Interaction of peristaltic motion with rheological properties of fluids in the symmetric and asymmetric channels

BY

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2011

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وَمَن يُؤْتَ الْحِكْمَةَ فَقَدْ أُوتِيَ خَيْرًا كَثِيرًا

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2011

بسم الله الرحمن الرحيم

In the name of Allah

the most Gracious,

the most Compassionate

Interaction of peristaltic motion with rheological properties of fluids in the symmetric and asymmetric channels

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Najma Saleem

A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF

**DOCTOR OF PHILOSOPHY
IN
MATHEMATICS**

Department of Mathematics
Quaid-i-Azam University
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CERTIFICATE

A THESIS SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE DOCTOR OF PHILOSOPHY

We accept this thesis as conforming to the required standard

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2011

Dedicated to

My parents, who have always been a source of prayers, love, and strength for me.

and also to my grandparents (late).

Preface

The range of peristaltic flows that occurs in physiology and engineering is very large. In particular such flows are encountered in the esophagus, bile ducts, the ureter, the gastrointestinal tract, small blood vessels and many other glandular ducts throughout the body. The principle of peristalsis is also quite common in the industrial applications for instance the transport of sanitary and corrosive fluids and blood pumps in the heart lung machine. It has been an established fact that most of the fluids in physiology are non-Newtonian. The motion of non-Newtonian fluids is an important topic in the field of chemical, biomedical and environmental engineering and science. The governing equations in the non-Newtonian fluids are of higher order than the Navier-Stokes equations. The mathematical models of peristaltic flows involving non-Newtonian fluids are of more intricate in nature. Such flows in the context of magnetohydrodynamics are of great interest for the movement of physiological fluids, for example, the blood and in view of analysis of peristaltic MHD compressor. However, very little has been reported yet on the peristaltic flows in the presence of an induced magnetic field. On the other hand, majority of available literature on the peristaltic flows analyzed the situation when no-slip condition has been considered. Such condition is not reliable especially in polymeric liquids with high molecular weight. No-slip condition is also not appropriate in physiological flows, thin film problems, rarefied fluid problems and flow on multiple interfaces.

Another aspect which is not yet given due attention in the literature is the heat and mass transfer effects on the peristaltic flows of non-Newtonian fluids. No doubt, heat transfer in tissues is subject to heat conduction in tissues, heat convection because of blood flow through the pores of tissues and heat radiation between surface and its environment. The heat transfer consideration in blood is very important in the oxygenation and hemodialysis processes. The simultaneous influence of heat and mass transfer in the peristalsis is also significant when one desires to analyze the Soret and Dufour effects.

Motivated by the above discussion, the main objective here is to examine the effects of rheological properties, an induced magnetic field, partial slip features and heat and mass transfer on the peristaltic flows. This thesis is structured as follows.

Chapter one is prepared for the brief review on the peristaltic flows and some relevant definitions. Analysis for the peristaltic flow of a Carreau fluid in a planar channel has been carried out in chapter two. Symmetric nature of flow is considered when the Reynolds number is low and wavelength is long. The results for different wave forms are established and compared. The pumping and trapping phenomena are given proper attention. It is noticed that the velocity at the center of channel and bolus size decrease when there is an increase in the Weissenberg number. The findings of this chapter have been published in **Numerical Methods for Partial Differential Equations 26, 519 (2010)**. Chapter three extends the research work of chapter two in the presence of an applied magnetic field. The governing equations are developed and analysis has been performed when magnetic Reynolds number is small. It is concluded that longitudinal velocity reduces in a magnetohydrodynamic (MHD) fluid. Further, the bolus size in MHD case also decreases when compared with the hydrodynamic fluid. These observations have been accepted for publications in **Zeitschrift**

Naturforschung A 66a, 215 (2011). The influence of an induced magnetic field on the flow analysis discussed in chapter three is seen in chapter four. Mathematical modelling is presented in detail. Besides the flow quantities constructed in the previous chapters, the expressions of magnetic force function and axial induced magnetic field have been developed additionally. It is found that an axial induced magnetic field exhibits symmetric nature about the origin. Moreover, axial induced magnetic field is decreased in MHD fluid. The behaviour of current density near the channel walls in MHD case is quite opposite to that of an induced magnetic field. Such conclusions have been published in **Comm. Nonlinear Sci. Numer. Simulation 15, 2407 (2010).** Chapter five describes the influence of an induced magnetic field on the peristaltic flow of a Carreau fluid in an asymmetric channel. The heat transfer is also taken into account. The walls of channel have different temperatures. The relevant equations are first modeled and then solved. It is shown that pumping rate in MHD fluid decreases. The axial induced magnetic field about the origin is not symmetric. This is because of the phase difference in the considered shapes of the channel walls. The temperature is an increasing function of Brinkman number. The contents of this chapter have been submitted for publication in **Comm. Nonlinear Sci. Numer. Simulation 16, 3559 (2011).** Chapter six is devoted to the peristaltic flow of hydrodynamic Carreau fluid in an asymmetric channel in the presence of partial slip and heat transfer. The associated equations and boundary conditions are developed. The partial slip condition in terms of shear stress is accounted. It is seen that there is a critical value of mean flow rate for which the frictional forces resists the flow along the channel walls. Below this critical value, the frictional force is an increasing function of slip parameter. This research material has been accepted for publication in **Zeitschrift Naturforschung A 65a, 1121 (2010).**

Chapter seven provides the analysis for peristaltic flow of a second order fluid in a planar channel. The flow is symmetric and fluid is electrically conducting. Induced magnetic field effect is included. The relevant equations are modeled and solved for small wave number. Trapping and pumping are also examined. Better pumping performance is achieved when there is an increase in the material parameters of second order fluid. The magnetic force function is an increasing function of viscoelastic parameter. The role of viscoelastic parameter on the current density distribution is qualitatively similar to that of the magnetic force function. The size of the trapped bolus decreases when the viscoelastic parameter increases. The contents of this chapter have been accepted for publication in **Int. J. Numerical Methods Fluids**. The considered flow problem in chapter seven in the presence of heat and mass transfer has been studied in chapter eight. The energy and concentration laws are examined additionally. Main conclusions of this chapter are accepted now in **Zeitschrift Naturforschung A**. The peristaltic flow of micropolar fluid in an asymmetric channel is discussed in chapter nine. The simultaneous effects of partial slip and heat and mass transfer are seen. Results for flow quantities like stream function, axial and microrotation velocities, temperature and concentration etc are presented and analyzed in detail. It is found that behaviour of slip and microrotation parameters on the longitudinal pressure gradient is qualitatively similar. The pressure gradient is an increasing function of coupling parameter. The pressure rise increases when slip parameter increases. The slip parameter on the temperature has opposite effect when compared with the pressure rise. However, the effect of slip parameter on the concentration field is similar to that of pressure rise in a qualitative sense. Such observations have been accepted for publication in **Chiense Phys. Letters**.

Acknowledgements

All praises for Allah Almighty, Creator, Sustainer, Lord of lords and Sovereign, who blessed me with courage and ability to achieve this task. I am nothing without my Allah but I can achieve everything with His assistance. All of my veneration and devotion goes to our beloved prophet Hazrat Muhammad (Pease be upon him), the source of kindness, humanity and guidance for the whole creatures. May Allah shower His countless blessings and salaam upon him, his family and companions.

First of all, I present my deepest gratitude and obliged to my supervisor **Dr. Tasawar Hayat** for his vigorous supervision, commitment, encouragement and guidance throughout my research period during Ph. D. I learned from him, how to live a committed, challenging and disciplined life.

My immense gratitude goes to all my teachers specially chair person of Mathematics department **Prof. Dr Ayub, Dr. M. Yousaf, Waseem Siddiqui, Qari Naseer Ahmad** (late) for their help and guidance. Also I would like to mention my school teachers **Mrs. Tasleem Pervaiz, Mrs. Shabana** and **Mrs. Nusrat Zahid**, they developed my interest in mathematics and always encouraged me.

The most important personalities in my life are my parents. I have no words to say them thanks. They are always a root of love, prayers, confidence and inspiration for me. More than that, they understand me like friends and support me in everywhere. I also pay special thanks to my Grandparents (late) for their affection and love. I owe my sincere thanks to my sisters (**Shazia, Tahira, ashii**) and my brothers (**Jamshaid** and **Ahmed**) for their great concern, love and close friendship. I would also like to say thanks **Safdar bhaii**, **Javid bhaii**, my **Khala** and all the well wishers.

I offer my cordial thanks to my roommate and sincere friends **Saira** and **Khalida** for their affection, care and support. During my stay at hostel, Saira helped and encouraged me a lot and I really enjoyed her company.

I am greatly indebted to **Nasir bhaii** for his help and guidance in research work. I also owe my sincere thanks to my research fellows **Rabia Noreen, Sadia, Saima, Hafiz Obaid, Yasir, Sehrish, Rahila, Maryam, Qasim, Safia, Ansa** and **Hafiz Sufian** for their suggestions and prolific discussions.

I would also offer a tribute and thanks to my friends **Tayyiba, Nusrat, Eman, Saba, Rabia, Hina, Tamkeen, Tabinda** and **Maryam** for their moral support and sincerety.

Najma Saleem
March 2011

Contents

1 Some relevant definitions and review 5

1.1 Peristaltic pumping 5

1.2 Review of literature 5

1.3 Some basic definitions 10

1.3.1 Retrograde pumping 10

1.3.2 Peristaltic pumping 10

1.3.3 Free pumping 10

1.3.4 Augmented pumping (or copumping) 11

1.3.5 Trapping 11

1.4 Basic equations 11

1.4.1 The continuity equation 11

1.4.2 The equation of motion 11

1.4.3 Maxwell's equations 12

1.4.4 (a) Gauss's law 12

1.4.5 (b) Gauss's law for magnetism 12

1.4.6 (c) Faraday's law 12

1.4.7 (d) Amperes law 12

1.4.8 (e) Ohms' law 12

1.4.9 Energy equation 13

1.4.10 Concentration equation 13

2 Peristaltic transport of a Carreau fluid in a channel with different wave forms 14

2.1 Governing equations 14
2.2 Problem statement 15
2.3 Perturbation solution 19
2.3.1 System for We^0 19
2.3.2 System for We^2 19
2.3.3 Solution for We^0 system 20
2.3.4 Solution for We^2 system 20
2.4 Expression for five wave shapes 21
2.5 Discussion 38
2.5.1 Pressure gradient 38
2.5.2 Pumping characteristics 39
2.5.3 Trapping 39

3 MHD peristaltic flow of a Carreau fluid in a channel **40**
3.1 Theory 40
3.2 Perturbation solution 42
3.2.1 $O(We^0)$ System 42
3.2.2 $O(We^2)$ System 43
3.2.3 Solution for $O(We^0)$ system 43
3.2.4 Solution for $O(We^2)$ system 43
3.3 Expression for wave shapes 45
3.4 Results and discussion 56
3.5 Concluding remarks 57

4 Influence of induced magnetic field on peristaltic flow in a Carreau fluid **58**
4.1 Development of mathematical problem 58
4.2 Perturbation solution 62
4.2.1 $O(We^0)$ System 62
4.2.2 $O(We^2)$ System 63
4.3 Results and discussion 84
4.4 Concluding remarks 86

5 **Influence of induced magnetic field and heat transfer on peristaltic transport of a Carreau fluid** 88
5.1 Development of the mathematical problem . . . 88
5.2 Perturbation solution . . . 92
5.2.1 $O(We^0)$ System . . . 92
5.2.2 $O(We^2)$ System . . . 93
5.3 Discussion . . . 115

6 **Peristaltic motion of a Carreau fluid in an asymmetric channel with partial slip and heat transfer characteristics** 117
6.1 Mathematical formulation . . . 117
6.2 Perturbation solution . . . 120
6.2.1 System of order We^0 . . . 121
6.2.2 System of order We^2 . . . 121
6.2.3 Solution for system of order We^0 . . . 122
6.2.4 Solution for system of order We^2 . . . 122
6.3 Discussion . . . 132

7 **Peristaltic flow of a second order fluid in the presence of an induced magnetic field** 138
7.1 Basic equations . . . 138
7.2 Perturbation solution . . . 143
7.2.1 Zeroth order system . . . 143
7.2.2 First order system . . . 144
7.2.3 Solution of zeroth order system . . . 144
7.2.4 Solution of first order system . . . 146
7.3 Analysis . . . 155

8 **Simultaneous effects of induced magnetic field and heat and mass transfer on the peristaltic motion of second order fluid in a channel** 165
8.1 Problem formulation . . . 165

8.2 Perturbation solution 170

8.2.1 Zeroth order system 171

8.2.2 First order system 171

8.3 Results and discussion 232

9 A mathematical model for the study of slip effects on the peristaltic motion of micropolar fluid with heat and mass transfer 234

9.1 Development of the mathematical problem 234

9.2 Exact solution 238

9.3 Discussion 280

Chapter 1

Some relevant definitions and review

The objective of this chapter is to present brief review for the dynamics of peristaltic flows and few standard definitions.

1.1 Peristaltic pumping

Peristaltic pumping is the process by which fluid can be transported due to travelling waves imposed on the walls of a tube or channel. Such process is quite useful in the situations where one prefers to avoid any internal moving parts for instance pistons in the pumping mechanism. There are many physiological and engineering phenomena where such process play a vital role. In the physiological world, the peristalsis has ample applications in the transport of urine from kidneys to the bladder, chyme in the gastrointestinal tract, spermatozoa in the ducts deferens of the male reproductive tract, ovum in the female fallopian tube, vasomation of blood vessels and several others. Roller and finger pumps also work according to this mechanism. The process of peristalsis is significant for water translocation in tall trees.

1.2 Review of literature

It is noticed from the available literature that peristaltic transport subject to various approximations have been discussed both theoretically and experimentally. Latham [1] initiated the research on the peristalsis by performing experiments. After such fundamental research, this topic has received great attention in different situations. As per our information, next attempt has been presented by Shapirom [2]. He discussed the peristaltic pumping and retrograde pumping phenomena especially. Metry and Chauvet [3] addressed the flow problem characterizing the intestinal peristaltic waves. Numerical solution has been obtained for the resulting problem. It is shown that the presented numerical solution has a very good agreement with the experimental data. The peristaltic activity regarding embryo transport within the uterine cavity in the tapered channel has been explained by Eytan et al. [4]. Here various characteristics of geometry, flow pattern, pressure distribution and reflux have been examined in detail. A novel electrostatic microchannel pump has been designed and simulated by Teymoori and Sani [5]. This microchannel pump works in view of peristalsis. They have studied drug delivery micropumps (which satisfy conditions such as drug compatibility, actuation safety, flow rate, self-priming, chip size, controlability of flow rate for all times and power consumption) and a novel micropump. The peristaltic motion of carrying and mixing chyme in small intestine under long wavelength assumption has been analyzed by Lew et al. [6]. They obtained two solutions of the problem (i) a peristaltic carrying without net pressure gradient and (ii) a peristaltic compression with net transport of the fluid. The total solution is shown as a linear combination of the two solutions. Lew and Fung [7] discussed the case for low Reynolds number in a valved vessel with special reference to flow in a valved vessel lie in veins and lymphatic ducts. They modeled the governing equations and obtained a series solution of the resulting problem. Particularly the change in the mean pressure and the velocity distribution along the tube axis for different orifice tube radius and the ratio of the inertia-orifice distance to the tube radius have been computed. The inertia and streamline curvature effects on peristaltic pumping have been seen by Jaffrin [8]. Such influence can be predicted in roller pump and gastrointestinal tracts. Walker and Shelley [9] discussed the shape optimization of peristaltic pumping. Rao and Mishra [10] have shared experimental conclusions and pattern of peristaltic pumping in porous tube. Zien and Ostrach [11] have reported that in human ureters (which are tubular organs connecting kidneys to the bladder) the wavelength of the peristaltic wave is very small when compared to half width of the channel. They studied an incompressible viscous fluid in a two dimensional channel when long wavelength approximation is accounted. Vajravelu et al. [12] considered the peristaltic pumping of a Herschel-Bulkley fluid in a channel. Gupta and Seof wall properties and heat transfer on peristaltic transport of an incompressible viscous fluid has been presented by Radhakrishnamacharya and Srinivasulu [35]. Vajravelu et al. [36] have investigated the peristaltic flow under long wavelength assumption by explaining the effect of heat transfer in a vertical porous annulus. Goldstein et al. [37] have presented a detail review regarding heat transfer upto 2003. Hayat et al. [38] have examined the effects of slip condition and heat transfer on peristaltic flow. Nadeem et al. [39] have investigated the influence of heat and mass transfer on peristaltic flow of a third order fluid in a diverging tube. In these investigations fluid considered is hydrodynamic. In magnetohydrodynamics, Kothandapani and Srinivas [40] studied the peristaltic flow of a Jeffery fluid under the effect of magnetic field in an asymmetric channel. Mekheimer [41] considered the peristaltic flow of blood under the effect of constant magnetic field in a non-uniform channel. The effect of an induced magnetic field is not taken into account. Hayat et al. [42] and Hayat and Ali [43] have studied the effect of magnetic field on peristaltic flow of third order fluid in an asymmetric channel and Jeffery fluid in a tube respectively. Hayat et al. [44] have seen the influence of applied magnetic field on the peristaltic flow of Johnson-Segalman fluid in a channel with complaint walls. Wang et al. [45] have studied the MHD peristaltic flow of a Sisko fluid in symmetric and asymmetric channels. Hakeem et al. [46] have examined the ffects of magnetic field on trapping through peristaltic motion of Carreau fluid in a uniform channel. This study is an extension of the work done by Misery et al. [47] and Elshehawey et al. [48]. Effects of endoscope and magnetic field on peristaltic transport of Jeffery fluid are explained by Hayat et al. [49]. A mathematical model for peristaltic flow of blood under the action of magnetic field is constructed by Tzirtzilakis [50]. Stud et al. [51] provided the behaviour and pattern of pumping action of blood due to a constant applied magnetic field. Elshehawey et al. [52] have studied the influence of an inclined magnetic field on peristaltic flow of an electrically conducting fluid through a porous medium between two inclined porous plates. They have presented a numerical study of their problem. Simultaneous effects of magnetic field and wall slip condition on peristaltic motion of a viscous fluid is studied by Ebaid [53]. Kothandapani and Srinivas [54] considered the same fluid in the presence of a constant magnetic field, wall properties, porous medium and heat and mass transfer. Ali et al. [55] have studied the combined effects of magnetic field and variable viscosity on peristaltic flow of a viscous fluid in symmetric channel. Hayat and Hina [56] have discussed the simultaneous effects of slip condition, constant magnetic field, wall properties, heat and mass transfer on peristaltic flow of a Maxwell fluid. Srinivas et al. [57] provided a detailed description of the influence of partial slip condition, magnetic field and heat transfer on peristaltic transport of viscous fluid. Srinivas and Kothandapani [58] have also studied the MHD peristaltic flow through a porous space in the presence of heat and mass transfer effects with complaint walls. Mekheimer and elmaboud [59] have examined the combined effects of heat transfer and magnetic field on peristaltic motion. They have presented an application on endoscope (a biomedical instrument) under the physical occurring assumptions of long wavelength and small Reynolds number. They have conducted a numerical study of pressure rise per wavelength and frictional forces. The peristaltic transport of a magneto viscous fluid in a two dimensional channel with porous boundaries is investigated theoretically by Elshehawey and Husseny [60]. Hayat et al. [61] discussed the influence of heat transfer on peristaltic motion of an electrically conducting fluid in a porous space. Nadeem and Akbar [62] have investigated the MHD peristaltic flow with variable viscosity and heat transfer. They obtained the solution of the problem by Admoian decomposition method. Hariharan et al. [63] reported the peristaltic flow of a non-Newtonian fluid in a diverging tube under different assumptions related to Reynolds number, periodicity of waves, wavelength amplitude ratio, wave shape and frame of reference. Moreover they considered five possible wave forms on the channel walls, namely, sinusoidal, multisinusoidal, triangular, square and trapezoidal waves and discussed the flow analysis in detail. A theoretical and experimental study for MHD flow is presented by Haim et al. [64]. In this study, they discussed mixer by considering only the effects of constant magnetic field and obtained some theoretical results. They also compared their results with the experimental data and found a very good agreement. Nadeem and Akram [65] have investigated the MHD peristaltic flow of viscous fluid in presence of partial slip effects.

It has been noticed from the existing literature that peristalsis has been reported mostly in the presence of an applied magnetic field. Only few studies are available which examines the induced magnetic field effects on peristalsis for instance Vishnyakov and Pavlov [66]. They have studied the peristaltic flow of electrically conductive fluid under the effect of transverse induced magnetic field. Eldabe et al. [67] have also investigated the peristaltic flow (of a bioviscosity fluid) under the action of induced magnetic field. Mekheimer [68, 69] have examined

$$\overline{\mathrm{T}} = -p\bar{\mathrm{I}} + \bar{\mathrm{S}}, \tag{1.4}$$

where $\overline{\mathrm{T}}$ is the Cauchy stress tensor, $\bar{\mathrm{S}}$ is the extra stress tensor, $\overline{\mathrm{b}}$ is the body force per unit mass, p is the pressure and $\bar{\mathrm{I}}$ is the identity tensor.

1.4.3 Maxwell's equations

1.4.4 (a) Gauss's law

$$\nabla.\mathrm{E} = \frac{\rho_1}{\varepsilon_0}, \tag{1.5}$$

where ρ_1 is the charge density and $\mathbf{E}$ is the electric field.

1.4.5 (b) Gauss's law for magnetism

$$\nabla.B = 0, \tag{1.6}$$

in which B is the magnetic field.

