A study of conjugate loops

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We accept this dissertation as conforming to the required standard

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Dedication

This thesis is dedicated to my parents. My father, *Mumtaz Ahmed* who not only raise and nurture me but also teached me the value of education. My mother, *Rahat Perveen* has been a source of motivation and strength during moments of despair and discouragement. I am deeply indebted to them for their continued support and unwavering faith in me.

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Notations

L	Loop
Q	Quasigroup
G	Group
e	Identity element of L
x^{λ}	Left inverse of x
x^{ρ}	Right inverse of x
$H \leq L$	H is a subloop of L
$H \trianglelefteq L$	H is a normal subloop of L
N	Normal subloop
L/N	Factor loop
L_x	Left translation
R_x	Right translation
$Mlt(L)$	Multiplication group of L
$Inn(L)$	Inner-mapping group of L
$N(L)$	Nucleus of L
$N_{\lambda}(L)$	Left nucleus of L
$N_{\mu}(L)$	Middle nucleus of L
$N_{\rho}(L)$	Right nucleus of L

$Z(L)$	Center of L
$C(L)$	Commutant of L
$[x, y]$	Commutator of x and y
$[x, y, z]$	Associator of x, y and z
(x, y)	Order pair of x and y
$L \times K$	Direct product of loops L and K
Id_L	Identity element of $Inn(L)$
$L_{x,y}$	Generator of $Inn(L)$ and $L_{x,y} = L_x L_y L_{yx}^{-1}$
$R_{x,y}$	Generator of $Inn(L)$ and $R_{x,y} = R_x R_y R_{xy}^{-1}$
T_x	Generator of $Inn(L)$ and $T_x = R_x L_x^{-1}$
$C_{x^{-1},x}$	Element of $Inn(L)$ and $C_{x^{-1},x} = L_{x^{-1}} R_x L_{x^{-1}}^{-1} R_x^{-1}$

Preface

It is well-known that a loop is a one-operational non-associative generalization of a group. The pioneer work of Moufang [17] and Bol [4] provided a motivation to the theory of loops, which provided a base to develop the research areas of algebra, geometry, topology and combinatorics. The development of loop theory remained hidden under the fast moving research horizon of the theory of groups. After the completion of the list of simple groups, the research environment is more appropriate for the structures of non-associative models like those of a loop and quasigroups. In the literature of loop theory, the groups are being used to derive new families of loops. For instant, construction of C-loops in [20]. In her famous article [17], Moufang derived that the alternative rule in algebra implies the well-known four Moufang identities. Afterwards, she considered loops satisfying these identities, which are called Moufang loops. In the present research environment it is called a Bol loop with left Bol property. The theory of Moufang loops has been developed by Bruck [7].

This thesis concerns one of the property of groups that is, existence of conjugate of each element. This property does not hold in the case of loops but we identify a class of loops having this property called conjugate loops. This is defined to be a class of loops satisfying the identity $x(yx^{-1}) = (xy)x^{-1}$.

Although these are the generalization of the groups, but conjugate loops fail to satisfy major properties of groups regarding conjugate of elements and conjugacy. It is observed, for example, that the conjugate of a subloop is not a subloop. Also unlike groups conjugacy is not an equivalence relation. Homomorphic image of a conjugate loop is again a conjugate loop and direct product of conjugate loops also defines a conjugate loop. Inverses are unique in conjugate loops. Smallest conjugate loop is of order 5 which is also a flexible loop.

An important part of this thesis is the relation of conjugate loops with other types of loops. Every Moufang loop is a conjugate loop. Also Steiner, C, LC and RC-loops define conjugate loops. It is also discussed in this dissertation, that CIP loops are conjugate loops and for the converse, every element of a conjugate loop must be self-conjugate. An IP loop is a conjugate loop if and only if it is flexible.

This dissertation consists of three chapters.

In chapter 1 we discussed the origin and history of the loop theory. It also contains basic definitions and introduction of quasigroups and loops, including a table of non-isomorphic quasigroups, groups and loops of different orders.

Chapter 2 deals with different types of loops and their relation. Identities of different loops are listed in the first section of this chapter. Second section consists of some results on Moufang, C, LC, RC-loops taken from [8], [10], [12] and [21]. In the last section of this chapter, a detailed picture of the relation of loops with each other is given.

The last chapter of this dissertation provides main theme and idea of the work. The layout of the chapter consists of three sections. Definition and counting of conjugate loops are the parts of first section. It also includes the properties of conjugate loops. Second section specifies the relation of conjugate loops with other loops. In this section we also look for the counter examples of the results which were recorded in the second chapter. In the third section, we construct the family of conjugate loops using two groups such that one is multiplicative group and other is additive abelian group.

AFSHAN BATOOL

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Contents

Chapter 1

Introduction

We began with some historical notes on loop theory, summarizing the period from 1920s through the 1960s. Also we give basic concepts of quasigroups and loops included the counting of quasigroups and loops of different orders. Although the whole subject could not be reviewed, nonetheless important and necessary topics, being helpful in the forthcoming literature are included. We follow the terminology used by Aleš Drápal in [9].

1.1 Origins and early history of loop theory

In his paper [19] H.O.Pflugfelder attempted to map, to fit together not only in a geographical and a chronological sense but also conceptually, the various areas where loop theory originated and through which it moved during the early part of its 70 years of history. 70 years is not very much compared to, say, over 300 years of differential calculus. But it is precisely because loop theory is a relatively young subject that it is often misinterpreted. Therefore, it is extremely important for us to acknowledge its distinctive origins.

When somebody asks, "*What is a loop?*", the simplest way to explain is to say, "*It is a group without Associativity*". This is true, but it is not the whole truth. It is essential to emphasize that loop theory is not just a generalization of group theory but a discipline of its own, originating from and still moving within four basic research areas — algebra, geometry, topology, and combinatorics.

One aim of his paper was to shed light on the original motivations for the first publications on quasigroups by Moufang and Bol. The events of those years aretoo far in the past for many people to know first-hand or to have heard about from witnesses. But they are also too recent to be found in math-history books or even on any of the new math-historical web-sites. Let us begin with period I, or rather with the prehistory of non-associativity.

I. 1920s — the first glimmering of non-associativity.

The oldest non-associative operation used by mankind was plain subtraction of natural numbers. But the first example of an abstract non-associative system was Cayley numbers, constructed by Arthur Cayley in 1845. Later they were generalized by Dickson to what we know as Cayley-Dickson algebras. They became the subject of vigorous study in the 1920s because of their prominent role in the structure theory of alternative rings.

Another class of non-associative structures was systems with one binary operation. One of the earliest publications dealing with binary systems that explicitly mentioned non-associativity was the paper On a Generalization of the Associative Law (1929) by Anton K. Suschkewitsch, who was a Russian professor of mathematics in Voronezh. In his paper, Suschkewitsch observes that, in the proof of the Lagrange theorem for groups, one does not make any use of the associative law. So he rightly conjectures that it could be possible to have non-associative binary systems which satisfy the Lagrange property. He constructs two types of such so-called "general groups", satisfying his Postulate A or Postulate B. In Suschkewitsch's approach, one can detect some early attempts in the direction of modern loop theory as a generalization of group-theoretical notions. His "general groups" seem to be the predecessors of modern quasigroups as isotopes of groups.

Binary systems with left and right division, which we now call quasigroups, were mentioned by Ernst Schroeder in his book Lehrbuch der Arithmethik und Algebra (1873), and in his Vorlesungenueber die Algebra der Logik (1890).

II. 1930s — the defining period.

On the algebraic scene, brilliant algebraists happened to be in Hamburg at the time, such as Erich Hecke, a student of Hilbert; Emil Artin; and Artin's students,Max Zorn and Hans

Zassenhaus. Algebraic interest in non-associativity first came not from binary systems, as was the case with Suschkevitsch, but from alternative algebras. It was around this time that Artin proved a theorem that Moufang would later use in her famous paper on quasigroups.

Artin's theorem: "*In an alternative algebra, if any three elements multiply associatively, they generate a subalgebra*".

From the point of view of loop theory, all these developments culminated in the appearance of two papers that defined the two most important classes of loops as we know them now, Moufang loops and Bol loops: *Zur Struktur von Alternativk oerpern* by Ruth Moufang (1935)[17], and *Gewebe und Gruppen* by Gerrit Bol (1937)[4]. Together, these papers marked the formal beginning of loop theory.

Let us first look at Moufang's paper, which was motivated by a publication by Max Zorn on alternative rings in which Zorn used Artin's theorem. Moufang starts with an alternative field and endeavors to prove Artin's theorem using the multiplicative system only. She defines a structure, which she calls a Quasigroup Q*, satisfying the following postulates:

(1), (2) closure, existence of an identity element and unique inverses

(3) a(a'b) = (aa')b and (ba')a = b(a'a)

(4) [a(ca)]b = a[c(ab)]

She also defines a system Q**, believing it to be different from Q*. Q** satisfies an additional identity:

(5) (ab)(ca) = a[(bc)a]

Bol soon showed that (4) implies (5), and Bruck later proved that they both are equivalent to two other identities:

(6) [(ab)c]b = a[b(cb)]

One can see that system Q* is what is now known as a Moufang loop, which can be defined by any one of the Moufang identities (4) through (6).

Moufang proves that Q* is diassociative—the subquasigroup generated by any two elements is associative — and satisfies a theorem that echoes Artin's theorem and is now known as Moufang's theorem.

Bol practically split the Moufang identity in two, showing that, in our language, a loop is Moufang if and only if it is both right and left Bol.

III. 1940s-60s — building the basic algebraic frame.

After the demise of quasigroups in Germany, it was the United States that became the new center of research on this subject.

In addition to alternative algebra research, there were already several American publications on quasigroups:

1937 *Theory of Quasi-Groups*, by Hausmann and Ore [13];

1939 *Quasi-Groups Which Satisfy Certain Generalized Associative Laws*, by Murdoch [18];

1940 *Quasi-Groups*, by Garrison [11].

The terminology of quasigroup theory then underwent a historic change. It became apparent that it was necessary to distinguish between two classes of quasigroups: those with and those without an identity element. A new name was needed to designate the system with identity. This occurred around 1942, among people of Albert's circle in Chicago, who coined the word "loop" after the Chicago Loop.

It was a brilliant choice in several senses. First, the word "loop" rhymes with"group". Second, it expresses a sense of closure. And third, it is short and simple, so that it could be easily adopted in other languages. Today, it is used in many languages, with slight variations: for example, DIE LOOP in German (first used by Pickert) and LUPA in Russian. The French are, of course, an original and non-conforming people, so in French it is LA BOUCLE.

The first publications introducing the term "loop" were the two very important papers that Albert wrote in 1943: *Quasigroups I and Quasigroups II.* In addition to introduction of the new term "*loop*", a highly significant aspect of the *Quasigroups I* paper, was the introduction of the concept of isotopy for quasigroups. Albert's papers were soon followed by two very important publications by Richard Hubert Bruck: *Some Results in the Theory of Quasigroups* (1944) [6] and *Contributions to the Theory of Loops* (1946) [7]. Without a doubt, in this American period of loop theory, stretching from the 1940s through the 1960s, the most important role has to be ascribed to Albert and Bruck and their schools.

Bruck's book [5], *A Survey of Binary Systems* appeared in 1958 and remains even today the most referred-to text on loops.

One can see that during this period, from the 1940s through the 1960s, the basic algebraic frame of loop theory was erected. Loop theory had gained a firm ground that would allow it to move in new directions and flourish in other places.

Belousov's role in the success of quasigroup and loop theory, and his book *Foundations of the Theory of Quasigroups and Loops (1967) [3]*, can rightly be compared with the role that Bruck and his Binary Systems. Leter on, new aspects and new approaches emerged in this field. Among them were the following:

New approaches to quasigroups — derivative operations,

New properties of known quasigroups — distributive being isotopic to commutative Moufang loops, left-distributive that are isotopic to groups, isotopes of totally symmetric quasigroups;,

Functional equations to express general laws of quasigroups (binary as well as n-ary),

Algebraic webs and their use in questions of isotopy of quasigroups and loops,

Generalized Moufang and Bol loops.

