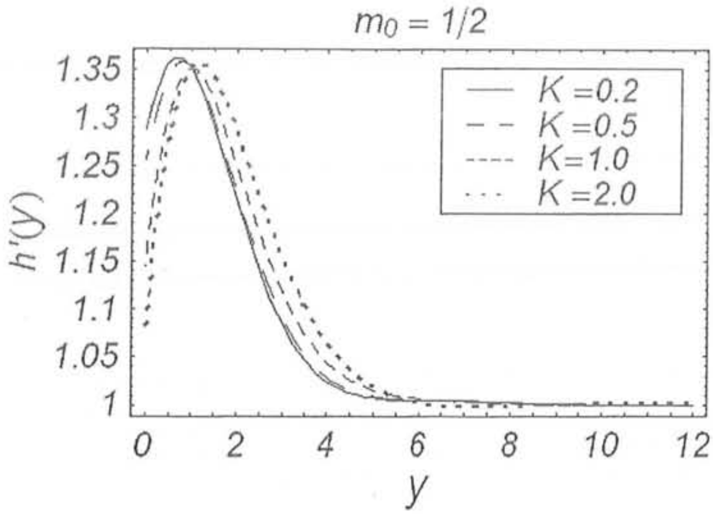

Fig: 3.5 Velocity profile h' for various values of K when

$m_0 = 0.5$

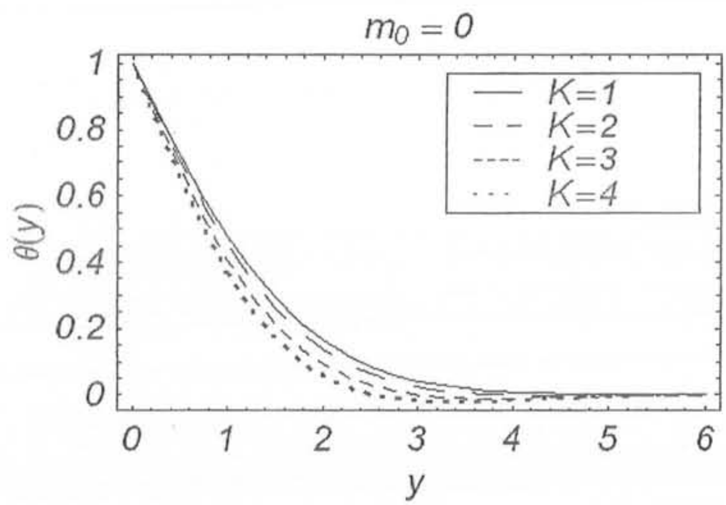

Fig: 3.6 Temprature profile θ for various values of K when

$m_0 = 0$

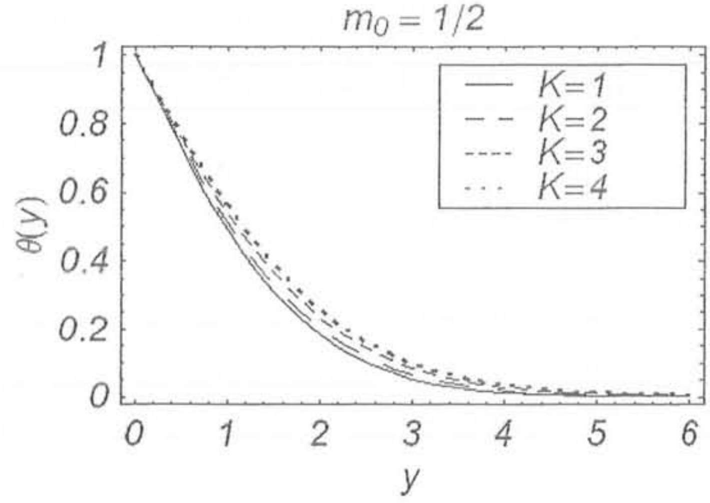

Fig: 3.7 Temprature profile θ for various values of K when

$m_0 = 0.5$

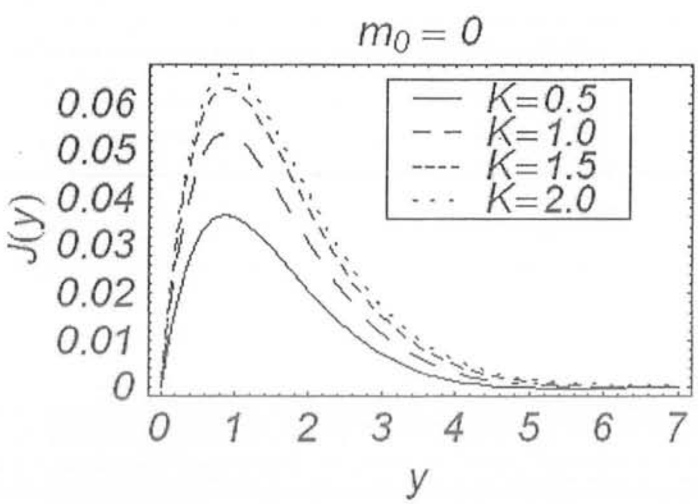

Fig: 3.8 Micro-rotation profile J for various values of K when

$m_0 = 0$