1.4.6 (c) Faraday's law

$$\nabla \times \mathrm{E} = -\frac{\partial \mathrm{B}}{\partial t}. \tag{1.7}$$

1.4.7 (d) Amperes law

The differential form is

$$\nabla \times \mathrm{B} = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial \mathrm{E}}{\partial t}, \tag{1.8}$$

in which μ_0 is the magnetic constant, ε_0 is the electric constant, $\mathbf{E}$ is the electric field and $\overline{\mathrm{J}}$ is the current density.

1.4.8 (e) Ohms' law

$$\overline{\mathrm{J}} = \sigma\left(\overline{\mathrm{E}} + \overline{\mathrm{V}} \times \overline{\mathrm{B}}\right), \tag{1.9}$$

where σ is the electrical conductivity of the fluid.

1.4.9 Energy equation

It is of the following form

$$\rho C_p \frac{dT}{dt} = k\nabla^2 T + \mathbf{T}.\mathbf{L}, \tag{1.10}$$

in which C_p is the specific heat, ρ the density, k the thermal conductivity, $\mathbf{L}$ the rate of strain tensor, T the temperature of fluid and $\mathbf{T}$ the Cauchy stress tensor.

1.4.10 Concentration equation

In mathematical form, we can express as follows

$$\frac{dC}{dt} = D\nabla^2 C + \frac{DK_T}{T_m}\nabla^2 T, \tag{1.11}$$

where D is coefficient of mass diffusivity, T_m the mean temperature, K_T the thermal diffusion ratio and C is the concentration of fluid.

Chapter 2

Peristaltic transport of a Carreau fluid in a channel with different wave forms

An analytic characterization of the rheological properties in the peristaltic flow of a Carreau fluid in a planar channel has been investigated in this chapter. Peristaltic wave trains on the channel walls are responsible for such flow. Five explicit wave shapes have been taken into consideration. Based on the two-dimensional analysis, the important variables are identified. The influence of such parameters is carefully analyzed. Trapping and pumping phenomena have been numerically studied and interesting conclusions are drawn.

2.1 Governing equations

We investigate the peristaltic flow of an incompressible Carreau fluid in a two-dimensional channel of width $2a$. The flow is induced by the periodic peristaltic wave of wavelength λ and amplitude b propagating with constant speed c along the channel walls. Its instantaneous height at any axial station $X^{'}$ is defined by the expression given below as

$$Y' = H\left(\frac{X' - ct'}{\lambda}\right) \tag{2.1}$$

Five possible wave forms namely sinusoidal (s), multisinusoidal (ms), triangular (t), square (sq), trapezoidal (tr) waves are considered in the present analysis.

In the laboratory frame (X', Y'), the flow is unsteady. However if observed in a coordinate system moving at speed c (wave frame), it behaves like a steady. The $X'-$axis is chosen parallel to the channel walls and $Y'-$axis is chosen normal to it. The coordinates and the velocities in the two frames have been associated by the following relations

$$x' = X' - ct', \quad y' = Y',$$
$$u'(x', y') = U' - c, \quad v'(x', y') = V', \tag{2.2}$$

where u' and v' are the velocity components in the wave frame.

The constitutive equations for Carreau fluid can be written as

$$\overline{\tau} = -\left[\eta_\infty + (\eta_0 - \eta_\infty)\left(1 + (\Gamma\bar{\dot{\gamma}})^2\right)^{\frac{n-1}{2}}\right]\bar{\dot{\gamma}}, \tag{2.3}$$

$$\bar{\dot{\gamma}} = \sqrt{\frac{1}{2}\sum_i\sum_j \bar{\dot{\gamma}}_{ij}\bar{\dot{\gamma}}_{ji}} = \sqrt{\frac{1}{2}\pi}. \tag{2.4}$$

Here $\overline{\tau}$ denotes an extra stress tensor, η_∞ infinite shear stress viscosity, η_0 zero shear-rate viscosity, Γ the time constant, n the dimensionless power law index and π the second invariant of strain-rate tensor.

Taking into account the case for which $\eta_\infty = 0$, one obtains the following expression

$$\overline{\tau} = -\left[\eta_0\left(1 + (\Gamma\bar{\dot{\gamma}})^2\right)^{\frac{n-1}{2}}\right]\bar{\dot{\gamma}} \tag{2.5}$$

which reduces to viscous fluid situation when $n = 1$ or $\Gamma = 0$.

2.2 Problem statement

The governing equations in the absence of body forces are

$$\nabla \cdot \overline{\mathbf{V'}} = 0, \tag{2.6}$$

discussed the simultaneous effects of slip condition, constant magnetic field, wall properties, heat and mass transfer on peristaltic flow of a Maxwell fluid. Srinivas et al. [57] provided a detailed description of the influence of partial slip condition, magnetic field and heat transfer on peristaltic transport of viscous fluid. Srinivas and Kothandapani [58] have also studied the MHD peristaltic flow through a porous space in the presence of heat and mass transfer effects with complaint walls. Mekheimer and elmaboud [59] have examined the combined effects of heat transfer and magnetic field on peristaltic motion. They have presented an application on endoscope (a biomedical instrument) under the physical occurring assumptions of long wavelength and small Reynolds number. They have conducted a numerical study of pressure rise per wavelength and frictional forces. The peristaltic transport of a magneto viscous fluid in a two dimensional channel with porous boundaries is investigated theoretically by Elshehawey and Husseny [60]. Hayat et al. [61] discussed the influence of heat transfer on peristaltic motion of an electrically conducting fluid in a porous space. Nadeem and Akbar [62] have investigated the MHD peristaltic flow with variable viscosity and heat transfer. They obtained the solution of the problem by Admoian decomposition method. Hariharan et al. [63] reported the peristaltic flow of a non-Newtonian fluid in a diverging tube under different assumptions related to Reynolds number, periodicity of waves, wavelength amplitude ratio, wave shape and frame of reference. Moreover they considered five possible wave forms on the channel walls, namely, sinusoidal, multisinusoidal, triangular, square and trapezoidal waves and discussed the flow analysis in detail. A theoretical and experimental study for MHD flow is presented by Haim et al. [64]. In this study, they discussed mixer by considering only the effects of constant magnetic field and obtained some theoretical results. They also compared their results with the experimental data and found a very good agreement. Nadeem and Akram [65] have investigated the MHD peristaltic flow of viscous fluid in presence of partial slip effects.

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1.3 Some basic definitions

This section includes some definitions which may be helpful for the understanding of flow analysis in this thesis.

1.3.1 Retrograde pumping

Retrograde means to "move backward". Retrograde pumping occurs when pressure rise per wavelength ($\triangle p_\lambda$) is positive and dimensionless mean flow rate (θ) is negative, i.e.

$$\triangle p_\lambda > 0 \quad \text{and} \quad \theta < 0.$$

1.3.2 Peristaltic pumping

A peristaltic pumping occurs when pressure rise per wavelength ($\triangle p_\lambda$) and dimensionless mean flow rate (θ) are both positive, i.e.

$$\triangle p_\lambda > 0 \quad \text{and} \quad \theta > 0.$$

This phenomenon occurs in many biological systems such as gastrointestinal tract.

1.3.3 Free pumping

Free pumping occurs when pressure rise per wavelength ($\triangle p_\lambda$) is equal to zero and the corresponding dimensionless mean flow rate (θ) is greater than zero, i.e.

$$\triangle p_\lambda = 0 \quad \text{and} \quad \theta > 0.$$

1.3.4 Augmented pumping (or copumping)

It occurs when pressure rise per wavelength ($\triangle p_{\lambda}$) is negative and dimensionless mean flow rate (θ) is positive, i.e.

$$\triangle p_{\lambda} < 0 \quad \text{and} \quad \theta > 0.$$

1.3.5 Trapping

The formation of internally circulating bolus of fluid by closed streamlines is called trapping and this trapped bolus is pushed ahead along with the peristaltic wave moving with constant wave speed c.

1.4 Basic equations

This section consists of some fundamental equations used in the mathematical models for the flows considered in this thesis.

1.4.1 The continuity equation

In mathematical form it is written as

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\left(\rho \overline{\mathbf{V}}\right) = 0 \tag{1.1}$$

which for an incompressible fluid reduces to

$$\operatorname{div} \overline{\mathbf{V}} = 0, \tag{1.2}$$

where ρ is the density, t is the time and $\overline{\mathbf{V}}$ is the velocity.

1.4.2 The equation of motion

Mathematically it is given by

$$\rho \frac{d\overline{\mathbf{V}}}{dt} = \operatorname{div} \overline{\mathbf{T}} + \rho \overline{\mathbf{b}}, \tag{1.3}$$

$$\overline{\mathrm{T}} = -p\bar{\mathrm{I}} + \bar{\mathrm{S}}, \tag{1.4}$$

where $\overline{\mathrm{T}}$ is the Cauchy stress tensor, $\bar{\mathrm{S}}$ is the extra stress tensor, $\bar{\mathrm{b}}$ is the body force per unit mass, p is the pressure and $\bar{\mathrm{I}}$ is the identity tensor.

1.4.3 Maxwell's equations

1.4.4 (a) Gauss's law

$$\nabla.\mathrm{E} = \frac{\rho_1}{\varepsilon_0}, \tag{1.5}$$

where ρ_1 is the charge density and $\mathbf{E}$ is the electric field.

1.4.5 (b) Gauss's law for magnetism

$$\nabla.B = 0, \tag{1.6}$$

in which B is the magnetic field.

1.4.6 (c) Faraday's law

$$\nabla \times \mathrm{E} = -\frac{\partial \mathrm{B}}{\partial t}. \tag{1.7}$$

1.4.7 (d) Amperes law

The differential form is

$$\nabla \times \mathrm{B} = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial \mathrm{E}}{\partial t}, \tag{1.8}$$

in which μ_0 is the magnetic constant, ε_0 is the electric constant, $\mathbf{E}$ is the electric field and $\overline{\mathrm{J}}$ is the current density.

1.4.8 (e) Ohms' law

$$\overline{\mathrm{J}} = \sigma \left(\overline{\mathrm{E}} + \overline{\mathrm{V}} \times \overline{\mathrm{B}} \right), \tag{1.9}$$

where σ is the electrical conductivity of the fluid.

$$\rho\left(\overline{\mathbf{V'}}\cdot\nabla\right)\overline{\mathbf{V'}} = -\overline{\nabla}p' + \operatorname{div}\overline{\tau'}, \tag{2.7}$$

where $\overline{\mathbf{V'}}$ is the velocity vector, p' the fluid pressure, $\overline{\tau'}$ an extra stress tensor and the body forces are taken absent.

The definition of velocity is

$$\overline{\mathbf{V'}} = \left(u', v', 0\right). \tag{2.8}$$

The scalar form of equations (2.6) and (2.7) are

$$\frac{\partial\overline{u}}{\partial\overline{x}} + \frac{\partial\overline{v}}{\partial\overline{y}} = 0, \tag{2.9}$$

$$\rho\left(u'\frac{\partial}{\partial x'} + v'\frac{\partial}{\partial y'}\right)u' = -\frac{\partial p'}{\partial x'} - \frac{\partial\tau'_{xx}}{\partial x'} - \frac{\partial\tau'_{xy}}{\partial y'}, \tag{2.10}$$

$$\rho\left(\overline{u}\frac{\partial}{\partial x'} + v'\frac{\partial}{\partial y'}\right)v' = -\frac{\partial p'}{\partial y'} - \frac{\partial\tau'_{xy}}{\partial x'} - \frac{\partial\tau'_{yy}}{\partial y'}. \tag{2.11}$$

The following variables are defined to obtain above dimensionless expressions

$$\begin{gathered} x = \frac{x'}{\lambda},\quad y = \frac{y'}{a},\quad u = \frac{u'}{c},\quad v = \frac{v'}{\delta c},\quad t = \frac{t'c}{\lambda},\quad h = \frac{H}{a}, \\ \delta = \frac{a}{\lambda},\quad \mathrm{Re} = \frac{\rho c a}{\eta_0},\quad p = \frac{a^2 p'}{c\lambda\eta_0},\quad We = \frac{\Gamma c}{a},\quad \Phi = \frac{a}{b}, \\ \tau_{xx} = \frac{\lambda\tau'_{xx}}{\eta_0 c},\quad \tau_{xy} = \frac{a\tau'_{xy}}{\eta_0 c},\quad \tau_{yy} = \frac{a\tau'_{yy}}{\eta_0 c}, \end{gathered} \tag{2.12}$$

where Re is the Reynolds number and δ is the wave number.

The non-dimensional resulting equations become

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.13}$$

$$Re\delta\left[u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right]u = -\frac{\partial p}{\partial x} - \delta^2\frac{\partial\tau_{xx}}{\partial x} - \frac{\partial\tau_{xy}}{\partial y}, \tag{2.14}$$

$$Re\delta^3\left[u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right]v = -\frac{\partial p}{\partial y} - \delta^3\frac{\partial\tau_{xy}}{\partial x} - \delta\frac{\partial\tau_{yy}}{\partial y}, \tag{2.15}$$

with

$$\tau_{xx} = -2\left[1 + \frac{(n-1)}{2}We^2\dot{\gamma}^2\right]\frac{\partial u}{\partial x}, \tag{2.16}$$

$$\tau_{xy} = -\left[1 + \frac{(n-1)}{2}We^2\dot{\gamma}^2\right]\left(\frac{\partial u}{\partial y} + \delta^2\frac{\partial v}{\partial x}\right), \tag{2.17}$$

$$\tau_{yy} = 2\delta\left[1 + \frac{(n-1)}{2}We^2\dot{\gamma}^2\right]\frac{\partial v}{\partial y} \tag{2.18}$$

in which We is the Weissenberg number.

The equations (2.14) and (2.15) subject to the long wavelength and low Reynolds number approximations take the forms

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\left[\left(1 + \frac{(n-1)}{2}We^2\left(\frac{\partial u}{\partial y}\right)^2\right)\frac{\partial u}{\partial y}\right], \tag{2.19}$$

$$\frac{\partial p}{\partial y} = 0. \tag{2.20}$$

From equations (2.19) and (2.20), we have

$$\frac{\partial^2}{\partial y^2}\left[\left(1 + \frac{(n-1)}{2}We^2\left(\frac{\partial u}{\partial y}\right)^2\right)\frac{\partial u}{\partial y}\right] = 0 \tag{2.21}$$

The equation (2.20) indicates that

$$p \neq p(y).$$

The dimensionless volume flow rate in the laboratory frame is expressed as

$$Q = \int_0^H U(X', Y', t')\, dY', \tag{2.22}$$

where $H = H(X', t')$. The above equation in the wave frame can be written as

$$q = \int_0^h u(x', y')\, dy', \tag{2.23}$$

in which $h = h(x')$.

Making use of Eqs.(2.2), (2.22) and (2.23), we obtain

$$Q = q + ch. \tag{2.24}$$

At fixed position X', the time-averaged flow over a period T is

$$Q' = \frac{1}{T}\int_0^T Q dt \tag{2.25}$$

which after employing equation (2.24) becomes

$$Q' = q + ac. \tag{2.26}$$

If θ and F are the dimensionless mean flows in the laboratory and wave frames defined by

$$\theta = \frac{Q'}{ac}, \quad F = \frac{q}{ac} \tag{2.27}$$

then Eq.(2.26) gives

$$\theta = F + 1, \tag{2.28}$$

$$F = \int_0^h u dy. \tag{2.29}$$

The appropriate boundary conditions of the problem are

$$\frac{\partial u}{\partial y} = 0 \text{ at } y = 0, \tag{2.30}$$

$$u = -1 \text{ at } y = h. \tag{2.31}$$

Expression of the non-dimensional pressure rise per wavelength (Δp_λ) is

$$\Delta p_\lambda = \int_0^1 \left(\frac{dp}{dx}\right) dx. \tag{2.32}$$

2.3 Perturbation solution

In this section, our interest is to find the analytic solution of Eqs. (2.19), (2.21) with boundary conditions (2.30) and (2.31). The closed form solution of the arising problem is not easy to obtain. Therefore we look for the perturbation solution. For such a solution, we write

$$u = u_0 + We^2 u_1 + O\left(We^4\right), \tag{2.33}$$

$$F = F_0 + We^2 F_1 + O\left(We^4\right), \tag{2.34}$$

$$p = p_0 + We^2 p_1 + O\left(We^4\right). \tag{2.35}$$

Substituting the above equations in Eqs. (2.19), (2.21), boundary conditions (2.30) and (2.31), one has the following systems:

2.3.1 System for We^0

$$\frac{dp_0}{dx} = \frac{\partial^2 u_0}{\partial y^2}, \tag{2.36}$$

$$\frac{\partial^3 u_0}{\partial y^3} = 0, \tag{2.37}$$

$$\frac{\partial u_0}{\partial y} = 0 \quad \text{at} \quad y = 0, \tag{2.38}$$

$$u_0 = -1 \quad \text{at} \quad y = h, \tag{2.39}$$

$$\Delta p_{\lambda_0} = \int_0^1 \frac{dp_0}{dx} dx. \tag{2.40}$$

2.3.2 System for We^2

$$\frac{dp_1}{dx} = \frac{\partial^2 u_1}{\partial y^2} + \left(\frac{n-1}{2}\right) \frac{\partial}{\partial y} \left\{ \left(\frac{\partial u_0}{\partial y}\right)^3 \right\}, \tag{2.41}$$

$$\frac{\partial^3 u_1}{\partial y^3} + \left(\frac{n-1}{2}\right) \frac{\partial^2}{\partial y^2} \left\{ \left(\frac{\partial u_0}{\partial y}\right)^3 \right\} = 0 \tag{2.42}$$

$$\frac{\partial u_1}{\partial y} = 0 \quad \text{at} \quad y = 0, \tag{2.43}$$

$$u_1 = 0 \quad \text{at} \quad y = h, \tag{2.44}$$

$$\Delta p_{\lambda_1} = \int_0^1 \frac{dp_1}{dx} dx. \tag{2.45}$$

2.3.3 Solution for We^0 system

The solutions of Eqs.(2.36) and (2.37) along with the boundary conditions are (2.38) and (2.39)

$$u_0 = \frac{1}{2}\frac{dp_0}{dx}\left(y^2 - h^2\right) - 1. \tag{2.46}$$

$$\frac{dp_0}{dx} = -\frac{3}{h^3}\left(F_0 + h\right). \tag{2.47}$$

2.3.4 Solution for We^2 system

Here the longitudinal velocity and pressure gradient are given by

$$u_1 = \frac{1}{2}\frac{dp_1}{dx}\left(y^2 - h^2\right) + \frac{(n-1)}{8}\left[\frac{3}{h^3}\left(F_0 + h\right)\right]^3 \left(y^4 - h^4\right). \tag{2.48}$$

$$\frac{dp_1}{dx} = -\frac{3}{h^3}\left[F_1 + \frac{(n-1)\, h^5}{10}\left(\frac{3}{h^3}\left(F_0 + h\right)\right)^3\right]. \tag{2.49}$$

Expression of longitudinal velocity, longitudinal pressure gradient and pressure rise per wavelength upto $O\left(We^2\right)$ are

$$u = u_0 + We^2 u_1, \tag{2.50}$$

$$\frac{dp}{dx} = \frac{dp_0}{dx} + We^2 \frac{dp_1}{dx}, \tag{2.51}$$

$$\Delta p_\lambda = \Delta p_{\lambda_0} + We^2 \Delta p_{\lambda_1}. \tag{2.52}$$

In the above solutions, we use

$$F = F_0 + We^2 F_1 \tag{2.53}$$

and neglect the terms which are greater than $O(We^2)$. The longitudinal velocity and pressure gradient after using Eq. (2.53) are

$$u = \frac{-3}{2h^3}(F+h)\left(y^2-h^2\right) - 1 + We^2 \begin{bmatrix} (n-1)\left(\frac{3}{h^3}(F+h)\right)^3 \\ \left(\frac{y^4-h^4}{8} - \frac{3h^2y^2-3h^4}{20}\right) \end{bmatrix}, \tag{2.54}$$

$$\frac{dp}{dx} = \frac{-3}{2h^3}(F+h) + We^2\left[\frac{-3h^2(n-1)}{10}\left(\frac{3}{h^3}(F+h)\right)^3\right]. \tag{2.55}$$

The stream function Ψ after utilizing $u = \frac{\partial \Psi}{\partial y}$ and $v = -\frac{\partial \Psi}{\partial x}$ yields

$$\Psi = \left[\frac{y}{2} + \frac{3(\theta-1)y}{2h} - \frac{(\theta-1)y^3}{2h^2} - \frac{y^3}{2h^2}\right] + We^2 \begin{bmatrix} -\frac{27(\theta-1)^3y}{40h^5} - \frac{81(\theta-1)^2y}{40h^4} - \frac{81(\theta-1)y}{40h^3} - \frac{81y}{40h^2} + \frac{27(\theta-1)^2ny}{40h^5} + \frac{81(\theta-1)^2ny}{40h^4} + \frac{81(\theta-1)y}{40h^3} \\ +\frac{27ny}{40h^2} + \frac{27(\theta-1)^3y^3}{40h^7} + \frac{81(\theta-1)^2y^3}{20h^6} + \frac{81(\theta-1)y^3}{20h^5} + \frac{27y^3}{40h^4} - \frac{27(\theta-1)^3ny^3}{40h^7} - \frac{81(\theta-1)^2ny^3}{40h6} - \frac{81(\theta-1)ny^3}{20h^5} \\ -\frac{27ny^3}{20h^4} - \frac{27(\theta-1)^3y^5}{20h^9} - \frac{81(\theta-1)^2y^5}{40h^8} - \frac{81(\theta-1)y^5}{40h^7} - \frac{27y^5}{40h^6} + \frac{27(\theta-1)^3ny^5}{40h^9} + \frac{81(\theta-1)^2ny^5}{40h^8} \\ +\frac{81(\theta-1)ny^5}{40h^7} + \frac{27ny^5}{40h^6} \end{bmatrix} - y. \tag{2.56}$$

2.4 Expression for five wave shapes

The nondimensional expressions of the considered wave forms are given by the following equations:

(1) Sinusoidal wave

$$h(x) = 1 + \Phi \sin 2\pi x.$$

(2) Multisinusoidal wave

$$h(x) = 1 + \Phi \sin 2N\pi x.$$

(3) Triangular wave

$$h(x) = 1 + \Phi\left[\frac{8}{\pi^3}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)^2}\sin\{2(2m-1)\pi x\}\right].$$

(4) Square wave

$$h(x) = 1 + \Phi \left[\frac{4}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)} \cos \{2(2m-1)\pi x\} \right].$$

(5) Trapezoidal wave

$$h(x) = 1 + \Phi \left[\frac{32}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} \sin \left\{ \frac{\pi}{3}(2m-1) \right\}}{(2m-1)^2} \sin \{2(2m-1)\pi x\} \right].$$

Total number of terms in the series that are incorporated in the analysis here are 50. Note that the expressions for triangular, square and trapezoidal waves are derived from Fourier series.