So, the theory of loops has its origins in geometry, combinatorics and nonassociative algebra. In geometry, the coordinatization of a projective plane leads to various loop structures on the set of labels from which coordinates are chosen. Any Latin square with first row and column in standard position is the multiplication table of a loop. In rings with identity for which there is a well defined notion of inverses, it is often the case that subsets closed under product and inverses are loops.

1.2 Quasigroups and loops

Quasigroups: A set of elements Q and a binary operation "." form a quasigroup if and only if the following are satisfied:

(1) If $a, b \in Q$, then there exists a unique $x, y \in Q$ such that $a.x = b$ or $y.a = b$.

(2) If $a, x, y \in Q$, then either $a.x = a.y$ or $x.a = y.a$ implies $x = y$.

Examples of quasigroups are:

(i) Set of integers $\mathbb{Z}$ under the binary operation of subtraction $(-)$.

(ii) Set of non-zero rationals Q under the binary operation of division $(\div)$.

Loop: A loop L is a quasigroup with an identity element e such that $x * e = x = e * x$ for all $x \in L$. It follows that the identity element e is unique, and that all elements of L have a unique left and right inverse which are need not to be same.

Examples of loops are:

(i)The set $\{\pm 1, \pm i, \pm j, \pm k\}$ where $ii = jj = kk = 1$ and with all other products as in the quaternion group forms a nonassociative loop of order 8.

(ii) Smallest nonassociative loop which is of order 5.

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	1	0	3	4	2
2	2	3	4	0	1
3	3	4	1	2	0
4	4	2	0	1	3

Every group is a loop but every loop is not a group i.e. loop is the generalization of group.

Every loop is a quasigroup but following example shows that the converse is not true.

$\cdot$	0	1	2	3
0	2	0	3	1
1	3	1	2	0
2	0	3	1	2
3	1	2	0	3

The following table shows non-isomorphic quasigroups, groups and loops of different order.

order	*quasigroups*	*groups*	*loops*
1	1	1	1
2	1	1	1
3	5	1	1
4	35	2	2
5	$1,411$	1	6
6	$1,13,051$	2	109
7	$12,19,84,55,835$	1	$23,746$
8	2.70×10^{15}	5	1.60×10^{8}
9	1.52×10^{22}	2	9.37×10^{12}
10	2.75×10^{30}	2	2.09×10^{19}

1.2.1 Definitions

Now we list some definition which we need in proving main results in coming chapters.

Subloop : A subloop H of a loop L is a subset of L which, under the inherited binary operation, is also a loop.

Normal subloop: A subloop H of a loop L is Normal subloop if and only if $x(yH) = (xy)H$, $(Hx)y = H(xy)$ and $xH = Hx$, for all $x, y \in L$.

Factor loop: The factor loop of a loop L to its normal subloop N is denoted by L/N and define as $L/N = \{xN : \forall\ x \in L\}$.

Binary operation defined on factor loop is "." such that

$$xN.yN = xyN \quad \forall x, y \in L$$

Homomorphism: Let K, H be two loops. Then a map $f : K \to H$ is a homomorphism if $f(x).f(y) = f(x.y)$ for every $x, y \in K$.

Isomorphism: If f is also a bijection, we speak of an isomorphism, and the two loops are called isomorphic.

Automorphism: Let L be a loop. Then a map $f : L \to L$ is a automorphism if $f(x).f(y) = f(x.y)$ for every $x, y \in L$ and f is a bijection.

Pseudo-automorphism: A (right) psuedo-automorphism of a loop L is a bijection θ of L with the property that, for some fixed $c \in L$.

$$(x\theta)(y\theta.c) = (xy)\theta.c$$

Homotopism: The ordered triple (α, β, γ) of maps $\alpha, \beta, \gamma : K \to H$ is a homotopism if $\alpha(x)\cdot\beta(y) = \gamma(x\cdot y)$ for every $x, y \in K$.

Autotopism: If the three maps are bijections, (α, β, γ) is an autotopism.

Left and Right translations: When x is an element of a loop L, the left translation L_x is a permutation of L such that $L_x(a) = xa$. Similarly right translation R_x is a permutation of L such that $R_x(a) = ax$.

Left and right multiplication group: The subgroups generated by $L_1 = \langle L_a : a \in L \rangle$ and $R_1 = \langle R_a : a \in L \rangle$ are called left and right multiplication groups respectively.

Multiplication group: The permutation group generated by left and right translations is called multiplication group and denoted by $\mathrm{Mlt}(L)$.

Inner mapping group: Let L be a loop and $\mathrm{Mlt}(L)$ be the multiplication group of L, then the subset of $\mathrm{Mlt}(L)$ consisting of all maps that fix the identity element of L is called the inner mapping group of L, denoted by $Inn(Q)$.

Left nucleus: The left nucleus of a loop L is $N_\lambda = \{l \in L : l(xy) = (lx)y$ for every $x, y \in L\}$.

Rightnucleus: The right nucleus of a loop L is the set $N_\rho = \{r \in L : (xy)r = x(yr)$ for every $x, y \in L\}$.

Middle nucleus: The middle nucleus of L is $N_\mu = \{m \in L : (ym)x = y(mx)$ for every $x, y \in L\}$.

Nucleus: The nucleus of L is the set $N = N_\rho \cap N_\lambda \cap N_\mu$. All nuclei are subloops.

Commutant: The commutant of a loop L is the set $C(L) = \{c \in L : cx = xc\ ;\ \forall\ x \in L\}$.

It is also known as *Moufang center* or *centrum*.

Commutator: The commutator of two elements x, y of a loop L is a unique element $[x, y]$ of L, such that

$$(xy) = (yx)[x, y]$$

Associator: The associator of three elements x, y, z of a loop L is a unique element $[x, y, z]$ of L,such that

$$(xy)z = x(yz)[x, y, z]$$

Center: The center of a loop L is the set $Z(L) = N(L) \cap C(L)$.

Antiautomorphic inverse property: Let $x, y \in L$,if x^{λ}, x^{ρ} and y^{λ}, y^{ρ} be the left and right inverses of x, y respectively.Then Loop L is said to have antiautomorphic property if

$$(xy)^{\lambda} = y^{\lambda}x^{\lambda} \text{ or } (xy)^{\rho} = y^{\rho}x^{\rho}$$

If loop L has unique inverses then this property becomes

$$(xy)^{-1} = y^{-1}x^{-1}$$

Automorphic inverse property: Let $x, y \in L$,if x^{λ}, x^{ρ} and y^{λ}, y^{ρ} be the left and right inverses of x, y respectively.Then Loop L is said to have antiautomorphic property if

$$(xy)^{\lambda} = x^{\lambda}y^{\lambda} \text{ or } (xy)^{\rho} = x^{\rho}y^{\rho}$$

If loop L has unique inverses then this property becomes

$$(xy)^{-1} = x^{-1}y^{-1}$$

Left inverse property: A loop L has the left inverse property if $x^{\lambda}(xy) = y$ for every $x, y \in L$.

Right inverse property: A loop L has Right inverse property if $(yx)x^{\rho} = y$ for every $x, y \in L$.

Weak inverse property: A loop has the weak inverse property if

$$x(yx)^{\lambda} = y^{\lambda} \text{ or } x(yx)^{\rho} = y^{\rho}$$

If loop has unique inverses then this property becomes

$$x(yx)^{-1} = y^{-1}$$

Cross inverse property: A loop has the Cross inverse property if

$$x(yx^{\lambda}) = (xy)x^{\lambda} = y \text{ or } x(yx^{\rho}) = (xy)x^{\rho} = y$$

If loop has unique inverses then this property becomes

$$x(yx^{-1}) = (xy)x^{-1} = y$$

1.2.2 Properties of Quasigroups

Semisymmetric quasigroups: A quasigroup Q is semisymmetric quasigroup if $(xy)x = x(yx) = y$ for every $x, y \in Q$.

Totally symmetric quasigroups:A semisymmetric commutative quasigroup is known as totally symmetric quasigroup.

Idempotent quasigroups: A quasigroup Q is idempotent quasigroup if $x^2 = x$ for every $x \in Q$. Idempotent totally symmetric quasigroups are known as *Steiner quasigroups.*

Unipotent quasigroups: A quasigroup Q is unipotent quasigroup if $x^2 = y^2$ for every $x, y \in Q$.

Left distributive quasigroups: A quasigroup Q is left distributive quasigroup if it satisfies $x(yz) = (xy)(xz)$ for all $x, y, z \in Q$.

Right distributive quasigroups: Similarly, Q is right distributive quasigroup if it satisfies $(xy)z = (xz)(yz)$.

Distributive quasigroups: A distributive quasigroup is a quasigroup that is both left and right distributive.

Entropic quasigroups: A quasigroup Q is called entropic quasigroup or medial quasigroup if it satisfies $(xy)(zw) = (xz)(yw)$ for all $x, y, z, w \in Q$.

Remark 1 *All these properties also hold in loops.*

Chapter 2

Different types of loops and Relation between their identities

As we discussed earlier that Loop Theory is not only the generalization of the group theory but a discipline in its own. Since Loops do not have associativity we classify the loops into different types, on the basis of weak associativity. In this chapter , we identify and discuss some identities of class of loops included a detailed picture of their relation with each other. For the collection of material we followed [2] and [7].

2.1 Types of loops

Extra loop:

Extra loop L satisfies $x(y(zx)) = ((xy)z)x \ \forall \ x, y, z \in L$.

Moufang loop:

Any loop L satisfying $(xy)(zx) = (x(yz))x \ \forall \ x, y, z \in L$, is called Moufang loop.

Left Alternative loop:

A loop L satisfying $x(xy) = (xx)y \ \forall \ x, y \in L$, is called left alternative loop.

Right Alternative loop:

A loop L satisfying $x(yy) = (xy)y \ \forall \ x, y \in L$ is called right alternative loop.

Flexible loop:

If a loop L satisfying $x(yx) = (xy)x \ \forall \ x, y \in L$, then L is called flexible loop.

Left Bol loop:

A loop satisfying $x(y(xz)) = (x(yz))x \ \forall \ x, y, z \in L$, is called left Bol loop.

Right Bol loop:

Right Bol loop L satisfies $x((yz)y) = ((xy)z)y \ \forall \ x, y, z \in L$.

LC-Loop:

Any loop L satisfying $(xx)(yz) = (x(xy))z \ \forall \ x, y, z \in L$, is called LC-loop.

RC-Loop:

If a loop L satisfying $x((yz)z) = (xy)(zz) \ \forall \ x, y, z \in L$ then is called RC-loop.

C-Loop:

A C-loop L satisfies $x(y(yz)) = ((xy)y)z \ \forall \ x, y, z \in L$.

Left Nuclear square loop:

Loop L satisfying $(xx)(yz) = ((xx)y)z \ \forall \ x, y, z \in L$, is called left nuclear square loop.

Middle Nuclear square loop:

Middle Nuclear square loop L satisfies $x((yy)z) = (x(yy))z \ \forall \ x, y, z \in L$.

Right Nuclear square loop:

A loop L is called right nuclear square loop if it satisfies $(y(zz)) = (xy)(zz) \ \forall \ x, y, z \in L$.

3-power associative loop;

If a loop L satisfies $(xx)x = x(xx)$, $\forall \ x \in L$ then, L is a 3-power associative loop.

Power associative loop:

A loop is said to be power associative loop if every element in it generates a group.

There is an example of a loop which is 3-power associative but not a power associative loop.

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	0	4	5	3
2	2	0	3	5	1	4
3	3	4	5	0	2	1
4	4	5	1	2	3	0
5	5	3	4	1	0	2

since $(1.1)(1.1) \neq 1(1(1.1))$.

Diassociative loop:

A loop is said to be diassociative lop if every two elements in it generate a group.

Antiautomorphic inverse property loop:

A loop having antiautomorphic inverse property is called antiautomorphic inverse loop.

Automorphic inverse property loop:

These loops satisfy the automorphic inverse property loop.

Two-sided inverse loop:

A loop L in having unique inverses i.e $x^{\lambda} = x^{\rho}\ \forall x \in L$ is called two-sided inverse loop.

