SOME STUDIES IN FUZZY HEMIRINGS

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A Thesis Submitted in the Partial Fulfillment of the requirement for the Degree of

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In
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Certificate

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We accept this thesis as conforming to the required standard.

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April 2011

Dedicated to

My Parents
My Husband
&
My Daughter
(Zahabia)

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0.2 Introduction

There are many concepts of universal algebra generalizing associative ring $(R, +, .)$. Some of them, in particular, nearrings and several kinds of semirings, have been proven very useful. Nearrings arise from rings by canceling either the axiom of left or those of right distributivity. The second type of these algebras $(R, +, .)$ called semirings (and sometimes halfrings), share the same properties as a ring except that $(R, +)$ is assumed to be a semigroup rather than an abelian group.

The notion of semiring was introduced by Vandiver in 1934 [42]. Semirings, ordered semirings and hemirings have been found useful for solving problems in different areas of applied mathematics and information sciences, since the structure of a semiring provides an algebraic framework for modeling and studying the key factors in these applied areas. They play an important role in studying optimization theory, graph theory, theory of discrete event dynamical systems, matrices, determinants, automata theory, formal languages and so on (see [8, 9, 17, 20, 40, 43]).

The theory of fuzzy sets, proposed by Zadeh [47] in 1965, has provided a useful mathematical tool for describing the behavior of systems that are too complex or ill-defined to admit precise mathematical analysis by classical methods and tools. Extensive applications of fuzzy set theory have been found in various fields such as artificial intelligence, computer science, control engineering, expert systems, management science, operations research, pattern recognition, robotics and others.

It was soon arise a natural question concerning a relation between fuzzy sets and algebraic systems. The study of the fuzzy algebraic structures has started in the pioneering paper of Rosenfeld [38] in 1971. He introduced the notion of fuzzy groups and showed that many results in groups can be extended to develop the theory of fuzzy groups in an elementary manner. After that the literature of various fuzzy algebraic concepts has been developing rapidly. Many authors fuzzified certain standard concepts and results on rings and modules. Investigations of fuzzy semirings were initiated in [5]. The relationship between the fuzzy sets and semirings (hemirings) has been considered by Dutta, Baik, Ghosh, Jun, Kim, Zhan and others [7, 14, 15, 18, 19, 25, 26, 29, 50].

Ideals play an important role in the structure theory of hemirings and are useful for many purposes. But they do not coincide with usual ring ideals. For this reason many results in ring theory have no analogues in semirings using only ideals. Henriksen defined [23] a more restricted class of ideals in semirings, which is called class of k-ideals, with the property that if the semiring R is the ring then a complex in R is a k-ideal if and only if it is a ring ideal. A still more restricted class of ideals in hemirings has been given by Iizuka [24]. However, a definition of ideal in any additively commutative semiring R can be given which coincides with Iizuka's definition provided R is a hemiring, and it is called h-ideal. La Torre [32] investigated h-ideals and k-ideals in hemirings in an effort to obtain analogues of familiar ring theorems. Fuzzy h-ideals and fuzzy k-ideals are studied in [6, 7, 26, 27, 34, 35, 46, 49].

0.3 Chapter-wise study

This thesis consists of eight chapters. Throughout this thesis, R will denote a hemiring, unless otherwise stated.

Chapter one, which is of introductory nature, provides basic definitions and results, which are needed for the subsequent chapters.

In Chapter two, we give the basic properties of k-product and k-sum of two fuzzy subsets and then characterize hemiring by the properties of k-ideals and by the properties of fuzzy k-ideals. In this chapter we give the primeness and semiprimeness of k-ideals and primeness and semiprimeness of fuzzy k-ideals.

In chapter three, we give some basic properties of h-intrinsic product and h-sum of two fuzzy subsets and then characterize hemiring by the properties of h-ideals and also by the properties of fuzzy h-ideals. In this chapter we give the primeness and semiprimeness of h-ideals and fuzzy h-ideals. In this chapter also the space of prime h-ideals and of fuzzy h-prime h-ideals is topologized.

In chapter four we define right k-weakly regular hemirings, which are generalization of k-regular hemirings. We characterize hemirings by the properties of their right k-ideals and by the properties of their fuzzy right k-ideals.

In chapter five, we characterize those hemirings for which each right h-ideal is idempotent. We also characterize those hemirings for which each fuzzy right h-ideal is idempotent. We have given the concept of right pure h-ideals, purely prime h-ideals, fuzzy right pure h-ideals and fuzzy purely prime h-ideals and characterize hemirings by these ideals.

In chapter six, we introduce the concepts of fuzzy k-bi-ideals and fuzzy k-quasi-ideals of hemirings. We characterize different classes of hemirings by the properties of k-bi-ideals and k-quasi-ideals.

In the chapter seven, we define prime, strongly prime and semiprime k-bi-ideals of a hemiring. We also define their fuzzy versions and characterize hemirings by the properties of these k-bi-ideals.

In the chapter eight, we define prime, strongly prime and semiprime h-bi-ideals of a hemiring. We also define their fuzzy versions and characterize hemirings by the properties of these h-bi-ideals.

Chapter 1

Preliminaries

The aim of this chapter is to provide the essential definitions and preliminaries results, concerning hemirings which are useful for our subsequent chapters. For undefined terms and notations, we refer to [20] and [21].

1.1 Definitions and Notations

A *semiring* is an algebraic system $(R, +, \cdot)$ consisting of a non-empty set R together with two binary operations called "addition" and "multiplication" (denoted in the usual manner) such that $(R, +)$ and $(R, \cdot)$ are semigroups and the following distributive laws:

$$a \cdot (b + c) = a \cdot b + a \cdot c, \text{ and } (b + c) \cdot a = b \cdot a + c \cdot a$$

are satisfied for all $a, b, c \in R$.

A semiring $(R, +, \cdot)$ is called a *hemiring* if $(R, +)$ is a commutative semigroup and R contains an absorbing *zero, 0*, i.e., an element $0 \in R$ such that $a + 0 = 0 + a = a$ and $a \cdot 0 = 0 \cdot a = 0$ for all $a \in R$.

By the *identity* of a hemiring $(R, +, \cdot)$ we mean an element $1 \in R$ (if it exists) such that $1 \cdot a = a \cdot 1 = a$ for all $a \in R$.

A hemiring $(R, +, \cdot)$ with commutative semigroup $(R, \cdot)$ is called *commutative.*

1. All rings are hemirings.

2. Let $\mathbb{N}_o$ be the set of whole numbers, then $\mathbb{N}_o$ is a commutative hemiring with identity under the ordinary addition and multiplication of numbers.

3. Let $\mathbb{R}^+$ be the set of all non-negative real numbers, then $\mathbb{R}^+$ is a commutative hemiring with identity under the ordinary addition and multiplication of numbers.

4. The set $R = \{0, a, 1\}$ with the following two operations:

$+$	0	a	1
0	0	a	1
a	a	a	a
1	1	a	1

$\cdot$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

is a commutative hemiring.

5. The set of all 2×2 matrices $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ with entries from $\mathbb{N}_o$ is a hemiring with identity under the usual addition and multiplication of matrices.

6. The set $R = \{0, 1, 2, 3\}$ with the following Cayley tables:

$+$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	2

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	1	1
3	0	1	1	1

is a commutative hemiring.

1.2 Ideals

A non-empty subset I of a hemiring R is called a *left (right) ideal* of R if (i) $a + b \in I$ and (ii) $ra \in I\,(ar \in I)$ for all $a, b \in I$, $r \in R$. Obviously $0 \in I$ in any left (right) ideal I of R.

A non-empty subset A of a hemiring R is called an *ideal* of R if it is both a left and a right ideal of R.

Let A and B be two non-empty subsets of a hemiring $(R, +, \cdot)$ then the *sum and product* of A and B are defined as

$$\begin{aligned} A + B &= \{a + b : a \in A \text{ and } b \in B\} \\ AB &= \{\Sigma_{finite} a_i b_i : a_i \in A \text{ and } b_i \in B\}. \end{aligned}$$

Definition 1 *A left (right) ideal A of a hemiring R is called a left (right) k-ideal of R if for any $a, b \in A$ and $x \in R$ from $x + a = b$ it follows $x \in A$.*

It is not necessary that every left (right) ideal of R is a left (right) k-ideal of R.

Example 2 *Let $R = \{0,a,b\}$ be a set with addition $(+)$ and multiplication $(\cdot)$ as follows:*

$+$	0	a	b
0	0	a	b
a	a	0	b
b	b	b	0

$\cdot$	0	a	b
0	0	0	0
a	0	0	0
b	0	0	b

Then R is a hemiring. Let $A = \{0, b\}$ is an ideal of R but it is not a k-ideal of R, since $a + b = b$ but $a \notin A$.

Definition 3 *A left (right) ideal I of a hemiring R is called a left (right) h-ideal of R if for any $a, b \in I$ and $x, y \in R$ from $x + a + y = b + y$ it follows $x \in I$.*

It is not necessary that every left (right) ideal of R is a left (right) h-ideal of R.

Example 4 *Let $R = \{0, a, b\}$ be a set with addition $(+)$ and multiplication $(\cdot)$ as follows:*

$+$	0	a	b
0	0	a	b
a	a	0	b
b	b	b	0

$\cdot$	0	a	b
0	0	0	0
a	0	0	0
b	0	0	b

Then R is a hemiring. Let $A = \{0, b\}$ is an ideal of R but it is not an h-ideal of R, since $a + 0 + b = 0 + b$ but $a \notin A$.

Every left (right) h-ideal is a left (right) k-ideal but the converse is not true.

Example 5 *Consider the semiring $R = \{0, 1, a, b, c\}$ defined by the following tables:*

$+$	0	1	a	b	c
0	0	1	a	b	c
1	1	b	1	a	1
a	a	1	a	b	a
b	b	a	b	1	b
c	c	1	a	b	c

$\cdot$	0	1	a	b	c
0	0	0	0	0	0
1	0	1	a	b	c
a	0	a	a	a	c
b	0	b	a	1	c
c	0	c	c	c	0

Ideals of R are $\{0\}, \{0, c\}, \{0, a, c\}$ and R. $\{0, c\}$ is a k-ideal but not an h-ideal because $a + c + b = 0 + b$ but $a \notin \{0, c\}$.

Lemma 6 *The intersection of any collection of left (right) h-ideals in a hemiring R is also a left (right) h-ideal of R.*

Lemma 7 *The intersection of any family of left (right) k-ideals of a hemiring R is a left (right) k-ideal of R.*

1.3 h-closure

By *h-closure* of a non-empty subset A of a hemiring R we mean the set

$$\overline{A} = \{x \in R \mid x + a + y = b + y \text{ for some } a, b \in A,\ y \in R\}.$$

It is clear that if A is a left (right) ideal of R, then $\overline{A}$ is the smallest left (right) h-ideal of R containing A. So, $\overline{A} = A$ for all left (right) h-ideals of R. Obviously $\overline{\overline{A}} = \overline{A}$ for each non-empty $A \subseteq R$. Also $\overline{A} \subseteq \overline{B}$ for all $A \subseteq B \subseteq R$.

Lemma 8 *[49] $\overline{AB} = \overline{\overline{A}\,\overline{B}}$ for any subsets A, B of a hemiring R.*

Lemma 9 *[49] If A and B are, respectively, right and left h-ideals of a hemiring R, then*

$$\overline{AB} \subseteq A \cap B.$$

Definition 10 *A subset A in a hemiring R is called h-idempotent if $A = \overline{A^2}$.*

1.4 k-closure

By *k-closure* of a non-empty subset A of a hemiring R we mean the set

$$\widehat{A} = \{x \in R : x + a = b \text{ for some } a, b \in A\}.$$

It is clear that if A is a left (right) ideal of R, then $\widehat{A}$ is the smallest left (right) k-ideal of R containing A. So, $\widehat{A} = A$ for all left (right) k-ideals of R. Obviously $\widehat{\widehat{A}} = \widehat{A}$ for each non-empty subset A of R. If A, B are subsets of R such that $A \subseteq B$, then $\widehat{A} \subseteq \widehat{B}$ Also $\widehat{AB} = \widehat{\widehat{A}\,\widehat{B}}$.

Lemma 11 *[39] If A and B are, respectively, right and left k-ideals of a hemiring R, then*

$$\widehat{AB} \subseteq A \cap B.$$

Definition 12 *A subset A in a hemiring R is called k-idempotent if $A = \widehat{A^2}$.*

1.5 h-quasi-ideals and h-bi-ideals [46]

A non-empty subset A in a hemiring R is called an *h-quasi-ideal* of R if A is closed under addition, $\overline{RA} \cap \overline{AR} \subseteq A$ and $x + a + y = b + y$ implies $x \in A$ for all $x, y \in R$ and $a, b \in A$.

A non-empty subset A in a hemiring R is called an *h-bi-ideal* of R if A is closed under addition and multiplication, $\overline{ARA} \subseteq A$ and $x + a + y = b + y$ implies $x \in A$ for all $x, y \in R$ and $a, b \in A$.

Proposition 13 *[46] Every quasi-ideal of a hemiring R is a bi-ideal of R.*

The converse is not true in general. Example 16 shows that AB is not a quasi-ideal but a bi-ideal of R.

Lemma 14 *[46] For any left (right) h-ideal, h-bi-ideal or h-quasi-ideal A of hemiring R, we have $\overline{A} = A$.*

Note that every left h-ideal (right h-ideal, h-bi-ideal, h-quasi-ideal) of a hemiring R is a left ideal (resp. right ideal, bi-ideal, quasi-ideal) of R but the converse is not true. Every left (right) h-ideal of R is an h-quasi-ideal of R and every h-quasi-ideal of R is an h-bi-ideal of R but the converse is not true.

Example 15 *[46] The set R of all 2×2 matrices $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is a hemiring with usual addition and multiplication of matrices, where $a_{ij} \in \mathbb{N}_o$, $\mathbb{N}_o$ is the set of all non-negative integers. Consider the set Q of all matrices of the form $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ $(a \in \mathbb{N}_o)$. Evidently Q is an h-quasi-ideal of R but not a left (right) h-ideal of R.*

Example 16 *[46] Let $\mathbb{N}$ and $\mathbb{R}^+$ denote the sets of all positive integers and positive real numbers respectively. The set R of of all matrices of the form $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ $(a, b \in \mathbb{R}^+, c \in \mathbb{N})$ together with $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a hemiring with respect to the usual addition and multiplication of matrices. Let A, B be the sets of all matrices $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ $(a,b \in \mathbb{R}^+, c \in \mathbb{N}, a < b)$ together with $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} p & 0 \\ q & k \end{pmatrix}$ $(p, q \in \mathbb{R}^+, k \in \mathbb{N}, 3 < q)$ together with $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, respectively. It is easy to show that A and B are right h-ideal and left h-ideal of R, respectively. Now the product AB is an h-bi-ideal of R but it is not an h-quasi-ideal of R. Indeed, the element*

$$\begin{pmatrix} 6 & 0 \\ 9 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 3 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \right) = \left(\begin{pmatrix} 1 & 0 \\ \frac{7}{6} & 1 \end{pmatrix} \begin{pmatrix} 24 & 0 \\ 4 & 1 \end{pmatrix} \right) \begin{pmatrix} \frac{1}{4} & 0 \\ 1 & 1 \end{pmatrix}$$

belongs to the intersection $\overline{R(AB)} \cap \overline{(AB)R}$, but it is not an element of AB. Hence $\overline{R(AB)} \cap \overline{(AB)R} \nsubseteq RL$.

1.6 Fuzzy subsets

Let X be a non-empty set. By a *fuzzy subset* μ of X we mean a membership function $\mu : X \to [0,1]$. Imμ denotes the set of all values of μ. A fuzzy subset $\mu : X \to [0,1]$ is non-empty if there exists at least one $x \in X$ such that $\mu(x) > 0$. For any fuzzy subsets λ and μ of X we define

$$\begin{aligned}
&\lambda \leq \mu \iff \lambda(x) \leq \mu(x), \\
&(\lambda \wedge \mu)(x) = \lambda(x) \wedge \mu(x) = \min\{\lambda(x), \mu(x)\}, \\
&(\lambda \vee \mu)(x) = \lambda(x) \vee \mu(x) = \max\{\lambda(x), \mu(x)\}
\end{aligned}$$

for all $x \in X$.

More generally, if $\{\lambda_i : i \in I\}$ is a collection of fuzzy subsets of X, then by the *intersection* and the *union* of this collection we mean fuzzy subsets

$$\begin{aligned}
\Big(\bigwedge_{i\in I} \lambda_i\Big)(x) &= \bigwedge_{i\in I} \lambda_i(x) = \inf_{i\in I}\{\lambda_i(x)\}, \\
\Big(\bigvee_{i\in I} \lambda_i\Big)(x) &= \bigvee_{i\in I} \lambda_i(x) = \sup_{i\in I}\{\lambda_i(x)\},
\end{aligned}$$

respectively.

Definition 17 *Let A be a non-empty subset of a hemiring R. Then a fuzzy set χ_A defined by*

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

is called the characteristic function of A.

1.7 Fuzzy ideals

A fuzzy subset λ of a semiring R is called a *fuzzy left (right) ideal* of R if for all $a, b \in R$ we have

(1) $\lambda(a+b) \geq \lambda(a) \wedge \lambda(b)$,

(2) $\lambda(ab) \geq \lambda(b)$, $(\lambda(ab) \geq \lambda(a))$.

Note that $\lambda(0) \geq \lambda(x)$ for all $x \in R$.

Definition 18 *A fuzzy left (right) ideal λ of a hemiring R is called a fuzzy left (right) k-ideal if $x + y = z \longrightarrow \lambda(x) \geq \lambda(y) \wedge \lambda(z)$ for all $x, y, z \in R$.*

Example 19 *[7] Every fuzzy ideal of a ring R is a fuzzy k-ideal.*

Example 20 *[7] Let μ be a fuzzy subset of the hemiring $\mathbb{N}_o$, where $\mathbb{N}_o$ is the set of whole numbers defined by*

$$\mu(x) = \begin{cases} 0.3 & \text{if } x \text{ is odd} \\ 0.5 & \text{if } x \text{ is non-zero even} \\ 1 & \text{if } x = 0 \end{cases}$$

Then μ is a fuzzy k-ideal of $\mathbb{N}_o$.

Example 21 *Let μ be a fuzzy subset of the hemiring $\mathbb{N}_o$, where $\mathbb{N}_o$ is the set of whole numbers defined by*

$$\mu(x) = \begin{cases} 1 & \text{if } 7 \leq x \\ 0.5 & \text{if } 5 \leq x \leq 7 \\ 0 & \text{if } 0 \leq x < 5 \end{cases}$$

Then it is easy to show that μ is a fuzzy ideal of $\mathbb{N}_o$ but not a fuzzy k-ideal of $\mathbb{N}_o$.

Definition 22 *[26] A fuzzy left (right) ideal λ of a hemiring R is called a fuzzy left (right) h-ideal if $x + a + y = b + y \Rightarrow \lambda(x) \geq \lambda(a) \wedge \lambda(b)$ for all $a, b, x, y \in R$.*

Example 23 *All fuzzy ideals of a ring are fuzzy h-ideals.*

Example 24 *The fuzzy subset*

$$\mu(n) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0.2 & \text{otherwise,} \end{cases}$$

defined on the set $\mathbb{N}_o$, where $\mathbb{N}_o$ is the set of whole numbers, is a fuzzy left h-ideal of the hemiring $(\mathbb{N}, +, \cdot)$.

Properties of fuzzy sets defined on an algebraic system $\mathfrak{A} = (X, \mathbb{F})$, where $\mathbb{F}$ is a family of operations (also partial) defined on X, can be characterized by the corresponding properties of some subsets of X. Namely, as it is proved in [30] the following Transfer Principle holds.

Lemma 25 *[30] A fuzzy set λ defined on $\mathfrak{A}$ has the property $\mathcal{P}$ if and only if all non-empty level subsets $U(\lambda; t) = \{x \in X \mid \lambda(x) \geq t\}$ have the property $\mathcal{P}$.*

For example, a fuzzy set λ of a hemiring R is a fuzzy left ideal if and only if each non-empty subset $U(\lambda;t)$ is a left ideal of R. Similarly, a fuzzy set λ in a hemiring R is a fuzzy left h-ideal of R if and only if each non-empty subset $U(\lambda;t)$ is a left h-ideal of R.

Proposition 26 *[26] Let A be a non-empty subset of a hemiring R. Then a fuzzy set λ_A defined by*

$$\lambda_A(x) = \begin{cases} t & \text{if } x \in A \\ s & \text{otherwise} \end{cases}$$

where $0 \leq s < t \leq 1$, is a fuzzy left h-ideal of R if and only if A is a left h-ideal of R.

Corollary 27 *Let A be a non-empty subset of a hemiring R. Then the characteristic function χ_A of A is a fuzzy left (right) h-ideal of R if and only if A is a left (right) h-ideal of R.*

Example 28 *Consider the semiring $R = \{0, 1, a, b, c\}$ defined by the following tables:*

$+$	0	1	a	b	c
0	0	1	a	b	c
1	1	b	1	a	1
a	a	1	a	b	a
b	b	a	b	1	b
c	c	1	a	b	c

$\cdot$	0	1	a	b	c
0	0	0	0	0	0
1	0	1	a	b	c
a	0	a	a	a	c
b	0	b	a	1	c
c	0	c	c	c	0

Ideals of R are $\{0\}, \{0, c\}, \{0, a, c\}$ and R. The h-ideals of R are $\{0, a, c\}$ and R. Define $\lambda : R \to [0, 1]$ such that $\lambda(0) = \lambda(c) = 0.8$, $\lambda(a) = 0.6$, $\lambda(b) = \lambda(1) = 0.5$ then

$$U(\lambda, t) = \begin{cases} R & \text{if } t \in (0, .5] \\ \{0, a, c\} & \text{if } t \in (.5, .6] \\ \{0\} & \text{if } t \in (.6, .8] \\ \phi & \text{if } t \in (.8, 1] \end{cases}$$

$\{0\}, \{0, c\}, \{0, a, c\}$ and R are ideals of R. So, $U(\lambda, t)$ is ideal of R. This implies that λ is a fuzzy ideal of R, but λ is not fuzzy h-ideal of R.

Proposition 29 *Let A, B be non-empty subsets of a hemiring R. Then fuzzy sets λ_A, λ_B defined by*

$$\lambda_A(x) = \begin{cases} t & \text{if } x \in A \\ s & \text{otherwise} \end{cases} \qquad \lambda_B(x) = \begin{cases} t & \text{if } x \in B \\ s & \text{otherwise} \end{cases}$$

where $0 \leq s < t \leq 1$, then

(1) $A \subseteq B \Leftrightarrow \lambda_A \leq \lambda_B$,

(2) $\lambda_A \wedge \lambda_B = \lambda_{A \cap B}$.

Proof. Let $A \subseteq B$. For $x \in A$ we have $\lambda_A(x) = t = \lambda_B(x)$. If $x \notin A$, then $\lambda_A(x) = s \leq \lambda_B(x)$. So, $\lambda_A \leq \lambda_B$. Conversely, if $\lambda_A \leq \lambda_B$, then for all $x \in A$ we obtain $t = \lambda_A(x) \leq \lambda_B(x)$. Thus $\lambda_B(x) = t$, i.e., $x \in B$. Consequently, $A \subseteq B$. This proves (1).

To prove (2) let $x \in A \cap B$. Then $x \in A$, $x \in B$ and $\lambda_A(x) \wedge \lambda_B(x) = t = \lambda_{A \cap B}(x)$. If $x \notin A \cap B$, then $\lambda_A(x) = s$ or $\lambda_B(x) = s$. So, $\lambda_A(x) \wedge \lambda_B(x) = s = \lambda_{A \cap B}(x)$, which completes the proof. ■

1.8 h-product of fuzzy subsets:

Definition 30 *[26]Let λ and μ be fuzzy subsets of a hemiring R. Then the h-product of λ and μ is defined by*

$$(\lambda \circ_h \mu)(x) = \begin{cases} \sup\limits_{x+a_1b_1+y=a_2b_2+y} (\lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2)) \\ 0 \quad \text{if } x \text{ is not expressed as } x + a_1b_1 + y = a_2b_2 + y. \end{cases}$$

One can prove that if λ and μ are fuzzy left (right) h-ideals in a hemiring R, then so is $\lambda \wedge \mu$. Moreover, if λ is a fuzzy right h-ideal and μ is a fuzzy left h-ideal of R then $\lambda \circ_h \mu \leq \lambda \wedge \mu$.

1.9 h-intrinsic product of fuzzy subsets

Generalizing the concept of h-product of two fuzzy subsets of R in [46] the following h-intrinsic product of two fuzzy subsets μ and ν on R is defined by

$$(\mu \odot_h \nu)(x) = \sup_{x+\sum\limits_{i=1}^{m} a_ib_i+z=\sum\limits_{j=1}^{n} a'_jb'_j+z} \left(\bigwedge_{i=1}^{m} (\mu(a_i) \wedge \nu(b_i)) \wedge \bigwedge_{j=1}^{n} (\mu(a'_j) \wedge \nu(b'_j)) \right)$$

and $(\mu \odot_h \nu)(x) = 0$ if x cannot be expressed as $x + \sum\limits_{i=1}^{m} a_ib_i + z = \sum\limits_{j=1}^{n} a'_jb'_j + z$.

Proposition 31 *[46] Let μ, ν, ω, λ be fuzzy subsets on R. Then*

(1) $\mu \leq \omega$ and $\nu \leq \lambda \Rightarrow \mu \odot_h \nu \leq \omega \odot_h \lambda$.

(2) $\chi_A \odot_h \chi_B = \chi_{\overline{AB}}$ for characteristic functions of subsets A, B of R.

Definition 32 *A fuzzy subset μ in a hemiring R is called idempotent if $\mu = \mu \odot_h \mu$.*

1.10 Fuzzy h-bi-ideals and fuzzy h-quasi-ideals [46]

A fuzzy subset λ of a hemiring R is called a *fuzzy h-bi-ideal* if for all $a, b, x, y, z \in R$ we have

(1) $\lambda(x+y) \geq \lambda(x) \wedge \lambda(y)$

(2) $\lambda(xy) \geq \lambda(x) \wedge \lambda(y)$

(3) $\lambda(xyz) \geq \lambda(x) \wedge \lambda(z)$

(4) $x+a+z=b+z \Rightarrow \lambda(x) \geq \lambda(a) \wedge \lambda(b)$.

A fuzzy subset λ of a hemiring R is called a *fuzzy h-quasi-ideal* if for all $a, b, x, y, z \in R$ we have

(1) $\lambda(x+y) \geq \lambda(x) \wedge \lambda(y)$,

(2) $(\lambda \odot_h \chi_R) \wedge (\chi_R \odot_h \lambda) \leq \lambda$,

(3) $x+a+z=b+z \Rightarrow \lambda(x) \geq \lambda(a) \wedge \lambda(b)$.

Example 33 *[46] The set $\mathbb{N}_o$ of whole numbers is a hemiring with respect to the usual addition and multiplication. Let $r, s \in [0,1]$ be such that $r \leq s$. Define a fuzzy subset μ in $\mathbb{N}_o$ by*

$$\mu(a) = \begin{cases} s & \text{if } a \in \langle 3 \rangle, \\ r & \text{otherwise} \end{cases}$$

for all $a \in \mathbb{N}_o$. Then μ is both a fuzzy h-bi-ideal and a fuzzy h-quasi-ideal of $\mathbb{N}_o$.

Lemma 34 *[46] Let R be a hemiring and $A \subseteq R$. Then the following conditions hold:*

(1) λ is a fuzzy h-bi-ideal of R if and only if all non-empty level subsets $U(\mu; t)$ are h-bi-ideal of R.

(2) λ is a fuzzy h-quasi-ideal of R if and only if all non-empty level subsets $U(\mu; t)$ are quasi-bi-ideal of R.

Lemma 35 *[46] Let R be a hemiring and $A \subseteq R$. Then the following conditions hold:*

(1) A is a left (resp. right) h-ideal of R if and only if χ_A is a fuzzy left (resp. right) h-ideal of R.

(2) A is an h-bi-ideal of R if and only if χ_A is a fuzzy h-bi-ideal of R.

(3) A is an h-quasi-ideal of R if and only if χ_A is a fuzzy h-quasi-ideal of R.

Lemma 36 *[46] A fuzzy subset μ in a hemiring R is a fuzzy left (resp. right) h-ideal of R if and only if for all $x, y, a, b, z \in R$, we have*

(1) $\mu(x+y) \geq \mu(x) \wedge \mu(y)$
(2) $\chi_R \odot_h \mu \leq \mu$ (*resp.* $\mu \odot_h \chi_R \leq \mu$)
(3) $x+a+z=b+z \Rightarrow \mu(x) \geq \mu(a) \wedge \mu(b)$.

Lemma 37 *[46] Let μ and ν be fuzzy right h-ideal and fuzzy left h-ideal of a hemiring R, respectively. Then $\mu \wedge \nu$ is a fuzzy h-quasi-ideal of R.*

Lemma 38 *[46] Any fuzzy h-quasi-ideal of a hemiring R is a fuzzy h-bi-ideal of R.*

1.11 h-hemiregular and k-regular hemirings

Definition 39 *Let $(R,+,\cdot)$ be a hemiring, then an element $x \in R$ is called regular if there exists an element $y \in R$ such that $x = xyx$.*

If every element of R is regular then R is called *regular*.
A hemiring R is called *fully idempotent* if each ideal of R is idempotent.

Theorem 40 *[3] A hemiring $(R,+,\cdot)$ is regular if and only if $I \cap J = IJ$ for every right ideal I and left ideal J of R.*

Definition 41 *[1, 39] A hemiring R is said to be k-regular if for each $a \in R$, there exist $x, y \in R$ such that $a + axa = aya$.*

It is obvious that every regular hemiring is a k-regular.

Theorem 42 *[39] A hemiring R is k-regular if and only if for any right k-ideal A and any left k-ideal B, we have*

$$\widehat{AB} = A \cap B.$$

Definition 43 *[49] A hemiring R is said to be h-hemiregular if for each $a \in R$, there exist $x, y, z \in R$ such that $a + axa + z = aya + z$.*

It is obvious that every regular hemiring is an h-hemiregular. Also every k-regular hemiring is an h-hemiregular but the converse is not true. Example 46 is an h-hemiregular but not k-regular.

Theorem 44 *[49] A hemiring R is h-hemiregular if and only if for any right h-ideal A and any left h-ideal B, we have*

$$\overline{AB} = A \cap B$$

Theorem 45 *[49] A hemiring R is h-hemiregular if and only if for any fuzzy right h-ideal μ and any fuzzy left h-ideal ν, we have*

$$\mu \odot_h \nu = \mu \wedge \nu.$$

Example 46 *[46] Let $R = \{0, a, b, c\}$ be a hemiring with addition " + " and multiplication " · " defined by the following table:*

+	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

·	0	a	b
0	0	0	0
a	0	a	a
b	0	a	a

Then R is h-hemiregular hemiring.

Example 47 *Let $R = \{0, x, 1\}$ be a hemiring with addition " + " and multiplication " · " defined by the following table:*

+	0	x	1
0	0	x	1
x	x	x	x
1	1	x	1

·	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

Then R is k-regular hemiring.

Lemma 48 *[46] Let R be a hemiring. Then the following assertions are equivalent:*

(1) R is h-hemiregular

(2) $B = \overline{BRB}$ for every h-bi-ideal B of R

(3) $Q = \overline{QRQ}$ for every h-quasi-ideal Q of R.

Lemma 49 *[46] Let R be a hemiring. Then the following assertions are equivalent:*

(1) R is h-hemiregular.

(2) $\mu \leq \mu \odot_h \chi_R \odot_h \mu$ for every fuzzy h-bi-ideal μ of R.

(3) $\mu \leq \mu \odot_h \chi_R \odot_h \mu$ for every fuzzy h-quasi-ideal μ of R.

Theorem 50 *[46] Let R be a hemiring. Then the following assertions are equivalent:*

(1) R is h-hemiregular.

(2) $\mu \wedge \nu \leq \mu \odot_h \nu \odot_h \mu$ for every fuzzy h-bi-ideal μ and every fuzzy h-ideal ν of R.

(3) $\mu \wedge \nu \leq \mu \odot_h \nu \odot_h \mu$ for every fuzzy h-quasi-ideal μ and every fuzzy h-ideal ν of R.

Corollary 51 *[46] Let R be a hemiring. Then the following assertions are equivalent:*

(1) R is h-hemiregular.

(2) $B \cap A = \overline{BAB}$ for every h-bi-ideal B and every h-ideal A of R.

(3) $Q \cap A = \overline{QAQ}$ for every h-quasi-ideal Q and every h-ideal A of R.

Theorem 52 *[46] Let R be a hemiring. Then the following assertions are equivalent:*

(1) R is h-hemiregular.

(2) $\mu \wedge \nu \leq \mu \odot_h \nu$ for every fuzzy h-bi-ideal μ and every fuzzy left h-ideal ν of R.

(3) $\mu \wedge \nu \leq \mu \odot_h \nu$ for every fuzzy h-quasi-ideal μ and every fuzzy left h-ideal ν of R.

(4) $\mu \wedge \nu \leq \mu \odot_h \nu$ for every fuzzy right h-ideal μ and every fuzzy h-bi-ideal ν of R.

(5) $\mu \wedge \nu \leq \mu \odot_h \nu$ for every fuzzy right h-ideal μ and every fuzzy h-quasi-ideal ν of R.

(6) $\mu \wedge \nu \wedge \omega \leq \mu \odot_h \nu \odot_h \omega$ for every fuzzy right h-ideal μ, every fuzzy h-bi-ideal ν and every fuzzy left h-ideal ω of R.

(7) $\mu \wedge \nu \wedge \omega \leq \mu \odot_h \nu \odot_h \omega$ for every fuzzy right h-ideal μ, every fuzzy h-quasi-ideal ν and every fuzzy left h-ideal ω of R.

Corollary 53 *[46] Let R be a hemiring then the following conditions are equivalent:*

(1) R is h-hemiregular.

(2) $B \cap C \subseteq \overline{BC}$ for every h-bi-ideal B and every left h-ideal C.

(3) $Q \cap C \subseteq \overline{QC}$ for every h-quasi-ideal Q and every left h-ideal C.

(4) $A \cap B \subseteq \overline{AB}$ for every right h-ideal A and for every h-bi-ideal B of R.

(5) $A \cap Q \subseteq \overline{AQ}$ for every right h-ideal A and for every h-quasi-ideal Q of R.

(6) $A \cap B \cap C \subseteq \overline{ABC}$ for every right h-ideal A, every h-bi-ideal B and every left h-ideal C of R.

(7) $A \cap Q \cap C \subseteq \overline{AQC}$ for every right h-ideal A, every h-quasi-ideal Q and every left h-ideal C of R.

Lemma 54 *[46] A hemiring R is h-hemiregular if and only if every right and left h-ideals of R are idempotent and for any right h-ideal A and any left h-ideal B of R, the set $\overline{AB}$ is an h-quasi-ideal of R.*

Theorem 55 *[46] A hemiring R is h-hemiregular if and only if the fuzzy right and fuzzy left h-ideals of R are idempotent and for any fuzzy right h-ideal μ and fuzzy left h-ideal ν of R, the set $\mu \odot_h \nu$ is a fuzzy h-quasi-ideal of R.*

1.12 h-intra-hemiregular hemirings

Definition 56 *A hemiring R is said to be intra-regular if $x \in Rx^2R$, that is, $x = \sum_{i=1}^{n} r_i x^2 r_i^{'}$ for some $r_i, r_i^{'} \in R$.*

Definition 57 *[46] A hemiring R is said to be h-intra hemiregular if for each $x \in R$ there exist $a_i, a_i^{'}, b_j, b_j^{'}, z \in R$ such that $x + \sum_{i=1}^{m} a_i x^2 b_i + z = \sum_{j=1}^{n} a_j^{'} x^2 b_j^{'} + z$.*

Also we can define its equivalent definitions (1) $x \in \overline{Rx^2R}$, $\forall x \in R$, (2) $A \subseteq \overline{RA^2R}$, $\forall A \subseteq R$.

In the case of rings the h-intra-hemiregularity coincides with the intra-regularitry of rings.

Example 58 *[46] Let $R = \{0,a,b,c\}$ be a hemiring with addition " + " and multiplication " · " defined by the following table:*

$+$	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

$\cdot$	0	a	b
0	0	0	0
a	0	a	a
b	0	a	a

Then R is h-intra-hemiregular hemiring.

Example 59 *[46] The set $\mathbb{N}$ of all non-negative integers with usual addition " + " and multiplication " · " is a hemiring, but it is not h-hemiregular and h-intra-hemiregular hemiring. Indeed $2 \in \mathbb{N}$ can not be written as $2+2a2+z = 2a^{'}2+z$ or $2+\sum_{i=1}^{m} a_i 2^2 b_i + z = \sum_{j=1}^{n} a_j^{'} 2^2 b_j^{'} + z$ for all $a_i, a_i^{'}, b_j, b_j^{'}, z \in \mathbb{N}$.*

Lemma 60 *[46] Let R be a hemiring then the following conditions are equivalent.*

(1) R is h-intra-hemiregular.

(2) $A \cap B \subseteq \overline{AB}$ for every left h-ideal A and every right h-ideal B of R.

Lemma 61 *[46] Let R be a hemiring then the following conditions are equivalent.*

(1) R is h-intra-hemiregular.

(2) $\mu \wedge \nu \leq \mu \odot_h \nu$ for every fuzzy left h-ideal μ and every fuzzy right h-ideal ν of R.

Theorem 62 *[46] Let R be a hemiring then the following conditions are equivalent.*

(1) R is h-intra-hemiregular.

(2) $\mu(x) = \mu(x^2)$ for all fuzzy h-ideal μ of R and $x \in R$.

Lemma 63 *[46] Let R be a hemiring then the following conditions are equivalent:*

(1) R is both h-hemiregular and h-intra-hemiregular.

(2) $B = \overline{B^2}$ for every h-bi-ideal B of R.

(3) $Q = \overline{Q^2}$ for every h-quasi-ideal Q of R.

Theorem 64 *[46] Let R be a hemiring then the following conditions are equivalent:*

(1) R is both h-hemiregular and h-intra-hemiregular.

(2) $\mu \odot_h \mu = \mu$ for each fuzzy h-bi-ideal μ of R.

(3) $\mu \odot_h \mu = \mu$ for each fuzzy h-quasi-ideal μ of R.

Theorem 65 *[46] Let R be a hemiring then the following conditions are equivalent:*

(1) R is both h-hemiregular and h-intra-hemiregular.

(2) $\mu \wedge \nu \leq \mu \odot_h \nu$ for all fuzzy h-bi-ideals μ, ν of R.

(3) $\mu \wedge \nu \leq \mu \odot_h \nu$ for fuzzy h-bi-ideal μ and h-quasi-ideal ν of R.

(4) $\mu \wedge \nu \leq \mu \odot_h \nu$ for fuzzy h-quasi-ideal μ and h-bi-ideal ν of R.

(5) $\mu \wedge \nu \leq \mu \odot_h \nu$ for all fuzzy h-quasi-ideals μ, ν of R.

1.13 Prime h-ideals

Definition 66 *[49] An h-ideal P of R is called prime if $P \neq R$ and for any h-ideals A, B of R from $AB \subseteq P$ it follows $A \subseteq P$ or $B \subseteq P$.*

Definition 67 *[49] A fuzzy left (right) h-ideal ξ of a hemiring R is said to be prime if ξ is a non-constant function and for any two fuzzy left (right) h-ideals μ, ν of R, $\mu \circ_h \nu \subseteq \xi$ implies $\mu \subseteq \xi$ or $\nu \subseteq \xi$.*

Example 68 *[49] The fuzzy subset*

$$\mu(n) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0.2 & \text{otherwise,} \end{cases}$$

defined on the set $\mathbb{N}_o$ of whole numbers is a prime fuzzy h-ideal of the hemiring $(\mathbb{N}_o, +, \cdot)$.

Chapter 2

Characterizations of hemirings by the properties of their k-ideals

In [23] Henriksen defined a more restricted class of ideals in semirings, which is called the class of k-ideals. These ideals have the property that if the semiring R is a ring then a subset of R is a k-ideal if and only if it is a ring ideal. Fuzzy k-ideals are studied in [6, 7, 15, 19]. In this chapter we characterize those hemirings for which each k-ideal is idempotent and also those hemirings for which each fuzzy k-ideal is idempotent.

2.1 Fuzzy k-ideals

Recall that a fuzzy subset λ of a hemiring R is called a fuzzy left (right) k-ideal of R if it satisfies the following conditions.

(1) $\lambda(x+y) \geq \lambda(x) \wedge \lambda(y)$

(2) $\lambda(xy) \geq \lambda(y)$ $(\lambda(xy) \geq \lambda(x))$

(3) $x+y=z \Rightarrow \lambda(x) \geq \lambda(y) \wedge \lambda(z)$, for all $x, y, z \in R$.

If λ is a fuzzy subset of R and $t \in [0,1]$, then the subset $U(\lambda; t) = \{x \in R : \lambda(x) \geq t\}$ is called the level subset of λ.

Proposition 69 *Let A be a non-empty subset of a hemiring R. Then a fuzzy subset λ_A of R defined by*

$$\lambda_A(x) = \begin{cases} t & \text{if } x \in A \\ s & \text{otherwise} \end{cases}$$

where $0 \leq s < t \leq 1$, is a fuzzy left (right) k-ideal of R if and only if A is a left (right) k-ideal of R.

Proof. Straightforward. ■

Proposition 70 *A fuzzy subset λ of a hemiring R is a fuzzy left (right) k-ideal of R if and only if each non-empty level subset of R is a left (right) k-ideal of R.*

Proof. Suppose that λ is a fuzzy left k-ideal of R and $t \in (0,1]$ such that $U(\lambda;t) \neq \phi$. Let $a, b \in U(\lambda;t)$, then $\lambda(a) \geq t$ and $\lambda(b) \geq t$. As $\lambda(a+b) \geq \lambda(a) \wedge \lambda(b)$, so $\lambda(a+b) \geq t$. Hence $a+b \in U(\lambda;t)$. For $r \in R$, $\lambda(ra) \geq \lambda(a)$ so $\lambda(ra) \geq t$. This implies $ra \in U(\lambda;t)$. Hence $U(\lambda;t)$ is a left ideal of R. Now let $x + a = b$ for some $a, b \in U(\lambda;t)$, then $\lambda(a) \geq t$ and $\lambda(b) \geq t$. Since $\lambda(x) \geq \lambda(a) \wedge \lambda(b)$, so $\lambda(x) \geq t$. Hence $x \in U(\lambda;t)$. Thus $U(\lambda;t)$ is a left k-ideal of R.

Conversely, assume that each non-empty subset $U(\lambda;t)$ of R is a left k-ideal of R. Let $a, b \in R$ be such that $\lambda(a+b) < \lambda(a) \wedge \lambda(b)$. Take $t \in (0,1]$ such that $\lambda(a+b) < t \leq \lambda(a) \wedge \lambda(b)$, then $a, b \in U(\lambda;t)$ but $a + b \notin U(\lambda;t)$, a contradiction. Hence $\lambda(a+b) \geq \lambda(a) \wedge \lambda(b)$.

Similarly we can show that $\lambda(ab) \geq \lambda(b)$.

Let $x, y, z \in R$ be such that $x + y = z$. If possible let $\lambda(x) < \lambda(y) \wedge \lambda(z)$. Take $t \in (0,1]$ such that $\lambda(x) < t \leq \lambda(y) \wedge \lambda(z)$, then $y, z \in U(\lambda;t)$ but $x \notin U(\lambda;t)$, a contradiction. Hence $\lambda(x) \geq \lambda(y) \wedge \lambda(z)$. Thus λ is a fuzzy left k-ideal of R. ■

Example 71 *The set $R = \{0,1,2,3\}$ with operations of addition and multiplication given by the following Cayley tables*

+	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	2

·	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	1	1
3	0	1	1	1

is a hemiring. Ideals in R are $\{0\}, \{0,1\}, \{0,1,2\}, \{0,1,2,3\}$. All ideals are k-ideals. Let $t_1, t_2, t_3, t_4 \in (0,1]$ such that $t_1 \geq t_2 \geq t_3 \geq t_4$.

Define $\lambda : R \Rightarrow [0,1]$ by $\lambda(0) = t_1$, $\lambda(1) = t_2$, $\lambda(2) = t_3$ and $\lambda(3) = t_4$

Then

$$U(\lambda;t) = \begin{cases} \{0,1,2,3\} & \text{if} \quad t \leq t_4 \\ \{0,1,2\} & \text{if} \quad t_4 < t \leq t_3 \\ \{0,1\} & \text{if} \quad t_3 < t \leq t_2 \\ \{0\} & \text{if} \quad t_2 < t \leq t_1 \\ \phi & \text{if} \quad t > t_1 \end{cases}$$

Thus by Proposition 70, λ is a fuzzy k-ideal of R.

2.2 k-product of fuzzy subsets

In this section we define k-product and k-sum of fuzzy subsets of a hemiring R. This product and sum has the property that if λ and μ are fuzzy left (right) k-ideals of R, then $\lambda \odot_k \mu$ and $\lambda +_k \mu$ are fuzzy left (right) k-ideals of R

Definition 72 *The k-product of two fuzzy subsets μ and ν of R is defined by*

$$(\mu \odot_k \nu)(x) = \bigvee_{x+\sum\limits_{i=1}^{m} a_i b_i = \sum\limits_{j=1}^{n} a_j' b_j'} \left[\begin{array}{c} \left[\bigwedge\limits_{i=1}^{m} \mu(a_i)\right] \wedge \left[\bigwedge\limits_{i=1}^{m} \nu(b_i)\right] \wedge \\ \left[\bigwedge\limits_{j=1}^{n} \mu(a_j')\right] \wedge \left[\bigwedge\limits_{j=1}^{n} \nu(b_j')\right] \end{array} \right]$$

$$\text{and } (\mu \odot_k \nu)(x) = 0 \text{ if } x \text{ can not be expressed as } x+\sum_{i=1}^{m} a_i b_i = \sum_{j=1}^{n} a_j' b_j'.$$

By direct calculations we obtain the following Proposition.

Proposition 73 *Let $\mu, \nu, \omega, \lambda$ be fuzzy subsets of R. If $\mu \leq \omega$ and $\nu \leq \lambda$, then $\mu \odot_k \nu \leq \omega \odot_k \lambda$.*

Lemma 74 *Let R be a hemiring and $A, B \subseteq R$. Then $\chi_A \odot_k \chi_B = \chi_{\widehat{AB}}$.*

Proof. Let $x \in R$. If $x \in \widehat{AB}$, then $\chi_{\widehat{AB}}(x) = 1$ and $x + \sum\limits_{i=1}^{m} p_i q_i = \sum\limits_{j=1}^{n} p_j' q_j'$ for some $p_i, p_j' \in A$ and $q_i, q_j' \in B$. Thus we have

$$(\chi_A \odot_k \chi_B)(x) = \bigvee_{x+\sum\limits_{i=1}^{m} a_i b_i = \sum\limits_{j=1}^{n} a_j' b_j'} \left[\begin{array}{c} [\bigwedge\limits_{i=1}^{m} \chi_A(a_i)] \wedge [\bigwedge\limits_{i=1}^{m} \chi_B(b_i)] \wedge \\ [\bigwedge\limits_{j=1}^{n} \chi_A(a_j')] \wedge [\bigwedge\limits_{j=1}^{n} \chi_B(b_j')] \end{array} \right]$$

$$\geq \left[\begin{array}{c} [\bigwedge\limits_{i=1}^{m} \chi_A(p_i)] \wedge [\bigwedge\limits_{i=1}^{m} \chi_B(q_i)] \wedge \\ [\bigwedge\limits_{j=1}^{n} \chi_A(p_j')] \wedge [\bigwedge\limits_{j=1}^{n} \chi_B(q_j')] \end{array} \right] = 1$$

and so

$$(\chi_A \odot_k \chi_B)(x) = 1 = \chi_{\widehat{AB}}$$

If $x \notin \widehat{AB}$, then $\chi_{\widehat{AB}} = 0$. If possible, let $(\chi_A \odot_k \chi_B)(x) \neq 0$. Then

$$(\chi_A \odot_k \chi_B)(x) = \bigvee_{x+\sum\limits_{i=1}^{m} a_i b_i = \sum\limits_{j=1}^{n} a_j' b_j'} \left[\begin{array}{c} [\bigwedge\limits_{i=1}^{m} \chi_A(a_i)] \wedge [\bigwedge\limits_{i=1}^{m} \chi_B(b_i)] \wedge \\ [\bigwedge\limits_{j=1}^{n} \chi_A(a_j')] \wedge [\bigwedge\limits_{j=1}^{n} \chi_B(b_j')] \end{array} \right] \neq 0.$$

Hence there exist $p_i, q_i, p_j^{'}, q_j^{'} \in R$ such that

$$x + \sum_{i=1}^{m} p_i q_i = \sum_{j=1}^{n} p_j^{'} q_j^{'}$$

and

$$\begin{bmatrix} [\bigwedge_{i=1}^{m} \chi_A(p_i)] \wedge [\bigwedge_{i=1}^{m} \chi_B(q_i)] \wedge \\ [\bigwedge_{j=1}^{n} \chi_A(p_j^{'})] \wedge [\bigwedge_{j=1}^{n} \chi_B(q_j^{'})] \end{bmatrix} \neq 0,$$

that is

$$\chi_A(p_i) = \chi_A(p_j^{'}) = \chi_B(q_i) = \chi_B(q_j^{'}) = 1,$$

hence $p_i, p_j^{'} \in A$ and $q_i, q_j^{'} \in B$, and so $x \in \widehat{AB}$ which is a contradiction. Thus we have $(\chi_A \odot_k \chi_B)(x) = 0 = \chi_{\widehat{AB}}(x)$.

Hence in any case, we have $(\chi_A \odot_k \chi_B)(x) = \chi_{\widehat{AB}}(x)$. ∎

Theorem 75 *If λ, μ are fuzzy k-ideals of R, then $\lambda \odot_k \mu$ is a fuzzy k-ideal of R and $\lambda \odot_k \mu \leq \lambda \wedge \mu$.*

Proof. Let λ, μ be fuzzy k-ideals of R. Let $x, y \in R$, then

$$(\lambda \odot_k \mu)(x) = \bigvee_{x + \sum_{i=1}^{m} a_i b_i = \sum_{j=1}^{n} a_j^{'} b_j^{'}} \begin{bmatrix} \left[\bigwedge_{i=1}^{m} \lambda(a_i)\right] \wedge \left[\bigwedge_{i=1}^{m} \mu(b_i)\right] \wedge \\ \left[\bigwedge_{j=1}^{n} \lambda(a_j^{'})\right] \wedge \left[\bigwedge_{j=1}^{n} \mu(b_j^{'})\right] \end{bmatrix}$$

and

$$(\lambda \odot_k \mu)(y) = \bigvee_{y + \sum_{k=1}^{p} c_k d_k = \sum_{l=1}^{q} c_l^{'} d_l^{'}} \begin{bmatrix} \left[\bigwedge_{k=1}^{p} \lambda(c_k)\right] \wedge \left[\bigwedge_{k=1}^{p} \mu(d_k)\right] \wedge \\ \left[\bigwedge_{l=1}^{q} \lambda(c_l^{'})\right] \wedge \left[\bigwedge_{l=1}^{q} \mu(d_l^{'})\right] \end{bmatrix}$$

Thus

$$(\lambda \odot_k \mu)(x) \wedge (\lambda \odot_k \mu)(y) = \left[\bigvee_{x + \sum_{i=1}^{m} a_i b_i = \sum_{j=1}^{n} a_j^{'} b_j^{'}} \begin{bmatrix} \left[\bigwedge_{i=1}^{m} \lambda(a_i)\right] \wedge \left[\bigwedge_{i=1}^{m} \mu(b_i)\right] \wedge \\ \left[\bigwedge_{j=1}^{n} \lambda(a_j^{'})\right] \wedge \left[\bigwedge_{j=1}^{n} \mu(b_j^{'})\right] \end{bmatrix} \right]$$

$$\wedge \left[\bigvee_{y + \sum_{k=1}^{p} c_k d_k = \sum_{l=1}^{q} c_l^{'} d_l^{'}} \begin{bmatrix} \left[\bigwedge_{k=1}^{p} \lambda(c_k)\right] \wedge \left[\bigwedge_{k=1}^{p} \mu(d_k)\right] \wedge \\ \left[\bigwedge_{l=1}^{q} \lambda(c_l^{'})\right] \wedge \left[\bigwedge_{l=1}^{q} \mu(d_l^{'})\right] \end{bmatrix} \right]$$

$$= \bigvee_{x+\sum\limits_{i=1}^{m} a_ib_i=\sum\limits_{j=1}^{n} a_j'b_j'} \left[\bigvee_{y+\sum\limits_{k=1}^{p} c_kd_k=\sum\limits_{l=1}^{q} c_l'd_l'} \left[\begin{array}{c} \left[\bigwedge\limits_{i=1}^{m} \lambda(a_i)\right] \wedge \left[\bigwedge\limits_{i=1}^{m} \mu(b_i)\right] \wedge \\ \left[\bigwedge\limits_{j=1}^{n} \lambda(a_j')\right] \wedge \left[\bigwedge\limits_{j=1}^{n} \mu(b_j')\right] \wedge \\ \left[\bigwedge\limits_{k=1}^{p} \lambda(c_k)\right] \wedge \left[\bigwedge\limits_{k=1}^{p} \mu(d_k)\right] \wedge \\ \left[\bigwedge\limits_{l=1}^{q} \lambda(c_l')\right] \wedge \left[\bigwedge\limits_{l=1}^{q} \mu(d_l')\right] \end{array} \right] \right]$$

Since for each expression $x+\sum\limits_{i=1}^{m} a_ib_i = \sum\limits_{j=1}^{n} a_j'b_j'$ and $y+\sum\limits_{k=1}^{p} c_kd_k = \sum\limits_{l=1}^{q} c_l'd_l'$ we have $x+y+\sum\limits_{i=1}^{m} a_ib_i + \sum\limits_{k=1}^{p} c_kd_k = \sum\limits_{j=1}^{n} a_j'b_j' + \sum\limits_{l=1}^{q} c_l'd_l'$, so we have

$$(\lambda \odot_k \mu)(x) \wedge (\lambda \odot_k \mu)(y) \leq \bigvee_{x+y+\sum\limits_{s=1}^{u} e_sf_s=\sum\limits_{t=1}^{v} e_t'f_t'} \left[\begin{array}{c} \left[\bigwedge\limits_{s=1}^{u} \lambda(e_s)\right] \wedge \left[\bigwedge\limits_{s=1}^{u} \mu(f_s)\right] \wedge \\ \left[\bigwedge\limits_{t=1}^{v} \lambda(e_t')\right] \wedge \left[\bigwedge\limits_{t=1}^{v} \mu(f_t')\right] \end{array} \right]$$
$$= (\lambda \odot_k \mu)(x+y).$$

$$\text{Similarly,} (\lambda \odot_k \mu)(x) = \bigvee_{x+\sum\limits_{i=1}^{m} a_ib_i=\sum\limits_{j=1}^{n} a_j'b_j'} \left[\begin{array}{c} \left[\bigwedge\limits_{i=1}^{m} \lambda(a_i)\right] \wedge \left[\bigwedge\limits_{i=1}^{m} \mu(b_i)\right] \wedge \\ \left[\bigwedge\limits_{j=1}^{n} \lambda(a_j')\right] \wedge \left[\bigwedge\limits_{j=1}^{n} \mu(b_j')\right] \end{array} \right]$$

$$\leq \bigvee_{x+\sum\limits_{i=1}^{m} a_ib_i=\sum\limits_{j=1}^{n} a_j'b_j'} \left[\begin{array}{c} \left[\bigwedge\limits_{i=1}^{m} \lambda(a_i)\right] \wedge \left[\bigwedge\limits_{i=1}^{m} \mu(b_ir)\right] \wedge \\ \left[\bigwedge\limits_{j=1}^{n} \lambda(a_j')\right] \wedge \left[\bigwedge\limits_{j=1}^{n} \mu(b_j'r)\right] \end{array} \right]$$

$$\leq \bigvee_{xr+\sum\limits_{k=1}^{p} g_kh_k=\sum\limits_{l=1}^{q} g_l'h_l'} \left[\begin{array}{c} \left[\bigwedge\limits_{k=1}^{p} \lambda(g_k)\right] \wedge \left[\bigwedge\limits_{k=1}^{p} \mu(h_k)\right] \wedge \\ \left[\bigwedge\limits_{l=1}^{q} \lambda(g_l')\right] \wedge \left[\bigwedge\limits_{l=1}^{q} \mu(h_l')\right] \end{array} \right]$$
$$= (\lambda \odot_k \mu)(xr).$$

Analogously we can verify that $(\lambda \odot_k \mu)(rx) \geq (\lambda \odot_k \mu)(x)$ for all $r, x \in R$. This means that $\lambda \odot_k \mu$ is a fuzzy ideal of R.