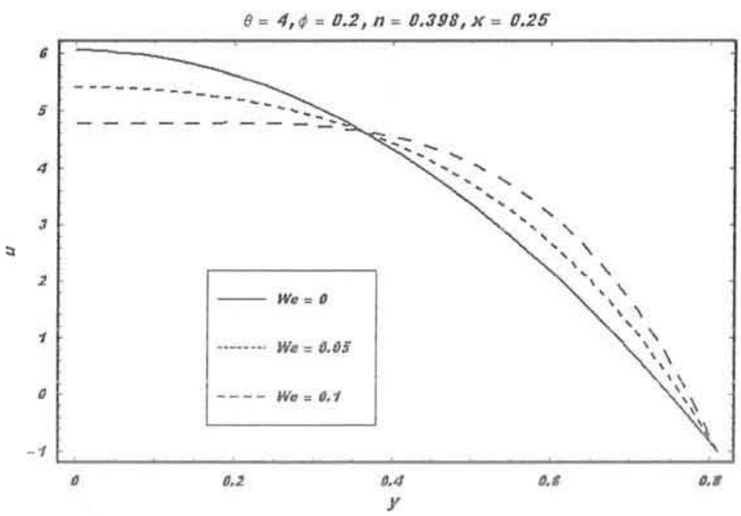

Figure 2.1a: Plot showing velocity u versus y for narrow part of the channel for sinusoidal wave.

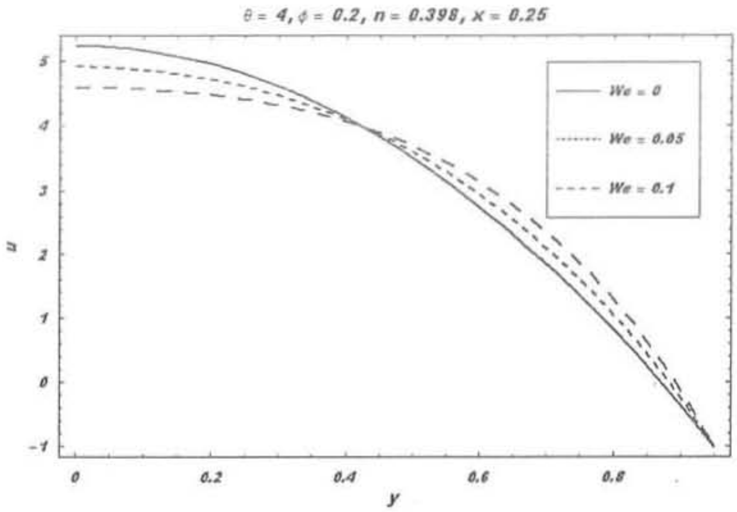

Figure 2.1b: Plot showing velocity u versus y for narrow part of the channel for triangular wave.

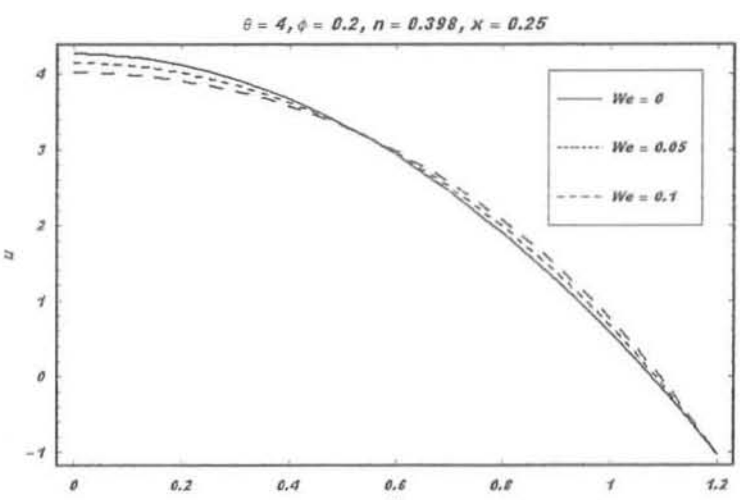

Figure 2.1c: Plot showing velocity u versus y for narrow part of the channel for square wave.

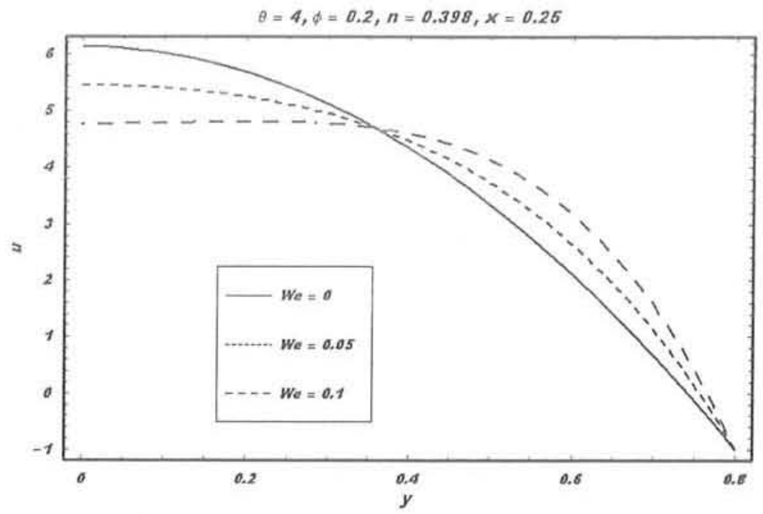

Figure 2.1d: Plot showing velocity u versus y for narrow part of the channel for trapezoidal wave.

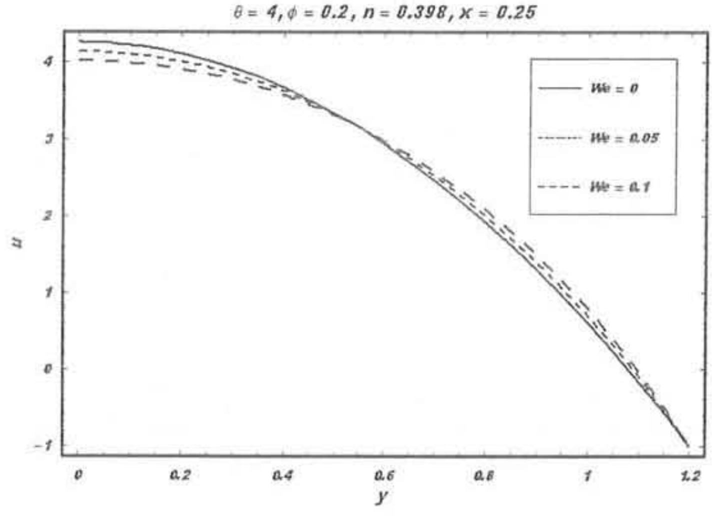

Figure 2.2a: Plot showing velocity u versus y for wider part of the channel for sinusoidal wave.

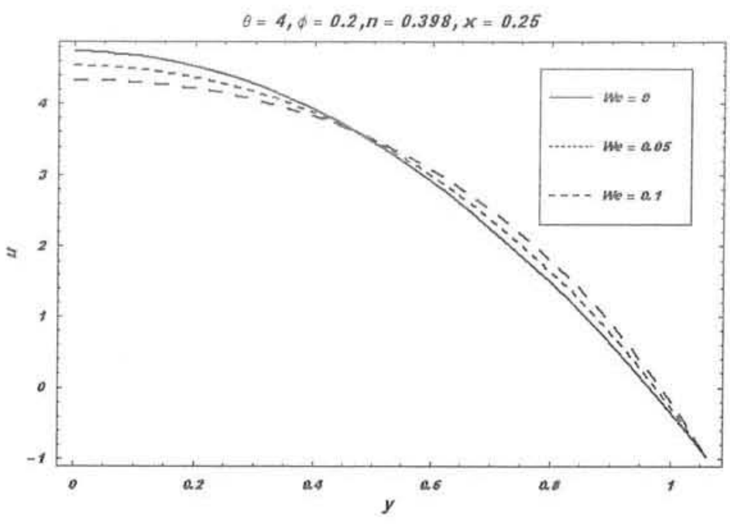

Figure 2.2b: Plot showing velocity u versus y for wider part of the channel for triangular wave.

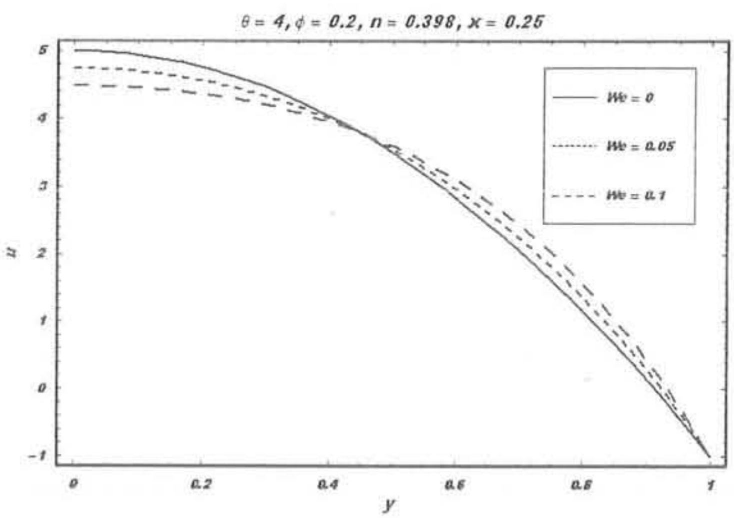

Figure 2.2c: Plot showing velocity u versus y for wider part of the channel for square wave.

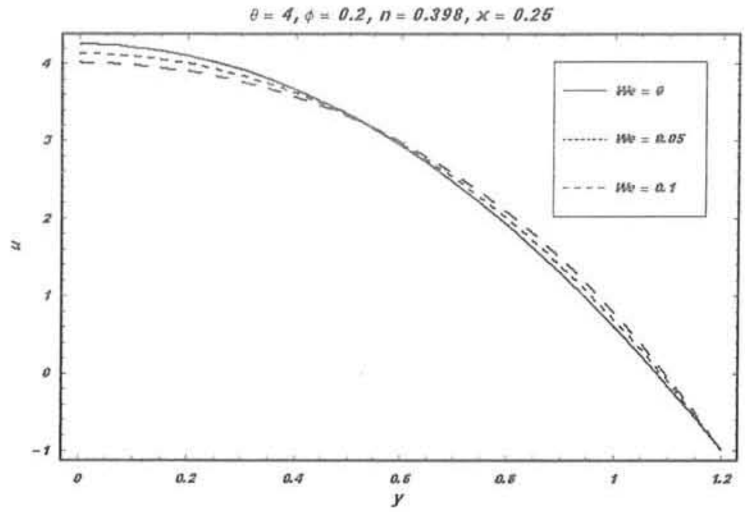

Figure 2.2d: Plot showing velocity u versus y for wider part of the channel for trapezoidal wave.

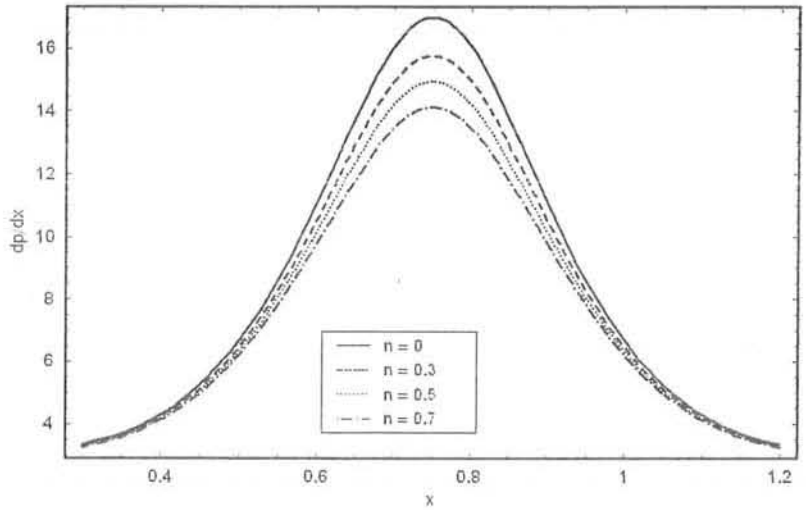

Figure 2.3a: Plot showing dp/dx versus x for sinusoidal wave. Here $\Phi = 0.2$, $\theta = -2$ and $We = 0.1$.

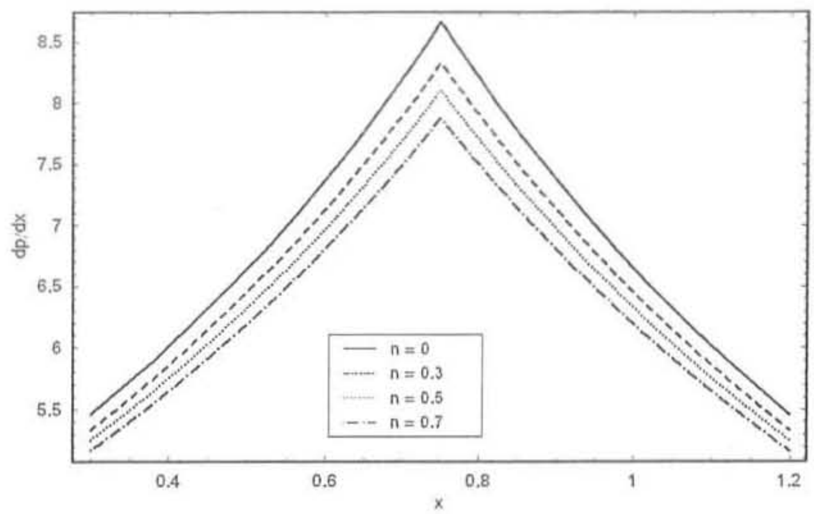

Figure 2.3b: Plot showing dp/dx versus x for triangular wave. Here $\Phi = 0.2$, $\theta = -2$ and $We = 0.1$.

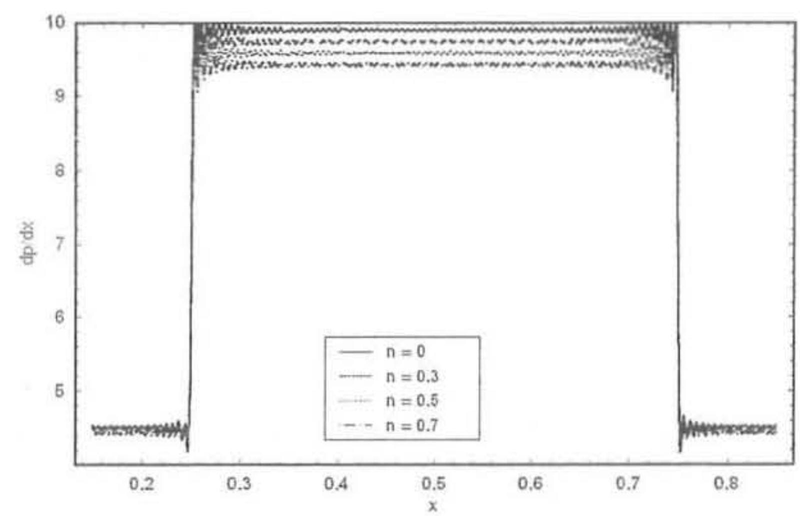

Figure 2.3c: Plot showing dp/dx versus x for square wave. Here $\Phi = 0.2$, $\theta = -2$ and $We = 0.1$.

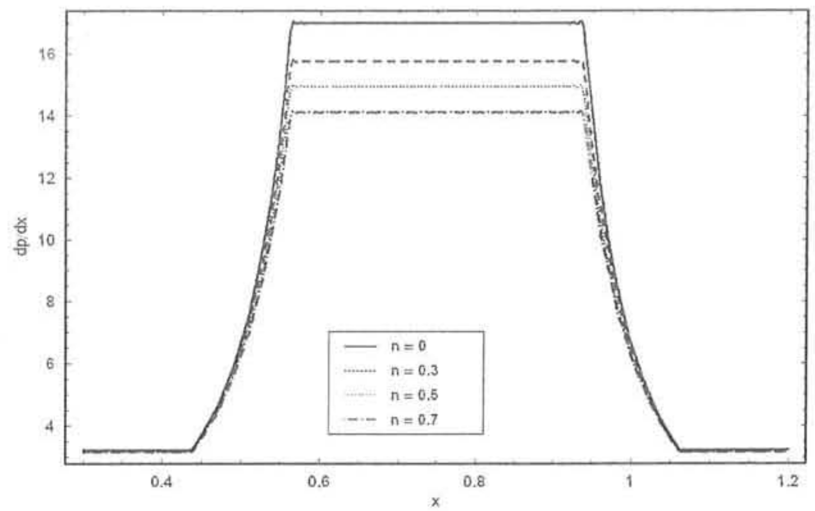

Figure 2.3d: Plot showing dp/dx versus x for trapezoidal wave. Here $\Phi = 0.2$, $\theta = -2$ and $We = 0.1$.

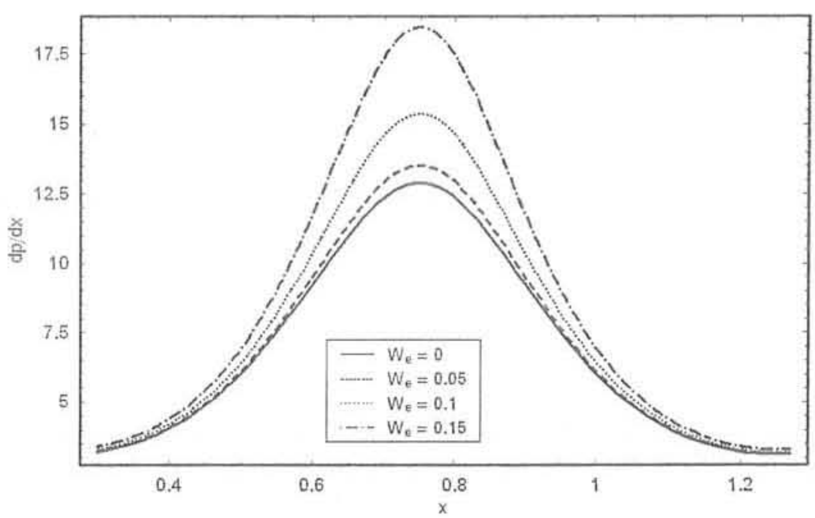

Figure 2.4a: Plot showing dp/dx versus x for sinusoidal wave. Here $\Phi = 0.2$, $\theta = -2$ and $n = 0.398$.

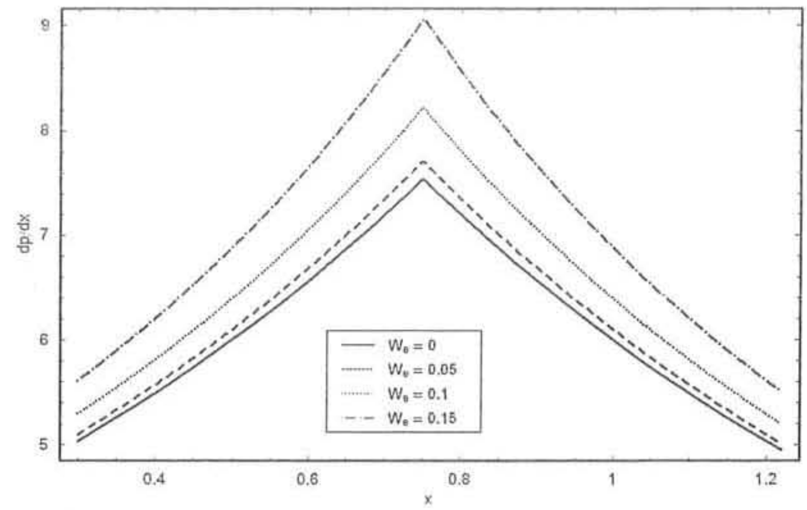

Figure 2.4b: Plot showing dp/dx versus x for triangular wave. Here $\Phi = 0.2$, $\theta = -2$ and $n = 0.398$.