Right inverse property loop:

A loop having right inverse property is called right inverse property loop.

Left inverse property loop:

A loop having right inverse property is called left inverse property loop.

Inverse property loop(IP Loop):

A loop having both left and right inverse property is called inverse property loop or it satisfies $x^{-1}(xy) = (yx)x^{-1} = y$

Steiner loop:

An Inverse property loop of exponent 2 is called a Steiner loop.

Weak inverse property loop:

A loop satisfying weak inverse propety is called weak inverse property loop.

Cross inverse property loop(CIP loop):

If a loop satisfies cross inverse property then it is called CIP loop.

Jordan Loop:

A loop L satisfying $x^2(yx) = (x^2y)x \ \forall x, y \in L$, is called Jordan loop.

Automorphic loop (A-loop):

An automorphic loop is a loop whose inner mappings are automorphisms.

2.2 Main results

The aim of this section is to record some results which will be needed in sequel.

Theorem 2 *[8, lemma 2.1] Every inner-mapping of a Moufang loop is a pseudo-automorphism.*

Theorem 3 *[8, Theorem 2] The nucleus of a Moufang loop is a normal subloop.*

Theorem 4 *[12, Lagrange's theorem] The order of any subloop of a finite Moufang loop M divide the order of M.*

In [10], F. Fenyves listed all the 60 identities of Bol-Moufang types.We consider here, the identities for LC and RC loops.

Remark 5 *List for LC and RC identities is the following*

$$
\begin{aligned}
xx.yz &= (x.xy)z \text{ LC-identities} \\
xx.yz &= (xx.y)z \\
xx.yz &= x(xy.z) \\
(x.xy)z &= x(x.yz) \\
(x.xy)z &= x(xy.z) \\
(x.xy)z &= x(x.yz) \\
(xx.y)z &= x(x.yz) \\
(xx.y)z &= x(xy.z) \\
x(x.yz) &= x(xy.z) \\
yz.xx &= (yz.x)x \\
yz.xx &= (y.zx)x \\
yz.xx &= y(zx.x) \text{ RC-identities} \\
yz.xx &= y(z.xx) \\
(yz.x)x &= (y.zx)x \\
(yz.x)x &= y(zx.x) \\
(yz.x)x &= y(z.xx) \\
(y.zx)x &= y(zx.x) \\
(y.zx)x &= y(z.xx) \\
y(zx.x) &= y(z.xx)
\end{aligned}
$$

Theorem 6 *[10, Theorem 4]A loop is a C-loop if and only if it is both LC- and RC-loop.*

Remark 7 *[10, Table 1] The multiplication table given below defines an LC-loop but not a*

C-loop.

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	5	4	3	2
2	2	4	0	5	1	3
3	3	2	1	0	5	4
4	4	5	3	2	0	1
5	5	3	4	1	2	0

Theorem 8 *[10, Corollary 3] Every LC loop is power-associative.*

Theorem 9 *[10, Theorem 2] If L is an LC-loop, then*

(i) L has left inverse property.

(ii) L is left alternative,

(iii) x^2 is in the left nucleus of L for all x in L.

(Analogous results holds for RC-loops)

Theorem 10 *[21, Lemma 5]The Following properties are true in any right central loop L.*

(i) If s is the square of some element of L and x is any element of L, then $(xs)^{-1} = s^{-1}x^{-1}$.

(ii) The order of any element is a divisor of the order of L.

Theorem 11 *[21, Proposition 6]Any RC of odd order is a group.*

Theorem 12 *[14, Theorem 4.5] If L is an IP loop then $N(L) = N_\lambda(L) = N_\mu(L) = N_\rho(L)$.*

Theorem 13 *[20, Lemma 3.3] Let $\mu : G \times G \to A$ be a factor set. Then (G, A, μ) is a C-loop if and only if*

$$\mu(h, k) + \mu(h, hk) + \mu(g, h, hk) = \mu(g, h) + \mu(gh, h) + \mu(gh.h, k)$$

for every $g, h, k \in G$.

Theorem 14 *[20, Proposition 3.4] Let $n > 2$ be an integer. Let A be an abelian group of order n,and $\alpha \in A$, an element of order bigger then 2. Let $G = \{1, u, v, w\}$ be a Klein group with* *neutral element 1. Then* (G, A, μ) *is a non-flexible C-loop with* $N = \{(1, a) : a \in A\}$, *where* μ *is a factor set.*

Exercise 15 *[20, Example 3.7] The smallest non commutative, non associative C-loop, which satisfied above condition is given by*

$\cdot$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2	0	4	5	3	7	8	6	10	11	9
2	2	0	1	5	3	4	8	6	7	11	9	10
3	3	4	5	0	1	2	9	10	11	6	7	8
4	4	5	3	1	2	0	10	11	9	7	8	6
5	5	3	4	2	0	1	11	9	10	8	6	7
6	6	7	8	10	11	9	0	1	2	5	3	4
7	7	8	6	11	9	10	1	2	0	3	4	5
8	8	6	7	9	10	11	2	0	1	4	5	3
9	9	10	11	8	6	7	3	4	5	2	0	1
10	10	11	9	6	7	8	4	5	3	0	1	2
11	11	9	10	7	8	6	5	3	4	1	2	0

In [15] Kinyon, Kunen and Phillips, proved that If A is, say, the 10-element Steiner loop, then it is not a group and hence not Moufang. Fig. 1 depicts the sub-varieties of diassociative loops discussed in that

paper.

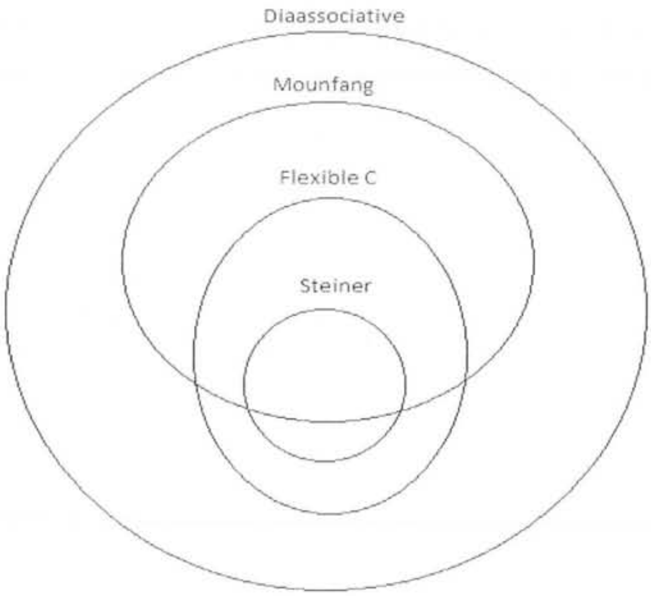

Fig.1

2.3 Implications

1) Diassociative loop ⇒ Power alternative loop.

2) Diassociative loop ⇒ Flexible loop.

3) Extra loop ⇒ Moufang loop.

4) C-loop ⇒ RC-loop.

5) C-loop ⇒ LC-loop.

6) C-loop and Flexible loop ⇒ Diassociative loop.

7) RC-loop and LC-loop ⇒ C-loop.

8) LC-loop ⇒ Left nuclear square loop.

9) LC-loop $\Rightarrow$ Middle nuclear square loop.

10) RC-loop $\Rightarrow$ Right nuclear square loop.

11) RC-loop $\Rightarrow$ Middle nuclear square loop.

12) Moufangloop $\Rightarrow$ Left Bol loop.

13) Moufang loop $\Rightarrow$ Right Bol loop.

14) Left Bol loop $\Rightarrow$ Left Power alternative loop.

15) Right Bol loop $\Rightarrow$ Right Power alternative loop.

16) Power alternative loop $\Rightarrow$ Alternative loop.

17) Left Bol loop and Right Bol loop $\Rightarrow$ Moufang loop.

18) Left Power alternative loop and commutative loop $\Rightarrow$ Right Power alternative loop.

19) LC-loop and commutative loop $\Rightarrow$ C-loop.

20) Commutative loop $\Rightarrow$ Flexible loop.

21) Moufang loop $\Rightarrow$ Flexible loop.

22) Moufang loop $\Rightarrow$ IP loop.

23) Moufang loop $\Rightarrow$ Stiener loop.

24) C-loop $\Rightarrow$ IP loop.

Figure 2 shows all varieties of loops of Bol-Moufang type and all inclusions among them, discussed by Kinyon, Phillips and Vojtechovsky in [16].

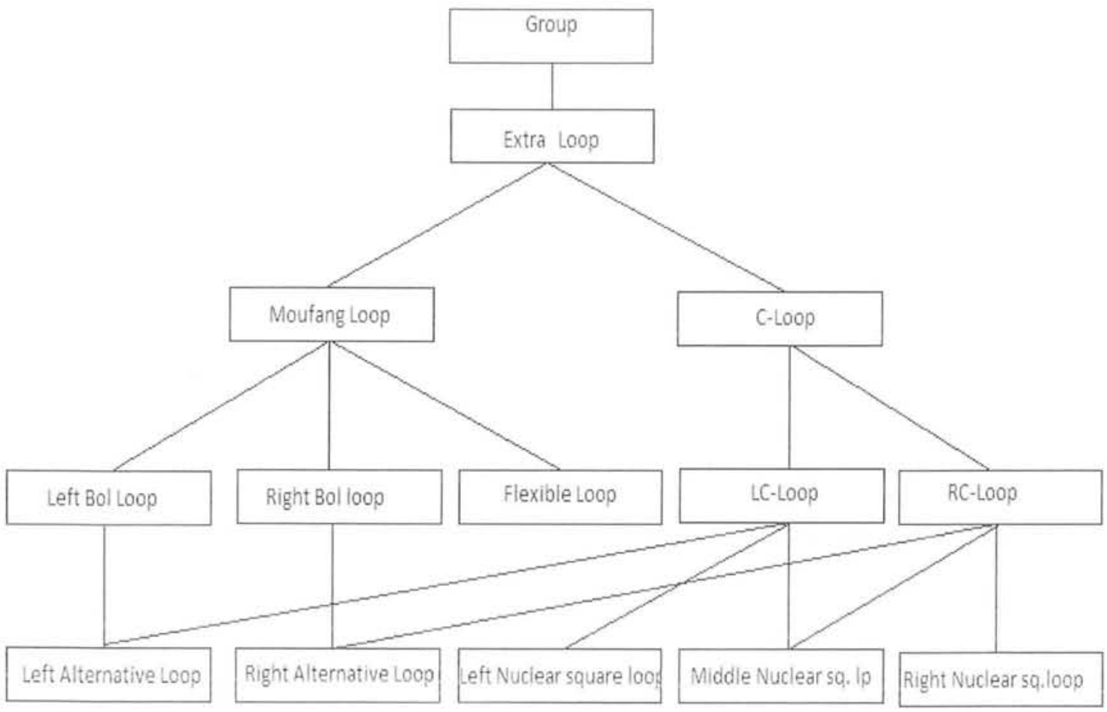

Fig.2

Chapter 3

A study of Conjugate loops

This chapter consists of three sections. In section1, the introduction and basic definition of conjugate loop is given. It also includes the counting of conjugate loops up to the order 8 and the properties of conjugate loops which distinguish this class of loops with other classes. Section 2 consists of the relation of conjugate loops with other loops. Here we define the exact location of conjugate loops among different types of loops. In the last section, we construct a family of conjugate loops using two groups such that one is multiplicative group and other is additive abelian group.

3.1 Conjugate loops

Definition 16 *A Loop L is said to be a conjugate loop if it satisfies the following identity*

$$x\left(yx^{-1}\right) = (xy)\,x^{-1} \;\; \forall\; x, y \in L$$

3.1.1 Counting of conjugate loops

Smallest conjugate loop is of order 5 given below.