To prove that $x + a = b$ implies $(\lambda \odot_k \mu)(x) \geq (\lambda \odot_k \mu)(a) \wedge (\lambda \odot_k \mu)(b)$, observe that

$$a + \sum_{i=1}^{m} a_ib_i = \sum_{j=1}^{n} a_j'b_j' \quad \text{and} \quad b + \sum_{k=1}^{l} c_kd_k = \sum_{q=1}^{p} c_q'd_q', \tag{2.1}$$

together with $x + a = b$, gives $x + a + \sum\limits_{i=1}^{m} a_ib_i = b + \sum\limits_{i=1}^{m} a_ib_i$. Thus

$$x + \sum_{j=1}^{n} a_j'b_j' = b + \sum_{i=1}^{m} a_ib_i$$

and, consequently,

$$\begin{aligned} x+\sum_{j=1}^{n} a_j' b_j' + \sum_{k=1}^{l} c_k d_k &= b+\sum_{k=1}^{l} c_k d_k + \sum_{i=1}^{m} a_i b_i \\ &= \sum_{q=1}^{p} c_q' d_q' + \sum_{i=1}^{m} a_i b_i \\ &= \sum_{i=1}^{m} a_i b_i + \sum_{q=1}^{p} c_q' d_q' \end{aligned}$$

Therefore

$$x+\sum_{j=1}^{n} a_j' b_j' + \sum_{k=1}^{l} c_k d_k = \sum_{i=1}^{m} a_i b_i + \sum_{q=1}^{p} c_q' d_q'. \tag{2.2}$$

Now, we have

$$\begin{aligned} (\lambda \odot_k \mu)(a) \wedge (\lambda \odot_k \mu)(b) &= \left[\bigvee_{a+\sum_{i=1}^{m} a_i b_i = \sum_{j=1}^{n} a_j' b_j'} \left[\begin{array}{c} \left[\bigwedge_{i=1}^{m} \lambda(a_i)\right] \wedge \left[\bigwedge_{i=1}^{m} \mu(b_i)\right] \wedge \\ \left[\bigwedge_{j=1}^{n} \lambda(a_j')\right] \wedge \left[\bigwedge_{j=1}^{n} \mu(b_j')\right] \end{array} \right] \right] \\ &\wedge \left[\bigvee_{b+\sum_{k=1}^{l} c_k d_k = \sum_{q=1}^{p} c_q' d_q'} \left[\begin{array}{c} \left[\bigwedge_{k=1}^{p} \lambda(c_k)\right] \wedge \left[\bigwedge_{k=1}^{p} \mu(d_k)\right] \wedge \\ \left[\bigwedge_{l=1}^{q} \lambda(c_l')\right] \wedge \left[\bigwedge_{l=1}^{q} \mu(d_l')\right] \end{array} \right] \right] \\ &= \bigvee_{a+\sum_{i=1}^{m} a_i b_i = \sum_{j=1}^{n} a_j' b_j'} \left(\bigvee_{b+\sum_{k=1}^{p} c_k d_k = \sum_{l=1}^{q} c_l' d_l'} \left(\begin{array}{c} \left[\bigwedge_{i=1}^{m} \lambda(a_i)\right] \wedge \left[\bigwedge_{i=1}^{m} \mu(b_i)\right] \wedge \\ \left[\bigwedge_{j=1}^{n} \lambda(a_j')\right] \wedge \left[\bigwedge_{j=1}^{n} \mu(b_j')\right] \wedge \\ \left[\bigwedge_{k=1}^{p} \lambda(c_k)\right] \wedge \left[\bigwedge_{k=1}^{p} \mu(d_k)\right] \wedge \\ \left[\bigwedge_{l=1}^{q} \lambda(c_l')\right] \wedge \left[\bigwedge_{l=1}^{q} \mu(d_l')\right] \end{array} \right) \right) \\ &\leq \bigvee_{x+\sum_{s=1}^{u} g_s h_s = \sum_{t=1}^{w} g_t' h_t'} \left(\begin{array}{c} \left[\bigwedge_{s=1}^{u} \lambda(g_s)\right] \wedge \left[\bigwedge_{s=1}^{u} \mu(h_s)\right] \wedge \\ \left[\bigwedge_{t=1}^{w} \lambda(g_t')\right] \wedge \left[\bigwedge_{t=1}^{w} \mu(h_t')\right] \end{array} \right) \\ &= (\lambda \odot_k \mu)(x) \end{aligned}$$

Thus $(\lambda \odot_k \mu)(a) \wedge (\lambda \odot_k \mu)(b) \leq (\lambda \odot_k \mu)(x)$. Hence $(\lambda \odot_k \mu)$ is a fuzzy k-ideal of R.

By simple calculations we can prove that $\lambda \odot_k \mu \leq \lambda \wedge \mu$. ■

Corollary 76 *If λ is fuzzy right k-ideal of R and μ a fuzzy left k-ideals of R, then $\lambda \odot_k \mu \leq \lambda \wedge \mu$.*

Definition 77 *The k-sum $\lambda +_k \mu$ of fuzzy subsets λ and μ of R is defined by*

$$(\lambda +_k \mu)(x) = \sup_{x+(a_1+b_1)=(a_2+b_2)} [\lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2)],$$

where $x, a_1, b_1, a_2, b_2 \in R$.

Theorem 78 *The k-sum of fuzzy k-ideals of R is also a fuzzy k-ideal of R.*

Proof. Let λ, μ be fuzzy k-ideals of R. Then for $x, y, r \in R$ we have

$$(\lambda +_k \mu)(x) \wedge (\lambda +_k \mu)(y) = \left[\bigvee_{x+(a_1+b_1)=(a_2+b_2)} [\lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \right]$$
$$\wedge \left[\bigvee_{y+(a_1'+b_1')=(a_2'+b_2')} [\lambda(a_1') \wedge \lambda(a_2') \wedge \mu(b_1') \wedge \mu(b_2')] \right]$$
$$= \bigvee_{\substack{x+(a_1+b_1)=(a_2+b_2) \\ y+(a_1'+b_1')=(a_2'+b_2')}} \begin{pmatrix} \lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2) \wedge \\ \lambda(a_1') \wedge \lambda(a_2') \wedge \mu(b_1') \wedge \mu(b_2') \end{pmatrix}$$
$$\leq \bigvee_{\substack{x+(a_1+b_1)=(a_2+b_2) \\ y+(a_1'+b_1')=(a_2'+b_2')}} \begin{pmatrix} \lambda(a_1+a_1') \wedge \lambda(a_2+a_2') \wedge \\ \mu(b_1+b_1') \wedge \mu(b_2+b_2') \end{pmatrix}$$
$$\leq \bigvee_{(x+y)+(c_1+d_1)=(c_2+d_2)} [\lambda(c_1) \wedge \lambda(c_2) \wedge \mu(d_1) \wedge \mu(d_2)]$$
$$= (\lambda +_k \mu)(x+y).$$

Similarly,

$$(\lambda +_k \mu)(x) = \bigvee_{x+(a_1+b_1)=(a_2+b_2)} [\lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2)]$$
$$\leq \bigvee_{x+(a_1+b_1)=(a_2+b_2)} [\lambda(ra_1) \wedge \lambda(ra_2) \wedge \mu(rb_1) \wedge \mu(rb_2)]$$
$$\leq \bigvee_{rx+(a_1''+b_1'')=(a_2''+b_2'')} [\lambda(a_1'') \wedge \lambda(a_2'') \wedge \mu(b_1'') \wedge \mu(b_2'')]$$
$$= (\lambda +_k \mu)(rx).$$

Analogously, we can verify $(\lambda +_k \mu)(x) \leq (\lambda +_k \mu)(xr)$. This proves that $(\lambda +_k \mu)$ is a fuzzy ideal of R.

Now we show that $x + a = b$ implies $(\lambda +_k \mu)(x) \geq (\lambda +_k \mu)(a) \wedge (\lambda +_k \mu)(b)$. For this let $a + (a_1 + b_1) = (a_2 + b_2)$ and $b + (c_1 + d_1) = (c_2 + d_2)$ Then,

$$x + a + (c_1 + d_1) = (c_2 + d_2)$$

whence

$$x + a + (c_1 + d_1) + (a_1 + b_1) = (c_2 + d_2) + (a_1 + b_1)$$

and

$$x + (a + a_1 + b_1) + (c_1 + d_1) = (c_2 + d_2) + (a_1 + b_1).$$

Then

$$x + (a_2 + b_2) + (c_1 + d_1) = (c_2 + d_2) + (a_1 + b_1).$$

Thus

$$x + (a_2 + c_1) + (b_2 + d_1) = (a_1 + c_2) + (b_1 + d_2).$$

Therefore

$$\begin{aligned}
(\lambda +_k \mu)(a) \wedge (\lambda +_k \mu)(b) &= \left[\bigvee_{a+(a_1+b_1)=(a_2+b_2)} [\lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \right] \\
&\quad \wedge \left[\bigvee_{b+(c_1+d_1)=(c_2+d_2)} [\lambda(c_1) \wedge \lambda(c_2) \wedge \mu(d_1) \wedge \mu(d_2)] \right] \\
&= \bigvee_{\substack{a+(a_1+b_1)=(a_2+b_2) \\ b+(c_1+d_1)=(c_2+d_2)}} \begin{pmatrix} \lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \\ \wedge \mu(b_2) \wedge \lambda(c_1) \wedge \lambda(c_2) \\ \wedge \mu(d_1) \wedge \mu(d_2) \end{pmatrix} \\
&\leq \bigvee_{\substack{a+(a_1+b_1)=(a_2+b_2) \\ b+(c_1+d_1)=(c_2+d_2)}} \begin{pmatrix} \lambda(a_2+c_1) \wedge \lambda(a_1+c_2) \wedge \\ \mu(b_2+d_1) \wedge \mu(b_1+d_2) \end{pmatrix} \\
&\leq \bigvee_{x+(a'+b')=(a''+b'')} [\lambda(a') \wedge \lambda(a'') \wedge \mu(b') \wedge \mu(b'')] \\
&= (\lambda +_k \mu)(x).
\end{aligned}$$

Thus $\lambda +_k \mu$ is a fuzzy k-ideal of R. ■

Theorem 79 *If μ is a fuzzy subset of a hemiring R, then the following are equivalent:*

(a) μ satisfies (1) $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$ and
(2) $x + a = b \Rightarrow \mu(x) \geq \min\{\mu(a), \mu(b)\}$

(b) $\mu +_k \mu \leq \mu$

Proof. (a) $\Rightarrow$ (b) Let $x \in R$, then

$$\begin{aligned}
(\mu +_k \mu)(x) &= \bigvee_{x+(a_1+b_1)=(a_2+b_2)} [\mu(a_1) \wedge \mu(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \\
&\leq \bigvee_{x+(a_1+b_1)=(a_2+b_2)} [\mu(a_1+a_2) \wedge \mu(b_1+b_2)] \quad \text{(by (1))} \\
&\leq \mu(x) \quad \text{(by (2))}
\end{aligned}$$

Thus $\mu +_k \mu \leq \mu$.

(b) $\Rightarrow$ (a) First we show that $\mu(0) \geq \mu(x)$ for all $x \in R$.

$$\begin{aligned}
\mu(0) \geq (\mu +_k \mu)(0) &= \bigvee_{0+(a_1+b_1)=(a_2+b_2)} [\mu(a_1) \wedge \mu(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \\
&\geq \mu(x) \wedge \mu(x) \wedge \mu(x) \wedge \mu(x) \quad \text{because } 0 + x + x = x + x \\
&= \mu(x).
\end{aligned}$$

Thus $\mu(0) \geq \mu(x)$ for all $x \in R$.

Now

$\mu(x+y) \geq (\mu +_k \mu)(x+y)$

$$= \bigvee_{x+y+(a_1+b_1)=(a_2+b_2)} [\mu(a_1) \wedge \mu(a_2) \wedge \mu(b_1) \wedge \mu(b_2)]$$
$$\geq \mu(0) \wedge \mu(0) \wedge \mu(x) \wedge \mu(y) \text{ because } x+y+0+0 = x+y$$
$$= \mu(x) \wedge \mu(y) \text{ (because } \mu(0) \geq \mu(x) \text{ for all } x \in R).$$

Again

$$\begin{aligned} \mu(x) &\geq (\mu +_k \mu)(x) \\ &= \bigvee_{x+(a_1+b_1)=(a_2+b_2)} [\mu(a_1) \wedge \mu(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \end{aligned}$$

If $x + a = b$ then $x + a + 0 = b + 0$ and so

$$\mu(x) \geq \mu(a) \wedge \mu(0) \wedge \mu(b) \wedge \mu(0) = \mu(a) \wedge \mu(b) \quad \left(\begin{array}{c} \text{because } \mu(0) \geq \mu(x) \\ \text{for all } x \in R \end{array} \right).$$

■

Lemma 80 *A fuzzy subset μ in a hemiring R is a fuzzy left (right) k-ideal if and only if*

(1) $\mu +_k \mu \leq \mu$ and

(2) $\chi_R \odot_k \mu \leq \mu \quad (\mu \odot_k \chi_R \leq \mu)$

Proof. Let μ be a fuzzy left k-ideal of R. By Theorem 79, μ satisfies (1). Now we prove condition (2). Let $x \in R$. If $(\chi_R \odot_k \mu)(x) = 0$, then $(\chi_R \odot_k \mu)(x) \leq (\mu)(x)$. Otherwise, there exist elements $a_i, b_i, a_j', b_j' \in R$ such that $x + \sum_{i=1}^{m} a_i b_i = \sum_{j=1}^{n} a_j' b_j'$. Then we have

$$\begin{aligned} (\chi_R \odot_k \mu)(x) &= \bigvee_{x+\sum_{i=1}^{m} a_i b_i = \sum_{j=1}^{n} a_j' b_j'} \left[\begin{array}{c} \left[\bigwedge_{i=1}^{m} \chi_R(a_i)\right] \wedge \left[\bigwedge_{i=1}^{m} \mu(b_i)\right] \wedge \\ \left[\bigwedge_{j=1}^{n} \chi_R(a_j')\right] \wedge \left[\bigwedge_{j=1}^{n} \mu(b_j')\right] \end{array} \right] \\ &= \bigvee_{x+\sum_{i=1}^{m} a_i b_i = \sum_{j=1}^{n} a_j' b_j'} \left[\left[\bigwedge_{i=1}^{m} \mu(b_i)\right] \wedge \left[\bigwedge_{j=1}^{n} \mu(b_j')\right]\right] \\ &\leq \bigvee_{x+\sum_{i=1}^{m} a_i b_i = \sum_{j=1}^{n} a_j' b_j'} \left[\left[\bigwedge_{i=1}^{m} \mu(a_i b_i)\right] \wedge \left[\bigwedge_{j=1}^{n} \mu(a_j' b_j')\right]\right] \\ &\leq \bigvee_{x+\sum_{i=1}^{m} a_i b_i = \sum_{j=1}^{n} a_j' b_j'} \left[\mu(\sum_{i=1}^{m} a_i b_i) \wedge \mu(\sum_{j=1}^{n} a_j' b_j')\right] \\ &\leq \bigvee_{x+\sum_{i=1}^{m} a_i b_i = \sum_{j=1}^{n} a_j' b_j'} \mu(x) = \mu(x). \end{aligned}$$

This implies that $\chi_R \odot_k \mu \leq \mu$.

Conversely, assume that the given conditions hold. In order to show that μ is a fuzzy left k-ideal of R it is sufficient to show that the condition $\mu(xy) \geq \mu(y)$ holds. Let $x, y \in R$. Then we have

$$\mu(xy) \geq (\chi_R \odot_k \mu)(xy) = \bigvee_{xy+\sum_{i=1}^{m} a_i b_i = \sum_{j=1}^{n} a_j' b_j'} \begin{bmatrix} \left[\bigwedge_{i=1}^{m} \mu(a_i)\right] \wedge \left[\bigwedge_{i=1}^{m} \mu(b_i)\right] \wedge \\ \left[\bigwedge_{j=1}^{n} \mu(a_j')\right] \wedge \left[\bigwedge_{j=1}^{n} \mu(b_j')\right] \end{bmatrix}$$

since $xy + 0y = xy$, so $\mu(xy) \geq \mu(y)$ and so μ is a fuzzy left k-ideal of R. ∎

Now we prove a characterization of k-regular hemirings.

Theorem 81 *A hemiring R is k-regular if and only if for any fuzzy right k-ideal μ and any fuzzy left k-ideal ν of R we have $\mu \odot_k \nu = \mu \wedge \nu$.*

Proof. Let R be a k-regular hemiring and μ, ν be fuzzy right k-ideal and fuzzy left k-ideal of R, respectively. Then by Lemma 80, we have $\mu \odot_k \nu \leq \mu \odot_k \chi_R \leq \mu$ and $\mu \odot_k \nu \leq \chi_R \odot_k \nu \leq \nu$. Thus $\mu \odot_k \nu \leq \mu \wedge \nu$. To show the converse inclusion, let $x \in R$. Since R is k-regular, so there exist $a, a' \in R$ such that $x + xax = xa'x$. Then we have

$$(\mu \odot_k \nu)(x) = \bigvee_{x+\sum_{i=1}^{m} a_i b_i = \sum_{j=1}^{n} a_j' b_j'} \left(\begin{array}{c} \left[\bigwedge_{i=1}^{m} \mu(a_i)\right] \wedge \left[\bigwedge_{i=1}^{m} \nu(b_i)\right] \wedge \\ \left[\bigwedge_{j=1}^{n} \mu(a_j')\right] \wedge \left[\bigwedge_{j=1}^{n} \nu(b_j')\right] \end{array} \right)$$

$$\geq \min\left\{\mu(xa), \mu(xa'), \nu(x)\right\} \geq \min\{\mu(x), \nu(x)\} = (\mu \wedge \nu)(x).$$

This implies that $\mu \odot_k \nu \geq \mu \wedge \nu$. Therefore $\mu \odot_k \nu = \mu \wedge \nu$.

Conversely, let C, D be any right k-ideal and left k-ideal of R, respectively. Then the characteristic functions χ_C, χ_D of C, D are fuzzy right k-ideal and fuzzy left k-ideal of R, respectively. Now, by the assumption and Lemma 74, we have

$$\chi_{\widehat{CD}} = \chi_C \odot_k \chi_D = \chi_C \wedge \chi_D = \chi_{C \cap D}.$$

So, $\widehat{CD} = C \cap D$. Hence by Theorem 42, R is k-regular hemiring. ∎

2.3 Idempotent k-ideals

From Theorem 42 it follows that in a k-regular hemiring every k-ideal A is k-idempotent, that is $\widehat{AA} = A$. On the other hand, in such hemirings we have $\lambda \odot_k \lambda = \lambda$ for all fuzzy k-ideals λ. Fuzzy k-ideal with this property will be called k-idempotent.

Proposition 82 *The following statements are equivalent for a hemiring R:*

1. Each k-ideal of R is idempotent.
2. $A \cap B = \widehat{AB}$ for each pair of k-ideals A, B of R.
3. $x \in \widehat{RxRxR}$ for every $x \in R$.
4. $X \subseteq \widehat{RXRXR}$ for every non empty subset X of R.
5. $A = \widehat{RARAR}$ for every k-ideal A of R.

If R is commutative, then the above assertions are equivalent to

6. R is k-regular.

Proof. (1) $\Rightarrow$ (2) Assume that each k-ideal of R is idempotent and A, B are k-ideals of R. By Lemma 11, $\widehat{AB} \subseteq A \cap B$. Since $A \cap B$ is a k-ideal of R, so by (1) $A \cap B = \widehat{(A \cap B)(A \cap B)} \subseteq \widehat{AB}$. Thus $A \cap B = \widehat{AB}$.

(2) $\Rightarrow$ (1) Obvious.

(1) $\Rightarrow$ (3) Let $x \in R$. The smallest k-ideal containing x has the form $\widehat{<x>} = \widehat{Rx + xR + RxR + \mathbb{N}_o x}$, where $\mathbb{N}_o$ is the set of whole numbers. By hypothesis $\widehat{<x>} = \widehat{\widehat{<x>}\ \widehat{<x>}} = \widehat{<x><x>}$.

Thus $x \in \widehat{(Rx + xR + RxR + \mathbb{N}_o x)(Rx + xR + RxR + \mathbb{N}_o x)} \subseteq \widehat{RxRRxR} \subseteq \widehat{RxRxR}$.

(3) $\Rightarrow$ (4) This is obvious.

(4) $\Rightarrow$ (5) Let A be a k-ideal of R. Then

$\widehat{A} = A \subseteq \widehat{RARAR} \subseteq \widehat{AA} \subseteq \widehat{A} = A$. Hence $A = \widehat{RARAR}$.

(5) $\Rightarrow$ (1) This is obvious.

If R is commutative then by Theorem 42, (2) $\Leftrightarrow$ (3). ∎

Proposition 83 *The following statements are equivalent for a hemiring R.*

1. Each fuzzy k-ideal of R is idempotent.
2. $\lambda \odot_k \mu = \lambda \wedge \mu$ for all fuzzy k-ideals of R.

If R is commutative, then the above assertions are equivalent to

3. R is k-regular.

Proof. (1) $\Rightarrow$ (2) Let λ and μ be fuzzy k-ideals of R. By Proposition 73, $(\lambda \wedge \mu) \odot_k (\lambda \wedge \mu) \leq \lambda \odot_k \mu$. Since $\lambda \wedge \mu$ is a fuzzy k-ideal of R, so by hypothesis $\lambda \wedge \mu$ is idempotent. Thus $\lambda \wedge \mu = (\lambda \wedge \mu) \odot_k (\lambda \wedge \mu) \leq \lambda \odot_k \mu$. By Theorem 75, $\lambda \odot_k \mu \leq \lambda \wedge \mu$. Thus $\lambda \odot_k \mu = \lambda \wedge \mu$.

(2) $\Rightarrow$ (1) Obvious.

If R is commutative then by Theorem 81, (2) $\Leftrightarrow$ (3). ■

Theorem 84 *Let R be a hemiring with identity 1, then the following assertions are equivalent:*

1. Each k-ideal of R is idempotent.
2. $A \cap B = \widehat{AB}$ for each pair of k-ideals A, B of R.
3. Each fuzzy k-ideal of R is idempotent.
4. $\lambda \odot_k \mu = \lambda \wedge \mu$ for all fuzzy k-ideals of R.

Proof. (1) $\Leftrightarrow$ (2) By Proposition 82.

(3) $\Leftrightarrow$ (4) By Proposition 83.

(1) $\Rightarrow$ (3) Let $x \in R$. The smallest k-ideal of R containing x has the form $\widehat{RxR}$. By hypothesis, we have $\widehat{RxR} = \widehat{(\widehat{RxR})(\widehat{RxR})} = \widehat{RxRRxR}$. Thus $x \in \widehat{RxR} = \widehat{RxRRxR}$, this implies

$$x + \sum_{i=1}^{m} r_i x s_i u_i x t_i = \sum_{j=1}^{n} r_j' x s_j' u_j' x t_j'$$

for some $r_i, s_i, u_i, t_i, r_j', s_j', u_j', t_j' \in R$.

As $\lambda(x) \leq \lambda(r_i x s_i)$ and $\lambda(x) \leq \lambda(u_i x t_i)$ for each $i \in \{1, 2, ...m\}$, so

$$\lambda(x) \leq \bigwedge_{i=1}^{m} \lambda(r_i x s_i) \text{ and } \lambda(x) \leq \bigwedge_{i=1}^{m} (u_i x t_i).$$

Therefore $\lambda(x) \leq \left[\bigwedge_{i=1}^{m} \lambda(r_i x s_i)\right] \wedge \left[\bigwedge_{i=1}^{m} (u_i x t_i)\right]$.

Similarly $\lambda(x) \leq \left[\bigwedge_{j=1}^{n} \lambda(r_j' x s_j')\right] \wedge \left[\bigwedge_{j=1}^{n} \lambda(u_j' x t_j')\right]$.

Therefore

$$\lambda(x) \leq \left[\bigwedge_{i=1}^{m} \lambda(r_i x s_i)\right] \wedge \left[\bigwedge_{i=1}^{m} (u_i x t_i)\right] \wedge \left[\bigwedge_{j=1}^{n} \lambda(r_j' x s_j')\right] \wedge \left[\bigwedge_{j=1}^{n} \lambda(u_j' x t_j')\right]$$

$$\leq \bigvee_{x + \sum_{i=1}^{m} r_i x s_i u_i x t_i = \sum_{j=1}^{n} r_j' x s_j' u_j' t_j'} \left(\begin{array}{c} \left[\bigwedge_{i=1}^{m} \lambda(r_i x s_i)\right] \wedge \left[\bigwedge_{i=1}^{m} (u_i x t_i)\right] \wedge \\ \left[\bigwedge_{j=1}^{n} \lambda(r_j' x s_j')\right] \wedge \left[\bigwedge_{j=1}^{n} \lambda(u_j' x t_j')\right] \end{array} \right)$$

$= (\lambda \odot_k \lambda)(x)$.

Hence $\lambda \leq \lambda \odot_k \lambda$. By Theorem 75, $\lambda \odot_k \lambda \leq \lambda$. Thus $\lambda \odot_k \lambda = \lambda$.

(3) $\Rightarrow$ (1) Let A be a k-ideal of R, then the characteristic function χ_A of A is a fuzzy k-ideal of R. Hence by hypothesis $\chi_A = \chi_A \odot_k \chi_A = \chi_{\widehat{AA}}$. Thus $A = \widehat{AA}$. ■

Theorem 85 *If each k-ideal of R is idempotent, then the collection of all k-ideals of R is a complete Brouwerian lattice.*

Proof. Let $\mathcal{L}_R$ be the collection of all k-ideals of R, then $\mathcal{L}_R$ is a poset under the inclusion of sets. It is not difficult to see that $\mathcal{L}_R$ is a complete lattice under the operations $\sqcup$, $\sqcap$ defined as $A \sqcup B = \widehat{A+B}$ and $A \sqcap B = A \cap B$.

We now show that $\mathcal{L}_R$ is a Brouwerian lattice, that is, for any $A, B \in \mathcal{L}_R$, the set $\mathcal{L}_R(A,B) = \{I \in \mathcal{L}_R \,|\, A \cap I \subseteq B\}$ contains a greatest element.

By Zorn's Lemma the set $\mathcal{L}_R(A,B)$ contains a maximal element M. Since each k-ideal of R is idempotent, so $\widehat{AI} = A \cap I \subseteq B$ and $\widehat{AM} = A \cap M \subseteq B$. Thus $\widehat{AI} + \widehat{AM} \subseteq B$. Consequently, $\widehat{\widehat{AI} + \widehat{AM}} \subseteq \widehat{B} = B$.

Since $\widehat{I+M} = I \sqcup M \in \mathcal{L}_R$, for every $x \in \widehat{I+M}$ there exist $i_1, i_2 \in I$, $m_1, m_2 \in M$ such that $x + i_1 + m_1 = i_2 + m_2$. Thus

$$dx + di_1 + dm_1 = di_2 + dm_2$$

for any $d \in D \in \mathcal{L}_R$. As $di_1, di_2 \in DI$, $dm_1, dm_2 \in DM$, we have $dx \in \widehat{DI + DM}$, which implies $D\widehat{(I+M)} \subseteq \widehat{DI+DM} \subseteq \widehat{\widehat{DI} + \widehat{DM}} \subseteq B$. Hence $\widehat{D\widehat{(I+M)}} \subseteq B$. This means that $D \cap \widehat{(I+M)} = \widehat{D\widehat{(I+M)}} \subseteq B$, i.e., $\widehat{I+M} \in \mathcal{L}_R(A,B)$, whence $\widehat{I+M} = M$ because M is maximal in $\mathcal{L}_R(A,B)$. Therefore $I \subseteq \widehat{I} \subseteq \widehat{I+M} = M$ for every $I \in \mathcal{L}_R(A,B)$. ■

Corollary 86 *If each k-ideal of R is idempotent, then the lattice $\mathcal{L}_R$ of all k-ideals of R is distributive.*

Proof. Each complete Brouwerian lattice is distributive (cf. [11], 11.11). ■

Theorem 87 *Each fuzzy k-ideal of R is idempotent if and only if the set of all fuzzy k-ideals of R (ordered by $\leq$) forms a distributive lattice under the k-sum and k-product of fuzzy k-ideals with $\lambda \odot_k \mu = \lambda \wedge \mu$.*

Proof. Suppose that each fuzzy k-ideal of R is idempotent. Then by Proposition 83, $\lambda \odot_k \mu = \lambda \wedge \mu$. Let $\mathcal{FL}_R$ be the collection of all fuzzy k-ideals of R. Then $\mathcal{FL}_R$ is a lattice (ordered by $\leq$) under the k-sum and k-product of fuzzy k-ideals.

We show that $(\lambda \odot_k \delta) +_k \mu = (\lambda +_k \mu) \odot_k (\delta +_k \mu)$ for all $\lambda, \mu, \delta \in \mathcal{FL}_R$. Let $x \in R$, then

$$\begin{aligned}
((\lambda \odot_k \delta) +_k \mu)(x) &= ((\lambda \wedge \delta) +_k \mu)(x) \\
&= \bigvee_{x+(a_1+b_1)=(a_2+b_2)} [(\lambda \wedge \delta)(a_1) \wedge (\lambda \wedge \delta)(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \\
&= \bigvee_{x+(a_1+b_1)=(a_2+b_2)} [\lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2) \wedge \delta(a_1) \wedge \delta(a_2)] \\
&= \left[\bigvee_{x+(a_1+b_1)=(a_2+b_2)} [\lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \right] \\
&\wedge \left[\bigvee_{x+(a_1+b_1)=(a_2+b_2)} [\delta(a_1) \wedge \delta(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \right] \\
&= (\lambda +_k \mu)(x) \wedge (\delta +_k \mu)(x) \\
&= [(\lambda +_k \mu) \wedge (\delta +_k \mu)](x) \\
&= ((\lambda +_k \mu) \odot_k (\delta +_k \mu))(x).
\end{aligned}$$

So, $\mathcal{FL}_R$ is a distributive lattice.

The converse is obvious. ■

2.4 Prime k-ideals

A proper (left, right) k-ideal P of R is called prime if for any (left, right) k-ideals A, B of R, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. A proper (left, right) k-ideal P of R is called *irreducible* if for any (left, right) k-ideals A, B of R, $A \cap B = P$ implies $A = P$ or $B = P$. By analogy a non-constant fuzzy k-ideal δ of R is called *prime* (in the first sense) if for any fuzzy k-ideals λ, μ of R, $\lambda \odot_k \mu \leq \delta$ implies $\lambda \leq \delta$ or $\mu \leq \delta$, and *irreducible* if $\lambda \wedge \mu = \delta$ implies $\lambda = \delta$ or $\mu = \delta$.

Theorem 88 *A left (right) k-ideal P of a hemiring R with identity is prime if and only if for all $a, b \in R$ from $aRb \subseteq P$ it follows $a \in P$ or $b \in P$.*

Proof. Assume that P is a prime left k-ideal of R and $aRb \subseteq P$ for some $a, b \in R$. Obviously, $A = \widehat{Ra}$ and $B = \widehat{Rb}$ are left k-ideals of R generated by a and b, respectively. So, $AB \subseteq \widehat{AB} = \widehat{\widehat{Ra}\,\widehat{Rb}} = \widehat{RaRb} \subseteq \widehat{RP} \subseteq P$, and consequently $A \subseteq P$ or $B \subseteq P$. If $A \subseteq P$, then $a \in P$. If $B \subseteq P$, then $b \in P$.

The converse is obvious. ■

Corollary 89 *A k-ideal P of a hemiring R with identity is prime if and only if for all $a, b \in R$ from $aRb \subseteq P$ it follows $a \in P$ or $b \in P$.*

Corollary 90 *A k-ideal P of a commutative hemiring R with identity is prime if and only if for all $a, b \in R$ from $ab \in P$ it follows $a \in P$ or $b \in P$.*

The result expressed by Corollary 89, suggests the following definition of prime fuzzy k-ideals.

Definition 91 *A non-constant fuzzy k-ideal δ of R is called prime (in the second sense) if for all $t \in [0, 1]$ and $a, b \in R$ the following condition is satisfied:*

if $\delta(axb) \geq t$ for every $x \in R$ then $\delta(a) \geq t$ or $\delta(b) \geq t$.

In other words, a non-constant fuzzy k-ideal δ is prime if from the fact that $axb \in U(\delta; t)$ for every $x \in R$ it follows $a \in U(\delta; t)$ or $b \in U(\delta; t)$. It is clear that any fuzzy k-ideal is prime in the first sense is prime in the second sense. The converse is not true.

Example 92 *In an ordinary hemiring of natural numbers the set of even numbers forms a k-ideal. A fuzzy set*

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0.5 & \text{if } n = 2k \neq 0, \\ 0.3 & \text{if } n = 2k + 1 \end{cases}$$

is a fuzzy k-ideal of this hemiring. It is prime in the second sense but it is not prime in the first sense.

Theorem 93 *A non-constant fuzzy k-ideal δ of a hemiring R with identity is prime in the second sense if and only if each its non-empty proper level set $U(\delta; t)$ is a prime k-ideal of R.*

Proof. Suppose δ is a prime fuzzy k-ideal of R in the second sense and let $U(\delta; t)$ be its arbitrary proper level set, i.e., $\emptyset \neq U(\delta; t) \neq R$. If $aRb \subseteq U(\delta; t)$, then $\delta(axb) \geq t$ for every $x \in R$. Hence $\delta(a) \geq t$ or $\delta(b) \geq t$, i.e., $a \in U(\delta; t)$ or $b \in U(\delta; t)$, which, by Corollary 89, means that $U(\delta; t)$ is a prime k-ideal of R.

To prove the converse, consider a non-constant fuzzy k-ideal δ of R. If it is not prime then there exist $a, b \in R$ such that $\delta(axb) \geq t$ for all $x \in R$, but $\delta(a) < t$ and $\delta(b) < t$. Thus, $aRb \subseteq U(\delta; t)$, but $a \notin U(\delta; t)$ and $b \notin U(\delta; t)$. Therefore $U(\delta; t)$ is not prime, which is a contradiction. Hence δ is a prime fuzzy k-ideal in the second sense. ∎

Corollary 94 *The fuzzy set λ_A defined in Proposition 69, is a prime fuzzy k-ideal of R (with identity) in the second sense if and only if A is a prime k-ideal of R.*

In view of the Transfer Principle the second definition of prime fuzzy k-ideal is better. Therefore fuzzy k-ideals which are prime in the first sense will be called *k-prime.*

Proposition 95 *A non-constant fuzzy k-ideal δ of a commutative hemiring R with identity is prime if and only if $\delta(ab) = \delta(a) \vee \delta(b)$ for all $a, b \in R$.*

Proof. Let δ be a non-constant fuzzy k-ideal of a commutative hemiring R with identity. If $\delta(ab) = t$, then for every $x \in R$, we have $\delta(axb) = \delta(xab) \geq \delta(x) \vee \delta(ab) \geq t$. Thus $\delta(axb) \geq t$ for every $x \in R$, which implies $\delta(a) \geq t$ or $\delta(b) \geq t$. If $\delta(a) \geq t$, then $t = \delta(ab) \geq \delta(a) \geq t$, whence $\delta(ab) = \delta(a)$. If $\delta(b) \geq t$, then, as in the previous case, $\delta(ab) = \delta(b)$. So, $\delta(ab) = \delta(a) \vee \delta(b)$.

Conversely, assume that $\delta(ab) = \delta(a) \vee \delta(b)$ for all $a, b \in R$. If $\delta(axb) \geq t$ for every $x \in R$, then replacing x by the identity of R, we obtain $\delta(ab) \geq t$. Thus $\delta(a) \vee \delta(b) \geq t$, i.e., $\delta(a) \geq t$ or $\delta(b) \geq t$, which means that δ is prime. ∎

Theorem 96 *Every proper k-ideal of a hemiring R is contained in some proper irreducible k-ideal of R.*

Proof. Let P be a proper k-ideal of R such that $a \notin P$. Let $\{P_\alpha \,|\, \alpha \in \Lambda\}$ be a family of all proper k-ideals of R containing P and not containing a. By Zorn's Lemma, this family contains a maximal element, say M. This maximal element is an irreducible k-ideal. Indeed, let $M = P_\beta \cap P_\delta$ for some k-ideals P_β, P_δ of R. If M is a proper subset of P_β and P_δ, then, according to the maximality of M, we have $a \in P_\beta$ and $a \in P_\delta$. Hence $a \in P_\beta \cap P_\delta = M$, which is impossible. Thus, either $M = P_\beta$ or $M = P_\delta$. ∎

Theorem 97 *If all k-ideals of R are idempotent, then a k-ideal P of R is irreducible if and only if it is prime.*

Proof. Assume that all k-ideals of R are idempotent. Let P be an irreducible k-ideal of R. If $AB \subseteq P$ for some k-ideals A, B of R, then by Proposition 82, $A \cap B = \widehat{AB} \subseteq \widehat{P} = P$. Thus $\widehat{(A \cap B) + P} = P$. Since $\mathcal{L}_R$ is a distributive lattice, so $P = \widehat{(A \cap B) + P} = \widehat{(A+P)} \cap \widehat{(B+P)}$. So either $\widehat{A+P} = P$ or $\widehat{B+P} = P$, that is either $A \subseteq P$ or $B \subseteq P$.

Conversely, if a k-ideal P is prime and $A \cap B = P$ for some $A, B \in \mathcal{L}_R$, then $AB \subseteq \widehat{AB} = A \cap B = P$. Thus $A \subseteq P$ or $B \subseteq P$. But $P \subseteq A$ and $P \subseteq B$. Hence $A = P$ or $B = P$. ∎

Corollary 98 *Let R be a hemiring in which all k-ideals are idempotent. Then each proper k-ideal of R is contained in some proper prime k-ideal.*

Theorem 99 *Let R be a hemiring in which all fuzzy k-ideals are idempotent. Then a fuzzy k-ideal of R is irreducible if and only if it is k-prime.*

Proof. Assume that all fuzzy k-ideals of R are idempotent and let δ be an arbitrary irreducible fuzzy k-ideal of R. We prove that it is k-prime. If $\lambda \odot_k \mu \leq \delta$ for some fuzzy k-ideals λ, μ of R then also $\lambda \wedge \mu \leq \delta$. Since the set $\mathcal{FL}_R$ of all fuzzy k-ideals of R is a distributive lattice, we have $\delta = (\lambda \wedge \mu) +_k \delta = (\lambda +_k \delta) \wedge (\mu +_k \delta)$. Thus $\lambda +_k \delta = \delta$ or $\mu +_k \delta = \delta$. This implies $\lambda \leq \delta$ or $\mu \leq \delta$. Hence δ is k-prime.

Conversely, if δ is a k-prime fuzzy k-ideal of R and $\lambda \wedge \mu = \delta$ for some $\lambda, \mu \in \mathcal{FL}_R$, then $\lambda \odot_k \mu = \delta$, which implies $\lambda \leq \delta$ or $\mu \leq \delta$. Since $\delta = \lambda \wedge \mu$, so we have also $\delta \leq \lambda$ and $\delta \leq \mu$. Thus $\lambda = \delta$ or $\mu = \delta$. So, δ is irreducible. ■

Theorem 100 *The following assertions for a hemiring R are equivalent:*

(1) Each k-ideal of R is idempotent.

(2) Each proper k-ideal P of R is the intersection of all prime k-ideals of R which contain P.

Proof. (1) $\Rightarrow$ (2) Let P be a proper k-ideal of R and let $\{P_\alpha \mid \alpha \in \Lambda\}$ be the family of all prime k-ideals of R which contain P. Theorem 96, guarantees the existence of such ideals. Clearly $P \subseteq \cap_{\alpha\in\Lambda} P_\alpha$. If $a \notin P$ then by Theorem 96, there exists an irreducible k-ideal P_a such that $P \subseteq P_a$ and $a \notin P_a$. By Theorem 97, P_a is prime. So there exists a prime k-ideal P_a such that $a \notin P_a$ and $P \subseteq P_a$. Hence $\cap P_\alpha \subseteq P$. Thus $P = \cap P_\alpha$.

(2) $\Rightarrow$ (1) Assume that each k-ideal of R is the intersection of all prime k-ideals of R which contain it. Let A be a k-ideal of R. If $\widehat{A^2} = R$, then we have $A = R$, which means that A is idempotent. If $\widehat{A^2} \neq R$, then $\widehat{A^2}$ is a proper k-ideal of R and so it is the intersection of all prime k-ideals of R containing $\widehat{A^2}$. Let $\widehat{A^2} = \cap P_\alpha$. Then $A^2 \subseteq P_\alpha$ for each α. Since P_α is prime, we have $A \subseteq P_\alpha$. Thus $A \subseteq \cap P_\alpha = \widehat{A^2}$. But $\widehat{A^2} \subseteq A$. Hence $A = \widehat{A^2}$. ■

Lemma 101 *Let R be a hemiring in which each fuzzy k-ideal is idempotent. If λ is a fuzzy k-ideal of R with $\lambda(a) = \alpha$, where a is any element of R and $\alpha \in (0,1]$, then there exists an irreducible k-prime fuzzy k-ideal δ of R such that $\lambda \leq \delta$ and $\delta(a) = \alpha$.*

Proof. Let λ be an arbitrary fuzzy k-ideal of R and $a \in R$ be fixed. Consider the following collection of fuzzy k-ideals of R

$$\mathcal{B} = \{\mu \mid \mu(a) = \lambda(a),\ \lambda \leq \mu\}.$$

B is non-empty since $\lambda \in B$. Let $\mathcal{F}$ be a totally ordered subset of B containing λ, say $\mathcal{F} = \{\lambda_i \mid i \in I\}$.

We claim that $\bigvee_{i\in I} \lambda_i$ is a fuzzy k-ideal of R.

For any $x, y \in R$, we have

$$\begin{aligned}\left(\bigvee_{i\in I}\lambda_i\right)(x) \wedge \left(\bigvee_{i\in I}\lambda_i\right)(y) &= \left(\bigvee_{i\in I}\lambda_i(x)\right) \wedge \left(\bigvee_{j\in I}\lambda_j(y)\right)\\ &= \bigvee_{i,j\in I}(\lambda_i(x)\wedge\lambda_j(y))\\ &\leq \bigvee_{i,j\in I}((\lambda_i(x)\vee\lambda_j(x))\wedge(\lambda_i(y)\vee\lambda_j(y)))\\ &\leq \bigvee_{i,j\in I}(\lambda_i(x+y)\vee\lambda_j(x+y))\\ &\leq \bigvee_{i\in I}\lambda_i(x+y) = \left(\bigvee_{i\in I}\lambda_i\right)(x+y).\end{aligned}$$

Similarly $\left(\bigvee_{i\in I}\lambda_i\right)(x) = \bigvee_{i\in I}\lambda_i(x) \leq \bigvee_{i\in I}\lambda_i(xr) = \left(\bigvee_{i\in I}\lambda_i\right)(xr)$

and $\left(\bigvee_{i\in I}\lambda_i\right)(x) \leq \left(\bigvee_{i\in I}\lambda_i\right)(rx)$ for all $x, r \in R$.

Thus $\bigvee_{i\in I}\lambda_i$ is a fuzzy ideal.

Now, let $x + a = b$, where $x, a, b \in R$. Then

$$\begin{aligned}\left(\bigvee_{i\in I}\lambda_i\right)(a) \wedge \left(\bigvee_{i\in I}\lambda_i\right)(b) &= \left(\bigvee_{i\in I}\lambda_i(a)\right) \wedge \left(\bigvee_{j\in I}(\lambda_j(b)\right)\\ &= \bigvee_{i,j\in I}(\lambda_i(a)\wedge\lambda_j(b))\\ &\leq \bigvee_{i,j\in I}(\lambda_i(a)\vee\lambda_j(a))\wedge(\lambda_i(b)\vee\lambda_j(b))\\ &\leq \bigvee_{i,j\in I}(\lambda_i(x)\vee\lambda_j(x))\\ &\leq \bigvee_{i\in I}\lambda_i(x) = \left(\bigvee_{i\in I}\lambda_i\right)(x).\end{aligned}$$

Thus $\bigvee_{i\in I}\lambda_i$ is a fuzzy k-ideal of R. Clearly $\lambda \leq \bigvee_{i\in I}\lambda_i$ and $(\bigvee_{i\in I}\lambda_i)(a) = \lambda(a) = \alpha$. Thus $\bigvee_{i\in I}\lambda_i$ is the least upper bound of $\mathcal{F}$. Hence by Zorn's lemma there exists a fuzzy k-ideal δ of R which is maximal with respect to the property that $\lambda \leq \delta$ and $\delta(a) = \alpha$.

We will show that δ is an irreducible fuzzy k-ideal of R. Let $\delta = \delta_1 \wedge \delta_2$, where δ_1, δ_2 are fuzzy k-ideals of R. Then $\delta \leq \delta_1$ and $\delta \leq \delta_2$. We claim that either $\delta = \delta_1$ or $\delta = \delta_2$. Suppose $\delta \neq \delta_1$ and $\delta \neq \delta_2$. Since δ is maximal with respect to the property that $\delta(a) = \alpha$ and since $\delta \lneqq \delta_1$ and $\delta \lneqq \delta_2$, so $\delta_1(a) \neq \alpha$ and $\delta_2(a) \neq \alpha$. Hence $\alpha = \delta(a) = (\delta_1 \wedge \delta_2)(a) = \delta_1(a) \wedge \delta_2(a) \neq \alpha$, which is impossible. Hence $\delta = \delta_1$ or $\delta = \delta_2$. Thus δ is an irreducible fuzzy k-ideal of R. By Theorem 99, δ is k-prime. ■

Theorem 102 *Each fuzzy k-ideal of R is idempotent if and only if each fuzzy k-ideal of R is the intersection of those k-prime fuzzy k-ideals of R which contain it.*

Proof. Suppose each fuzzy k-ideal of R is idempotent. Let λ be a fuzzy k-ideal of R and let $\{\lambda_\alpha \mid \alpha \in \Lambda\}$ be the family of all k-prime fuzzy k-ideals of R which contain λ. Obviously $\lambda \leq \bigwedge_{\alpha\in\Lambda} \lambda_\alpha$. We now show that $\bigwedge_{\alpha\in\Lambda} \lambda_\alpha \leq \lambda$. Let a be an arbitrary element of R. Then, by Lemma 101, there exists an irreducible k-prime fuzzy k-ideal δ such that $\lambda \leq \delta$ and $\lambda(a) = \delta(a)$. Hence $\delta \in \{\lambda_\alpha \mid \alpha \in \Lambda\}$ and $\bigwedge_{\alpha\in\Lambda} \lambda_\alpha \leq \delta$. So, $\bigwedge_{\alpha\in\Lambda} \lambda_\alpha(a) \leq \delta(a) = \lambda(a)$. Thus $\bigwedge_{\alpha\in\Lambda} \lambda_\alpha \leq \lambda$. Therefore $\bigwedge_{\alpha\in\Lambda} \lambda_\alpha = \lambda$.

Conversely, assume that each fuzzy k-ideal of R is the intersection of those k-prime fuzzy k-ideals of R which contain it. Let λ be a fuzzy k-ideal of R then $\lambda \odot_k \lambda$ is also a fuzzy k-ideal of R, so

$\lambda \odot_k \lambda = \bigwedge_{\alpha\in\Lambda} \lambda_\alpha$ where λ_α are k-prime fuzzy k-ideals of R. Thus each λ_α contains $\lambda \odot_k \lambda$, and hence λ. So $\lambda \leq \bigwedge_{\alpha\in\Lambda} \lambda_\alpha = \lambda \odot \lambda$, but $\lambda \odot_k \lambda \leq \lambda$ always. Hence $\lambda = \lambda \odot_k \lambda$. ■

2.5 Semiprime k-ideals

Definition 103 *A proper (left, right) k-ideal A of R is called semiprime if for any (left, right) k-ideal B of R, $B^2 \subseteq A$ implies $B \subseteq A$. Similarly, a non-constant fuzzy k-ideal λ of R is called semiprime if for any fuzzy k-ideal δ of R, $\delta \odot_k \delta \leq \lambda$ implies $\delta \leq \lambda$.*

Theorem 104 *A (left, right) k-ideal P of a hemiring R with identity is semiprime if and only if for every $a \in R$ from $aRa \subseteq P$ it follows $a \in P$.*

Proof. Proof is similar to the proof of Theorem 88. ■

Corollary 105 *A k-ideal P of a commutative hemiring R with identity is semiprime if and only if for all $a \in R$ from $a^2 \in P$ it follows $a \in P$.*

Theorem 106 *The following assertions for a hemiring R are equivalent:*

(1) Each k-ideal of R is idempotent.

(2) Each k-ideal of R is semiprime.

Proof. Suppose that each k-ideal of R is idempotent. Let A, B be k-ideals of R such that $B^2 \subseteq A$. Then $\widehat{B^2} \subseteq \widehat{A} = A$. By hypothesis $B = \widehat{B^2}$, so $B \subseteq A$. Hence A is semiprime.

Conversely, assume that each k-ideal of R is semiprime. Let A be a k-ideal of R, then $\widehat{A^2}$ is a k-ideal of R. Also $A^2 \subseteq \widehat{A^2}$. Hence by hypothesis $A \subseteq \widehat{A^2}$. But $\widehat{A^2} \subseteq A$ always. Hence $A = \widehat{A^2}$. ■

Theorem 107 *Each fuzzy k-ideal of R is idempotent if and only if each fuzzy k-ideal of R is semiprime.*

Proof. For any fuzzy k-ideal λ of R we have $\lambda \odot_h \lambda \leq \lambda$. If each fuzzy k-ideal of R is semiprime, then $\lambda \odot_k \lambda \leq \lambda \odot_k \lambda$ implies $\lambda \leq \lambda \odot_k \lambda$. Hence $\lambda \odot_k \lambda = \lambda$.

The converse is obvious. ∎

Theorem 104, suggest the following definition of semiprime fuzzy k-ideals.

Definition 108 *A non-constant fuzzy k-ideal δ of R is called semiprime (in the second sense) if for all $t \in [0,1]$ and $a \in R$ the following condition is satisfied:*

if $\delta(axa) \geq t$ for every $x \in R$ then $\delta(a) \geq t$.

Theorem 109 *A non-constant fuzzy k-ideal δ of R is semiprime in the second sense if and only if each its proper non-empty level set $U(\delta;t)$ is a semiprime k-ideal of R.*

Proof. Proof is similar to the proof of Theorem 93 ∎

Corollary 110 *A fuzzy set λ_A defined in Proposition 69 is a semiprime fuzzy k-ideal of R in the second sense if and only if A is a semiprime k-ideal of R.*

In view of the Transfer Principle the second definition of semiprime fuzzy k-ideal is better. Therefore fuzzy k-ideals which are semiprime in the first sense should be called k-semiprime.

Proposition 111 *A non-constant fuzzy k-ideal δ of a commutative hemiring R with identity is semiprime if and only if $\delta(a^2) = \delta(a)$ for every $a \in R$.*

Proof. Proof is similar to the proof of Proposition 95. ∎

Every fuzzy k-prime k-ideal is fuzzy k-semiprime k-ideal but the converse is not true.

Example 112 *Consider the hemiring $R = \{0, a, b, c\}$ defined by the following tables:*

$+$	0	a	b	c
0	0	a	b	c
a	a	b	c	a
b	b	c	a	b
c	c	a	b	c

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	b	c
c	0	c	c	c

This hemiring has two k-ideals $\{0, c\}$ and R. Obviously these k-ideals are idempotent.

For any fuzzy ideal λ of R and any $x \in R$ we have $\lambda(0) \geq \lambda(x) \geq \lambda(a)$. Indeed, $\lambda(0) = \lambda(0x) \geq \lambda(x) = \lambda(xa) \geq \lambda(a)$. This together with $\lambda(a) = \lambda(b+b) \geq \lambda(b) \wedge \lambda(b) = \lambda(b)$ implies $\lambda(a) = \lambda(b)$. Consequently, $\lambda(c) = \lambda(a+b) \geq \lambda(a) \wedge \lambda(b) = \lambda(b)$. Therefore $\lambda(0) \geq \lambda(c) \geq \lambda(b) = \lambda(a)$ for every fuzzy k-ideal of this hemiring.

Now we prove that each fuzzy k-ideal of R is idempotent. Since $\lambda \odot_k \lambda \leq \lambda$ always, so we have to show that $\lambda \odot_k \lambda \geq \lambda$. Obviously, for every $x \in R$ we have

$$\begin{aligned}(\lambda \odot_k \lambda)(x) &= \sup_{x+\sum\limits_{i=1}^{m} a_i b_i = \sum\limits_{j=1}^{n} a'_j b'_j} \left[\bigwedge_{i=1}^{n} (\lambda(a_i) \wedge \lambda(b_i)) \wedge \bigwedge_{j=1}^{m} \left(\lambda(a'_j) \wedge \lambda(b'_j) \right) \right] \\ &\geq \sup_{x+cd=c'd'} [\lambda(c) \wedge \lambda(d) \wedge \lambda(c') \wedge \lambda(d')] = \lambda(c) \wedge \lambda(d) \wedge \lambda(c') \wedge \lambda(d').\end{aligned}$$

So, $x + cd = c'd'$ implies $(\lambda \odot_k \lambda)(x) \geq \lambda(c) \wedge \lambda(d) \wedge \lambda(c') \wedge \lambda(d')$. Hence $0 + 00 = 00$ implies $(\lambda \odot_k \lambda)(0) \geq \lambda(0)$. Similarly $a + bb = bc$ implies $(\lambda \odot_k \lambda)(a) \geq \lambda(b) \wedge \lambda(c) = \lambda(b) = \lambda(a)$, $b + aa = bc$ implies $(\lambda \odot_k \lambda)(b) \geq \lambda(a) \wedge \lambda(b) \wedge \lambda(c) = \lambda(b)$. Analogously, from $c + 00 = cc$ it follows $(\lambda \odot_k \lambda)(c) \geq \lambda(0) \wedge \lambda(c) = \lambda(c)$. This proves that $(\lambda \odot_k \lambda)(x) \geq \lambda(x)$ for every $x \in R$. Therefore $\lambda \odot_k \lambda = \lambda$ for every fuzzy k-ideal of R, which, by Theorem 106, means that each fuzzy k-ideal of R is semiprime.

Consider the following three fuzzy sets:

$$\begin{aligned}\lambda(0) = \lambda(c) = 0.8, \quad \lambda(a) = \lambda(b) = 0.4, \\ \mu(0) = \mu(c) = 0.6, \quad \mu(a) = \mu(b) = 0.5, \\ \delta(0) = \delta(c) = 0.7, \quad \delta(a) = \delta(b) = 0.45.\end{aligned}$$

These three fuzzy sets are idempotent fuzzy k-ideals. Since all fuzzy k-ideal of this hemiring are idempotent, by Proposition 82, we have $\lambda \odot_k \mu = \lambda \wedge \mu$. Thus $(\lambda \odot_k \mu)(0) = (\lambda \odot_k \mu)(c) = 0.6$ and $(\lambda \odot_k \mu)(a) = (\lambda \odot_k \mu)(b) = 0.4$. So, $\lambda \odot_k \mu \leq \delta$ but neither $\lambda \leq \delta$ nor $\mu \leq \delta$, that is δ is not a k-prime fuzzy k-ideal.

2.6 Prime Spectrum

Let R be a hemiring in which each k-ideal is idempotent. Let $\mathcal{L}(R)$ be the lattice of all k-ideals of R and $\mathcal{P}(R)$ be the set of all proper prime k-ideals of R. For each k-ideal I of R define $\theta_I = \{J \in \mathcal{P}(R) : I \nsubseteq J\}$ and $\Im(\mathcal{P}(R)) = \{\theta_I : I \in \mathcal{L}(R)\}$.

Theorem 113 *The set $\Im(\mathcal{P}(R))$ forms a topology on the set $\mathcal{P}(R)$.*

Proof. Since $\theta_{\{0\}} = \{J \in \mathcal{P}(R) : \{0\} \nsubseteq J\} = \phi$, where ϕ is the usual empty set, because 0 belongs to each k-ideal. So empty set belongs to $\Im(\mathcal{P}(R))$.

Also $\theta_R = \{J \in \mathcal{P}(R) : R \nsubseteq J\} = \mathcal{P}(R)$, because $\mathcal{P}(R)$ is the set of all proper prime k-ideals of R. Thus $\mathcal{P}(R)$ belongs to $\Im(\mathcal{P}(R))$.

Suppose $\theta_{I_1}, \theta_{I_2} \in \Im(\mathcal{P}(R))$ where I_1 and I_2 are in $\mathcal{L}(R)$. Then

$\theta_{I_1} \cap \theta_{I_2} = \{J \in \mathcal{P}(R) : I_1 \nsubseteq J \text{ and } I_2 \nsubseteq J\}$. Since each k-ideal of R is idempotent so $I_1 I_2 = I_1 \cap I_2$. Thus $\theta_{I_1} \cap \theta_{I_2} = \theta_{I_1 \cap I_2}$. So $\theta_{I_1} \cap \theta_{I_2}$ belongs to $\Im(\mathcal{P}(R))$.

Let $\{\theta_{I_i}\}_{i\in\Sigma}$ be an arbitrary family of members of $\Im(\mathcal{P}(R))$. Then

$\bigcup\limits_{i\in\Omega} \theta_{I_i} = \bigcup\limits_{i\in\Omega} \{J \in \mathcal{P}(R) : I_i \nsubseteq J\} = \{J \in \mathcal{P}(R) : \exists l \in \Omega \text{ so that } I_l \nsubseteq J\} =$
$\theta_{\sum\limits_{i\in\Omega} I_i}$, where $\sum\limits_{i\in\Omega} I_i$ is the k-ideal generated by $\bigcup\limits_{i\in\Omega} I_i$.

Hence $\Im(\mathcal{P}(R))$ is a topology on $\mathcal{P}(R)$. ■

Definition 114 *A fuzzy k-ideal μ of a hemiring R is said to be normal if there exists $x \in R$ such that $\mu(x) = 1$. If μ is a normal fuzzy k-ideal of R, then $\mu(0) = 1$, hence μ is normal if and only if $\mu(0) = 1$.*

Theorem 115 *A fuzzy subset δ of a hemiring R is a k-prime fuzzy left (right) k-ideal of R if and only if*

(*i*) $\delta^\circ = \{x \in R : \delta(x) = \delta(0)\}$ is a prime left (right) k-ideal of R.

(*ii*) $\mathrm{Im}\delta = \{\delta(x) : x \in R\}$ contains exactly two elements

(*iii*) $\delta(0) = 1$.

Proof. Let δ be a k-prime fuzzy left k-ideal.

(*i*) Infact δ° is a prime left k-ideal, because for $x, y \in \delta^\circ$, $\delta(x+y) \geq \delta(x) \wedge \delta(y) = \delta(0)$ implies that $x + y \in \delta^\circ$. Also for each $x \in R$ and $y \in \delta^\circ$, $\delta(xy) \geq \delta(y) = \delta(0)$ implies that $xy \in \delta^\circ$. Now for $a, b \in \delta^\circ$ and $x \in R$, $x + a = b$. Since δ is fuzzy left k-ideal so $\delta(x) \geq \delta(a) \wedge \delta(b) = \delta(0)$, implies that $\delta(x) \geq \delta(0)$ and $\delta(x) \leq \delta(0)$ always hold, so $\delta(x) = \delta(0)$ implies that $x \in \delta^\circ$. Hence δ° is a left k-ideal of R.

Let A, B be left k-ideals of R such that $\widehat{AB} \subseteq \delta^\circ$, then χ_A, χ_B, the characteristic function of A, B, are fuzzy left k-ideals of R, such that

$$\chi_A \odot_k \chi_B = \chi_{\widehat{AB}} \leq \chi_{\delta^\circ}$$

Let us define

$\chi_A(x) = \begin{cases} \delta(0) & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$, $\quad \chi_B(x) = \begin{cases} \delta(0) & \text{if } x \in B \\ 0 & \text{otherwise} \end{cases}$ and

$\chi_{\delta^\circ}(x) = \begin{cases} \delta(0) & \text{if } x \in \delta^\circ \\ 0 & \text{otherwise} \end{cases}$

then $\chi_{\delta^\circ} \leq \delta$, implies that $\chi_A \leq \delta$ or $\chi_B \leq \delta$, since δ is k-prime.

Thus $\chi_A \leq \chi_{\delta^\circ}$ or $\chi_B \leq \chi_{\delta^\circ}$. Hence $A \subseteq \delta^\circ$ or $B \subseteq \delta^\circ$. Thus δ° is a prime left k-ideal of R.