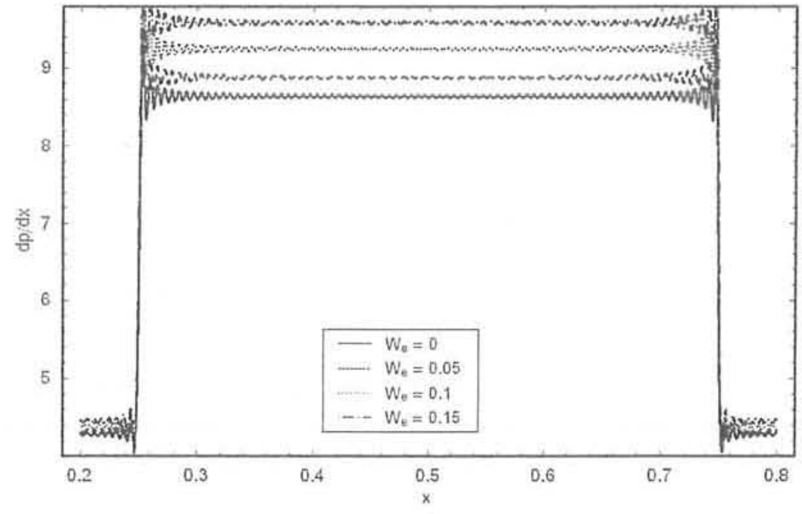

Figure 2.4c: Plot showing dp/dx versus x for square wave. Here $\Phi = 0.2$, $\theta = -2$ and $n = 0.398$.

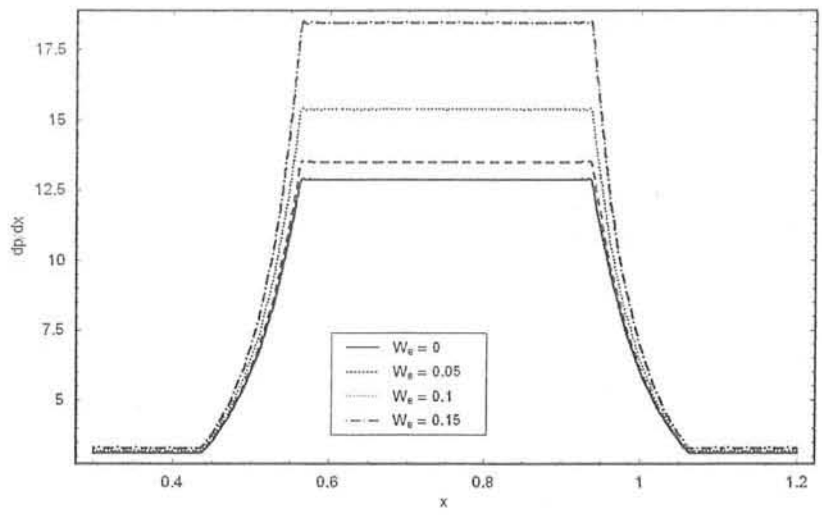

Figure 2.4d: Plot showing dp/dx versus x for trapezoidal wave. Here $\Phi = 0.2$, $\theta = -2$ and $n = 0.398$.

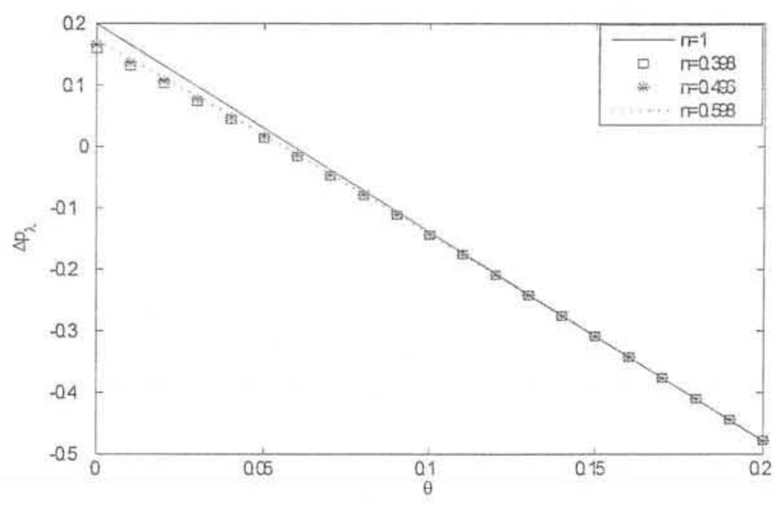

Figure 2.5a: Plot showing Δp_λ versus flow rate θ for sinusoidal wave form. Here $\Phi = 0.2$, $We = 0.1$.

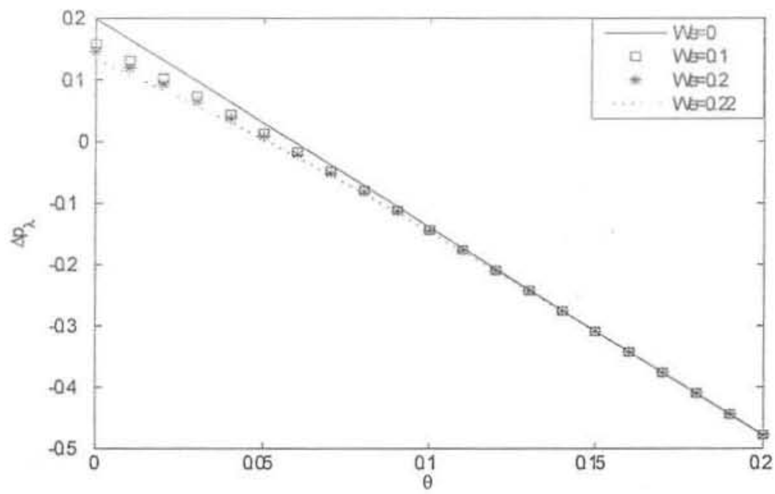

Figure 2.5b: Plot showing Δp_λversus flow rate θ for sinusoidal wave form. Here $\Phi = 0.2$, $n = 0.398$.

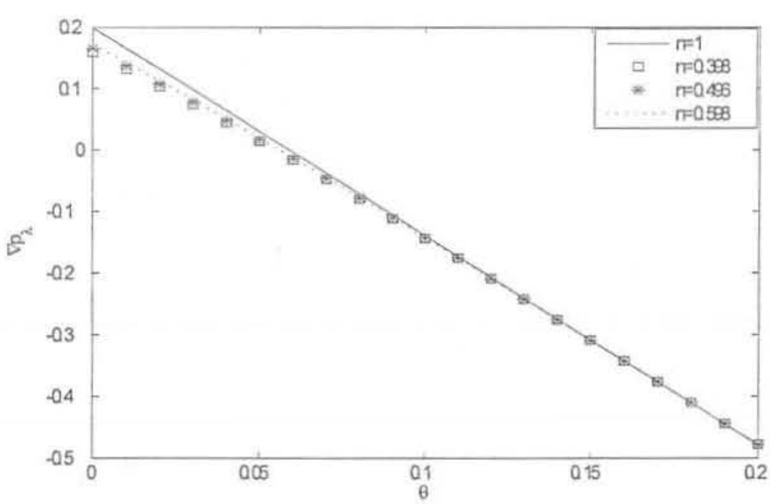

Figure 2.6: Plot showing Δp_λversus flow rate θ for multisinusoidal wave form. Here $\Phi = 0.2$, $n = 0.398$, $N = 2$.

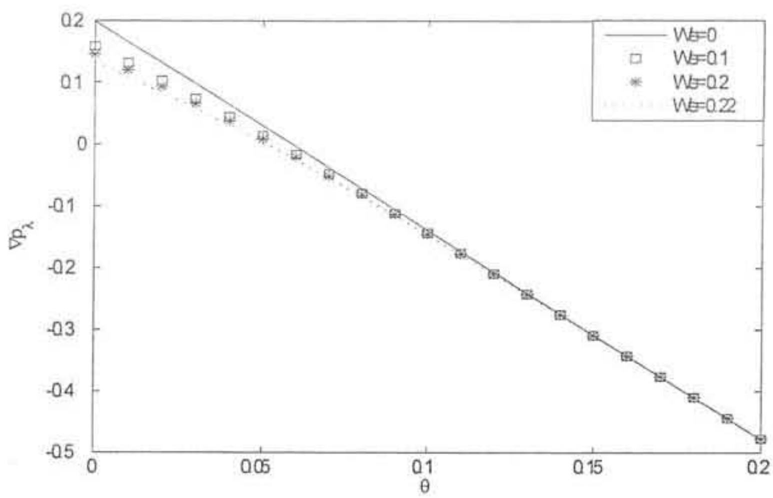

Figure 2.7a: Plot showing Δp_λ versus flow rate θ for triangular wave form. Here $\Phi = 0.2$, $We = 0.1$.

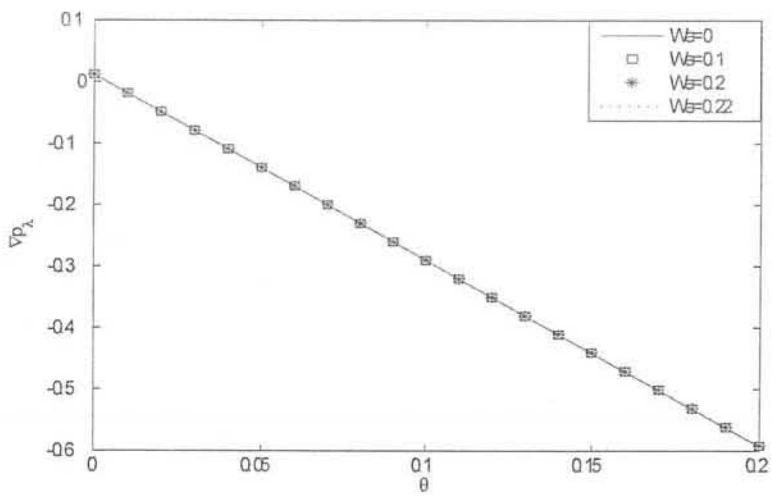

Figure 2.7b: Plot showing Δp_λversus flow rate θ for triangular wave form. Here $\Phi = 0.2$, $n = 0.398$.

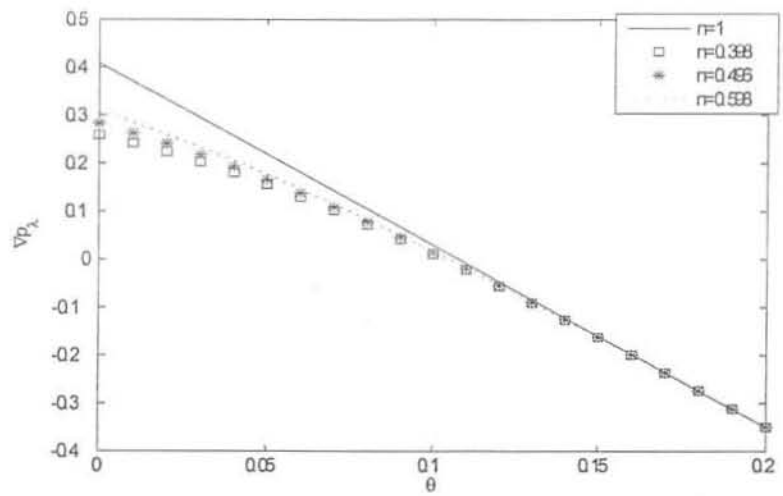

Figure 2.8a: Plot showing Δp_λ versus flow rate θ for square wave form. Here $\Phi = 0.2$, $We = 0.1$.

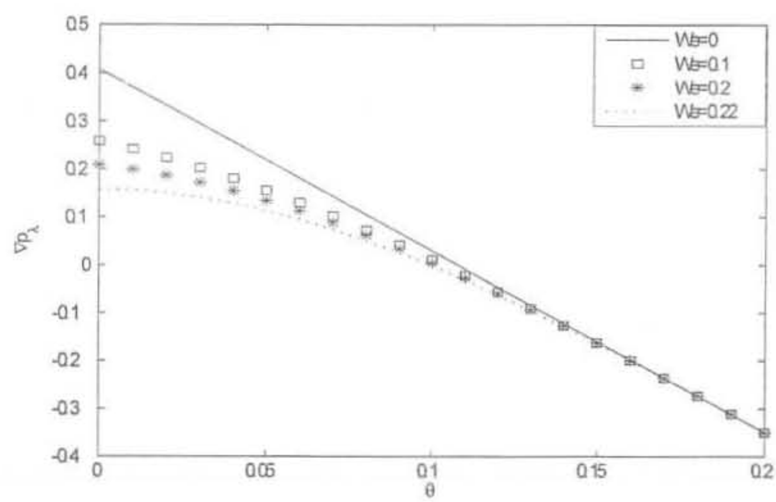

Figure 2.8b: Plot showing Δp_λversus flow rate θ for square wave form. Here $\Phi = 0.2$, $n = 0.398$.

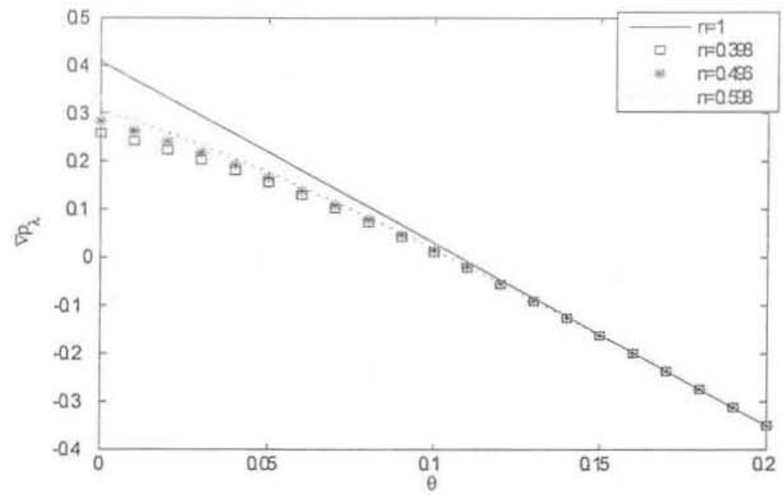

Figure 2.9a: Plot showing Δp_λ versus flow rate θ for trapezoidal wave form. Here $\Phi = 0.2$, $We = 0.1$.

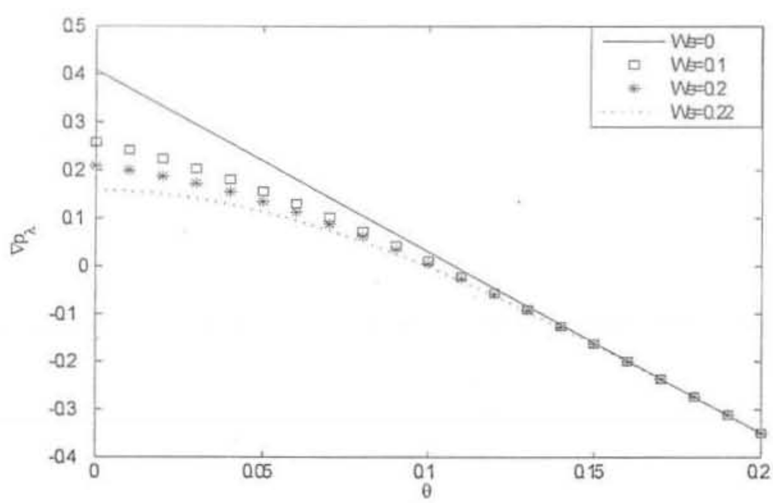

Figure 2.9b: Plot showing Δp_λversus flow rate θ for trapezoidal wave form. Here $\Phi = 0.2$, $n = 0.398$.

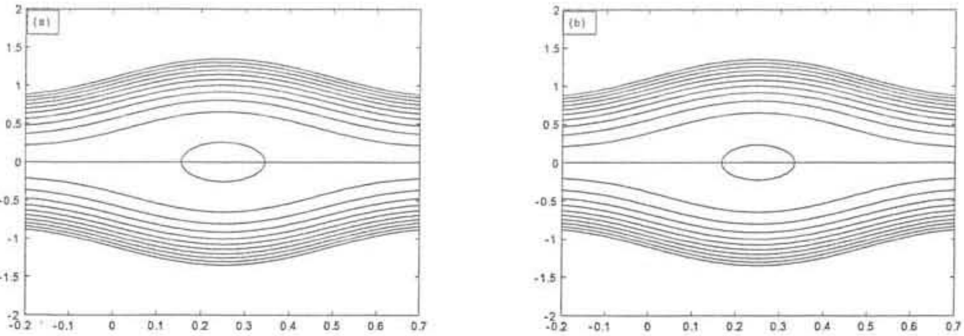

Figure 2.10a: Streamlines for $n = 1$, (panel(a)), $n = 0.496$ (panel (b)). The other parameters are $\Phi = 0.2$, $We = 0.1$, $\theta = 0.61$.

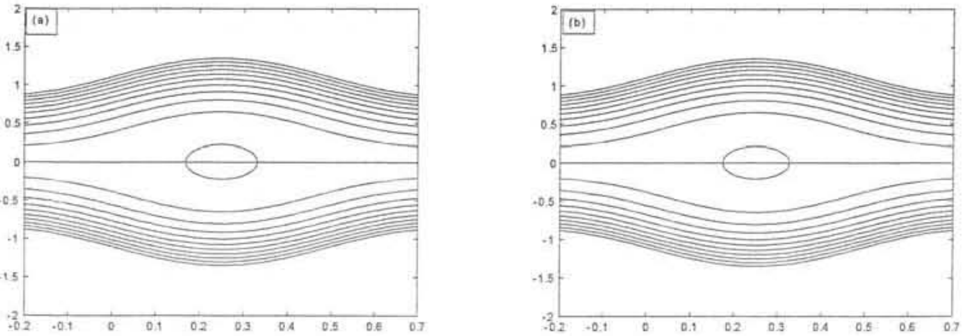

Figure 2.10b: Streamlines for $We = 0.1$, (panel(a)), $We = 0.2$ (panel (b)). The other parameters are $\Phi = 0.2$, $n = 0.398$, $\theta = 0.61$.

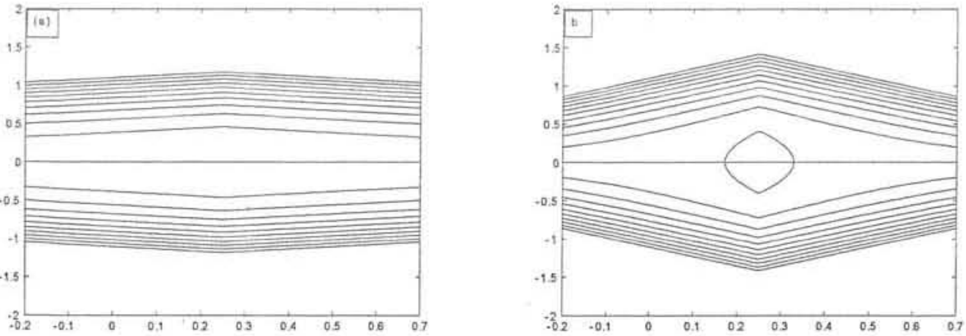

Figure 2.11: Streamlines for $\Phi = 0.2$, (panel(a)), $\Phi = 0.8$ (panel (b)). The other parameters are $n = 0.398$, $We = 0.1$, $\theta = 0.61$.

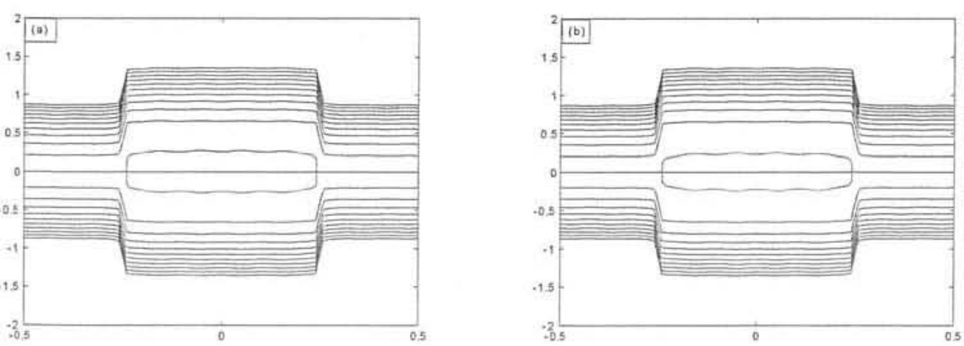

Figure 2.12a: Streamlines for $n = 1$, (panel(a)), $n = 0.496$ (panel (b)). The other parameters are $\Phi = 0.2$, $We = 0.1$, $\theta = 0.61$.

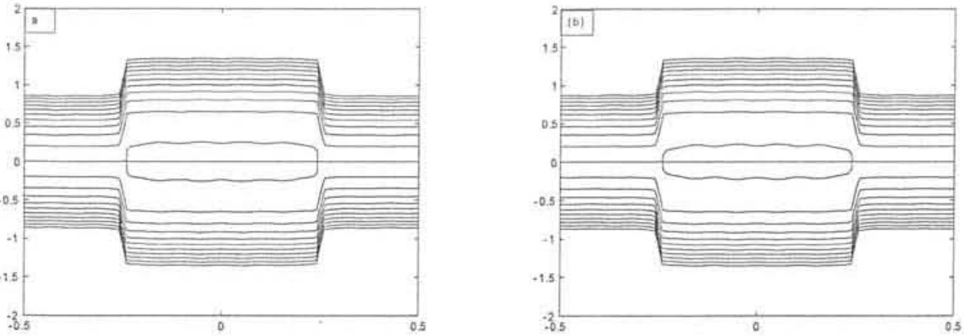

Figure 2.12b: Streamlines for $We = 0$, (panel(a)), $We = 0.2$ (panel (b)). The other parameters are $\Phi = 0.2$, $\theta = 0.61$, $n = 0.398$.