.	0	1	2	3	4
0	0	1	2	3	4
1	1	0	3	4	2
2	2	4	0	1	3
3	3	2	4	0	1
4	4	3	1	2	0

Above loop is non-diaassociative, non-alternative and does not satisfy inverse property. Also it does not satisfy the identity $(xy)^{-1} = x^{-1}y^{-1}(Automorphic\ Inverse\ Property)$ but it satisfies $(xy)^{-1} = y^{-1}x^{-1}(Antiautomorphic\ Inverse\ Property)$.This loop is also a flexible loop and power-associative.

order	conjugate loops	flexible
5	1	1
6	4	4
7	4	4
8	53	51

Clearly smallest conjugate loop which is not flexible is of order 8. Also smallest non power-associative conjugate loop is of order 8.

Smallest non flexible, non power-associative conjugate loop is

.	0	1	2	3	4	5	6	7
0	0	1	2	3	5	5	6	7
1	1	0	4	5	6	7	3	2
2	2	7	5	0	3	1	4	6
3	3	6	0	4	1	2	7	5
4	4	2	3	6	7	0	5	1
5	5	3	7	2	0	6	1	4
6	6	4	1	7	5	3	2	0
7	7	5	6	1	2	4	0	3

3.1.2 Properties of conjugate loops

Conjugate of an element

Let L be a conjugate loop then y is said to be the conjugate of x, where $x, y \in L$ if there exists some $g \in L$ such that $g(xg^{-1}) = y$.

Conjugate of every element exists in conjugate loops.

Since $x(e.x^{-1}) = x.x^{-1} = e \; \forall \; x \in L$ where e is the identity element of L. So, conjugate of identity is identity itself.

Also, if $x^{-1} = y$ then $y(xy^{-1}) = x$. So for all $x \in L$, there exists $x^{-1} \in L$ such that $x^{-1}(x.(x^{-1})^{-1}) = (x^{-1}x).(x^{-1})^{-1} = e.x = x$

Conjugacy is not an equivalence relation

Unlike groups, conjugacy is not an equivalence relation in conjugate loops.

Reflexive:

Let L be a conjugate loop.

Since $(xx)x^{-1} = x(xx^{-1}) = x(e) = x \;\; \forall \; x \in L$, where e is the identity of L. So L is reflexive.

Symmetric:

Following example shows that symmetric property does not hold in conjugate loops.

Example 17

.	0	1	2	3	4	5	6	7
0	0	1	2	3	5	5	6	7
1	1	0	3	2	5	6	7	4
2	2	3	1	0	6	7	4	5
3	3	2	0	1	7	4	5	6
4	4	5	6	7	3	0	1	2
5	5	6	7	4	0	1	2	3
6	6	7	4	5	1	2	3	0
7	7	4	5	6	2	3	0	1

In this loop 4 is the conjugate of 7, because there exists $2 \in L$ such that $2(\ 72^{-1}) = 4$. But we cannot find any $g \in L$ such that $g(\ 4g^{-1}) = 7$. So, 7 is not a conjugate of 4 hence symmetric property does not hold in conjuagte loops.

Transitive:

Above loop also shows that the transitive property does not hold in conjugate loops.

here 7 is the conjugate of 5 and 5 is the conjugate of 4 but clearly 7 is not a conjugate of 4.

hence conjugacy is not an equivalence relation in conjugate loops.

Conjugate of a subloop is not a subloop

We know that in groups, conjugate of a subgroup is again a subgroup. But following example shows that it is not true in the case of conjugate loops.

Example 18 *Consider the following conjugate loop of order 20.*

.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	2	1	4	3	6	5	8	7	10	9	12	11	20	19	18	17	16	15	14	13
3	3	10	5	2	7	4	9	6	1	8	13	14	15	16	17	18	19	20	11	12
4	4	9	6	1	8	3	10	5	2	7	14	13	12	11	20	19	18	17	16	15
5	5	8	7	10	9	2	1	4	3	6	15	16	17	18	19	20	11	12	13	14
6	6	7	8	9	10	1	2	3	4	5	16	15	14	13	12	11	20	19	18	17
7	7	6	9	8	1	10	3	2	5	4	17	18	19	20	11	12	13	14	15	16
8	8	5	10	7	2	9	4	1	6	3	18	17	16	15	14	13	12	11	20	19
9	9	4	1	6	3	8	5	10	7	2	19	20	11	12	13	14	15	16	17	18
10	10	3	2	5	4	7	6	9	8	1	20	19	18	17	16	15	14	13	12	11
11	11	12	19	14	17	16	15	18	13	20	1	2	9	4	7	6	5	8	3	10
12	12	11	20	13	18	15	16	17	14	19	2	1	4	9	6	7	8	5	10	3
13	13	20	11	12	19	14	17	16	15	18	3	4	1	6	9	8	7	10	5	2
14	14	19	12	11	20	13	18	15	16	17	4	3	6	1	8	9	10	7	2	5
15	15	18	13	20	11	12	19	14	17	16	5	6	3	8	1	10	9	2	7	4
16	16	17	14	19	12	11	20	13	18	15	6	5	8	3	10	1	2	9	4	7
17	17	16	15	18	13	20	11	12	19	14	7	8	5	10	3	2	1	4	9	6
18	18	15	16	17	14	19	12	11	20	13	8	7	10	5	2	3	4	1	6	9
19	19	14	17	16	15	18	13	20	11	12	9	10	7	2	5	4	3	6	1	8
20	20	13	18	15	16	17	14	19	12	11	10	9	2	7	4	5	6	3	8	1

Here consider the subloop H= $\{1, 2, 16, 17\}$,

$9H9^{-1}$=$\{1,6,12,13\}$ is not a subloop of the above loop.

Theorem 19 *In a Conjugate loop L, if $H \trianglelefteq L$ then $x\left(Hx^{-1}\right) = H$.*

Proof. Let L be a conjugate loop and H $\trianglelefteq L$, then by the defination of normal subloop

$$aH = Ha \tag{3.1}$$

$$a(bH) = (ab)H \tag{3.2}$$

Now

$$\begin{aligned} x\left(Hx^{-1}\right) &= x(x^{-1}H) \quad \text{from equation (3.1)} \\ &= (xx^{-1})H \quad \text{from equation (3.2)} \\ &= H \end{aligned}$$

■

Theorem 20 *Quotient loop of a conjugate loop is again a conjugate loop.*

Proof. Let L be a conjugate loop and N be its normal subloop. we have to show that L/N is a conjugate loop.

Now let $xN, yN \in L/N$

also, we know that $(xN)^{-1} = x^{-1}N$

$$\begin{aligned} xN(yN.x^{-1}N) &= xN(yx^{-1}N) \\ &= xN.yx^{-1}N \\ &= x(yx^{-1})N \\ &= (xy)x^{-1}N \\ &= xyN.x^{-1}N \\ &= (xNyN)x^{-1}N \end{aligned}$$

so L/N is a conjugate loop.

Theorem 21 *Homomorphic image of conjugate loop is again a conjugate loop.*

■

Proof. Let L be a conjugate loop and f be a homomorphism from L to K i.e $f : L \to K$. We have to show that $f(L)$ is also a conjugate loop.
Now let $x, y \in L$ and L is a conjugate loop so $(x^{-1}y)x = x^{-1}(yx)$

$$\begin{aligned}
(x^{-1}y)x &= x^{-1}(yx) \\
&\Rightarrow f((x^{-1}y)x) = f((x^{-1}(yx)) \\
&\Rightarrow f(x^{-1}y)f(x) = f(x^{-1})f(yx) \\
&\Rightarrow (f(x^{-1})f(y))f(x) = f(x^{-1})(f(y)f(x) \\
&\Rightarrow (f(x)^{-1}f(y))f(x) = f(x)^{-1}(f(y)f(x)) \\
\text{since } f(x^{-1}) &= f(x^{-1})
\end{aligned}$$

hence $f(L)$ is a conjugate loop. ■

Theorem 22 *Direct product of conjugate loop is a conjugate loop.*

Proof. Let L_1 and L_2 be two conjugate loops. We have to prove that $L_1 \times L_2$ is again a conjugate loop i.e $(x_1, x_2)^{-1}((y_1, y_2)(z_1, z_2)) = ((x_1, x_2)^{-1}(y_1, y_2))(z_1, z_2)$

$$\begin{aligned}
LHS \quad (x_1, x_2)^{-1}((y_1, y_2)(z_1, z_2)) &= (x_1^{-1}, x_2^{-1})(y_1 z_2, y_2 z_2) \\
&= (x_1^{-1}(y_1 z_2), x_2^{-1}(y_2 z_2)) \\
&= ((x_1^{-1}y_1)z_2), (x_2^{-1}y_2)z_2)) \text{ since L is conjugate loop} \\
RHS \quad 3(x_1, x_2)^{-1}(y_1, y_2))(z_1, z_2) &= ((x_1^{-1}, x_2^{-1})(y_1, y_2))(z_1, z_2) \\
&= (x_1^{-1}y_1, x_2^{-1}y_2)(z_1, z_2) \\
&= ((x_1^{-1}y_1)z_1, (x_2^{-1}y_2)z_2) \\
So, \quad (x_1, x_2)^{-1}((y_1, y_2)(z_1, z_2)) &= ((x_1, x_2)^{-1}(y_1, y_2))(z_1, z_2)
\end{aligned}$$

So, Direct product of conjugate loop is again a conjugate loop. ■

Corollary 23 Let $L_1, L_2, L_3, ...L_n$ be n conjugate loops, then $L_1 \times L_2 \times L_3 ... \times L_n$ is also a conjugate loop.

Corollary 24 Let L *be a conjugate loop* then $L \times L \times L ... \times L$ *(n-times)* is again a conjugate loop.

Theorem 25 *In Conjugate Loop L,* $L_x R_{x^{-1}} = R_{x^{-1}} L_x$.*Similarly* $R_x L_{x^{-1}} = L_{x^{-1}} R_x$.

Proof. *Let* $y \in L$

$$\begin{aligned} yL_x R_{x-1} &= (xy)\, R_{x^{-1}} \\ &= (xy)x^{-1} \\ &= x\left(yx^{-1}\right) \\ &= x(yR_{x^{-1}}) \\ &= yR_{x^{-1}} L_x \\ So, \;\; L_x R_{x^{-1}} &= R_{x^{-1}} L_x \end{aligned}$$

■

Theorem 26 *If* $y \in N(L)$ *where L is a commutative conjugate loop then* $L_{x^{-1}y} L_x = L_x L_y L_{x^{-1}}$.

Proof. Let $z \in L$ then

$$\begin{aligned}
zL_{x^{-1}y}L_x &= ((x^{-1}y)z)L_x \\
&= x((x^{-1}y)z) \\
&= x(x^{-1}(yz)) \text{ since } y \in N(L) \\
&= (x^{-1}(yz))x \text{ commutativity} \\
&= x^{-1}((yz)x) \text{ conjugate property} \\
&= x^{-1}(y(zx)) \quad \text{since } y \in N(L) \\
&= y(zx)L_{x^{-1}} \\
&= zxL_yL_{x^{-1}} \\
&= zL_xL_yL_{x^{-1}} \\
So,\ L_{x^{-1}y}L_x &= L_xL_yL_{x^{-1}}
\end{aligned}$$

■

Theorem 27 *In Conjugate loop L,*

(i) $L_{y,x}$ *fix* x^{-1}.

(ii) $L_{y,x^{-1}}$ *fix* x.

Proof. *Let L be a conjugate loop and* $x, y \in L$.

i)

$$\begin{aligned}
x^{-1}L_{y,x} &= x^{-1}L_yL_xL_{xy}^{-1} \\
&= (yx^{-1})L_xL_{xy}^{-1} \\
&= x(yx^{-1})L_{xy}^{-1} \\
&= (xy)x^{-1}L_{xy}^{-1} \quad \textit{identity of conjugate loops} \\
&= x^{-1}L_{xy}L_{xy}^{-1} \\
&= x^{-1}
\end{aligned}$$

(ii)

$$
\begin{aligned}
xL_{y,x^{-1}} &= xL_yL_{x^{-1}}L^{-1}_{x^{-1}y} \\
&= (yx)L_{x-1}L^{-1}_{x^{-1}y} \\
&= x^{-1}(yx)L^{-1}_{x^{-1}y} \\
&= (x^{-1}y)xL^{-1}_{x^{-1}y} \quad \textit{identity of conjugate loops} \\
&= xL_{x^{-1}y}L^{-1}_{x^{-1}y} \\
&= x
\end{aligned}
$$

■

Theorem 28 *In any two sided inverse loop L, if $L_{y,x}$ fix every elements of L then L is a conjugate loop.*

Proof. *Let $L_{y,x}$ fixes every element of L then*

$$
\begin{aligned}
zL_{y,x} &= z \quad \forall z \in L \\
put\ z &= x^{-1} \\
x^{-1}L_{y,x} &= x^{-1} \\
x^{-1}L_yL_xL^{-1}_{xy} &= x^{-1} \\
x^{-1}L_yL_x &= x^{-1}L_{xy} \\
\left(yx^{-1}\right)L_x &= (xy)\,x^{-1} \\
x\left(yx^{-1}\right) &= (xy)\,x^{-1}
\end{aligned}
$$

Hence, L is a conjugate loop.