(ii) Suppose that Imδ contains more than two elements. Then there exist two elements $a, b \in R\backslash\delta^{\circ}$ such that $\delta(a) \neq \delta(b)$. We assume that $\delta(a) < \delta(b)$. Since δ is a fuzzy left k-ideal and $b \notin \delta^{\circ}$, it follows that $\delta(a) < \delta(b) < \delta(0)$. So there exist $r, t \in [0,1]$ such that

$$\delta(a) < r < \delta(b) < t < \delta(0) \tag{2.3}$$

Let ν and ω be fuzzy left k-ideals defined by $\mu = r\chi_{\langle a\rangle}$ and $\nu = t\chi_{\langle b\rangle}$, where $\chi_{\langle a\rangle}, \chi_{\langle b\rangle}$ are characteristic functions of ideals generated by a and b, respectively. Then, for any $x \in R$, which cannot be expressed in the form $x + \sum\limits_{i=1}^{m} a_i b_i = \sum\limits_{j=1}^{n} a_j' b_j'$, where $a_i, a_j' \in \langle a\rangle$ and $b_i, b_j' \in \langle b\rangle$, we have $(\mu \odot_k \nu)(x) = 0$. Otherwise

$$(\mu \odot_k \nu)(x) = \bigvee_{x+\sum\limits_{i=1}^{m} a_i b_i=\sum\limits_{j=1}^{n} a_j' b_j'} \left[\begin{array}{c} \left[\bigwedge\limits_{i=1}^{m} \mu(a_i)\right] \wedge \left[\bigwedge\limits_{i=1}^{m} \nu(b_i)\right] \wedge \\ \left[\bigwedge\limits_{j=1}^{n} \mu(a_j')\right] \wedge \left[\bigwedge\limits_{j=1}^{n} \nu(b_j')\right] \end{array}\right] = r \wedge t = r.$$

Since δ is a fuzzy left k-ideal, from $x + \sum\limits_{i=1}^{m} a_i b_i = \sum\limits_{j=1}^{n} a_j' b_j'$ it follows that

$$\delta(x) \geq \delta\left(\sum_{i=1}^{m} a_i b_i\right) \wedge \delta\left(\sum_{j=1}^{n} a_j' b_j'\right) \geq \delta(a_i b_i) \wedge \delta\left(a_j' b_j'\right) \geq \delta(b_i) \wedge \delta\left(b_j'\right) \geq r.$$

So, $\mu \odot_k \nu \leq \delta$, implies that $\mu \leq \delta$ or $\nu \leq \delta$ because δ is a k-prime fuzzy left k-ideal. Therefore $\mu(a) = r \leq \delta(a)$ or $\nu(b) = t \leq \delta(b)$ which contradicts to 2.3. Thus, Imδ contains exactly two elements.

(iii) Suppose that $\delta(0) \neq 1$. Then, according to (ii), Im$\delta = \{\alpha, \beta\}$, where $0 \leq \alpha < \beta < 1$. Since $\delta(0) = \delta(0 \cdot x) \geq \delta(x)$ for all $x \in R$, we have $\delta(0) = \beta$. Thus

$$\delta(x) = \begin{cases} \beta & \text{if } x \in \delta^{\circ}, \\ \alpha & \text{otherwise.} \end{cases}$$

Consider, for fixed $a \in \delta^{\circ}$ and $b \in R\backslash\delta^{\circ}$, two fuzzy subsets

$$\mu(x) = \begin{cases} t & \text{if } x \in \langle a\rangle, \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad \nu(x) = \begin{cases} t & \text{if } x \in \langle b\rangle, \\ 0 & \text{otherwise.} \end{cases}$$

where $0 \leq \alpha < r < \beta < t \leq 1$.

It is clear that μ and ν are fuzzy k-ideals of R.

If x does not satisfy the equality $x + \sum\limits_{i=1}^{m} a_i b_i = \sum\limits_{j=1}^{n} a_j' b_j'$, where $a_i, a_j' \in \langle a\rangle$ and $b_i, b_j' \in \langle b\rangle$, then we have $(\mu \odot_k \nu)(x) = 0$. Otherwise

$$(\mu \odot_k \nu)(x) = \bigvee_{x+\sum\limits_{i=1}^{m} a_i b_i=\sum\limits_{j=1}^{n} a_j' b_j'} \left[\begin{array}{c} \left[\bigwedge\limits_{i=1}^{m} \mu(a_i)\right] \wedge \left[\bigwedge\limits_{i=1}^{m} \nu(b_i)\right] \wedge \\ \left[\bigwedge\limits_{j=1}^{n} \mu(a_j')\right] \wedge \left[\bigwedge\limits_{j=1}^{n} \nu(b_j')\right] \end{array}\right] = r \wedge t = r.$$

By (i) δ° is a prime left k-ideal of R. If $a_i, a_j^{'} \in \langle a \rangle$, then $a_i, a_j^{'} \in \delta^\circ$ because $a \in \delta^\circ$ and $\langle a \rangle \subseteq \delta^\circ$. This implies $x \in \delta^\circ$. Thus $\delta(x) = \beta > r = (\mu \odot_k \nu)(x)$. Therefore, $\mu \odot_k \nu \leq \delta$. But $\mu(a) = t > \beta = \delta(a)$ and $\nu(b) = r > \alpha = \delta(b)$, which gives $\mu \nleq \delta$ and $\nu \nleq \delta$. This contradicts to the assumption that δ is k-prime fuzzy left k-ideal of R. Hence $\delta(0) = 1$.

Conversely, assume that the above conditions are satisfied. Then $\delta(0) = 1$ and $\mathrm{Im}\delta = \{\alpha, 1\}$ for some $0 \leq \alpha < 1$. Moreover, $\delta(x+y) \geq \delta(x) \wedge \delta(y)$ for $x, y \in R$ because if $\delta(x+y) < \delta(x) \wedge \delta(y)$ then $\delta(x) = \delta(y) = 1$, that is, $x, y \in \delta^\circ$, implies that $x+y \in \delta^\circ$ implies that $\delta(x+y) = 1$, which is impossible. Similarly $\delta(xy) \geq \delta(y)$ since $\delta(y) = 1$ implies $xy \in \delta^\circ$, whence $\delta(xy) = 1$. This means that δ is a fuzzy left ideal of R. If $x+a = b$ for $x, a, b \in R$ then $\delta(x) \geq \delta(a) \wedge \delta(b)$ because if $\delta(x) < \delta(a) \wedge \delta(b)$ then $\delta(a) = \delta(b) = 1$, that is, $a, b \in \delta^\circ$, implies that $x \in \delta^\circ$ implies that $\delta(x) = 1$, which is impossible. Hence δ is a fuzzy left k-ideal. δ is prime. Let μ, ν be two fuzzy left k-ideals of R such that $\mu \odot_k \nu \leq \delta$, and $\mu \nleq \delta$ and $\nu \nleq \delta$. Assume that $\mu(c) > \delta(c)$ and $\nu(d) > \delta(d)$ for some $c, d \in R$. It is possible only in the case when $\delta(c) = \delta(d) = \alpha$, i.e. when $c, d \notin \delta^\circ$. Since δ° is prime, then there exists $r \in R$ such that $crd \notin \delta^\circ$. Otherwise, $cRd \subseteq \delta^\circ$, whence $(Rc)(Rd) \subseteq \delta^\circ$. So, $\widehat{(Rc)(Rd)} \subseteq \widehat{\delta^\circ} = \delta^\circ$, because δ° is a left k-ideal of R. Moreover, $\widehat{(Rc)}\widehat{(Rd)} \subseteq \widehat{\widehat{(Rc)}\widehat{(Rd)}} = \widehat{(Rc)(Rd)}$. Thus $\widehat{(Rc)}\widehat{(Rd)} \subseteq \delta^\circ$, and consequently $\widehat{(Rc)} \subseteq \delta^\circ$ or $\widehat{(Rd)} \subseteq \delta^\circ$. In the first case $\langle c \rangle \langle c \rangle \subseteq \widehat{(Rc)} \subseteq \delta^\circ$, whence $\widehat{\langle c \rangle} \subseteq \delta^\circ$. So, $c \in \langle c \rangle \in \widehat{\langle c \rangle} \subseteq \delta^\circ$. This is contradiction. Also the second case yields a contradiction.

Let $a = crd$. Then $\delta(a) = \alpha$. Thus, by the assumption

$$(\mu \odot_k \nu)(a) \leq \delta(a) = \alpha. \tag{2.4}$$

Obviously $a + crd = 2crd$. Thus $a + (c)(rd) = (2c)(rd)$. Therefore for $a = crd$ we have

$$\begin{aligned}(\mu \odot_k \nu)(a) &= \bigvee_{x+\sum_{i=1}^{m} a_i b_i = \sum_{j=1}^{n} a_j' b_j'} \left[\begin{array}{c} \left[\bigwedge_{i=1}^{m} \mu(a_i)\right] \wedge \left[\bigwedge_{i=1}^{m} \nu(b_i)\right] \wedge \\ \left[\bigwedge_{j=1}^{n} \mu(a_j')\right] \wedge \left[\bigwedge_{j=1}^{n} \nu(b_j')\right] \end{array} \right] \\ &\geq \mu(c) \wedge \mu(2c) \wedge \nu(rd) \geq \mu(c) \wedge \nu(d) > \alpha,\end{aligned}$$

since $\mu(c) > \alpha$ and $\nu(d) > \alpha$.

This contradicts 2.4. Hence for any fuzzy left k-ideals μ and ν of R, $\mu \odot_k \nu \leq \delta$ implies $\mu \leq \delta$ or $\nu \leq \delta$. This completes the proof. ■

Corollary 116 *Every k-prime fuzzy k-ideal of a hemiring is normal.*

Let R be a hemiring in which each fuzzy k-ideal is idempotent, $\mathcal{L}_R$ the lattice of fuzzy normal k-ideals of R and $\mathcal{F}P_R$ the set of all proper fuzzy k-prime k-ideals of R. For any fuzzy normal k-ideal λ of R, we define $\theta_\lambda = \{\mu \in \mathcal{F}P_R : \lambda \nleq \mu\}$ and $\tau(\mathcal{F}P_R) = \{\theta_\lambda : \lambda \in \mathcal{L}_R\}$.

A fuzzy k-ideal λ of R is called proper if $\lambda \neq \mathbb{R}$, where $\mathbb{R}$ is the fuzzy k-ideal of R defined by $\mathbb{R}(x) = 1$, $\forall x \in R$.

Theorem 117 *The set $\tau(\mathcal{F}P_R)$ forms a topology on the set $\mathcal{F}P_R$.*

Proof. (1) $\theta_\Phi = \left\{\mu \in \mathcal{F}P_R : \Phi \nleq \mu\right\} = \phi$, where ϕ is the usual empty set and Φ is the characteristic function of k-ideal $\{0\}$. This follows since each k-prime fuzzy k-ideal of R is normal. Thus the empty subset belongs to $\tau(\mathcal{F}P_R)$.

(2) $\theta_{\mathbb{R}} = \left\{\mu \in \mathcal{F}P_R : \mathbb{R} \nleq \mu\right\} = \mathcal{F}P_R$. This is true, since $\mathcal{F}P_R$ is the set of proper k-prime fuzzy k-ideals of R. So $\theta_{\mathbb{R}} = \mathcal{F}P_R$ is an element of $\tau(\mathcal{F}P_R)$.

(3) Let $\theta_{\delta_1}, \theta_{\delta_2} \in \tau(\mathcal{F}P_R)$ with $\delta_1, \delta_2 \in \mathcal{L}_R$.

Then

$$\theta_{\delta_1} \cap \theta_{\delta_2} = \left\{\mu \in \mathcal{F}P_R : \delta_1 \nleq \mu \text{ and } \delta_2 \nleq \mu\right\}$$

Since each fuzzy k-ideal of R is idempotent, this implies $\delta_1\delta_2 = \delta_1 \wedge \delta_2$. Thus

$$\theta_{\delta_1} \cap \theta_{\delta_2} = \left\{\mu \in \mathcal{F}P_R : \delta_1 \nleq \mu \text{ and } \delta_2 \nleq \mu\right\} = \theta_{\delta_1 \wedge \delta_2}.$$

(4) Let us consider an arbitrary family $\{\delta_i\}_{i\in I}$ of fuzzy k-ideals of R. Since

$$\cup_{i\in I}\theta_{\delta_i} = \cup_{i\in I}\left\{\mu \in \mathcal{F}P_R : \delta_i \nleq \mu\right\} = \left\{\mu \in \mathcal{F}P_R : \exists' s\ k \in I \text{ so that } \delta_k \nleq \mu\right\}$$

Note that

$$\left(\sum_{i\in I}\delta_i\right)(x) = \bigvee_{x+a_1+a_2+....=b_1+b_2+....}\{\delta_1(a_1) \wedge \delta_2(a_2) \wedge ....\delta_1(b_1) \wedge \delta_2(b_2) \wedge ...\}$$

where $a_1, a_2, ...b_1, b_2, ... \in R$ and only a finite number of the $a_i's$ and $b_i's$ are not zero. Since $\delta_i(0) = 1$, therefore we are considering the infimum of a finite number of terms because $1's$ are effectively not being considered. Now, if for some $k \in I$, $\delta_k \nleq \mu$, then there exists $x \in R$ such that $\delta_k(x) > \mu(x)$. Consider the particular expression for x in which $a_k = x$, $b_k = 0$ and $a_i = b_i = 0$ for all $i \neq k$. We see that $\delta_k(x)$ is an element of the set whose supremum is defined to be $\left(\sum_{i\in I}\delta_i\right)(x)$.

Thus $\left(\sum_{i\in I}\delta_i\right)(x) \geq \delta_k(x) > \mu(x)$. This implies $\left(\sum_{i\in I}\delta_i\right)(x) > \mu(x)$ that is $\sum_{i\in I}\delta_i \nleq \mu$.

Hence $\delta_k \nleq \mu$ for some $k \in I$ implies $\sum_{i\in I}\delta_i \nleq \mu$.

Conversely, suppose that $\sum_{i\in I}\delta_i \nleq \mu$ then there exists an element $x\in R$ such that $\left(\sum_{i\in I}\delta_i\right)(x) > \mu(x)$.

This means that

$$\bigvee_{x+a_1+a_2+....=b_1+b_2+....}\{\delta_1(a_1)\wedge\delta_2(a_2)\wedge....\delta_1(b_1)\wedge\delta_2(b_2)\wedge...\} > \mu(x).$$

Now, if all the elements of the set (whose supremum we are taking) are individually less than are equal to $\mu(x)$, then we have

$$\begin{aligned}\left(\sum_{i\in I}\delta_i\right)(x) &= \bigvee_{x+a_1+a_2+....=b_1+b_2+....}\{\delta_1(a_1)\wedge\delta_2(a_2)\wedge....\delta_1(b_1)\wedge\delta_2(b_2)\wedge...\}\\ &\leq \mu(x)\end{aligned}$$

which does not agree with what we have assumed. Thus, there is at least one element of the set (whose supremum we are taking), say,

$$\delta_1\left(a_1'\right)\wedge\delta_2\left(a_2'\right)\wedge.......\delta_1\left(b_1'\right)\wedge\delta_2\left(b_2'\right)\wedge...... > \mu(x)$$

($x+a_1'+a_2'+....=b_1'+b_2'+....$ being the corresponding breakup of x, where only a finite number of $a_i's$ and $b_i's$ are not zero.)

Thus,

$$\delta_1\left(a_1'\right)\wedge\delta_2\left(a_2'\right)\wedge.....\delta_1\left(b_1'\right)\wedge\delta_2\left(b_2'\right)\wedge.... > \mu(x) \geq \mu_1\left(a_1'\right)\wedge\mu_2\left(a_2'\right)\wedge....\mu_1\left(b_1'\right)\wedge\mu_2\left(b_2'\right)\wedge....$$

Let

$$\delta_1\left(a_1'\right)\wedge\delta_2\left(a_2'\right)\wedge.......\delta_1\left(b_1'\right)\wedge\delta_2\left(b_2'\right)\wedge...... = \delta_p\left(x_p'\right)$$

and

$$\mu_1\left(a_1'\right)\wedge\mu_2\left(a_2'\right)\wedge.......\mu_1\left(b_1'\right)\wedge\mu_2\left(b_2'\right)\wedge...... = \mu_p\left(x_p'\right) \text{ where } p\in I$$

So, $\delta_p\left(x_p'\right) > \mu_p\left(x_p'\right)$ it follows that $\delta_p \nleq \mu$ for some $p\in I$.

Hence $\sum_{i\in I}\delta_i \nleq \mu$ implies that $\delta_p \nleq \mu$ for some $p\in I$.

Hence the two statements (i) $\sum_{i\in I}\delta_i \nleq \mu$ and (ii) $\delta_p \nleq \mu$ for some $p\in I$ are equivalent.

Hence

$$\cup_{i\in I}\theta_{\delta_i} = \cup_{i\in I}\left\{\mu\in\mathcal{F}P_R : \delta_i \nleq \mu\right\} = \cup_{i\in I}\left\{\mu\in\mathcal{F}P_R : \sum_{i\in I}\delta_i \nleq \mu\right\} = \theta_{\sum_{i\in I}\theta_i}$$

because, $\sum_{i\in I}\delta_i$ is also a fuzzy k-ideal of R .

Thus, $\cup_{i\in I}\theta_{\delta_i}\in\tau(\mathcal{F}P_R)$.

Hence it follows that $\tau(\mathcal{F}P_R)$ forms a topology on the set $\mathcal{F}P_R$. ■

Chapter 3

Characterizations of hemirings by the properties of their h-ideals

In [3], J.Ahsan studied those hemirings for which each ideal is idempotent. In [5] those hemirings are studied for which each fuzzy ideal is idempotent. In this chapter we characterize hemirings in which each h-ideal is idempotent. We also characterize hemirings for which each fuzzy h-ideal is idempotent.

3.1 h-intrinsic product of fuzzy subsets

Recall that the *h-intrinsic product* of two fuzzy subsets μ and ν on R is

$$(\mu \odot_h \nu)(x) = \sup_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a_j' b_j' + z} \left(\bigwedge_{i=1}^{m} (\mu(a_i) \wedge \nu(b_i)) \wedge \bigwedge_{j=1}^{n} (\mu(a_j') \wedge \nu(b_j')) \right)$$

and $(\mu \odot_h \nu)(x) = 0$ if x can not be expressed as $x + \sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a_j' b_j' + z$.

Theorem 118 *If λ and μ are fuzzy h-ideals of R, then so is $\lambda \odot_h \mu$. Moreover, $\lambda \odot_h \mu \leq \lambda \wedge \mu$.*

Proof. Let λ and μ be fuzzy h-ideals of R. Let $x, y \in R$, then

$$(\lambda \odot_h \mu)(x) = \sup_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a_j' b_j' + z} \left(\bigwedge_{i=1}^{m} (\lambda(a_i) \wedge \mu(b_i)) \wedge \bigwedge_{i=1}^{n} (\lambda(a_j') \wedge \mu(b_j')) \right)$$

and

$$(\lambda \odot_h \mu)(y) = \sup_{y+\sum_{k=1}^{p} c_k d_k + z' = \sum_{l=1}^{q} c_l' d_l' + z'} \left(\bigwedge_{k=1}^{p} (\lambda(c_k) \wedge \mu(d_k)) \wedge \bigwedge_{l=1}^{q} (\lambda(c_l') \wedge \mu(d_l')) \right)$$

Thus

$$
\begin{aligned}
(\lambda \odot_h \mu)(x+y) &= \sup_{x+y+\sum\limits_{s=1}^{u} e_s f_s + z = \sum\limits_{t=1}^{v} e'_t f'_t + z} \Big(\bigwedge_{s=1}^{u} (\lambda(e_s) \wedge \mu(f_s)) \wedge \bigwedge_{t=1}^{v} (\lambda(e'_t) \wedge \mu(f'_t)) \Big) \\
&\geq \sup_{x+\sum\limits_{i=1}^{m} a_i b_i + z = \sum\limits_{j=1}^{n} a'_j b'_j + z} \left(\sup_{y+\sum\limits_{k=1}^{p} c_k d_k + z' = \sum\limits_{l=1}^{q} c'_l d'_l + z'} \left(\begin{array}{l} \bigwedge\limits_{i=1}^{m} \Big(\lambda(a_i) \wedge \mu(b_i)\Big) \wedge \\ \bigwedge\limits_{j=1}^{n} \Big(\lambda(a'_j) \wedge \mu(b'_j)\Big) \wedge \\ \bigwedge\limits_{k=1}^{p} \Big(\lambda(c_k) \wedge \mu(d_k)\Big) \wedge \\ \bigwedge\limits_{l=1}^{q} \Big(\lambda(c'_l) \wedge \mu(d'_l)\Big) \end{array} \right) \right) \\
&= \sup_{x+\sum\limits_{i=1}^{m} a_i b_i + z = \sum\limits_{j=1}^{n} a'_j b'_j + z} \Big(\bigwedge_{i=1}^{m} (\lambda(a_i) \wedge \mu(b_i)) \wedge \bigwedge_{i=1}^{n} (\lambda(a'_j) \wedge \mu(b'_j)) \Big) \\
&\wedge \sup_{y+\sum\limits_{k=1}^{p} c_k d_k + z' = \sum\limits_{l=1}^{q} c'_l d'_l + z'} \Big(\bigwedge_{k=1}^{p} (\lambda(c_k) \wedge \mu(d_k)) \wedge \bigwedge_{l=1}^{q} (\lambda(c'_l) \wedge \mu(d'_l)) \Big) \\
&= (\lambda \odot_h \mu)(x) \wedge (\lambda \odot_h \mu)(y).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(\lambda \odot_h \mu)(xr) &= \sup_{xr+\sum\limits_{k=1}^{p} g_k h_k + z = \sum\limits_{l=1}^{q} g'_l h'_l + z} \Big(\bigwedge_{k=1}^{p} (\lambda(g_k) \wedge \mu(h_k)) \wedge \bigwedge_{l=1}^{q} (\lambda(g'_l) \wedge \mu(h'_l)) \Big) \\
&\geq \sup_{x+\sum\limits_{i=1}^{m} a_i b_i + z = \sum\limits_{j=1}^{n} a'_j b'_j + z} \Big(\bigwedge_{i=1}^{m} (\lambda(a_i) \wedge \mu(b_i r)) \wedge \bigwedge_{j=1}^{n} (\lambda(a'_j) \wedge \mu(b'_j r)) \Big) \\
&\geq \sup_{x+\sum\limits_{i=1}^{m} a_i b_i + z = \sum\limits_{j=1}^{n} a'_j b'_j + z} \Big(\bigwedge_{i=1}^{m} (\lambda(a_i) \wedge \mu(b_i)) \wedge \bigwedge_{j=1}^{n} (\lambda(a'_j) \wedge \mu(b'_j)) \Big) \\
&= (\lambda \odot_h \mu)(x)
\end{aligned}
$$

Analogously we can verify that $(\lambda \odot_h \mu)(rx) \geq (\lambda \odot_h \mu)(x)$ for all $r \in R$. This means that $\lambda \odot_h \mu$ is a fuzzy ideal of R.

To prove that $x + a + y = b + y$ implies $(\lambda \odot_h \mu)(x) \geq (\lambda \odot_h \mu)(a) \wedge (\lambda \odot_h \mu)(b)$, observe that

$$
a + \sum_{i=1}^{m} a_i b_i + z_1 = \sum_{j=1}^{n} a'_j b'_j + z_1 \text{ and } b + \sum_{k=1}^{l} c_k d_k + z_2 = \sum_{q=1}^{p} c'_q d'_q + z_2, \tag{3.1}
$$

together with $x + a + y = b + y$, gives

$$
x + a + (\sum_{i=1}^{m} a_i b_i + z_1) + y = b + (\sum_{i=1}^{m} a_i b_i + z_1) + y.
$$

Thus,

$$
x + \sum_{j=1}^{n} a'_j b'_j + z_1 + y = b + \sum_{i=1}^{m} a_i b_i + z_1 + y
$$

and, consequently,

$$x+\sum_{j=1}^{n}a_j'b_j'+(\sum_{k=1}^{l}c_kd_k+z_2)+z_1+y=b+(\sum_{k=1}^{l}c_kd_k+z_2)+\sum_{i=1}^{m}a_ib_i+z_1+y$$
$$=\sum_{q=1}^{p}c_q'd_q'+z_2+\sum_{i=1}^{m}a_ib_i+z_1+y$$
$$=\sum_{i=1}^{m}a_ib_i+\sum_{q=1}^{p}c_q'd_q'+z_2+z_1+y.$$

Therefore,

$$x+\sum_{j=1}^{n}a_j'b_j'+\sum_{k=1}^{l}c_kd_k+z_2+z_1+y=\sum_{i=1}^{m}a_ib_i+\sum_{q=1}^{p}c_q'd_q'+z_2+z_1+y. \quad (3.2)$$

Now, in view of (3.1) and (3.2), we have

$$(\lambda\odot_h\mu)(a)\wedge(\lambda\odot_h\mu)(b)=\left(\sup_{a+\sum_{i=1}^{m}a_ib_i+z=\sum_{j=1}^{n}a_j'b_j'+z}\left[\begin{array}{c}\left[\bigwedge_{i=1}^{m}(\lambda(a_i)\wedge\mu(b_i))\right]\\ \wedge\left[\bigwedge_{i=1}^{n}\left(\lambda(a_j')\wedge\mu(b_j')\right)\right]\end{array}\right]\right)$$
$$\wedge\left(\sup_{b+\sum_{k=1}^{p}c_kd_k+z'=\sum_{l=1}^{q}c_l'd_l'+z'}\left(\bigwedge_{k=1}^{p}(\lambda(c_k)\wedge\mu(d_k))\wedge\bigwedge_{l=1}^{q}(\lambda(c_l')\wedge\mu(d_l'))\right)\right)$$
$$=\sup_{a+\sum_{i=1}^{m}a_ib_i+z=\sum_{j=1}^{n}a_j'b_j'+z}\left(\sup_{b+\sum_{k=1}^{p}c_kd_k+z'=\sum_{l=1}^{q}c_l'd_l'+z'}\left(\begin{array}{c}\bigwedge_{i=1}^{m}\left(\lambda(a_i)\wedge\mu(b_i)\right)\wedge\\ \bigwedge_{j=1}^{n}\left(\lambda(a_j')\wedge\mu(b_j')\right)\wedge\\ \bigwedge_{k=1}^{p}\left(\lambda(c_k)\wedge\mu(d_k)\right)\wedge\\ \bigwedge_{l=1}^{q}\left(\lambda(c_l')\wedge\mu(d_l')\right)\end{array}\right)\right)$$
$$\leq\sup_{x+\sum_{s=1}^{u}g_sh_s+z=\sum_{t=1}^{w}g_t'h_t'+z}\left(\bigwedge_{s=1}^{u}\left(\lambda(g_s)\wedge\mu(h_s)\right)\wedge\bigwedge_{t=1}^{w}\left(\lambda(g_t')\wedge\mu(h_t')\right)\right)$$
$$=(\lambda\odot_h\mu)(x).$$

Thus $(\lambda\odot_h\mu)(a)\wedge(\lambda\odot_h\mu)(b)\leq(\lambda\odot_h\mu)(x)$. This completes the proof that $(\lambda\odot_h\mu)$ is a fuzzy h-ideal of R.

By simple calculations we can prove that $\lambda\odot_h\mu\leq\lambda\wedge\mu$. ■

3.2 Idempotent h-ideals

The concept of h-hemiregular of a hemiring was introduced in [49] as a generalization of the concept of regular semiring. From results proved in [49] (see Theorem 44) it follows that in an h-hemiregular hemiring every h-ideal A is *h-idempotent*, that is $\overline{AA}=A$. On the other hand, Theorem 45 implies that in such hemirings we have $\lambda\odot_h\lambda=\lambda$ for all fuzzy h-ideals λ. Fuzzy h-ideals with this property is called *h-idempotent*.

Proposition 119 *The following statements are equivalent:*

1. Each h-ideal of R is h-idempotent.
2. $A \cap B = \overline{AB}$ for each pair A, B of h-ideals of R.
3. $x \in \overline{RxRxR}$ for every $x \in R$.
4. $A \subseteq \overline{RARAR}$ for every non-empty $A \subseteq R$.
5. $A = \overline{RARAR}$ for every h-ideal A of R.

Proof. Indeed, by Lemma 9, $\overline{AB} \subseteq A \cap B$ for all h-ideals A, B of R. Since $A \cap B$ is an h-ideal of R, (1) implies $A \cap B = \overline{(A \cap B)(A \cap B)} \subseteq \overline{AB}$. Thus $A \cap B = \overline{AB}$. So, (1) implies (2). The converse implication is obvious.

It is clear that the smallest h-ideal of R containing $x \in R$ has the form

$$\langle x \rangle = \overline{\langle x \rangle} = \overline{Rx + xR + RxR + Sx},$$

where Sx is a finite sum of x's. If (1) holds, then $\overline{\langle x \rangle} = \overline{\overline{\langle x \rangle}\,\overline{\langle x \rangle}} = \overline{\langle x \rangle \langle x \rangle}$. Consequently,

$$\begin{aligned} x = 0 + x &\in \overline{Rx + xR + RxR + Sx} \\ &= \overline{(Rx + xR + RxR + Sx)(Rx + xR + RxR + Sx)} \subseteq \overline{RxRRxR} \subseteq \overline{RxRxR} \end{aligned}$$

for every $x \in R$. So, (1) implies (3). Clearly (3) implies (4). If (4) holds, then for every h-ideal A of R we have $\overline{A} = A \subseteq \overline{RARAR} \subseteq \overline{AA} \subseteq \overline{A} = A$, which proves (5). The implication (5) $\Rightarrow$ (1) is obvious. ■

As a consequence of the above result and Theorem 44 we obtain the following characterization of h-hemiregularity of commutative hemirings.

Corollary 120 *A commutative hemiring is h-hemiregular if and only if all its h-ideals are h-idempotent.*

Proposition 121 *The following statements are equivalent:*

1. Each fuzzy h-ideal of R is idempotent.
2. $\lambda \odot_h \mu = \lambda \wedge \mu$ for all fuzzy h-ideals λ, μ of R.

Proof. Let λ and μ be fuzzy h-ideals of R. Since $\lambda \wedge \mu$ is a fuzzy h-ideal of R such that $\lambda \wedge \mu \leq \lambda$ and $\lambda \wedge \mu \leq \mu$, Proposition 31 implies $(\lambda \wedge \mu) \odot_h (\lambda \wedge \mu) \leq \lambda \odot_h \mu$. So, if $\lambda \wedge \mu$ is an idempotent fuzzy h-ideal, then $\lambda \wedge \mu \leq \lambda \odot_h \mu$, which together with Theorem 118 gives $\lambda \odot_h \mu = \lambda \wedge \mu$. This means that (1) implies (2). The converse implication is obvious. ■

Comparing this Proposition with Theorem 45 we obtain

Corollary 122 *A commutative hemiring is h-hemiregular if and only if all its fuzzy h-ideals are idempotent, or equivalently, if and only if $\lambda \odot_h \mu = \lambda \wedge \mu$ holds for all its fuzzy h-ideals λ, μ.*

Theorem 123 *For hemirings with identity the following statements are equivalent:*

1. Each h-ideal of R is h-idempotent.
2. $A \cap B = \overline{AB}$ for each pair A, B of h-ideals of R.
3. Each fuzzy h-ideal of R is idempotent.
4. $\lambda \odot_h \mu = \lambda \wedge \mu$ for all fuzzy h-ideals λ, μ of R.

Proof. (1) $\Leftrightarrow$ (2) by Proposition 119, (3) $\Leftrightarrow$ (4) by Proposition 121. To prove that (1) and (3) are equivalent observe that the smallest h-ideal containing $x \in R$ has the form RxR. Its h-closure $\overline{RxR}$ is an h-ideal. Since, by (1), all h-ideals of R are h-idempotent, we have $\overline{RxR} = \overline{(\overline{RxR})(\overline{RxR})} = \overline{RxRRxR}$ (Lemma 8). Thus $x \in \overline{RxR} = \overline{RxRRxR}$ implies $x + \sum_{i=1}^{m} r_i x s_i u_i x t_i + z = \sum_{j=1}^{n} r_j' x s_j' u_j' t_j' + z$, .for some $r_i, s_i, u_i, t_i, r_j', s_j', u_j', t_j', z \in R$. But, by Theorem 118, for every fuzzy h-ideal of R we have $\lambda \odot_h \lambda \leq \lambda$. Now

$$\lambda(x) = \lambda(x) \wedge \lambda(x) \leq \bigwedge_{i=1}^{m} \Big(\lambda(r_i x s_i) \wedge \lambda(u_i x t_i)\Big).$$

Also

$$\lambda(x) = \lambda(x) \wedge \lambda(x) \leq \bigwedge_{j=1}^{n} \Big(\lambda(r_j' x s_j') \wedge \lambda(u_j' x t_j')\Big).$$

Therefore

$$\begin{aligned}
\lambda(x) &\leq \left[\bigwedge_{i=1}^{m} (\lambda(r_i x s_i) \wedge \lambda(u_i x t_i))\right] \wedge \left[\bigwedge_{j=1}^{n} \Big(\lambda(r_j' x s_j') \wedge \lambda(u_j' x t_j')\Big)\right] \\
&\leq \sup_{x+\sum_{i=1}^{m} r_i x s_i u_i x t_i + z = \sum_{j=1}^{n} r_j' x s_j' u_j' t_j' + z} \left[\begin{array}{c} \left[\bigwedge_{i=1}^{m} [\lambda(r_i x s_i) \wedge \lambda(u_i x t_i)]\right] \\ \wedge \left[\bigwedge_{j=1}^{n} \Big(\lambda(r_j' x s_j') \wedge \lambda(u_j' x t_j')\Big)\right] \end{array}\right] \\
&= (\lambda \odot_h \lambda)(x).
\end{aligned}$$

Hence $\lambda \leq \lambda \odot_h \lambda$, which proves $\lambda \odot_h \lambda = \lambda$. So, (1) implies (3).

Conversely, according to Proposition 26, the characteristic function χ_A of any h-ideal A of R is a fuzzy h-ideal of R. If it is idempotent, then $\chi_A = \chi_A \odot_h \chi_A = \chi_{\overline{AA}}$ (Proposition 31). Thus $A = \overline{AA}$ and so (3) implies (1). ∎

Now we define the h-sum of fuzzy subsets of a hemiring R.

Definition 124 *The h-sum $\lambda +_h \mu$ of fuzzy subsets λ and μ of R is defined by*

$$(\lambda +_h \mu)(x) = \sup_{x+(a_1+b_1)+z=(a_2+b_2)+z} \Big(\lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2)\Big),$$

where $x, a_1, b_1, a_2, b_2, z \in R$.

Theorem 125 *The h-sum of fuzzy h-ideals of R is a fuzzy h-ideal of R.*

Proof. Let λ, μ be fuzzy h-ideals of R. Then for $x, y \in R$ we have

$$\begin{aligned}
(\lambda +_h \mu)(x) \wedge (\lambda +_h \mu)(y) &= \sup_{x+(a_1+b_1)+z=(a_2+b_2)+z} \Big(\lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2)\Big) \\
&\wedge \sup_{y+(a_1'+b_1')+z'=(a_2'+b_2')+z'} \Big(\lambda(a_1') \wedge \lambda(a_2') \wedge \mu(b_1') \wedge \mu(b_2')\Big) \\
&= \sup_{\substack{x+(a_1+b_1)+z=(a_2+b_2)+z \\ y+(a_1'+b_1')+z'=(a_2'+b_2')+z'}} \begin{pmatrix} \lambda(a_1) \wedge \lambda(a_2) \\ \wedge\mu(b_1) \wedge \mu(b_2) \\ \wedge\lambda(a_1') \wedge \lambda(a_2') \\ \wedge\mu(b_1') \wedge \mu(b_2') \end{pmatrix} \\
&\leq \sup_{\substack{x+(a_1+b_1)+z=(a_2+b_2)+z \\ y+(a_1'+b_1')+z'=(a_2'+b_2')+z'}} \begin{pmatrix} \lambda(a_1+a_1') \wedge \lambda(a_2+a_2')\wedge \\ \mu(b_1+b_1') \wedge \mu(b_2+b_2') \end{pmatrix} \\
&\leq \sup_{(x+y)+(c_1+d_1)+z''=(c_2+d_2)+z''} [\lambda(c_1) \wedge \lambda(c_2) \wedge \mu(d_1) \wedge \mu(d_2)] \\
&= (\lambda +_h \mu)(x+y).
\end{aligned}$$

Similarly,

$$\begin{aligned}
(\lambda +_h \mu)(x) &= \sup_{x+(a_1+b_1)+z=(a_2+b_2)+z} \Big(\lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2)\Big) \\
&\leq \sup_{x+(a_1+b_1)+z=(a_2+b_2)+z} \Big(\lambda(ra_1) \wedge \lambda(ra_2) \wedge \mu(rb_1) \wedge \mu(rb_2)\Big) \\
&\leq \sup_{rx+(a_1''+b_1'')+z''=(a_2''+b_2'')+z''} \Big(\lambda(a_1'') \wedge \lambda(a_2'') \wedge \mu(b_1'') \wedge \mu(b_2'')\Big) \\
&= (\lambda +_h \mu)(rx).
\end{aligned}$$

Analogously $(\lambda +_h \mu)(x) \leq (\lambda +_h \mu)(xr)$. This proves that $(\lambda +_h \mu)$ is a fuzzy ideal of R.

Now we show that $x+a+z = b+z$ implies $(\lambda +_h \mu)(x) \geq (\lambda +_h \mu)(a) \wedge (\lambda +_h \mu)(b)$. For this let $a + (a_1 + b_1) + z_1 = (a_2 + b_2) + z_1$ and $b + (c_1 + d_1) + z_2 = (c_2 + d_2) + z_2$. Then,

$$a + (c_2 + d_2 + z_2) + (a_1 + b_1 + z_1) = (a_2 + b_2 + z_1) + (b + c_1 + d_1 + z_2),$$

whence

$$a + (a_1 + c_2) + (b_1 + d_2) + (z_1 + z_2) = b + (a_2 + c_1) + (b_2 + d_1) + (z_1 + z_2).$$

Consequently

$$a+(a_1+c_2)+(b_1+d_2)+(z_1+z_2+z)=b+z+(a_2+c_1)+(b_2+d_1)+(z_1+z_2)$$

and

$$a+(a_1+c_2)+(b_1+d_2)+(z_1+z_2+z)=x+a+z+(a_2+c_1)+(b_2+d_1)+(z_1+z_2).$$

Thus

$$x+(a_2+c_1)+(b_2+d_1)+(z_1+z_2+z+a)=(a_1+c_2)+(b_1+d_2)+(z_1+z_2+z+a),$$

i.e., $x+(a'+b')+z'=(a''+b'')+z'$ for some $a',b',a'',b''\in R$. Terefore

$$\begin{aligned}
(\lambda+_h\mu)(a)\wedge(\lambda+_h\mu)(b) &= \left(\sup_{a+(a_1+b_1)+z_1=(a_2+b_2)+z_1}\Big(\lambda(a_1)\wedge\lambda(a_2)\wedge\mu(b_1)\wedge\mu(b_2)\Big)\right)\\
&\wedge\left(\sup_{b+(c_1+d_1)+z_2=(c_2+d_2)+z_2}\Big(\lambda(c_1)\wedge\lambda(c_2)\wedge\mu(d_1)\wedge\mu(d_2)\Big)\right)\\
&= \sup_{\substack{a+(a_1+b_1)+z_1=(a_2+b_2)+z_1\\ b+(c_1+d_1)+z_2=(c_2+d_2)+z_2}}\left(\begin{array}{c}\lambda(a_1)\wedge\lambda(a_2)\wedge\mu(b_1)\wedge\mu(b_2)\wedge\\ \lambda(c_1)\wedge\lambda(c_2)\wedge\mu(d_1)\wedge\mu(d_2)\end{array}\right)\\
&\le \sup_{\substack{a+(a_1+b_1)+z_1=(a_2+b_2)+z_1\\ b+(c_1+d_1)+z_2=(c_2+d_2)+z_2}}\left(\begin{array}{c}\lambda(a_1+c_2)\wedge\lambda(a_2+c_1)\wedge\\ \mu(b_1+d_2)\wedge\mu(b_2+d_1)\end{array}\right)\\
&\le \sup_{x+(a'+b')+z'=(a''+b'')+z'}[\lambda(a')\wedge\lambda(a'')\wedge\mu(b')\wedge\mu(b'')]\\
&= (\lambda+_h\mu)(x).
\end{aligned}$$

Thus $\lambda+_h\mu$ is a fuzzy h-ideal of R. ■

Theorem 126 *If all h-ideals of R are h-idempotent, then the collection of all h-ideals of R forms a complete Brouwerian lattice.*

Proof. The collection $\mathcal{L}_R$ of all h-ideals of R is a poset under the inclusion of sets. It is not difficult to see that $\mathcal{L}_R$ is a complete lattice under operations $\sqcup$, $\sqcap$ defined as $A\sqcup B=\overline{A+B}$ and $A\sqcap B=A\cap B$.

We show that $\mathcal{L}_R$ is a Brouwerian lattice, that is, for any $A,B\in\mathcal{L}_R$, the set $\mathcal{L}_R(A,B)=\{I\in\mathcal{L}_R\,|\,A\cap I\subseteq B\}$ contains a greatest element.

By Zorn's Lemma the set $\mathcal{L}_R(A,B)$ contains a maximal element M. Since each h-ideal of R is h-idempotent, $\overline{AI}=A\cap I\subseteq B$ and $\overline{AM}=A\cap M\subseteq B$ (Proposition 119). Thus $\overline{AI}+\overline{AM}\subseteq B$. Consequently, $\overline{\overline{AI}+\overline{AM}}\subseteq\overline{B}=B$.

Since $\overline{I+M}=I\sqcup M\in\mathcal{L}_R$, for every $x\in\overline{I+M}$ there exist $i_1,i_2\in I$, $m_1,m_2\in M$ and $z\in R$ such that $x+i_1+m_1+z=i_2+m_2+z$. Thus

$$dx+di_1+dm_1+dz=di_2+dm_2+dz$$

for any $d \in D \in \mathcal{L}_R$. As $di_1, di_2 \in DI$, $dm_1, dm_2 \in DM$, $dz \in R$, we have $dx \in \overline{DI + DM}$, which implies $D\left(\overline{I+M}\right) \subseteq \overline{DI+DM} \subseteq \overline{\overline{DI} + \overline{DM}} \subseteq B$. Hence $\overline{D\left(\overline{I+M}\right)} \subseteq B$. This means that $D \cap \left(\overline{I+M}\right) = \overline{D\left(\overline{I+M}\right)} \subseteq B$, i.e., $\overline{I+M} \in \mathcal{L}_R(A,B)$, whence $\overline{I+M} = M$ because M is maximal in $\mathcal{L}_R(A,B)$. Therefore $I \subseteq \overline{I} \subseteq \overline{I+M} = M$ for every $I \in \mathcal{L}_R(A,B)$. ∎

Corollary 127 *If all h-ideals of R are idempotent, then the lattice $\mathcal{L}_R$ is distributive.*

Proof. Each complete Brouwerian lattice is distributive (cf. [11], 11.11). ∎

Theorem 128 *Each fuzzy h-ideal of R is h-idempotent if and only if the set of all fuzzy h-ideals of R (ordered by $\leq$) forms a distributive lattice under the h-sum and h-intrinsic product of fuzzy h-ideals with $\lambda \odot_h \mu = \lambda \wedge \mu$.*

Proof. Assume that all fuzzy h-ideals of R are idempotent. Then $\lambda \odot_h \mu = \lambda \wedge \mu$ (Proposition 121) and, as it is not difficult to see, the set $\mathcal{FL}_R$ of all fuzzy h-ideals of R (ordered by $\leq$) is a lattice under the h-sum and h-intrinsic product of fuzzy h-ideals.

We show that $(\lambda \odot_h \delta) +_h \mu = (\lambda +_h \mu) \odot_h (\delta +_h \mu)$ for all $\lambda, \mu, \delta \in \mathcal{FL}_R$. Indeed, for any $x \in R$ we have

$$
\begin{aligned}
&((\lambda \odot_h \delta) +_h \mu)(x) = ((\lambda \wedge \delta) +_h \mu)(x) \\
&= \sup_{x+(a_1+b_1)+z=(a_2+b_2)+z} \begin{bmatrix} (\lambda \wedge \delta)(a_1) \wedge (\lambda \wedge \delta)(a_2) \\ \wedge \mu(b_1) \wedge \mu(b_2) \end{bmatrix} \\
&= \sup_{x+(a_1+b_1)+z=(a_2+b_2)+z} \begin{bmatrix} \lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \\ \mu(b_2) \wedge \delta(a_1) \wedge \delta(a_2) \end{bmatrix} \\
&= \sup_{x+(a_1+b_1)+z=(a_2+b_2)+z} [\lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \\
&\wedge \sup_{x+(a_1+b_1)+z=(a_2+b_2)+z} [\delta(a_1) \wedge \delta(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \\
&= (\lambda +_h \mu)(x) \wedge (\delta +_h \mu)(x) \\
&= [(\lambda +_h \mu) \wedge (\delta +_h \mu)](x) \\
&= ((\lambda +_h \mu) \odot_h (\delta +_h \mu))(x).
\end{aligned}
$$

So, $\mathcal{FL}_R$ is a distributive lattice.

The converse statement is a consequence of Proposition 121. ∎

3.3 Prime h-ideals

An h-ideal P of R is called *prime* if $P \neq R$ and for any h-ideals A, B of R from $AB \subseteq P$ it follows $A \subseteq P$ or $B \subseteq P$, and *irreducible* if $P \neq R$ and $A \cap B = P$ implies $A = P$ or $B = P$. By analogy a non-constant fuzzy h-ideal δ of R is called *prime* (in the first sense) if for any fuzzy h-ideals λ, μ of R from $\lambda \odot_h \mu \leq \delta$ it follows $\lambda \leq \delta$ or $\mu \leq \delta$, and *irreducible* if $\lambda \wedge \mu = \delta$ implies $\lambda = \delta$ or $\mu = \delta$.

Theorem 129 *A left (right) h-ideal P of R is prime if and only if for all $a, b \in R$ from $aRb \subseteq P$ it follows $a \in P$ or $b \in P$.*

Proof. Assume that P is a prime left h-ideal of R and $aRb \subseteq P$ for some $a, b \in R$. Obviously, $A = \overline{Ra}$ and $B = \overline{Rb}$ are left h-ideals of R. So, $AB \subseteq \overline{AB} = \overline{\overline{Ra}\,\overline{Rb}} = \overline{RaRb} \subseteq \overline{RP} \subseteq P$, and consequently $A \subseteq P$ or $B \subseteq P$. Let $\langle x \rangle$ be a left h-ideal generated by $x \in R$. If $A \subseteq P$, then $\langle a \rangle \subseteq \overline{Ra} = A \subseteq P$, whence $a \in P$. If $B \subseteq P$, then $\langle b \rangle \subseteq \overline{Rb} = B \subseteq P$, whence $b \in P$.

The converse is obvious. ■

Corollary 130 *An h-ideal P of R is prime if and only if for all $a, b \in R$ from $aRb \subseteq P$ it follows $a \in P$ or $b \in P$.*

Corollary 131 *An h-ideal P of a commutative hemiring R with identity is prime if and only if for all $a, b \in R$ from $ab \in P$ it follows $a \in P$ or $b \in P$.*

The result expressed by Corollary 130 suggest the following definition of prime fuzzy h-ideals.

Definition 132 *A non-constant fuzzy h-ideal δ of R is called prime (in the second sense) if for all $t \in [0,1]$ and $a, b \in R$ the following condition is satisfied:*
if $\delta(axb) \geq t$ for every $x \in R$ then $\delta(a) \geq t$ or $\delta(b) \geq t$.

In other words, a non-constant fuzzy h-ideal δ is prime if from the fact that $axb \in U(\delta;t)$ for every $x \in R$ it follows $a \in U(\delta;t)$ or $b \in U(\delta;t)$. It is clear that any fuzzy h-ideal is prime in the first sense is prime in the second sense. The converse is not true.

Example 133 *In an ordinary hemiring of natural numbers the set of even numbers forms an h-ideal. A fuzzy set*

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0.8 & \text{if } n = 2k \neq 0, \\ 0.4 & \text{if } n = 2k+1 \end{cases}$$

is a fuzzy h-ideal of this hemiring. It is prime in the second sense but it is not prime in the first sense.

Theorem 134 *A non-constant fuzzy h-ideal δ of R is prime in the second sense if and only if each its proper level set $U(\delta;t)$ is a prime h-ideal of R.*

Proof. Let a fuzzy h-ideal δ of R be prime in the second sense and let $U(\delta;t)$ be its arbitrary proper level set, i.e., $\emptyset \neq U(\delta;t) \neq R$. If $aRb \subseteq U(\delta;t)$, then $\delta(axb) \geq t$ for every $x \in R$. Hence $\delta(a) \geq t$ or $\delta(b) \geq t$, i.e., $a \in U(\delta;t)$ or $b \in U(\delta;t)$, which, by Corollary 130, means that $U(\delta;t)$ is a prime h-ideal of R.

To prove the converse consider a non-constant fuzzy h-ideal δ of R. If it is not prime then there exists $a, b \in R$ such that $\delta(axb) \geq t$ for all $x \in R$, but $\delta(a) < t$ and $\delta(b) < t$. Thus, $aRb \subseteq U(\delta;t)$, but $a \notin U(\delta;t)$ and $b \notin U(\delta;t)$. Therefore $U(\delta;t)$ is not prime which is a contradiction. Hence δ is prime. ∎

Corollary 135 *A fuzzy set λ_A defined in Proposition 26 is a prime fuzzy h-ideal of R if and only if A is a prime h-ideal of R.*

In view of the Transfer Principle (Lemma 25) the second definition of prime fuzzy h-ideals is better. Therefore fuzzy h-ideals which are prime in the first sense will be called *h-prime*.

Proposition 136 *A non-constant fuzzy h-ideal δ of a commutative hemiring R with identity is prime if and only if $\delta(ab) = \delta(a) \vee \delta(b)$ for all $a, b \in R$.*

Proof. Let δ be a non-constant fuzzy h-ideal of a commutative hemiring R with identity. If $\delta(ab) = t$, then, for every $x \in R$, we have $\delta(axb) = \delta(xab) \geq \delta(x) \vee \delta(ab) \geq t$. Thus $\delta(axb) \geq t$ for every $x \in R$, which implies $\delta(a) \geq t$ or $\delta(b) \geq t$. If $\delta(a) \geq t$, then $t = \delta(ab) \geq \delta(a) \geq t$, whence $\delta(ab) = \delta(a)$. If $\delta(b) \geq t$, then, as in the previous case, $\delta(ab) = \delta(b)$. So, $\delta(ab) = \delta(a) \vee \delta(b)$.

Conversely, assume that $\delta(ab) = \delta(a) \vee \delta(b)$ for all $a, b \in R$. If $\delta(axb) \geq t$ for every $x \in R$, then, replacing in this inequality x by the identity of R, we obtain $\delta(ab) \geq t$. Thus $\delta(a) \vee \delta(b) \geq t$, i.e., $\delta(a) \geq t$ or $\delta(b) \geq t$, which means that fuzzy h-ideal δ is prime. ∎

Theorem 137 *Every proper h-ideal is contained in some proper irreducible h-ideal.*

Proof. Let P be a proper h-ideal of R and let $\{P_\alpha \mid \alpha \in \Lambda\}$ be a family of all proper h-ideals of R containing P. By Zorn's Lemma, for any fixed $a \notin P$, the family of h-ideals P_α such that $P \subseteq P_\alpha$ and $a \notin P_\alpha$ contains a maximal element M. This maximal element is an irreducible h-ideal. Indeed, let $M = P_\beta \cap P_\delta$ for some h-ideals of R. If M is a proper subset of P_β and P_δ, then, according to the maximality of M, we have $a \in P_\beta$ and $a \in P_\delta$. Hence $a \in P_\beta \cap P_\delta = M$, which is impossible. Thus, either $M = P_\beta$ or $M = P_\delta$. ∎

Theorem 138 *If all h-ideals of R are h-idempotent, then an h-ideal P of R is irreducible if and only if it is h-prime.*

Proof. Assume that all h-ideals of R are h-idempotent. Let P be a fixed irreducible h-ideal. If $AB \subseteq P$ for some h-ideals A, B of R, then $A \cap B = \overline{AB} \subseteq \overline{P} = P$, by Proposition 119. Thus $\overline{(A \cap B) + P} = P$. Since $\mathcal{L}_R$ is a distributive lattice, $P = \overline{(A \cap B) + P} = \overline{(A + P)} \cap \overline{(B + P)}$. So either $\overline{A + P} = P$ or $\overline{B + P} = P$, that is, either $A \subseteq P$ or $B \subseteq P$.

Conversely, if an h-ideal P is prime and $A \cap B = P$ for some $A, B \in \mathcal{L}_R$, then $AB \subseteq \overline{AB} = A \cap B = P$. Thus $A \subseteq P$ or $B \subseteq P$. But $P \subseteq A$ and $P \subseteq B$. Hence $A = P$ or $B = P$. ∎

Corollary 139 *In hemirings in which all h-ideals are h-idempotent each proper h-ideal is contained in some proper prime h-ideal.*

Theorem 140 *In hemirings in which all fuzzy h-ideals are idempotent a fuzzy h-ideal is irreducible if and only if it is h-prime.*

Proof. Suppose all fuzzy h-ideals of R are idempotent and let δ be an arbitrary irreducible fuzzy h-ideal of R. We prove that it is prime. If $\lambda \odot_h \mu \leq \delta$ for some fuzzy h-ideals λ, μ of R, then also $\lambda \wedge \mu \leq \delta$. Since the set $\mathcal{FL}_R$ of all fuzzy h-ideals of R is a distributive lattice (Theorem 128) we have $\delta = (\lambda \wedge \mu) +_h \delta = (\lambda +_h \delta) \wedge (\mu +_h \delta)$. Thus $\lambda +_h \delta = \delta$ or $\mu +_h \delta = \delta$. But $\leq$ is a lattice order, so $\lambda \leq \delta$ or $\mu \leq \delta$. This proves that a fuzzy h-ideal δ is h-prime.

Conversely, if δ is an h-prime fuzzy h-ideal of R and $\lambda \wedge \mu = \delta$ for some $\lambda, \mu \in \mathcal{FL}_R$, then $\lambda \odot_h \mu = \delta$, which implies $\lambda \leq \delta$ or $\mu \leq \delta$. Since $\leq$ is a lattice order and $\delta = \lambda \wedge \mu$ we have $\delta \leq \lambda$ and $\delta \leq \mu$. Thus $\lambda = \delta$ or $\mu = \delta$. So, δ is irreducible. ∎

Theorem 141 *The following assertions for a hemiring R are equivalent:*

(1) Each h-ideal of R is h-idempotent.

(2) Each proper h-ideal P of R is the intersection of all h-prime h-ideals of R which contain P.

Proof. Let P be a proper h-ideal of R and let $\{P_\alpha \mid \alpha \in \Lambda\}$ be the family of all h-prime h-ideals of R containing P. Clearly $P \subseteq \cap_{\alpha\in\Lambda} P_\alpha$. By Zorn's Lemma, for any fixed $a \notin P$, the family of h-ideals P_α such that $P \subseteq P_\alpha$ and $a \notin P_\alpha$ contains a maximal element M_a. We will show that this maximal element is an irreducible h-ideal. Let $M_a = K \cap L$. If M_a is a proper subset of K and L, then, according to the maximality of M_a, we have $a \in K$ and $a \in L$. Hence $a \in K \cap L = M_a$, which is impossible. Thus, either $M_a = K$ or $M_a = L$. By Theorem 138, M_a is a prime h-ideal. So there exists an h-prime h-ideal M_a such that $a \notin M_a$ and $P \subseteq M_a$. Hence $\cap P_\alpha \subseteq P$. Thus $P = \cap P_\alpha$.

Assume that each h-ideal of R is the intersection of all h-prime h-ideals of R which contain it. Let A be an h-ideal of R. If $\overline{A^2} = R$, then, by Lemma 9, we have $A = R$, which means such h-ideal is h-idempotent. If $\overline{A^2} \neq R$, then $\overline{A^2}$ is a proper h-ideal of R and so it is the intersection of all h-prime h-ideals of R containing A. Let $\overline{A^2} = \cap P_\alpha$. Then $A^2 \subseteq P_\alpha$ for each α. Since P_α is prime, we have $A \subseteq P_\alpha$. Thus $A \subseteq \cap P_\alpha = \overline{A^2}$. But $\overline{A^2} \subseteq A$ for every h-ideal. Hence $A = \overline{A^2}$. ∎

Lemma 142 *Let R be a hemiring in which each fuzzy h-ideal is idempotent. If λ is a fuzzy h-ideal of R with $\lambda(a) = \alpha$, where a is any element of R and $\alpha \in [0,1]$, then there exists an irreducible h-prime fuzzy h-ideal δ of R such that $\lambda \leq \delta$ and $\delta(a) = \alpha$.*

Proof. Let λ be an arbitrary fuzzy h-ideal of R and let $a \in R$ be fixed. Consider the following collection of fuzzy h-ideals of R

$$\mathcal{B} = \{\mu \mid \mu(a) = \lambda(a),\ \lambda \leq \mu\}.$$

$\mathcal{B}$ is non-empty since $\lambda \in \mathcal{B}$. Let $\mathcal{F}$ be a totally ordered subset of $\mathcal{B}$ containing λ, say $\mathcal{F} = \{\lambda_i \mid i \in I\}$. Obviously $\lambda_i \vee \lambda_j \in \mathcal{F}$ for any $\lambda_i, \lambda_j \in \mathcal{F}$. So, for example, $(\lambda_i(x) \vee \lambda_j(x)) \wedge (\lambda_i(y) \vee \lambda_j(y)) \leq \lambda_i(x+y) \vee \lambda_j(x+y)$ for any $\lambda_i, \lambda_j \in \mathcal{F}$ and $x, y \in R$.

We claim that $\bigvee_{i\in I} \lambda_i$ is a fuzzy h-ideal of R.

For any $x, y \in R$, we have

$$\begin{aligned}
(\bigvee_{i\in I} \lambda_i)(x) \wedge (\bigvee_{i\in I} \lambda_i)(y) &= (\bigvee_{i\in I} \lambda_i(x)) \wedge (\bigvee_{j\in I} \lambda_j(y)) \\
&= \bigvee_{i,j\in I} (\lambda_i(x) \wedge \lambda_j(y)) \\
&\leq \bigvee_{i,j\in I} \Big((\lambda_i(x) \vee \lambda_j(x)) \wedge (\lambda_i(y) \vee \lambda_j(y))\Big) \\
&\leq \bigvee_{i,j\in I} (\lambda_i(x+y) \vee \lambda_j(x+y)) \\
&\leq \bigvee_{i\in I} \lambda_i(x+y) = (\bigvee_{i\in I} \lambda_i)(x+y).
\end{aligned}$$

Similarly

$$(\bigvee_{i\in I} \lambda_i)(x) = \bigvee_{i\in I} \lambda_i(x) \leq \bigvee_{i\in I} \lambda_i(xr) = (\bigvee_{i\in I} \lambda_i)(xr)$$

and

$$(\bigvee_{i\in I} \lambda_i)(x) \leq (\bigvee_{i\in I} \lambda_i)(rx)$$

for all $x, r \in R$. Thus $\bigvee_{i\in I} \lambda_i$ is a fuzzy ideal.