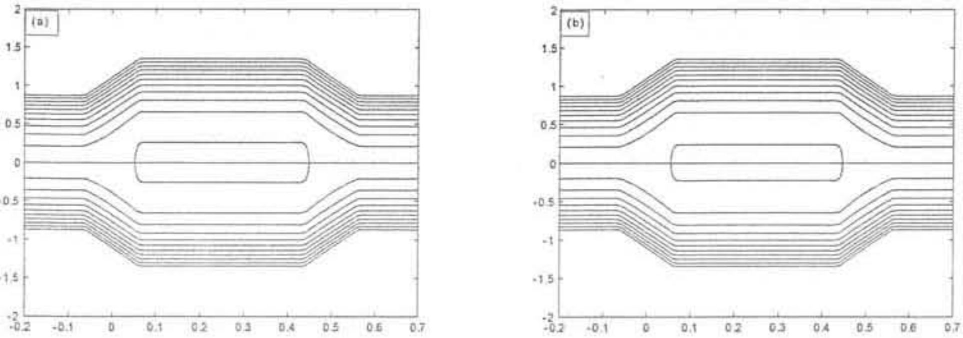

Figure 2.13a: Streamlines for $n = 1$, (panel(a)), $n = 0.496$ (panel (b)). The other parameters are $\Phi = 0.2$, $We = 0.1$, $\theta = 0.61$.

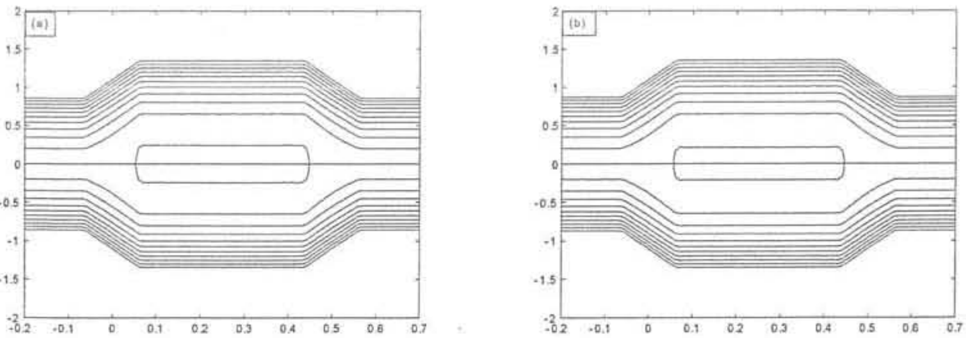

Figure 2.13b: Streamlines for $We = 0$, (panel(a)), $We = 0.2$ (panel (b)). The other parameters are $\Phi = 0.2$, $\theta = 0.61$, $n = 0.398$.

2.5 Discussion

This section monitors the variations of the influential parameters on the longitudinal velocity and pressure gradient. In addition, the trapping and pumping processes have been analyzed by performing numerical integration.

The longitudinal velocity $u(y)$ over two different cross-sections (narrow and wider parts) of the channel for different wave forms have been plotted in the Figures 2.1 and 2.2. It can be seen from these Figures that longitudinal velocity decreases at the center of the channel for all the considered wave forms when We is increased. The magnitude of the velocity is less in the wider part of the channel when compared to the narrow part in all the such considered wave forms. In all these figures, the value of θ is equal to 4. For small values of θ, the effects of We on the velocity can not be noticed.

2.5.1 Pressure gradient

The pressure gradients versus x for different wave forms have been potrayed in the Figures $2.3 - 2.4$. It is seen from Figures $2.3(a - d)$ that in all wave forms, the dp/dx increases with a decrease in n. This decrease is maximum for triangular wave and minimum for sinusoidal and trapezoidal waves. Figures $2.4(a - d)$ show the variation of We on dp/dx. It is found that the behaviour is quite opposite to that what we noticed in Figure 2.3. Here we can see that dp/dx increases for large values of We. A careful analysis indicates that dp/dx is small in the wider part of the channel. In this type of region, it is possible for flow to occur with such small dp/dx. However, in the narrow part of the channel, it is required to apply much greater pressure gradient to maintain such flow.

2.5.2 Pumping characteristics

The expression for pressure rise per wavelength is evaluated numerically and is plotted for the five considered wave forms. When pressure difference $\Delta p_\lambda = 0$ which is the case of free pumping, the corresponding time mean flow rate is denoted by θ_0. The maximum pressure against which the peristalsis works as a pump, that is, Δp_λ corresponding to $\theta = 0$ is denoted by p_0. When $\Delta p_\lambda < 0$, the pressure assists the flow and it is known as copumping. Figures 2.5a and 2.5b analyze the role of Δp_λ with flow rate θ for different dimensionless power law index n and Weissenberg number We, respectively, for sinusoidal wave form. It is clear that $\theta_0 | p_0$ decreases upon increasing n and We. In copumping case, the pumping rate is independent upon n and We. Similar results can be achieved for multisinusoidal wave form (Figure 2.6). For triangular wave, the pumping rate is independent of values of n and We (Figure 2.7). It is also observed from this figure that there is no peristaltic pumping for triangular wave. From Figures 2.8 and 2.9, we conclude that peristaltic pumping rate and free pumping flux are decreasing functions of n and We while in copumping, the effects of n and We are not noticeable. Moreover, the pumping curve for square and trapezoidal waves have similar variations.

2.5.3 Trapping

The formation of internally circulating bolus of fluid by closed streamlines is called trapping and this trapped bolus pushed ahead along with the peristaltic wave. The effects of n and We on trapping for different wave forms have been arranged in the Figures 2.10 − 2.13. These figures depict that by increasing (decreasing) We (n), the bolus squeezes (i.e. its size reduces) for all the considered wave forms. Interestingly for all the considered wave forms, trapping occurs at $\theta = 0.61$ and $\Phi = 0.2$. However for triangular wave, this is not the case. For triangular wave, the bolus appears only for large values of Φ or θ.

Chapter 3

MHD peristaltic flow of a Carreau fluid in a channel

The main purpose of this chapter is to put forward the MHD flow analysis of peristaltic transport in the non-Newtonian fluids. For this aim, our interest here is to discuss the flow analysis of previous chapter in the regime of magnetohydrodynamics. The Carreau fluid is electrically conducting in the presence of a constant applied magnetic field. The expressions of longitudinal velocity, pressure gradient, stream function are obtained. Results of these flow quantities , trapping and pumping phenomena have been examined graphically for various values of Hartman number.

3.1 Theory

The physical model in the problem is quite similar to that which considered in chapter 1. Besides this, the fluid is electrically conducting and the channel walls are non-conducting. A uniform magnetic field B_0 is applied in the $Y^{'}-$direction. No electric field is taken into account. The magnetic Reynolds number is chosen small and hence the induced magnetic field is neglected. The fundamental equations which can govern the magnetohydrodynamic (MHD) flow are Eqs. (2.6), (2.8) and

$$\mathrm{div}\,\mathbf{B} = 0, \quad curl\mathbf{B} = \mu_e \mathbf{J}, \quad \mathrm{curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \mathbf{J} = \sigma\left(\mathbf{E} + \mathbf{V} \times \mathbf{B}\right), \tag{3.1}$$

$$\rho\left[\frac{\partial}{\partial t}+\left(\overline{\mathbf{V}'}\cdot\nabla\right)\right]\overline{\mathbf{V}'}=-\overline{\nabla}p'+\operatorname{div}\overline{\tau'}+\mathbf{J}\times\mathbf{B}. \tag{3.2}$$

In the above expressions, p' is the fluid pressure, μ_e the magnetic permeability, σ the electrically conductivity, $\mathbf{E}$ the electric field, $\mathbf{J}$ the current density, $\mathbf{B}\,(=\mathbf{B}_0+\mathbf{B}_1)$ the total magnetic field, $\mathbf{B}_0$ constant and $\mathbf{B}_1$ is the induced magnetic field. Under low magnetic Reynolds number approximation, the Lorentz force reduces to

$$\mathbf{J}\times\mathbf{B}=-\sigma B_0^2\left(\overline{U},0,0\right). \tag{3.3}$$

Using the non-dimensional quantities in Eq. (2.12) along with

$$u=\frac{\partial\Psi}{\partial y},\ v=-\delta\frac{\partial\Psi}{\partial x},\ M^2=\frac{\sigma B_0'^2a^2}{\eta_0}, \tag{3.4}$$

equations (3.2) and (3.3) give

$$\delta\operatorname{Re}\left[\left(\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial\Psi}{\partial y}\right]=-\frac{\partial p}{\partial x}-\frac{\delta^2}{2}\frac{\partial\tau_{xx}}{\partial x}-\frac{\partial\tau_{xy}}{\partial y}-M^2\frac{\partial\Psi}{\partial y}, \tag{3.5}$$

$$-\delta^3\operatorname{Re}\left[\left(\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial\Psi}{\partial x}\right]=-\frac{\partial p}{\partial y}-\delta^2\frac{\partial\tau_{xy}}{\partial x}-\delta\frac{\partial\tau_{yy}}{\partial y}, \tag{3.6}$$

where

$$\tau_{xx}=-2\left[1+\frac{(n-1)}{2}We^2\dot{\gamma}^2\right]\frac{\partial^2\Psi}{\partial x\partial y}, \tag{3.7}$$

$$\tau_{xy}=-\left[1+\frac{(n-1)}{2}We^2\dot{\gamma}^2\right]\left(\frac{\partial^2\Psi}{\partial y^2}-\delta^2\frac{\partial^2\Psi}{\partial x^2}\right), \tag{3.8}$$

$$\tau_{yy}=2\delta\left[1+\frac{(n-1)}{2}We^2\dot{\gamma}^2\right]\frac{\partial^2\Psi}{\partial x\partial y}, \tag{3.9}$$

$$\dot{\gamma}=\left[2\delta^2\left(\frac{\partial^2\Psi}{\partial x\partial y}\right)^2+\left(\frac{\partial^2\Psi}{\partial y^2}-\delta^2\frac{\partial^2\Psi}{\partial x^2}\right)^2+2\delta^2\left(\frac{\partial^2\Psi}{\partial x\partial y}\right)^2\right]^{\frac{1}{2}}. \tag{3.10}$$

In the foregoing equations, We is the Wessinberg number, Re the Reynolds number and M the Hartman number.

To facilitate the analysis, we adopt the assumptions of long wavelength and low Reynolds number [21 − 30] and therefore Eqs. (3.5) and (3.6) after using Eqs. (3.10) take the forms

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\left[1 + \frac{(n-1)}{2} We^2 \left(\frac{\partial^2 \Psi}{\partial y^2}\right)^2\right] \frac{\partial^2 \Psi}{\partial y^2} - M^2 \frac{\partial \Psi}{\partial y}, \tag{3.11}$$

$$\frac{\partial p}{\partial y} = 0. \tag{3.12}$$

The above equations imply that

$$\frac{\partial^2}{\partial y^2}\left[1 + \frac{(n-1)}{2} We^2 \left(\frac{\partial^2 \Psi}{\partial y^2}\right)^2\right] \frac{\partial^2 \Psi}{\partial y^2} - M^2 \frac{\partial^2 \Psi}{\partial y^2} = 0 \tag{3.13}$$

and by Eq. (3.12), $p \neq p(y)$.

The dimensionless boundary conditions and pressure rise per wavelength Δp_λ are analogous to those which presented in Eqs. (2.30)-(2.32). For the convenience of readers, we again write these as

$$\Psi = 0, \quad \frac{\partial^2 \Psi}{\partial y^2} = 0 \ \text{ at } y = 0, \tag{3.14}$$

$$\Psi = F, \quad \frac{\partial \Psi}{\partial y} = -1 \ \text{ at } y = h, \tag{3.15}$$

$$\Delta p_\lambda = \int_0^1 \left(\frac{dp}{dx}\right) dx, \tag{3.16}$$

$$\theta = F + 1, \tag{3.17}$$

$$F = \int_0^h \frac{\partial \Psi}{\partial y} dy. \tag{3.18}$$

3.2 Perturbation solution

Taking into account the procedure of section (2.3), we have

3.2.1 O(We^0) System

$$\frac{\partial^4 \Psi_0}{\partial y^4} - M^2 \frac{\partial^2 \Psi_0}{\partial y^2} = 0,$$

$$\frac{dp_0}{dx} = \frac{\partial^3 \Psi_0}{\partial y^3} - M^2 \frac{\partial \Psi_0}{\partial y},$$

$$\Psi_0 = 0, \qquad \frac{\partial^2 \Psi_0}{\partial y^2} = 0 \quad \text{at} \quad y = 0,$$

$$\Psi_0 = F_0, \qquad \frac{\partial \Psi_0}{\partial y} = -1 \quad \text{at} \quad y = h,$$

$$\Delta p_{\lambda_0} = \int_0^1 \frac{dp_0}{dx} dx.$$

3.2.2 O(We^2) System

$$\frac{\partial^4 \Psi_1}{\partial y^4} - M^2 \frac{\partial^2 \Psi_1}{\partial y^2} + \left(\frac{n-1}{2}\right) \frac{\partial^2}{\partial y^2} \left[\left(\frac{\partial^2 \Psi_0}{\partial y^2} \right)^3 \right] = 0,$$

$$\frac{dp_1}{dx} = \frac{\partial^3 \Psi_1}{\partial y^3} - M^2 \frac{\partial \Psi_1}{\partial y} + \left(\frac{n-1}{2}\right) \frac{\partial}{\partial y} \left[\left(\frac{\partial^2 \Psi_0}{\partial y^2} \right)^3 \right],$$

$$\Psi_1 = 0, \qquad \frac{\partial^2 \Psi_1}{\partial y^2} = 0 \quad \text{at} \quad y = 0,$$

$$\Psi_1 = F_1, \qquad \frac{\partial \Psi_0}{\partial y} = 0 \quad \text{at} \quad y = h,$$

$$\Delta p_{\lambda_1} = \int_0^1 \frac{dp_1}{dx} dx.$$

3.2.3 Solution for O(We^0) system

At this order, we have

$$\Psi_0 = \left(\frac{F_0 M + \tanh Mh}{Mh - \tanh Mh} \right) \left\{ y - \frac{\sinh My}{M \cosh Mh} \right\} - \frac{\sinh My}{M \cosh Mh}, \tag{3.19}$$

$$\frac{dp_0}{dx} = -M^2 \left(\frac{MF_0 + \tanh Mh}{Mh - \tanh Mh} \right). \tag{3.20}$$

3.2.4 Solution for O(We^2) system

The solutions of first order system are

$$
\begin{aligned}
\Psi_1 \;=\; & \frac{F_1 y M \cosh Mh}{Mh\cosh Mh-\sinh Mh}-\frac{(n-1)\left(\frac{dp_0}{dx}-M^2\right)^3 My\cosh Mh}{\cosh Mh^3\left(Mh\cosh Mh-\sinh Mh\right)} \\
& \left\{\frac{3h\cosh 3Mh}{64M^4}-\frac{\sinh 3Mh}{64M^5}-\frac{3h^2\sinh Mh}{16M^3}\right\}-\frac{(n-1)\left(\frac{dp_0}{dx}-M^2\right)^3 y}{\cosh Mh^3 Mh} \\
& \left\{\frac{-3\cosh 3Mh}{64M^4}+\frac{3\cosh Mh}{16M^4}+\frac{3h\sinh Mh}{16M^3}\right\}-\frac{F_1\sinh My}{Mh\cosh Mh-\sinh Mh} \\
& +\frac{(n-1)\left(\frac{dp_0}{dx}-M^2\right)^3\sinh My}{\cosh Mh^3\left(Mh\cosh Mh-\sinh Mh\right)}\left\{\frac{3h\cosh 3Mh}{64M^4}-\frac{\sinh 3Mh}{64M^5}-\frac{3h^2\sinh Mh}{16M^3}\right\} \\
& +\frac{(n-1)\left(\frac{dp_0}{dx}-M^2\right)^3}{\cosh Mh^3 Mh}\left\{-\frac{\sinh 3My}{64M^5}+\frac{3y\cosh My}{16M^4}\right\} \qquad (3.21)
\end{aligned}
$$

and the expression of an axial pressure gradient is

$$
\begin{aligned}
\frac{dp_1}{dx} \;=\; & -\frac{F_1 M^3\cosh Mh}{Mh\cosh Mh-\sinh Mh}+\frac{(n-1)\left(\frac{dp_0}{dx}-M^2\right)^3}{\cosh Mh^2\left(Mh\cosh Mh-\sinh Mh\right)} \\
& \left\{\frac{3h\cosh 3Mh}{64M}-\frac{\sinh 3Mh}{64M^2}-\frac{3h^2\sinh Mh}{16}\right\}+\frac{(n-1)\left(\frac{dp_0}{dx}-M^2\right)^3}{\cosh Mh^3 Mh} \\
& \left\{\frac{3\cosh Mh}{16M^2}-\frac{3\cosh 3Mh}{64M^2}+\frac{3h\sinh Mh}{16M}\right\}. \qquad (3.22)
\end{aligned}
$$

Expression of stream function and longitudinal pressure gradient and pressure rise per wavelength upto $O\left(We^2\right)$ are

$$
u = u_0 + We^2 u_1, \qquad (3.23)
$$

$$
\frac{dp}{dx} = \frac{dp_0}{dx} + We^2\frac{dp_1}{dx}, \qquad (3.24)
$$

$$
\Delta p_\lambda = \Delta p_{\lambda_0} + We^2\Delta p_{\lambda_1}. \qquad (3.25)
$$

In the above solutions, we use

$$F = F_0 + We^2 F_1 \tag{3.26}$$

and neglect the terms which are greater than $O(We^2)$. The stream function and longitudinal pressure gradient after using Eq. (3.26) reduce to the following results

$$\begin{aligned}
\Psi &= \left(\frac{FM + \tanh Mh}{Mh - \tanh Mh}\right)\left\{y - \frac{\sinh My}{M\cosh Mh}\right\} - \frac{\sinh My}{M\cosh Mh} \\
&- \frac{We^2(n-1)\left(\frac{dp_0}{dx} - M^2\right)^3 My\cosh Mh}{\cosh Mh^3(Mh\cosh Mh - \sinh Mh)}\left\{\frac{3h\cosh 3Mh}{64M^4} - \frac{\sinh 3Mh}{64M^5} - \frac{3h^2\sinh Mh}{16M^3}\right\} \\
&- \frac{We^2(n-1)\left(\frac{dp_0}{dx} - M^2\right)^3 y}{\cosh Mh^3 Mh}\left\{\frac{-3\cosh 3Mh}{64M^4} + \frac{3\cosh Mh}{16M^4} + \frac{3h\sinh Mh}{16M^3}\right\} \\
&+ \frac{We^2(n-1)\left(\frac{dp_0}{dx} - M^2\right)^3 \sinh My}{\cosh Mh^3(Mh\cosh Mh - \sinh Mh)}\left\{\frac{3h\cosh 3Mh}{64M^4} - \frac{\sinh 3Mh}{64M^5} - \frac{3h^2\sinh Mh}{16M^3}\right\} \\
&+ \frac{We^2(n-1)\left(\frac{dp_0}{dx} - M^2\right)^3}{\cosh Mh^3 Mh}\left\{\frac{3y\cosh My}{16M^4} - \frac{\sinh 3My}{64M^5}\right\},
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
\frac{dp}{dx} &= -\frac{FM^3\cosh Mh + M^2\sinh Mh}{Mh\cosh Mh - \sinh Mh} - We^2(F+h)^3 M^7(n-1) \\
&\quad \frac{12Mh - 8\sinh 2Mh + \sinh 4Mh}{64\sinh Mh - hM\sinh Mh}.
\end{aligned} \tag{3.28}$$

3.3 Expression for wave shapes

This section presents the four wave forms given by

(1) Sinusoidal wave

$$h(x) = 1 + \Phi\sin 2\pi x.$$

(2) Triangular wave

$$h(x) = 1 + \Phi\left[\frac{8}{\pi^3}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)^2}\sin\{2(2m-1)\pi x\}\right].$$

(3) Square wave

$$h(x) = 1 + \Phi \left[\frac{4}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)} \cos\{2(2m-1)\pi x\} \right].$$

(4) Trapezoidal wave

$$h(x) = 1 + \Phi \left[\frac{32}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} \sin\left\{\frac{\pi}{3}(2m-1)\right\}}{(2m-1)^2} \sin\{2(2m-1)\pi x\} \right].$$

Note that in the present analysis, the 50 terms have been accounted.

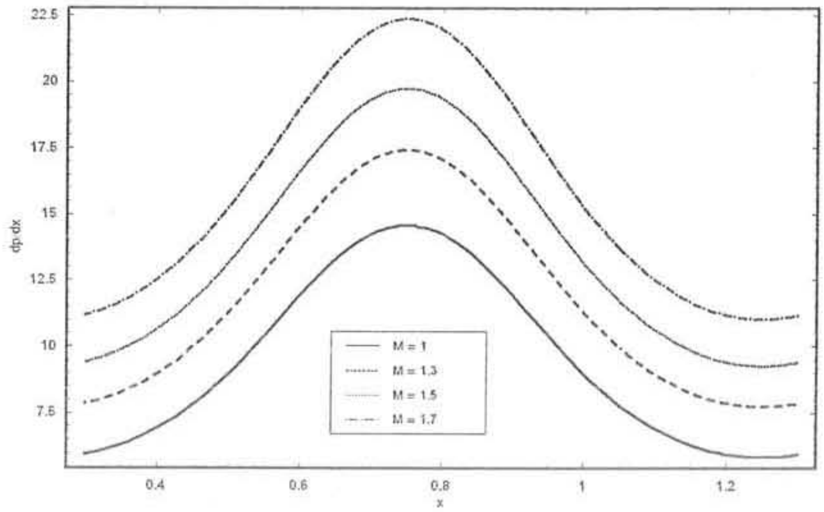

Figure 3.1a: Plot showing dp/dx versus x for sinusoidal wave. Here $\Phi = 0.2$, $\theta = -2$ and $We = 0.1$ and n=0.398.