Theorem 29 *Let L be a conjugate loop then $L_{x,x^{-1}}$ and $R_{x,x^{-1}}$ fix $N(L)$.*

Proof. *Let $a \in N(L)$*

$$\begin{aligned}
(a)L_{x,x^{-1}} &= (a)L_xL_{x^{-1}}L^{-1}_{x^{-1}x} \\
&= (a)L_xL_{x^{-1}} \\
&= x^{-1}(xa) \\
&= (x^{-1}x)a \quad \textit{using conjugate property} \\
&= a
\end{aligned}$$

■

Also,

$$\begin{aligned}
(a)R_{x,x^{-1}} &= (a)R_xR_{x^{-1}}R^{-1}_{xx^{-1}} \\
&= (a)R_xR_{x^{-1}} \\
&= (ax)x^{-1} \\
&= a(xx^{-1} \quad \textit{since } a \in N(L) \\
&= a
\end{aligned}$$

■

3.1.3 Conjugate loops and Autotopisms

Theorem 30 *Let L be a two sided inverse loop and $(L_x, L_{x^{-1}}, R_{x^{-1}}L_x)$ is an autotopism for all $x, y \in L$ then L is a conjugate loop.*

Proof. *Let L be a two sided inverse loop and $(L_x, L_{x^{-1}}, R_{x^{-1}}L_x)$ is an autotopism for all*

$x, y \in L$. Then by definition

$$\begin{aligned}
(u)\, L_x\, (v)\, L_{x^{-1}} &= (uv) R_{x^{-1}} L_x \ \forall u, v \in L \\
(xu)(x^{-1} v) &= x\left((uv)\, x^{-1}\right) \\
put\ u &= y\ and\ v = e \\
(xy)(x^{-1} e) &= x\left((ye)\, x^{-1}\right) \\
x(yx^{-1}) &= (xy)x^{-1}
\end{aligned}$$

Theorem 31 *Let L be a two sided inverse loop and (L_x, Id_L, L_x) is an autotopism for all $x, y \in L$ then L is a conjugate loop.*

■

Proof. *Let L be a two sided inverse loop and (L_x, Id_L, L_x) is an autotopism for all $x, y \in L$. Then by definition*

$$\begin{aligned}
(u)\, L_x\, (v)\, Id_L &= (uv) L_x \ \forall\ u, v \in L \\
(xu)(v) &= x\, (uv) \\
put\ u &= y\ and\ v = x^{-1} \\
(xy)x^{-1} &= x\left(yx^{-1}\right)
\end{aligned}$$

so L is a conjugate loop. ■

Theorem 32 *Let L be a two sided inverse loop and $(L_{x^{-1}}, Id_L, L_{x^{-1}})$ is an autotopism for all $x, y \in L$ then L is a conjugate loop.*

Proof. *Let L be a two sided inverse loop and $(L_{x^{-1}}, Id_L, L_{x^{-1}})$ is an autotopism for all $x, y \in L$. Then by definition*

$$
\begin{aligned}
(u)\, L_{x^{-1}}\, (v)\, Id_L &= (uv)L_{x^{-1}}\ \forall\ u, v \in L \\
(x^{-1}u)(v) &= x^{-1}\,(uv) \\
put\ u &= y\ and\ v = x \\
(x^{-1}y)x &= x^{-1}\,(yx)
\end{aligned}
$$

so L is a conjugate loop. ■

Theorem 33 *Let L be a two sided inverse loop and $(L_x, R_{x^{-1}}, L_x R_{x^{-1}})$ is an autotopism for all $x, y \in L$ then L is a conjugate loop.*

Proof. Let L be a two sided inverse loop and $(L_x, R_{x^{-1}}, L_x R_{x^{-1}})$ is an autotopism for all $x, y \in L$.Then by definition.

$$
\begin{aligned}
(u)\, L_x\, (v)\, R_{x^{-1}} &= (uv)L_x R_{x^{-1}}\ \forall u, v \in L \\
(xu)(vx^{-1}) &= (x\,(uv))x^{-1} \\
put\ u &= e\ and\ v = y \\
(xe)(yx^{-1}) &= (x\,(ey))x^{-1} \\
x(yx^{-1}) &= (xy)x^{-1}
\end{aligned}
$$

so L is a conjugate loop.

Theorem 34 *Let L be a two sided inverse loop and (L_x, L_y, L_{xy}) is an autotopism for all $x, y \in L$ then L is a conjugate loop.*

■

Proof. Let L be a two sided inverse loop and (L_x, L_y, L_{xy}) is an autotopism for all $x, y \in L$.Then by definition

$$\begin{aligned}
(u)\,L_x\,(v)\,L_y &= (uv)L_{xy}\;\forall u, v \in L \\
(xu)(yv) &= xy\,(uv) \\
put\ u &= e\ and\ v = x^{-1} \\
(xe)(yx^{-1}) &= xy(ex^{-1}) \\
x(yx^{-1}) &= (xy)x^{-1}
\end{aligned}$$

so L is a conjugate loop. ■

Theorem 35 *In conjugate loop L,* $C_{x^{-1},x} = L_{x^{-1}}R_xL_{x^{-1}}^{-1}R_x^{-1}$ *is an automorphism.*

Proof. Let $a, b \in L$.

$$\begin{aligned}
(a)C_{x^{-1},x}(b)C_{x^{-1},x} &= (a)L_{x^{-1}}R_xL_{x^{-1}}^{-1}R_x^{-1}(b)L_{x^{-1}}R_xL_{x^{-1}}^{-1}R_x^{-1} \\
&= ((x^{-1}a)x)L_{x^{-1}}^{-1}R_x^{-1}((x^{-1}b)x)L_{x^{-1}}^{-1}R_x^{-1} \\
&= (x^{-1}(ax))L_{x^{-1}}^{-1}R_x^{-1}((x^{-1}(bx))L_{x^{-1}}^{-1}R_x^{-1} \text{ using conjugate property} \\
&= (ax)L_{x^{-1}}L_{x^{-1}}^{-1}R_x^{-1}(bx)L_{x^{-1}}L_{x^{-1}}^{-1}R_x^{-1} \\
&= (ax)R_x^{-1}(bx)R_x^{-1} \\
&= aR_xR_x^{-1}bR_xR_x^{-1} \\
&= ab \\
(ab)C_{x^{-1},x} &= (ab)L_{x^{-1}}R_xL_{x^{-1}}^{-1}R_x^{-1} \\
&= (x^{-1}.ab)x)L_{x^{-1}}^{-1}R_x^{-1} \\
&= (x^{-1}(ab.x))L_{x^{-1}}^{-1}R_x^{-1} \qquad \text{using conjugate property} \\
&= (ab.x)L_{x^{-1}}L_{x^{-1}}^{-1}R_x^{-1} \\
&= (ab.x)R_x^{-1} \\
&= abR_xR_x^{-1} \\
&= ab
\end{aligned}$$

So,

$$(a)C_{x^{-1},x}(b)C_{x^{-1},x} = (ab)C_{x^{-1},x}$$

hence, $C_{x^{-1},x}$ is an automorphism. ∎

Corollary 36 *In conjugate loop L, $(C_{x^{-1},x})^{-1} = R_x L_{x^{-1}} L_{x^{-1}}^{-1} R_x^{-1}$ is an automorphism.*

Theorem 37 *Let L be a conjugate loop and $x \in N(L)$ then following are the autotopisms in L.*

$(i)(C_{x^{-1},x}, R_x, R_x)$

$(ii)(L_x, C_{x^{-1},x}, L_x)$

$(iii)(L_x C_{x^{-1},x}, Id_L, L_x)$

$(iv)((C_{x^{-1},x})^{-1}, R_x, R_x)$

$(v)(L_x, (C_{x^{-1},x})^{-1}, L_x)$

Proof. Let $y, z \in L$

(i)

$$\begin{aligned}
(y)C_{x^{-1},x}.(z)R_x &= (y)L_{x^{-1}} R_x L_{x^{-1}}^{-1} R_x^{-1}.(z)R_x \\
&= ((x^{-1}y)x)L_{x^{-1}}^{-1} R_x^{-1}.zx \\
&= (x^{-1}(yx))L_{x^{-1}}^{-1} R_x^{-1}.zx \quad \text{using conjugate property} \\
&= (yx)L_{x^{-1}} L_{x^{-1}}^{-1} R_x^{-1}.zx \\
&= (yx)R_x^{-1}.zx \\
&= yR_x R_x^{-1}.zx \\
&= y(zx) \\
&= (yz)x \qquad \text{since } x \in N(L) \\
&= (yz)R_x
\end{aligned}$$

so, $(C_{x^{-1},x}, R_x, R_x)$ is an autotopism.

(ii)

$$
\begin{aligned}
(y)L_x(z)C_{x^{-1},x} &= (y)L_x.(z)L_{x^{-1}}R_xL_{x^{-1}}^{-1}R_x^{-1} \\
&= xy.((x^{-1}z)x)L_{x^{-1}}^{-1}R_x^{-1} \\
&= xy.(x^{-1}(zx))L_{x^{-1}}^{-1}R_x^{-1} \quad \text{using conjugate property} \\
&= xy.(zx)L_{x^{-1}}L_{x^{-1}}^{-1}R_x^{-1} \\
&= xy.(zx)R_x^{-1} \\
&= xy.zR_xR_x^{-1} \\
&= (xy)z \\
&= x(yz) \qquad \text{since } x \in N(L) \\
&= (yz)L_x
\end{aligned}
$$

so, $(L_x, C_{x^{-1},x}, L_x)$ is an autotopism.

(iii)

$$
\begin{aligned}
(y)L_xC_{x^{-1},x}.(z)Id_L &= (y)L_xL_{x^{-1}}R_xL_{x^{-1}}^{-1}R_x^{-1}.z \\
&= ((x^{-1}.xy)x)L_{x^{-1}}^{-1}R_x^{-1}.z \\
&= (x^{-1}(xy.x))L_{x^{-1}}^{-1}R_x^{-1}.z \quad \text{using conjugate property} \\
&= (xy.x)L_{x^{-1}}L_{x^{-1}}^{-1}R_x^{-1}.z \\
&= (xy.x)R_x^{-1}.z \\
&= xy.R_xR_x^{-1}.z \\
&= (xy)z \\
&= x(yz) \qquad \text{since } x \in N(L) \\
&= (yz)L_x
\end{aligned}
$$

so, $(L_xC_{x^{-1},x}, Id_L, L_x)$ is an autotopism.

(iv)

$$
\begin{aligned}
(y)(C_{x^{-1},x})^{-1}.(z)R_x &= (y)(L_{x^{-1}}R_xL_{x^{-1}}^{-1}R_x^{-1})^{-1}.(z)R_x \\
&= (y)R_xL_{x^{-1}}R_x^{-1}L_{x^{-1}}^{-1}.zx \\
&= (x^{-1}(yx))R_x^{-1}L_{x^{-1}}^{-1}.zx \\
&= ((x^{-1}y)x)R_x^{-1}L_{x^{-1}}^{-1}.zx \quad \text{using conjugate property} \\
&= (x^{-1}y)R_xR_x^{-1}L_{x^{-1}}^{-1}.zx \\
&= (x^{-1}y)L_{x^{-1}}^{-1}.zx \\
&= yL_{x^{-1}}L_{x^{-1}}^{-1}(zx) \\
&= y(zx) \\
&= (yz)x \qquad \text{since } x \in N(L) \\
&= (yz)R_x
\end{aligned}
$$

so, $((C_{x^{-1},x})^{-1}, R_x, R_x)$ is an autotopism.