Now, let $x + a + z = b + z$, where $a, b, z \in R$. Then

$$\begin{aligned}
(\bigvee_{i\in I} \lambda_i)(a) \wedge (\bigvee_{i\in I} \lambda_i)(b) &= (\bigvee_{i\in I} \lambda_i(a)) \wedge (\bigvee_{j\in I} (\lambda_j(b)) \\
&= \bigvee_{i,j\in I} (\lambda_i(a) \wedge \lambda_j(b)) \\
&\leq \bigvee_{i,j\in I} \Big((\lambda_i(a) \vee \lambda_j(a)) \wedge (\lambda_i(b) \vee \lambda_j(b))\Big)
\end{aligned}$$

$$\leq \bigvee_{i,j} \left(\lambda_i(x) \vee \lambda_j(x)\right) \leq \bigvee_{i\in i} \lambda_i(x) = \big(\bigvee_{i\in I} \lambda_i\big)(x).$$

This means that $\bigvee_{i\in I} \lambda_i$ is a fuzzy h-ideal of R. Clearly $\lambda \leq \bigvee_{i\in I} \lambda_i$ and $(\bigvee_{i\in I} \lambda_i)(a) = \lambda(a) = \alpha$. Thus $\bigvee_{i\in I} \lambda_i$ is the least upper bound of $\mathcal{F}$. Hence by Zorn's lemma there exists a fuzzy h-ideal δ of R which is maximal with respect to the property that $\lambda \leq \delta$ and $\delta(a) = \alpha$.

We will show that δ is an irreducible fuzzy h-ideal of R. Let $\delta = \delta_1 \wedge \delta_2$, where δ_1, δ_2 are fuzzy h-ideals of R. Then $\delta \leq \delta_1$ and $\delta \leq \delta_2$ since $\mathcal{FL}_R$ is a lattice. We claim that either $\delta = \delta_1$ or $\delta = \delta_2$. Suppose $\delta \neq \delta_1$ and $\delta \neq \delta_2$. Since δ is maximal with respect to the property that $\delta(a) = \alpha$ and since $\delta \lneqq \delta_1$ and $\delta \lneqq \delta_2$, so $\delta_1(a) \neq \alpha$ and $\delta_2(a) \neq \alpha$. Hence $\alpha = \delta(a) = (\delta_1 \wedge \delta_2)(a) = \delta_1(a) \wedge \delta_2(a) \neq \alpha$, which is impossible. Hence $\delta = \delta_1$ or $\delta = \delta_2$. Thus δ is an irreducible fuzzy h-ideal of R. By Theorem 140, it is also h-prime. ■

Theorem 143 *Each fuzzy h-ideal of R is idempotent if and only if each fuzzy h-ideal of R is the intersection of those h-prime fuzzy h-ideals of R which contain it.*

Proof. Suppose each fuzzy h-ideal of R is idempotent. Let λ be a fuzzy h-ideal of R and let $\{\lambda_\alpha \mid \alpha \in \Lambda\}$ be the family of all h-prime fuzzy h-ideals of R which contain λ. Obviously $\lambda \leq \bigwedge_{\alpha\in\Lambda} \lambda_\alpha$. We now show that $\bigwedge_{\alpha\in\Lambda} \lambda_\alpha \leq \lambda$. Let a be an arbitrary element of R. Then, according to Lemma 142, there exists an irreducible and h-prime fuzzy h-ideal δ such that $\lambda \leq \delta$ and $\lambda(a) = \delta(a)$. Hence $\delta \in \{\lambda_\alpha \mid \alpha \in \Lambda\}$ and $\bigwedge_{\alpha\in\Lambda} \lambda_\alpha \leq \delta$. So, $\bigwedge_{\alpha\in\Lambda} \lambda_\alpha(a) \leq \delta(a) = \lambda(a)$. Thus $\bigwedge_{\alpha\in\Lambda} \lambda_\alpha \leq \lambda$. Therefore $\bigwedge_{\alpha\in\Lambda} \lambda_\alpha = \lambda$.

Conversely, assume that each fuzzy h-ideal of R is the intersection of those h-prime fuzzy h-ideals of R which contain it. Let λ be a fuzzy h-ideal of R then $\lambda \odot_h \lambda$ is also fuzzy h-ideal of R, so $\lambda \odot_h \lambda = \bigwedge_{\alpha\in\Lambda} \lambda_\alpha$ where λ_α are h-prime fuzzy h-ideals of R. Thus each λ_α contains $\lambda \odot_h \lambda$, and hence λ. So $\lambda \leq \bigwedge_{\alpha\in\Lambda} \lambda_\alpha = \lambda \odot_h \lambda$, but $\lambda \odot_h \lambda \leq \lambda$ always. Hence $\lambda = \lambda \odot_h \lambda$. ■

3.4 Semiprime h-ideals

Definition 144 *An h-ideal A of R is called semiprime if $A \neq R$ and for any h-ideal B of R, $B^2 \subseteq A$ implies $B \subseteq A$. Similarly, a non-constant fuzzy h-ideal λ of R is called semiprime if for any fuzzy h-ideal δ of R, $\delta \odot_h \delta \leq \lambda$ implies $\delta \leq \lambda$.*

Obviously, each h-prime h-ideal is semiprime. Each h-prime fuzzy h-ideal is semiprime. The converse is not true (see Example 150).

Using the same method as in the proof of Theorem 129 we can prove

Theorem 145 *A (left, right) h-ideal P of R is semiprime if and only if for every $a \in R$ from $aRa \subseteq P$ it follows $a \in P$.*

Corollary 146 *An h-ideal P of a commutative hemiring R with identity is semiprime if and only if for all $a \in R$ from $a^2 \in P$ it follows $a \in P$.*

Theorem 147 *The following assertions for a hemiring R are equivalent:*

(1) Each h-ideal of R is h-idempotent.

(2) Each h-ideal of R is semiprime.

Proof. Suppose that each h-ideal of R is idempotent. Let A, B be h-ideals of R such that $B^2 \subseteq A$. Thus $\overline{B^2} \subseteq \overline{A} = A$. By hypothesis $B = \overline{B^2}$, so $B \subseteq A$. Hence A is semiprime.

Conversely, assume that each h-ideal of R is semiprime. Let A be an h-ideal of R. Then $\overline{A^2}$ is also an h-ideal of R. Also $A^2 \subseteq \overline{A^2}$. Hence by hypothesis $A \subseteq \overline{A^2}$. But $\overline{A^2} \subseteq A$ always. Hence $A = \overline{A^2}$. ∎

Theorem 148 *Each fuzzy h-ideal of R is idempotent if and only if each fuzzy h-ideal of R is semiprime.*

Proof. For any h-ideal of R we have $\lambda \odot_h \lambda \leq \lambda$ (Theorem 118). If each h-ideal of R is semiprime, then $\lambda \odot_h \lambda \leq \lambda \odot_h \lambda$ implies $\lambda \leq \lambda \odot_h \lambda$. Hence $\lambda \odot_h \lambda = \lambda$.

The converse is obvious. ∎

Below we present two examples of hemirings in which all fuzzy h-ideals are semiprime.

Example 149 *Consider the set $R = \{0, a, 1\}$ with the following two operations:*

$+$	0	a	1
0	0	a	1
a	a	a	a
1	1	a	1

$\cdot$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

Then $(R, +, \cdot)$ is a commutative hemiring with identity. It has only one proper ideal $\{0, a\}$. This ideal is not an h-ideal. The only h-ideal of R is $\{0, a, 1\}$, which is clearly h-idempotent.

Since $0 = 0a = a0 = 01 = 10$, for any fuzzy ideal λ of this hemiring we have $\lambda(0) \geq \lambda(a)$ and $\lambda(0) \geq \lambda(1)$ and $\lambda(a) = \lambda(1a) \geq \lambda(1)$. Thus $\lambda(0) \geq \lambda(a) \geq \lambda(1)$. If λ is a fuzzy h-ideal, then $1 + 0 + 1 = 0 + 1$ implies $\lambda(1) \geq \lambda(0) \wedge \lambda(0) = \lambda(0)$, which proves that each fuzzy h-ideal of this hemiring is a constant function. So, $\lambda \odot_h \lambda = \lambda$ for each fuzzy h-ideal λ of R. This, by Theorem 148, means that each fuzzy h-ideal of R is semiprime.

Example 150 *Now, consider the hemiring $R = \{0, a, b, c\}$ defined by the following tables:*

$+$	0	a	b	c
0	0	a	b	c
a	a	b	c	a
b	b	c	a	b
c	c	a	b	c

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	b	c
c	0	c	b	c

This hemiring has only one h-ideal $A = R$. Obviously this h-ideal is h-idempotent.

For any fuzzy ideal λ of R and any $x \in R$ we have $\lambda(0) \geq \lambda(x) \geq \lambda(a)$. Indeed, $\lambda(0) = \lambda(0x) \geq \lambda(x) = \lambda(xa) \geq \lambda(a)$. This together with $\lambda(a) = \lambda(b+b) \geq \lambda(b) \wedge \lambda(b) = \lambda(b)$ implies $\lambda(a) = \lambda(b)$. Consequently, $\lambda(c) = \lambda(a+b) \geq \lambda(a) \wedge \lambda(b) = \lambda(b)$. Therefore $\lambda(0) \geq \lambda(c) \geq \lambda(b) = \lambda(a)$. Moreover, if λ is a fuzzy h-ideal, then $c+0+a = 0+a$, which implies $\lambda(c) \geq \lambda(0) \wedge \lambda(0) = \lambda(0)$. Thus $\lambda(0) = \lambda(c) \geq \lambda(b) = \lambda(a)$ for every fuzzy h-ideal of this hemiring.

Now we prove that each fuzzy h-ideal of R is idempotent. Since $\lambda \odot_h \lambda \leq \lambda$ always, so we have to show that $\lambda \odot_h \lambda \geq \lambda$. Obviously, for every $x \in R$ we have

$$\begin{aligned}(\lambda \odot_h \lambda)(x) &= \sup_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \left(\bigwedge_{i=1}^{n} (\lambda(a_i) \wedge \lambda(b_i)) \wedge \bigwedge_{j=1}^{m} (\lambda(a'_j) \wedge \lambda(b'_j)) \right) \\ &\geq \sup_{x+cd+z=c'd'+z} (\lambda(c) \wedge \lambda(d) \wedge \lambda(c') \wedge \lambda(d')) = \lambda(c) \wedge \lambda(d) \wedge \lambda(c') \wedge \lambda(d').\end{aligned}$$

So, $x + cd + z = c'd' + z$ implies $(\lambda \odot_h \lambda)(x) \geq \lambda(c) \wedge \lambda(d) \wedge \lambda(c') \wedge \lambda(d')$. Hence $0 + 00 + z = 00 + z$ implies $(\lambda \odot_h \lambda)(0) \geq \lambda(0)$. Similarly $a + bb + z = bc + z$ implies $(\lambda \odot_h \lambda)(a) \geq \lambda(b) \wedge \lambda(c) = \lambda(b) = \lambda(a)$, $b + aa + z = bc + z$ implies $(\lambda \odot_h \lambda)(b) \geq \lambda(a) \wedge \lambda(b) \wedge \lambda(c) = \lambda(b)$. Analogously, from $c + 00 + z = cc + z$ it follows $(\lambda \circ_h \lambda)(c) \geq \lambda(0) \wedge \lambda(c) = \lambda(c)$. This proves that $(\lambda \odot_h \lambda)(x) \geq \lambda(x)$ for every $x \in R$. Therefore $\lambda \odot_h \lambda = \lambda$ for every fuzzy h-ideal of R, which, by Theorem 148, means that each fuzzy h-ideal of R is semiprime.

Consider the following three fuzzy sets:

$$\begin{aligned} \lambda(0) = \lambda(c) = 0.8, &\quad \lambda(a) = \lambda(b) = 0.4, \\ \mu(0) = \mu(c) = 0.6, &\quad \mu(a) = \mu(b) = 0.5, \\ \delta(0) = \delta(c) = 0.7, &\quad \delta(a) = \delta(b) = 0.45. \end{aligned}$$

These three fuzzy sets are idempotent fuzzy h-ideals. Since all fuzzy h-ideal of this hemiring are idempotent, by Proposition 121, we have $\lambda \odot_h \mu = \lambda \wedge \mu$. Thus $(\lambda \odot_h \mu)(0) = (\lambda \odot_h \mu)(c) = 0.6$ and $(\lambda \odot_h \mu)(a) = (\lambda \odot_h \mu)(b) = 0.4$. So, $\lambda \odot_h \mu \leq \delta$ but neither $\lambda \leq \delta$ nor $\mu \leq \delta$, that is δ is not an h-prime fuzzy h-ideal.

Theorem 145 suggest the following definition of semiprime fuzzy h-ideals.

Definition 151 *A non-constant fuzzy h-ideal δ of R is called semiprime (in the second sense) if for all $t \in [0,1]$ and $a \in R$ the following condition is satisfied:*

if $\delta(axa) \geq t$ for every $x \in R$ then $\delta(a) \geq t$.

In other words, a non-constant fuzzy h-ideal δ is semiprime if from the fact that $axa \in U(\delta;t)$ for every $x \in R$ it follows $a \in U(\delta;t)$. It is clear that any fuzzy h-ideal semiprime in the first sense is semiprime in the second sense. The converse is not true (see Example 133).

Theorem 152 *A non-constant fuzzy h-ideal δ of R is semiprime in the second sense if and only if each its proper level set $U(\delta;t)$ is a semiprime h-ideal of R.*

The proof is analogous to the proof of Theorem 134.

Corollary 153 *A fuzzy set λ_A defined in Proposition 26 is a semiprime fuzzy h-ideal of R if and only if A is a semiprime h-ideal of R.*

In view of the Transfer Principle (Lemma 25) the second definition of semiprime fuzzy h-ideals is better. Therefore fuzzy h-ideals which are prime in the first sense should be called *h-semiprime.*

Proposition 154 *A non-constant fuzzy h-ideal δ of a commutative hemiring R with identity is semiprime if and only if $\delta(a^2) = \delta(a)$ for every $a \in R$.*

Proof. The proof is similar to the proof of Proposition 136. ∎

3.5 Prime Spectrum

Let R be a hemiring in which each h-ideal is idempotent. Let $\mathcal{L}(R)$ be the lattice of all h-ideals of R and $\mathcal{P}(R)$ be the set of all proper prime h-ideals of R. For each h-ideal I of R define $\theta_I = \{J \in \mathcal{P}(R) : I \nsubseteq J\}$ and $\Im(\mathcal{P}(R)) = \{\theta_I : I \in \mathcal{L}(R)\}$.

Theorem 155 *The set $\Im(\mathcal{P}(R))$ forms a topology on the set $\mathcal{P}(R)$.*

Proof. Since $\theta_{\{0\}} = \{J \in \mathcal{P}(R) : \{0\} \nsubseteq J\} = \phi$, where ϕ is the usual empty set, because 0 belongs to each h-ideal. So empty set belongs to $\Im(\mathcal{P}(R))$.

Also $\theta_R = \{J \in \mathcal{P}(R) : R \nsubseteq J\} = \mathcal{P}(R)$, because $\mathcal{P}(R)$ is the set of all proper prime h-ideals of R. Thus $\mathcal{P}(R)$ belongs to $\Im(\mathcal{P}(R))$.

Suppose $\theta_{I_1}, \theta_{I_2} \in \Im(\mathcal{P}(R))$ where I_1 and I_2 are in $\mathcal{L}(R)$. Then

$\theta_{I_1} \cap \theta_{I_2} = \{J \in \mathcal{P}(R) : I_1 \nsubseteq J \text{ and } I_2 \nsubseteq J\}$. Since each h-ideal of R is idempotent so $I_1 I_2 = I_1 \cap I_2$. Thus $\theta_{I_1} \cap \theta_{I_2} = \theta_{I_1 \cap I_2}$. So $\theta_{I_1} \cap \theta_{I_2}$ belongs to $\Im(\mathcal{P}(R))$.

Let $\{\theta_{I_i}\}_{i\in\Sigma}$ be an arbitrary family of members of $\Im(\mathcal{P}(R))$. Then

$$\begin{aligned}\underset{i\in\Omega}{\cup}\theta_{I_i} &= \underset{i\in\Omega}{\cup}\{J\in\mathcal{P}(R) : I_i \nsubseteq J\} = \{J\in\mathcal{P}(R) : \exists l\in\Omega \text{ so that } I_l \nsubseteq J\} \\ &= \theta_{\underset{i\in\Omega}{\Sigma} I_i}\end{aligned}$$

where $\underset{i\in\Omega}{\Sigma} I_i$ is the h-ideal generated by $\underset{i\in\Omega}{\cup} I_i$.
Hence $\Im(\mathcal{P}(R))$ is a topology on $\mathcal{P}(R)$. ■

Definition 156 [49] *A fuzzy h-ideal μ of a hemiring R is said to be normal if there exists $x\in R$ such that $\mu(x)=1$. If μ is a normal fuzzy h-ideal of R, then $\mu(0)=1$, hence μ is normal if and only if $\mu(0)=1$.*

Theorem 157 [49] *A fuzzy subset λ of a hemiring R is an h-prime fuzzy h-ideal of R if and only if*

(*i*) $\lambda^0=\{x\in R : \lambda(x)=\lambda(0)\}$ is a prime h-ideal of R.
(*ii*) $\mathrm{Im}\lambda=\{\lambda(x) : x\in R\}$ contains exactly two elements
(*iii*) $\lambda(0)=1$.

Corollary 158 [49] *Every h-prime fuzzy h-ideal of a hemiring is normal.*

Let R be a hemiring in which each fuzzy h-ideal is idempotent, $\mathcal{L}_R$ the lattice of fuzzy normal h-ideals of R and $\mathcal{F}P_R$ the set of all proper fuzzy h-prime h-ideals of R. For any fuzzy normal h-ideal λ of R, we define $\theta_\lambda=\{\mu\in\mathcal{F}P_R : \lambda\nleq\mu\}$ and $\tau(\mathcal{F}P_R)=\{\theta_\lambda : \lambda\in\mathcal{L}_R\}$.

A fuzzy h-ideal λ of R is called proper if $\lambda\neq\mathbb{R}$, where $\mathbb{R}$ is the fuzzy h-ideal of R defined by $\mathbb{R}(x)=1$, $\forall x\in R$.

Theorem 159 *The set $\tau(\mathcal{F}P_R)$ forms a topology on the set $\mathcal{F}P_R$.*

Proof. (1) $\theta_\Phi=\left\{\mu\in\mathcal{F}P_R : \Phi\nleq\mu\right\}=\phi$, where ϕ is the usual empty set and Φ is the characteristic function of h-ideal $\{0\}$. This follows since each h-prime fuzzy h-ideal of R is normal. Thus the empty subset belongs to $\tau(\mathcal{F}P_R)$.

(2) $\theta_{\mathbb{R}}=\left\{\mu\in\mathcal{F}P_R : \mathbb{R}\nleq\mu\right\}=\mathcal{F}P_R$. This is true, since $\mathcal{F}P_R$ is the set of proper h-prime fuzzy h-ideals of R. So $\theta_{\mathbb{R}}=\mathcal{F}P_R$ is an element of $\tau(\mathcal{F}P_R)$.

(3) Let $\theta_{\delta_1},\theta_{\delta_2}\in\tau(\mathcal{F}P_R)$ with $\delta_1,\delta_2\in\mathcal{L}_R$.
Then

$$\theta_{\delta_1}\wedge\theta_{\delta_2}=\left\{\mu\in\mathcal{F}P_R : \delta_1\nleq\mu \text{ and } \delta_2\nleq\mu\right\}$$

Since each fuzzy h-ideal of R is idempotent, this implies $\delta_1\delta_2=\delta_1\wedge\delta_2$. Thus

$$\theta_{\delta_1}\wedge\theta_{\delta_2}=\left\{\mu\in\mathcal{F}P_R : \delta_1\nleq\mu \text{ and } \delta_2\nleq\mu\right\}=\theta_{\delta_1\wedge\delta_2}.$$

(4) Let us consider an arbitrary family $\{\delta_i\}_{i\in I}$ of fuzzy h-ideals of R. Since

$\cup_{i\in I}\theta_{\delta_i} = \cup_{i\in I}\left\{\mu \in \mathcal{FP}_R : \delta_i \nleq \mu\right\} = \left\{\mu \in \mathcal{FP}_R : \exists' s\ k \in I \text{ so that } \delta_k \nleq \mu\right\}$

Note that

$$\left(\sum_{i\in I}\delta_i\right)(x) = \bigvee_{x+a_1+a_2+....=b_1+b_2+....}\{\delta_1(a_1)\wedge\delta_2(a_2)\wedge....\delta_1(b_1)\wedge\delta_2(b_2)\wedge...\}$$

where $a_1, a_2, ...b_1, b_2, ... \in R$ and only a finite number of the $a_i's$ and $b_i's$ are not zero. Since $\delta_i(0) = 1$, therefore we are considering the infimum of a finite number of terms because $1's$ are effectively not being considered. Now, if for some $k \in I$, $\delta_k \nleq \mu$, then there exists $x \in R$ such that $\delta_k(x) > \mu(x)$. Consider the particular expression for X in which $a_k = x$, $b_k = 0$ and $a_i = b_i = 0$ for all $i \neq k$. We see that $\delta_k(x)$ is an element of the set whose supremum is defined to be $\left(\sum_{i\in I}\delta_i\right)(x)$.

Thus $\left(\sum_{i\in I}\delta_i\right)(x) \geq \delta_k(x) > \mu(x)$. This implies $\left(\sum_{i\in I}\delta_i\right)(x) > \mu(x)$ that is $\sum_{i\in I}\delta_i \nleq \mu$.

Hence $\delta_k \nleq \mu$ for some $k \in I$ implies $\sum_{i\in I}\delta_i \nleq \mu$.

Conversely, suppose that $\sum_{i\in I}\delta_i \nleq \mu$ then there exists an element $x \in R$ such that $\left(\sum_{i\in I}\delta_i\right)(x) > \mu(x)$.

This means that

$$\bigvee_{x+a_1+a_2+....+z=b_1+b_2+....+z}\{\delta_1(a_1)\wedge\delta_2(a_2)\wedge....\delta_1(b_1)\wedge\delta_2(b_2)\wedge...\} > \mu(x).$$

Now, if all the elements of the set (whose supremum we are taking) are individually less than are equal to $\mu(x)$, then we have

$$\begin{aligned}\left(\sum_{i\in I}\delta_i\right)(x) &= \bigvee_{x+a_1+a_2+....+z=b_1+b_2+....+z}\{\delta_1(a_1)\wedge\delta_2(a_2)\wedge....\delta_1(b_1)\wedge\delta_2(b_2)\wedge...\}\\ &\leq \mu(x)\end{aligned}$$

which does not agree with what we have assumed. Thus, there is at least one element of the set (whose supremum we are taking), say,

$$\delta_1\left(a_1'\right)\wedge\delta_2\left(a_2'\right)\wedge ....\delta_1\left(b_1'\right)\wedge\delta_2\left(b_2'\right)\wedge ...... > \mu(x)$$

($x + a_1' + a_2' + ... = b_1' + b_2' + ...$ being the corresponding breakup of x, where only a finite number of $a_i's$ and $b_i's$ are not zero.)

Thus,
$\delta_1\left(a_1^{'}\right)\wedge\delta_2\left(a_2^{'}\right)\wedge.......\delta_1\left(b_1^{'}\right)\wedge\delta_2\left(b_2^{'}\right)\wedge......$
$>\mu(x)\geq\mu_1\left(a_1^{'}\right)\wedge\mu_2\left(a_2^{'}\right)\wedge.......\mu_1\left(b_1^{'}\right)\wedge\mu_2\left(b_2^{'}\right)\wedge......$
Let

$$\delta_1\left(a_1^{'}\right)\wedge\delta_2\left(a_2^{'}\right)\wedge.......\delta_1\left(b_1^{'}\right)\wedge\delta_2\left(b_2^{'}\right)\wedge......=\delta_p\left(x_p^{'}\right)$$

and

$$\mu_1\left(a_1^{'}\right)\wedge\mu_2\left(a_2^{'}\right)\wedge.......\mu_1\left(b_1^{'}\right)\wedge\mu_2\left(b_2^{'}\right)\wedge......=\mu_p\left(x_p^{'}\right)\text{ where } p\in I$$

So, $\delta_p\left(x_p^{'}\right)>\mu_p\left(x_p^{'}\right)$ it follows that $\delta_p\nleq\mu$ for some $p\in I$.

Hence $\sum_{i\in I}\delta_i\nleq\mu$ implies that $\delta_p\nleq\mu$ for some $p\in I$.

Hence the two statements (i) $\sum_{i\in I}\delta_i\nleq\mu$ and (ii) $\delta_p\nleq\mu$ for some $p\in I$ are equivalent.

Hence

$$\cup_{i\in I}\theta_{\delta_i}=\cup_{i\in I}\left\{\mu\in\mathcal{F}P_R:\delta_i\nleq\mu\right\}=\cup_{i\in I}\left\{\mu\in\mathcal{F}P_R:\sum_{i\in I}\delta_i\nleq\mu\right\}=\theta_{\sum_{i\in I}\theta_i}$$

because, $\sum_{i\in I}\delta_i$ is also a fuzzy h-ideal of R .

Thus, $\cup_{i\in I}\theta_{\delta_i}\in\tau\left(\mathcal{F}P_R\right)$.

Hence it follows that $\tau\left(\mathcal{F}P_R\right)$ forms a topology on the set $\mathcal{F}P_R$. ■

Chapter 4

Right k-weakly regular hemirings

Generalizing the concept of k-regular hemiring we define right k-weakly regular hemiring. We characterize right k-weakly regular hemirings by the properties of their right k-ideals and fuzzy right k-ideals. We also define right pure k-ideal and right pure fuzzy k-ideals of R and prove that a hemiring R is right k-weakly regular if and only if each k-ideal of R is right pure if and only if each fuzzy k-ideal of R is right pure.

4.1 Right k-weakly regular hemirings

Definition 160 *A hemiring R is called right (left) k-weakly regular hemiring if for each $x \in R$, $x \in \widehat{(xR)^2}$ $\left(res.\ x \in \widehat{(Rx)^2}\right)$.*

That is for each $x \in R$ we have $r_i, s_i, t_j, p_j \in R$ such that $x + \sum\limits_{i=1}^{n} xr_ixs_i = \sum\limits_{j=1}^{m} xt_jxp_j$ $\left(x + \sum\limits_{i=1}^{n} r_ixs_ix = \sum\limits_{j=1}^{m} t_jxp_jx\right)$. Thus each k-regular hemiring with 1 is right k-weakly regular but the converse is not true. However for a commutative hemiring both the concepts coincide.

Proposition 161 *The following statements are equivalent for a hemiring R with identity :*

1. R is right k-weakly regular hemiring.
2. All right k-ideals of R are k-idempotent (A right k-ideal B of R is k-idempotent if $\widehat{B^2} = B$).
3. $\widehat{BA} = B \cap A$ for all right k-ideals B and two-sided k-ideals A of R.

Proof. (1) $\Rightarrow$ (2) Let R be a right k-weakly regular hemiring and B be a right k-ideal of R. Clearly $\widehat{B^2} \subseteq B$.

Let $x \in B$. Since R is right k-weakly regular, so $x \in \widehat{(xR)^2}$ where xR is the right ideal of R generated by x and so $\widehat{xR}$ is the right k-ideal of R generated by x. Thus $xR \subseteq B$, this implies

$$x \in \widehat{(xR)(xR)} \subseteq \widehat{BB} = \widehat{B^2}.$$

Thus

$$B \subseteq \widehat{B^2}.$$

So, $\widehat{B^2} = B$.

(2) $\Rightarrow$ (3) Let B be a right k-ideal of R and A a two-sided k-ideal of R, then by Lemma 11, $\widehat{BA} \subseteq B \cap A$. To prove the reverse inclusion, let $x \in B \cap A$ and xR and RxR are right ideal and two-sided ideal of R generated by x, respectively. Thus $xR \subseteq B$ and $RxR \subseteq A$.

$$x \in xR \subseteq \widehat{xR} = \widehat{\widehat{xR}\ \widehat{xR}} = \widehat{xRxR} = \widehat{(xR)(xR)} = \widehat{x(RxR)} \subseteq \widehat{xA} \subseteq \widehat{BA}$$

Hence $B \cap A \subseteq \widehat{BA}$ and so $B \cap A = \widehat{BA}$.

(3) $\Rightarrow$ (1) Let $x \in R$ and RxR and xR be the two-sided ideal and right ideal of R generated by x, respectively. Then

$$x \in xR \cap RxR \subseteq \widehat{xR} \cap \widehat{RxR} = \widehat{\widehat{xR}\ \widehat{RxR}} = \widehat{xRRxR} = \widehat{xR^2xR} = \widehat{(xR)^2}.$$

Hence R is right k-weakly regular hemiring. ■

Theorem 162 *For a hemiring R with* 1, *the following assertions are equivalent:*

1. R is right k-weakly regular hemiring.
2. All fuzzy right k-ideals of R are k-idempotent (A fuzzy right k-ideal λ of R is k-idempotent if $\lambda \odot_k \lambda = \lambda$).
3. $\lambda \odot_k \mu = \lambda \wedge \mu$ for all fuzzy right k-ideals λ and all fuzzy two-sided k-ideals μ of R.

Proof. (1) $\Rightarrow$ (2) Let λ be a fuzzy right k-ideal of R, then we have $\lambda \odot_k \lambda \leq \lambda$.

For the reverse inclusion, let $x \in R$. Since R is right k-weakly regular so there exist $s_i, t_i, s_j^{'}, t_j^{'} \in R$ such that

$$x + \sum_{i=1}^{m} xs_ixt_i = \sum_{j=1}^{n} xs_j^{'}xt_j^{'}.$$

Hence

$$\lambda(x) = \lambda(x) \wedge \lambda(x) \leq \bigwedge_{i=1}^{m} (\lambda(xs_i) \wedge \lambda(xt_i)).$$

Also

$$\lambda(x) = \lambda(x) \wedge \lambda(x) \leq \bigwedge_{j=1}^{n} \left(\lambda(xs_j^{'}) \wedge \lambda(xt_j^{'})\right).$$

Therefore

$$\lambda(x) \leq \left[\bigwedge_{i=1}^{m} (\lambda(xs_i) \wedge \lambda(xt_i))\right] \wedge \left[\bigwedge_{j=1}^{n} \left(\lambda(xs_j^{'}) \wedge \lambda(xt_j^{'})\right)\right]$$

$$\leq \bigvee_{x + \sum_{i=1}^{m} xs_ixt_i = \sum_{j=1}^{n} xs_j^{'}xt_j^{'}} \left[\begin{array}{c} \left[\bigwedge_{i=1}^{m} (\lambda(xs_i) \wedge \lambda(xt_i))\right] \\ \wedge \left[\bigwedge_{j=1}^{n} \left(\lambda(xs_j^{'}) \wedge \lambda(xt_j^{'})\right)\right] \end{array}\right]$$

$$= (\lambda \odot_k \lambda)(x).$$

Hence $\lambda \leq \lambda \odot_k \lambda$, which proves $\lambda \odot_k \lambda = \lambda$.

(2) $\Rightarrow$ (3) Let λ and μ be fuzzy right and two sided k-ideal of R, respectively. Then $\lambda \wedge \mu$ is a fuzzy right k-ideal of R. By Corollary 76 $\lambda \odot_k \mu \leq \lambda \wedge \mu$. By hypothesis,

$$(\lambda \wedge \mu) = (\lambda \wedge \mu) \odot_k (\lambda \wedge \mu) \leq \lambda \odot_k \mu$$

Hence $\lambda \odot_k \mu = \lambda \wedge \mu$.

(3) $\Rightarrow$ (1) Let B be a right k-ideal of R and A be a two-sided k-ideal of R, then the characteristic functions χ_B and χ_A of B and A are fuzzy right and fuzzy two-sided k-ideals of R, repectively. Hence by the hypothesis and Propositions 29 and Lemma 74, we have

$$\chi_B \odot_k \chi_A = \chi_B \wedge \chi_A \quad \Rightarrow \quad \chi_{\widehat{BA}} = \chi_{B \cap A} \quad \Rightarrow \quad \widehat{BA} = B \cap A.$$

Thus by Proposition 161, R is right k-weakly regular hemiring. ∎

Theorem 163 *For a hemiring R with 1, the following assertions are equivalent:*

1. R is right k-weakly regular hemiring.

2. All right k-ideals of R are k-idempotent.

3. $\widehat{BA} = B \cap A$ for all right k-ideals B and two-sided k-ideals A of R.

4. All fuzzy right k-ideals of R are k-idempotent.

5. $\lambda \odot_k \mu = \lambda \wedge \mu$ for all fuzzy right k-ideals λ and all fuzzy two-sided k-ideals μ of R.

If R is commutative, then the above assertions are equivalent to

6. R is k-regular.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) by Proposition 161.
(1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) by Theorem 162.
Finally If R is commutative, then by Theorem 42, (1) $\Leftrightarrow$ (6). ∎

Theorem 164 *The collection of all k-ideals of a right k-weakly regular hemiring R forms a complete distributive lattice.*

Proof. Let $\mathcal{L}_R$ be the collection of all k-ideals of right k-weakly regular hemiring R, then $\mathcal{L}_R$ is a poset under the inclusion of sets. It is not difficult to see that $\mathcal{L}_R$ is a complete lattice under the operations $\sqcup$, $\sqcap$ defined as $A \sqcup B = \widehat{A+B}$ and $A \sqcap B = A \cap B$.

We now show that $\mathcal{L}_R$ is a Brouwerian lattice, that is, for any $A, B \in \mathcal{L}_R$, the set $\mathcal{L}_R(A,B) = \{I \in \mathcal{L}_R \mid A \cap I \subseteq B\}$ contains a greatest element.

By Zorn's Lemma the set $\mathcal{L}_R(A,B)$ contains a maximal element M. Since R is right k-weakly regular hemiring , so $\widehat{AI} = A \cap I \subseteq B$ and $\widehat{AM} = A \cap M \subseteq B$. Thus $\widehat{AI} + \widehat{AM} \subseteq B$. Consequently, $\widehat{\widehat{AI} + \widehat{AM}} \subseteq \widehat{B} = B$.

Since $\widehat{I+M} = I \sqcup M \in \mathcal{L}_R$, for every $x \in \widehat{I+M}$ there exist $i_1, i_2 \in I$, $m_1, m_2 \in M$ such that $x + i_1 + m_1 = i_2 + m_2$. Thus

$$dx + di_1 + dm_1 = di_2 + dm_2$$

for any $d \in D \in \mathcal{L}_R$. As $di_1, di_2 \in DI$, $dm_1, dm_2 \in DM$, we have $dx \in \widehat{DI + DM}$, which implies $D\left(\widehat{I+M}\right) \subseteq \widehat{DI+DM} \subseteq \widehat{\widehat{DI} + \widehat{DM}} \subseteq B$. Hence $\widehat{D\left(\widehat{I+M}\right)} \subseteq B$. This means that $D \cap \left(\widehat{I+M}\right) = \widehat{D\left(\widehat{I+M}\right)} \subseteq B$, i.e., $\widehat{I+M} \in \mathcal{L}_R(A,B)$, whence $\widehat{I+M} = M$ because M is maximal in $\mathcal{L}_R(A,B)$. Therefore $I \subseteq \widehat{I} \subseteq \widehat{I+M} = M$ for every $I \in \mathcal{L}_R(A,B)$. Each complete Brouwerian lattice is distributive (cf. [11], 11.11). Hence $\mathcal{L}_R$ is complete distributive lattice. ∎

The following example shows that if the collection of all k-ideals of a hemiring R is a distributive lattice then R is not necessarily a right k-weakly regular hemiring.

Example 165 *Consider the hemiring $R = \{0, a, b\}$ with the following operations*

+	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

$\cdot$	0	a	b
0	0	0	0
a	0	0	0
b	0	0	b

The k-ideals of R are $\{0\}, \{0, a\}$ and R. Since $\{0\} \subseteq \{0, a\} \subseteq R$. So the collection of k-ideals is a distributive lattice but R is not right k-weakly regular hemiring.

Theorem 166 *If R is a right k-weakly regular hemiring, then the set $\mathbb{FI}_R$ of all fuzzy k-ideals of R (ordered by $\leq$) is a distributive lattice.*

Proof. The set $\mathbb{FI}_R$ of all fuzzy k-ideals of R (ordered by $\leq$) is clearly a lattice under the k-sum and intersection of fuzzy k-ideals. Now we show that $\mathbb{FI}_R$ is a distributive lattice, that is for any fuzzy k-ideals λ, μ, δ of R we have $(\lambda \wedge \delta) + \mu = (\lambda + \mu) \wedge (\delta + \mu)$.

For any $x \in R$

$$\begin{aligned}
[(\lambda \wedge \delta) + \mu](x) &= \bigvee_{x+(a_1+b_1)=(a_2+b_2)} \begin{bmatrix} (\lambda \wedge \delta)(a_1) \wedge (\lambda \wedge \delta)(a_2) \wedge \\ (\mu)(b_1) \wedge (\mu)(b_2) \end{bmatrix} \\
&= \bigvee_{x+(a_1+b_1)=(a_2+b_2)} \begin{bmatrix} \lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \\ \mu(b_2) \wedge \delta(a_1) \wedge \delta(a_2) \end{bmatrix} \\
&= \bigvee_{x+(a_1+b_1)=(a_2+b_2)} \begin{bmatrix} [\lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \wedge \\ [\delta(a_1) \wedge \delta(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \end{bmatrix} \\
&= \left(\bigvee_{x+(a_1+b_1)=(a_2+b_2)} [\lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \right) \\
&\wedge \left(\bigvee_{x+(a_1+b_1)=(a_2+b_2)} [\delta(a_1) \wedge \delta(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \right) \\
&= (\lambda + \mu)(x) \wedge (\delta + \mu)(x) = [(\lambda + \mu) \wedge (\delta + \mu)](x). \quad \blacksquare
\end{aligned}$$

4.2 Prime and Fuzzy prime right k-ideals

Definition 167 *A right k-ideal P of a hemiring R is called k-prime (k-semiprime) right k-ideal of R if for any right k-ideals A, B of R,*

$$AB \subseteq P \Rightarrow A \subseteq P \text{ or } B \subseteq P \quad \left(A^2 \subseteq P \Rightarrow A \subseteq P\right).$$

P is called a *k-irreducible* (*k-strongly irreducible*) right k-ideal of R if for any right k-ideals A, B of R

$$A \cap B = P \Rightarrow A = P \text{ or } B = P \quad (A \cap B \subseteq P \Rightarrow A \subseteq P \text{ or } B \subseteq P).$$

A fuzzy right k-ideal μ of a hemiring R is called a fuzzy *k-prime* (*k-semiprime*) right k-ideal of R if for any fuzzy k-right ideals λ, δ of R,

$$\lambda \odot_k \delta \leq \mu \Rightarrow \lambda \leq \mu \text{ or } \delta \leq \mu \ (\lambda \odot_k \lambda \leq \mu \Rightarrow \lambda \leq \mu).$$

μ is called a fuzzy *k-irreducible* (*k-strongly irreducible*) if for any fuzzy right k-ideals λ, δ of R,

$$\lambda \wedge \delta = \mu \Rightarrow \lambda = \mu \text{ or } \delta = \mu \ (\lambda \wedge \delta \leq \mu \Rightarrow \lambda \leq \mu \text{ or } \delta \leq \mu).$$

Lemma 168 (*a*) *Every k-prime right k-ideal (fuzzy k-prime right k-ideal) of a hemiring R is a k-semiprime right k-ideal (fuzzy k-semiprime right k-ideal) of R.*

(*b*) Intersection of k-prime right k-ideals (fuzzy k-prime right k-ideals) of R is k-semiprime right k-ideal (fuzzy k-semiprime right k-ideal) of R.

Proof. Straightforward. ■

Theorem 169 *Let R be a right k-weakly regular hemiring. Then each proper right k-ideal of R is the intersection of right k-irreducible k-ideals which contain it.*

Proof. Let I be a proper right k-ideal of R and let $\{I_\alpha : \alpha \in \Lambda\}$ be a family of right k-irreducible k-ideals of R which contain I. Clearly $I \subseteq \cap_{\alpha\in\Lambda} I_\alpha$. Suppose $a \notin I$. Then by Zorn's Lemma there exists a right k-ideal I_β such that I_β is maximal with respect to the property $I \subseteq I_\beta$ and $a \notin I_\beta$. We will show that I_β is k-irreducible. Let A, B be right k-ideals of R such that $I_\beta = B \cap A$. Suppose $I_\beta \subset B$ and $I_\beta \subset A$. Then by the maximality of I_β, we have $a \in B$ and $a \in A$. But this implies $a \in B \cap A = I_\beta$, which is a contradiction. Hence either $I_\beta = B$ or $I_\beta = A$. So there exists a k-irreducible k-ideal I_β such that $a \notin I_\beta$ and $I \subseteq I_\beta$. Hence $\cap I_\alpha \subseteq I$. Thus $I = \cap I_\alpha$. ■

Proposition 170 *Let R be a right k-weakly regular hemiring. If λ is a fuzzy right k-ideal of R with $\lambda(a) = \alpha$, where a is any element of R and $\alpha \in (0,1]$, then there exists a fuzzy k-irreducible right k-ideal δ of R such that $\lambda \leq \delta$ and $\delta(a) = \alpha$.*

Proof. Let $X = \{\mu : \mu \text{ is a fuzzy right } k\text{-ideal of } R, \mu(a) = \alpha \text{ and } \lambda \leq \mu\}$. Then $X \neq \phi$, since $\lambda \in X$. Let $\mathcal{F}$ be a totally ordered subset of X, say $\mathcal{F} = \{\lambda_i : i \in I\}$. We claim that $\bigvee_{i\in I} \lambda_i$ is a fuzzy right k-ideal of R. For any $x, r \in R$, we have

$$\left(\bigvee_i \lambda_i\right)(x) = \bigvee_i (\lambda_i(x)) \leq \bigvee_i (\lambda_i(xr)) = \left(\bigvee_i \lambda_i\right)(xr)$$

Let $x, y \in R$, consider

$$\begin{aligned}
\left(\bigvee_i \lambda_i\right)(x) \wedge \left(\bigvee_i \lambda_i\right)(y) &= \left(\bigvee_i (\lambda_i(x))\right) \wedge \left(\bigvee_j (\lambda_j(y))\right) \\
&= \bigvee_j \left[\bigvee_i (\lambda_i(x)) \wedge \lambda_j(y)\right] \\
&= \bigvee_j \left[\bigvee_i (\lambda_i(x) \wedge \lambda_j(y))\right] \\
&\leq \bigvee_j \left[\bigvee_i \left(\lambda_i^j(x) \wedge \lambda_i^j(y)\right)\right] \\
&\text{where } \lambda_i^j = \max\{\lambda_i, \lambda_j\}, \text{ note that } \lambda_i^j \in \{\lambda_i : i \in I\} \\
&\leq \bigvee_j \left[\bigvee_i \left[\lambda_i^j(x+y)\right]\right] \\
&= \bigvee_{i,j} \left[\lambda_i^j(x+y)\right] \\
&\leq \bigvee_i [\lambda_i(x+y)] = \left(\bigvee_i \lambda_i\right)(x+y)
\end{aligned}$$

Now, let $x + a = b$ where $x, a, b \in R$. Then

$$\begin{aligned}
\left(\bigvee_i \lambda_i\right)(a) \wedge \left(\bigvee_i \lambda_i\right)(b) &= \left(\bigvee_i (\lambda_i(a))\right) \wedge \left(\bigvee_j (\lambda_j(b))\right) \\
&= \bigvee_j \left[\left(\bigvee_i (\lambda_i(a))\right) \wedge \lambda_j(b)\right] \\
&= \bigvee_j \left[\bigvee_i (\lambda_i(a) \wedge \lambda_j(b))\right] \\
&\leq \bigvee_j \left[\bigvee_i \left(\lambda_i^j(a) \wedge \lambda_i^j(b)\right)\right] \\
&\text{where } \lambda_i^j = \max\{\lambda_i, \lambda_j\}, \text{ note that } \lambda_i^j \in \{\lambda_i : i \in I\} \\
&\leq \bigvee_j \left[\bigvee_i \left(\lambda_i^j(x)\right)\right] \text{ because } \lambda_i^j \text{ is a fuzzy } k\text{-ideal} \\
&= \bigvee_{i,j} \left[\lambda_i^j(x)\right] \leq \bigvee_i [\lambda_i(x)] = \left(\bigvee_i \lambda_i\right)(x)
\end{aligned}$$

Thus $\bigvee_i \lambda_i$ is a fuzzy right k-ideal of R. Clearly $\lambda \leq \bigvee_i \lambda_i$ and $\bigvee_i \lambda_i(a) = \bigvee_i (\lambda_i(a)) = \alpha$. Thus $\bigvee_i \lambda_i$ is the l.u.b of $\mathcal{F}$. Hence by Zorn's lemma there exists a fuzzy right k-ideal δ of R which is maximal with respect to the property that $\lambda \leq \delta$ and $\delta(a) = \alpha$.

We will show that δ is fuzzy k-irreducible right k-ideal of R. Let $\delta = \delta_1 \wedge \delta_2$, where δ_1, δ_2 are fuzzy right k-ideals of R. Thus $\delta \leq \delta_1$ and $\delta \leq \delta_2$. We claim that either $\delta = \delta_1$ or $\delta = \delta_2$. Suppose $\delta \neq \delta_1$ and $\delta \neq \delta_2$. Since δ is maximal with respect to the property that $\delta(a) = \alpha$ and since $\delta \subsetneq \delta_1$ and $\delta \subsetneq \delta_2$, so $\delta_1(a) \neq \alpha$ and $\delta_2(a) \neq \alpha$. Hence $\alpha = \delta(a) = (\delta_1 \wedge \delta_2)(a) = (\delta_1)(a) \wedge (\delta_2)(a) \neq \alpha$, which is impossible. Hence $\delta = \delta_1$ or $\delta = \delta_2$. Thus δ is fuzzy k-irreducible right k-ideal of R. ■

Theorem 171 *Every fuzzy right k-ideal of a hemiring R is the intersection of all fuzzy k-irreducible right k-ideals of R which contain it.*

Proof. Let λ be the fuzzy right k-ideal of R and let $\{\lambda_\alpha : \alpha \in \Lambda\}$ be the family of all fuzzy k-irreducible right k-ideals of R which contain λ. Obviously $\lambda \leq \bigwedge_{\alpha\in\Lambda} \lambda_\alpha$. We show that $\bigwedge_{\alpha\in\Lambda} \lambda_\alpha \leq \lambda$. Let a be any element of R, then by Proposition 170, there exists a fuzzy k-irreducible right k-ideal λ_β such that $\lambda \leq \lambda_\beta$ and $\lambda(a) = \lambda_\beta(a)$. Hence $\lambda_\beta \in \{\lambda_\alpha : \alpha \in \Lambda\}$. Hence $\bigwedge_{\alpha\in\Lambda} \lambda_\alpha \leq \lambda_\beta$, so $\bigwedge_{\alpha\in\Lambda} \lambda_\alpha(a) \leq \lambda_\beta(a) = \lambda(a) \Rightarrow \bigwedge_{\alpha\in\Lambda} \lambda_\alpha \leq \lambda$. Hence $\bigwedge_{\alpha\in\Lambda} \lambda_\alpha = \lambda$. ∎

Theorem 172 *The following assertions for a hemiring R with 1 are equivalent:*

1. R is right k-weakly regular hemiring.

2. Each right k-ideal of R is k-semiprime right k-ideal of R.

Proof. (1) $\Rightarrow$ (2) Suppose every right k-ideal is idempotent. Let I, J be right k-ideals of R, such that $J^2 \subseteq I$. Thus $\widehat{J^2} \subseteq \widehat{I}$. By Proposition 161, $J = \widehat{J^2}$, so $J \subseteq I$. Hence I is a k-semiprime right k-ideal of R.

(2) $\Rightarrow$ (1) Assume that each right k-ideal of R is k-semiprime. Let I be right k-ideal of R. Then $\widehat{I^2}$ is also a right k-ideal of R and $I^2 \subseteq \widehat{I^2}$. Hence by hypothesis $I \subseteq \widehat{I^2}$. But $\widehat{I^2} \subseteq I$ always. Hence $I = \widehat{I^2}$. Thus by Proposition 161, R is right k-weakly regular hemiring. ∎

Theorem 173 *The following assertions for a hemiring R with 1 are equivalent:*

1. R is right k-weakly regular hemiring.

2. All fuzzy right k-ideals of R are k-idempotent (A fuzzy right k-ideal λ of R is idempotent if $\lambda \odot_k \lambda = \lambda$).

3. $\lambda \odot_k \mu = \lambda \wedge \mu$ for all fuzzy right k-ideals λ and all fuzzy two-sided k-ideals μ of R.

4. Each fuzzy right k-ideal of R is a fuzzy k-semiprime right k-ideal of R.

Proof. (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) by Theorem 162

(2) $\Rightarrow$ (4) Let δ be any fuzzy right k-ideal of R then $\lambda \odot_k \lambda \leq \delta$, where λ is a fuzzy right k-ideal of R. By (2) $\lambda \odot_k \lambda = \lambda$, so $\lambda \leq \delta$. Thus δ is a fuzzy k-semiprime right k-ideal of R.

(4) $\Rightarrow$ (2) Let δ be any fuzzy right k-ideal of R then $\delta \odot_k \delta$ is also a fuzzy right k-ideal of R and so by (4) $\delta \odot_k \delta$ is a fuzzy k-semiprime right k-ideal of R. As $\delta \odot_k \delta \leq \delta \odot_k \delta \Rightarrow \delta \leq \delta \odot_k \delta$ but $\delta \odot_k \delta \leq \delta$ always. So $\delta \odot_k \delta = \delta$. ∎

Theorem 174 *If every right k-ideal of a hemiring R is k-prime right k-ideal then R is right k-weakly regular hemiring and the set of k-ideals of R is totally ordered.*

Proof. Suppose R is a hemiring in which each right k-ideal is prime right k-ideal. Let A be a right k-ideal of R then $\widehat{A^2}$ is a right k-ideal of R. As $A^2 \subseteq \widehat{A^2}$ $\Rightarrow A \subseteq \widehat{A^2}$. But $\widehat{A^2} \subseteq A$ always. Hence $A = \widehat{A^2}$. Thus R is right k-weakly regular hemiring.

Let A, B be any k-ideals of R then $AB \subseteq A \cap B$. As $A \cap B$ is a k-ideal of R, so a k-prime right k-ideal. Thus either $A \subseteq A \cap B$ or $B \subseteq A \cap B$. That is either $A \subseteq B$ or $B \subseteq A$. ∎

Theorem 175 *If R is right k-weakly regular hemiring and the set of all right k-ideals of R is totally ordered then every right k-ideal of R is a k-prime right k-ideal of R.*

Proof. Let A, B, C be right k-ideals of R such that $AB \subseteq C$. Since the set of all right k-ideals of R is totally ordered, so we have $A \subseteq B$ or $B \subseteq A$. If $A \subseteq B$ then $A = \widehat{AA} \subseteq \widehat{AB} \subseteq C$. If $B \subseteq A$ then $B = \widehat{BB} \subseteq \widehat{AB} \subseteq C$. Thus C is a k-prime right k-ideal. ∎

Theorem 176 *If every fuzzy right k-ideal of a hemiring R is fuzzy k-prime right k-ideal then R is right k-weakly regular hemiring and the set of fuzzy k-ideals of R is totally ordered.*

Proof. Suppose R is a hemiring in which each fuzzy right k-ideal is fuzzy prime right k-ideal. Let λ be a fuzzy right k-ideal of R then $\lambda \odot_k \lambda$ is also a fuzzy right k-ideal of R. As $\lambda \odot_k \lambda \leq \lambda \odot_k \lambda \Rightarrow \lambda \leq \lambda \odot_k \lambda$. But $\lambda \odot_k \lambda \leq \lambda$ always. Hence $\lambda = \lambda \odot_k \lambda$. Thus R is right k-weakly regular hemiring.

Let λ, μ be any fuzzy k-ideals of R then $\lambda \odot_k \mu \leq \lambda \wedge \mu$. As $\lambda \wedge \mu$ is a fuzzy k-ideal of R so a fuzzy k-prime right k-ideal. Thus either $\lambda \leq \lambda \wedge \mu$ or $\mu \leq \lambda \wedge \mu$. That is either $\lambda \leq \mu$ or $\mu \leq \lambda$. ∎

Theorem 177 *If R is right k-weakly regular hemiring and the set of all fuzzy right k-ideals of R is totally ordered then every fuzzy right k-ideal of R is a fuzzy k-prime right k-ideal of R.*

Proof. Let λ, μ, ν be fuzzy right k-ideals of R such that $\lambda \odot_k \mu \leq \nu$. Since the set of all fuzzy right k-ideals of R is totally ordered, so we have $\lambda \leq \mu$ or $\mu \leq \lambda$. If $\lambda \leq \mu$ then $\lambda = \lambda \odot_k \lambda \leq \lambda \odot_k \mu \leq \nu$. If $\mu \leq \lambda$ then $\mu = \mu \odot_k \mu \leq \lambda \odot_k \mu \leq \nu$. Thus ν is a fuzzy k-prime right k-ideal. ∎

Example 178 *Consider the set $R = \{0, x, 1\}$ in which the "sup"($\vee$) and "inf" ($\wedge$) are defined by the chains $0 < 1 < x$ and $0 < x < 1$. On the set R, define $+ = \vee$ and $\cdot = \wedge$. Then $(R, +, \cdot)$ is a hemiring with the following tables:*

$+$	0	x	1
0	0	x	1
x	x	x	x
1	1	x	1

$\cdot$	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

The right ideals of R are $\{0\}, \{0, x\}, \{0, x, 1\}$. The right k-ideals of R are $\{0\}$ $\{0, x, 1\}$, which are idempotent. Obviously each right k-ideal of R is k-prime

In order to examine the right fuzzy k-ideals of R, we observe the following facts concerning R.

Fact 1.

Let $\lambda : R \Rightarrow [0, 1]$ be a fuzzy subset of R. Then λ is a fuzzy right ideal of R if and only if $\lambda(0) \geq \lambda(x) \geq \lambda(1)$.

Proof. Suppose $\lambda : R \Rightarrow [0, 1]$ is a fuzzy right ideal of R. Since $0 = x \cdot 0 = 1 \cdot 0$ so $\lambda(0) \geq \lambda(x)$ and $\lambda(0) \geq \lambda(1)$. Also $\lambda(x) = \lambda(1 \cdot x) \geq \lambda(1)$. Thus $\lambda(0) \geq \lambda(x) \geq \lambda(1)$.

Conversely, suppose that $\lambda : R \Rightarrow [0, 1]$ is a fuzzy subset of R such that $\lambda(0) \geq \lambda(x) \geq \lambda(1)$. By the definition of "+" defined on R, we have $m + m' = m$ or m' for every $m, m' \in R$, and certainly $\lambda(m) \wedge \lambda\left(m'\right) \leq \lambda(m)$ and $\lambda(m) \wedge \lambda\left(m'\right) \leq \lambda\left(m'\right)$. Thus $\lambda\left(m + m'\right) \geq \lambda(m) \wedge \lambda\left(m'\right)$. By the definition of "$\cdot$" defined on R, it is easy to verify that $\lambda(ma) \geq \lambda(m)$ for all m, a in R. Hence λ is a fuzzy right ideal of R. ■

Fact 2.

Let $\lambda : R \Rightarrow [0, 1]$ be a fuzzy subset of R. Then λ is a fuzzy right k-ideal of R if and only if $\lambda(0) \geq \lambda(x) = \lambda(1)$.

Proof. Suppose $\lambda : R \Rightarrow [0, 1]$ is a fuzzy right k-ideal of R. Then by the Fact 1 $\lambda(0) \geq \lambda(x) \geq \lambda(1)$. Since $1 + x = x$, so $\lambda(1) \geq \lambda(x) \wedge \lambda(x) = \lambda(x)$. Thus $\lambda(0) \geq \lambda(x) = \lambda(1)$.

Conversely, suppose that $\lambda : R \Rightarrow [0, 1]$ is a fuzzy subset of R such that $\lambda(0) \geq \lambda(x) = \lambda(1)$ then by the Fact 1, λ is a fuzzy right ideal of R.

If $x + a = b$ for $a, b, x \in R$ then $\lambda(x) \geq \lambda(a) \wedge \lambda(b)$. So λ is a fuzzy right k-ideal of R. ■

Obviously R is right k-weakly regular hemiring. But each fuzzy right k-ideal of R is not k-prime. Because λ, μ, ν defined by $\lambda(0) = 0.8$, $\lambda(x) = \lambda(1) = 0.6$, $\mu(0) = 0.9$, $\mu(x) = \mu(1) = 0.5$ and $\nu(0) = 0.85$, $\nu(x) = \nu(1) = 0.55$ are fuzzy k-ideals of R such that $\lambda \odot_k \mu \leq \nu$ but neither $\lambda \leq \nu$ nor $\mu \leq \nu$.

4.3 Right pure k-ideals

In this section we define right pure k-ideals of a hemiring R and also right pure fuzzy k-ideals of R. We prove that a two-sided k-ideal I of a hemiring R is right pure if and only if for every right k-ideal A of R, we have $A \cap I = \widehat{AI}$.

Definition 179 *A k-ideal I of a hemiring R is called right pure if for each $x \in I$, $x \in \widehat{xI}$, that is for each $x \in I$ there exist $y, z \in I$ such that $x + xy = xz$.*

Lemma 180 *A k-ideal I of a hemiring R is right pure if and only if $A \cap I = \widehat{AI}$ for every right k-ideal A of R.*

Proof. Suppose that I is a right pure k-ideal of R and A is a right k-ideal of R. Then

$$\widehat{AI} \subseteq A \cap I.$$

Let $a \in A \cap I$, then $a \in A$ and $a \in I$. Since I is right pure so $a \in \widehat{aI} \subseteq \widehat{AI}$. Thus $A \cap I \subseteq \widehat{AI}$. Hence $A \cap I = \widehat{AI}$.

Conversely, assume that $A \cap I = \widehat{AI}$ for every right k-ideal A of R. Let $x \in I$. Take A, the principal right k-ideal generated by x, that is, $A = \widehat{xR + \mathbb{N}_o x}$, where $\mathbb{N}_o = \{0, 1, 2, ...\}$. By hypothesis $A \cap I = \widehat{AI} = \widehat{\widehat{(xR + \mathbb{N}_o x)}\, I}$
$= \widehat{(xR + \mathbb{N}_o x)I} \subseteq \widehat{xI}$. So $x \in \widehat{xI}$.

Hence I is a right pure k-ideal of R. ■

Definition 181 *A fuzzy k-ideal λ of a hemiring R is called right pure fuzzy k-ideal of R if $\mu \wedge \lambda = \mu \odot_k \lambda$ for every fuzzy right k-ideal μ of R.*

Proposition 182 *Let A be a non-empty subset of a hemiring R. Then χ_A, the characteristic function of A, is right pure fuzzy k-ideal of R if and only if A is right pure k-ideal of R.*

Proof. Let A be a right pure k-ideal of R. Then χ_A is a fuzzy k-ideal of R. To prove that χ_A is right pure we have to show that for any fuzzy right k-ideal μ of R, $\mu \wedge \chi_A = \mu \odot_h \chi_A$. Now if $x \notin A$, then

$(\mu \wedge \chi_A)(x) = \mu(x) \wedge \chi_A(x) = 0 \leq (\mu \odot_h \chi_A)(x)$.