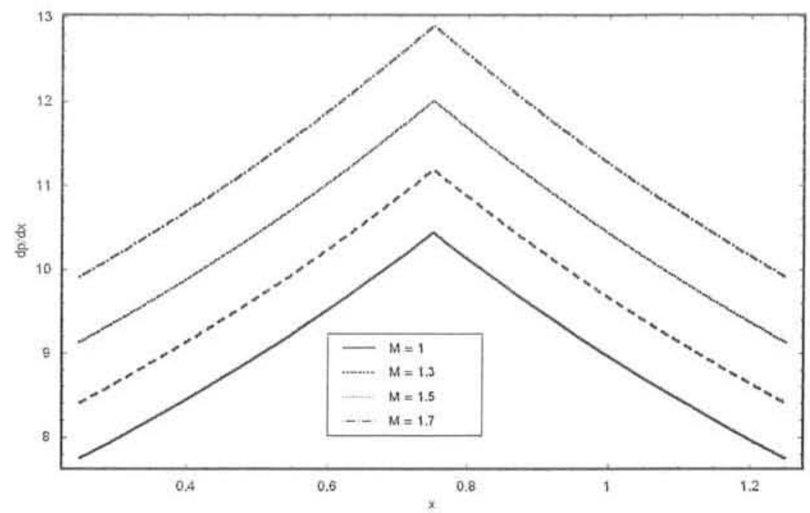

Figure 3.1b: Plot showing dp/dx versus x for triangular wave. Here $\Phi = 0.2$, $\theta = -2$ and $We = 0.1$ and n=0.398.

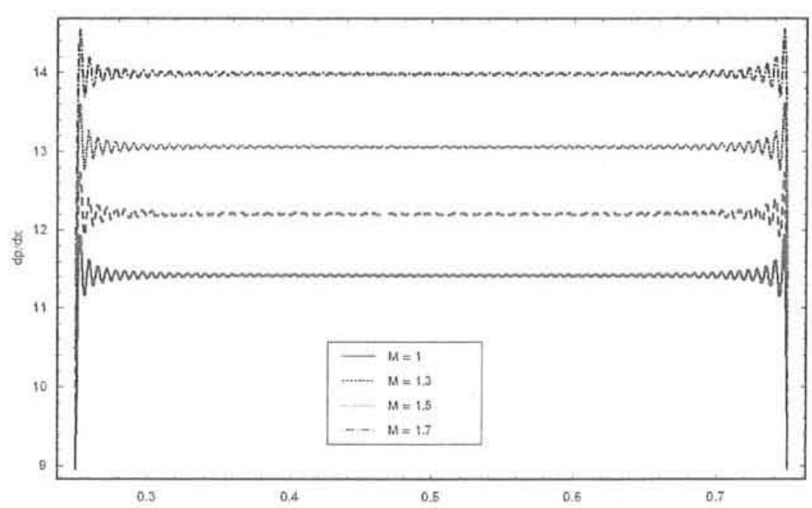

Figure 3.1c: Plot showing dp/dx versus x for square wave. Here $\Phi = 0.2$, $\theta = -2$ and $We = 0.1$ and n=0.398.

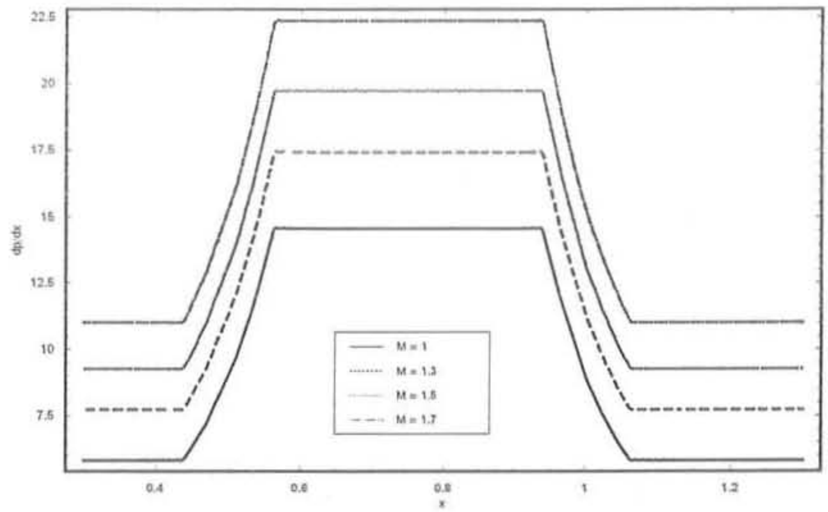

Figure 3.1d: Plot showing dp/dx versus x for trapezoidal wave. Here $\Phi = 0.2$, $\theta = -2$ and $We = 0.1$ and n=0.398.

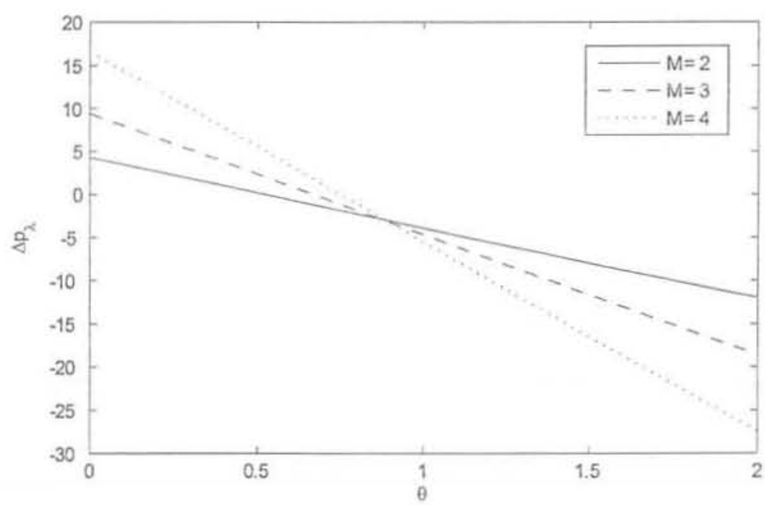

Figure 3.2: Plot showing Δp_λ versus flow rate θ for sinusoidal wave. Here $\Phi = 0.2$, $We = 0.4$ and $n = 0.398$.

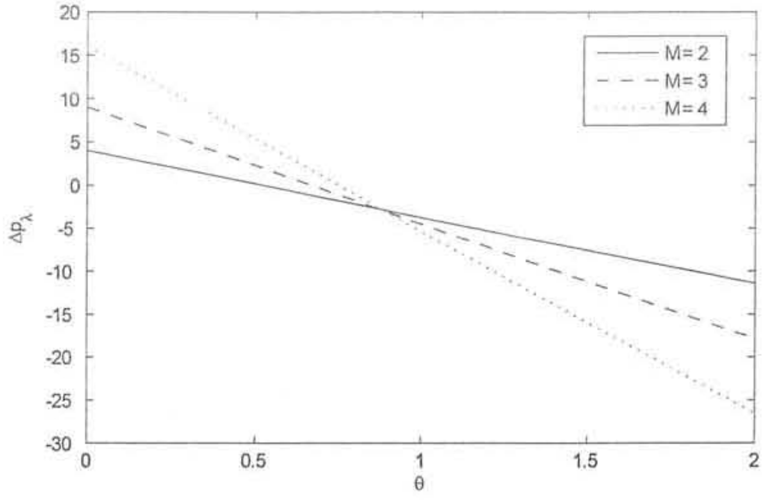

Figure 3.3: Plot showing Δp_λ versus flow rate θ for triangular wave. Here $\Phi = 0.2$, $We = 0.4$ and $n = 0.398$.

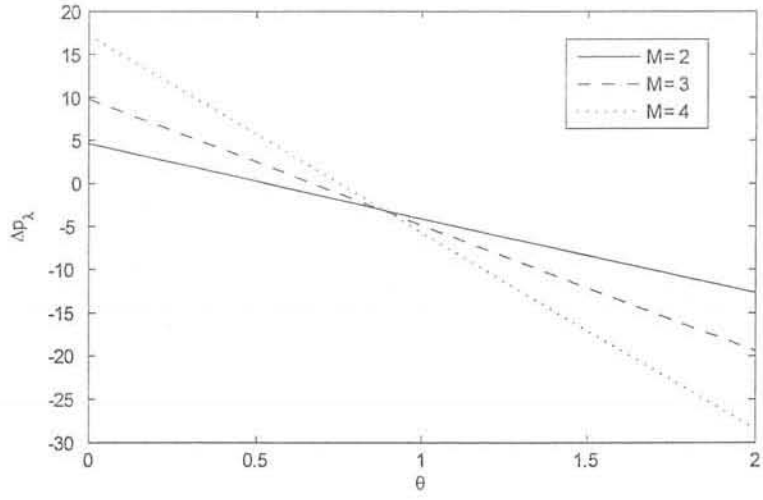

Figure 3.4: Plot showing Δp_λ versus flow rate θ for square wave. Here $\Phi = 0.2$, $We = 0.4$ and $n = 0.398$.

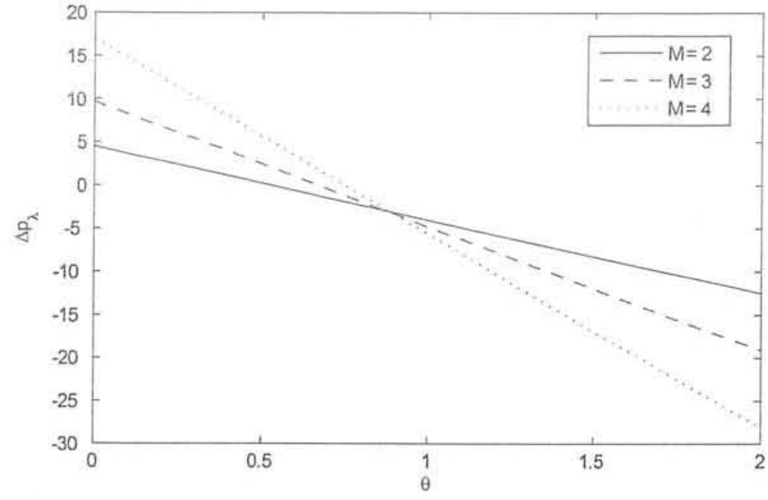

Figure 3.5: Plot showing Δp_λ versus flow rate θ for trapezoidal wave. Here $\Phi = 0.2$, $We = 0.4$ and $n = 0.398$.

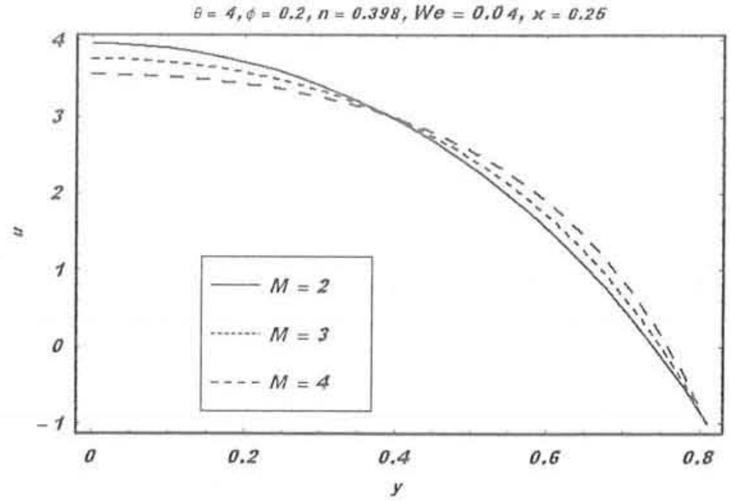

Figure 3.6a: Plot showing velocity u versus y for narrow part of the channel for sinusoidal wave.

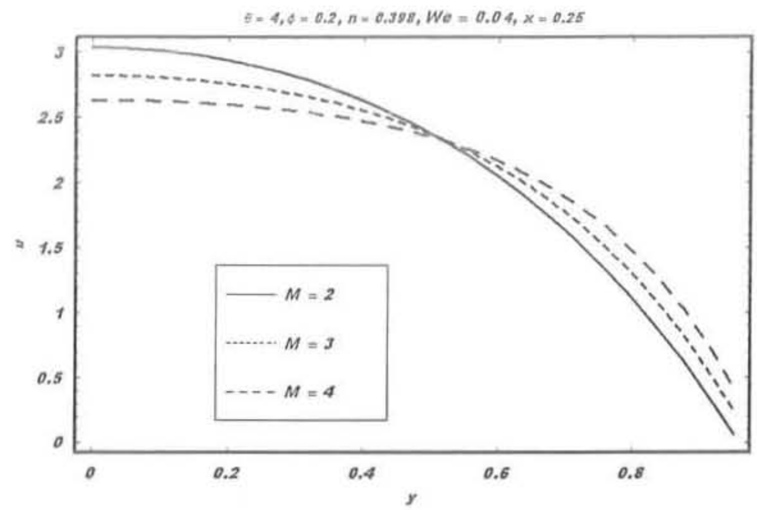

Figure 3.6b: Plot showing velocity u versus y for narrow part of the channel for triangular wave.

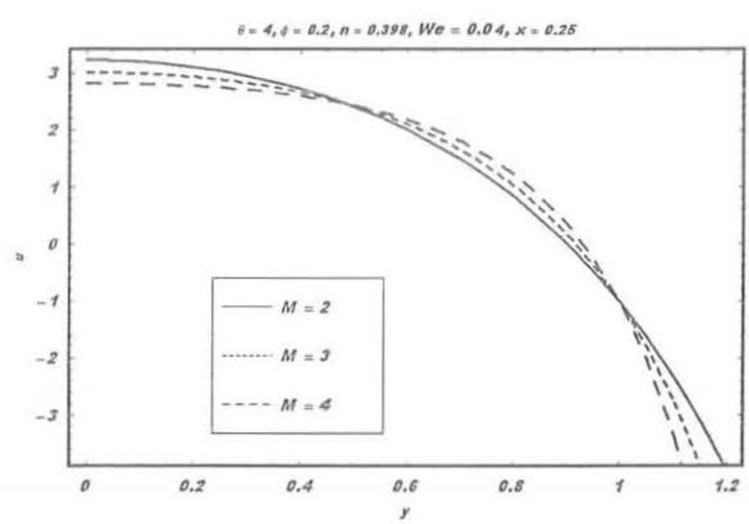

Figure 3.6c: Plot showing velocity u versus y for narrow part of the channel for square wave.

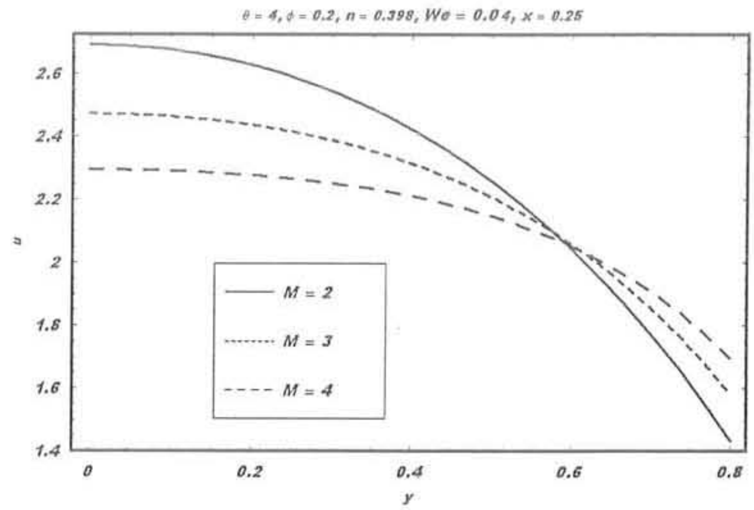

Figure 3.6d: Plot showing velocity u versus y for narrow part of the channel for trapezoidal wave.

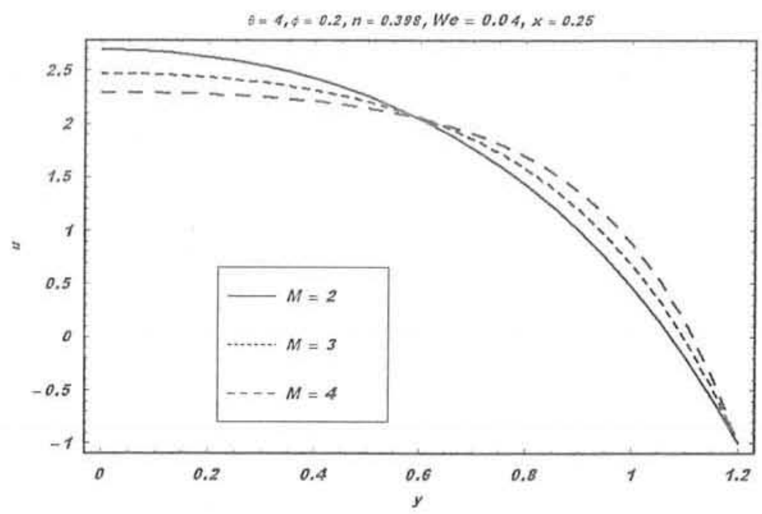

Figure 3.7a: Plot showing velocity u versus y for wider part of the channel for sinusoidal wave.

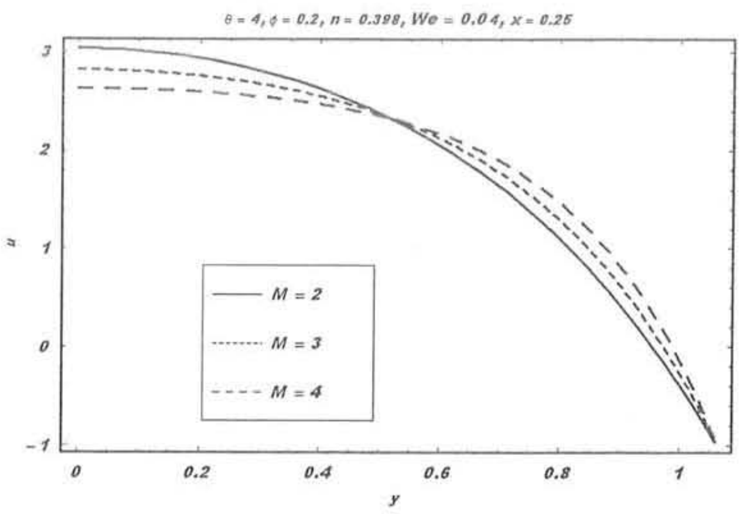

Figure 3.7b: Plot showing velocity u versus y for wider part of the channel for triangular wave.

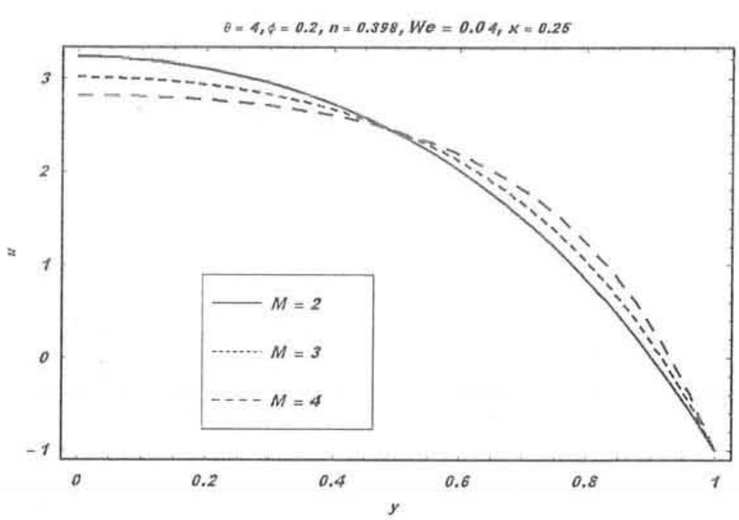

Figure 3.7c: Plot showing velocity u versus y for wider part of the channel for square wave.

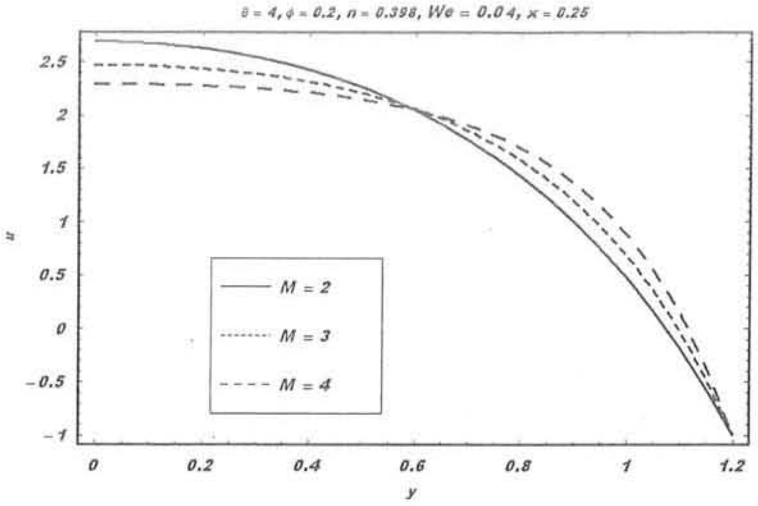

Figure 3.7d: Plot showing velocity u versus y for wider part of the channel for trapezoidal wave.