(v)

$$
\begin{aligned}
(y)L_x.(z)(C_{x^{-1},x})^{-1} &= (y)L_x.(z)(L_{x^{-1}}R_xL_{x^{-1}}^{-1}R_x^{-1})^{-1} \\
&= xy.(z)R_xL_{x^{-1}}R_x^{-1}L_{x^{-1}}^{-1}. \\
&= xy.(x^{-1}(zx))R_x^{-1}L_{x^{-1}}^{-1} \\
&= xy.((x^{-1}z)x)R_x^{-1}L_{x^{-1}}^{-1} \quad \text{using conjugate property} \\
&= xy.(x^{-1}z)R_xR_x^{-1}L_{x^{-1}}^{-1} \\
&= xy.(x^{-1}z)L_{x^{-1}}^{-1} \\
&= xy.zL_{x^{-1}}L_{x^{-1}}^{-1} \\
&= (xy)z \\
&= x(yz) \qquad \text{since } x \in N(L) \\
&= (yz)L_x
\end{aligned}
$$

so, $(L_x, (C_{x^{-1},x})^{-1}, L_x)$ is an autotopism. ■

3.2 Relation of conjugate loops with other types of loop

In this section we investigate connections between conjugate loops and other branches of loops.

Theorem 38 *Every Steiner loop is a conjugate loop.*

Proof. As we know that commutative IP loop of exponent 2 is a Steiner loop so it satisfies

$$x^{-1}(xy) = y$$

and

$$(yx)x^{-1} = y$$

Now

$$\begin{aligned} (x^{-1}y)x &= x(x^{-1}y) \\ &= y \\ x^{-1}(yx) &= x^{-1}(xy) \\ &= y \end{aligned}$$

so, every Steiner loop is a conjugate loop. ■

Theorem 39 *Every Moufang loop is a conjugate loop.*

Proof. Consider a Moufang loop identity

$$\begin{aligned} (xy)(zx) &= (x(yz))x \\ \text{putting } z &= x^{-1} \\ xy &= (x(yx^{-1}))x \\ (xy)x^{-1} &= ((x(yx^{-1}))x)x^{-1} \\ &= x(yx^{-1}) \quad \text{by inverse property} \end{aligned}$$

so, every Moufang loop is a conjugate loop. ■

Remark 40 *Following loop is a conjugate loop but not a Steiner neither Moufang. So, converses of above theorems are not true.*

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	1	0	3	4	2
2	2	3	4	0	1
3	3	4	1	2	0
4	4	2	0	1	3

Theorem 41 *Every LC-loop is a conjugate loop.*

Proof. The identity of LC-loop, given in the remark(5) is

$$\begin{aligned} x(x.yz) &= x(xy.z) \\ \text{replacing } z &= x^{-1} \\ x(x.yx^{-1}) &= x(xy.x^{-1}) \\ x^{-1}(x(x.yx^{-1})) &= x^{-1}(x(xy.x^{-1})) \text{ pre-multiplying by } x^{-1} \\ x.yx^{-1} &= xy.x^{-1} \end{aligned}$$

Hence, every LC-loop is a conjugate loop. ∎

Theorem 42 *Every RC-loop is a conjugate loop.*

Proof. The identity of RC-loop, given in the remark (5) is

$$\begin{aligned} (x.yz)z &= (xy.z)z \\ \text{replacing } z &= x^{-1} \\ (x.yx^{-1})x^{-1} &= (xy.x^{-1})x^{-1} \\ ((x.yx^{-1})x^{-1})x &= ((xy.x^{-1})x^{-1})x \quad \text{post-multiplying by } x \\ x.yx^{-1} &= xy.x^{-1} \end{aligned}$$

Hence every RC-loop is a conjugate loop. ∎

Corollary 43 *According to Theorem(6), A loop L is a C-loop if and only if it is both LC- and RC-loop. so,*

$$C\text{-}loop \Rightarrow LC\text{-}loop \Rightarrow Conjugate\ loop$$
$$C\text{-}loop \Rightarrow RC\text{-}loop \Rightarrow Conjugate\ loop$$

or

$$C\text{-}loop \Rightarrow Conjugate\ loop$$

Therefore, every C-loop is a conjugate loop.

Exercise 44 *Consider the followind loop.*

·	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	4	5	3	2
2	2	5	0	4	1	3
3	3	4	5	0	2	1
4	4	2	3	1	5	0
5	5	3	1	2	0	4

This conjugate loop is not a C-loop neither LC nor RC.

Now it can be seen that conjugate loops find their place in Fig.1 of chapter 2, as shown below in Fig.3

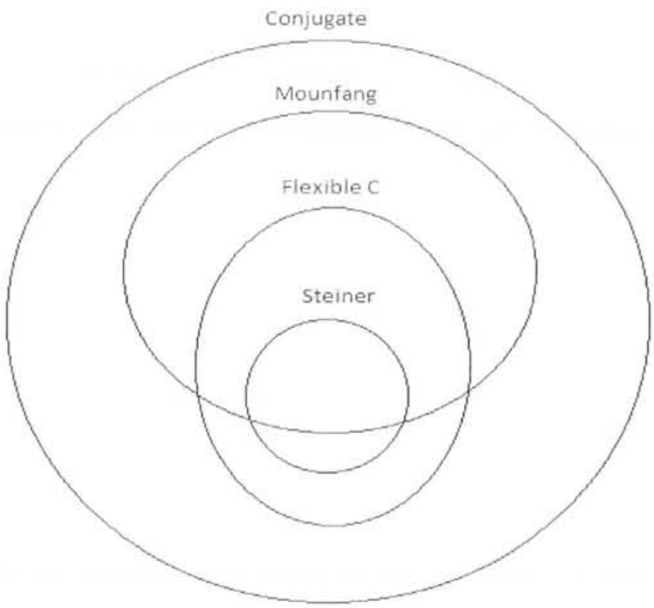

Fig. 3

Theorem 45 *Let L be a loop of exponent 2 then L is a conjugate loop iff it is flexible.*

Proof. Suppose L is a conjugate loop then

$$x(yx^{-1}) = (xy)x^{-1} \; \forall x, y \in L \tag{3.3}$$

since L is of exponent 2 so,

$$x^2 = e$$

$$or$$

$$x^{-1} = x \tag{3.4}$$

so, above equation(3.3) becomes

$$x(yx) = (xy)x \ \forall x, y \in L$$

which proves that L is a flexible loop.

Now conversely suppose that L is a flexible loop.Then it satisfies

$$x(yx) = (xy)x \ \forall x, y \in L$$

From equation(3.4),we have

$$x(yx^{-1}) = (xy)x^{-1} \ \forall x, y \in L$$

so, L is a conjugate loop. ■

Counter Examples:

Example 46 *$L_{x,y}$ is not a pseudo-automorphism in conjugate loops.Consider the following conjugate loop of order 7.*

·	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	0	5	6	4	3
2	2	0	1	6	5	3	4
3	3	6	5	4	0	1	2
4	4	5	6	0	3	2	1
5	5	3	4	2	1	6	0
6	6	4	3	1	2	0	5

consider

$$\begin{aligned} L_{1,3} &= (1\ 6) \\ (1.2)L_{1,3} &= (0)L_{1,3} = 0 \\ (1)L_{1,3.}(2)L_{1,3} &= 6.2 = 3 \\ (1.2)L_{1,3} &\neq (1)L_{1,3}(2)L_{1,3} \end{aligned}$$

And we cannot find any companion $c \in L$ such that

$$(1.2)L_{1,3}.c = (1)L_{1,3}(2.c)L_{1,3}$$

By Theorem(2),in moufang loops every innermapping is a pseudo-automorphism, shows that this result is not true for conjugate loops.

Example 47 *According to Theorem (3), in Moufang loops Nucleus is a normal subloop but it is not true in the case of conjugate loops. Consider following conjugate loop L of order 8.*

·	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	4	5	2	3	7	6
2	2	5	0	6	7	1	3	4
3	3	4	6	0	1	7	2	5
4	4	3	1	7	6	0	5	2
5	5	2	7	1	0	6	4	3
6	6	7	3	2	5	4	1	0
7	7	6	5	4	3	2	0	1

here N(L)={0,1} which is not normal in L.

Example 48 *Following conjugate loop contradicts Theorem(4).*

.	0	1	2	3	4
0	0	1	2	3	4
1	1	0	3	4	2
2	2	4	0	1	3
3	3	2	4	0	1
4	4	3	1	2	0

Example 49 *Following conjugate loop is not a power-associative, hence a counter example of*

Theorem(8) .

·	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	6	7	4
2	2	3	1	0	6	7	4	5
3	3	2	0	1	7	4	5	6
4	4	5	6	7	3	0	1	2
5	5	6	7	4	0	1	2	3
6	6	7	4	5	1	2	3	0
7	7	4	5	6	2	3	0	1

Example 50 *Following non associative conjugate loop is of order 5 so Theorem*(11) *fails for conjugate loops.This loop also contradicts Theorem*(10) (*ii*).

.	0	1	2	3	4
0	0	1	2	3	4
1	1	0	3	4	2
2	2	4	0	1	3
3	3	2	4	0	1
4	4	3	1	2	0

Example 51 *Conjugate loop given below contradicts Theorem*(10) (*i*).

.	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	3	4	5	2
2	2	4	0	5	3	1
3	3	5	1	0	2	4
4	4	2	5	1	0	3
5	5	3	4	2	1	0

Theorem 52 *Every CIP(cross inverse property) loop is a conjugate loop.*

Following example shows that the converse of above theorem does not hold.

Example 53

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	4	5	3	2
2	2	5	0	4	1	3
3	3	4	5	0	2	1
4	4	2	3	1	5	0
5	5	3	1	2	0	4

This conjugate loop of order 6 is not a CIP loop.

Theorem 54 *A conjugate loop L is a CIP loop iff every element in L is self conjugate.*

Theorem 55 *Every loop of exponent 3 is Jordan iff it is conjugate loop.*

Proof. Let L be a loop of exponent 3 and suppose that it is Jordan.Then

$$\begin{aligned} x^2(yx) &= (x^2y)x \\ x^{-1}(yx) &= (x^{-1}y)x \quad \text{since L is of exponent 3} \end{aligned}$$

So L is a conjugate loop.

Conversely suppose that L is a conjugate loop of exponent 3.

$$\begin{aligned} x^{-1}(yx) &= (x^{-1}y)x \\ x^2(yx) &= (x^2y)x \quad \text{since L is of exponent 3} \end{aligned}$$

So, L is a Jordan loop. ■

Theorem 56 *Let L be an IP loop then L is a conjugate iff it is flexible.*

Proof. *Suppose L is a conjugate loop, we have to show that it is flexible*

$$
\begin{aligned}
(x(yx))x^{-1} &= x((yx.x^{-1}) && \textit{since } L \textit{ is a conjugate loop} \\
&= xy && \textit{by inverse property} \\
(x(yx).x^{-1})x &= (xy)x && \textit{post multiplying by } x \\
(x(yx) &= (xy)x && \textit{by inverse property}
\end{aligned}
$$

so L is flexible loop

now, conversely suppose that L is flexible loop

$$
\begin{aligned}
((x(yx^{-1}))x &= x(yx^{-1}.x) && \textit{by flexibility} \\
&= xy && \textit{by inverse property} \\
((x(yx^{-1}).x)x^{-1} &= (xy)x^{-1} && \textit{post multiplying by } x^{-1} \\
x(yx^{-1}) &= (xy)x^{-1}
\end{aligned}
$$

So L is a conjugate loop. ■

Fig .4 describe these results diagrammatically.