For the case $x \in A$, as A is right pure k-ideal of R, so there exist $a, b \in A$, such that $x + xa = xb$.

As $x, a, b \in A$, this implies $\chi_A(x) = \chi_A(a) = \chi_A(b) = 1$.

Now,

$$
\begin{aligned}
(\mu \odot_k \chi_A)(x) &= \bigvee_{x+\sum_{i=1}^{m} a_i b_i = \sum_{j=1}^{n} a'_j b'_j} \left[\bigwedge_{i=1}^{m} [\mu(a_i) \wedge \chi_A(b_i)] \wedge \bigwedge_{j=1}^{n} \left[\mu(a'_j) \wedge \chi_A(b'_j) \right] \right] \\
&\geq \min [\mu(x) \wedge \chi_A(a) \wedge \mu(x) \wedge \chi_A(b)] \\
&\geq \min [\mu(x) \wedge \chi_A(x) \wedge \mu(x) \wedge \chi_A(x)] \\
&\geq \mu(x) \wedge \chi_A(x) = (\mu \wedge \chi_A)(x).
\end{aligned}
$$

So, in both the cases $\mu \odot_k \chi_A \geq \mu \wedge \chi_A$. But $\mu \odot_k \chi_A \leq \mu \wedge \chi_A$ is always true.
Thus, $\mu \wedge \chi_A = \mu \odot_k \chi_A$.
So, χ_A is right pure fuzzy k-ideal of R.

Conversely, let χ_A be a right pure fuzzy k-ideal of R. Then A is a k-ideal of R. Let B be a right k-ideal of R, then χ_B is a fuzzy right k-ideal of R. Hence by hypothesis $\chi_B \odot_k \chi_A = \chi_B \wedge \chi_A = \chi_{B \cap A}$. By Lemma 74, $\chi_B \odot_k \chi_A = \chi_{\widehat{BA}}$. This implies that $B \cap A = \widehat{BA}$. Therefore A is a right pure k-ideal of R. ■

Proposition 183 *The intersection of right pure k-ideals of R is a right pure k-ideal of R.*

Proof. Let A, B be right pure k-ideals of R and I be any right k-ideal of R. Then
$I \cap (A \cap B) = (I \cap A) \cap B = (\widehat{IA}) \cap B$ because A is right pure
$= \widehat{(\widehat{IA})B}$ because B is right pure and $(\widehat{IA})$ is a right k-ideal
$= \widehat{(IA)B} = \widehat{I(AB)} = \widehat{I(A \cap B)}$.
Hence $A \cap B$ is a right pure k-ideal of R. ■

Proposition 184 *Let λ_1, λ_2 be right pure fuzzy k-ideals of R, then so is $\lambda_1 \wedge \lambda_2$.*

Proof. Let λ_1 and λ_2 be right pure fuzzy k-ideals of R then $\lambda_1 \wedge \lambda_2$ is a fuzzy k-ideal of R. We have to show that, for any fuzzy right k-ideal μ of R, $\mu \odot_k (\lambda_1 \wedge \lambda_2) = \mu \wedge (\lambda_1 \wedge \lambda_2)$.

Since λ_2 is right pure fuzzy k-ideal of R so it follows that $\lambda_1 \odot_k \lambda_2 = \lambda_1 \wedge \lambda_2$.
Hence

$$\mu \odot_k (\lambda_1 \odot_k \lambda_2) = \mu \odot_k (\lambda_1 \wedge \lambda_2).$$

Also,

$$
\begin{aligned}
\mu \wedge (\lambda_1 \wedge \lambda_2) &= (\mu \wedge \lambda_1) \wedge \lambda_2 \\
&= (\mu \odot_k \lambda_1) \wedge \lambda_2 \quad \text{since } \lambda_1 \text{ is right pure} \\
&= (\mu \odot_k \lambda_1) \odot_k \lambda_2 \quad \text{since } \mu \odot_k \lambda_1 \text{ is a fuzzy right } k\text{-ideal of } R \\
&= \mu \odot_k (\lambda_1 \odot_k \lambda_2).
\end{aligned}
$$

Thus

$$\mu \wedge (\lambda_1 \wedge \lambda_2) = \mu \odot_k (\lambda_1 \wedge \lambda_2).$$

■

Proposition 185 *The following statements are equivalent for a hemiring R with identity :*

1. R is right k-weakly regular hemiring.
2. All right k-ideals of R are k-idempotent (A right k-ideal B of R is k-idempotent if $\widehat{B^2} = B$).
3. Every k-ideal of R is right pure.

Proof. (1) $\Leftrightarrow$ (2) By Proposition 161.

(1) $\Rightarrow$ (3) Let R be a right k-weakly regular hemiring. Let I and A be k-ideal and right k-ideal of R, respectively. Then $A \cap I = \widehat{AI}$.

Thus by Lemma 180, A is right pure.

(3) $\Rightarrow$ (1) Let I be a k-ideal of R and A a right k-ideal of R, then by hypothesis I is right pure and so $A \cap I = \widehat{AI}$. Thus by Proposition 161, R is right k-weakly regular hemiring. ■

Proposition 186 *The following assertions are equivalent for a hemiring R with identity:*

1. R is right k-weakly regular hemiring.
2. All right k-ideals of R are k-idempotent.
3. Every k-ideal of R is right pure.
4. All fuzzy right k-ideals of R are k-idempotent.
5. Every fuzzy k-ideal of R is right pure.

If R is commutative, then the above assertions are equivalent to

6. R is k-regular.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) by Proposition 185.

(1) $\Leftrightarrow$ (4) by Theorem 162.

(4) $\Rightarrow$ (5) Let λ and μ be fuzzy right and two sided k-ideals of R, respectively. Then $\lambda \wedge \mu$ is a fuzzy right k-ideal of R. By Corollary 76, $\lambda \odot_k \mu \leq \lambda \wedge \mu$. By hypothesis,

$$(\lambda \wedge \mu) = (\lambda \wedge \mu) \odot_k (\lambda \wedge \mu) \leq \lambda \odot_k \mu.$$

Hence $\lambda \odot_k \mu = \lambda \wedge \mu$. Thus μ is right pure.

(5) $\Rightarrow$ (1) Let B be a right k-ideal of R and A be a two-sided k-ideal of R then the characteristic functions χ_B and χ_A of B and A are fuzzy right and fuzzy two-sided k-ideals of R, repectively. Hence by hypothesis

$$\chi_B \odot_h \chi_A = \chi_B \wedge \chi_A \Rightarrow \chi_{\widehat{BA}} = \chi_{B\cap A} \Rightarrow \widehat{BA} = B \cap A.$$

Thus by Proposition 161, R is right k-weakly regular hemiring.

Finally If R is commutative then by Theorem 81, (1) $\Leftrightarrow$ (6). ■

4.4 Purely prime k-ideals

In this section we define purely prime k-ideals and purely prime fuzzy k-ideals of a hemiring and study some basic properties of these ideals.

Definition 187 *A proper right pure k-ideal I of a hemiring R is called purely prime if for any right pure k-ideals A, B of R, $A \cap B \subseteq I \Rightarrow A \subseteq I$ or $B \subseteq I$.*

If A, B are right pure k-ideals of R then $A \cap B = \widehat{AB}$. Thus the above definition is equivalent to $\widehat{AB} \subseteq I \Rightarrow A \subseteq I$ or $B \subseteq I$.

Definition 188 *A proper right pure k-ideal μ of a hemiring R is called purely prime if for any right pure fuzzy k-ideals λ, δ of R, $\lambda \wedge \delta \leq \mu \Rightarrow \lambda \leq \mu$ or $\delta \leq \mu$.*

If λ, δ are right pure fuzzy k-ideals of R, then $\lambda \wedge \delta = \lambda \odot_k \delta$. Thus the above definition is equivalent to $\lambda \odot_k \delta \leq \mu \Rightarrow \lambda \leq \mu$ or $\delta \leq \mu$.

Proposition 189 *Let R be a right k-weakly regular hemiring with 1 and I be a k-ideal of R. Then the following assertions are equivalent:*

1. For k-ideals A, B of R, $A \cap B = I \Rightarrow A = I$ or $B = I$.

2. $A \cap B \subseteq I \Rightarrow A \subseteq I$ or $B \subseteq I$.

Proof. (1) $\Rightarrow$ (2) Suppose A, B are k-ideals of R such that $A \cap B \subseteq I$. Then by Theorem 164, $I = \widehat{(A \cap B) + I} = \widehat{(A + I)} \cap \widehat{(B + I)}$. Hence by the hypothesis $I = \widehat{(A + I)}$ or $I = \widehat{(B + I)}$, i.e., $A \subseteq I$ or $B \subseteq I$.

(2) $\Rightarrow$ (1) Suppose A, B are k-ideals of R such that $A \cap B = I$. Then $I \subseteq A$ and $I \subseteq B$. On the other hand by hypothesis $A \subseteq I$ or $B \subseteq I$. Thus $A = I$ or $B = I$. ■

Proposition 190 *Let R be a right k-weakly regular hemiring. Then any proper right pure k-ideal of R is contained in a purely prime k-ideal of R.*

Proof. Let I be a proper right pure k-ideal of a right k-weakly regular hemiring R and $a \in R$ such that $a \notin I$. Consider the set

$$X = \{J_p : J_p \text{ is a proper right pure } k\text{-ideal of } R \text{ such that } I \subseteq J_p \text{ and } a \notin J_p\}.$$

Then $X \neq \phi$ because $I \in X$. By Zorn's Lemma this family contains a maximal element, say M. This maximal element is purely prime. Indeed, let $A \cap B = M$ for some right pure k-ideals A, B of R. If A, B both properly contains M, then by the maximality of M, $a \in A$ and $a \in B$. Thus $a \in A \cap B = M$, which is a contradiction. Hence either $A = M$ or $B = M$. ∎

Proposition 191 *Let R be a right k-weakly regular hemiring. Then each proper right pure k-ideal is the intersection of all purely prime k-ideals of R which contain it.*

Proof. Proof is similar to the proof of Theorem 169. ∎

Proposition 192 *Let R be a right k-weakly regular hemiring. If λ is a right pure fuzzy k-ideal of R with $\lambda(a) = t$ where $a \in R$ and $t \in (0,1]$, then there exists a purely prime fuzzy k-ideal μ of R such that $\lambda \leq \mu$ and $\mu(a) = t$.*

Proof. The proof is similar to the proof of Proposition 170. ∎

Proposition 193 *Let R be a right k-weakly regular hemiring. Then each proper fuzzy right pure k-ideal is the intersection of all purely prime fuzzy k-ideals of R which contain it.*

Proof. Proof is similar to the proof of Theorem 171. ∎

Chapter 5

Right h-weakly regular hemirings

In this chapter we define right h-weakly regular hemirings and characterize these hemirings by the properties of their right h-ideals and fuzzy right h-ideals.

5.1 Right h-weakly regular hemirings

Definition 194 *A hemiring R is called right (left) h-weakly regular hemiring if for each $x \in R$, $x \in \overline{(xR)^2}$ $\left(res.\ x \in \overline{(Rx)^2}\right)$.*

That is for each $x \in R$ we have $r_i, s_i, t_j, p_j, z \in R$ such that $x + \sum_{i=1}^{n} xr_ixs_i + z = \sum_{j=1}^{m} xt_jxp_j + z$ $\left(x + \sum_{i=1}^{n} r_ixs_ix + z = \sum_{j=1}^{m} t_jxp_jx + z \right)$. Thus each h-hemiregular hemiring with identity is right h-weakly regular but the converse is not true. However for a commutative hemiring both the concepts coincide.

Proposition 195 *The following statements are equivalent for a hemiring R with identity:*

1. R is right h-weakly regular.
2. All right h-ideals of R are h-idempotent (A right h-ideal B of R is h-idempotent if $\overline{B^2} = B$).
3. $\overline{BA} = B \cap A$ for all right h-ideals B and two-sided h-ideals A of R.

Proof. (1) $\Rightarrow$ (2) Let R be a right h-weakly regular hemiring and B be a right h-ideal of R. Clearly $\overline{B^2} \subseteq B$.

Let $x \in B$. Since R is right h-weakly regular, so $x \in \overline{(xR)^2}$ where xR is the right ideal of R generated by x and so $\overline{xR}$ is the right h-ideal of R generated by x. Thus $xR \subseteq B$, this implies

$$x \in \overline{(xR)(xR)} \subseteq \overline{BB} = \overline{B^2}.$$

Thus $B \subseteq \overline{B^2}$. So, $\overline{B^2} = B$.

(2) $\Rightarrow$ (3) Let B be a right h-ideal of R and A be a two-sided h-ideal of R. Then by Lemma 9, $\overline{BA} \subseteq B \cap A$. To prove the reverse inclusion, let $x \in B \cap A$ and xR and RxR are the right ideal and two-sided ideal of R generated by x, respectively. Thus $xR \subseteq B$ and $RxR \subseteq A$. Now

$$x \in xR \subseteq \overline{xR} = \overline{\overline{xR}\,\overline{xR}} = \overline{xRxR} = \overline{(xR)(xR)} = \overline{x(RxR)} \subseteq \overline{xA} \subseteq \overline{BA}.$$

Hence $B \cap A \subseteq \overline{BA}$ and so $B \cap A = \overline{BA}$.

(3) $\Rightarrow$ (1) Let $x \in R$ and RxR and xR be the two-sided ideal and right ideal of R generated by x, repectively. Then

$$x \in xR \cap RxR \subseteq \overline{xR} \cap \overline{RxR} = \overline{\overline{xR}\,\overline{RxR}} = \overline{xRRxR} = \overline{xR^2xR} = \overline{(xR)^2}.$$

Hence R is right h-weakly regular hemiring. ∎

Theorem 196 *The following assertions are equivalent for a hemiring R with identity:*

1. R is right h-weakly regular hemiring.
2. All fuzzy right h-ideals of R are h- idempotent (A fuzzy right h-ideal λ of R is idempotent if $\lambda \odot_h \lambda = \lambda$).
3. $\lambda \odot_h \mu = \lambda \wedge \mu$ for all fuzzy right h-ideals λ and all fuzzy two-sided h-ideals μ of R.

Proof. (1) $\Rightarrow$ (2) Let λ be a fuzzy right h-ideal of R, then we have $\lambda \odot_h \lambda \leq \lambda$.

For the reverse inclusion, let $x \in R$. Since R is right h-weakly regular, so there exist $s_i, t_i, s_j', t_j', z \in R$ such that

$$x + \sum_{i=1}^{m} xs_ixt_i + z = \sum_{j=1}^{n} xs_j'xt_j' + z.$$

Hence

$$\lambda(x) = \lambda(x) \wedge \lambda(x) \leq \bigwedge_{i=1}^{m} \left(\lambda(xs_i) \wedge \lambda(xt_i)\right).$$

Also

$$\lambda(x) = \lambda(x) \wedge \lambda(x) \leq \bigwedge_{j=1}^{n} \left(\lambda(xs_j') \wedge \lambda(xt_j')\right).$$

Therefore

$$\begin{aligned}\lambda(x) &\leq \left[\bigwedge_{i=1}^{m}\left(\lambda(xs_i)\wedge\lambda(xt_i)\right)\right]\wedge\left[\bigwedge_{j=1}^{n}\left(\lambda(xs_j')\wedge\lambda(xt_j')\right)\right]\\ &\leq \bigvee_{x+\sum_{i=1}^{m}xs_ixt_i+z=\sum_{j=1}^{n}xs_j'xt_j'+z}\left[\begin{array}{c}\left[\bigwedge_{i=1}^{m}\left(\lambda(xs_i)\wedge\lambda(xt_i)\right)\right]\\ \wedge\left[\bigwedge_{j=1}^{n}\left(\lambda(xs_j')\wedge\lambda(xt_j')\right)\right]\end{array}\right]\\ &= (\lambda\odot_h\lambda)(x).\end{aligned}$$

Hence $\lambda \leq \lambda \odot_h \lambda$. Thus $\lambda \odot_h \lambda = \lambda$.

(2) $\Rightarrow$ (3) Let λ and μ be fuzzy right and two sided h-ideals of R, respectively. Then $\lambda \wedge \mu$ is a fuzzy right h-ideal of R and $\lambda \odot_h \mu \leq \lambda \wedge \mu$ is always true. By hypothesis,

$$(\lambda \wedge \mu) = (\lambda \wedge \mu) \odot_h (\lambda \wedge \mu) \leq \lambda \odot_h \mu$$

Hence $\lambda \odot_h \mu = \lambda \wedge \mu$.

(3) $\Rightarrow$ (1) Let B be a right h-ideal of R and A be a two-sided h-ideal of R. Then the characteristic functions χ_B and χ_A of B and A are fuzzy right and fuzzy two-sided h-ideal of R, repectively. Hence by hypothesis, Propositions 31 and 29,

$$\chi_B \odot_h \chi_A = \chi_B \wedge \chi_A \Rightarrow \chi_{\overline{BA}} = \chi_{B\cap A} \Rightarrow \overline{BA} = B \cap A.$$

Thus by Proposition 195, R is right h-weakly regular hemiring. ■

Theorem 197 *The following assertions are equivalent for a hemiring R with identity:*

1. R is right h-weakly regular hemiring.
2. All right h-ideals of R are h-idempotent.
3. $\overline{BA} = B \cap A$ for all right h-ideals B and two-sided h-ideals A of R.
4. All fuzzy right h-ideals of R are h-idempotent.
5. $\lambda \odot_h \mu = \lambda \wedge \mu$ for all fuzzy right h-ideals λ and all fuzzy two-sided h-ideals μ of R.

If R is commutative, then the above assertions are equivalent to

6. R is h-hemiregular.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) by Proposition 195.
(1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) by Theorem 196.
Finally If R is commutative, then by Theorem 45, (1) $\Leftrightarrow$ (6). ■

Theorem 198 *If R is right h-weakly regular hemiring, then the collection of all h-ideals forms a complete Brouwerian lattice.*

Proof. The collection $\mathcal{L}_R$ of all h-ideals of right h-weakly regular hemiring R is a poset under the inclusion of sets. It is not difficult to see that $\mathcal{L}_R$ is a complete lattice under the operations $\sqcup$, $\sqcap$ defined as $A \sqcup B = \overline{A+B}$ and $A \sqcap B = A \cap B$.

We show that $\mathcal{L}_R$ is a Brouwerian lattice, that is, for any $A, B \in \mathcal{L}_R$, the set $\mathcal{L}_R(A,B) = \{I \in \mathcal{L}_R \mid A \cap I \subseteq B\}$ contains a greatest element.

By Zorn's Lemma the set $\mathcal{L}_R(A,B)$ contains a maximal element M. Since R is right h-weakly regular hemiring, so $\overline{AI} = A \cap I \subseteq B$ and $\overline{AM} = A \cap M \subseteq B$. Thus $\overline{AI} + \overline{AM} \subseteq B$. Consequently, $\overline{\overline{AI} + \overline{AM}} \subseteq \overline{B} = B$.

Since $\overline{I+M} = I \sqcup M \in \mathcal{L}_R$, for every $x \in \overline{I+M}$ there exist $i_1, i_2 \in I$, $m_1, m_2 \in M$ and $z \in R$ such that $x + i_1 + m_1 + z = i_2 + m_2 + z$. Thus

$$dx + di_1 + dm_1 + dz = di_2 + dm_2 + dz$$

for any $d \in D \in \mathcal{L}_R$. As $di_1, di_2 \in DI$, $dm_1, dm_2 \in DM$, $dz \in R$, we have $dx \in \overline{DI + DM}$, which implies $D\left(\overline{I+M}\right) \subseteq \overline{DI+DM} \subseteq \overline{\overline{DI} + \overline{DM}} \subseteq B$. Hence $\overline{D\left(\overline{I+M}\right)} \subseteq B$. This means that $D \cap \left(\overline{I+M}\right) = \overline{D\left(\overline{I+M}\right)} \subseteq B$, i.e., $\overline{I+M} \in \mathcal{L}_R(A,B)$, whence $\overline{I+M} = M$ because M is maximal in $\mathcal{L}_R(A,B)$. Therefore $I \subseteq \overline{I} \subseteq \overline{I+M} = M$ for every $I \in \mathcal{L}_R(A,B)$. ■

Corollary 199 *If R is right h-weakly regular hemiring, then the lattice $\mathcal{L}_R$ is distributive.*

Proof. Each complete Brouwerian lattice is distributive (cf. [11], 11.11). ■

The following example shows that if the collection of all h-ideals of a hemiring R is a distributive lattice then R is not necessarily a right h-weakly regular hemiring.

Example 200 *Consider the hemiring $R = \{0, a, b\}$ with the following operations*

$+$	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

$\cdot$	0	a	b
0	0	0	0
a	0	0	0
b	0	0	b

The ideals of R are $\{0\}, \{0,a\}, \{0,b\}$ and R. Only R itself is an h-ideal of R. The collection of h-ideals is a distributive lattice but R is not right h-weakly regular hemiring, since $\{0, a, b\}$ is not h-idempotent.

Theorem 201 *If R is right h-weakly regular hemiring, then the set FI_R of all fuzzy h-ideals of R (ordered by $\leq$) is a distributive lattice.*

Proof. The set FI_R of all fuzzy h-ideals of R (ordered by $\leq$) is clearly a lattice under the h-sum and intersection of fuzzy h-ideals. Now we show that FI_R is a distributive lattice, that is for any fuzzy h-ideals λ, μ, δ of R we have $(\lambda \wedge \delta) + \mu = (\lambda + \mu) \wedge (\delta + \mu)$.

For any $x \in R$,

$$\begin{aligned}
[(\lambda \wedge \delta) + \mu](x) &= \bigvee_{x+(a_1+b_1)+z=(a_2+b_2)+z} \begin{bmatrix} (\lambda \wedge \delta)(a_1) \wedge (\lambda \wedge \delta)(a_2) \wedge \\ (\mu)(b_1) \wedge (\mu)(b_2) \end{bmatrix} \\
&= \bigvee_{x+(a_1+b_1)+z=(a_2+b_2)+z} \begin{bmatrix} \lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \\ \mu(b_2) \wedge \delta(a_1) \wedge \delta(a_2) \end{bmatrix} \\
&= \bigvee_{x+(a_1+b_1)+z=(a_2+b_2)+z} \begin{bmatrix} [\lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \wedge \\ [\delta(a_1) \wedge \delta(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \end{bmatrix} \\
&= \left(\bigvee_{x+(a_1+b_1)+z=(a_2+b_2)+z} [\lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \right) \\
&\quad \wedge \left(\bigvee_{x+(a_1+b_1)+z=(a_2+b_2)+z} [\delta(a_1) \wedge \delta(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \right) \\
&= (\lambda + \mu)(x) \wedge (\delta + \mu)(x) \\
&= [(\lambda + \mu) \wedge (\delta + \mu)](x). \quad \blacksquare
\end{aligned}$$

5.2 Prime and Fuzzy prime right h-ideals

Definition 202 *A right h-ideal P of a hemiring R is called h-prime (h-semiprime) right h-ideal of R if for any right h-ideals A, B of R,*

$$AB \subseteq P \Rightarrow A \subseteq P \text{ or } B \subseteq P \ \left(A^2 \subseteq P \Rightarrow A \subseteq P\right).$$

P is called an h-irreducible (h-strongly irreducible) right h-ideal of R if for any right h-ideals A, B of R

$$A \cap B = P \Rightarrow A = P \text{ or } B = P \ (A \cap B \subseteq P \Rightarrow A \subseteq P \text{ or } B \subseteq P).$$

A fuzzy right h-ideal μ of a hemiring R is called a fuzzy h-prime (h-semiprime) right h-ideal of R if for any fuzzy right h-ideals λ, δ of R,

$$\lambda \odot_h \delta \leq \mu \Rightarrow \lambda \leq \mu \text{ or } \delta \leq \mu \ (\lambda \odot_h \lambda \leq \mu \Rightarrow \lambda \leq \mu).$$

μ is called a fuzzy h-irreducible (h-strongly irreducible) if for any fuzzy right h-ideals λ, δ of R,

$$\lambda \wedge \delta = \mu \Rightarrow \lambda = \mu \text{ or } \delta = \mu \ (\lambda \wedge \delta \leq \mu \Rightarrow \lambda \leq \mu \text{ or } \delta \leq \mu).$$

Lemma 203 (a) *Every h-prime right h-ideal (fuzzy h-prime right h-ideal) of a hemiring R is an h-semiprime right h-ideal (fuzzy h-semiprime right h-ideal) of R.*

(b) The intersection of h-prime right h-ideal (fuzzy h-prime right h-ideal) of R is an h-semiprime right h-ideal (fuzzy h-semiprime right h-ideal)of R.

Proof. Straightforward. ■

Theorem 204 *Let R be a right h-weakly regular hemiring. Then each proper right h-ideal of R is the intersection of right h-irreducible h-ideals which contain it.*

Proof. Let I be a proper right h-ideal of R and let $\{I_\alpha : \alpha \in \Lambda\}$ be a family of right h-irreducible h-ideals of R which contain I. Clearly $I \subseteq \cap_{\alpha\in\Lambda} I_\alpha$. Suppose $a \notin I$. Then by Zorn's Lemma there exists a right h-ideal I_β such that I_β is maximal with respect to the property $I \subseteq I_\beta$ and $a \notin I_\beta$. We will show that I_β is h-irreducible. Let A, B be right h-ideals of R such that $I_\beta = B \cap A$. Suppose $I_\beta \subset B$ and $I_\beta \subset A$. Then by the maximality of I_β, we have $a \in B$ and $a \in A$. But this implies $a \in B \cap A = I_\beta$, which is a contradiction. Hence either $I_\beta = B$ or $I_\beta = A$. So there exists an h-irreducible h-ideal I_β such that $a \notin I_\beta$ and $I \subseteq I_\beta$. Hence $\cap I_\alpha \subseteq I$. Thus $I = \cap I_\alpha$. ■

Proposition 205 *Let R be a right h-weakly regular hemiring. If λ is a fuzzy right h-ideal of R with $\lambda(a) = \alpha$, where a is any element of R and $\alpha \in (0, 1]$, then there exists a fuzzy h-irreducible right h-ideal δ of R such that $\lambda \leq \delta$ and $\delta(a) = \alpha$.*

Proof. Let $X = \{\mu : \mu$ is a fuzzy right h-ideal of R, $\mu(a) = \alpha$ and $\lambda \leq \mu\}$. Then $X \neq \phi$, since $\lambda \in X$. Let $\mathcal{F}$ be a totally ordered subset of X, say $\mathcal{F} = \{\lambda_i : i \in I\}$. We claim that $\bigvee_{i\in I} \lambda_i$ is a fuzzy right h-ideal of R. For any $x, r \in R$, we have

$$\left(\bigvee_i \lambda_i\right)(x) = \bigvee_i (\lambda_i(x)) \leq \bigvee_i (\lambda_i(xr)) = \left(\bigvee_i \lambda_i\right)(xr)$$

Let $x, y \in R$, consider

$$\begin{aligned}
\left(\bigvee_i \lambda_i\right)(x) \wedge \left(\bigvee_i \lambda_i\right)(y) &= \left(\bigvee_i (\lambda_i(x))\right) \wedge \left(\bigvee_j (\lambda_j(y))\right) \\
&= \bigvee_j \left[\bigvee_i (\lambda_i(x)) \wedge \lambda_j(y)\right] \\
&= \bigvee_j \left[\bigvee_i (\lambda_i(x) \wedge \lambda_j(y))\right] \\
&\leq \bigvee_j \left[\bigvee_i \left(\lambda_i^j(x) \wedge \lambda_i^j(y)\right)\right] \\
&\text{where } \lambda_i^j = \max\{\lambda_i, \lambda_j\}, \text{ note that } \lambda_i^j \in \{\lambda_i : i \in I\} \\
&\leq \bigvee_j \left[\bigvee_i \left[\lambda_i^j(x+y)\right]\right] \\
&= \bigvee_{i,j} \left[\lambda_i^j(x+y)\right]
\end{aligned}$$

$$\leq \bigvee_i [\lambda_i (x+y)] = \left(\bigvee_i \lambda_i\right)(x+y)$$

Now, let $x+a+z=b+z$ where $x,a,b,z \in R$. Then

$$\begin{aligned}
\left(\bigvee_i \lambda_i\right)(a) \wedge \left(\bigvee_i \lambda_i\right)(b) &= \left(\bigvee_i (\lambda_i (a))\right) \wedge \left(\bigvee_j (\lambda_j (b))\right) \\
&= \bigvee_j \left[\left(\bigvee_i (\lambda_i (a))\right) \wedge \lambda_j (b)\right] \\
&= \bigvee_j \left[\bigvee_i (\lambda_i (a) \wedge \lambda_j (b))\right] \\
&\leq \bigvee_j \left[\bigvee_i \left(\lambda_i^j (a) \wedge \lambda_i^j (b)\right)\right] \\
&\text{where } \lambda_i^j = \max\{\lambda_i, \lambda_j\}, \text{ note that } \lambda_i^j \in \{\lambda_i : i \in I\} \\
&\leq \bigvee_j \left[\bigvee_i \left(\lambda_i^j (x)\right)\right] \text{ because } \lambda_i^j \text{ is a fuzzy right } h\text{-ideal} \\
&= \bigvee_{i,j} \left[\lambda_i^j (x)\right] \leq \bigvee_i [\lambda_i (x)] = \left(\bigvee_i \lambda_i\right)(x).
\end{aligned}$$

Thus $\bigvee_i \lambda_i$ is a fuzzy right h-ideal of R. Clearly $\lambda \leq \bigvee_i \lambda_i$ and $\bigvee_i \lambda_i (a) = \bigvee_i (\lambda_i (a)) = \alpha$. Thus $\bigvee_i \lambda_i$ is the l.u.b of $\mathcal{F}$. Hence by Zorn's lemma there exists a fuzzy right h-ideal δ of R which is maximal with respect to the property that $\lambda \leq \delta$ and $\delta(a) = \alpha$.

We will show that δ is fuzzy h-irreducible right h-ideal of R. Let $\delta = \delta_1 \wedge \delta_2$, where δ_1, δ_2 are fuzzy right h-ideals of R. Thus $\delta \leq \delta_1$ and $\delta \leq \delta_2$. We claim that either $\delta = \delta_1$ or $\delta = \delta_2$. Suppose $\delta \neq \delta_1$ and $\delta \neq \delta_2$. Since δ is maximal with respect to the property that $\delta(a) = \alpha$ and since $\delta \lneqq \delta_1$ and $\delta \lneqq \delta_2$, so $\delta_1(a) \neq \alpha$ and $\delta_2(a) \neq \alpha$. Hence $\alpha = \delta(a) = (\delta_1 \wedge \delta_2)(a) = (\delta_1)(a) \wedge (\delta_2)(a) \neq \alpha$, which is impossible. Hence $\delta = \delta_1$ or $\delta = \delta_2$. Thus δ is fuzzy h-irreducible right h-ideal of R. ■

Theorem 206 *Every fuzzy right h-ideal of a hemiring R is the intersection of all fuzzy h-irreducible right h-ideals of R which contain it.*

Proof. Let λ be the fuzzy right h-ideal of R and let $\{\lambda_\alpha : \alpha \in \Lambda\}$ be the family of all fuzzy h-irreducible right h-ideals of R which contain λ. Obviously $\lambda \leq \bigwedge_{\alpha\in\Lambda} \lambda_\alpha$. We now show that $\bigwedge_{\alpha\in\Lambda} \lambda_\alpha \leq \lambda$. Let a be any element of R, then by Proposition 205, there exists a fuzzy h-irreducible right h-ideal λ_β such that $\lambda \leq \lambda_\beta$ and $\lambda(a) = \lambda_\beta(a)$. Hence $\lambda_\beta \in \{\lambda_\alpha : \alpha \in \Lambda\}$ and so $\bigwedge_{\alpha\in\Lambda} \lambda_\alpha \leq \lambda_\beta$. Thus $\bigwedge_{\alpha\in\Lambda} \lambda_\alpha(a) \leq \lambda_\beta(a) = \lambda(a)$ $\Rightarrow \bigwedge_{\alpha\in\Lambda} \lambda_\alpha \leq \lambda$. Hence $\bigwedge_{\alpha\in\Lambda} \lambda_\alpha = \lambda$. ■

Theorem 207 *The following assertions for a hemiring R are equivalent:*

1. R is right h-weakly regular hemiring.

2. Each right h-ideal of R is h-semiprime right h-ideal of R.

Proof. (1) $\Rightarrow$ (2) Suppose R is right h-weakly regular hemiring. Let I, J be right h-ideals of R, such that $J^2 \subseteq I \Rightarrow \overline{J^2} \subseteq I$. By Theorem 197, $J = \overline{J^2}$, so $J \subseteq I$. Hence I is an h-semiprime right h-ideal of R.

(2) $\Rightarrow$ (1) Assume that each right h-ideal of R is h-semiprime. Let I be a right h-ideal of R. Then $\overline{I^2}$ is also a right h-ideal of R. Also $I^2 \subseteq \overline{I^2}$. Hence by hypothesis $I \subseteq \overline{I^2}$. But $\overline{I^2} \subseteq I$ always. Hence $I = \overline{I^2}$. Thus by Theorem 197, R is right h-weakly regular hemiring. ■

Theorem 208 *The following assertions for a hemiring R with identity are equivalent:*

1. R is right h-weakly regular hemiring.
2. All fuzzy right h-ideals of R are h- idempotent (A fuzzy right h-ideal λ of R is idempotent if $\lambda \odot_h \lambda = \lambda$)
3. $\lambda \odot_h \mu = \lambda \wedge \mu$ for all fuzzy right h-ideals λ and all fuzzy two-sided h-ideals μ of R.
4. Each fuzzy right h-ideal of R is a fuzzy h-semiprime right h-ideal of R.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) Theorem 196.

(2) $\Rightarrow$ (4) Let δ be any fuzzy right h-ideal of R, then $\lambda \odot_h \lambda \leq \delta$, where λ is a fuzzy right h-ideal of R . By (2) $\lambda \odot_h \lambda = \lambda$, so $\lambda \leq \delta$. Thus δ is a fuzzy h-semiprime right h-ideal of R.

(4) $\Rightarrow$ (2) Let δ be any fuzzy right h-ideal of R, then $\delta \odot_h \delta$ is also a fuzzy right h-ideal of R and so by (4) $\delta \odot_h \delta$ is a fuzzy h-semiprime right h-ideal of R. As $\delta \odot_h \delta \leq \delta \odot_h \delta \Rightarrow \delta \leq \delta \odot_h \delta$ but $\delta \odot_h \delta \leq \delta$ always. So $\delta \odot_h \delta = \delta$. ■

Theorem 209 *If every right h-ideal of a hemiring R is h-prime right h-ideal then R is right h-weakly regular hemiring and the set of h-ideals of R is totally ordered.*

Proof. Suppose R is a hemiring in which each right h-ideal is prime right h-ideal. Let A be a right h-ideal of R then $\overline{A^2}$ is a right h-ideal of R. As $A^2 \subseteq \overline{A^2} \Rightarrow A \subseteq \overline{A^2}$. But $\overline{A^2} \subseteq A$ always. Hence $A = \overline{A^2}$. Thus R is right h-weakly regular hemiring.

Let A, B be any h-ideals of R then $AB \subseteq A \cap B$. As $A \cap B$ is an h-ideal of R, so an h-prime right h-ideal. Thus either $A \subseteq A \cap B$ or $B \subseteq A \cap B$. That is either $A \subseteq B$ or $B \subseteq A$. ■

Theorem 210 *If R is right h-weakly regular hemiring and the set of all right h-ideals of R is totally ordered then every right h-ideal of R is an h-prime right h-ideal of R.*

Proof. Let A, B, C be right h-ideals of R such that $AB \subseteq C$. Since the set of all right h-ideals of R is totally ordered, so we have $A \subseteq B$ or $B \subseteq A$. If $A \subseteq B$ then $A = \overline{AA} \subseteq \overline{AB} \subseteq C$. If $B \subseteq A$ then $B = \overline{BB} \subseteq \overline{AB} \subseteq C$. Thus C is an h-prime right h-ideal. ■

Theorem 211 *If every fuzzy right h-ideal of a hemiring R is fuzzy h-prime right h-ideal, then R is right h-weakly regular hemiring and the set of fuzzy h-ideals of R is totally ordered.*

Proof. Suppose R is a hemiring in which each fuzzy right h-ideal is fuzzy prime right h-ideal. Let λ be a fuzzy right h-ideal of R then $\lambda \odot_h \lambda$ is also a fuzzy right h-ideal of R. As $\lambda \odot_h \lambda \leq \lambda \odot_h \lambda \Rightarrow \lambda \leq \lambda \odot_h \lambda$. But $\lambda \odot_h \lambda \leq \lambda$ always. Hence $\lambda = \lambda \odot_h \lambda$. Thus R is right h-weakly regular hemiring.

Let λ, μ be any fuzzy h-ideals of R then $\lambda \odot_h \mu \leq \lambda \wedge \mu$. As $\lambda \wedge \mu$ is a fuzzy right h-ideal of R so a fuzzy h-prime right h-ideal. Thus either $\lambda \leq \lambda \wedge \mu$ or $\mu \leq \lambda \wedge \mu$. That is either $\lambda \leq \mu$ or $\mu \leq \lambda$. ■

Theorem 212 *If R is a right h-weakly regular hemiring and the set of all fuzzy right h-ideals of R is totally ordered then every fuzzy right h-ideal of R is a fuzzy h-prime right h-ideal of R.*

Proof. Let λ, μ, ν be fuzzy right h-ideals of R such that $\lambda \odot_h \mu \leq \nu$. Since the set of all fuzzy right h-ideals of R is totally ordered, we have $\lambda \leq \mu$ or $\mu \leq \lambda$. If $\lambda \leq \mu$ then $\lambda = \lambda \odot_h \lambda \leq \lambda \odot_h \mu \leq \nu$. If $\mu \leq \lambda$, then $\mu = \mu \odot_h \mu \leq \lambda \odot_h \mu \leq \nu$. Thus ν is a fuzzy h-prime right h-ideal. ■

Example 213 *Consider the set $R = \{0, x, 1\}$ in which the "sup"($\vee$) and "inf" ($\wedge$) are defined by the chains $0 < 1 < x$ and $0 < x < 1$. On the set R, define $+ = \vee$ and $\cdot = \wedge$. Then $(R, +, \cdot)$ is a hemiring with the following tables:*

$+$	0	x	1
0	0	x	1
x	x	x	x
1	1	x	1

$\cdot$	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

The right ideals of R are $\{0\}, \{0, x\}, \{0, x, 1\}$. The only right h-ideal of R is $\{0, x, 1\}$, which is idempotent. Obviously R is right h-weakly regular hemiring and $\{0, x, 1\}$ is h-prime and thus h-semiprime.

In order to examine the right fuzzy h-ideals of R, we observe the following facts concerning R.

Fact 1.

Let $\lambda : R \to [0,1]$ be a fuzzy subset of R. Then λ is a fuzzy right ideal of R if and only if $\lambda(0) \geq \lambda(x) \geq \lambda(1)$.

Proof. Suppose $\lambda : R \to [0,1]$ is a fuzzy right ideal of R. Since $0 = x \cdot 0 = 1 \cdot 0$ so $\lambda(0) \geq \lambda(x)$ and $\lambda(0) \geq \lambda(1)$. Also $\lambda(x) = \lambda(1 \cdot x) \geq \lambda(1)$. Thus $\lambda(0) \geq \lambda(x) \geq \lambda(1)$.

Conversely, suppose that $\lambda : R \to [0,1]$ is a fuzzy subset of R such that $\lambda(0) \geq \lambda(x) \geq \lambda(1)$. By the definition of $+$ defined on R, we have $m + m' = m$ or m' for every $m, m' \in R$, and certainly $\lambda(m) \wedge \lambda\left(m'\right) \leq \lambda(m)$ and $\lambda(m) \wedge \lambda\left(m'\right) \leq \lambda\left(m'\right)$. Thus $\lambda\left(m + m'\right) \geq \lambda(m) \wedge \lambda\left(m'\right)$. By the definition of $\cdot$ defined on R, it is easy to verify that $\lambda(ma) \geq \lambda(m)$ for all m, a in R. Hence λ is a fuzzy right ideal of R. ■

Fact 2.

Let $\lambda : R \to [0,1]$ be a fuzzy subset of R. Then λ is a fuzzy right h-ideal of R if and only if $\lambda(0) = \lambda(x) = \lambda(1)$.

Proof. Suppose $\lambda : R \to [0,1]$ is a fuzzy right h-ideal of R. Then by the Fact 1 $\lambda(0) \geq \lambda(x) \geq \lambda(1)$. Since $1 + 0 + 1 = 0 + 1$, so $\lambda(1) \geq \lambda(0) \wedge \lambda(0) = \lambda(0)$. Thus $\lambda(0) = \lambda(x) = \lambda(1)$.

Conversely, suppose that $\lambda : R \to [0,1]$ is a fuzzy subset of R such that $\lambda(0) = \lambda(x) = \lambda(1)$ then by the Fact 1, λ is a fuzzy right ideal of R.

If $x + a + z = b + z$ for $a, b, x, z \in R$ then $\lambda(x) = \lambda(a) \wedge \lambda(b)$. So λ is a fuzzy right h-ideal of R. ■

Fact 3.

All fuzzy right h-ideal of R in the above example are idempotent.

Proof. Since each $x \in R$ can be expressed as $x + a_1b_1 + z = a_2b_2 + z$ for some $a_1, b_1, a_2, b_2, z \in R$ and each fuzzy right h-ideal of R is a constant function, so $\lambda \odot_h \lambda = \lambda$ for each fuzzy right h-ideal of R. ■

Thus each fuzzy right h-ideal of R is fuzzy h-semiprime. Also each fuzzy right h-ideal of R is fuzzy h-prime. Because $\lambda \odot_h \mu = \lambda \wedge \mu$ and $\lambda \odot_h \mu \leq \nu \Rightarrow \lambda \wedge \mu \leq \nu$. As each fuzzy h-ideal is constant so either $\lambda \wedge \mu = \lambda$ or $\lambda \wedge \mu = \mu$. Thus $\lambda \leq \nu$ or $\mu \leq \nu$.

5.3 Right pure h-ideals

In this section we define right pure h-ideals of a hemiring R and also right pure fuzzy h-ideals of hemiring R. We prove that every two-sided h-ideal I of a hemiring R is right pure if and only if for every right h-ideal A of R, we have $A \cap I = \overline{AI}$.

Definition 214 *An h-ideal I of a hemiring R is called right pure if for each $x \in I$, $x \in \overline{xI}$, that is for each $x \in I$ there exist $a, b \in I$ and $z \in R$ such that $x + xa + z = xb + z$.*

Example 215 *Consider the hemiring $R = \{0, a, b\}$ with the following operations*

$+$	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

$\cdot$	0	a	b
0	0	0	0
a	0	0	0
b	0	0	b

The only h-ideal of R is R itself which is right pure.

Lemma 216 *An h-ideal I of a hemiring R is right pure if and only if $A \cap I = \overline{AI}$ for every right h-ideal A of R.*

Proof. Suppose that I is a right pure h-ideal of R and A is a right h-ideal of R. Then

$$\overline{AI} \subseteq A \cap I.$$

Let $a \in A \cap I$, then $a \in A$ and $a \in I$. Since I is right pure so $a \in \overline{aI} \subseteq \overline{AI}$. Thus $A \cap I \subseteq \overline{AI}$. Hence $A \cap I = \overline{AI}$.

Conversely, assume that $A \cap I = \overline{AI}$ for every right h-ideal A of R. Let $x \in I$. Take A, the principal right h-ideal generated by x, that is, $A = \overline{xR + \mathbb{N}_o x}$, where $\mathbb{N}_o = \{0, 1, 2, ...\}$. By hypothesis $A \cap I = \overline{AI} = \overline{\overline{(xR + \mathbb{N}_o x)}I}$
$= \overline{(xR + \mathbb{N}_o x)I} \subseteq \overline{xI}$. So $x \in \overline{xI}$.

Hence I is a right pure h-ideal of R. ∎

Definition 217 *A fuzzy h-ideal λ of a hemiring R is called right pure fuzzy h-ideal of R if $\mu \wedge \lambda = \mu \odot_h \lambda$ for every fuzzy right h-ideal μ of R.*

Proposition 218 *Let A be a non-empty subset of a hemiring R. Then χ_A, the characteristic function of A, is right pure fuzzy h-ideal of R if and only if A is right pure h-ideal of R.*

Proof. Let A be a right pure h-ideal of R. By Corollary 27, χ_A is a fuzzy h-ideal of R.

To prove that χ_A is right pure we have to show that for any fuzzy right h-ideal μ of R, $\mu \wedge \chi_A = \mu \odot_h \chi_A$. We know that $\mu \odot_h \chi_A \leq \mu \wedge \chi_A$ is always true.

Now, if $x \notin A$, then

$$(\mu \wedge \chi_A)(x) = \mu(x) \wedge \chi_A(x) = 0 \leq (\mu \odot_h \chi_A)(x).$$

For the case $x \in A$, as A is right pure h-ideal of R, so there exist $a, b \in A$ and $z \in R$, such that $x + xa + z = xb + z$

As $x, a, b \in A$, this implies $\chi_A(x) = \chi_A(a) = \chi_A(b) = 1$.

Now,

$$\begin{aligned}(\mu \odot_h \chi_A)(x) &= \sup_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \left[\bigwedge_{i=1}^{m} [\mu(a_i) \wedge \chi_A(b_i)] \wedge \bigwedge_{j=1}^{n} \left[\mu(a'_j) \wedge \chi_A(b'_j)\right] \right] \\ &\geq \bigvee_{x+xa+z=xb+z} \min[\mu(x) \wedge \chi_A(a) \wedge \mu(x) \wedge \chi_A(b)] \\ &\geq \min[\mu(x) \wedge \chi_A(x) \wedge \mu(x) \wedge \chi_A(x)] \\ &\geq \mu(x) \wedge \chi_A(x) \\ &= (\mu \wedge \chi_A)(x)\end{aligned}$$

So, in both cases $\mu \odot_h \chi_A \geq \mu \wedge \chi_A$.

Thus, $\mu \wedge \chi_A = \mu \odot_h \chi_A$.

So, χ_A is right pure fuzzy h-ideal of R.

Conversely, let χ_A be right pure fuzzy h-ideal of R. Then by Corollary 27, A is an h-ideal of R. Let I be a right h-ideal of R, then χ_I is a fuzzy right h-ideal of R. Hence by hypothesis and Propositions 29 and 31,

$$\chi_{\overline{IA}} = \chi_I \odot_h \chi_A = \chi_I \wedge \chi_A = \chi_{I \cap A}.$$

Thus $\overline{IA} = I \cap A$. So A is right pure h-ideal of R. ■

Proposition 219 *Let R be a hemiring then the intersection of right pure h-ideals of R is a right pure h-ideal of R.*

Proof. Let A, B be right pure h-ideals of R and I be any right h-ideal of R. Then

$$\begin{aligned}I \cap (A \cap B) &= (I \cap A) \cap B \\ &= (\overline{IA}) \cap B \quad \text{because } A \text{ is right pure} \\ &= \overline{(\overline{IA})B} \quad \text{because } B \text{ is right pure and } (\overline{IA}) \text{ is a right } h\text{-ideal} \\ &= \overline{(IA)B} = \overline{I(AB)} = \overline{I(A \cap B)}\end{aligned}$$

Hence $A \cap B$ is a right pure h-ideal of R. ■

Proposition 220 *Let λ_1, λ_2 be right pure fuzzy h-ideals of R, then so is $\lambda_1 \wedge \lambda_2$.*

Proof. Let λ_1 and λ_2 be right pure fuzzy h-ideals of R. Then $\lambda_1 \wedge \lambda_2$ is a fuzzy h-ideal of R. We have to show that, for any fuzzy right h-ideal μ of R, $\mu \odot_h (\lambda_1 \wedge \lambda_2) = \mu \wedge (\lambda_1 \wedge \lambda_2)$.

Since λ_2 is right pure fuzzy h-ideal of R so it follows that $\lambda_1 \odot_h \lambda_2 = \lambda_1 \wedge \lambda_2$.

Hence

$$\mu \odot_h (\lambda_1 \odot_k \lambda_2) = \mu \odot_h (\lambda_1 \wedge \lambda_2).$$

Also,

$$\begin{aligned}\mu \wedge (\lambda_1 \wedge \lambda_2) &= (\mu \wedge \lambda_1) \wedge \lambda_2 \\ &= (\mu \odot_h \lambda_1) \wedge \lambda_2 \quad \text{since } \lambda_1 \text{ is right pure} \\ &= (\mu \odot_h \lambda_1) \odot_h \lambda_2 \quad \text{since } \mu \odot_h \lambda_1 \text{ is a fuzzy right } h\text{-ideal of } R \\ &= \mu \odot_h (\lambda_1 \odot_h \lambda_2).\end{aligned}$$

Thus

$$\mu \wedge (\lambda_1 \wedge \lambda_2) = \mu \odot_h (\lambda_1 \wedge \lambda_2).$$

■

Proposition 221 *The following statements are equivalent for a hemiring R with identity :*

1. R is right h-weakly regular hemiring.
2. All right h-ideals of R are h-idempotent (A right h-ideal B of R is h-idempotent if $\overline{B^2} = B$).
3. Every h-ideal of R is right pure.

Proof. (1) $\Leftrightarrow$ (2) By Proposition 195.

(1) $\Rightarrow$ (3) Let R be right h-weakly regular hemiring. Let I and A be h-ideal and right h-ideal of R, respectively. Then $A \cap I = \overline{AI}$.

Thus by Lemma 216, A is right pure.

(3) $\Rightarrow$ (1) Let I be an h-ideal of R and A a right h-ideal of R, then by hypothesis I is right pure and so $A \cap I = \overline{AI}$. Thus by Proposition 195, R is right h-weakly regular hemiring. ■

Theorem 222 *The following statements are equivalent for a hemiring R with identity :*

1. R is right h-weakly regular hemiring.
2. All right h-ideals of R are h-idempotent.
3. Every h-ideal of R is right pure.
4. All fuzzy right h-ideals of R are h-idempotent.
5. Every fuzzy h-ideal of R is right pure.

If R is commutative, then the above assertions are equivalent to

6. R is h-hemiregular.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) by Proposition 221.

(1) $\Leftrightarrow$ (4) by Theorem 196.

(4) $\Rightarrow$ (5) Let λ and μ be fuzzy right and two sided h-ideals of R, respectively. Then $\lambda \wedge \mu$ is a fuzzy right h-ideal of R. Also $\lambda \odot_h \mu \leq \lambda \odot_h \chi_R \leq \lambda$ and $\lambda \odot_h \mu \leq \chi_R \odot_h \mu \leq \mu$. Thus $\lambda \odot_h \mu \leq \lambda \wedge \mu$. By hypothesis,

$$(\lambda \wedge \mu) = (\lambda \wedge \mu) \odot_h (\lambda \wedge \mu) \leq \lambda \odot_h \mu$$

Hence $\lambda \odot_h \mu = \lambda \wedge \mu$. Thus λ is right pure.

(5) $\Rightarrow$ (1) Let B be a right h-ideal of R and A a two-sided h-ideal of R then the characteristic functions χ_B and χ_A of B and A are fuzzy right and fuzzy two-sided h-ideals of R, repectively. Hence by hypothesis

$$\chi_B \odot_h \chi_A = \chi_B \wedge \chi_A \Rightarrow \chi_{\overline{BA}} = \chi_{B \cap A} \Rightarrow \overline{BA} = B \cap A.$$

Thus by Proposition 195, R is right h-weakly regular hemiring.

Finally If R is commutative, then by Theorem 45, (1) $\Leftrightarrow$ (6) . ■

Example 223 *Consider the hemiring $R = \{0, a, b\}$ with the following operations*

$+$	0	a	b
0	0	a	b
a	a	b	0
b	b	0	a

$\cdot$	0	a	b
0	0	0	0
a	0	a	b
b	0	b	a

R is right weakly regular hemiring, so each fuzzy h-ideal of R is right pure.

5.4 Purely prime h-ideals

In this section we define purely prime h-ideals and purely prime fuzzy h-ideals of a hemiring R and study some basic properties of these ideals.

Definition 224 *A proper right pure h-ideal I of a hemiring R is called purely prime if for any right pure h-ideals A, B of R, $A \cap B \subseteq I \Rightarrow A \subseteq I$ or $B \subseteq I$.*

If A, B are right pure h-ideals of R then $A \cap B = \overline{AB}$. Thus the above definition is equivalent to $\overline{AB} \subseteq I \Rightarrow A \subseteq I$ or $B \subseteq I$.

Definition 225 *A proper right pure h-ideal μ of a hemiring R is called purely prime if for any right pure fuzzy h-ideals λ, δ of R, $\lambda \wedge \delta \leq \mu \Rightarrow \lambda \leq \mu$ or $\delta \leq \mu$.*

If λ, δ are right pure fuzzy h-ideals of R, then $\lambda \wedge \delta = \lambda \odot_h \delta$. Thus the above definition is equivalent to $\lambda \odot_h \delta \leq \mu \Rightarrow \lambda \leq \mu$ or $\delta \leq \mu$.

Proposition 226 *Let R be a right h-weakly regular hemiring with identity and I be an h-ideal of R. Then the following assertions are equivalent:*

1. For h-ideals A, B of R, $A \cap B = I \Rightarrow A = I$ or $B = I$.
2. $A \cap B \subseteq I \Rightarrow A \subseteq I$ or $B \subseteq I$.

Proof. (1) $\Rightarrow$ (2) Suppose A, B are h-ideals of R such that $A \cap B \subseteq I$. Then by Theorem 198, $I = \overline{(A \cap B) + I} = \overline{(A + I)} \cap \overline{(B + I)}$. Hence by hypothesis $I = \overline{(A + I)}$ or $I = \overline{(B + I)}$, i.e., $A \subseteq I$ or $B \subseteq I$.

(2) $\Rightarrow$ (1) Suppose A, B are h-ideals of R such that $A \cap B = I$. Then $I \subseteq A$ and $I \subseteq B$. On the other hand by hypothesis $A \subseteq I$ or $B \subseteq I$. Thus $A = I$ or $B = I$. ∎

Proposition 227 *Let R be a right h-weakly regular hemiring. Then any proper right pure h-ideal of R is contained in a purely prime h-ideal of R.*

Proof. Let I be a proper right pure h-ideal of an h-weakly regular hemiring R and $a \in R$ such that $a \notin I$. Consider the set

$$X = \{J_p : J_p \text{ is a proper right pure } h\text{-ideal of } R \text{ such that } I \subseteq J_p \text{ and } a \notin J_p\}.$$

Then $X \neq \phi$ because $I \in X$. By Zorn's Lemma this family contains a maximal element, say M. This maximal element is purely prime. Indeed, let $A \cap B = M$ for some right pure h-ideals A, B of R. If A, B both properly contains M, then by the maximality of M, $a \in A$ and $a \in B$. Thus $a \in A \cap B = M$, which is a contradiction. Hence either $A = M$ or $B = M$. ∎

Proposition 228 *Let R be a right h-weakly regular hemiring. Then each proper right pure h-ideal is the intersection of all purely prime h-ideals of R which contain it.*

Proof. Proof is similar to the proof of Theorem 204. ∎

Proposition 229 *Let R be a right h-weakly regular hemiring. If λ is a right pure fuzzy h-ideal of R with $\lambda(a) = t$ where $a \in R$ and $t \in [0, 1]$, then there exists a purely prime fuzzy h-ideal μ of R such that $\lambda \leq \mu$ and $\mu(a) = t$.*

Proof. The proof is similar to the proof of Proposition 205. ∎

Proposition 230 *Let R be a right h-weakly regular hemiring. Then each proper fuzzy right pure h-ideal is the intersection of all purely prime fuzzy h-ideals of R which contain it.*

Proof. Proof is similar to the proof of Theorem 206. ∎

Chapter 6

k-regular and k-intra-regular hemirings

In this chapter we introduce the concepts of fuzzy k-bi-ideals and fuzzy k-quasi-ideals of hemirings. We characterized different classes of hemirings by the properties of k-bi-ideals and k-quasi-ideals.

6.1 k-quasi-ideals

A non-empty subset A of a hemiring R is called a k-quasi-ideal of R if A is closed under addition, $\widehat{RA} \cap \widehat{AR} \subseteq A$ and $x+a=b$ implies $x \in A$ for all $x \in R$ and $a, b \in A$.

A non-empty subset A of a hemiring R is called a k-bi-ideal of R if A is closed under addition and multiplication, $\widehat{ARA} \subseteq A$ and $x+a=b$ implies $x \in A$ for all $x \in R$ and $a, b \in A$.

Lemma 231 *Every left (right) k-ideal of a hemiring R is a k-quasi-ideal of R and every k-quasi-ideal of R is a k-bi-ideal of R.*

Proof. Straightforward. ∎

The converse of the above Lemma does not hold as shown in the following examples.

Example 232 *The set R of all 2×2 matrices $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is a hemiring with usual addition and multiplication of matrices, where $a_{ij} \in \mathbb{N}_o$, $\mathbb{N}_o$ is the set of all non-negative integers. Consider the set Q of all matrices of the form $\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$ $(a \in \mathbb{N}_o)$. Evidently Q is a k-quasi-ideal of R but not a left (right) k-ideal of R.*

Example 233 *Let $\mathbb{N}^+$ and $\mathbb{R}^+$ denote the sets of all positive integers and positive real numbers, respectively. The set R of of all matrices of the form* $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ $(a, b \in \mathbb{R}, c \in \mathbb{N})$ *together with* $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ *is a hemiring with respect to the usual addition and multiplication of matrices. Let A, B be the sets of all matrices* $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ $(a, b \in \mathbb{R}, c \in \mathbb{N}, a < b)$ *together with* $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} p & 0 \\ q & d \end{pmatrix}$ $(p, q \in \mathbb{R}, d \in \mathbb{N}, 3 < q)$ *together with* $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, *respectively. It is easy to show that A and B are right k-ideal and left k-ideal of R, respectively. Now the product AB is a k-bi-ideal of R but it is not a k-quasi-ideal of R. Indeed, the element*

$$\begin{pmatrix} 6 & 0 \\ 9 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 3 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \right) = \left(\begin{pmatrix} 1 & 0 \\ \frac{7}{6} & 1 \end{pmatrix} \begin{pmatrix} 24 & 0 \\ 4 & 1 \end{pmatrix} \right) \begin{pmatrix} \frac{1}{4} & 0 \\ 1 & 1 \end{pmatrix}$$

belongs to the intersection $\widehat{R(AB)} \cap \widehat{(AB)R}$, but it is not an element of AB. Hence $\widehat{R(AB)} \cap \widehat{(AB)R} \nsubseteq AB$.