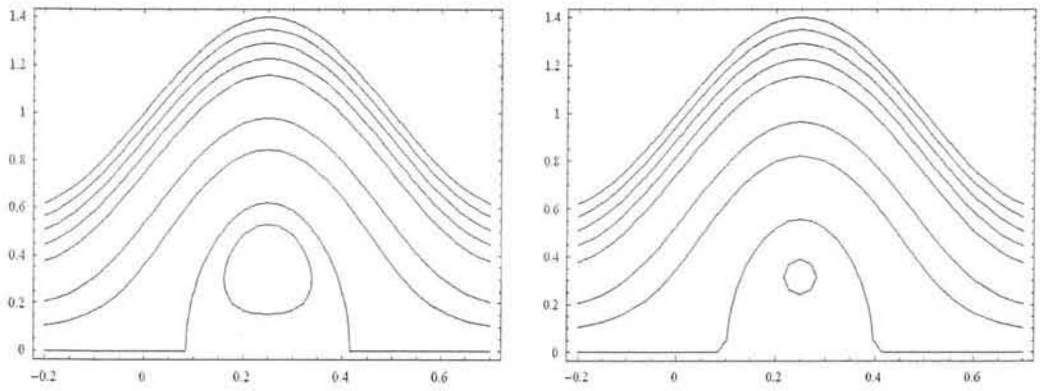

Figure 3.8: Streamlines (sinusoidal wave) for $M = 0.2$, (panel(a)), $M = 0.8$ (panel (b)). The other parameters are $\Phi = 0.4$, $n = 0.398$, $We = 0.04$, $\theta = 0.6$.

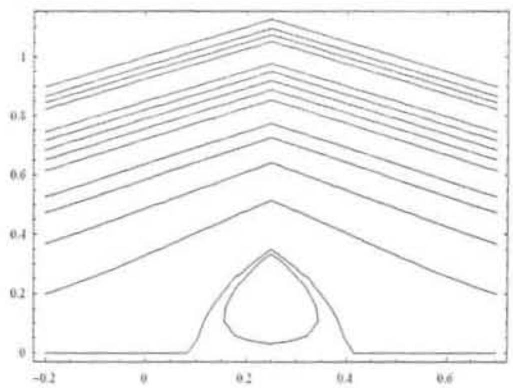

Figure 3.9: Streamlines (triangular wave) for $M = 0.2$, (panel(a)), $M = 0.8$ (panel (b)). The other parameters are $\Phi = 0.4$, $n = 0.398$, $We = 0.04$, $\theta = 0.6$.

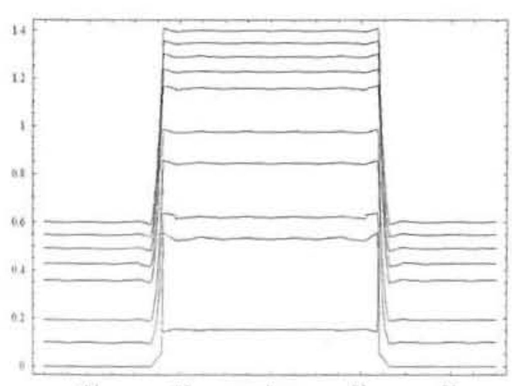

Figure 3.10: Streamlines (square wave) for $M = 0.2$, (panel(a)), $M = 0.8$ (panel (b)). The other parameters are $\Phi = 0.4$, $n = 0.398$, $We = 0.04$, $\theta = 0.6$.

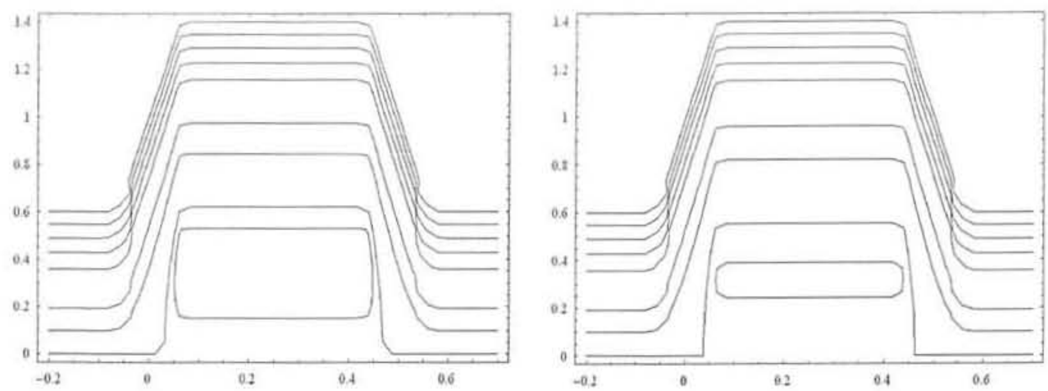

Figure 3.11: Streamlines (trapezoidal wave) for $M = 0.2$, (panel(a)), $M = 0.8$ (panel (b)). The other parameters are $\Phi = 0.4$, $n = 0.398$, $We = 0.04$, $\theta = 0.6$.

3.4 Results and discussion

Our primary interest in this study is to discuss the salient features of Hartman number (M) on various flow quantities such as pressure gradient (dp/dx), pressure rise per wavelength (Δp_λ), longitudinal velocity (u) and stream function (Ψ). Graphical results presented in the Figures 3.1 − 3.11 illustrate these effects.

Figures $3.1(a-d)$ present the variations in dp/dx versus x for four possible wave forms when different values of M are taken into account. Here we can see that dp/dx increases with an increase in M both in the wider and narrow parts of the channel. Moreover a much greater pressure gradient is required for the narrow part of the channel in order to maintain the same flux as compared to that in the wider part of channel. A comparative study shows that dp/dx is greater for sinusoidal and trapezoidal waves and smaller for triangular and square wave forms.

In Figures 3.2 − 3.5, the variation of (Δp_λ) with flow rate θ is displayed for different values of M. These Figures show that in all the considered wave forms, the Hartman number causes an increase in Δp_λ in pumping as well as copumping regions. The peristaltic pumping rate and free pumping rate increases with an increase in M. However, in copumping for an appropriate negative value of Δp_λ, the flow rate decreases by increasing M. A close look at Figure 3.3 (which is for triangular wave) reveals that Δp_λ for triangular wave is less in magnitude when compared with the other waveforms.

Figures 3.6$(a-d)$ are sketched just to see the influence of M on longitudinal velocity in the narrow part of the channel for all the considered wave forms. From these figures, it is concluded that the longitudinal velocity near the centre of the channel decreases by increasing M. However, the opposite behaviour is seen near the wall. A comparison of these figures further reveals that at the channel centre, the longitudinal velocity is maximum in the case of sinusoidal and trapezoidal waves.

Figures 3.7$(a-d)$ illustrate the variation of u in the wider part of the channel for all the considered wave forms. We observe from these figures that the behaviour of velocity in wider part of the channel is quite similar to that of in the narrow part.

To discuss the effects of M on the phenomenon of trapping we have prepared Figures $(3.8-3.11)$. These figures reveal that by increasing M the size of the trapped bolus decreases and it vanishes when large values of M are taken into account. We have noticed from these figures that the lower trapping limit for triangular wave is less when compared with the other wave forms.

3.5 Concluding remarks

An analysis of peristaltic flow of MHD Carreau fluid is presented in a two dimensional channel under long wavelength and low Reynolds number approximations. Four different wave forms are examined. The effects of Hartman number (M) on pressure rise per wavelength, longitudinal velocity and trapping phenomenon are seen through graphs. It is observed that Δp_λ increases by increasing M and for triangular wave, its magnitude is less when compared with the others waves forms. The size of the trapped bolus is a decreasing function of M. The lower trapping limit for triangular wave is less in comparison to the other wave forms.

Chapter 4

Influence of induced magnetic field on peristaltic flow in a Carreau fluid

In this chapter, the influence of an induced magnetic field on the peristaltic motion of a Carreau fluid in planar channel is studied. The associated mathematical modelling is developed. The solution expressions are derived using small Weissenberg number and compared in different wave forms. The results for the pressure gradient, the stream function, the magnetic force function, the axial induced magnetic field and the distribution of current density are obtained. Graphical illustrations are presented and discussed.

4.1 Development of mathematical problem

We investigate the MHD flow of an incompressible Carreau fluid in a two-dimensional channel of uniform thickness. Four possible wave forms namely sinusoidal (s), triangular (t), square (sq), trapezoidal (tr) travelling down on the channel walls are considered. We consider a wave of amplitude b that propagates on the channel walls with constant speed c. The wave shape and the transformation between the laboratory and wave frames are given in section (3.3) and Eq. (2.2) respectively. The system is stressed by a constant magnetic field of strength H_0' in the transverse direction. This gives rise to an induced magnetic field $H'(h_{X'}'(X', Y', t')$, $h_{Y'}'(X', Y', t'), 0)$ and hence the total magnetic field is $H'^{+}(h_{X'}'(X', Y', t'), H_0'+h_{Y'}'(X', Y', t'), 0)$.

The equations governing the present flow may be put in the form

$$\nabla \cdot \mathbf{H}' = 0, \quad \nabla \cdot \mathbf{E}' = 0, \tag{4.1}$$

$$\nabla \wedge \mathbf{H}' = \mathbf{J}', \quad \mathbf{J}' = \sigma \left\{ \mathbf{E}' + \mu_e \left(\mathbf{V}' \wedge \mathbf{H}'^{+} \right) \right\}, \tag{4.2}$$

$$\nabla \wedge \mathbf{E}' = -\mu_e \frac{\partial \mathbf{H}'}{\partial t'}, \tag{4.3}$$

$$\nabla \cdot \overline{\mathbf{V}'} = 0, \tag{4.4}$$

$$\rho \left[\frac{\partial}{\partial t'} + (\mathbf{V}' \cdot \nabla) \right] \overline{\mathbf{V}'} = -\nabla p' + \operatorname{div} \overline{\tau'} - \mu_e \left\{ (\mathbf{H}'^{+} \cdot \nabla) - \frac{1}{2} \left(H'^{+} \right)^2 \nabla \right\}. \tag{4.5}$$

In the above expressions, p' is the fluid pressure, $\mathbf{J}$ the current density, μ_e magnetic permeability, σ the electrical conductivity, $\mathbf{E}'$ an induced magnetic field and the velocity $\overline{\mathbf{V}'}$ and extra stress tensor $\overline{\tau'}$ are defined by expressions (2.8) and (2.5) respectively.

With Eqs. (4.1)-(4.3), the induction equation takes the following form

$$\frac{\partial \mathbf{H}'^{+}}{\partial t'} = \nabla \wedge \left\{ \mathbf{V}' \wedge \mathbf{H}'^{+} \right\} + \frac{1}{\zeta} \nabla^2 \mathbf{H}'^{+} \tag{4.6}$$

in which $\zeta = 1/\sigma\mu_e$ is the magnetic diffusivity.

The scalar equations in the laboratory frame are

$$\begin{aligned} \rho \left[\frac{\partial}{\partial t'} + U' \frac{\partial}{\partial X'} + V' \frac{\partial}{\partial Y'} \right] U' &= -\frac{\partial p'}{\partial X'} - \frac{\partial \tau'_{X'X'}}{\partial X'} - \frac{\partial \tau'_{X'Y'}}{\partial Y'} - \frac{\mu_e}{2} \left(\frac{\partial H^{+2}}{\partial X'} \right) \\ &\quad + \mu_e \begin{pmatrix} h'_{X'} \frac{\partial}{\partial X'} + h'_{Y'} \frac{\partial}{\partial Y'} \\ + H_0 \frac{\partial}{\partial Y'} \end{pmatrix} h'_{X'}, \end{aligned} \tag{4.7}$$

$$\begin{aligned} \rho \left[\frac{\partial}{\partial t'} + U' \frac{\partial}{\partial X'} + V' \frac{\partial}{\partial Y'} \right] V' &= -\frac{\partial p'}{\partial Y'} - \frac{\partial \tau'_{X'Y'}}{\partial X'} - \frac{\partial \tau'_{Y'Y'}}{\partial Y'} - \frac{\mu_e}{2} \left(\frac{\partial H^{+2}}{\partial Y'} \right) \\ &\quad + \mu_e \begin{pmatrix} h'_{X'} \frac{\partial}{\partial X'} + h'_{Y'} \frac{\partial}{\partial Y'} \\ + H_0 \frac{\partial}{\partial Y'} \end{pmatrix} h'_{Y'}. \end{aligned} \tag{4.8}$$

The above equations along with the transformations (2.2) may be written into the following forms

$$\rho\left(u'\frac{\partial}{\partial x'}+v'\frac{\partial}{\partial y'}\right)u' = -\frac{\partial p'}{\partial x'}-\frac{\partial \tau'_{xx}}{\partial x'}-\frac{\partial \tau'_{xy}}{\partial y'} +\mu_e\left(h'_x\frac{\partial}{\partial x'}+h'_y\frac{\partial}{\partial y'}+H_0\frac{\partial}{\partial y'}\right)h'_x-\frac{\mu_e}{2}\left(\frac{\partial H^{+^2}}{\partial x'}\right), \quad (4.9)$$

$$\rho\left(u'\frac{\partial}{\partial x'}+v'\frac{\partial}{\partial y'}\right)v' = -\frac{\partial p'}{\partial x'}-\frac{\partial \tau'_{yx}}{\partial x'}-\frac{\partial \tau'_{yy}}{\partial y'} +\mu_e\left(h'_x\frac{\partial}{\partial x'}+h'_y\frac{\partial}{\partial y'}+H_0\frac{\partial}{\partial y'}\right)h'_y-\frac{\mu_e}{2}\left(\frac{\partial H^{+^2}}{\partial y'}\right). \quad (4.10)$$

Considering

$$x=\frac{2\pi x'}{\lambda},\quad y=\frac{y'}{a},\quad u=\frac{u'}{c},\quad v=\frac{v'}{c},\quad t=\frac{2\pi t'c}{\lambda},\quad p=\frac{2\pi a^2 p'}{c\lambda\mu},$$
$$\tau=\frac{a\tau'}{\mu c},\quad h=\frac{h'}{a},\quad \Psi=\frac{\Psi'}{ca},\quad \phi=\frac{\phi'}{H_0 a},\quad \delta=\frac{a}{\lambda},\quad \mathrm{Re}=\frac{\rho c a}{\mu},$$
$$R_m=\sigma\mu_e a c,\quad S_t=\frac{H_0}{c}\sqrt{\frac{\mu_e}{\rho}} \quad (4.11)$$

one can express Eqs. (4.6), (4.9) and (4.10) as

$$\mathrm{Re}\,\delta\left\{\left(\Psi_y\frac{\partial}{\partial x}-\Psi_x\frac{\partial}{\partial y}\right)\Psi_y\right\} = -\frac{\partial p_m}{\partial x}+\delta\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y} +\mathrm{Re}\,\delta S_t^2\left(\phi_y\frac{\partial}{\partial x}-\phi_x\frac{\partial}{\partial y}\right)\phi_y+\mathrm{Re}\,S_t^2\phi_{yy}, \quad (4.12)$$

$$-\mathrm{Re}\,\delta^3\left\{\left(\Psi_y\frac{\partial}{\partial x}-\Psi_x\frac{\partial}{\partial y}\right)\Psi_x\right\} = -\frac{\partial p_m}{\partial y}+\delta^2\frac{\partial \tau_{xy}}{\partial x}+\delta\frac{\partial \tau_{yy}}{\partial x} -\mathrm{Re}\,\delta^3 S_t^2\left(\phi_y\frac{\partial}{\partial x}-\phi_x\frac{\partial}{\partial y}\right)\phi_x-\mathrm{Re}\,\delta^2 S_t^2\phi_{xy}, \quad (4.13)$$

$$\Psi_y - \delta\left(\Psi_y \phi_x - \Psi_x \phi_y\right) + \frac{1}{R_m}\nabla^2 \phi = E, \tag{4.14}$$

where in terms of stream function Ψ and magnetic force function ϕ, we have

$$u = \frac{\partial \Psi}{\partial y}, \quad v = -\delta\frac{\partial \Psi}{\partial x}, \quad h_x = \frac{\partial \phi}{\partial y}, \quad h_y = -\delta\frac{\partial \phi}{\partial x}$$

$$p_m = p + \frac{1}{2}\mathrm{Re}\,\delta\frac{\mu_e\left(H'^{+}\right)^2}{\rho c^2}, \tag{4.15}$$

$$\nabla^2 = \delta^2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

where δ is the wave number, We the Wessinberg number, S_t the Strommer's number, Re the Reynolds number, R_m the magnetic Reynolds number, p_m is the magnetic pressure, M the Hartman number and τ_{xx}, τ_{xy}, τ_{yy} and $\dot{\gamma}$ have been presented in Eqs. (3.7)-(3.10).

Utilizing long wavelength and low Reynolds number approximations one may express Eqs. (4.12)-(4.14) as

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\left[\left\{1 + \frac{(n-1)}{2}We^2\left(\frac{\partial^2 \Psi}{\partial y^2}\right)^2\right\}\frac{\partial^2 \Psi}{\partial y^2}\right] + M^2\left(E - \frac{\partial \Psi}{\partial y}\right), \tag{4.16}$$

$$\frac{\partial p}{\partial y} = 0, \tag{4.17}$$

$$\frac{\partial \Psi}{\partial y} + \frac{1}{R_m}\frac{\partial^2 \phi}{\partial y^2} = E, \tag{4.18}$$

where Eq. (4.17) shows that $p \neq p(y)$ and hence $p = p(x)$. Furthermore, Eqs. (4.16) and (4.17) after cross differentiation give

$$\frac{\partial^2}{\partial y^2}\left[\left\{1 + \frac{(n-1)}{2}We^2\left(\frac{\partial^2 \Psi}{\partial y^2}\right)^2\right\}\frac{\partial^2 \Psi}{\partial y^2}\right] - M^2\frac{\partial^2 \Psi}{\partial y^2} = 0. \tag{4.19}$$

We pose dimensionless boundary conditions and pressure rise per wavelength (Δp_λ) as follows

$$\Psi = 0, \quad \frac{\partial^2 \Psi}{\partial y^2} = 0, \quad \frac{\partial \phi}{\partial y} = 0, \;\; \text{at } y = 0, \tag{4.20}$$

$$\Psi = F, \quad \frac{\partial \Psi}{\partial y} = -1, \quad \phi = 0 \;\; \text{at } y = h, \tag{4.21}$$

$$\Delta p_\lambda = \int_0^{2\pi} \left(\frac{dp}{dx}\right) dx, \tag{4.22}$$

where the dimensionless mean flows in laboratory (θ) and wave (F) frames are defined as

$$\theta = F + 1, \tag{4.23}$$

$$F = \int_0^h \frac{\partial \Psi}{\partial y} dy. \tag{4.24}$$

4.2 Perturbation solution

In order to find the series solution, we expand Ψ, F and p as

$$\Psi = \Psi_0 + We^2 \Psi_1 + O\left(We^4\right), \tag{4.25}$$

$$F = F_0 + We^2 F_1 + O\left(We^4\right), \tag{4.26}$$

$$p = p_0 + We^2 p_1 + O\left(We^4\right). \tag{4.27}$$

Substituting Eqs. (4.25)–(4.27) into Eqs. (4.16) and (4.19)-(4.22), we have the following systems.