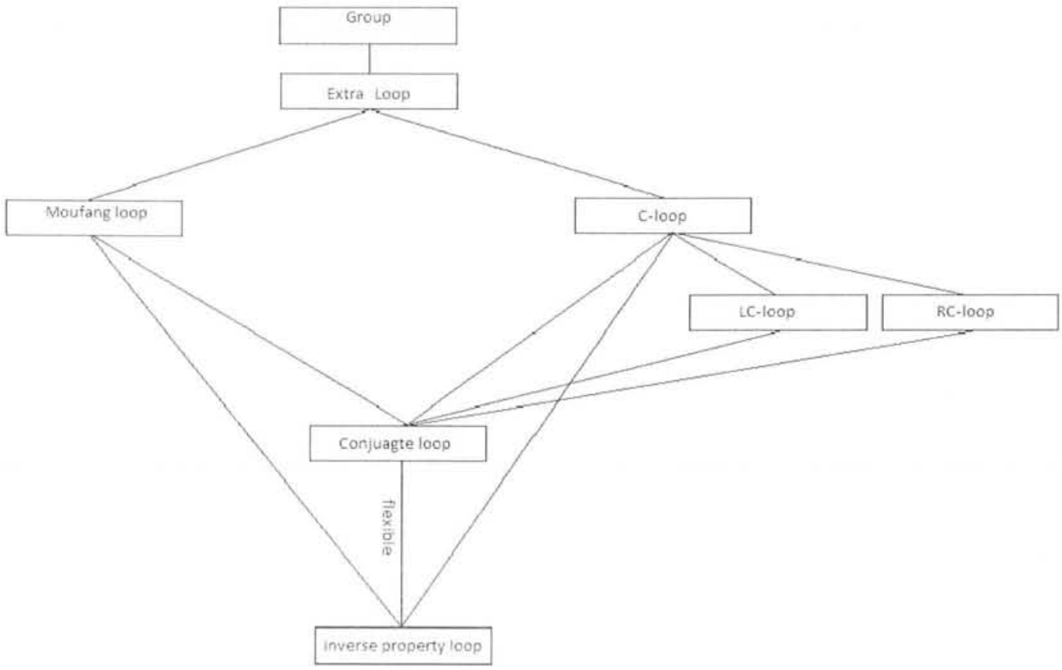

Fig.4

3.2.1 Inverse property Conjugate Flexible loop(IPCF loop)

Theorem (56) shows that an IP loop is conjugate loop iff it is flexible. Here we now define a class of loop which has all these three properties together called IPCF loops.

Counting of IPCF loops

Smallest non associative IPCF loop is of order 7, given below

.	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	0	5	6	4	3
2	2	0	1	6	5	3	4
3	3	6	5	4	0	1	2
4	4	5	6	0	3	2	1
5	5	3	4	2	1	6	0
6	6	4	3	1	2	0	5

This loop is not a dia-associative loop.

Since every Moufang is an IP, conjugate and flexible loop so, every moufang loop is a IPCF loop.Also steiner loops are IPCF loops.

In [1] J. Slaney and A. Ali listed the IP loops upto the order 13. Here we added to that list, IPCF loops of the respective orders.

size	IP loops	IPCF loops
7	1	1
8	3	0
9	5	0
10	45	7
11	48	2
12	2679	1
13	10341	5

Properties of IPCF loops

In commutative conjugate loop L,

$$g(ag^{-1}) \neq a \; \forall \; g, a \in L$$

but in commutative IPCF loops

$$\begin{aligned} g(ag^{-1}) &= g(g^{-1}a) \text{ by commutativity} \\ &= a \qquad\quad \text{ by inverse property} \end{aligned}$$

so we have a following result

Theorem 57 *An IPCF loop L is commutative iff every element in L is self conjugate.*

Proof. Suppose L is commutative and $g, a \in L$ then

$$\begin{aligned} g(ag^{-1}) &= g(g^{-1}a) \text{ by commutativity} \\ &= a \qquad\quad \text{ by inverse property} \end{aligned}$$

which implies that every element in L is self-conjugate.

Now, conversely suppose that in L every element is self-conjugate then

$$\begin{aligned} g(ag^{-1}) &= a \\ ((ga)g^{-1})g &= ag \text{ post multiplying by g} \\ ga &= ag \text{ by inverse property} \end{aligned}$$

So, L is commutative. Hence the result follows. ■

Theorem 58 *Let L be a IPCF loop then $x \in Z(L)$ iff following conditions hold.*

(i) $x \in C(L)$

(ii) $(R_x, L_x^{-1}R_x, R_x)$ *is an autotopism.*

Proof. Suppose first that the (i) and (ii) holds.

$$\begin{aligned}
& \textit{since } (R_x, L_x^{-1}R_x, R_x) \textit{ is an autotopism} \\
\Rightarrow\ & (y)R_x.(z)L_x^{-1}R_x = (yz)R_x \\
\Rightarrow\ & (y)R_x.(z)L_{x^{-1}}R_x = (yz)R_x \\
\Rightarrow\ & yx.(x^{-1}z)x = (yz)x \\
\Rightarrow\ & yx.x^{-1}(zx) = (yz)x \text{ using conjugate property} \\
\Rightarrow\ & yx.x^{-1}(xz) = (yz)x \text{ since } x \in C(L) \\
\Rightarrow\ & yx.z = yz.x \text{ using inverse property} \\
\Rightarrow\ & xy.z = x.yz \text{ since } x \in C(L) \\
\Rightarrow\ & x \in N_\lambda(L)
\end{aligned}$$

we know that in IP loops

$$\begin{aligned}
N(L) &= N_\lambda(L) = N_\mu(L) = N_\rho(L) \text{ using theorem (12)} \\
&\Rightarrow x \in N(L) \\
&\Rightarrow x \in N(L) \cap C(L) \text{ as } x \in C(L) \\
&\Rightarrow x \in Z(L) \text{ as } Z(L) = N(L) \cap C(L)
\end{aligned}$$

Conversely, let

$$\begin{aligned}
x &\in Z(L) \\
&\Rightarrow x \in N(L) \cap C(L) \\
&\Rightarrow x \in N(L) \text{ and } x \in C(L)
\end{aligned}$$

As

$$
\begin{aligned}
N(L) &= N_\lambda(L) = N_\mu(L) = N_\rho(L) \text{ using Theorem (12)} \\
&\Rightarrow x \in N_\lambda(L) \\
&\Rightarrow xy.z = x.yz \ \forall\ y, z \in L \\
&\Rightarrow yx.z = yz.x \text{ since } x \in C(L) \\
&\Rightarrow yx.x(x^{-1}z) = yz.x \text{ using inverse property} \\
&\Rightarrow yx.(x^{-1}z)x = yz.x \text{ since } x \in C(L) \\
&\Rightarrow (y)R_x(z)L_x^{-1}R_x = (yz)R_x \\
&\Rightarrow (R_x, L_x^{-1}R_x, R_x) \text{ is an autotopism}
\end{aligned}
$$

■

Corollary 59 *Let L be a IPCF loop and $x \in C(L)$ then following are the autotopisms in L.*

(i) $(R_x, L_x^{-1}R_x, R_x)$

(ii) $(L_x^{-1}R_x, L_x, L_x)$.

3.3 Construction of conjugate loops

Let G be a multiplicative group with neutral element 1, and A be an abelian group written additively with neutral element 0. Any map

$$\mu : G \times G \to A$$

satisfying

$$\mu(1, g) = \mu(g, 1) = 0 \text{ for every } g \in G$$

is called a factor set. When $\mu : G \times G \to A$ is a factor set, we can define multiplication on $G \times A$ by

$$(g, a)(h, b) = (gh, a + b + \mu(g, h)). \tag{3.5}$$

The resulting groupoid is clearly a loop with neutral element $(1, 0)$. It will be denoted by (G, A, μ). Additional properties of (G, A, μ) can be enforced by additional requirements on μ.

We construct conjugate loop with the help of two groups such that one is multiplicative group and other is additive abelian group.

Theorem 60 *Let $\mu : G \times G \to A$ be a factor set then (G, A, μ) is a conjugate loop if and only if*

$$\mu(g, h) + \mu(gh, g^{-1}) = \mu(h, g^{-1}) + \mu(g, hg^{-1}) \tag{3.6}$$

Proof. *By definition the loop (G, A, μ) is conjugate if and only if*

$$\{(g, a)(h, b)\}(g, a)^{-1} = (g, a)\{(h, b)(g, a)^{-1}\}$$

consider L.H.S

$$\begin{aligned}
\{(g, a)(h, b)\}(g, a)^{-1} &= (gh, a + b + \mu(g, h))(g^{-1}, -a - \mu(g, g^{-1})) \\
&= ((gh)g^{-1}, a + b + \mu(g, h)) - a - \mu(g, g^{-1}) + \mu(gh, g^{-1})) \\
&= ((gh)g^{-1}, b + \mu(g, h)) - \mu(g, g^{-1}) + \mu(gh, g^{-1}))
\end{aligned}$$

Now consider R.H.S

$$
\begin{aligned}
(g,a)\{(h,b)(g,a)^{-1}\} &= (g,a)\{(h,b)g^{-1}, -a-\mu(g,g^{-1})\} \\
&= (g,a)(hg^{-1}, b-a-\mu(g,g^{-1})+\mu(h,g^{-1})) \\
&= (g(hg^{-1}), a+b-a-\mu(g,g^{-1})+\mu(h,g^{-1})+\mu(g,hg^{-1})) \\
&= ((gh)g^{-1}, b-\mu(g,g^{-1})+\mu(h,g^{-1})+\mu(g,hg^{-1}))
\end{aligned}
$$

Comparing both sides we get

$$\mu(g,h))+\mu(gh,g^{-1})=\mu(h,g^{-1})+\mu(g,hg^{-1})$$

■

Theorem 61 *Let $n>2$ be an integer and let A be an abelian group of order n, and $\alpha\in A$ an element of order bigger than 2. Let $G=\{1,u,v,w\}$ be the multiplicative group with neutral element 1. We define $\mu: G\times G\to A$ by*

$$\mu(x,y)=\begin{cases}\alpha \text{ if } (x,y)=(u,w),(w,u)\\ -\alpha \text{ if } (x,y)=(u,v),(v,u)\\ 0 \text{ otherwise}\end{cases}$$

Then (G,A,μ) is a conjugate loop with $N(L)=\{(1,a): a\in A\}$.

Proof. Let $L=(G,A,\mu)$. The map μ is conjugate factor set. It can be depicted as follows:

μ	1	u	v	w
1	0	0	0	0
u	0	0	$-\alpha$	α
v	0	$-\alpha$	0	0
w	0	α	0	0

where we know that the group G has the following multiplication table.