Lemma 234 *Let Q_1 and Q_2 be k-quasi-ideals of a hemiring R, then $Q_1 \cap Q_2$ is a k-quasi-ideal of R.*

Proof. Straightforward. ■

Corollary 235 *If A is a left k-ideal and B is a right k-ideal of a hemiring R, then $A \cap B$ is a k-quasi-ideal of R.*

Definition 236 *A fuzzy subset λ of a hemiring R is called a fuzzy k-bi-ideal of R if for all $x, y, z \in R$ we have*

1. $\lambda(x + y) \geq \lambda(x) \wedge \lambda(y)$
2. $\lambda(xy) \geq \lambda(x) \wedge \lambda(y)$
3. $\lambda(xyz) \geq \lambda(x) \wedge \lambda(z)$
4. $x + y = z \Rightarrow \lambda(x) \geq \lambda(y) \wedge \lambda(z)$.

Theorem 237 *A fuzzy subset μ in a hemiring R is a fuzzy k-bi-ideal of R if and only if*

1. $\mu +_k \mu \leq \mu$
2. $\mu \odot_k \mu \leq \mu$
3. $\mu \odot_k \chi_R \odot_k \mu \leq \mu$.

Proof. Let μ be a fuzzy k-bi-ideal of R. By Theorem 79, μ satisfies (1).

Let $x \in R$. If $(\mu \odot_k \mu)(x) = 0$ then $\mu \odot_k \mu \leq \mu$. Otherwise, there exist $a_i, b_i, a_j^{'}, b_j^{'} \in R$ such that $x + \sum\limits_{i=1}^{m} a_i b_i = \sum\limits_{j=1}^{n} a_j^{'} b_j^{'}$. Then we have

$$\begin{aligned}
(\mu \odot_k \mu)(x) &= \bigvee_{x+\sum\limits_{i=1}^{m} a_i b_i = \sum\limits_{j=1}^{n} a_j^{'} b_j^{'}} \left[\bigwedge_{i=1}^{m} [\mu(a_i) \wedge \mu(b_i)] \wedge \bigwedge_{j=1}^{n} \left[\mu(a_j^{'}) \wedge \mu(b_j^{'})\right] \right] \\
&\leq \bigvee_{x+\sum\limits_{i=1}^{m} a_i b_i = \sum\limits_{j=1}^{n} a_j^{'} b_j^{'}} \left[\bigwedge_{i=1}^{m} [\mu(a_i b_i)] \wedge \bigwedge_{j=1}^{n} \left[\mu(a_j^{'} b_j^{'})\right] \right] \\
&\leq \bigvee_{x+\sum\limits_{i=1}^{m} a_i b_i = \sum\limits_{j=1}^{n} a_j^{'} b_j^{'}} \left[\mu(\sum_{i=1}^{m} a_i b_i) \wedge \mu(\sum_{j=1}^{n} a_j^{'} b_j^{'}) \right] \\
&\leq \mu(x).
\end{aligned}$$

Hence $\mu \odot_k \mu \leq \mu$.

Let $x \in R$. If $(\mu \odot_k \chi_R \odot_k \mu)(x) = 0$ then $\mu \odot_k \chi_R \odot_k \mu \leq \mu$. Otherwise, there exist $a_i, b_i, a_j^{'}, b_j^{'} \in R$ such that $x + \sum\limits_{i=1}^{m} a_i b_i = \sum\limits_{j=1}^{n} a_j^{'} b_j^{'}$. Then we have

$$(\mu \odot_k \chi_R \odot_k \mu)(x) = \bigvee_{x+\sum\limits_{i=1}^{m} a_i b_i = \sum\limits_{j=1}^{n} a_j^{'} b_j^{'}} \left[\begin{array}{c} \bigwedge\limits_{i=1}^{m} [(\mu \odot_k \chi_R)(a_i) \wedge \mu(b_i)] \wedge \\ \bigwedge\limits_{j=1}^{n} \left[(\mu \odot_k \chi_R)(a_j^{'}) \wedge \mu(b_j^{'})\right] \end{array} \right]$$

$$= \bigvee_{x+\sum\limits_{i=1}^{m} a_i b_i = \sum\limits_{j=1}^{n} a_j^{'} b_j^{'}} \left[\begin{array}{c} \bigwedge\limits_{i=1}^{m} \left[\bigvee\limits_{a_i+\sum\limits_{k=1}^{p} c_k d_k = \sum\limits_{q=1}^{r} c_q^{'} d_q^{'}} \left\{ \bigwedge\limits_{k=1}^{p} \mu(c_k) \wedge \bigwedge\limits_{q=1}^{r} \mu\left(c_q^{'}\right) \right\} \wedge \mu(b_i) \right] \wedge \\ \bigwedge\limits_{j=1}^{n} \left[\bigvee\limits_{a_j^{'}+\sum\limits_{l=1}^{s} e_l f_l = \sum\limits_{u=1}^{t} e_u^{'} f_u^{'}} \left\{ \bigwedge\limits_{l=1}^{s} \mu(e_l) \wedge \bigwedge\limits_{u=1}^{t} \mu\left(e_u^{'}\right) \right\} \wedge \mu\left(b_j^{'}\right) \right] \end{array} \right]$$

$$= \bigvee_{x+\sum_{i=1}^{m} a_ib_i=\sum_{j=1}^{n} a_j'b_j'} \left[\begin{array}{l} \bigwedge_{i=1}^{m} \left[\bigvee_{a_i+\sum_{k=1}^{p} c_kd_k=\sum_{q=1}^{r} c_q'd_q'} \left\{ \begin{array}{l} \bigwedge_{k=1}^{p} [\mu(c_k) \wedge \mu(b_i)] \wedge \\ \bigwedge_{q=1}^{r} \left[\mu\left(c_q'\right) \wedge \mu(b_i)\right] \end{array} \right\} \right] \wedge \\ \bigwedge_{j=1}^{n} \left[\bigvee_{a_j'+\sum_{l=1}^{s} e_lf_l=\sum_{u=1}^{t} e_u'f_u'} \left\{ \begin{array}{l} \bigwedge_{l=1}^{s} \left[\mu(e_l) \wedge \mu\left(b_j'\right)\right] \wedge \\ \bigwedge_{u=1}^{t} \left[\mu\left(e_u'\right) \wedge \mu\left(b_j'\right)\right] \end{array} \right\} \right] \end{array} \right]$$

Since $a_i + \sum_{k=1}^{p} c_kd_k = \sum_{q=1}^{r} c_q'd_q'$ and $a_j' + \sum_{l=1}^{s} e_lf_l = \sum_{u=1}^{t} e_u'f_u'$ so,

$a_ib_i + \sum_{k=1}^{p} c_kd_kb_i = \sum_{q=1}^{r} c_q'd_q'b_i$ and $a_j'b_j' + \sum_{l=1}^{s} e_lf_lb_j' = \sum_{u=1}^{t} e_u'f_u'b_j'$

$(\mu \odot_k \chi_R \odot_k \mu)(x)$

$$\leq \bigvee_{x+\sum_{i=1}^{m} a_ib_i=\sum_{j=1}^{n} a_j'b_j'} \left[\begin{array}{l} \bigwedge_{i=1}^{m} \left[\bigvee_{a_i+\sum_{k=1}^{p} c_kd_k=\sum_{q=1}^{r} c_q'd_q'} \left\{ \mu\left(\sum_{k=1}^{p} c_kd_kb_i\right) \wedge \mu\left(\sum_{q=1}^{r} c_q'd_q'b_i\right) \right\} \right] \wedge \\ \bigwedge_{j=1}^{n} \left[\bigvee_{a_j'+\sum_{l=1}^{s} e_lf_l=\sum_{u=1}^{t} e_u'f_u'} \left\{ \mu\left(\sum_{l=1}^{s} e_lf_lb_j'\right) \wedge \mu\left(\sum_{u=1}^{t} e_u'f_u'b_j'\right) \right\} \right] \end{array} \right]$$

Since μ is a fuzzy k-bi-ideal of R, so

$\mu(a_ib_i) \geq \mu\left(\sum_{k=1}^{p} c_kd_kb_i\right) \wedge \mu\left(\sum_{q=1}^{r} c_q'd_q'b_i\right)$ and

$\mu\left(a_j'b_j'\right) \geq \mu\left(\sum_{l=1}^{s} e_lf_lb_j'\right) \wedge \mu\left(\sum_{u=1}^{t} e_u'f_u'b_j'\right)$

Hence,

$$\begin{aligned} (\mu \odot_k \chi_R \odot_k \mu)(x) &\leq \bigvee_{x+\sum_{i=1}^{m} a_ib_i=\sum_{j=1}^{n} a_j'b_j'} \left[\bigwedge_{i=1}^{m} [\mu(a_ib_i)] \wedge \bigwedge_{j=1}^{n} \left[\mu\left(a_j'b_j'\right)\right] \right] \\ &= \bigvee_{x+\sum_{i=1}^{m} a_ib_i=\sum_{j=1}^{n} a_j'b_j'} \left[\mu\left(\sum_{i=1}^{m} a_ib_i\right) \wedge \mu\left(\sum_{j=1}^{n} a_j'b_j'\right) \right] \\ &\leq \mu(x). \end{aligned}$$

Hence $\mu \odot_k \chi_R \odot_k \mu \leq \mu$.

Conversely, let μ be a fuzzy subset of R. By Theorem 79, μ satisfies $\mu(x+y) \geq \mu(x) \wedge \mu(y)$ and $x + y = z \Rightarrow \mu(x) \geq \lambda(y) \wedge \lambda(z)$

Now

$$\begin{aligned}
\mu(xy) &\geq ((\mu \odot_k \chi_R) \odot_k \mu)(xy) \\
&= \bigvee_{xy+\sum_{i=1}^{m} a_i b_i = \sum_{j=1}^{n} a_j' b_j'} \left[\begin{array}{c} \bigwedge_{i=1}^{m} [(\mu \odot_k \chi_R)(a_i) \wedge \mu(b_i)] \wedge \\ \bigwedge_{j=1}^{n} \left[(\mu \odot_k \chi_R)(a_j') \wedge \mu(b_j')\right] \end{array} \right] \\
&\geq (\mu \odot_k \chi_R)(0) \wedge \mu(x) \wedge (\mu \odot_k \chi_R)(0) \wedge \mu(y) \\
&\quad (\text{since } \; xy + 00 = xy) \\
&= \bigvee_{0+\sum_{i=1}^{m} c_i d_i = \sum_{j=1}^{n} c_j' d_j'} \left[\bigwedge_{i=1}^{m} [\mu(c_i) \wedge \chi_R(d_i)] \wedge \bigwedge_{j=1}^{n} \left[\mu(c_j') \wedge \chi_R(d_j')\right] \right] \wedge \mu(x) \wedge \mu(y) \\
&\geq \mu(0) \wedge \mu(0) \wedge \mu(x) \wedge \mu(y) \quad (\text{ since } 0 + 00 = 00) \\
&= \mu(x) \wedge \mu(y)
\end{aligned}$$

Similarly we can show that $\mu(xyz) \geq \mu(x) \wedge \mu(z)$.

Hence μ is a fuzzy k-bi-ideal of R. ∎

Definition 238 *A fuzzy subset λ of a hemiring R is called a fuzzy k-quasi-ideal if for all $x, y, z \in R$ we have*

1. $\lambda(x + y) \geq \lambda(x) \wedge \lambda(y)$,
2. $(\lambda \odot_k \chi_R) \wedge (\chi_R \odot_k \lambda) \leq \lambda$,
3. $x + y = z \Rightarrow \lambda(x) \geq \lambda(y) \wedge \lambda(z)$.

Every fuzzy left (right) k-ideal of R is a fuzzy k-quasi-ideal of R and every fuzzy k-quasi-ideal of R is a fuzzy k-bi-ideal of R. But the converse does not hold.

Theorem 239 *A fuzzy subset λ of a hemiring R is a fuzzy k-bi-ideal of R if and only if each non-empty level subset $U(\lambda; t)$ of λ is a k-bi-ideal of R.*

Proof. Suppose λ is a fuzzy k-bi-ideal of R and $t \in (0, 1]$ such that $U(\lambda; t) \neq \phi$. Let $a, b \in U(\lambda; t)$, then $\lambda(a) \geq t$ and $\lambda(b) \geq t$. As $\lambda(a + b) \geq \lambda(a) \wedge \lambda(b)$, so $\lambda(a + b) \geq t$. Hence $a + b \in U(\lambda; t)$. Also, $\lambda(ab) \geq \lambda(a) \wedge \lambda(b)$ so $\lambda(ab) \geq t$. This implies $ab \in U(\lambda; t)$.

Let $x \in \overbrace{U(\lambda; t) RU(\lambda; t)}$, then $x + \sum_{i=1}^{m} a_i r_i b_i = \sum_{j=1}^{n} a_j' r_j' b_j'$, for $a_i, b_i, a_j', b_j' \in U(\lambda; t)$ and $r_i, r_j' \in R$. Since λ is a fuzzy k-bi-ideal of R, so $\lambda(a_i r_i b_i) \geq \lambda(a_i) \wedge \lambda(b_i) \geq t \Rightarrow a_i r_i b_i \in U(\lambda; t)$. Hence $\sum_{i=1}^{m} a_i r_i b_i \in U(\lambda; t)$. Similarly $\sum_{j=1}^{n} a_j' r_j' b_j' \in U(\lambda; t)$. Hence $x \in U(\lambda; t)$. Thus $\overbrace{U(\lambda; t) RU(\lambda; t)} \subseteq U(\lambda; t)$.

Now let $x+a=b$ for some $a,b\in U(\lambda;t)$, then $\lambda(a)\geq t$ and $\lambda(b)\geq t$. Since $\lambda(x)\geq\lambda(a)\wedge\lambda(b)$, so $\lambda(x)\geq t$. Hence $x\in U(\lambda;t)$. Thus $U(\lambda;t)$ is a k-bi-ideal of R.

Conversely, assume that each non-empty subset $U(\lambda;t)$ of R is a k-bi-ideal of R. Let $a,b\in R$ be such that $\lambda(a+b)<\lambda(a)\wedge\lambda(b)$. Take $t\in(0,1]$ such that $\lambda(a+b)<t\leq\lambda(a)\wedge\lambda(b)$, then $a,b\in U(\lambda;t)$ but $a+b\notin U(\lambda;t)$, a contradiction. Hence $\lambda(a+b)\geq\lambda(a)\wedge\lambda(b)$.

Similarly we can show that $\lambda(ab)\geq\lambda(a)\wedge\lambda(b)$ and $\lambda(abc)\geq\lambda(a)\wedge\lambda(c)$.

Let $x,y,z\in R$ be such that $x+y=z$. If possible let $\lambda(x)<\lambda(y)\wedge\lambda(z)$. Take $t\in(0,1]$ such that $\lambda(x)<t\leq\lambda(y)\wedge\lambda(z)$, then $y,z\in U(\lambda;t)$ but $x\notin U(\lambda;t)$, a contradiction. Hence $\lambda(x)\geq\lambda(y)\wedge\lambda(z)$. Thus λ is a fuzzy k-bi-ideal of R. ■

Corollary 240 *Let A be a non-empty subset of a hemiring R. Then A is a k-bi-ideal of R if and only if the characteristic function χ_A of A is a fuzzy k-bi-ideal of R.*

Theorem 241 *A fuzzy subset λ of a hemiring R is a fuzzy k-quasi-ideal of R if and only if each non-empty level subset $U(\lambda;t)$ of λ is a k-quasi-ideal of R.*

Proof. Suppose λ is a fuzzy k-quasi-ideal of R and $t\in(0,1]$ such that $U(\lambda;t)\neq\phi$. Let $a,b\in U(\lambda;t)$, then $\lambda(a)\geq t$ and $\lambda(b)\geq t$. As $\lambda(a+b)\geq\lambda(a)\wedge\lambda(b)$, so $\lambda(a+b)\geq t$. Hence $a+b\in U(\lambda;t)$.

Let $x\in\widehat{U(\lambda;t)R}\cap\widehat{RU(\lambda;t)}$ then $x\in\widehat{U(\lambda;t)R}$ and $x\in\widehat{RU(\lambda;t)}$. Then $x+\sum_{i=1}^{m}u_ir_i=\sum_{j=1}^{n}u_j'r_j'$ and $x+\sum_{k=1}^{p}v_ks_k=\sum_{l=1}^{q}v_l's_l'$, for some $u_i,u_j',s_k,s_l'\in U(\lambda;t)$ and $r_i,r_j',v_k,v_l'\in R$. Now

$$\begin{aligned}\lambda(x)&\geq[(\lambda\odot_k\chi_R)\wedge(\chi_R\odot_k\lambda)](x)\\&=(\lambda\odot_k\chi_R)(x)\wedge(\chi_R\odot_k\lambda)(x)\\&=\bigvee_{x+\sum_{i=1}^{m}u_ir_i=\sum_{j=1}^{n}u_j'r_j'}\left[\lambda(u_i)\wedge\lambda\left(u_j'\right)\right]\wedge\bigvee_{x+\sum_{k=1}^{p}v_ks_k=\sum_{l=1}^{q}v_l's_l'}\left[\lambda(s_k)\wedge\lambda\left(s_l'\right)\right]\\&\geq t\wedge t=t\end{aligned}$$

So, $\lambda(x)\geq t$. Thus, $x\in U(\lambda;t)$. Hence $\widehat{U(\lambda;t)R}\cap\widehat{RU(\lambda;t)}\subseteq U(\lambda;t)$.

Now let $x+a=b$ for some $a,b\in U(\lambda;t)$, then $\lambda(a)\geq t$ and $\lambda(b)\geq t$. Since $\lambda(x)\geq\lambda(a)\wedge\lambda(b)$, so $\lambda(x)\geq t$. Hence $x\in U(\lambda;t)$. Thus $U(\lambda;t)$ is a k-quasi-ideal of R.

Conversely, assume that each non-empty subset $U(\lambda;t)$ of R is a k-quasi-ideal of R. Let $a,b\in R$ such that $\lambda(a+b)<\lambda(a)\wedge\lambda(b)$. Take $t\in(0,1]$ such that $\lambda(a+b)<t\leq\lambda(a)\wedge\lambda(b)$, then $a,b\in U(\lambda;t)$ but $a+b\notin U(\lambda;t)$, a contradiction. Hence $\lambda(a+b)\geq\lambda(a)\wedge\lambda(b)$.

Let $x \in R$. If possible let $\lambda(x) < [(\lambda \odot_k \chi_R) \wedge (\chi_R \odot_k \lambda)](x)$. Take $t \in (0,1]$ such that $\lambda(x) < t \leq [(\lambda \odot_k \chi_R) \wedge (\chi_R \odot_k \lambda)](x)$. If $[(\lambda \odot_k \chi_R) \wedge (\chi_R \odot_k \lambda)](x) \geq t$ then there exist expression forms $x + \sum_{i=1}^{m} u_i r_i = \sum_{j=1}^{n} u_j' r_j'$ and $x + \sum_{k=1}^{p} v_k s_k = \sum_{l=1}^{q} v_l' s_l'$. Now,

$$(\lambda \odot_k \chi_R)(x) \wedge (\chi_R \odot_k \lambda)(x) = \left[\bigvee_{x+\sum_{i=1}^{m} u_i r_i = \sum_{j=1}^{n} u_j' r_j'} \left[\lambda(u_i) \wedge \lambda\left(u_j'\right)\right] \right] \wedge \left[\bigvee_{x+\sum_{k=1}^{p} v_k s_k = \sum_{l=1}^{q} v_l' s_l'} \left[\lambda(s_k) \wedge \lambda\left(s_l'\right)\right] \right]$$

Hence $\bigvee_{x+\sum_{i=1}^{m} u_i r_i = \sum_{j=1}^{n} u_j' r_j'} \left[\lambda(u_i) \wedge \lambda\left(u_j'\right)\right] \geq t$ and $\bigvee_{x+\sum_{k=1}^{p} v_k s_k = \sum_{l=1}^{q} v_l' s_l'} \left[\lambda(s_k) \wedge \lambda\left(s_l'\right)\right] \geq t$, so, $\lambda(u_i) \geq t, \lambda\left(u_j'\right) \geq t, \lambda(s_k) \geq t, \lambda\left(s_l'\right) \geq t$, that is, $u_i, u_j', s_k, s_l' \in U(\lambda;t)$. Since $U(\lambda;t)$ is a k-quasi-ideal of R, so $\sum_{i=1}^{m} u_i r_i, \sum_{j=1}^{n} u_j' r_j' \in U(\lambda;t)R$ and $\sum_{k=1}^{p} v_k s_k, \sum_{l=1}^{q} v_l' s_l' \in RU(\lambda;t)$. This implies $x \in \widehat{U(\lambda;t)R}$ and $x \in \widehat{RU(\lambda;t)}$. Hence $x \in \widehat{U(\lambda;t)R} \cap \widehat{RU(\lambda;t)} \subseteq U(\lambda;t)$, and hence $x \in U(\lambda;t)$, that is $\lambda(x) \geq t$, a contradiction. Hence $(\lambda \odot_k \chi_R) \wedge (\chi_R \odot_k \lambda) \leq \lambda$. Let $x, y, z \in R$ such that $x + y = z$. If possible let $\lambda(x) < \lambda(y) \wedge \lambda(z)$. Take $t \in (0,1]$ such that $\lambda(x) < t \leq \lambda(y) \wedge \lambda(z)$, then $y, z \in U(\lambda;t)$ but $x \notin U(\lambda;t)$, a contradiction. Hence $\lambda(x) \geq \lambda(y) \wedge \lambda(z)$. Thus λ is a fuzzy k-quasi-ideal of R. ■

Corollary 242 *Let A be a non-empty subset of a hemiring R. Then A is a k-quasi-ideal of R if and only if the characteristic function χ_A of A is a fuzzy k-quasi-ideal of R.*

Proposition 243 *The intersection of fuzzy k-quasi-ideals of a hemiring R is a fuzzy k-quasi-ideal of R.*

Proof. Let μ, ν be fuzzy k-quasi-ideals of hemiring R.
Let $x, y \in R$. Then

$$\begin{aligned}(\mu \wedge \nu)(x+y) &= \mu(x+y) \wedge \nu(x+y) \geq [\mu(x) \wedge \mu(y)] \wedge [\nu(x) \wedge \nu(y)] \\ &= [\mu(x) \wedge \nu(x)] \wedge [\mu(y) \wedge \nu(y)] = (\mu \wedge \nu)(x) \wedge (\mu \wedge \nu)(y)\end{aligned}$$

Now let $a, b, x \in R$ such that $x + a = b$. Then

$$\begin{aligned}(\mu \wedge \nu)(x) &= \mu(x) \wedge \nu(x) \geq [\mu(a) \wedge \mu(b)] \wedge [\nu(a) \wedge \nu(b)] \\ &= [\mu(a) \wedge \nu(a)] \wedge [\mu(b) \wedge \nu(b)] = (\mu \wedge \nu)(a) \wedge (\mu \wedge \nu)(b)\end{aligned}$$

Also,

$$\begin{aligned}((\mu \wedge \nu) \odot_k \chi_R) \wedge (\chi_R \odot_k (\mu \wedge \nu)) &= ((\mu \wedge \nu) \odot_k \chi_R) \wedge (\chi_R \odot_k (\mu \wedge \nu)) \\ &\quad \wedge ((\mu \wedge \nu) \odot_k \chi_R) \wedge (\chi_R \odot_k (\mu \wedge \nu)) \\ &\leq (\mu \odot_k \chi_R) \wedge (\chi_R \odot_k \nu) \wedge (\nu \odot_k \chi_R) \wedge (\chi_R \odot_k \mu) \\ &= [(\mu \odot_k \chi_R) \wedge (\chi_R \odot_k \mu)] \wedge [(\nu \odot_k \chi_R) \wedge (\chi_R \odot_k \nu)] \\ &\leq \mu \wedge \nu.\end{aligned}$$

This completes the proof. ■

Corollary 244 *Let μ and ν be fuzzy right k-ideal and fuzzy left k-ideal of a hemiring R, respectively. Then $\mu \wedge \nu$ is a fuzzy k-quasi-ideal of R.*

6.2 k-regular hemirings

Theorem 245 *Let R be a hemiring. Then the following assertions are equivalent:*

1. R is k-regular
2. $B = \widehat{BRB}$ for every k-bi-ideal B of R
3. $Q = \widehat{QRQ}$ for every k-quasi-ideal of R.

Proof. (1) $\Rightarrow$ (2) Let R be a k-regular hemiring and B be any k-bi-ideal of R. For $x \in B$ there exist $a, a' \in R$ such that $x + xax = xa'x$. Then $xax, xa'x \in BRB$ and so $x \in \widehat{BRB}$. This implies $B \subseteq \widehat{BRB}$. On the other hand, since B is a k-bi-ideal of R, we have $\widehat{BRB} \subseteq B$. Thus $B = \widehat{BRB}$.

(2) $\Rightarrow$ (3) Straight forward.

(3) $\Rightarrow$ (1) Let B and A be any right k-ideal and left k-ideal of R , respectively. Then $B \cap A$ is a k-quasi-ideal of R.

By the hypothesis we have

$$B \cap A = \widehat{(B \cap A) R (B \cap A)} \subseteq \widehat{BRA} \subseteq \widehat{BA}$$

Also, $\widehat{BA} \subseteq B \cap A$ always.

So $\widehat{BA} = B \cap A$. Thus by Theorem 42 R is k-regular hemiring. ■

Now we prove the fuzzy version of the above Theorem.

Theorem 246 *Let R be a hemiring. Then the following assertions are equivalent:*

1. R is k-regular.

2. $\mu \leq \mu \odot_k \chi_R \odot_k \mu$ for every fuzzy k-bi-ideal μ of R.

3. $\mu \leq \mu \odot_k \chi_R \odot_k \mu$ for every fuzzy k-quasi-ideal μ of R.

Proof. (1) $\Rightarrow$ (2) Let R be a k-regular hemiring. Let μ be any fuzzy k-bi-ideal of R and $x \in R$. Since R is k-regular, there exist $a, a' \in R$ such that $x + xax = xa'x$. Thus we have

$$\begin{aligned}
(\mu \odot_k \chi_R \odot_k \mu)(x) &= \bigvee_{x+\sum_{i=1}^{m} a_i b_i = \sum_{j=1}^{n} a_j' b_j'} \left[\begin{array}{c} \bigwedge_{i=1}^{m} [(\mu \odot_k \chi_R)(a_i) \wedge \mu(b_i)] \wedge \\ \bigwedge_{j=1}^{n} \left[(\mu \odot_k \chi_R)(a_j') \wedge \mu(b_j')\right] \end{array} \right] \\
&\geq (\mu \odot_k \chi_R)(xa) \wedge (\mu \odot_k \chi_R)\left(xa'\right) \wedge \mu(x) \\
&= \left\{ \bigvee_{xa+\sum_{i=1}^{m} c_i d_i = \sum_{j=1}^{n} c_j' d_j'} \left[\bigwedge_{i=1}^{m} [\mu(c_i)] \wedge \bigwedge_{j=1}^{n} \left[\mu(c_j')\right] \right] \right\} \\
&\quad \wedge \left\{ \bigvee_{xa'+\sum_{i=1}^{m} l_i f_i = \sum_{j=1}^{n} l_j' f_j'} \left[\bigwedge_{i=1}^{m} [\mu(l_i)] \wedge \bigwedge_{j=1}^{n} \left[\mu(l_j')\right] \right] \right\} \wedge \mu(x) \\
&\geq \left[\mu(xax) \wedge \mu\left(xa'x\right)\right] \wedge \left[\mu(xax) \wedge \mu\left(xa'x\right)\right] \wedge \mu(x) \\
&\quad \left(\begin{array}{c} \text{because } x + xax = xa'x \text{ implies } xa + (xax)a = \left(xa'x\right)a \\ \text{and } xa' + (xax)a' = \left(xa'x\right)a' \end{array} \right) \\
&\geq \mu(x) \wedge \mu(x) \wedge \mu(x) = \mu(x).
\end{aligned}$$

This implies that $\mu \leq \mu \odot_k \chi_R \odot_k \mu$.

(2) $\Rightarrow$ (3) Straightforward.

(3) $\Rightarrow$ (1) Let λ, μ be fuzzy right k-ideal and fuzzy left k-ideal of R, respectively. Then $\lambda \wedge \mu$ is a fuzzy k-quasi-ideal of R. Hence by hypothesis

$$\begin{aligned}
\lambda \wedge \mu &\leq (\lambda \wedge \mu) \odot_k \chi_R \odot_k (\lambda \wedge \mu) \\
&\leq \lambda \odot_k \chi_R \odot_k \mu \\
&\leq \lambda \odot_k \mu
\end{aligned}$$

But $\lambda \odot_k \mu \leq \lambda \wedge \mu$ always. Hence $\lambda \odot_k \mu = \lambda \wedge \mu$.
Thus by Theorem 81 R is k-regular hemiring. ∎

Theorem 247 *Let R be a hemiring. Then the following assertions are equivalent:*

1. R is k-regular.

2. $\mu \wedge \nu \leq \mu \odot_k \nu \odot_k \mu$ for every fuzzy k-bi-ideal μ and every fuzzy k-ideal ν of R.

3. $\mu \wedge \nu \leq \mu \odot_k \nu \odot_k \mu$ for every fuzzy k-quasi-ideal μ and every fuzzy k-ideal ν of R.

Proof. (1) $\Rightarrow$ (2) Let μ and ν be any fuzzy k-bi-ideal and fuzzy k-ideal of R, respectively. For $x \in R$, there exist $a, a' \in R$ such that $x + xax = xa'x$. Thus we have

$$\begin{aligned}
(\mu \odot_k \nu \odot_k \mu)(x) &= \bigvee_{x+\sum_{i=1}^{m} a_i b_i = \sum_{j=1}^{n} a'_j b'_j} \left[\begin{array}{c} \bigwedge_{i=1}^{m} [(\mu \odot_k \nu)(a_i) \wedge \mu(b_i)] \\ \wedge \bigwedge_{j=1}^{n} \left[(\mu \odot_k \nu)(a'_j) \wedge \mu(b'_j) \right] \end{array} \right] \\
&\geq (\mu \odot_k \nu)(xa) \wedge (\mu \odot_k \nu)\left(xa'\right) \wedge \mu(x) \\
&\left\{ \bigvee_{xa+\sum_{i=1}^{m} c_i d_i = \sum_{j=1}^{n} c'_j d'_j} \left[\bigwedge_{i=1}^{m} [\mu(c_i) \wedge \nu(d_i)] \wedge \bigwedge_{j=1}^{n} \left[\mu(c'_j) \wedge \nu(d'_j) \right] \right] \right\} \\
&\wedge \left\{ \bigvee_{xa'+\sum_{i=1}^{m} l_i f_i = \sum_{j=1}^{n} l'_j f'_j} \left[\bigwedge_{i=1}^{m} [\mu(l_i) \wedge \nu(f_i)] \wedge \bigwedge_{j=1}^{n} \left[\mu(l'_j) \wedge \nu(f'_j) \right] \right] \right\} \wedge \mu(x) \\
&\geq \left[\mu(x) \wedge \nu(axa) \wedge \nu\left(a'xa\right) \right] \wedge \left[\mu(x) \wedge \nu\left(axa'\right) \wedge \nu\left(a'xa'\right) \right] \wedge \mu(x) \\
&\left(\begin{array}{c} \text{because } x + xax = xa'x \text{ implies } xa + xaxa = xa'xa \\ \text{and } xa' + xaxa' = xa'xa' \end{array} \right) \\
&\geq \mu(x) \wedge \nu(x) \wedge \mu(x) \wedge \nu(x) \\
&= \mu(x) \wedge \nu(x) = (\mu \wedge \nu)(x)
\end{aligned}$$

This implies that $\mu \wedge \nu \leq \mu \odot_k \nu \odot_k \mu$.

(2) $\Rightarrow$ (3) Straightforward.

(3) $\Rightarrow$ (1) Let μ be any fuzzy k-quasi-ideal of R. Since χ_R is a fuzzy k-ideal of R, we have by hypothesis $\mu = \mu \wedge \chi_R \leq \mu \odot_k \chi_R \odot_k \mu$. Therefore by Theorem 246, R is k-regular. ■

Corollary 248 *Let R be a hemiring. Then the following assertions are equivalent:*

1. R is k-regular.

2. $B \cap A = \widehat{BAB}$ for every k-bi-ideal B and every k-ideal A of R.

3. $Q \cap A = \widehat{QAQ}$ for every k-quasi-ideal Q and every k-ideal A of R.

Proof. (1) $\Rightarrow$ (2) Let B and A be any k-bi-ideal and any k-ideal of R, respectively. Then the characteristic functions χ_B and χ_A of B and A are fuzzy k-bi-ideal and fuzzy k-ideal of R, respectively. Thus by Theorem 247, we have

$$\chi_{B\cap A} = \chi_B \wedge \chi_A \leq \chi_B \odot_k \chi_A \odot_k \chi_B = \chi_{\widehat{BAB}}$$

This implies that $B \cap A \subseteq \widehat{BAB}$. On the other hand, since B and A are k-bi-ideal and k-ideal of R, respectively, we have $\widehat{BAB} \subseteq \widehat{BRB} \subseteq \widehat{B} = B$ and $\widehat{BAB} \subseteq \widehat{RAR} \subseteq \widehat{A} = A$. Thus $\widehat{BAB} \subseteq B \cap A$ and so $B \cap A = \widehat{BAB}$.

(2) $\Rightarrow$ (3) Straight forward.

(3) $\Rightarrow$ (1) Let Q be any k-quasi-ideal of R. Since R is itself a k-ideal of R, we have $Q = Q \cap R = \widehat{QRQ}$. Therefore by Theorem 245 R is k-regular. ■

Theorem 249 *The following assertions are equivalent for a hemiring R:*

1. R is k-regular.

2. $\mu \wedge \nu \leq \mu \odot_k \nu$ for every fuzzy k-bi-ideal μ and every fuzzy left k-ideal ν of R.

3. $\mu \wedge \nu \leq \mu \odot_k \nu$ for every fuzzy k-quasi-ideal μ and every fuzzy left k-ideal ν of R.

4. $\mu \wedge \nu \leq \mu \odot_k \nu$ for every fuzzy right k-ideal μ and every fuzzy k-bi-ideal ν of R.

5. $\mu \wedge \nu \leq \mu \odot_k \nu$ for every fuzzy right k-ideal μ and every fuzzy k-quasi-ideal ν of R.

6. $\mu \wedge \nu \wedge \omega \leq \mu \odot_k \nu \odot_k \omega$ for every fuzzy right k-ideal μ, every fuzzy k-bi-ideal ν and every fuzzy left k-ideal ω of R.

7. $\mu \wedge \nu \wedge \omega \leq \mu \odot_k \nu \odot_k \omega$ for every fuzzy right k-ideal μ, every fuzzy k-quasi-ideal ν and every fuzzy left k-ideal ω of R.

Proof. (1) $\Rightarrow$ (2) Let μ and ν be any fuzzy k-bi-ideal and fuzzy left k-ideal of R, respectively. Now let $x \in R$. Since R is k-regular, there exist $a, a' \in R$ such that $x + xax = xa'x$. Then we have

$$(\mu \odot_k \nu)(x) = \bigvee_{x+\sum\limits_{i=1}^{m} a_i b_i = \sum\limits_{j=1}^{n} a'_j b'_j} \left[\bigwedge_{i=1}^{m} [\mu(a_i) \wedge \nu(b_i)] \wedge \bigwedge_{j=1}^{n} \left[\mu(a'_j) \wedge \nu(b'_j)\right] \right]$$

$$\geq \mu(x) \wedge \nu(ax) \wedge \nu\left(a'x\right) \geq \mu(x) \wedge \nu(x) = (\mu \wedge \nu)(x)$$

This implies that $\mu \wedge \nu \leq \mu \odot_k \nu$.

(2) $\Rightarrow$ (3) Straightforward.

(3) $\Rightarrow$ (1) Let A and B be any right k-ideal and left k-ideal of R, respectively. Then A is a k-quasi-ideal of R. The characteristic functions χ_A and χ_B of A and B are fuzzy k-quasi-ideal and fuzzy left k-ideal of R, respectively. By the assumption we have $\chi_A \wedge \chi_B \leq \chi_A \odot_k \chi_B$. This implies that, $\chi_{A\cap B} \leq \chi_{\widehat{AB}}$ that is $A \cap B \subseteq \widehat{AB}$. But $\widehat{AB} \subseteq A \cap B$ always. Thus $\widehat{AB} = A \cap B$. Therefore by Theorem 42 R is k-regular.

Similarly we can prove that (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5).

(1) $\Rightarrow$ (6) Let μ, ν, ω be any fuzzy right k-ideal, any fuzzy k-bi-ideal and any fuzzy left k-ideal of R, respectively. Now let $x \in R$. Since R is k-regular, there exist $a, a' \in R$ such that $x + xax = xa'x$. Then we have

$$(\mu \odot_k \nu \odot_k \omega)(x) = \bigvee_{x+\sum_{i=1}^{m} a_i b_i = \sum_{j=1}^{n} a_j' b_j'} \left[\begin{array}{c} \bigwedge_{i=1}^{m} [(\mu \odot_k \nu)(a_i) \wedge \omega(b_i)] \\ \wedge \bigwedge_{j=1}^{n} \left[(\mu \odot_k \nu)(a_j') \wedge \omega(b_j')\right] \end{array} \right]$$

$$\geq (\mu \odot_k \nu)(x) \wedge \omega(ax) \wedge \omega(a'x)$$

$$= \bigvee_{x+\sum_{i=1}^{m} a_i b_i = \sum_{j=1}^{n} a_j' b_j'} \left[\bigwedge_{i=1}^{m} [\mu(a_i) \wedge \nu(b_i)] \wedge \bigwedge_{j=1}^{n} \left[\mu(a_j') \wedge \nu(b_j')\right] \right]$$

$$\wedge \omega(ax) \wedge \omega(a'x)$$

$$\geq \mu(xa) \wedge \mu\left(xa'\right) \wedge \nu(x) \wedge \omega(ax) \wedge \omega(a'x)$$

$$\geq \mu(x) \wedge \nu(x) \wedge \omega(x) = (\mu \wedge \nu \wedge \omega)(x)$$

So, $\mu \wedge \nu \wedge \omega \leq \mu \odot_k \nu \odot_k \omega$.

(6) $\Rightarrow$ (7) Straightforward.

(7) $\Rightarrow$ (1) Let A and B be any right k-ideal and left k-ideal of R, respectively. The characteristic functions χ_A and χ_B of A and B are fuzzy right k-ideal and fuzzy left k-ideal of R, respectively. Since χ_R is a fuzzy k-quasi-ideal of R, by assumption, we have

$$\chi_A \wedge \chi_R \wedge \chi_B \leq \chi_A \odot \chi_R \odot \chi_B \Rightarrow \chi_{A\cap R\cap B} \leq \chi_{\widehat{ARB}}$$

This implies $A \cap R \cap B \subseteq \widehat{ARB}$. Thus $A \cap B \subseteq \widehat{AB}$. But $A \cap B \supseteq \widehat{AB}$ is always true. Therefore $A \cap B = \widehat{AB}$. Hence R is k-regular ■

Corollary 250 *Let R be a hemiring then the following conditions are equivalent:*

1. R is k-regular.

2. $B \cap C \subseteq \widehat{BC}$ for every k-bi-ideal B and every left k-ideal C.

3. $Q \cap C \subseteq \widehat{QC}$ for every k-quasi-ideal Q and every left k-ideal C.

4. $A \cap B \subseteq \widehat{AB}$ for every right k-ideal A and for every k-bi-ideal B of R.

5. $A \cap Q \subseteq \widehat{AQ}$ for every right k-ideal A and for every k-quasi-ideal Q of R.

6. $A \cap B \cap C \subseteq \widehat{ABC}$ for every right k-ideal A, every k-bi-ideal B and every left k-ideal C of R.

7. $A \cap Q \cap C \subseteq \widehat{AQC}$ for every right k-ideal A, every k-quasi-ideal Q and every left k-ideal C of R.

Proof. (1) $\Rightarrow$ (2) Let B and C be any k-bi-ideal and left k-ideal of R, respectively. Then characteristic functions χ_B and χ_C of B and C are fuzzy k-bi-ideal and fuzzy left k-ideal of R, respectively. By Theorem 249, we have

$$\chi_{B \cap C} = \chi_B \wedge \chi_C \leq \chi_B \odot_k \chi_C = \chi_{\widehat{BC}}$$

This implies $B \cap C \subseteq \widehat{BC}$.

(2) $\Rightarrow$ (3) Straight forward.

(3) $\Rightarrow$ (1) Let A and B be any right k-ideal and left k-ideal of R, respectively. As A is a k-quasi-ideal of R. By the assumption $A \cap B \subseteq \widehat{AB}$. But $\widehat{AB} \subseteq A \cap B$ always. Therefore $A \cap B = \widehat{AB}$. Hence R is k-regular.

Similarly we can show that (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) and (1) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7). ∎

Lemma 251 *A hemiring R is k-regular if and only if the right and left k-ideals of R are idempotent and for any right k-ideal A and any left k-ideal B of R, the set $\widehat{AB}$ is a k-quasi-ideal of R.*

Proof. Suppose R is k-regular hemiring. Let A and B be right k-ideal and left k-ideal of R, respectively. Then we have $\widehat{A^2} \subseteq \widehat{A} = A$. Let $x \in A$. Since R is k-regular, there exist $a, a' \in R$ such that $x + xax = xa'x$. Since A is right k-ideal of R, so $xa, xa' \in A$ and so $xax, xa'x \in A^2$. This implies $x \in \widehat{A^2}$, that is, $A \subseteq \widehat{A^2}$. Thus $A = \widehat{A^2}$ and so A is idempotent. Similarly, we can show that B is idempotent. Since R is k-regular we have $A \cap B = \widehat{AB}$. As $A \cap B$ is k-quasi-ideal of R so $\widehat{AB}$ is a k-quasi-ideal of R.

Conversely, let Q be a k-quasi-ideal of R. Then it is easy to check that $Q+RQ$ is a left ideal of R. This implies $\widehat{Q+RQ}$ is a left k-ideal of R. Thus by the assumption, we have

$$Q \subseteq \widehat{Q+RQ} = \widehat{(Q+RQ)(Q+RQ)} = \widehat{Q^2+QRQ+RQ^2+RQRQ}$$
$$\subseteq \widehat{RQ+RQ+RQ+RQ} = \widehat{RQ}$$

That is $Q \subseteq \widehat{RQ}$. Similarly, we can show that $Q \subseteq \widehat{QR}$.

Thus $Q \subseteq \widehat{RQ} \cap \widehat{QR} \subseteq Q$. So $Q = \widehat{QR} \cap \widehat{RQ}$.

On the other hand, it is clear that $\widehat{QR}$ and $\widehat{RQ}$ are right k-ideal and left k-ideal of R, respectively. Then, by the assumption, $\left(\widehat{QR}\right)^2 = \widehat{QR}$, $\left(\widehat{RQ}\right)^2 = \widehat{RQ}$ and the set $\widehat{(QR)(RQ)}$ is a k-quasi-ideal of R. Thus we have

$$Q = \widehat{QR} \cap \widehat{RQ} = \left(\widehat{QR}\right)^2 \cap \left(\widehat{RQ}\right)^2 = \widehat{(QR)(QR)} \cap \widehat{(RQ)(RQ)}$$
$$= \widehat{Q(RR)QR} \cap \widehat{RQ(RRQ)} = \widehat{(QR)(RQ)R} \cap \widehat{R(QR)(RQ)}$$
$$= \widehat{\widehat{(QR)(RQ)}\,\widehat{R}} \cap \widehat{\widehat{R}\,\widehat{(QR)(RQ)}} = \widehat{\widehat{(QR)(RQ)}R} \cap \widehat{R\widehat{(QR)(RQ)}}$$
$$\subseteq \widehat{\widehat{(QR)(RQ)}} = \widehat{QRRQ} \subseteq \widehat{QRQ} \subseteq Q$$

So, $Q = \widehat{QRQ}$ and so R is k-regular. ■

Theorem 252 *A hemiring R is k-regular if and only if the fuzzy right and fuzzy left k-ideals of R are idempotent and for any fuzzy right k-ideal μ and fuzzy left k-ideal ν of R, $\mu \odot_k \nu$ is a fuzzy k-quasi-ideal of R.*

Proof. Let R be k-regular and μ be a fuzzy right k-ideal of R. Then $\mu \odot_k \mu \leq \mu \odot_k \chi_R \leq \mu$.

Let $x \in R$. Since R is k-regular, there exist $a, a' \in R$ such that $x + xax = xa'x$. Then we have

$$(\mu \odot_k \mu)(x) = \bigvee_{x+\sum_{i=1}^{m} a_i b_i = \sum_{j=1}^{n} a'_j b'_j} \left[\bigwedge_{i=1}^{m} [\mu(a_i) \wedge \mu(b_i)] \wedge \bigwedge_{j=1}^{n} \left[\mu(a'_j) \wedge \mu(b'_j)\right]\right]$$
$$\geq \mu(xa) \wedge \mu(x) \wedge \mu\left(xa'\right) \geq \mu(x).$$

This implies that $(\mu \odot_k \mu) \geq \mu$. Hence $\mu = \mu \odot_k \mu$, so μ is idempotent. Similarly we can prove that every fuzzy left k-ideal of R is idempotent. Now let μ and ν be any fuzzy right k-ideal and fuzzy left k-ideal of R, respectively. By Theorem 81, we have $\mu \odot_k \nu = \mu \wedge \nu$ and it follows from Corollary 244, that $\mu \odot_k \nu$ is a fuzzy k-quasi-ideal of R.

Conversely, let A be a right k-ideal of R. Then χ_A, the characteristic function of

A, is a fuzzy right k-ideal of R. And $\chi_A = \chi_A \odot_k \chi_A = \chi_{\widehat{A^2}}$ implies that $A = \widehat{A^2}$, that is, A is idempotent. Similarly we can show that left k-ideals of R are idempotent. Now let A be a right ideal and B be a left k-ideal of R. Then $\chi_{\widehat{AB}} = \chi_A \odot_k \chi_B$ is a fuzzy k-quasi-ideal of R, that is, $\widehat{AB}$ is a k-quasi-ideal of R. Therefore by Lemma 251, R is k-regular. ■

6.3 k-intra-regular hemirings

Definition 253 *A hemiring R is said to be k-intra-regular if for each $x \in R$, there exists $a_i, a_i^{'}, b_j, b_j^{'} \in R$ such that $x + \sum\limits_{i=1}^{m} a_i x^2 b_i = \sum\limits_{j=1}^{n} a_j^{'} x^2 b_j^{'}$.*

Also we can define its equivalent definitions (1) $x \in \widehat{Rx^2R}$ $\forall x \in R$, (2) $A \subseteq \widehat{RA^2R}$ $\forall A \subseteq R$.

In the case of rings the k-intra-regularity coincides with the intra-regularitry of rings.

Example 254 *Let R be a hemiring defined by the following Caylay's table:*

$+$	0	x	1
0	0	x	1
x	x	x	x
1	1	x	1

$\cdot$	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

Then R is k-intra-regular hemiring.

Example 255 *The set $\mathbb{N}_o$ of all non-negative integers with usual addition "$+$" and multiplication "$\cdot$" is a hemiring, but it is not k-regular and k-intra-regular hemiring. Indeed $2 \in \mathbb{N}_o$ can not be written as $2 + 2a2 = 2a^{'}2$ or $2 + \sum\limits_{i=1}^{m} a_i 2^2 b_i = \sum\limits_{j=1}^{n} a_j^{'} 2^2 b_j^{'}$ for all $a_i, a_i^{'}, b_j, b_j^{'} \in \mathbb{N}_o$.*

Lemma 256 *The following assertions are equivalent for a hemiring R:*

1. R is k-intra-regular.
2. $A \cap B \subseteq \widehat{AB}$ for every left k-ideal A and every right k-ideal B of R.

Proof. (1) $\Rightarrow$ (2) Let A and B be any left k-ideal and right k-ideal of R, respectively. Since R is k-intra-regular, we have

$$A \cap B \subseteq \widehat{R(A\cap B)^2 R} = \widehat{(R(A\cap B))((A\cap B)R)} \subseteq \widehat{(RA)(BR)} \subseteq \widehat{AB}.$$

(2) $\Rightarrow$ (1) Let $a \in R$. Then it is easy to check that $\widehat{Rx+\mathbb{N}x}$ and $\widehat{xR+\mathbb{N}x}$, where $\mathbb{N} = \{0,1,2,.....\}$, are the principal left k-ideal and principal right k-ideal of R generated by x, respectively. By the hypothesis, we have

$$x \in \widehat{Rx+\mathbb{N}x} \cap \widehat{xR+\mathbb{N}x} \subseteq \widehat{\widehat{(Rx+\mathbb{N}x)}\ \widehat{(xR+\mathbb{N}x)}}$$
$$= \widehat{(Rx+\mathbb{N}x)(xR+\mathbb{N}x)} = \widehat{(RxxR)+(Rx\mathbb{N}x)+(\mathbb{N}xxR)+(\mathbb{N}x\mathbb{N}x)}$$
$$\subseteq \widehat{(RxxR)}$$

Thus we have $x + \sum_{i=1}^{m} a_i x^2 b_i = \sum_{j=1}^{n} a_j' x^2 b_j'$ for some $a_i, a_i', b_j, b_j' \in R$. Thus R is k-intra-regular. ■

Lemma 257 *Let R be a hemiring then the following conditions are equivalent.*

1. R is k-intra-regular.

2. $\mu \wedge \nu \leq \mu \odot_k \nu$ for every fuzzy left k-ideal μ and every fuzzy right k-ideal ν of R.

Proof. (1) $\Rightarrow$ (2) Let R be a k-intra-regular hemiring. Let μ and ν be any fuzzy left k-ideal and fuzzy right k-ideal of R, respectively. Now let $x \in R$. Since R is k-intra-regular, there exist $a_i, a_i', b_j, b_j' \in R$ such that

$$x + \sum_{i=1}^{m} a_i x^2 b_i = \sum_{j=1}^{n} a_j' x^2 b_j'$$

that is

$$x + \sum_{i=1}^{m} (a_i x)(x b_i) = \sum_{j=1}^{n} \left(a_j' x\right)\left(x b_j'\right)$$

Then we have

$$(\mu \odot_k \nu)(x) = \bigvee_{x+\sum_{i=1}^{m} a_i b_i = \sum_{j=1}^{n} a_j' b_j'} \left[\bigwedge_{i=1}^{m} [\mu(a_i) \wedge \nu(b_i)] \wedge \bigwedge_{j=1}^{n} \left[\mu(a_j') \wedge \nu(b_j')\right]\right]$$
$$\geq \bigwedge_{i=1}^{m} [\mu(a_i x) \wedge \nu(x b_i)] \wedge \bigwedge_{j=1}^{n} \left[\mu\left(a_j' x\right) \wedge \nu\left(x b_j'\right)\right]$$
$$\geq \mu(x) \wedge \nu(x) = (\mu \wedge \nu)(x)$$

This implies that $\mu \wedge \nu \leq \mu \odot_k \nu$.

(2) $\Rightarrow$ (1) Let A, B be left k-ideal and right k-ideal of R, respectively. Then χ_A and χ_B, the characteristic functions of A and B are fuzzy left k-ideal and fuzzy right k-ideal of R, respectively. Now, by the hypothesis,

$$\chi_{A\cap B} = \chi_A \wedge \chi_B \leq \chi_A \odot_k \chi_B = \chi_{\widehat{AB}}.$$

Thus $A \cap B \subseteq \widehat{AB}$. So R is k-intra-regular. ■

Theorem 258 *The following conditions are equivalent for a hemiring R.*

1. R is k-intra-regular.

2. For all $x \in R$, $\mu(x) = \mu(x^2)$ for all fuzzy k-ideals μ of R.

Proof. (1) $\Rightarrow$ (2) Assume that R is k-intra-regular. Let μ be a fuzzy k-ideal of R and $x \in R$. Since R is k-intra-regular, there exist $a_i, a_i^{'}, b_j, b_j^{'} \in R$ such that

$$x + \sum_{i=1}^{m} a_i x^2 b_i = \sum_{j=1}^{n} a_j^{'} x^2 b_j^{'}$$

Then we have
$\mu(x) \geq \left[\mu\left(\sum\limits_{i=1}^{m} a_i x^2 b_i\right)\right] \wedge \left[\mu\left(\sum\limits_{j=1}^{n} a_j^{'} x^2 b_j^{'}\right)\right]$
$\geq \left[\bigwedge_{i=1}^{m} \mu(a_i x^2 b_i)\right] \wedge \left[\bigwedge_{j=1}^{n} \mu\left(a_j^{'} x^2 b_j^{'}\right)\right]$
$\geq \mu(x^2) = \mu(xx) \geq \mu(x)$
implies that $\mu(x) = \mu(x^2)$.

(2) $\Rightarrow$ (1) Let $x \in R$. Then $\widehat{Nx^2 + Rx^2 + x^2R + Rx^2R}$ is the principal k-ideal of R generated by x^2, where $N = \{0, 1, 2, ...\}$. Now, the characteristic function $\chi_{\widehat{Nx^2 + Rx^2 + x^2R + Rx^2R}}$ of $\widehat{Nx^2 + Rx^2 + x^2R + Rx^2R}$ is a fuzzy k-ideal of R. Since

$$x^2 \in \widehat{Nx^2 + Rx^2 + x^2R + Rx^2R},$$

we have

$$\chi_{\widehat{Nx^2 + Rx^2 + x^2R + Rx^2R}}(x) = \chi_{\widehat{Nx^2 + Rx^2 + x^2R + Rx^2R}}(x^2) = 1,$$

so $x \in \widehat{Nx^2 + Rx^2 + x^2R + Rx^2R}$. Thus we have $x + \sum\limits_{i=1}^{m} a_i x^2 b_i = \sum\limits_{j=1}^{n} a_j^{'} x^2 b_j^{'}$ for some $a_i, a_i^{'}, b_j, b_j^{'} \in R$. Therefore R is k-intra-regular. ■

Lemma 259 *The following assertions are equivalent for a hemiring R:*

1. R is both k-regular and k-intra-regular.

2. $B = \widehat{B^2}$ for every k-bi-ideal B of R.

3. $Q = \widehat{Q^2}$ for every k-quasi-ideal Q of R.

Proof. (1) $\Rightarrow$ (2) Let R be both k-regular and k-intra-regular. Let B be a k-bi-ideal of R and $x \in B$. Then $\widehat{B^2} \subseteq \widehat{B} = B$. Since R is both k-regular and k-intra-regular, there exist elements $p_1, p_2, a_i, a_i^{'}, b_j, b_j^{'} \in R$ such that

$$x + xp_1x = xp_2x \tag{6.1}$$

and

$$x + \sum_{i=1}^{m} a_i x^2 b_i = \sum_{j=1}^{n} a_j^{'} x^2 b_j^{'} \tag{6.2}$$

so we have

$$xp_1x + xp_1xp_1x = xp_2xp_1x \tag{6.3}$$

and

$$xp_2x + xp_1xp_2x = xp_2xp_2x \tag{6.4}$$

Adding xp_1xp_1x, xp_1xp_2x on both sides of equation 6.1 we get

$$x + xp_1x + xp_1xp_1x + xp_1xp_2x = xp_2x + xp_1xp_1x + xp_1xp_2x$$

Using equations 6.3 and 6.4, we get

$$x + xp_2xp_1x + xp_1xp_2x = xp_2xp_2x + xp_1xp_1x \tag{6.5}$$

Now, multiply equation 6.2 by p_1x from the right and by xp_2 from the left

$$xp_2xp_1x + xp_2\left(\sum_{i=1}^{m} a_i x^2 b_i\right)p_1x = xp_2\left(\sum_{j=1}^{n} a_j^{'} x^2 b_j^{'}\right)p_1x, \tag{6.6}$$

multiply equation 6.2 by p_2x from the right and by xp_1 from the left

$$xp_1xp_2x + xp_1\left(\sum_{i=1}^{m} a_i x^2 b_i\right)p_2x = xp_1\left(\sum_{j=1}^{n} a_j^{'} x^2 b_j^{'}\right)p_2x, \tag{6.7}$$

multiply equation 6.2 by p_2x from the right and by xp_2 from the left

$$xp_2xp_2x + xp_2\left(\sum_{i=1}^{m} a_i x^2 b_i\right)p_2x = xp_2\left(\sum_{j=1}^{n} a_j^{'} x^2 b_j^{'}\right)p_2x, \tag{6.8}$$

multiply equation 6.2 by p_1x from the right and by xp_1 from the left

$$xp_1xp_1x + xp_1\left(\sum_{i=1}^{m} a_ix^2b_i\right)p_1x = xp_1\left(\sum_{j=1}^{n} a_j'x^2b_j'\right)p_1x. \tag{6.9}$$

Adding $\sum\limits_{i=1}^{m}(xp_2a_ix)(xb_ip_1x)$, $\sum\limits_{i=1}^{m}(xp_1a_ix)(xb_ip_2x)$, $\sum\limits_{i=1}^{m}(xp_2a_ix)(xb_ip_2x)$, and $\sum\limits_{i=1}^{m}(xp_1a_ix)(xb_ip_1x)$ on both sides of equation 6.5, we get

$x + xp_2xp_1x + \sum\limits_{i=1}^{m}(xp_2a_ix)(xb_ip_1x) + xp_1xp_2x + \sum\limits_{i=1}^{m}(xp_1a_ix)(xb_ip_2x)$
$+ \sum\limits_{i=1}^{m}(xp_2a_ix)(xb_ip_2x) + \sum\limits_{i=1}^{m}(xp_1a_ix)(xb_ip_1x)$
$= xp_2xp_2x + \sum\limits_{i=1}^{m}(xp_2a_ix)(xb_ip_2x) + xp_1xp_1x + \sum\limits_{i=1}^{m}(xp_1a_ix)(xb_ip_1x)$
$+ \sum\limits_{i=1}^{m}(xp_2a_ix)(xb_ip_1x) + \sum\limits_{i=1}^{m}(xp_1a_ix)(xb_ip_2x)$

Using equations 6.6, 6.7, 6.8, 6.9

$x + \sum\limits_{j=1}^{n}\left(xp_2a_j'x\right)\left(xb_j'p_1x\right) + \sum\limits_{j=1}^{n}\left(xp_1a_j'x\right)\left(xb_j'p_2x\right)$
$+ \sum\limits_{i=1}^{m}(xp_2a_ix)(xb_ip_2x) + \sum\limits_{i=1}^{m}(xp_1a_ix)(xb_ip_1x)$
$= \sum\limits_{j=1}^{n}\left(xp_2a_j'x\right)\left(xb_j'p_2x\right) + \sum\limits_{j=1}^{n}\left(xp_1a_j'x\right)\left(xb_j'p_1x\right)$
$+ \sum\limits_{i=1}^{m}(xp_2a_ix)(xb_ip_2x) + \sum\limits_{i=1}^{m}(xp_1a_ix)(xb_ip_1x)$

Since B is a k-bi-ideal of R, so

$\sum\limits_{j=1}^{n}\left(xp_2a_j'x\right)\left(xb_j'p_1x\right) + \sum\limits_{j=1}^{n}\left(xp_1a_j'x\right)\left(xb_j'p_2x\right) + \sum\limits_{i=1}^{m}(xp_2a_ix)(xb_ip_2x)$
$+ \sum\limits_{i=1}^{m}(xp_1a_ix)(xb_ip_1x) \in B^2$

and $\sum\limits_{j=1}^{n}\left(xp_2a_j'x\right)\left(xb_j'p_2x\right) + \sum\limits_{j=1}^{n}\left(xp_1a_j'x\right)\left(xb_j'p_1x\right) + \sum\limits_{i=1}^{m}(xp_2a_ix)(xb_ip_2x)$
$+ \sum\limits_{i=1}^{m}(xp_1a_ix)(xb_ip_1x) \in B^2$

So we have $x \in \widehat{B^2}$ and so $B \subseteq \widehat{B^2}$.