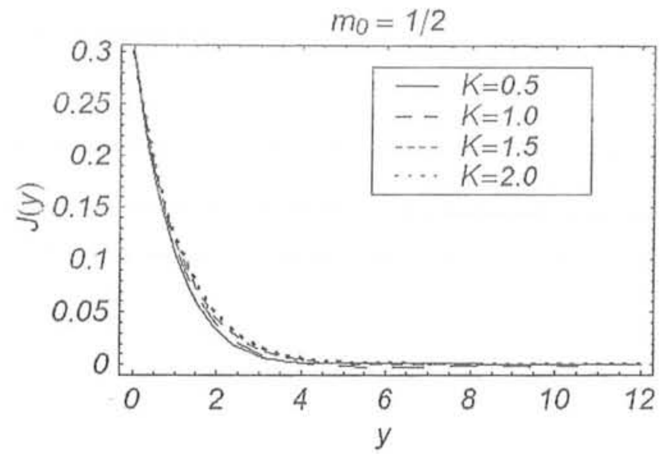

Fig: 3.9 Micro-rotation profile J for various values of K when $m_0 = 0.5$

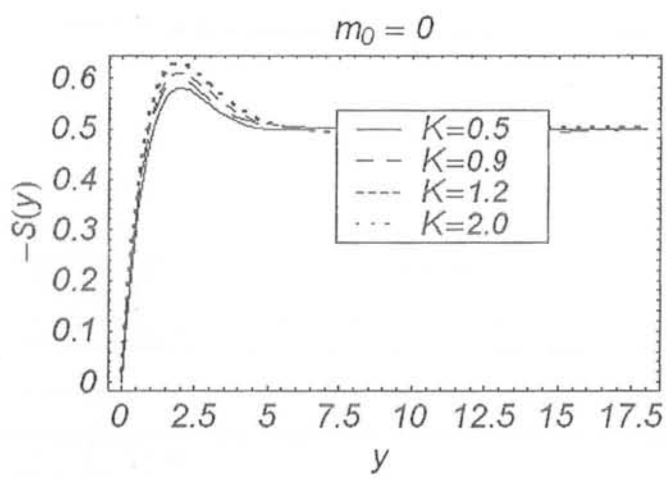

Fig: 3.10 Micro-rotation profile s for various values of K when $m_0 = 0$

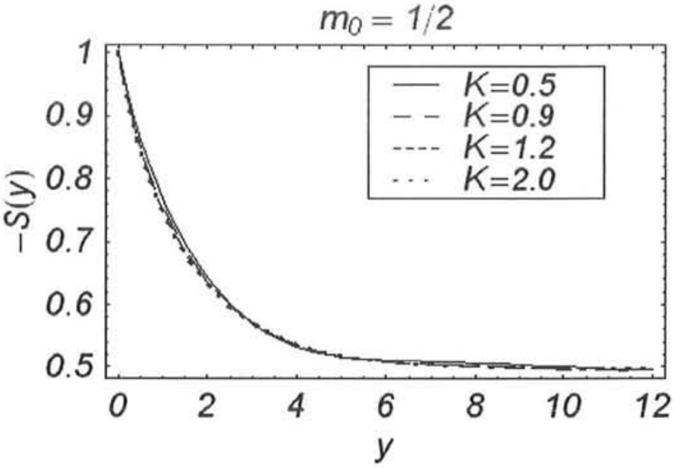

Fig: 3.11 Micro-rotation profile s for various values of K when

$m_0 = 0.5$

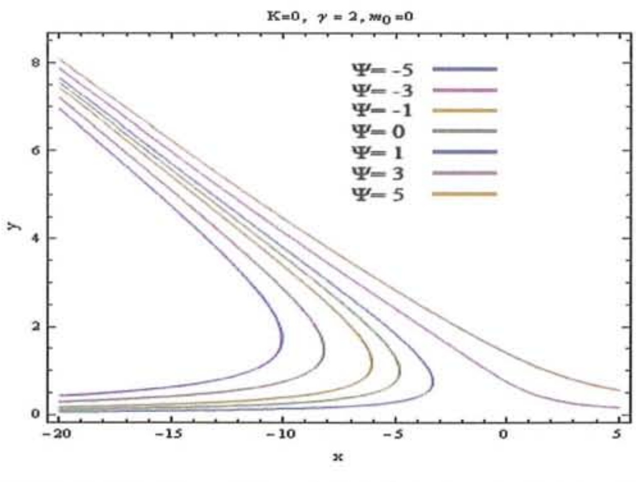

Fig: 3.12 Streamline pattern flow for $K = 0$, when $m_0 = 0$