.	1	u	v	w
1	1	u	v	w
u	u	1	w	v
v	v	w	1	u
w	w	v	u	1

To show that $L = (G, A, \mu)$ is conjugate loop, we verify equation (3.6) as follows.
Take $h = 1$ in equation (3.6) we have

$$\begin{aligned} \mu(g,1) + \mu(g,g^{-1}) &= \mu(1,g^{-1}) + \mu(g,g^{-1}) \\ \mu(g,1) &= \mu(1,g^{-1}) \end{aligned}$$

which is true for all $g \in G$.
Take $h = u$ in equation (3.6) we have

$$\mu(g,u) + \mu(gu,g^{-1}) = \mu(u,g^{-1}) + \mu(g,ug^{-1})$$

Now, putting $g = 1, u, v, w$

$$\begin{aligned} \mu(1,u) + \mu(u,1) &= \mu(u,1) + \mu(1,u) = 0 \\ \mu(u,u) + \mu(1,u) &= \mu(u,u) + \mu(u,1) = 0 \\ \mu(v,u) + \mu(w,v) &= \mu(u,v) + \mu(v,w) \\ -\alpha + 0 &= -\alpha + 0 = 0 \\ \mu(w,u) + \mu(v,w) &= \mu(u,w) + \mu(w,v) \\ \alpha + 0 &= \alpha + 0 = 0 \end{aligned}$$

Now, we put $h = v$ in equation (3.6) we have

$$\mu(g,v) + \mu(gv,g^{-1}) = \mu(v,g^{-1}) + \mu(g,vg^{-1})$$

putting $g = 1, u, v, w$

$$\begin{aligned}
\mu(1,v) + \mu(v,1) &= \mu(v,1) + \mu(1,v) = 0 \\
\mu(u,v) + \mu(w,u) &= \mu(v,v) + \mu(v,1) \\
-\alpha + \alpha &= 0 + 0 = 0 \\
\mu(w,v) + \mu(u,w) &= \mu(v,w) + \mu(w,u) \\
0 + \alpha &= 0 + \alpha = 0
\end{aligned}$$

We put now $h = w$ in equation (3.6) we have

$$\mu(g,w) + \mu(gw,g^{-1}) = \mu(w,g^{-1}) + \mu(g,wg^{-1})$$

putting $g = 1, u, v, w$

$$\begin{aligned}
\mu(1,w) + \mu(w,1) &= \mu(w,1) + \mu(1,w) = 0 \\
\mu(u,w) + \mu(v,u) &= \mu(w,u) + \mu(u,v) \\
\alpha - \alpha &= \alpha - \alpha = 0 \\
\mu(g,w) + \mu(gw,g^{-1}) &= \mu(w,g^{-1}) + \mu(g,wg^{-1}) \\
\mu(v,w) + \mu(u,v) &= \mu(w,v) + \mu(v,u) \\
0 - \alpha &= 0 - \alpha = 0 \\
\mu(w,w) + \mu(1,w) &= \mu(w,w) + \mu(w,1) = 0
\end{aligned}$$

hence equation (3.6) is true for all values of G. Now we will show that associative law does not hold in $L = (G, A, \mu)$. For this consider

$$(u,a)((v,a)(v,a) = (u,a)(1,2a) = (u,3a)$$

and

$$((u,a)(v,a))(v,a) = (w,2a-\alpha)(v,a) = (u,3a-\alpha)$$

$$\Rightarrow (u,a)((v,a)(v,a) \neq ((u,a)(v,a))(v,a)$$

Now it remains to show that $N(L) = \{(1,a) : a \in A]\}$. For this consider

$$\begin{aligned}
((g,b)(1,a))(h,c) &= (g, b+a+\mu(g,1))(h,c) \\
&= (g, b+a+0)(h,c) \\
&= (gh, b+a+c+\mu(g,h)) \\
(g,b)((1,a)(h,c)) &= (g,b)(h, a+c+\mu(1,h)) \\
&= (g,b)(h, a+c+0) \\
&= (gh, b+a+c+\mu(g,h))
\end{aligned}$$

so,

$$\begin{aligned}
&\Rightarrow ((g,b)(1,a))(h,c) = (g,b)((1,a)(h,c)) \\
&\Rightarrow (1,a) \in N_{\mu}(L)
\end{aligned}$$

Similarly we can show that

$$(1,a) \in N_{\lambda}(L) \text{ and } (1,a) \in N_{\rho}(L)$$

hence

$$\begin{aligned}
(1,a) &\in N(L) \\
&\Rightarrow N(L) = \{(1,a) : a \in A\}
\end{aligned}$$

Which is the required result. ■

Corollary 62 *For any abelian group A of order $n > 2$ there exists a conjugate L such that $order(A) = order(N(L))$.*

Example 63 *Let $G = \{1_G, u, v, w\}$ and $A = \{0, 1, 2\}$, then (G, A, μ) is the conjugate loop whose multiplication table is given by*

$\cdot$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2	0	4	5	3	7	8	6	10	11	9
2	2	0	1	5	3	4	8	6	7	11	9	10
3	3	4	5	0	1	2	10	11	9	8	6	7
4	4	5	3	1	2	0	11	9	10	6	7	8
5	5	3	4	2	0	1	9	10	11	7	8	6
6	6	7	8	10	11	9	0	1	2	3	4	5
7	7	8	6	11	9	10	1	2	0	4	5	3
8	8	6	7	9	10	11	2	0	1	5	3	4
9	9	10	11	8	6	7	3	4	5	0	1	2
10	10	11	9	6	7	8	4	5	3	1	2	0
11	11	9	10	7	8	6	5	3	4	2	0	1

where

$$(G, A, \mu) = G \times A = \{(1_G, 0), (1_G, 1), (1_G, 2), (u, 0), (u, 1), (u, 2), (v, 0), (v, 1), (v, 2), (w, 0), (w, 1), (w, 2)\}$$

and

$$\begin{aligned} (1_G, 0) &= 0, (1_G, 1) = 1, (1_G, 2) = 2, (u, 0) = 3, (u, 1) = 4, (u, 2) = 5, \\ (v, 0) &= 6, (v, 1) = 7, (v, 2) = 8, (w, 0) = 9, (w, 1) = 10, (w, 2) = 11 \end{aligned}$$

Also

$$N((G, A, \mu)) = \{(1_G, 0), (1_G, 1), (1_G, 2)\}$$

This is the smallest non-associative non alternative commutative conjugate loop constracted through this process.By changing the order of additive abelian group A, we can constract the conjugate loops of high orders.

Remark 64 *Conjugate loop formed by taking* $G = \{1, u, u^2\}$ *and* $A = \{0, 1\}$, *is associative loop(group) of order 6.*

Lemma 65 *Let* $\mu : G \times G \to A$ *is a factor set, then the loop* (G, A, μ) *is commutative if and only if*

$$gh = hg \text{ and } \mu(g, h) = \mu(h, g) \text{ for all } g, h \in G.$$

Proof. *Let* $(g, a), (h, b) \in (G, A, \mu)$, *then,*$(G, A, \mu)$ *is commutative*

$$\begin{aligned}
&\Leftrightarrow \quad (g, a)(h, b) = (h, b)(g, a) \\
&\Leftrightarrow \quad (gh, a + b + \mu(g, h)) = (hg, b + a + \mu(h, g)) \text{ using equation (3.5)} \\
&\Leftrightarrow \quad gh = hg \text{ and } \mu(g, h) = \mu(h, g)
\end{aligned}$$

■

Next we will prove that the loop constructed by Theorem(60) is also an IP loop.

Theorem 66 *Let* $\mu : G \times G \to A$ *be a factor set defined in the same way as in Theorem*(60), *where G is a Klein group and A is any abelian group of order* $n > 2$,*then* (G, A, μ) *is a conjugate,flexible and IP loop.*

Proof. Conjugate property of (G, A, μ) follows from Theorem(60),

As G is kelien group, so $g = g^{-1}$ and by lemma(65) hg=gh for all $g, h \in G$.Using these values in equation (3.6) we have

$$\begin{aligned}
\mu(g, h) + \mu(gh, g) &= \mu(h, g) + \mu(g, hg) \\
b + 2a + \mu(g, h) + \mu(gh, g) &= b + 2a + \mu(h, g) + \mu(g, hg) \text{ where a,b} \in \text{A} \\
b + a + a + \mu(g, h) + \mu(gh, g) &= b + a + a + \mu(h, g) + \mu(g, hg) \because \text{A is a group} \\
a + b + \mu(g, h) + a + \mu(gh, g) &= a + b + a + \mu(h, g) + \mu(g, hg) \\
((gh)g, a + b + \mu(g, h) + a + \mu(gh, g)) &= ((gh)g, a + b + a + \mu(h, g) + \mu(g, hg)) \\
((gh)g, a + b + \mu(g, h) + a + \mu(gh, g)) &= (g(hg), a + b + a + \mu(h, g) + \mu(g, hg)) \because \text{G is a group} \\
(gh, a + b + \mu(g, h)(g, a) &= (g, a)(hg, b + a + \mu(h, g) \\
((g, a)(h, b))(g, a) &= (g, a)((h, b)(g, a))
\end{aligned}$$

This implies that (G, A, μ) is a flexible loop.

Now, w,ll show that it is also an IP loop.For this again considering equation (3.6).

$$\begin{aligned}
&\Rightarrow \mu(g,h)+\mu(gh,g^{-1}) \\
&= \mu(h,g^{-1})+\mu(g,hg^{-1}) \\
&\Rightarrow \mu(g,h)+\mu(g^{-1},gh) \\
&= \mu(h,g)+\mu(hg,g^{-1}) \\
&\Rightarrow -a-\mu(g,g^{-1})+a+b+\mu(g,h)+\mu(g^{-1},gh) \\
&= b+a+\mu(h,g)-a-\mu(g,g^{-1})+\mu(hg,g^{-1}) \\
&\Rightarrow (h,-a-\mu(g,g^{-1})+a+b+\mu(g,h)+\mu(g^{-1},gh)) \\
&= (h,b+a+\mu(h,g)-a-\mu(g,g^{-1})+\mu(hg,g^{-1})) \\
&\Rightarrow (1.h,-a-\mu(g,g^{-1})+a+b+\mu(g,h)+\mu(g^{-1},gh)) \\
&= (h.1,b+a+\mu(h,g)-a-\mu(g,g^{-1})+\mu(hg,g^{-1})) \\
&\Rightarrow (g^{-1}(gh),-a-\mu(g,g^{-1})+a+b+\mu(g,h)+\mu(g^{-1},gh)) \\
&= ((hg)g^{-1},b+a+\mu(h,g)-a-\mu(g,g^{-1})+\mu(hg,g^{-1})) \\
&\Rightarrow (g^{-1},-a-\mu(g,g^{-1}))(gh,a+b+\mu(g,h) \\
&= (hg,b+a+\mu(h,g))(g^{-1},-a-\mu(g,g^{-1}))
\end{aligned}$$

by using lemma(65) we can write

$$\begin{aligned}
&\Rightarrow (g,a)^{-1}(gh,a+b+\mu(g,h) = (hg,b+a+\mu(h,g))(g,a)^{-1} \\
&\Rightarrow (g,a)^{-1}((g,a)(h,b)) = ((h,b)(g,a))(g,a)^{-1}
\end{aligned}$$

This implies that (G, A, μ) is an IP loop. hence the result follows.

Theorem 67 *Let $\mu : G \times G \to A$ be a factor set defined in the same way as in Theorem(60), where G is a klein group and A is any abelian group of order $n > 2$,then (G, A, μ) is a conjugate Jordan loop.*

Proof. *Conjugate property of (G, A, μ) follows from Theorem(60) and from the definition of $\mu : G \times G \to A$ we can write*

$$
\begin{aligned}
\mu(h,g) &= \mu(h,g) \text{ for all } h,g \in G \\
&\Rightarrow \mu(h,g) + \mu(1,hg) = \mu(1,h) + \mu(1.h,g) \\
&\Rightarrow \mu(h,g) + \mu(g^2,hg) = \mu(g^2,h) + \mu(g^2h,g) \\
&\Rightarrow a + a + \mu(g,g) + b + a + \mu(h,g) + \mu(g^2,hg) \\
&= a + a + \mu(g,g) + b + \mu(g^2,h) + \mu(g^2.h,g) \\
&\Rightarrow (g^2(hg), a + a + \mu(g,g) + b + a + \mu(h,g) + \mu(g^2,hg)) \\
&= (g^2(hg), a + a + \mu(g,g) + b + \mu(g^2,h) + \mu(g^2.h,g)) \\
&\Rightarrow (g^2(hg), a + a + \mu(g,g) + b + a + \mu(h,g) + \mu(g^2,hg)) \\
&= (g^2h)g, a + a + \mu(g,g) + b + \mu(g^2,h) + \mu(g^2h,g)) \text{ since } G \text{ is a group} \\
&\Rightarrow (g^2, a + a + \mu(g,g))(hg, b + a + \mu(h,g)) \\
&= (g^2h, a + a + \mu(g,g) + b + \mu(g^2,h))(g,a) \\
&\Rightarrow [(g,a)(g,a)][(h,b)(g,a)] = [(g^2, a + a + \mu(g,g))(h,b)](g,a) \\
&\Rightarrow (g,a)^2[(h,b)(g,a)] = [\{(g,a)(g,a)\}(h,b)](g,a) \\
&\Rightarrow (g,a)^2[(h,b)(g,a)] = [(g,a)^2(h,b)](g,a)
\end{aligned}
$$

which is a Jordan identity also (G, A, μ) is commutative. Hence (G, A, μ) is a Jordan loop.

■

■

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