Therefore $B = \widehat{B^2}$.

(2) $\Rightarrow$ (3) Straightforward.

(3) $\Rightarrow$ (1) Let A, B be the left k-ideal and right k-ideal of R, respectively. Then $A \cap B$ is a k-quasi-ideal of R. By the assumption, we have $B \cap A = \widehat{(B \cap A)(B \cap A)} \subseteq \widehat{BA} \subseteq \widehat{B} \cap \widehat{A} = B \cap A$ and $A \cap B = \widehat{(A \cap B)(A \cap B)} \subseteq \widehat{AB}$. Therefore R is both k-regular and k-intra-regular. ■

Theorem 260 *Let R be a hemiring. Then the following conditions are equivalent:*

1. R is both k-regular and k-intra-regular.

2. $\mu \odot_k \mu = \mu$ for each fuzzy k-bi-ideal μ of R.

3. $\mu \odot_k \mu = \mu$ for each fuzzy k-quasi-ideal μ of R.

Proof. (1) $\Rightarrow$ (2) Let R be both k-regular and k-intra-regular hemiring. Let μ be a fuzzy k-bi-ideal of R and $x \in R$. Then $\mu \odot_k \mu \leq \mu$. Since R is both k-regular and k-intra-regular, then there exist elements $p_1, p_2, a_i, a_i^{'}, b_j, b_j^{'} \in R$ such that

$$x + xp_1x = xp_2x$$

and

$$x + \sum_{i=1}^{m} a_i x^2 b_i = \sum_{j=1}^{n} a_j^{'} x^2 b_j^{'}$$

As proved in the proof of Lemma 259 we have

$$x + \sum_{j=1}^{n} \left(xp_2a_j^{'}x\right)\left(xb_j^{'}p_1x\right) + \sum_{j=1}^{n} \left(xp_1a_j^{'}x\right)\left(xb_j^{'}p_2x\right)$$
$$+ \sum_{i=1}^{m} (xp_2a_ix)(xb_ip_2x) + \sum_{i=1}^{m} (xp_1a_ix)(xb_ip_1x)$$
$$= \sum_{j=1}^{n} \left(xp_2a_j^{'}x\right)\left(xb_j^{'}p_2x\right) + \sum_{j=1}^{n} \left(xp_1a_j^{'}x\right)\left(xb_j^{'}p_1x\right)$$
$$+ \sum_{i=1}^{m} (xp_2a_ix)(xb_ip_2x) + \sum_{i=1}^{m} (xp_1a_ix)(xb_ip_1x)$$

Thus,

$$\begin{aligned}(\mu \odot_k \mu)(x) &= \bigvee_{x+\sum_{i=1}^{m} a_ib_i=\sum_{j=1}^{n} a_j^{'}b_j^{'}} \left[\bigwedge_{i=1}^{m} [\mu(a_i) \wedge \mu(b_i)] \wedge \bigwedge_{j=1}^{n} \left[\mu(a_j^{'}) \wedge \mu(b_j^{'})\right]\right] \\ &\geq \mu\left(xp_2a_j^{'}x\right) \wedge \mu\left(xb_j^{'}p_1x\right) \wedge \mu\left(xp_1a_j^{'}x\right) \wedge \mu\left(xb_j^{'}p_2x\right) \\ &\quad \wedge \mu(xp_2a_ix) \wedge \mu(xb_ip_2x) \wedge \mu(xp_1a_ix) \wedge \mu(xb_ip_1x) \\ &\geq \mu(x)\end{aligned}$$

This implies that $\mu \leq \mu \odot_k \mu$.

Therefore $\mu \odot_k \mu = \mu$.

(2) $\Rightarrow$ (3) Straightforward.

(3) $\Rightarrow$ (1) Let Q be any k-quasi-ideal of R. Then χ_Q, the characteristic function of Q, is a fuzzy k-quasi-ideal of R. By the assumption $\chi_Q = \chi_Q \odot_k \chi_Q = \chi_{\widehat{Q^2}}$. Thus $Q = \widehat{Q^2}$. Hence by Lemma 259, R is both k-regular and k-intra-regular. ∎

Theorem 261 *Let R be a hemiring then the following conditions are equivalent:*

1. R is both k-regular and k-intra-regular.

2. $\mu \wedge \nu \leq \mu \odot_k \nu$ for all fuzzy k-bi-ideal μ, ν of R.

3. $\mu \wedge \nu \leq \mu \odot_k \nu$ for fuzzy k-bi-ideal μ and k-quasi-ideal ν of R.

4. $\mu \wedge \nu \leq \mu \odot_k \nu$ for fuzzy k-quasi-ideal μ and k-bi-ideal ν of R.

5. $\mu \wedge \nu \leq \mu \odot_k \nu$ for all fuzzy k-quasi-ideal μ, ν of R.

Proof. (1) $\Rightarrow$ (2) Let μ and ν be fuzzy k-bi-ideals of R and $x \in R$. Since R is both k-regular and k-intra-regular, there exist elements $p_1, p_2, a_i, a_i', b_j, b_j' \in R$ such that

$$x + xp_1x = xp_2x$$

and

$$x + \sum_{i=1}^{m} a_i x^2 b_i = \sum_{j=1}^{n} a_j' x^2 b_j'$$

As proved in the proof of Lemma 259 we have

$$x + \sum_{j=1}^{n} \left(xp_2a_j'x\right)\left(xb_j'p_1x\right) + \sum_{j=1}^{n}\left(xp_1a_j'x\right)\left(xb_j'p_2x\right)$$
$$+ \sum_{i=1}^{m}(xp_2a_ix)(xb_ip_2x) + \sum_{i=1}^{m}(xp_1a_ix)(xb_ip_1x)$$
$$= \sum_{j=1}^{n}\left(xp_2a_j'x\right)\left(xb_j'p_2x\right) + \sum_{j=1}^{n}\left(xp_1a_j'x\right)\left(xb_j'p_1x\right)$$
$$+ \sum_{i=1}^{m}(xp_2a_ix)(xb_ip_2x) + \sum_{i=1}^{m}(xp_1a_ix)(xb_ip_1x)$$

Thus we have

$$\begin{aligned}(\mu \odot_k \nu)(x) &= \bigvee_{x+\sum_{i=1}^{m} a_ib_i = \sum_{j=1}^{n} a_j'b_j'} \left[\bigwedge_{i=1}^{m}\left[\mu(a_i) \wedge \nu(b_i)\right] \wedge \bigwedge_{j=1}^{n}\left[\mu(a_j') \wedge \nu(b_j')\right]\right] \\ &\geq \mu\left(xp_2a_j'x\right) \wedge \nu\left(xb_j'p_1x\right) \wedge \mu\left(xp_1a_j'x\right) \wedge \nu\left(xb_j'p_2x\right) \\ &\wedge \mu(xp_2a_ix) \wedge \nu(xb_ip_2x) \wedge \mu(xp_1a_ix) \wedge \nu(xb_ip_1x) \\ &\geq (\mu \wedge \nu)(x).\end{aligned}$$

Hence $\mu \wedge \nu \leq \mu \odot_k \nu$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5), since every fuzzy k-quasi-ideal of R is a fuzzy k-bi-ideal of R.

(5) $\Rightarrow$ (1) Let Q be any k-quasi-ideal of R. Then χ_Q, the characteristic function of Q, is a fuzzy k-quasi-ideal of R. By the assumption

$$\chi_Q = \chi_Q \wedge \chi_Q \leq \chi_Q \odot_k \chi_Q = \chi_{\widehat{Q^2}}.$$

So $Q \subseteq \widehat{Q^2}$. Since $Q \supseteq \widehat{Q^2}$ always true, therefore $Q = \widehat{Q^2}$. Hence R is both k-regular and k-intra-regular. ■

Chapter 7

Prime k-bi-ideals in hemirings

In this chapter we define prime, strongly prime and semiprime k-bi-ideals of a hemiring. We also define their fuzzy versions and characterize hemirings by the properties of these k-bi-ideals.

Recall the following definitions.

A hemiring R is said to be k-regular if for each $x \in R$, there exist $a, b \in R$ such that $x + xax = xbx$.

A hemiring R is said to be k-intra-regular if for each $x \in R$, there exist $a_i, a_i', b_j, b_j' \in R$ such that $x + \sum_{i=1}^{m} a_i x^2 a_i' = \sum_{j=1}^{n} b_j x^2 b_j'$.

A hemiring R is said to be right k-weakly regular if for each $x \in R$, $x \in \widehat{(xR)^2}$. That is for each $x \in R$ there exist $a_i, a_i', b_j, b_j' \in R$ such that $x + \sum_{i=1}^{m} x a_i x a_i' = \sum_{j=1}^{n} x b_j x b_j'$.

7.1 Right k-weakly regular hemirings

In chapter 4 we characterized right k-weakly regular hemirings in terms of their right k-ideals and fuzzy right k-ideals. In this section we characterize right k-weakly regular hemirings in terms of k-bi-ideals and fuzzy k-bi-ideals.

Theorem 262 *The following assertions are equivalent for a hemiring R with identity:*

1. R is right k-weakly regular hemiring.

2. $B \cap I \subseteq \widehat{BI}$ for every k-bi-ideal B and every k-ideal I of R.

3. $Q \cap I \subseteq \widehat{QI}$ for every k-quasi-ideal Q and every k-ideal I of R.

Proof. (1) $\Rightarrow$ (2) Let R be a right k-weakly regular hemiring and B, I are k-bi-ideal and k-ideal of R, respectively. Let $x \in B \cap I$. Then $x \in B$ and $x \in I$.

Since R is right k-weakly regular hemiring, there exist $a_i, a_i', b_j, b_j' \in R$ such that $x + \sum_{i=1}^{m} xa_ixa_i' = \sum_{j=1}^{n} xb_jxb_j'$. Since $x \in B$ and $x \in I$, we have $\sum_{i=1}^{m} xa_ixa_i'$ and $\sum_{j=1}^{n} xb_jxb_j'$ are in BI. Thus from $x + \sum_{i=1}^{m} xa_ixa_i' = \sum_{j=1}^{n} xb_jxb_j'$ it follows that $x \in \widehat{BI}$. Hence $B \cap I \subseteq \widehat{BI}$.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Since every right k-ideal is a k-quasi-ideal, choose Q a right k-ideal of R. Then by hypothesis $Q \cap I \subseteq \widehat{QI}$ for every right k-ideal Q and every k-ideal I of R. But $\widehat{QI} \subseteq Q \cap I$ always holds. Hence $Q \cap I = \widehat{QI}$ for every right k-ideal Q and every k-ideal I of R. Thus by Proposition 161, R is right k-weakly regular hemiring. ■

Theorem 263 *The following assertions are equivalent for a hemiring R with identity:*

1. R is right k-weakly regular hemiring.
2. $\lambda \wedge \mu \leq \lambda \odot_k \mu$ for every fuzzy k-bi-ideal λ and every fuzzy k-ideal μ of R.
3. $\lambda \wedge \mu \leq \lambda \odot_k \mu$ for every fuzzy k-quasi-ideal λ and every fuzzy k-ideal μ of R.

Proof. (1) $\Rightarrow$ (2) Let R be a right k-weakly regular hemiring and λ, μ be fuzzy k-bi-ideal and fuzzy k-ideal of R, respectively. Let $x \in R$. Then there exist $a_i, a_i', b_j, b_j' \in R$ such that $x + \sum_{i=1}^{m} xa_ixa_i' = \sum_{j=1}^{n} xb_jxb_j'$. Now

$$\begin{aligned}
(\lambda \odot_k \mu)(x) &= \bigvee_{x+\sum_{i=1}^{m} c_id_i = \sum_{j=1}^{n} c_j'd_j'} \left[\bigwedge_{i=1}^{m} (\lambda(c_i) \wedge \mu(d_i)) \wedge \bigwedge_{j=1}^{n} \left(\lambda(c_j') \wedge \mu(d_j')\right) \right] \\
&\geq \left[\bigwedge_{i=1}^{m} (\lambda(x) \wedge \mu(a_ixa_i')) \wedge \bigwedge_{j=1}^{n} \left(\lambda(x) \wedge \mu(b_jxb_j')\right) \right] \\
&\geq \left[\bigwedge_{i=1}^{m} (\lambda(x) \wedge \mu(x)) \wedge \bigwedge_{j=1}^{n} (\lambda(x) \wedge \mu(x)) \right] \\
&= (\lambda \wedge \mu)(x).
\end{aligned}$$

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Since every fuzzy right k-ideal is a fuzzy k-quasi-ideal of R, choose λ a fuzzy right k-ideal of R and μ any fuzzy k-ideal of R. Then by hypothesis, $\lambda \wedge \mu \leq \lambda \odot_k \mu$. But $\lambda \odot_k \mu \leq \lambda \wedge \mu$ always true. Thus $\lambda \odot_k \mu = \lambda \wedge \mu$ for every fuzzy right k-ideal λ and fuzzy k-ideal μ of R. Hence by Theorem 162, R is right k-weakly regular hemiring. ■

Theorem 264 *The following assertions are equivalent for a hemiring R with identity:*

1. R is right k-weakly regular hemiring.

2. $B \cap I \cap J \subseteq \widehat{BIJ}$ for every k-bi-ideal B, every k-ideal I and every right k-ideal J of R.

3. $Q \cap I \cap J \subseteq \widehat{QIJ}$ for every k-quasi-ideal Q, every k-ideal I and every right k-ideal J of R.

Proof. (1) $\Rightarrow$ (2) Let R be a right k-weakly regular hemiring and B, I, J are k-bi-ideal, k-ideal and right k-ideal of R, respectively. Let $x \in B \cap I \cap J$. Then $x \in B$, $x \in I$ and $x \in J$. Since R is right k-weakly regular hemiring, there exist $a_i, a_i', b_j, b_j' \in R$ such that $x + \sum_{i=1}^{m} xa_ixa_i' = \sum_{j=1}^{n} xb_jxb_j'$. Now $xa_ixa_i' + \sum_{i=1}^{m} xa_ixa_i'a_ixa_i' = \sum_{j=1}^{n} xb_jxb_j'a_ixa_i'$. Since $x \in B$, $a_ixa_i'a_i \in I$, $xa_i' \in J$ and $b_jxb_j'a_i \in I$, we have $\sum_{i=1}^{m} xa_ixa_i'a_ixa_i'$, $\sum_{j=1}^{n} xb_jxb_j'a_ixa_i' \in BIJ$. Thus $xa_ixa_i' \in \widehat{BIJ}$. Similarly $xb_jxb_j' \in \widehat{BIJ}$. Hence from $x + \sum_{i=1}^{m} xa_ixa_i' = \sum_{j=1}^{n} xb_jxb_j'$ it follows that $x \in \widehat{BIJ}$. This shows $B \cap I \cap J \subseteq \widehat{BIJ}$.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Since every right k-ideal is a k-quasi-ideal, choose Q a right k-ideal of R and $J = R$. Then by hypothesis $Q \cap I \cap J = Q \cap I \cap R = Q \cap I \subseteq \widehat{QIR} \subseteq \widehat{QI}$. But $\widehat{QI} \subseteq Q \cap I$ always true. Hence $Q \cap I = \widehat{QI}$ for every right k-ideal Q and every k-ideal I of R. Thus by Proposition 161, R is right k-weakly regular hemiring. ■

Theorem 265 *The following assertions are equivalent for a hemiring R with identity:*

1. R is right k-weakly regular hemiring.

2. $\lambda \wedge \mu \wedge \nu \leq \lambda \odot_k \mu \odot_k \nu$ for every fuzzy k-bi-ideal λ, every fuzzy k-ideal μ and every fuzzy right k-ideal ν of R.

3. $\lambda \wedge \mu \wedge \nu \leq \lambda \odot_k \mu \odot_k \nu$ for every fuzzy k-quasi-ideal λ, every fuzzy k-ideal μ and every fuzzy right k-ideal ν of R.

Proof. (1) $\Rightarrow$ (2) Let R be a right k-weakly regular hemiring and λ, μ, ν be fuzzy k-bi-ideal, fuzzy k-ideal and fuzzy right k-ideal of R, respectively. Let $x \in R$. Then there exist $a_i, a_i', b_j, b_j' \in R$ such that $x + \sum_{i=1}^{m} xa_ixa_i' = \sum_{j=1}^{n} xb_jxb_j'$. Now

$$[(\lambda \odot_k \mu) \odot_k \nu](x) = \bigvee_{x+\sum_{i=1}^{m} c_id_i = \sum_{j=1}^{n} c_j'd_j'} \left[\begin{array}{c} \bigwedge_{i=1}^{m} ((\lambda \odot_k \mu)(c_i) \wedge \nu(d_i)) \\ \wedge \bigwedge_{j=1}^{n} \left((\lambda \odot_k \mu)(c_j') \wedge \nu(d_j')\right) \end{array} \right]$$

$$\geq \left[\bigwedge_{i=1}^{m}((\lambda \odot_k \mu)(xa_i) \wedge \nu(xa_i')) \wedge \bigwedge_{j=1}^{n}\left((\lambda \odot_k \mu)(xb_j) \wedge \nu(xb_j')\right)\right]$$
$$\geq \left[\bigwedge_{i=1}^{m}((\lambda \odot_k \mu)(xa_i) \wedge \nu(x)) \wedge \bigwedge_{j=1}^{n}((\lambda \odot_k \mu)(xb_j) \wedge \nu(x))\right]$$

As

$$(\lambda \odot_k \mu)(xa_i) = \bigvee_{xa_i + \sum_{i=1}^{m} c_i d_i = \sum_{j=1}^{n} c_j' d_j'} \left[\bigwedge_{i=1}^{m}(\lambda(c_i) \wedge \mu(d_i)) \wedge \bigwedge_{j=1}^{n}\left(\lambda(c_j') \wedge \mu(d_j')\right)\right]$$
$$\geq \left[\bigwedge_{i=1}^{m}(\lambda(x) \wedge \mu(a_i x a_i' a_i)) \wedge \bigwedge_{j=1}^{n}\left(\lambda(x) \wedge \mu(b_j x b_j' a_i)\right)\right]$$
$$\left(\text{because } xa_i + \sum_{i=1}^{m} xa_i x a_i' a_i = \sum_{j=1}^{n} xb_j x b_j' a_i\right)$$
$$\geq \left[\bigwedge_{i=1}^{m}(\lambda(x) \wedge \mu(x)) \wedge \bigwedge_{j=1}^{n}(\lambda(x) \wedge \mu(x))\right] = (\lambda \wedge \mu)(x)$$

Similarly $(\lambda \odot_k \mu)(xb_j) \geq (\lambda \wedge \mu)(x)$.

Thus

$$[(\lambda \odot_k \mu) \odot_k \nu](x) \geq \left[\bigwedge_{i=1}^{m}((\lambda \odot_k \mu)(xa_i) \wedge \nu(x)) \wedge \bigwedge_{j=1}^{n}((\lambda \odot_k \mu)(xb_j) \wedge \nu(x))\right]$$
$$\geq (\lambda \wedge \mu)(x) \wedge \nu(x) = ((\lambda \wedge \mu) \wedge \nu)(x)$$

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Since every fuzzy right k-ideal is a fuzzy k-quasi-ideal of R, choose λ a fuzzy right k-ideal of R, μ any fuzzy k-ideal and χ_R fuzzy right k-ideal of R. Then by hypothesis $\lambda \wedge \mu = \lambda \wedge \mu \wedge \chi_R \leq \lambda \odot_k \mu \odot_k \chi_R \leq \lambda \odot_k \mu$. But $\lambda \odot_k \mu \leq \lambda \wedge \mu$ always true. Thus $\lambda \odot_k \mu = \lambda \wedge \mu$ for every fuzzy right k-ideal λ and fuzzy k-ideal μ of R. Hence by Theorem 162, R is right k-weakly regular hemiring. ■

7.2 Prime and semiprime k-bi-ideals

In Chapter 6 many characterizations of k-regular, k-intra-regular and of both k-regular and k-intra-regular hemirings are given. Here we give some more characterizations of these hemirings.

Proposition 266 *Let B_1 and B_2 be k-bi-ideals of a hemiring R. Then $\widehat{B_1B_2}$ is a k-bi-ideal of R.*

Proof. Let B_1 and B_2 be k-bi-ideals of R and $x, y \in \widehat{B_1B_2}$. Then

$$x + \sum_{i=1}^{n} a_i a_i' = \sum_{j=1}^{m} b_j b_j' \tag{7.1}$$

and

$$y+\sum_{g=1}^{p}c_gc_g^{'}=\sum_{l=1}^{q}d_ld_l^{'} \tag{7.2}$$

for $a_i, b_j, c_g, d_l \in B_1$, $a_i^{'}, b_j^{'}, c_g^{'}, d_l^{'} \in B_2$.

Now

$$x+y+\left(\sum_{i=1}^{n}a_ia_i^{'}+\sum_{g=1}^{p}c_gc_g^{'}\right)=\left(\sum_{j=1}^{m}b_jb_j^{'}+\sum_{l=1}^{q}d_ld_l^{'}\right).$$

As $\left(\sum_{i=1}^{n}a_ia_i^{'}+\sum_{g=1}^{p}c_gc_g^{'}\right)$ and $\left(\sum_{j=1}^{m}b_jb_j^{'}+\sum_{l=1}^{q}d_ld_l^{'}\right)$ are in B_1B_2 so $x+y\in\widehat{B_1B_2}$.

Multiplying the equation (7.1) by y we get

$$xy+\sum_{i=1}^{n}a_ia_i^{'}y=\sum_{j=1}^{m}b_jb_j^{'}y \tag{7.3}$$

Multiplying the equation (7.2) by $\sum_{i=1}^{n}a_ia_i^{'}$ we get

$$\sum_{i=1}^{n}a_ia_i^{'}y+\left(\sum_{i=1}^{n}a_ia_i^{'}\right)\left(\sum_{g=1}^{p}c_gc_g^{'}\right)=\left(\sum_{i=1}^{n}a_ia_i^{'}\right)\left(\sum_{l=1}^{q}d_ld_l^{'}\right) \tag{7.4}$$

As each term in the expression of $\left(\sum_{i=1}^{n}a_ia_i^{'}\right)\left(\sum_{g=1}^{p}c_gc_g^{'}\right)$ is the form $a_ia_i^{'}c_gc_g^{'}$ where $a_i, c_g\in B_1$ and $a_i^{'}, c_g^{'}\in B_2$. Thus $\left(a_ia_i^{'}c_g\right)c_g^{'}\in(B_1RB_1)B_2\subseteq B_1B_2$. This implies $\left(\sum_{i=1}^{n}a_ia_i^{'}\right)\left(\sum_{g=1}^{p}c_gc_g^{'}\right)\in B_1B_2$. Similarly $\left(\sum_{i=1}^{n}a_ia_i^{'}\right)\left(\sum_{l=1}^{q}d_ld_l^{'}\right)\in B_1B_2$. Hence from equation (7.4) we get $\sum_{i=1}^{n}a_ia_i^{'}y\in\widehat{B_1B_2}$. Similarly $\sum_{j=1}^{m}b_jb_j^{'}y\in\widehat{B_1B_2}$. Thus from equation (7.3) we get $xy\in\widehat{B_1B_2}$.

Hence $\widehat{B_1B_2}$ is closed under addition and multiplication.

Now

$$\widehat{\left(\widehat{B_1B_2}\right)R\left(\widehat{B_1B_2}\right)}=\widehat{(B_1B_2)R(B_1B_2)}\subseteq\widehat{B_1B_2}$$

Let $x+a=b$ for some $a,b\in\widehat{B_1B_2}$, $x\in R$. Then $x\in\widehat{\widehat{B_1B_2}}=\widehat{B_1B_2}$.

Hence $\widehat{B_1B_2}$ is a k-bi-ideal of R. ■

Proposition 267 *Let λ,μ be fuzzy k-bi-ideal of a hemiring R. Then $\lambda\odot_k\mu$ is a fuzzy k-bi-ideal of R.*

Proof. Let λ,μ be fuzzy k-bi-ideal of R and $x,y\in R$. Then

$$(\lambda\odot_k\mu)(x)=\bigvee_{x+\sum_{i=1}^{n}a_ia_i^{'}=\sum_{j=1}^{m}b_jb_j^{'}}\left[\bigwedge_{i=1}^{n}\left[\lambda(a_i)\wedge\mu\left(a_i^{'}\right)\right]\wedge\bigwedge_{j=1}^{m}\left[\lambda(b_j)\wedge\mu\left(b_j^{'}\right)\right]\right] \tag{7.5}$$

and

$$(\lambda \odot_k \mu)(y) = \bigvee_{y+\sum_{g=1}^{p} c_g c_g' = \sum_{l=1}^{q} d_l d_l'} \left[\bigwedge_{k=1}^{p} \left[\lambda(c_g) \wedge \mu\left(c_g'\right) \right] \wedge \bigwedge_{l=1}^{q} \left[\lambda(d_l) \wedge \mu\left(d_l'\right) \right] \right] \tag{7.6}$$

where $a_i, b_j, c_g, d_l, a_i', b_j', c_g', d_l' \in R$.

Now

$$(\lambda \odot_k \mu)(x+y) = \bigvee_{x+y+\sum_{s=1}^{u} e_s f_s = \sum_{t=1}^{v} e_t' f_t'} \left[\left[\bigwedge_{s=1}^{u} (\lambda(e_s) \wedge \mu(f_s)) \right] \wedge \left[\bigwedge_{t=1}^{v} \left(\lambda(e_t') \wedge \mu(f_t') \right) \right] \right]$$

$$\geq \bigvee_{x+\sum_{i=1}^{n} a_i a_i' = \sum_{j=1}^{m} b_j b_j'} \left(\bigvee_{y+\sum_{g=1}^{p} c_g c_g' = \sum_{l=1}^{q} d_l d_l'} \begin{pmatrix} \bigwedge_{i=1}^{n} \left[\lambda(a_i) \wedge \mu\left(a_i'\right) \right] \wedge \\ \bigwedge_{j=1}^{m} \left[\lambda(b_j) \wedge \mu\left(b_j'\right) \right] \wedge \\ \bigwedge_{g=1}^{p} \left[\lambda(c_g) \wedge \mu\left(c_g'\right) \right] \wedge \\ \bigwedge_{l=1}^{q} \left[\lambda(d_l) \wedge \mu\left(d_l'\right) \right] \end{pmatrix} \right)$$

$$= \bigvee_{x+\sum_{i=1}^{n} a_i a_i' = \sum_{j=1}^{m} b_j b_j'} \left[\bigwedge_{i=1}^{n} \left[\lambda(a_i) \wedge \mu\left(a_i'\right) \right] \wedge \bigwedge_{j=1}^{m} \left[\lambda(b_j) \wedge \mu\left(b_j'\right) \right] \right]$$

$$\wedge \bigvee_{y+\sum_{g=1}^{p} c_g c_g' = \sum_{l=1}^{q} d_l d_l'} \left[\bigwedge_{g=1}^{p} \left[\lambda(c_g) \wedge \mu\left(c_g'\right) \right] \wedge \bigwedge_{l=1}^{q} \left[\lambda(d_l) \wedge \mu\left(d_l'\right) \right] \right]$$

$= (\lambda \odot_k \mu)(x) \wedge (\lambda \odot_k \mu)(y)$.

To prove that $x + a = b$ implies $(\lambda \odot_k \mu)(x) \geq (\lambda \odot_k \mu)(a) \wedge (\lambda \odot_k \mu)(b)$, observe that

$$a + \sum_{i=1}^{m} a_i b_i = \sum_{j=1}^{n} a_j' b_j' \quad \text{and} \quad b + \sum_{g=1}^{l} c_g d_g = \sum_{q=1}^{p} c_q' d_q', \tag{7.7}$$

together with $x + a = b$, gives

$$x + a + \sum_{i=1}^{m} a_i b_i = b + \sum_{i=1}^{m} a_i b_i.$$

Thus,

$$x + \sum_{j=1}^{n} a_j' b_j' = b + \sum_{i=1}^{m} a_i b_i$$

and, consequently,

$$\begin{aligned} x + \sum_{j=1}^{n} a_j' b_j' + \sum_{g=1}^{l} c_g d_g &= b + \sum_{g=1}^{l} c_g d_g + \sum_{i=1}^{m} a_i b_i \\ &= \sum_{q=1}^{p} c_q' d_q' + \sum_{i=1}^{m} a_i b_i \\ &= \sum_{i=1}^{m} a_i b_i + \sum_{q=1}^{p} c_q' d_q'. \end{aligned}$$ Therefore,

$$x + \sum_{j=1}^{n} a_j' b_j' + \sum_{g=1}^{l} c_g d_g = \sum_{i=1}^{m} a_i b_i + \sum_{q=1}^{p} c_q' d_q'. \tag{7.8}$$

Now, in view of equations (7.7) and (7.8), we have

$$(\lambda \odot_k \mu)(a) \wedge (\lambda \odot_k \mu)(b) = \left(\bigvee_{a+\sum_{i=1}^{m} a_i b_i = \sum_{j=1}^{n} a'_j b'_j} \left(\begin{array}{c} \left[\bigwedge_{i=1}^{m} (\lambda(a_i) \wedge \mu(b_i)) \right] \\ \wedge \left[\bigwedge_{j=1}^{n} \left(\lambda(a'_j) \wedge \mu(b'_j) \right) \right] \end{array} \right) \right)$$

$$\wedge \left(\bigvee_{b+\sum_{g=1}^{p} c_g d_g = \sum_{l=1}^{q} c'_l d'_l} \left(\begin{array}{c} \left[\bigwedge_{g=1}^{p} (\lambda(c_g) \wedge \mu(d_g)) \right] \\ \wedge \left[\bigwedge_{l=1}^{q} \left(\lambda(c'_l) \wedge \mu(d'_l) \right) \right] \end{array} \right) \right)$$

$$= \bigvee_{a+\sum_{i=1}^{m} a_i b_i = \sum_{j=1}^{n} a'_j b'_j} \left(\bigvee_{b+\sum_{g=1}^{p} c_g d_g = \sum_{l=1}^{q} c'_l d'_l} \left(\begin{array}{c} \bigwedge_{i=1}^{m} (\lambda(a_i) \wedge \mu(b_i)) \wedge \\ \bigwedge_{j=1}^{n} \left(\lambda(a'_j) \wedge \mu(b'_j) \right) \wedge \\ \bigwedge_{g=1}^{p} (\lambda(c_g) \wedge \mu(d_g)) \wedge \\ \bigwedge_{l=1}^{q} \left(\lambda(c'_l) \wedge \mu(d'_l) \right) \end{array} \right) \right)$$

$$\leq \bigvee_{x+\sum_{s=1}^{u} r_s h_s = \sum_{t=1}^{w} r'_t h'_t} \left(\bigwedge_{s=1}^{u} (\lambda(r_s) \wedge \mu(h_s)) \wedge \bigwedge_{t=1}^{w} \left(\lambda(r'_t) \wedge \mu(h'_t) \right) \right)$$

$$= (\lambda \odot_k \mu)(x).$$

Now

$$\begin{aligned} (\lambda \odot_k \mu) \odot_k (\lambda \odot_k \mu) &= (\lambda \odot_k \mu \odot_k \lambda) \odot_k \mu \\ &\leq (\lambda \odot_k \chi_R \odot_k \lambda) \odot_k \mu \\ &\leq \lambda \odot_k \mu \end{aligned}$$

Also

$$\begin{aligned} (\lambda \odot_k \mu) \odot_k \chi_R \odot_k (\lambda \odot_k \mu) &= (\lambda \odot_k (\mu \odot_k \chi_R) \odot_k \lambda) \odot_k \mu \\ &\leq (\lambda \odot_k \chi_R \odot_k \lambda) \odot_k \mu \leq \lambda \odot_k \mu \end{aligned}$$

Thus $\lambda \odot_k \mu$ is a fuzzy k-bi-ideal of R. ∎

Definition 268 *Let R be a hemiring. A k-bi-ideal B of R is called prime (resp. semiprime) if $\widehat{B_1 B_2} \subseteq B$ $\left(\text{resp. } \widehat{B_1^2} \subseteq B \right)$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ (resp. $B_1 \subseteq B$) for all k-bi-ideals B_1, B_2 of R.*

Definition 269 *Let R be a hemiring. A k-bi-ideal B of R is called strongly prime if $\widehat{B_1 B_2} \cap \widehat{B_2 B_1} \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ for all k-bi-ideals B_1, B_2 of R.*

Obviously every strongly prime k-bi-ideal is a prime k-bi-ideal and every prime k-bi-ideal is a semiprime k-bi-ideal.

Definition 270 *A fuzzy k-bi-ideal λ of a hemiring R is called prime (semiprime) if $\mu \odot_k \nu \leq \lambda$ $(\mu \odot_k \mu \leq \lambda)$ implies $\mu \leq \lambda$ or $\nu \leq \lambda$ $(\mu \leq \lambda)$ for all fuzzy k-bi-ideals μ, ν of R.*

A fuzzy k-bi-ideal λ of a hemiring R is called strongly prime if $\mu \odot_k \nu \wedge \nu \odot_k \mu \leq \lambda$ implies $\mu \leq \lambda$ or $\nu \leq \lambda$ for all fuzzy k-bi-ideals μ, ν of R.

Lemma 271 *Let R be a hemiring, $\{B_i : i \in I\}$ a family of prime k-bi-ideal of R. Then $\bigcap_{i \in I} B_i$ is a semiprime k-bi-ideal of R.*

Proof. Straightforward. ■

Proposition 272 *Let R be a hemiring and μ, ν be fuzzy k-bi-ideals of R then $\mu \wedge \nu$ is also fuzzy k-bi-ideal of R.*

Proof. Let $x, y, z \in R$. Then

(i)

$$\begin{aligned} (\mu \wedge \nu)(x+y) &= \mu(x+y) \wedge \nu(x+y) \geq [\mu(x) \wedge \mu(y)] \wedge [\nu(x) \wedge \nu(y)] \\ &= [\mu(x) \wedge \nu(x)] \wedge [\mu(y) \wedge \nu(y)] = (\mu \wedge \nu)(x) \wedge (\mu \wedge \nu)(y). \end{aligned}$$

(ii)

$$\begin{aligned} (\mu \wedge \nu)(xy) &= \mu(xy) \wedge \nu(xy) \geq \mu(x) \wedge \mu(y) \wedge \nu(x) \wedge \nu(y) \\ &= (\mu \wedge \nu)(x) \wedge (\mu \wedge \nu)(y). \end{aligned}$$

(iii)

$$\begin{aligned} (\mu \wedge \nu)(xyz) &= \mu(xyz) \wedge \nu(xyz) \geq \mu(x) \wedge \mu(z) \wedge \nu(x) \wedge \nu(z) \\ &= (\mu \wedge \nu)(x) \wedge (\mu \wedge \nu)(z). \end{aligned}$$

(iv) Now let $a, b, x \in R$ such that $x + a = b$. Then

$$\begin{aligned} (\mu \wedge \nu)(x) &= \mu(x) \wedge \nu(x) \geq [\mu(a) \wedge \mu(b)] \wedge [\nu(a) \wedge \nu(b)] \\ &= [\mu(a) \wedge \nu(a)] \wedge [\mu(b) \wedge \nu(b)] = (\mu \wedge \nu)(a) \wedge (\mu \wedge \nu)(b). \end{aligned}$$

Hence $\mu \wedge \nu$ is a fuzzy k-bi-ideal of R. ■

Proposition 273 *Let R be a hemiring and $\{\lambda_i : i \in I\}$ a family of fuzzy prime k-bi-ideal of R. Then $\bigwedge_{i \in I} \lambda_i$ is a semiprime fuzzy k-bi-ideal of R.*

Proof. Straightforward. ■

Definition 274 *Let R be a hemiring. A k-bi-ideal B of R is called irreducible (resp. strongly irreducible) if $B_1 \cap B_2 = B$ (resp. $B_1 \cap B_2 \subseteq B$) implies $B_1 = B$ or $B_2 = B$ (resp. $B_1 \subseteq B$ or $B_2 \subseteq B$) for all k-bi-ideals B_1, B_2 of R.*

Proposition 275 *Every strongly irreducible semiprime k-bi-ideal of a hemiring R is a strongly prime k-bi-ideal of R.*

Proof. Let B be a strongly irreducible semiprime k-bi-ideal of R. Let B_1, B_2 be any k-bi-ideals of R such that $\widehat{B_1B_2} \cap \widehat{B_2B_1} \subseteq B$. Since $B_1 \cap B_2$ is a k-bi-ideal and $\widehat{(B_1 \cap B_2)(B_1 \cap B_2)} \subseteq \widehat{B_1B_2}$, $\widehat{(B_1 \cap B_2)(B_1 \cap B_2)} \subseteq \widehat{B_2B_1}$.

Thus $\widehat{(B_1 \cap B_2)(B_1 \cap B_2)} \subseteq \widehat{B_1B_2} \cap \widehat{B_2B_1} \subseteq B$. Since B is a semiprime k-bi-ideal of R, we have $B_1 \cap B_2 \subseteq B$. As B is strongly irreducible, we have $B_1 \subseteq B$ or $B_2 \subseteq B$. Thus B is strongly prime k-bi-ideal of R. ■

Proposition 276 *Let R be a hemiring and B be a k-bi-ideal of R. Let $a \in R$ be such that $a \notin B$. Then there exists an irreducible k-bi-ideal A of R such that $B \subseteq A$ and $a \notin A$.*

Proof. Let $\mathcal{F}$ be the collection of all k-bi-ideals of R which contains B but does not contain a. Then $\mathcal{F} \neq \phi$, because $B \in \mathcal{F}$. The collection $\mathcal{F}$ is a partially ordered set under inclusion. As every totally ordered subset in $\mathcal{F}$ is bounded above, so by Zorn's Lemma there exists a maximal element say $A \in \mathcal{F}$. We will show that A is an irreducible k-bi-ideal of R. Let C, D be two k-bi-ideals of R such that $C \cap D = A$. If both C and D properly contains A then $a \in C$ and $a \in D$, then $a \in A$. This contradicts the fact that $a \notin A$. Thus $A = C$ or $A = D$. Hence A is an irreducible k-bi-ideal of R such that $B \subseteq A$ and $a \notin A$. ■

Definition 277 *Let R be a hemiring and λ a fuzzy k-bi-ideal of R then λ is called irreducible (resp. strongly irreducible) fuzzy k-bi-ideal of R if $\mu \wedge \nu = \lambda$ (resp. $\mu \wedge \nu \leq \lambda$) implies $\mu = \lambda$ or $\nu = \lambda$ (resp. $\mu \leq \lambda$ or $\nu \leq \lambda$) for all fuzzy k-bi-ideals μ, ν of R.*

Proposition 278 *Let R be a hemiring. Then every strongly irreducible semiprime fuzzy k-bi-ideal of R is a strongly prime fuzzy k-bi-ideal of R.*

Proof. Let λ be a strongly irreducible semiprime k-bi-ideal of R. Let μ, ν be any fuzzy k-bi-ideals of R such that $(\mu \odot_k \nu) \wedge (\nu \odot_k \mu) \leq \lambda$. As $\mu \wedge \nu$ is a fuzzy k-bi-ideal of R and $(\mu \wedge \nu) \odot_k (\nu \wedge \mu) \leq \mu \odot_k \nu$ and $(\mu \wedge \nu) \odot_k (\nu \wedge \mu) \leq \nu \odot_k \mu$. Thus $(\mu \wedge \nu) \odot_k (\nu \wedge \mu) \leq (\mu \odot_k \nu) \wedge (\nu \odot_k \mu) \leq \lambda$. That is $(\mu \wedge \nu) \odot_k (\nu \wedge \mu) \leq \lambda$.

As λ is semiprime, we have $\mu \wedge \nu \leq \lambda$. Since λ is strongly irreducible, we have $\mu \leq \lambda$ or $\nu \leq \lambda$. Hence λ is strongly prime. ■

Proposition 279 *Let R be a hemiring, λ a fuzzy k-bi-ideal of R with $\lambda(a)=\alpha$, where a is any element of R and $\alpha\in(0,1]$. Then there exists a fuzzy irreducible k-bi-ideal δ of R such that $\lambda\leq\delta$ and $\delta(a)=\alpha$.*

Proof. Let $X=\{\mu:\mu$ is a fuzzy k-bi-ideal of R, $\mu(a)=\alpha$ and $\lambda\leq\mu\}$. Then $X\neq\phi$, because $\lambda\in X$. Let $\mathcal{F}$ be a totally ordered subset of X, say $\mathcal{F}=\{\lambda_i:i\in I\}$. We claim that $\bigvee\limits_{i\in I}\lambda_i$ is a fuzzy k-bi-ideal of R containing λ.

Let $x,y\in R$, consider

(1)

$$\begin{aligned}
\left(\bigvee_i\lambda_i\right)(x)\wedge\left(\bigvee_i\lambda_i\right)(y) &= \left(\bigvee_i(\lambda_i(x))\right)\wedge\left(\bigvee_j(\lambda_j(y))\right)\\
&= \bigvee_j\left[\bigvee_i(\lambda_i(x))\wedge\lambda_j(y)\right]\\
&= \bigvee_j\left[\bigvee_i(\lambda_i(x)\wedge\lambda_j(y))\right]\\
&\leq \bigvee_j\left[\bigvee_i\left(\lambda_i^j(x)\wedge\lambda_i^j(y)\right)\right]\\
&\text{where } \lambda_i^j=\max\{\lambda_i,\lambda_j\}, \text{ note that } \lambda_i^j\in\{\lambda_i:i\in I\}\\
&\leq \bigvee_j\left[\bigvee_i\left[\lambda_i^j(x+y)\right]\right]=\bigvee_{i,j}\left[\lambda_i^j(x+y)\right]\\
&\leq \bigvee_i[\lambda_i(x+y)]=\left(\bigvee_i\lambda_i\right)(x+y)
\end{aligned}$$

(2)

$$\begin{aligned}
\left(\bigvee_i\lambda_i\right)(xy) &= \bigvee_i(\lambda_i(xy))\\
&\geq \bigvee_i(\lambda_i(x)\wedge\lambda_j(y))\\
&= \bigvee_i(\lambda_i(x))\wedge\bigvee_i(\lambda_j(y))\\
&= \left(\bigvee_i\lambda_i\right)(x)\wedge\left(\bigvee_i\lambda_i\right)(y)
\end{aligned}$$

(3)

$$\begin{aligned}
\left(\bigvee_i\lambda_i\right)(xyz) &= \bigvee_i(\lambda_i(xyz))\\
&\geq \bigvee_i(\lambda_i(x)\wedge\lambda_j(z))\\
&= \bigvee_i(\lambda_i(x))\wedge\bigvee_i(\lambda_j(z))\\
&= \left(\bigvee_i\lambda_i\right)(x)\wedge\left(\bigvee_i\lambda_i\right)(z)
\end{aligned}$$

(4) Now, let $x+a=b$, where $a,b\in R$. Then

$$\begin{aligned}
\left(\bigvee_i\lambda_i\right)(a)\wedge\left(\bigvee_i\lambda_i\right)(b) &= \left(\bigvee_i(\lambda_i(a))\right)\wedge\left(\bigvee_j(\lambda_j(b))\right)\\
&= \bigvee_j\left[\left(\bigvee_i(\lambda_i(a))\right)\wedge\lambda_j(b)\right]
\end{aligned}$$

$$= \bigvee_j \left[\bigvee_i (\lambda_i(a) \wedge \lambda_j(b)) \right]$$
$$\leq \bigvee_j \left[\bigvee_i \left(\lambda_i^j(a) \wedge \lambda_i^j(b) \right) \right]$$
where $\lambda_i^j = \max\{\lambda_i, \lambda_j\}$, note that $\lambda_i^j \in \{\lambda_i : i \in I\}$
$$\leq \bigvee_j \left[\bigvee_i \left(\lambda_i^j(x) \right) \right] \text{ because } \lambda_i^j \text{ is a fuzzy } k\text{-bi-ideal}$$
$$= \bigvee_{i,j} \left[\lambda_i^j(x) \right] \leq \bigvee_i [\lambda_i(x)] = \left(\bigvee_i \lambda_i \right)(x)$$

Thus $\bigvee_i \lambda_i$ is a fuzzy k-bi-ideal of R. Clearly $\lambda \leq \bigvee_i \lambda_i$ and $\bigvee_i \lambda_i(a) = \bigvee_i (\lambda_i(a)) = \alpha$. Thus $\bigvee_i \lambda_i$ is the l.u.b of $\mathcal{F}$. Hence by Zorn's lemma there exists a fuzzy k-bi-ideal δ of R which is maximal with respect to the property that $\lambda \leq \delta$ and $\delta(a) = \alpha$.

We will show that δ is fuzzy irreducible k-bi-ideal of R. Let $\delta = \delta_1 \wedge \delta_2$, where δ_1, δ_2 are fuzzy k-bi-ideals of R. Thus $\delta \leq \delta_1$ and $\delta \leq \delta_2$. We claim that either $\delta = \delta_1$ or $\delta = \delta_2$. Suppose $\delta \neq \delta_1$ and $\delta \neq \delta_2$. Since δ is maximal with respect to the property that $\delta(a) = \alpha$ and since $\delta \lneqq \delta_1$ and $\delta \lneqq \delta_2$, so $\delta_1(a) \neq \alpha$ and $\delta_2(a) \neq \alpha$. Hence $\alpha = \delta(a) = (\delta_1 \wedge \delta_2)(a) = (\delta_1)(a) \wedge (\delta_2)(a) \neq \alpha$, which is impossible. Hence $\delta = \delta_1$ or $\delta = \delta_2$. Thus δ is fuzzy irreducible k-bi-ideal of R. ■

Theorem 280 *For a hemiring R, the following assertions are equivalent:*

1. R is both k-regular and k-intra-regular.
2. $\widehat{B^2} = B$ for every k-bi-ideal B of R.
3. $B_1 \cap B_2 \subseteq \widehat{B_1B_2} \cap \widehat{B_2B_1}$ for all k-bi-ideals B_1, B_2 of R.
4. Each k-bi-ideal of R is semiprime.
5. Every proper k-bi-ideal of R is the intersection of all irreducible semiprime k-bi-ideals of R which contain it.

Proof. (1) $\Leftrightarrow$ (2) This is Lemma 259.

(2) $\Rightarrow$ (3) Let B_1, B_2 be k-bi-ideals of R. Then $B_1 \cap B_2$ is also a k-bi-ideal of R. By hypothesis $B_1 \cap B_2 = \widehat{(B_1 \cap B_2)^2} \subseteq \widehat{B_1B_2}$. Similarly $B_1 \cap B_2 = \widehat{(B_1 \cap B_2)^2} \subseteq \widehat{B_2B_1}$. Thus $B_1 \cap B_2 \subseteq \widehat{B_1B_2} \cap \widehat{B_2B_1}$. By Proposition 266 $\widehat{B_1B_2}$ and $\widehat{B_2B_1}$ are k-bi-ideals of R and so $\widehat{B_1B_2} \cap \widehat{B_2B_1}$ is a k-bi-ideal of R. Thus by the hypothesis

$$\widehat{B_1B_2} \cap \widehat{B_2B_1} = \widehat{\left(\widehat{B_1B_2} \cap \widehat{B_2B_1} \right) \left(\widehat{B_1B_2} \cap \widehat{B_2B_1} \right)}$$
$$\subseteq \widehat{\widehat{B_1B_2}\ \widehat{B_2B_1}} = \widehat{B_1B_2B_2B_1} \subseteq \widehat{B_1RB_1} \subseteq B_1$$

Similarly $\widehat{B_1B_2} \cap \widehat{B_2B_1} \subseteq B_2$. Hence $\widehat{B_1B_2} \cap \widehat{B_2B_1} \subseteq B_1 \cap B_2$.

This implies $\widehat{B_1B_2} \cap \widehat{B_2B_1} = B_1 \cap B_2$.

(3) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (4) Let B_1, B_2 be k-bi-ideals of R, such that $B_1^2 \subseteq B_2$. Then $\widehat{B_1^2} \subseteq B_2$. By hypothesis $B_1 = \widehat{B_1^2} \subseteq B_2$. Thus B_2 is semiprime.

(4) $\Rightarrow$ (2) Obvious.

(4) $\Rightarrow$ (5) Let B be a proper k-bi-ideal of R. Then B is contained in the intersection of all irreducible k-bi-ideals of R which contain it. Proposition 276, guarantees the existence of such irreducible k-bi-ideals. If $a \notin B$, then there exists an irreducible k-bi-ideal of R which contain it but does not contain a. Hence B is the intersection of all irreducible k-bi-ideals of R which contain it. By hypothesis each k-bi-ideal is semiprime, so B is the intersection of all irreducible semiprime k-bi-ideals of R which contain it.

(5) $\Rightarrow$ (2) Let B be a k-bi-ideal of R. If $\widehat{B^2} = R$ then clearly $B = R$. If $\widehat{B^2} \neq R$, then $\widehat{B^2}$ is proper k-bi-ideal of R containing $\widehat{B^2}$ and so by our hypothesis,

$$\widehat{B^2} = \bigcap \{B_\alpha : B_\alpha \text{ irreducible semiprime } k\text{-bi-ideals of } R\}.$$

Since each B_α is a semiprime k-bi-ideal, $B \subseteq B_\alpha$ for all α, and so $B \subseteq \bigcap_\alpha B_\alpha = \widehat{B^2}$. Thus $\widehat{B^2} = B$. ■

Theorem 281 *Let R be a k-regular and k-intra-regular hemiring and B be a k-bi-ideal of R. Then B is strongly irreducible if and only if B is strongly prime.*

Proof. Proof follows from Theorem 280. ■

Theorem 282 *Each k-bi-ideal of a hemiring R is strongly prime if and only if R is k-regular, k-intra-regular and the set of k-bi-ideals of R is totally ordered by inclusion.*

Proof. Suppose that each k-bi-ideal of R is strongly prime. Then each k-bi-ideal of R is semiprime. Thus by Theorem 280, R is both k-regular and k-intra-regular. Now we show that the set of k-bi-ideals of R is totally ordered. Let B_1 and B_2 be any two k-bi-ideals of R. Then by Theorem 280, $B_1 \cap B_2 = \widehat{B_1B_2} \cap \widehat{B_2B_1}$. As each k-bi-ideal is strongly prime, $B_1 \cap B_2$ is strongly prime. Hence either $B_1 \subseteq B_1 \cap B_2$ or $B_2 \subseteq B_1 \cap B_2$ that is either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.

Conversely, assume that R is k-regular, k-intra-regular and the set of k-bi-ideals of R is totally ordered. We show that each k-bi-ideal of R is strongly prime. Let B, B_1, B_2 be k-bi-ideals of R such that $\widehat{B_1B_2} \cap \widehat{B_2B_1} \subseteq B$. Since R is both k-regular and k-intra-regular, by Theorem 280, $B_1 \cap B_2 = \widehat{B_1B_2} \cap \widehat{B_2B_1}$. Since $\widehat{B_1B_2} \cap \widehat{B_2B_1} \subseteq B$, so $B_1 \cap B_2 \subseteq B$. As the set of k-bi-ideals of R is totally ordered, so either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$, that is, either $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$. Thus either $B_1 \subseteq B$ or $B_2 \subseteq B$. ■

Theorem 283 *If the set of k-bi-ideals of R is totally ordered, then R is both k-regular and k-intra-regular if and only if each k-bi-ideal of R is prime.*

Proof. Suppose that R is both k-regular and k-intra-regular. Let B, B_1, B_2 be k-bi-ideals of R such that $\widehat{B_1 B_2} \subseteq B$. Since the set of k-bi-ideals of R is totally ordered, either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Suppose $B_1 \subseteq B_2$. Then $\widehat{B_1^2} \subseteq \widehat{B_1 B_2} \subseteq B$. By Theorem 280, B is semiprime so $B_1 \subseteq B$. Hence B is prime k-bi-ideal of R.

Conversely, assume that every k-bi-ideal of R is prime. Thus every k-bi-ideal of R is semiprime. Hence by Theorem 280, R is both k-regular and k-intra-regular. ■

Theorem 284 *Let R be a hemiring. Then the followings are equivalent:*

1. The set of k-bi-ideals of R is totally ordered under inclusion.

2. Each k-bi-ideal of R is strongly irreducible.

3. Each k-bi-ideal of R is irreducible.

Proof. (1) $\Rightarrow$ (2) Let B, B_1, B_2 be k-bi-ideals of R such that $B_1 \cap B_2 \subseteq B$. Since the set of k-bi-ideals of R is totally ordered under inclusion, so either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Thus either $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$. Hence $B_1 \cap B_2 \subseteq B \Rightarrow B_1 \subseteq B$ or $B_2 \subseteq B$. Thus B is strongly irreducible k-bi-ideal of R.

(2) $\Rightarrow$ (3) Let B, B_1, B_2 be k-bi-ideals of R such that $B_1 \cap B_2 = B$. Then $B \subseteq B_1$ or $B \subseteq B_2$. By the hypothesis, either $B_1 \subseteq B$ or $B_2 \subseteq B$. Hence either $B_1 = B$ or $B_2 = B$. Thus B is irreducible.

(3) $\Rightarrow$ (1) Let B_1, B_2 be two k-bi-ideals of R. Then $B_1 \cap B_2$ is a k-bi-ideal of R. Also $B_1 \cap B_2 = B_1 \cap B_2$. So by hypothesis either $B_1 = B_1 \cap B_2$ or $B_2 = B_1 \cap B_2$, i.e.; either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Hence the set of k-bi-ideals of R is totally ordered. ■

Theorem 285 *For a hemiring R, the following assertions are equivalent:*

1. R is both k-regular and k-intra-regular.

2. $\lambda \odot_k \lambda = \lambda$ for every fuzzy k-bi-ideal λ of R.

3. $\lambda \wedge \mu = (\lambda \odot_k \mu) \wedge (\mu \odot_k \lambda)$ for all fuzzy k-bi-ideals λ, μ of R.

4. Each fuzzy k-bi-ideal of R is semiprime.

5. Each proper fuzzy k-bi-ideal of R is the intersection of all irreducible semiprime fuzzy k-bi-ideals of R which contain it.

Proof. (1) $\Rightarrow$ (2) This is Theorem 260.

(2) $\Rightarrow$ (3) Let λ, μ be fuzzy k-bi-ideals of R. Then $\lambda \wedge \mu$ is also fuzzy k-bi-ideal of R. By hypothesis $\lambda \wedge \mu = (\lambda \wedge \mu) \odot_k (\lambda \wedge \mu) \leq \lambda \odot_k \mu$. Similarly $\lambda \wedge \mu \leq \mu \odot_k \lambda$. Thus $\lambda \wedge \mu \leq (\lambda \odot_k \mu) \wedge (\mu \odot_k \lambda)$.

Now by Proposition 267, $(\lambda \odot_k \mu)$ and $(\mu \odot_k \lambda)$ are fuzzy k-bi-ideals of R and so $(\lambda \odot_k \mu) \wedge (\mu \odot_k \lambda)$ is a fuzzy k-bi-ideal of R. Thus by hypothesis

$$\begin{aligned} ((\lambda \odot_k \mu) \wedge (\mu \odot_k \lambda)) &= ((\lambda \odot_k \mu) \wedge (\mu \odot_k \lambda)) \odot_k ((\lambda \odot_k \mu) \wedge (\mu \odot_k \lambda)) \\ &\leq (\lambda \odot_k \mu) \odot_k (\mu \odot_k \lambda) \leq \lambda \odot_k \chi_R \odot_k \mu \leq \lambda. \end{aligned}$$

Similarly $((\lambda \odot_k \mu) \wedge (\mu \odot_k \lambda)) \leq \mu$. Thus $(\lambda \odot_k \mu) \wedge (\mu \odot_k \lambda) \leq \lambda \wedge \mu$.
Hence $(\lambda \odot_k \mu) \wedge (\mu \odot_k \lambda) = \lambda \wedge \mu$.

(3) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (4) Let λ, μ be fuzzy k-bi-ideals of R such that $\lambda \odot_k \lambda \leq \mu$. Since by (2) $\lambda \odot_k \lambda = \lambda$, so $\lambda \leq \mu$. Thus μ is semiprime.

(4) $\Rightarrow$ (2) Obvious.

(4) $\Rightarrow$ (5) Let λ be a proper fuzzy k-bi-ideal of R. Then λ is contained in the intersection of all irreducible fuzzy k-bi-ideals of R which contain it. Proposition 279, guarantees the existence of such irreducible fuzzy k-bi-ideals. If $a \in R$ and $t \in (0,1]$ such that $\lambda(a) = t$, then there exists an irreducible fuzzy k-bi-ideal μ_α such that $\lambda \leq \mu_\alpha$ and $\mu_\alpha(a) = t$. Hence λ is the intersection of all irreducible fuzzy k-bi-ideals of R which contain it. By hypothesis each fuzzy k-bi-ideal is semiprime. Thus λ is the intersection of all irreducible, semiprime fuzzy k-bi-ideals of R which contain it.

(5) $\Rightarrow$ (2) Let λ be a fuzzy k-bi-ideal of R. Then $\lambda \odot_k \lambda$ is a fuzzy k-bi-ideal of R, so $\lambda \odot_k \lambda = \bigwedge \lambda_\alpha$ where λ_α are irreducible, semiprime fuzzy k-bi-ideals of R which contain $\lambda \odot_k \lambda$. Since each λ_α is semiprime, so $\lambda \leq \lambda_\alpha$ for all α. Thus $\lambda \leq \bigwedge \lambda_\alpha = \lambda \odot_k \lambda$. But $\lambda \odot_k \lambda \leq \lambda$ always. Hence $\lambda = \lambda \odot_k \lambda$. ∎

Theorem 286 *Let R be a k-regular and k-intra-regular hemiring and λ be a fuzzy k-bi-ideal of R. Then λ is strongly irreducible if and only if λ is strongly prime.*

Proof. Proof follows from Theorem 285. ∎

Theorem 287 *Each fuzzy k-bi-ideal of a hemiring R is strongly prime if and only if R is k-regular and k-intra-regular and the set of fuzzy k-bi-ideals of R is totally ordered.*

Proof. Suppose that each fuzzy k-bi-ideal of R is strongly prime. Then each fuzzy k-bi-ideal of R is semiprime. Thus by Theorem 285, R is both k-regular and k-intra-regular. Now we show that the set of fuzzy k-bi-ideals of R is totally ordered. Let λ and μ be any two fuzzy k-bi-ideals of R. Then by Theorem 285, $\lambda \wedge \mu = (\lambda \odot_k \mu) \wedge (\mu \odot_k \lambda)$. As each fuzzy k-bi-ideal is strongly prime, $\lambda \leq \lambda \wedge \mu$ or $\mu \leq \lambda \wedge \mu$ that is either $\lambda \leq \mu$ or $\mu \leq \lambda$.