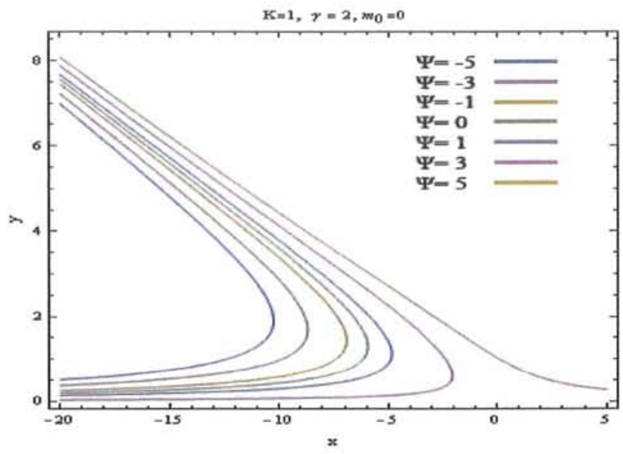

Fig: 3.13 Streamline pattern flow for $K = 1$, when $m_0 = 0$

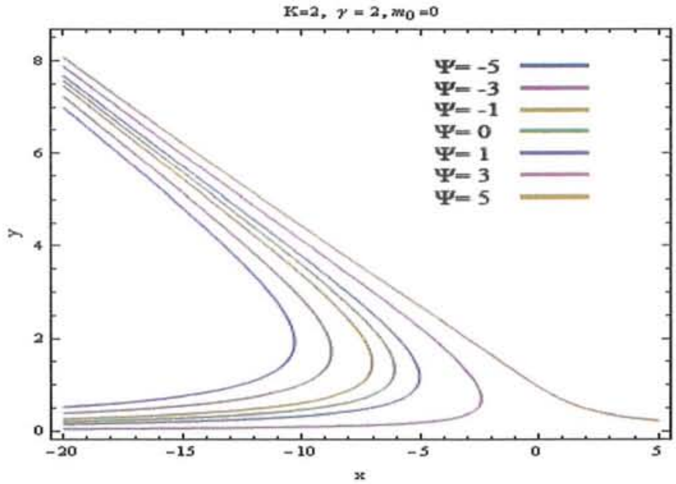

Fig: 3.14 Streamline pattern flow for $K = 2$, when $m_0 = 0$

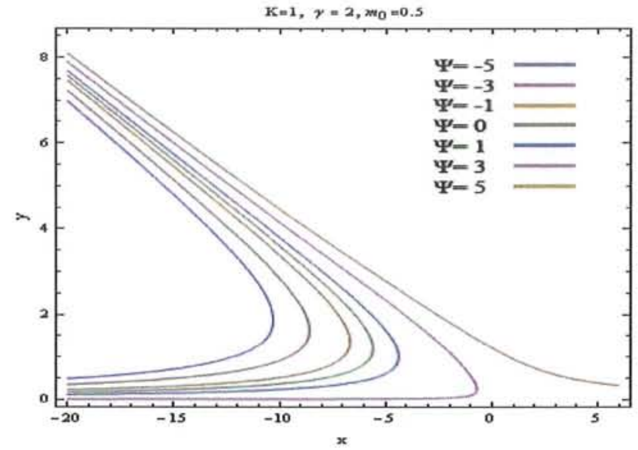

Fig: 3.15 Streamline pattern flow for $K = 1$, when $m_0 = 0.5$

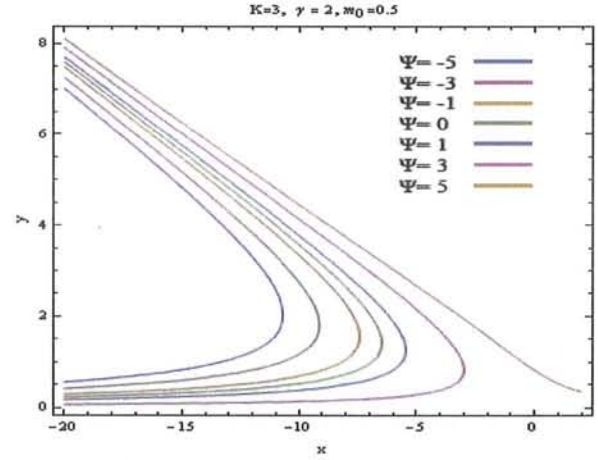

Fig: 3.16 Streamline pattern flow for $K = 3$, when $m_0 = 0.5$

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