Conversely, assume that R is k-regular, k-intra-regular and the set of fuzzy k-bi-ideals of R is totally ordered. We show that each fuzzy k-bi-ideal of R is strongly prime. Let λ, μ, ν be fuzzy k-bi-ideals of R such that $(\mu \odot_k \nu) \wedge (\nu \odot_k \mu) \leq \lambda$. Since R is both k-regular and k-intra-regular, by Theorem 285, $\mu \wedge \nu = (\mu \odot_k \nu) \wedge (\nu \odot_k \mu)$. Since $(\mu \odot_k \nu) \wedge (\nu \odot_k \mu) \leq \lambda$, so $\mu \wedge \nu \leq \lambda$. As the set of fuzzy k-bi-ideals of R is totally ordered, so either $\mu \leq \nu$ or $\nu \leq \mu$, that is, either $\mu \wedge \nu = \mu$ or $\mu \wedge \nu = \nu$. Thus either $\mu \leq \lambda$ or $\nu \leq \lambda$. ■

Theorem 288 *If the set of fuzzy k-bi-ideals of R is totally ordered, then R is both k-regular and k-intra-regular if and only if each fuzzy k-bi-ideal of R is prime.*

Proof. Suppose that R is both k-regular and k-intra-regular. Let λ, μ, ν be fuzzy k-bi-ideals of R such that $\mu \odot_k \nu \leq \lambda$. Since the set of fuzzy k-bi-ideals of R is totally ordered, either $\mu \leq \nu$ or $\nu \leq \mu$. Suppose $\mu \leq \nu$. Then $\mu \odot_k \mu \leq \mu \odot_k \nu \leq \lambda$. By Theorem 285, λ is semiprime so $\mu \leq \lambda$. Hence λ is prime fuzzy k-bi-ideal of R.

Conversely, assume that every fuzzy k-bi-ideal of R is prime. Thus every fuzzy k-bi-ideal of R is semiprime. Hence by Theorem 285, R is both k-regular and k-intra-regular ■

Theorem 289 *Let R be a hemiring. Then the following are equivalent:*

1. The set of all fuzzy k-bi-ideals of R is totally ordered under inclusion.
2. Each fuzzy k-bi-ideal of R is strongly irreducible.
3. Each fuzzy k-bi-ideal of R is irreducible.

Proof. (1) $\Rightarrow$ (2) Let μ, ν, λ be any fuzzy k-bi-ideals of R such that $\mu \wedge \nu \leq \lambda$. Since the set of fuzzy k-bi-ideals of R is totally ordered, so either $\mu \leq \nu$ or $\nu \leq \mu$. Therefore $\mu \wedge \nu = \mu$ or $\mu \wedge \nu = \nu$. Hence $\mu \wedge \nu \leq \lambda \Rightarrow$ either $\mu \leq \lambda$ or $\nu \leq \lambda$. Hence λ is strongly irreducible.

(2) $\Rightarrow$ (3) Let μ, ν, λ be any fuzzy k-bi-ideals of R such that $\mu \wedge \nu = \lambda$. Then $\lambda \leq \mu$ or $\lambda \leq \nu$. By hypothesis, either $\mu \leq \lambda$ or $\nu \leq \lambda$. So either $\mu = \lambda$ or $\nu = \lambda$. Thus λ is irreducible.

(3) $\Rightarrow$ (1) Let μ, λ be any fuzzy k-bi-ideals of R. Then $\lambda \wedge \mu$ is a fuzzy k-bi-ideal of R. Also $\lambda \wedge \mu = \lambda \wedge \mu$. So by hypothesis, either $\lambda = \lambda \wedge \mu$ or $\mu = \lambda \wedge \mu$, that is either $\lambda \leq \mu$ or $\mu \leq \lambda$. Therefore the set of fuzzy k-bi-ideals of R is totally ordered. ■

Chapter 8

Prime h-bi-ideals in hemirings

In this chapter we define prime, strongly prime and semiprime h-bi-ideals of a hemiring. We also define their fuzzy versions and characterize hemirings by the properties of these h-bi-ideals.

Recall the following definitions.

A hemiring R is said to be h-hemiregular if for each $x \in R$, there exist $a, b, z \in R$ such that $x + xax + z = xbx + z$.

A hemiring R is said to be h-intra-hemiregular if for each $x \in R$, there exist $a_i, a_i', b_j, b_j', z \in R$ such that $x + \sum_{i=1}^{m} a_i x^2 a_i' + z = \sum_{j=1}^{n} b_j x^2 b_j' + z$.

A hemiring R is said to be right h-weakly regular if for each $x \in R$, $x \in \overline{(xR)^2}$. That is for each $x \in R$ there exist $a_i, a_i', b_j, b_j', z \in R$ such that $x + \sum_{i=1}^{m} x a_i x a_i' + z = \sum_{j=1}^{n} x b_j x b_j' + z$.

8.1 Right h-weakly regular hemirings

In chapter 5 we characterized right h-weakly regular hemirings in terms of their right h-ideals and fuzzy right h-ideals. Now we characterize right h-weakly regular hemirings in terms of h-bi-ideals and fuzzy h-bi-ideals.

Theorem 290 *The following assertions are equivalent for a hemiring R with identity:*

1. R is right h-weakly regular hemiring.
2. $B \cap I \subseteq \overline{BI}$ for every h-bi-ideal B and every h-ideal I of R.
3. $Q \cap I \subseteq \overline{QI}$ for every h-quasi-ideal Q and every h-ideal I of R.

Proof. (1) $\Rightarrow$ (2) Let R be a right h-weakly regular hemiring and B, I are h-bi-ideal and h-ideal of R, respectively. Let $x \in B \cap I$. Then $x \in B$ and $x \in I$. Since R is right h-weakly regular hemiring, there exist $a_i, a_i', b_j, b_j', z \in R$ such that $x + \sum_{i=1}^{m} xa_ixa_i' + z = \sum_{j=1}^{n} xb_jxb_j' + z$. Since $x \in B$ and $x \in I$, we have $\sum_{i=1}^{m} xa_ixa_i'$ and $\sum_{j=1}^{n} xb_jxb_j'$ are in BI. Thus from $x + \sum_{i=1}^{m} xa_ixa_i' + z = \sum_{j=1}^{n} xb_jxb_j' + z$ it follows that $x \in \overline{BI}$. Hence $B \cap I \subseteq \overline{BI}$.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Since every right h-ideal is an h-quasi-ideal, choose Q a right h-ideal of R. Then by hypothesis $Q \cap I \subseteq \overline{QI}$ for every right h-ideal Q and every h-ideal I of R. But $\overline{QI} \subseteq Q \cap I$ always holds. Hence $Q \cap I = \overline{QI}$ for every right h-ideal Q and every h-ideal I of R. Thus by Proposition 195, R is right h-weakly regular hemiring. ■

Theorem 291 *The following assertions are equivalent for a hemiring R with identity:*

1. R is right h-weakly regular hemiring.

2. $\lambda \wedge \mu \leq \lambda \odot_h \mu$ for every fuzzy h-bi-ideal λ and every fuzzy h-ideal μ of R.

3. $\lambda \wedge \mu \leq \lambda \odot_h \mu$ for every fuzzy h-quasi-ideal λ and every fuzzy h-ideal μ of R.

Proof. (1) $\Rightarrow$ (2) Let R be a right h-weakly regular hemiring and λ, μ be fuzzy h-bi-ideal and fuzzy h-ideal of R, respectively. Let $x \in R$. Then there exist $a_i, a_i', b_j, b_j', z \in R$ such that $x + \sum_{i=1}^{m} xa_ixa_i' + z = \sum_{j=1}^{n} xb_jxb_j' + z$. Now

$$\begin{aligned}
(\lambda \odot_h \mu)(x) &= \bigvee_{x + \sum_{i=1}^{m} c_id_i + z = \sum_{j=1}^{n} c_j'd_j' + z} \left[\bigwedge_{i=1}^{m} (\lambda(c_i) \wedge \mu(d_i)) \wedge \bigwedge_{j=1}^{n} \left(\lambda(c_j') \wedge \mu(d_j') \right) \right] \\
&\geq \left[\bigwedge_{i=1}^{m} (\lambda(x) \wedge \mu(a_ixa_i')) \wedge \bigwedge_{j=1}^{n} \left(\lambda(x) \wedge \mu(b_jxb_j') \right) \right] \\
&\geq \left[\bigwedge_{i=1}^{m} (\lambda(x) \wedge \mu(x)) \wedge \bigwedge_{j=1}^{n} (\lambda(x) \wedge \mu(x)) \right] \\
&= (\lambda \wedge \mu)(x).
\end{aligned}$$

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Since every fuzzy right h-ideal is a fuzzy h-quasi-ideal of R, choose λ a fuzzy right h-ideal of R and μ any fuzzy h-ideal of R. Then by hypothesis $\lambda \wedge \mu \leq \lambda \odot_h \mu$. But $\lambda \odot_h \mu \leq \lambda \wedge \mu$ always true. Thus $\lambda \odot_h \mu = \lambda \wedge \mu$ for every fuzzy right h-ideal λ and fuzzy h-ideal μ of R. Hence by Theorem 196, R is right h-weakly regular hemiring. ■

Theorem 292 *The following assertions are equivalent for a hemiring R with identity:*

1. R is right h-weakly regular hemiring.

2. $B \cap I \cap J \subseteq \overline{BIJ}$ for every h-bi-ideal B, every h-ideal I and every right h-ideal J of R.

3. $Q \cap I \cap J \subseteq \overline{QIJ}$ for every h-quasi-ideal Q, every h-ideal I and every right h-ideal J of R.

Proof. (1) $\Rightarrow$ (2) Let R be a right h-weakly regular hemiring and B, I, J are h-bi-ideal, h-ideal and right h-ideal of R, respectively. Let $x \in B \cap I \cap J$. Then $x \in B$, $x \in I$ and $x \in J$. Since R is right h-weakly regular hemiring, there exist $a_i, a_i', b_j, b_j', z \in R$ such that $x + \sum_{i=1}^{m} xa_ixa_i' + z = \sum_{j=1}^{n} xb_jxb_j' + z$. Now $xa_ixa_i' + \sum_{i=1}^{m} xa_ixa_i'a_ixa_i' + za_ixa_i' = \sum_{j=1}^{n} xb_jxb_j'a_ixa_i' + za_ixa_i'$. Since $x \in B$, $a_ixa_i'a_i \in I, xa_i' \in J$ and $b_jxb_j'a_i \in I$, we have $\sum_{i=1}^{m} xa_ixa_i'a_ixa_i', \sum_{j=1}^{n} xb_jxb_j'a_ixa_i' \in BIJ$. Thus $xa_ixa_i' \in \overline{BIJ}$. Similarly $xb_jxb_j' \in \overline{BIJ}$. Hence from $x + \sum_{i=1}^{m} xa_ixa_i' + z = \sum_{j=1}^{n} xb_jxb_j' + z$ it follows that $x \in \overline{BIJ}$. This shows $B \cap I \cap J \subseteq \overline{BIJ}$.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Since every right h-ideal is an h-quasi-ideal, choose Q a right h-ideal of R and $J = R$. Then by hypothesis $Q \cap I \cap J = Q \cap I \cap R = Q \cap I \subseteq \overline{QIR} \subseteq \overline{QI}$. But $\overline{QI} \subseteq Q \cap I$ always true. Hence $Q \cap I = \overline{QI}$ for every right h-ideal Q and every h-ideal I of R. Thus by Lemma 195, R is right h-weakly regular hemiring. ■

Theorem 293 *The following assertions are equivalent for a hemiring R with identity:*

1. R is right h-weakly regular hemiring.

2. $\lambda \wedge \mu \wedge \nu \leq \lambda \odot_h \mu \odot_h \nu$ for every fuzzy h-bi-ideal λ, every fuzzy h-ideal μ and every fuzzy right h-ideal ν of R.

3. $\lambda \wedge \mu \wedge \nu \leq \lambda \odot_h \mu \odot_h \nu$ for every fuzzy h-quasi-ideal λ, every fuzzy h-ideal μ and every fuzzy right h-ideal ν of R.

Proof. (1) $\Rightarrow$ (2) Let R be a right h-weakly regular hemiring and λ, μ, ν be fuzzy h-bi-ideal, fuzzy h-ideal and fuzzy right h-ideal of R, respectively. Let $x \in R$. Then there exist $a_i, a_i', b_j, b_j', z \in R$ such that $x + \sum_{i=1}^{m} xa_ixa_i' + z = \sum_{j=1}^{n} xb_jxb_j' + z$. Now

$$[(\lambda \odot_h \mu) \odot_h \nu](x) = \bigvee_{x + \sum_{i=1}^{m} c_id_i + z = \sum_{j=1}^{n} c_j'd_j' + z} \left[\begin{array}{c} \bigwedge_{i=1}^{m} ((\lambda \odot_h \mu)(c_i) \wedge \nu(d_i)) \\ \wedge \bigwedge_{j=1}^{n} \left((\lambda \odot_h \mu)(c_j') \wedge \nu(d_j')\right) \end{array} \right]$$

$$\geq \left[\bigwedge_{i=1}^{m}((\lambda \odot_h \mu)(xa_i) \wedge \nu(xa_i')) \wedge \bigwedge_{j=1}^{n}\left((\lambda \odot_h \mu)(xb_j) \wedge \nu(xb_j')\right)\right]$$
$$\geq \left[\bigwedge_{i=1}^{m}((\lambda \odot_h \mu)(xa_i) \wedge \nu(x)) \wedge \bigwedge_{j=1}^{n}((\lambda \odot_h \mu)(xb_j) \wedge \nu(x))\right]$$

As

$$(\lambda \odot_h \mu)(xa_i) = \bigvee_{xa_i+\sum_{i=1}^{m} c_id_i+z=\sum_{j=1}^{n} c_j'd_j'+z} \left[\bigwedge_{i=1}^{m}(\lambda(c_i) \wedge \mu(d_i)) \wedge \bigwedge_{j=1}^{n}\left(\lambda(c_j') \wedge \mu(d_j')\right)\right]$$
$$\geq \left[\bigwedge_{i=1}^{m}(\lambda(x) \wedge \mu(a_ixa_i'a_i)) \wedge \bigwedge_{j=1}^{n}\left(\lambda(x) \wedge \mu(b_jxb_j'a_i)\right)\right]$$
$$\left(\text{because } xa_i + \sum_{i=1}^{m} xa_ixa_i'a_i + za_i = \sum_{j=1}^{n} xb_jxb_j'a_i + za_i\right)$$
$$\geq \left[\bigwedge_{i=1}^{m}(\lambda(x) \wedge \mu(x)) \wedge \bigwedge_{j=1}^{n}(\lambda(x) \wedge \mu(x))\right] = (\lambda \wedge \mu)(x)$$

Similarly $(\lambda \odot_h \mu)(xb_j) \geq (\lambda \wedge \mu)(x)$.

Thus

$$[(\lambda \odot_h \mu) \odot_h \nu](x) \geq \left[\bigwedge_{i=1}^{m}((\lambda \odot_h \mu)(xa_i) \wedge \nu(x)) \wedge \bigwedge_{j=1}^{n}((\lambda \odot_h \mu)(xb_j) \wedge \nu(x))\right]$$
$$\geq (\lambda \wedge \mu)(x) \wedge \nu(x) = ((\lambda \wedge \mu) \wedge \nu)(x).$$

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Since every fuzzy right h-ideal is a fuzzy h-quasi-ideal of R, choose λ a fuzzy right h-ideal of R, μ any fuzzy h-ideal and χ_R fuzzy right h-ideal of R. Then by hypothesis $\lambda \wedge \mu = \lambda \wedge \mu \wedge \chi_R \leq \lambda \odot_h \mu \odot_h \chi_R \leq \lambda \odot_h \mu$. But $\lambda \odot_h \mu \leq \lambda \wedge \mu$ always true. Thus $\lambda \odot_h \mu = \lambda \wedge \mu$ for every fuzzy right h-ideal λ and fuzzy h-ideal μ of R. Hence by Theorem 196, R is right h-weakly regular hemiring. ■

8.2 Prime and semiprime h-bi-ideals

In [46] many characterizations of h-hemiregular, h-intra-hemiregular and of both h-hemiregular and h-intra-hemiregular hemirings are given. In this section we give some more characterizations of these hemirings.

Proposition 294 *Let B_1 and B_2 be h-bi-ideals of a hemiring R. Then $\overline{B_1B_2}$ is an h-bi-ideal of R.*

Proof. Let B_1 and B_2 be h-bi-ideals of R and $x, y \in \overline{B_1B_2}$. Then

$$x + \sum_{i=1}^{n} a_ia_i' + z = \sum_{j=1}^{m} b_jb_j' + z \tag{8.1}$$

and

$$y+\sum_{k=1}^{p} c_k c_k^{'} + z_1 = \sum_{l=1}^{q} d_l d_l^{'} + z_1 \tag{8.2}$$

for $a_i, b_j, c_k, d_l \in B_1$, $a_i^{'}, b_j^{'}, c_k^{'}, d_l^{'} \in B_2$ and $z, z_1 \in R$.

Now

$$x+y+\left(\sum_{i=1}^{n} a_i a_i^{'} + \sum_{k=1}^{p} c_k c_k^{'}\right) + z + z_1 = \left(\sum_{j=1}^{m} b_j b_j^{'} + \sum_{l=1}^{q} d_l d_l^{'}\right) + z + z_1.$$

As $\left(\sum_{i=1}^{n} a_i a_i^{'} + \sum_{k=1}^{p} c_k c_k^{'}\right)$ and $\left(\sum_{j=1}^{m} b_j b_j^{'} + \sum_{l=1}^{q} d_l d_l^{'}\right)$ are in B_1B_2 so $x+y \in \overline{B_1B_2}$.

Multiplying the equation (8.1) by y we get

$$xy+\sum_{i=1}^{n} a_i a_i^{'} y + zy = \sum_{j=1}^{m} b_j b_j^{'} y + zy \tag{8.3}$$

Multiplying the equation (8.2) by $\sum_{i=1}^{n} a_i a_i^{'}$ we get

$$\sum_{i=1}^{n} a_i a_i^{'} y + \left(\sum_{i=1}^{n} a_i a_i^{'}\right)\left(\sum_{k=1}^{p} c_k c_k^{'}\right) + \sum_{i=1}^{n} a_i a_i^{'} z_1 = \left(\sum_{i=1}^{n} a_i a_i^{'}\right)\left(\sum_{l=1}^{q} d_l d_l^{'}\right) + \sum_{i=1}^{n} a_i a_i^{'} z_1 \tag{8.4}$$

As each term in the expression of $\left(\sum_{i=1}^{n} a_i a_i^{'}\right)\left(\sum_{k=1}^{p} c_k c_k^{'}\right)$ is the form $a_i a_i^{'} c_k c_k^{'}$ where $a_i, c_k \in B_1$ and $a_i^{'}, c_k^{'} \in B_2$. Thus $\left(a_i a_i^{'} c_k\right) c_k^{'} \in (B_1RB_1)\, B_2 \subseteq B_1B_2$. This implies $\left(\sum_{i=1}^{n} a_i a_i^{'}\right)\left(\sum_{k=1}^{p} c_k c_k^{'}\right) \in B_1B_2$. Similarly $\left(\sum_{i=1}^{n} a_i a_i^{'}\right)\left(\sum_{l=1}^{q} d_l d_l^{'}\right) \in B_1B_2$. Hence from equation (8.4) we get $\sum_{i=1}^{n} a_i a_i^{'} y \in \overline{B_1B_2}$. Similarly $\sum_{j=1}^{m} b_j b_j^{'} y \in \overline{B_1B_2}$. Thus from equation (8.3) we get $xy \in \overline{B_1B_2}$.

Hence $\overline{B_1B_2}$ is closed under addition and multiplication.

Now

$$\overline{\left(\overline{B_1B_2}\right) R \left(\overline{B_1B_2}\right)} = \overline{(B_1B_2)\, R\, (B_1B_2)} \subseteq \overline{B_1B_2}$$

Let $x+a+z = b+z$ for some $a, b \in \overline{B_1B_2}, x, z \in R$. Then $x \in \overline{\overline{B_1B_2}} = \overline{B_1B_2}$.

Hence $\overline{B_1B_2}$ is an h-bi-ideal of R. ■

Proposition 295 *Let λ, μ be fuzzy h-bi-ideal of a hemiring R. Then $\lambda \odot_h \mu$ is a fuzzy h-bi-ideal of R.*

Proof. Let λ, μ be fuzzy h-bi-ideals of R and $x, y \in R$. Then

$$(\lambda \odot_h \mu)(x) = \bigvee_{x+\sum_{i=1}^{n} a_i a_i^{'} + z = \sum_{j=1}^{m} b_j b_j^{'} + z} \left[\bigwedge_{i=1}^{n}\left[\lambda(a_i) \wedge \mu\left(a_i^{'}\right)\right] \wedge \bigwedge_{j=1}^{m}\left[\lambda(b_j) \wedge \mu\left(b_j^{'}\right)\right]\right] \tag{8.5}$$

and

$$(\lambda \odot_h \mu)(y) = \bigvee_{y+\sum_{k=1}^{p} c_k c_k' + z_1 = \sum_{l=1}^{q} d_l d_l' + z_1} \left[\bigwedge_{k=1}^{p} \left[\lambda(c_k) \wedge \mu\left(c_k'\right) \right] \wedge \bigwedge_{l=1}^{q} \left[\lambda(d_l) \wedge \mu\left(d_l'\right) \right] \right] \quad (8.6)$$

where $a_i, b_j, c_k, d_l, a_i', b_j', c_k', d_l', z, z_1 \in R$.

Now

$$(\lambda \odot_h \mu)(x+y) = \bigvee_{x+y+\sum_{s=1}^{u} e_s f_s + z = \sum_{t=1}^{v} e_t' f_t' + z} \left[\left[\bigwedge_{s=1}^{u} (\lambda(e_s) \wedge \mu(f_s)) \right] \wedge \left[\bigwedge_{t=1}^{v} \left(\lambda(e_t') \wedge \mu(f_t') \right) \right] \right]$$

$$\geq \bigvee_{x+\sum_{i=1}^{n} a_i a_i' + z = \sum_{j=1}^{m} b_j b_j' + z} \left(\bigvee_{y+\sum_{k=1}^{p} c_k c_k' + z_1 = \sum_{l=1}^{q} d_l d_l' + z_1} \left(\begin{array}{l} \bigwedge_{i=1}^{n} \left[\lambda(a_i) \wedge \mu\left(a_i'\right) \right] \wedge \\ \bigwedge_{j=1}^{m} \left[\lambda(b_j) \wedge \mu\left(b_j'\right) \right] \wedge \\ \bigwedge_{k=1}^{p} \left[\lambda(c_k) \wedge \mu\left(c_k'\right) \right] \wedge \\ \bigwedge_{l=1}^{q} \left[\lambda(d_l) \wedge \mu\left(d_l'\right) \right] \end{array} \right) \right)$$

$$= \bigvee_{x+\sum_{i=1}^{n} a_i a_i' + z = \sum_{j=1}^{m} b_j b_j' + z} \left[\bigwedge_{i=1}^{n} \left[\lambda(a_i) \wedge \mu\left(a_i'\right) \right] \wedge \bigwedge_{j=1}^{m} \left[\lambda(b_j) \wedge \mu\left(b_j'\right) \right] \right]$$

$$\wedge \bigvee_{y+\sum_{k=1}^{p} c_k c_k' + z_1 = \sum_{l=1}^{q} d_l d_l' + z_1} \left[\bigwedge_{k=1}^{p} \left[\lambda(c_k) \wedge \mu\left(c_k'\right) \right] \wedge \bigwedge_{l=1}^{q} \left[\lambda(d_l) \wedge \mu\left(d_l'\right) \right] \right]$$

$= (\lambda \odot_h \mu)(x) \wedge (\lambda \odot_h \mu)(y)$.

To prove that $x + a + y = b + y$ implies $(\lambda \odot_h \mu)(x) \geq (\lambda \odot_h \mu)(a) \wedge (\lambda \odot_h \mu)(b)$, observe that

$$a + \sum_{i=1}^{m} a_i b_i + z_1 = \sum_{j=1}^{n} a_j' b_j' + z_1 \text{ and } b + \sum_{k=1}^{l} c_k d_k + z_2 = \sum_{q=1}^{p} c_q' d_q' + z_2, \quad (8.7)$$

together with $x + a + y = b + y$, gives

$$x + a + (\sum_{i=1}^{m} a_i b_i + z_1) + y = b + (\sum_{i=1}^{m} a_i b_i + z_1) + y.$$

Thus,

$$x + \sum_{j=1}^{n} a_j' b_j' + z_1 + y = b + \sum_{i=1}^{m} a_i b_i + z_1 + y$$

and, consequently,

$$\begin{aligned} x + \sum_{j=1}^{n} a_j' b_j' + (\sum_{k=1}^{l} c_k d_k + z_2) + z_1 + y &= b + (\sum_{k=1}^{l} c_k d_k + z_2) + \sum_{i=1}^{m} a_i b_i + z_1 + y \\ &= \sum_{q=1}^{p} c_q' d_q' + z_2 + \sum_{i=1}^{m} a_i b_i + z_1 + y \\ &= \sum_{i=1}^{m} a_i b_i + \sum_{q=1}^{p} c_q' d_q' + z_2 + z_1 + y. \end{aligned}$$

Therefore,

$$x + \sum_{j=1}^{n} a_j' b_j' + \sum_{k=1}^{l} c_k d_k + z_2 + z_1 + y = \sum_{i=1}^{m} a_i b_i + \sum_{q=1}^{p} c_q' d_q' + z_2 + z_1 + y. \quad (8.8)$$

Now, in view of equations (8.7) and (8.8), we have

$$(\lambda \odot_h \mu)(a) \wedge (\lambda \odot_h \mu)(b)$$
$$= \left(\bigvee_{a+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \left(\left[\bigwedge_{i=1}^{m} (\lambda(a_i) \wedge \mu(b_i)) \right] \wedge \left[\bigwedge_{i=1}^{n} \left(\lambda(a'_j) \wedge \mu(b'_j) \right) \right] \right) \right)$$
$$\wedge \left(\bigvee_{b+\sum_{k=1}^{p} c_k d_k + z' = \sum_{l=1}^{q} c'_l d'_l + z'} \left(\left[\bigwedge_{k=1}^{p} (\lambda(c_k) \wedge \mu(d_k)) \right] \wedge \left[\bigwedge_{l=1}^{q} \left(\lambda(c'_l) \wedge \mu(d'_l) \right) \right] \right) \right)$$
$$= \bigvee_{a+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \left(\bigvee_{b+\sum_{k=1}^{p} c_k d_k + z' = \sum_{l=1}^{q} c'_l d'_l + z'} \begin{pmatrix} \bigwedge_{i=1}^{m} (\lambda(a_i) \wedge \mu(b_i)) \wedge \\ \bigwedge_{j=1}^{n} \left(\lambda(a'_j) \wedge \mu(b'_j) \right) \wedge \\ \bigwedge_{k=1}^{p} (\lambda(c_k) \wedge \mu(d_k)) \wedge \\ \bigwedge_{l=1}^{q} \left(\lambda(c'_l) \wedge \mu(d'_l) \right) \end{pmatrix} \right)$$
$$\leq \bigvee_{x+\sum_{s=1}^{u} g_s h_s + z = \sum_{t=1}^{w} g'_t h'_t + z} \left(\bigwedge_{s=1}^{u} (\lambda(g_s) \wedge \mu(h_s)) \wedge \bigwedge_{t=1}^{w} \left(\lambda(g'_t) \wedge \mu(h'_t) \right) \right)$$
$$= (\lambda \odot_h \mu)(x).$$

Now

$$\begin{aligned} (\lambda \odot_h \mu) \odot_h (\lambda \odot_h \mu) &= (\lambda \odot_h \mu \odot_h \lambda) \odot_h \mu \\ &\leq (\lambda \odot_h \chi_R \odot_h \lambda) \odot_h \mu \\ &\leq \lambda \odot_h \mu \end{aligned}$$

Also

$$\begin{aligned} (\lambda \odot_h \mu) \odot_h \chi_R \odot_h (\lambda \odot_h \mu) &= (\lambda \odot_h (\mu \odot_h \chi_R) \odot_h \lambda) \odot_h \mu \\ &\leq (\lambda \odot_h \chi_R \odot_h \lambda) \odot_h \mu \\ &\leq \lambda \odot_h \mu \end{aligned}$$

Thus $\lambda \odot_h \mu$ is a fuzzy h-bi-ideal of R. ■

Definition 296 *Let R be a hemiring. An h-bi-ideal B of R is called prime (resp. semiprime) if $\overline{B_1 B_2} \subseteq B$ $\left(\text{resp. } \overline{B_1^2} \subseteq B\right)$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ (resp. $B_1 \subseteq B$) for all h-bi-ideals B_1, B_2 of R.*

Definition 297 *Let R be a hemiring. An h-bi-ideal B of R is called strongly prime if $\overline{B_1 B_2} \cap \overline{B_2 B_1} \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ for all h-bi-ideals B_1, B_2 of R.*

Obviously every strongly prime h-bi-ideal is a prime h-bi-ideal and every prime h-bi-ideal is a semiprime h-bi-ideal.

Definition 298 *A fuzzy h-bi-ideal λ of a hemiring R is called prime (semiprime) if $\mu \odot_h \nu \leq \lambda$ $(\mu \odot_h \mu \leq \lambda)$ implies $\mu \leq \lambda$ or $\nu \leq \lambda$ $(\mu \leq \lambda)$ for all fuzzy h-bi-ideals μ, ν of R.*

A fuzzy h-bi-ideal λ of a hemiring R is called strongly prime if $\mu \odot_h \nu \wedge \nu \odot_h \mu \leq \lambda$ implies $\mu \leq \lambda$ or $\nu \leq \lambda$ for all fuzzy h-bi-ideals μ, ν of R.

Lemma 299 *Let R be a hemiring, $\{B_i : i \in I\}$ a family of prime h-bi-ideal of R. Then $\bigcap\limits_{i \in I} B_i$ is a semiprime h-bi-ideal of R.*

Proof. Straightforward. ■

Proposition 300 *Let R be a hemiring and μ, ν be fuzzy h-bi-ideals of R then $\mu \wedge \nu$ is also fuzzy h-bi-ideal of R.*

Proof. Let $x, y, z \in R$. Then

(i)

$$\begin{aligned}(\mu \wedge \nu)(x+y) &= \mu(x+y) \wedge \nu(x+y) \geq [\mu(x) \wedge \mu(y)] \wedge [\nu(x) \wedge \nu(y)] \\ &= [\mu(x) \wedge \nu(x)] \wedge [\mu(y) \wedge \nu(y)] = (\mu \wedge \nu)(x) \wedge (\mu \wedge \nu)(y).\end{aligned}$$

(ii)

$$\begin{aligned}(\mu \wedge \nu)(xy) &= \mu(xy) \wedge \nu(xy) \geq \mu(x) \wedge \mu(y) \wedge \nu(x) \wedge \nu(y) \\ &= (\mu \wedge \nu)(x) \wedge (\mu \wedge \nu)(y).\end{aligned}$$

(iii)

$$\begin{aligned}(\mu \wedge \nu)(xyz) &= \mu(xyz) \wedge \nu(xyz) \geq \mu(x) \wedge \mu(z) \wedge \nu(x) \wedge \nu(z) \\ &= (\mu \wedge \nu)(x) \wedge (\mu \wedge \nu)(z).\end{aligned}$$

(iv) Now let $a, b, x, z \in R$ such that $x + a + z = b + z$. Then

$$\begin{aligned}(\mu \wedge \nu)(x) &= \mu(x) \wedge \nu(x) \geq [\mu(a) \wedge \mu(b)] \wedge [\nu(a) \wedge \nu(b)] \\ &= [\mu(a) \wedge \nu(a)] \wedge [\mu(b) \wedge \nu(b)] = (\mu \wedge \nu)(a) \wedge (\mu \wedge \nu)(b).\end{aligned}$$

Hence $\mu \wedge \nu$ is a fuzzy h-bi-ideal of R. ■

Proposition 301 *Let R be a hemiring and $\{\lambda_i : i \in I\}$ a family of fuzzy prime h-bi-ideal of R. Then $\bigwedge_{i\in I} \lambda_i$ is a semiprime fuzzy h-bi-ideal of R.*

Proof. Straight forward. ■

Definition 302 *Let R be a hemiring. An h-bi-ideal B of R is called irreducible (res. strongly irreducible) if $B_1 \cap B_2 = B$ (resp. $B_1 \cap B_2 \subseteq B$) implies $B_1 = B$ or $B_2 = B$ (resp. $B_1 \subseteq B$ or $B_2 \subseteq B$) for all h-bi-ideals B_1, B_2 of R.*

Proposition 303 *Every strongly irreducible semiprime h-bi-ideal of a hemiring R is a strongly prime h-bi-ideal of R.*

Proof. Let B be a strongly irreducible semiprime h-bi-ideal of R. Let B_1, B_2 be any h-bi-ideals of R such that $\overline{B_1B_2} \cap \overline{B_2B_1} \subseteq B$. Since $B_1 \cap B_2$ is an h-bi-ideal and $\overline{(B_1 \cap B_2)(B_1 \cap B_2)} \subseteq \overline{B_1B_2}$, $\overline{(B_1 \cap B_2)(B_1 \cap B_2)} \subseteq \overline{B_2B_1}$.

Thus $\overline{(B_1 \cap B_2)(B_1 \cap B_2)} \subseteq \overline{B_1B_2} \cap \overline{B_2B_1} \subseteq B$. Since B is a semiprime h-bi-ideal of R, we have $B_1 \cap B_2 \subseteq B$. As B is strongly irreducible, we have $B_1 \subseteq B$ or $B_2 \subseteq B$. Thus B is strongly prime h-bi-ideal of R. ■

Proposition 304 *Let R be a hemiring and B be an h-bi-ideal of R. Let $a \in R$ be such that $a \notin B$. Then there exists an irreducible h-bi-ideal A of R such that $B \subseteq A$ and $a \notin A$.*

Proof. Let $\mathcal{F}$ be the collection of all h-bi-ideals of R which contains B but don't contain a. Then $\mathcal{F} \neq \phi$, because $B \in \mathcal{F}$. The collection $\mathcal{F}$ is a partially ordered set under inclusion. As every totally ordered subset in $\mathcal{F}$ is bounded above, so by Zorn's Lemma there exists a maximal element say $A \in \mathcal{F}$. We will show that A is an irreducible h-bi-ideal of R. Let C, D be two h-bi-ideals of R such that $C \cap D = A$. If both C and D properly contains A then $a \in C$ and $a \in D$, then $a \in A$. This contradicts the fact that $a \notin A$. Thus $A = C$ or $A = D$. Hence A is an irreducible h-bi-ideal of R such that $B \subseteq A$ and $a \notin A$. ■

Definition 305 *Let R be a hemiring and λ a fuzzy h-bi-ideal of R then λ is called irreducible (res. strongly irreducible) fuzzy h-bi-ideal of R if $\mu \wedge \nu = \lambda$ (res. $\mu \wedge \nu \leq \lambda$) implies $\mu = \lambda$ or $\nu = \lambda$ (res. $\mu \leq \lambda$ or $\nu \leq \lambda$) for all fuzzy h-bi-ideals μ, ν of R.*

Proposition 306 *Let R be a hemiring. Then every strongly irreducible semiprime fuzzy h-bi-ideal of R is a strongly prime fuzzy h-bi-ideal of R.*

Proof. Let λ be a strongly irreducible semiprime h-bi-ideal of R. Let μ, ν be any fuzzy h-bi-ideals of R such that $(\mu \odot_h \nu) \wedge (\nu \odot_h \mu) \leq \lambda$. As $\mu \wedge \nu$ is a fuzzy h-bi-ideal of R and $(\mu \wedge \nu) \odot_h (\nu \wedge \mu) \leq \mu \odot_h \nu$ and $(\mu \wedge \nu) \odot_h (\nu \wedge \mu) \leq \nu \odot_h \mu$. Thus $(\mu \wedge \nu) \odot_h (\nu \wedge \mu) \leq (\mu \odot_h \nu) \wedge (\nu \odot_h \mu) \leq \lambda$. That is $(\mu \wedge \nu) \odot_h (\nu \wedge \mu) \leq \lambda$.

As λ is semiprime, we have $\mu \wedge \nu \leq \lambda$. Since λ is strongly irreducible, we have $\mu \leq \lambda$ or $\nu \leq \lambda$. Hence λ is strongly prime. ■

Proposition 307 *Let R be a hemiring, λ a fuzzy h-bi-ideal of R with $\lambda(a) = \alpha$, where a is any element of R and $\alpha \in (0,1]$. Then there exists a fuzzy irreducible h-bi-ideal δ of R such that $\lambda \leq \delta$ and $\delta(a) = \alpha$.*

Proof. Let $X = \{\mu : \mu \text{ is a fuzzy } h\text{-bi-ideal of } R,\ \mu(a) = \alpha \text{ and } \lambda \leq \mu\}$. Then $X \neq \phi$, because $\lambda \in X$. Let $\mathcal{F}$ be a totally ordered subset of X, say $\mathcal{F} = \{\lambda_i : i \in I\}$. We claim that $\bigvee_{i \in I} \lambda_i$ is a fuzzy h-bi-ideal of R containing λ.

Let $x, y, z \in R$, consider

(1)

$$\begin{aligned}
\left(\bigvee_i \lambda_i\right)(x) \wedge \left(\bigvee_i \lambda_i\right)(y) &= \left(\bigvee_i (\lambda_i(x))\right) \wedge \left(\bigvee_j (\lambda_j(y))\right) \\
&= \bigvee_j \left[\bigvee_i (\lambda_i(x)) \wedge \lambda_j(y)\right] \\
&= \bigvee_j \left[\bigvee_i (\lambda_i(x) \wedge \lambda_j(y))\right] \\
&\leq \bigvee_j \left[\bigvee_i \left(\lambda_i^j(x) \wedge \lambda_i^j(y)\right)\right] \\
&\text{where } \lambda_i^j = \max\{\lambda_i, \lambda_j\}, \text{ note that } \lambda_i^j \in \{\lambda_i : i \in I\} \\
&\leq \bigvee_j \left[\bigvee_i \left[\lambda_i^j(x+y)\right]\right] \\
&= \bigvee_{i,j} \left[\lambda_i^j(x+y)\right] \\
&\leq \bigvee_i [\lambda_i(x+y)] = \left(\bigvee_i \lambda_i\right)(x+y)
\end{aligned}$$

(2)

$$\begin{aligned}
\left(\bigvee_i \lambda_i\right)(xy) = \bigvee_i (\lambda_i(xy)) &\geq \bigvee_i (\lambda_i(x) \wedge \lambda_j(y)) \\
&= \bigvee_i (\lambda_i(x)) \wedge \bigvee_i (\lambda_j(y)) \\
&= \left(\bigvee_i \lambda_i\right)(x) \wedge \left(\bigvee_i \lambda_i\right)(y)
\end{aligned}$$

(3)

$$\begin{aligned}
\left(\bigvee_i \lambda_i\right)(xyz) &= \bigvee_i (\lambda_i(xyz)) \geq \bigvee_i (\lambda_i(x) \wedge \lambda_j(z)) \\
&= \bigvee_i (\lambda_i(x)) \wedge \bigvee_i (\lambda_j(z))
\end{aligned}$$

$$= \left(\bigvee_i \lambda_i\right)(x) \wedge \left(\bigvee_i \lambda_i\right)(z)$$

(4) Now, let $x + a + z = b + z$, where $a, b \in R$. Then

$$\left(\bigvee_i \lambda_i\right)(a) \wedge \left(\bigvee_i \lambda_i\right)(b) = \left(\bigvee_i (\lambda_i(a))\right) \wedge \left(\bigvee_j (\lambda_j(b))\right)$$
$$= \bigvee_j \left[\left(\bigvee_i (\lambda_i(a))\right) \wedge \lambda_j(b)\right]$$
$$= \bigvee_j \left[\bigvee_i (\lambda_i(a) \wedge \lambda_j(b))\right]$$
$$\leq \bigvee_j \left[\bigvee_i \left(\lambda_i^j(a) \wedge \lambda_i^j(b)\right)\right]$$

where $\lambda_i^j = \max\{\lambda_i, \lambda_j\}$, note that $\lambda_i^j \in \{\lambda_i : i \in I\}$

$$\leq \bigvee_j \left[\bigvee_i \left(\lambda_i^j(x)\right)\right] \text{ because } \lambda_i^j \text{ is a fuzzy } h\text{-bi-ideal}$$
$$= \bigvee_{i,j} \left[\lambda_i^j(x)\right] \leq \bigvee_i [\lambda_i(x)] = \left(\bigvee_i \lambda_i\right)(x)$$

Thus $\bigvee_i \lambda_i$ is a fuzzy h-bi-ideal of R. Clearly $\lambda \leq \bigvee_i \lambda_i$ and $\bigvee_i \lambda_i(a) = \bigvee_i (\lambda_i(a)) = \alpha$. Thus $\bigvee_i \lambda_i$ is the l.u.b of $\mathcal{F}$. Hence by Zorn's lemma there exists a fuzzy h-bi-ideal δ of R which is maximal with respect to the property that $\lambda \leq \delta$ and $\delta(a) = \alpha$.

We will show that δ is fuzzy irreducible h-bi-ideal of R. Let $\delta = \delta_1 \wedge \delta_2$, where δ_1, δ_2 are fuzzy h-bi-ideals of R. Thus $\delta \leq \delta_1$ and $\delta \leq \delta_2$. We claim that either $\delta = \delta_1$ or $\delta = \delta_2$. Suppose $\delta \neq \delta_1$ and $\delta \neq \delta_2$. Since δ is maximal with respect to the property that $\delta(a) = \alpha$ and since $\delta \lneq \delta_1$ and $\delta \lneq \delta_2$, so $\delta_1(a) \neq \alpha$ and $\delta_2(a) \neq \alpha$. Hence $\alpha = \delta(a) = (\delta_1 \wedge \delta_2)(a) = (\delta_1)(a) \wedge (\delta_2)(a) \neq \alpha$, which is impossible. Hence $\delta = \delta_1$ or $\delta = \delta_2$. Thus δ is fuzzy irreducible h-bi-ideal of R. ■

Theorem 308 *For a hemiring R, the following assertions are equivalent:*

1. R is both h-hemiregular and h-intra-hemiregular.
2. $\overline{B^2} = B$ for every h-bi-ideal B of R.
3. $B_1 \cap B_2 = \overline{B_1B_2} \cap \overline{B_2B_1}$ for all h-bi-ideals B_1, B_2 of R.
4. Each h-bi-ideal of R is semiprime.
5. Every proper h-bi-ideal of R is the intersection of all irreducible semiprime h-bi-ideals of R which contain it.

Proof. (1) $\Leftrightarrow$ (2) This is Lemma 63.

(2) $\Rightarrow$ (3) Let B_1, B_2 be h-bi-ideals of R. Then $B_1 \cap B_2$ is also an h-bi-ideal of R. By hypothesis $B_1 \cap B_2 = \overline{(B_1 \cap B_2)^2} \subseteq \overline{B_1B_2}$. Similarly $B_1 \cap B_2 = \overline{(B_1 \cap B_2)^2} \subseteq \overline{B_2B_1}$.

Thus $B_1 \cap B_2 \subseteq \overline{B_1B_2} \cap \overline{B_2B_1}$. By Proposition 294 $\overline{B_1B_2}$ and $\overline{B_2B_1}$ are h-bi-ideals of R and so $\overline{B_1B_2} \cap \overline{B_2B_1}$ is an h-bi-ideal of R. Thus by the hypothesis

$$\begin{aligned}\overline{B_1B_2} \cap \overline{B_2B_1} &= \overline{(\overline{B_1B_2} \cap \overline{B_2B_1})\,(\overline{B_1B_2} \cap \overline{B_2B_1})} \\ &\subseteq \overline{\overline{B_1B_2}\;\overline{B_2B_1}} = \overline{B_1B_2B_2B_1} \subseteq \overline{B_1RB_1} \subseteq B_1\end{aligned}$$

Similarly $\overline{B_1B_2} \cap \overline{B_2B_1} \subseteq B_2$. Hence $\overline{B_1B_2} \cap \overline{B_2B_1} \subseteq B_1 \cap B_2$.

This implies $\overline{B_1B_2} \cap \overline{B_2B_1} = B_1 \cap B_2$.

(3) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (4) Let B_1, B_2 be h-bi-ideals of R such that $B_1^2 \subseteq B_2$. Then $\overline{B_1^2} \subseteq B_2$. By hypothesis $B_1 = \overline{B_1^2} \subseteq B_2$. Thus B_2 is semiprime.

(4) $\Rightarrow$ (2) Obvious.

(4) $\Rightarrow$ (5) Let B be a proper h-bi-ideal of R. Then B is contained in the intersection of all irreducible h-bi-ideals of R which contain it. Proposition 304, guarantees the existence of such irreducible h-bi-ideals. If $a \notin B$, then there exists an irreducible h-bi-ideal of R which contain it but does not contain a. Hence B is the intersection of all irreducible h-bi-ideals of R which contain it. By hypothesis each h-bi-ideal is semiprime, so B is the intersection of all irreducible semiprime h-bi-ideals of R which contain it.

(5) $\Rightarrow$ (2) Let B be an h-bi-ideal of R. If $\overline{B^2} = R$ then clearly $B = R$. If $\overline{B^2} \neq R$, then $\overline{B^2}$ is a proper h-bi-ideal of R containing $\overline{B^2}$ and so by our hypothesis,

$$\overline{B^2} = \bigcap \{B_\alpha : B_\alpha \text{ irreducible semiprime } h\text{-bi-ideals of } R\}.$$

Since each B_α is a semiprime h-bi-ideal, $B \subseteq B_\alpha$ for all α, and so $B \subseteq \bigcap_\alpha B_\alpha = \overline{B^2}$. Thus $\overline{B^2} = B$. ■

Theorem 309 *Let R be an h-hemiregular and h-intra-hemiregular hemiring and B be an h-bi-ideal of R. Then B is strongly irreducible if and only if B is strongly prime.*

Proof. Proof follows from Theorem 308. ■

Theorem 310 *Each h-bi-ideal of a hemiring R is strongly prime if and only if R is h-hemiregular, h-intra-hemiregular and the set of h-bi-ideals of R is totally ordered by inclusion.*

Proof. Suppose that each h-bi-ideal of R is strongly prime. Then each h-bi-ideal of R is semiprime. Thus by Theorem 308, R is both h-hemiregular and h-intra-hemiregular. Now we show that the set of h-bi-ideals of R is totally ordered. Let B_1 and B_2 be any two h-bi-ideals of R. Then by Theorem 308, $B_1 \cap B_2 = \overline{B_1B_2} \cap \overline{B_2B_1}$. As each h-bi-ideal is strongly prime, $B_1 \cap B_2$ is strongly prime. Hence either $B_1 \subseteq B_1 \cap B_2$ or $B_2 \subseteq B_1 \cap B_2$ that is either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.

Conversely, assume that R is h-hemiregular, h-intra-hemiregular and the set of h-bi-ideals of R is totally ordered. We show that each h-bi-ideal of R is strongly prime. Let B, B_1, B_2 be h-bi-ideals of R such that $\overline{B_1B_2} \cap \overline{B_2B_1} \subseteq B$. Since R is both h-hemiregular and h-intra-hemiregular, by Theorem 308, $B_1 \cap B_2 = \overline{B_1B_2} \cap \overline{B_2B_1}$. Since $\overline{B_1B_2} \cap \overline{B_2B_1} \subseteq B$, so $B_1 \cap B_2 \subseteq B$. As the set of h-bi-ideals of R is totally ordered, so either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$, that is, either $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$. Thus either $B_1 \subseteq B$ or $B_2 \subseteq B$. ■

Theorem 311 *If the set of h-bi-ideals of R is totally ordered, then R is both h-hemiregular and h-intra-hemiregular if and only if each h-bi-ideal of R is prime.*

Proof. Suppose that R is both h-hemiregular and h-intra-hemiregular. Let B, B_1, B_2 be h-bi-ideals of R such that $\overline{B_1B_2} \subseteq B$. Since the set of h-bi-ideals of R is totally ordered, either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Suppose $B_1 \subseteq B_2$. Then $\overline{B_1^2} \subseteq \overline{B_1B_2} \subseteq B$. By Theorem 308, B is semiprime so $B_1 \subseteq B$. Hence B is prime h-bi-ideal of R.

Conversely, assume that every h-bi-ideal of R is prime. Thus every h-bi-ideal of R is semiprime. Hence by Theorem 308, R is both h-hemiregular and h-intra-hemiregular. ■

Theorem 312 *Let R be a hemiring. Then the following are equivalent:*

1. The set of h-bi-ideals of R is totally ordered under inclusion.

2. Each h-bi-ideal of R is strongly irreducible.

3. Each h-bi-ideal of R is irreducible.

Proof. (1) $\Rightarrow$ (2) Let B, B_1, B_2 be h-bi-ideals of R such that $B_1 \cap B_2 \subseteq B$. Since the set of h-bi-ideals of R is totally ordered under inclusion, so either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Thus either $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$. Hence $B_1 \cap B_2 \subseteq B \Rightarrow B_1 \subseteq B$ or $B_2 \subseteq B$. Thus B is strongly irreducible h-bi-ideal of R.

(2) $\Rightarrow$ (3) Let B, B_1, B_2 be h-bi-ideals of R such that $B_1 \cap B_2 = B$. Then $B \subseteq B_1$ or $B \subseteq B_2$. By the hypothesis, either $B_1 \subseteq B$ or $B_2 \subseteq B$. Hence either $B_1 = B$ or $B_2 = B$. Thus B is irreducible.

(3) $\Rightarrow$ (1) Let B_1, B_2 be two h-bi-ideals of R. Then $B_1 \cap B_2$ is an h-bi-ideal of R. Also $B_1 \cap B_2 = B_1 \cap B_2$. So by hypothesis either $B_1 = B_1 \cap B_2$ or $B_2 = B_1 \cap B_2$, i.e.; either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Hence the set of h-bi-ideals of R is totally ordered. ■

Theorem 313 *For a hemiring R, the following assertions are equivalent:*

1. R is both h-hemiregular and h-intra-hemiregular.

2. $\lambda \odot_h \lambda = \lambda$ for every fuzzy h-bi-ideal λ of R.

3. $\lambda \wedge \mu = (\lambda \odot_h \mu) \wedge (\mu \odot_h \lambda)$ for all fuzzy h-bi-ideals λ, μ of R.

4. Each fuzzy h-bi-ideal of R is semiprime.

5. Each proper fuzzy h-bi-ideal of R is the intersection of all irreducible semiprime fuzzy h-bi-ideals of R which contain it.

Proof. (1) $\Rightarrow$ (2) This is Theorem 64.

(2) $\Rightarrow$ (3) Let λ, μ be fuzzy h-bi-ideals of R. Then $\lambda \wedge \mu$ is also fuzzy h-bi-ideal of R. By hypothesis $\lambda \wedge \mu = (\lambda \wedge \mu) \odot_h (\lambda \wedge \mu) \leq \lambda \odot_h \mu$. Similarly $\lambda \wedge \mu \leq \mu \odot_h \lambda$. Thus $\lambda \wedge \mu \leq (\lambda \odot_h \mu) \wedge (\mu \odot_h \lambda)$.

Now by Proposition 295, $(\lambda \odot_h \mu)$ and $(\mu \odot_h \lambda)$ are fuzzy h-bi-ideals of R and so $(\lambda \odot_h \mu) \wedge (\mu \odot_h \lambda)$ is a fuzzy h-bi-ideal of R. Thus by hypothesis

$$\begin{aligned} ((\lambda \odot_h \mu) \wedge (\mu \odot_h \lambda)) &= ((\lambda \odot_h \mu) \wedge (\mu \odot_h \lambda)) \odot_h ((\lambda \odot_h \mu) \wedge (\mu \odot_h \lambda)) \\ &\leq (\lambda \odot_h \mu) \odot_h (\mu \odot_h \lambda) \leq \lambda \odot_h \chi_R \odot_h \lambda \leq \lambda. \end{aligned}$$

Similarly $((\lambda \odot_h \mu) \wedge (\mu \odot_h \lambda)) \leq \mu$. Thus $(\lambda \odot_h \mu) \wedge (\mu \odot_h \lambda) \leq \lambda \wedge \mu$. Hence $(\lambda \odot_h \mu) \wedge (\mu \odot_h \lambda) = \lambda \wedge \mu$.

(3) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (4) Let λ, μ be fuzzy h-bi-ideals of R such that $\lambda \odot_h \lambda \leq \mu$. Since by (2) $\lambda \odot_h \lambda = \lambda$, so $\lambda \leq \mu$. Thus μ is semiprime.

(4) $\Rightarrow$ (2) Obvious.

(4) $\Rightarrow$ (5) Let λ be a proper fuzzy h-bi-ideal of R. Then λ is contained in the intersection of all irreducible fuzzy h-bi-ideals of R which contain it. Proposition 307, guarantees the existence of such irreducible fuzzy h-bi-ideals. If $a \in R$ and $t \in (0,1]$ such that $\lambda(a) = t$, then there exists an irreducible fuzzy h-bi-ideal μ_α such that $\lambda \leq \mu_\alpha$ and $\mu_\alpha(a) = t$. Hence λ is the intersection of all irreducible fuzzy h-bi-ideals of R which contain it. By hypothesis each fuzzy h-bi-ideal is semiprime. Thus λ is the intersection of all irreducible, semiprime fuzzy h-bi-ideals of R which contain it.

(5) $\Rightarrow$ (2) Let λ be a fuzzy h-bi-ideal of R. Then $\lambda \odot_h \lambda$ is a fuzzy h-bi-ideal of R, so $\lambda \odot_h \lambda = \bigwedge \lambda_\alpha$ where λ_α are irreducible, semiprime fuzzy h-bi-ideals of R which contain $\lambda \odot_h \lambda$. Since each λ_α is semiprime, so $\lambda \leq \lambda_\alpha$ for all α. Thus $\lambda \leq \bigwedge \lambda_\alpha = \lambda \odot_h \lambda$. But $\lambda \odot_h \lambda \leq \lambda$ always. Hence $\lambda = \lambda \odot_h \lambda$. ■

Theorem 314 *Let R be an h-hemiregular and h-intra-hemiregular hemiring and λ be a fuzzy h-bi-ideal of R. Then λ is strongly irreducible if and only if λ is strongly prime.*

Proof. Proof follows from Theorem 313. ■

Theorem 315 *Each fuzzy h-bi-ideal of a hemiring R is strongly prime if and only if R is h-hemiregular, h-intra-hemiregular and the set of fuzzy h-bi-ideals of R is totally ordered.*

Proof. Suppose that each fuzzy h-bi-ideal of R is strongly prime. Then each fuzzy h-bi-ideal of R is semiprime. Thus by Theorem 313, R is both h-hemiregular and h-intra-hemiregular. Now we show that the set of fuzzy h-bi-ideals of R is totally ordered. Let λ and μ be any two fuzzy h-bi-ideals of R. Then by Theorem 313, $\lambda \wedge \mu = (\lambda \odot_h \mu) \wedge (\mu \odot_h \lambda)$. As each fuzzy h-bi-ideal is strongly prime, $\lambda \wedge \mu$ is strongly prime. Hence either $\lambda \leq \lambda \wedge \mu$ or $\mu \leq \lambda \wedge \mu$ that is either $\lambda \leq \mu$ or $\mu \leq \lambda$.

Conversely, assume that R is h-hemiregular, h-intra-hemiregular and the set of fuzzy h-bi-ideals of R is totally ordered. We show that each fuzzy h-bi-ideal of R is strongly prime. Let λ, μ, ν be fuzzy h-bi-ideals of R such that $(\mu \odot_h \nu) \wedge (\nu \odot_h \mu) \leq \lambda$. Since R is both h-hemiregular and h-intra-hemiregular, by Theorem 313, $\mu \wedge \nu = (\mu \odot_h \nu) \wedge (\nu \odot_h \mu)$. Since $(\mu \odot_h \nu) \wedge (\nu \odot_h \mu) \leq \lambda$, so $\mu \wedge \nu \leq \lambda$. As the set of fuzzy h-bi-ideals of R is totally ordered, so either $\mu \leq \nu$ or $\nu \leq \mu$, that is, either $\mu \wedge \nu = \mu$ or $\mu \wedge \nu = \nu$. Thus either $\mu \leq \lambda$ or $\nu \leq \lambda$. ■

Theorem 316 *If the set of fuzzy h-bi-ideals of R is totally ordered, then R is both h-hemiregular and h-intra-hemiregular if and only if each fuzzy h-bi-ideal of R is prime.*

Proof. Suppose that R is both h-hemiregular and h-intra-hemiregular. Let λ, μ, ν be fuzzy h-bi-ideals of R such that $\mu \odot_h \nu \leq \lambda$. Since the set of fuzzy h-bi-ideals of R is totally ordered, either $\mu \leq \nu$ or $\nu \leq \mu$. Suppose $\mu \leq \nu$. Then $\mu \odot_h \mu \leq \mu \odot_h \nu \leq \lambda$. By Theorem 313, λ is semiprime so $\mu \leq \lambda$. Hence λ is prime.

Conversely, assume that every fuzzy h-bi-ideal of R is prime. Thus every fuzzy h-bi-ideal of R is semiprime. Hence by Theorem 313, R is both h-hemiregular and h-intra-hemiregular. ■

Theorem 317 *Let R be a hemiring. Then the following are equivalent:*

1. The set of all fuzzy h-bi-ideals of R is totally ordered under inclusion.
2. Each fuzzy h-bi-ideal of R is strongly irreducible.
3. Each fuzzy h-bi-ideal of R is irreducible.

Proof. (1) $\Rightarrow$ (2) Let μ, ν, λ be any fuzzy h-bi-ideals of R such that $\mu \wedge \nu \leq \lambda$. Since the set of fuzzy h-bi-ideals of R is totally ordered, so either $\mu \leq \nu$ or $\nu \leq \mu$.

Therefore $\mu \wedge \nu = \mu$ or $\mu \wedge \nu = \nu$. Hence $\mu \wedge \nu \leq \lambda \Rightarrow$ either $\mu \leq \lambda$ or $\nu \leq \lambda$. Hence λ is strongly irreducible.

(2) $\Rightarrow$ (3) Let μ, ν, λ be any fuzzy h-bi-ideals of R such that $\mu \wedge \nu = \lambda$. Then $\lambda \leq \mu$ or $\lambda \leq \nu$. By hypothesis, either $\mu \leq \lambda$ or $\nu \leq \lambda$. So either $\mu = \lambda$ or $\nu = \lambda$. Thus λ is irreducible.

(3) $\Rightarrow$ (1) Let μ, λ be any fuzzy h-bi-ideals of R. Then $\lambda \wedge \mu$ is a fuzzy h-bi-ideal of R. Also $\lambda \wedge \mu = \lambda \wedge \mu$. So by hypothesis, either $\lambda = \lambda \wedge \mu$ or $\mu = \lambda \wedge \mu$, that is either $\lambda \leq \mu$ or $\mu \leq \lambda$. Therefore the set of fuzzy h-bi-ideals of R is totally ordered. ■

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