4.2.1 O(We^0) System

$$\frac{\partial p_0}{\partial x} = \frac{\partial^3 \Psi_0}{\partial y^3} + M^2 \left(E - \frac{\partial \Psi_0}{\partial y}\right),$$

$$\frac{\partial^4 \Psi_0}{\partial y^4} - M^2 \frac{\partial^2 \Psi_0}{\partial y^2} = 0,$$

$$\Psi_0 = 0, \quad \frac{\partial^2 \Psi_0}{\partial y^2} = 0 \quad \text{at} \quad y = 0,$$
$$\Psi_0 = F_0, \quad \frac{\partial \Psi_0}{\partial y} = -1 \quad \text{at} \quad y = h.$$

4.2.2 O(We^2) System

$$\frac{\partial p_1}{\partial x} = \frac{\partial^3 \Psi_1}{\partial y^3} - M^2 \frac{\partial \Psi_1}{\partial y} + \left(\frac{n-1}{2}\right) \frac{\partial}{\partial y} \left\{ \left(\frac{\partial^2 \Psi_0}{\partial y^2} \right)^3 \right\},$$

$$\frac{\partial^4 \Psi_1}{\partial y^4} - M^2 \frac{\partial^2 \Psi_1}{\partial y^2} + \left(\frac{n-1}{2}\right) \frac{\partial^2}{\partial y^2} \left\{ \left(\frac{\partial^2 \Psi_0}{\partial y^2} \right)^3 \right\} = 0,$$

$$\Psi_1 = 0, \quad \frac{\partial^2 \Psi_1}{\partial y^2} = 0 \quad \text{at} \quad y = 0,$$
$$\Psi_1 = F_1, \quad \frac{\partial \Psi_1}{\partial y} = 0 \quad \text{at} \quad y = h.$$

Solving the resulting systems and then neglecting the terms of order greater than We^2, one has

$$\Psi = \Psi_0 + We^2 \Psi_1, \tag{4.28}$$

$$\frac{dp}{dx} = \frac{dp_0}{dx} + We^2 \frac{dp_1}{dx}, \tag{4.29}$$

$$\Delta p_\lambda = \Delta p_{\lambda_0} + We^2 \Delta p_{\lambda_1}, \tag{4.30}$$

whence

$$\Psi_0 = \frac{F_0 y M \cosh Mh + y \sinh Mh}{Mh \cosh Mh - \sinh Mh} - \frac{(F_0 + h) \sinh My}{Mh \cosh Mh - \sinh Mh}, \tag{4.31}$$

$$\Psi_1 = \frac{F_1 y M \cosh Mh}{Mh\cosh Mh - \sinh Mh} - \frac{(n-1)(F_0+h)^3 M^7 y \cosh Mh}{(Mh\cosh Mh - \sinh Mh)^4}$$
$$\left\{\frac{3h\cosh 3Mh}{64M} - \frac{\sinh 3Mh}{64M^2} - \frac{3h^2\sinh Mh}{16}\right\} + \frac{(n-1)(F_0+h)^3 M^6 y}{(Mh\cosh Mh - \sinh Mh)^3}$$
$$\left\{\frac{3\cosh 3Mh}{64M} - \frac{3\cosh Mh}{16M} - \frac{3h\sinh Mh}{16}\right\} - \frac{F_1\sinh My}{Mh\cosh Mh - \sinh Mh}$$
$$+ \frac{(n-1)(F_0+h)^3 M^6 \sinh My}{(Mh\cosh Mh - \sinh Mh)^4}\left\{\frac{3h\cosh 3Mh}{64M} - \frac{\sinh 3Mh}{64M^2} - \frac{3h^2\sinh Mh}{16}\right\}$$
$$- \frac{(n-1)(F_0+h)^3 M^6}{(Mh\cosh Mh - \sinh Mh)^3}\left\{\frac{\sinh 3My}{64M^2} - \frac{3y\cosh My}{16M}\right\}, \tag{4.32}$$

$$\frac{dp_0}{dx} = -\left(\frac{F_0 M^3\cosh Mh + M^2\sinh Mh}{Mh\cosh Mh - \sinh Mh} - M^2 E\right), \tag{4.33}$$

$$\frac{dp_1}{dx} = -\frac{F_1 M^3\cosh Mh}{Mh\cosh Mh - \sinh Mh} + \frac{(n-1)(F_0+h)^3 M^9\cosh Mh}{(Mh\cosh Mh - \sinh Mh)^4}$$
$$\left\{\frac{3h\cosh 3Mh}{64M} - \frac{\sinh 3Mh}{64M^2} - \frac{3h^2\sinh Mh}{16}\right\} - \frac{(n-1)(F_0+h)^3 M^8}{(Mh\cosh Mh - \sinh Mh)^3}$$
$$\left\{\frac{3\cosh 3Mh}{64M} - \frac{3\cosh Mh}{16M} - \frac{3h\sinh Mh}{16}\right\}, \tag{4.34}$$

$$\Delta p_{\lambda_0} = \int_0^{2\pi}\frac{dp_0}{dx}dx, \tag{4.35}$$

$$\Delta p_{\lambda_1} = \int_0^{2\pi}\frac{dp_1}{dx}dx. \tag{4.36}$$

The stream function Ψ and magnetic force function ϕ are

$$\Psi = \left(\frac{FM\cosh Mh + \sinh Mh}{Mh\cosh Mh - \sinh Mh}\right)y - \frac{(F+h)\sinh My}{Mh\cosh Mh - \sinh Mh}$$
$$- \frac{We^2(n-1)(F+h)^3 M^7 y\cosh Mh}{(Mh\cosh Mh - \sinh Mh)^4}\left\{\frac{3h\cosh 3Mh}{64M} - \frac{\sinh 3Mh}{64M^2} - \frac{3h^2\sinh Mh}{16}\right\}$$
$$+ \frac{We^2(n-1)(F+h)^3 M^6\sinh My}{(Mh\cosh Mh - \sinh Mh)^4}\left\{\frac{3h\cosh 3Mh}{64M} - \frac{\sinh 3Mh}{64M^2} - \frac{3h^2\sinh Mh}{16}\right\}$$
$$- \frac{We^2(n-1)(F+h)^3 M^6}{(Mh\cosh Mh - \sinh Mh)^3}\left\{\frac{\sinh 3My}{64M^2} - \frac{3y\cosh My}{16M}\right\}$$
$$+ \frac{We^2(n-1)(F+h)^3 M^6 y}{(Mh\cosh Mh - \sinh Mh)^3}\left\{\frac{3\cosh 3Mh}{64M} - \frac{3\cosh Mh}{16M} - \frac{3h\sinh Mh}{16}\right\}, \tag{4.37}$$

$$
\begin{aligned}
\phi \;=\; & \frac{1}{384M\left(-Mh\cosh Mh+\sinh Mh\right)^{4}}((-36M(-2y^2+F^3hM^6(-1+n)We^2(h-y)\\
& (h+y)+3F^2h^2M^6(-1+n)We^2(h-y)(h+y)+h^2(-2+M^2(2y^2+h^2(2\\
& We^2(h-y)(h+y))))+Fh(-4+M^2(-2y^2+h^2(6+M^2(-2+3M^2(-1+n)\\
& We^2(h-y)(h+y))))+2(-1+h^2M^2)^2(h-y)(h+y)E)-32M(-3Fh^5M^4\\
& +2F^3hM^4(-1+n)We^2+3Fh^3M^2(2+M^2(2(-1+n)We^2+y^2))+3h^6M^4E\\
& +3y^2(1+E)+h^2(6F^2M^4(-1+n)We^2-3(1+E))+h^4M^2(6+M^2(2\\
& (-1+n)We^2-3y^2E)))\cosh 2Mh+8M(3y^2+h(F^3M^4(-1+n)We^2+3F^2hM^4\\
& (-1+n)We^2+h(-21+M^2(h^2(-15+M^2(-1+n)We^2)+9y^2))+3F(-6\\
& +M^2(-3y^2+h^2(1+M^2(h^2+(-1+n)We^2-y^2)))))-3(1+6h^2M^2+h^4M^4)\\
& (h-y)(h+y)E\cosh 4Mh+2(F+h)^3M^4(-1+n)We^2\cosh 3My(hM\cosh Mh-\sinh Mh)\\
& +8F^3M^4(-1+n)We^2(4+3M^2(h-y)(h+y)+3F^2hM^4(-1+n)We^2(4+3M^2\\
& (h-y)(h+y)+3F(-4+M^2(-2y^2+h^2(14+M^2(3h^2(-2+M^2(-1+n)We^2)\\
& +6y^2+(-1+n)We^2(4-3M^2y^2)))))+h(-12+M^2(6y^2(3+4E)+3h^4M^2\\
& (2+M^2(-1+n)We^2+8E)-h^2(6(-3+4E)+M^2((-1+n)We^2(-4+3M^2y^2)\\
& +6y^2(1+4E))))))\sinh 2Mh+6(F+h)\cosh My(12hM(-4+4h^2M^2+(F+h)^2M^4\\
& (-1+n)We^2)\cosh Mh+hM(48+16h^2M^2-3(F+h)^2M^4(-1+n)We^2)\cosh 3Mh\\
& +12(-1+h^2M^2)(-4+(F+h)^2M^4(-1+n)We^2\sinh Mh+(-16-48h^2M^2\\
& +(F+h)^2M^4(-1+n)We^2)\sinh 3Mh+(-3F^2hM^4(-1+n)We^2(2+3M^2(h-y)(h+y)\\
& +F^3M^3(-1+n)We^2(-2+3M^2(-h^2+y^2))+3F(16+M^2(8y^2+h^2(40+M^2\\
& (-8-M^2(-1+n)We^2+24y^2+(-1+n)We^2(-2+3M^2y^2)))))+h(48+M^2\\
& (-72y^2+h^2(216+M^2(3h^2(8-M^2(-1+n)We^2)-24y^2+(-1+n)We^2(-2\\
& +3M^2y^2)))+96(1+h^2M^2)(h-y)(h+y)E)))\sinh 4Mh+72(F+h)^3M^5\\
& (-1+n)We^2y(-hM\cosh Mh+\sinh Mh)\sinh My)R_{m)}. \qquad (4.38)
\end{aligned}
$$

The axial induced magnetic field is given by

$$
\begin{aligned}
h_x(x,y) &= \frac{1}{384M\left(-Mh\cosh Mh+\sinh Mh\right)^4}((-36M(F^3hM^6(-1+n)We^2(h-y)\\
&+3F^2h^2M^6(-1+n)We^2(h-y)-4y-F^3hM^6(-1+n)We^2(h+y)\\
&-3F^2h^2M^6(-1+n)We^2(h+y)+h^2M^2(4y+h^2(M^4(-1+n)We^2(h-y)\\
&-M^4(-1+n)We^2(h+y)))+FhM^2(-4y+h^2(M^2(-2+3M^2(-1+n)We^2)\\
&(h-y)-M^2(-2+3M^2(-1+n)We^2)(h+y)))+2(-1+h^2M^2)^2(h-y)E\\
&-2(-1+h^2M^2)^2(h+y)E)-32M(6Fh^3M^4y-6h^4M^4yE+6y(1+E))\\
&\cosh 2Mh+8M(6y+h(18hM^2y+3FM^2(-6y-2h^2M^2y))-3(1+6h^2M^2\\
&+h^4M^4)(h-y)E+3(1+6h^2M^2+h^4M^4)(h+y)E)\cosh 4Mh+72(F+h)^3M^6\\
&(-1+n)We^2y\cosh My\left(-Mh\cosh Mh+\sinh Mh\right)+8(F^3M^4(-1+n)We^2(3M^2\\
&(h-y)-3M^2(h+y))+3F^2hM^4(-1+n)We^2(3M^2(h-y)-3M^2(h+y))\\
&+3FM^2(-4y+h^2M^2(12y-6M^2(-1+n)We^2y))+hM^2(12y(3+4E)-h^2M^2\\
&(6M^2(-1+n)We^2y+12y(1+4E))))\sinh 2Mh(6F^3M^6(-1+n)We^2y-3F^2hM^4\\
&(-1+n)We^2(3M^2(h-y)-3M^2(h+y))+3FM^2(16y+h^2M^2(48y+6M^2\\
&(-1+n)We^2y))+hM^2(-144y+h^2M^2(-48y+6M^2(-1+n)We^2y)+96\\
&(1+h^2M^2)(h-y)E-96(1+h^2M^2)(h+y)E))\sinh 4Mh+72(F+h)^3M^5\\
&(-1+n)We^2\sinh My\left(-Mh\cosh Mh+\sinh Mh\right)+6(F+h)M(12hM(-4+4h^2M^2\\
&+(F+h)^2M^4(-1+n)We^2\cosh Mh+hM(48+16h^2M^2-3(F+h)^2M^4(-1+n)\\
&We^2)\cosh 3Mh+12(-1+h^2M^2)(-4+(F+h)^2M^4(-1+n)We^2)\sinh Mh\\
&+(-16-48h^2M^2+(F+h)^2M^4(-1+n)We^2)\sinh 3Mh)\sinh My+6(F+h)^3M^5\\
&(-1+n)We^2\left(Mh\cosh Mh-\sinh Mh\right)\sinh 3My)R_m. \qquad (4.39)
\end{aligned}
$$

Like the previous chapters, the graphical results and comparison have been sought here by considering 50 terms for four different wave forms namely Sinusoidal, Triangular, Square and Trapezoidal waves.

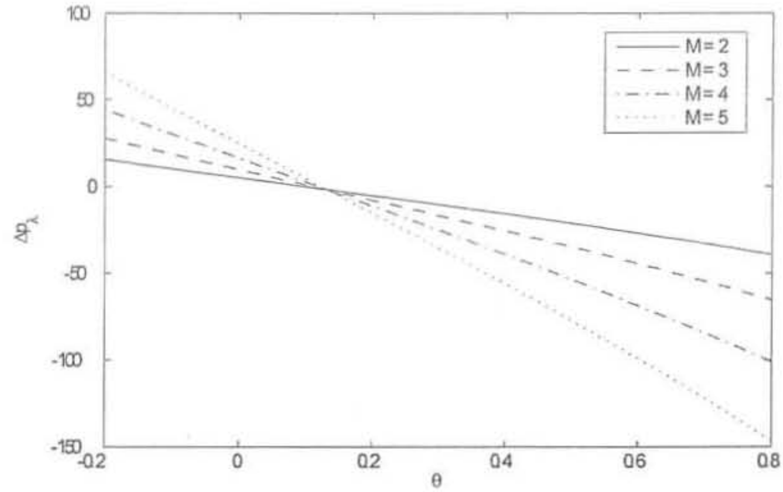

Figure 4.1: Plot showing Δp_λ versus flow rate θ for sinusoidal wave. Here $\Phi = 0.2$, $We = 0.4$, $E = -0.87$, $N = 1$ and $n = 0.398$.

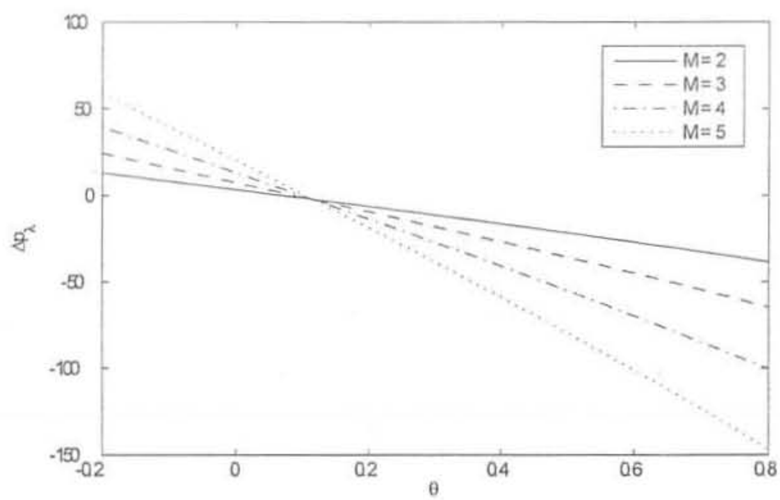

Figure 4.2: Plot showing Δp_λ versus flow rate θ for triangular wave. Here $\Phi = 0.2$, $We = 0.4$, $E = -0.87$, $N = 1$ and $n = 0.398$.

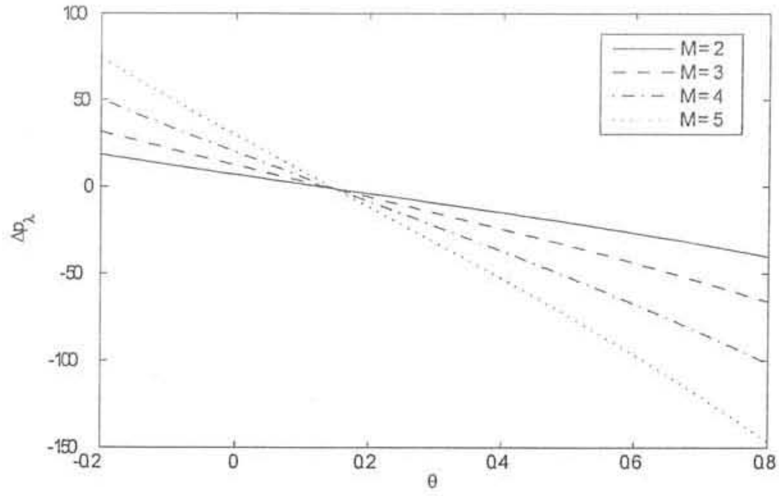

Figure 4.3: Plot showing Δp_λ versus flow rate θ for square wave. Here $\Phi = 0.2$, $We = 0.4$, $E = -0.87$, $N = 1$ and $n = 0.398$.

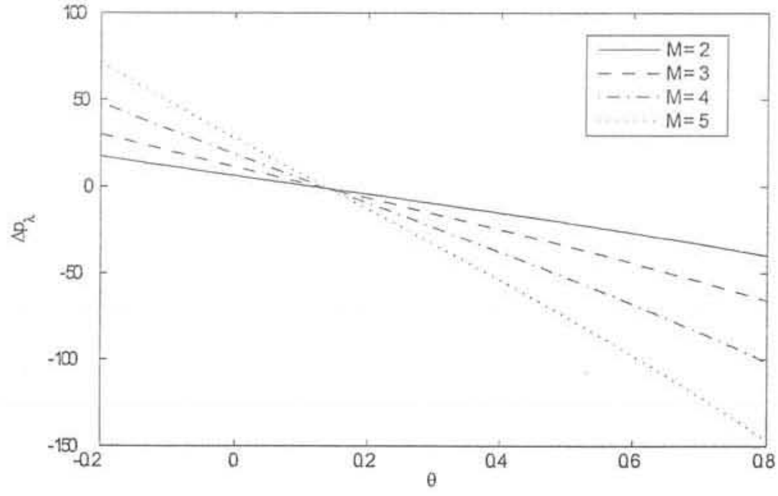

Figure 4.4: Plot showing Δp_λ versus flow rate θ for trapezoidal wave. Here $\Phi = 0.2$, $We = 0.4$, $E = -0.87$, $N = 1$ and $n = 0.398$.

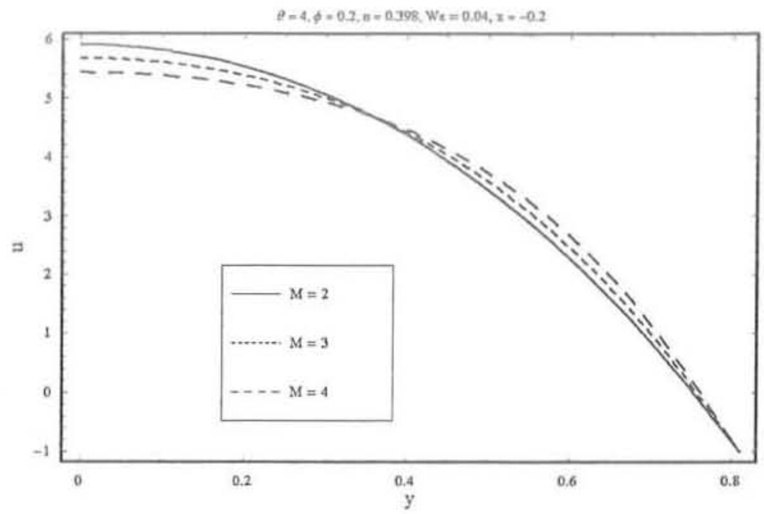

Figure 4.5a: Plot showing velocity u versus y in narrow part of the channel for sinusoidal wave.

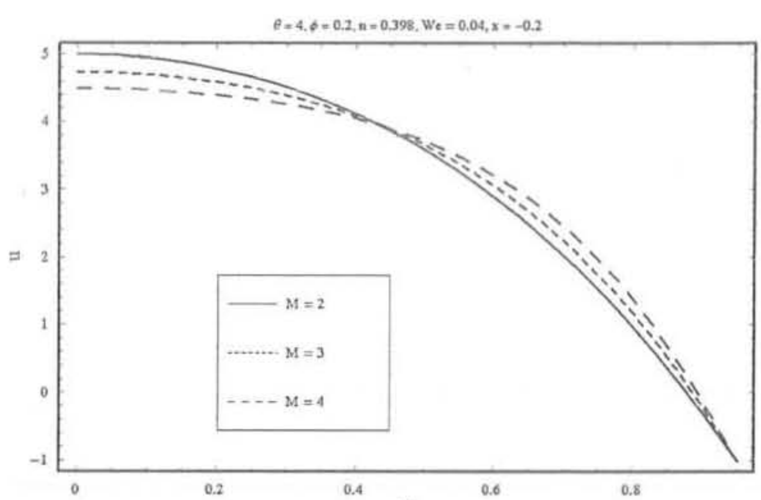

Figure 4.5b: Plot showing velocity u versus y in narrow part of the channel for triangular wave.

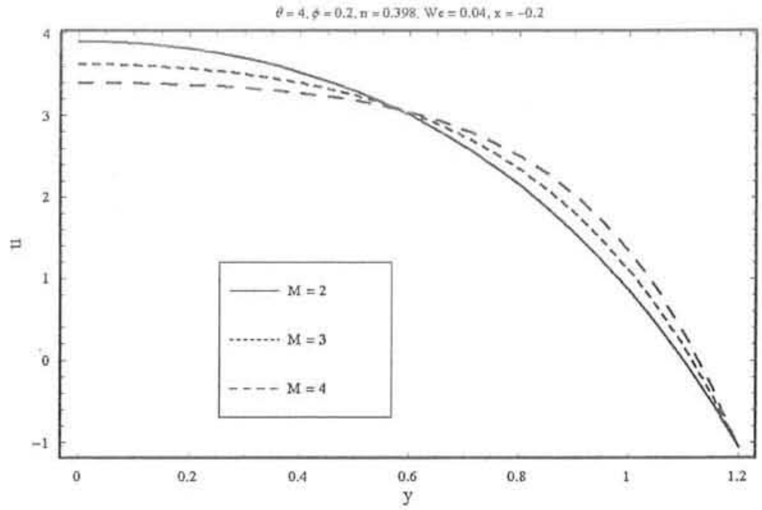

Figure 4.5c: Plot showing velocity u versus y in narrow part of the channel for square wave.

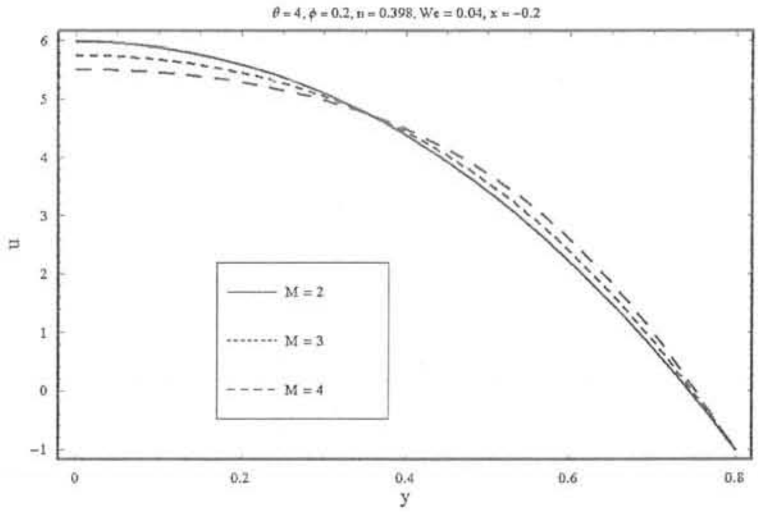

Figure 4.5d: Plot showing velocity u versus y in narrow part of the channel for trapezoidal wave.

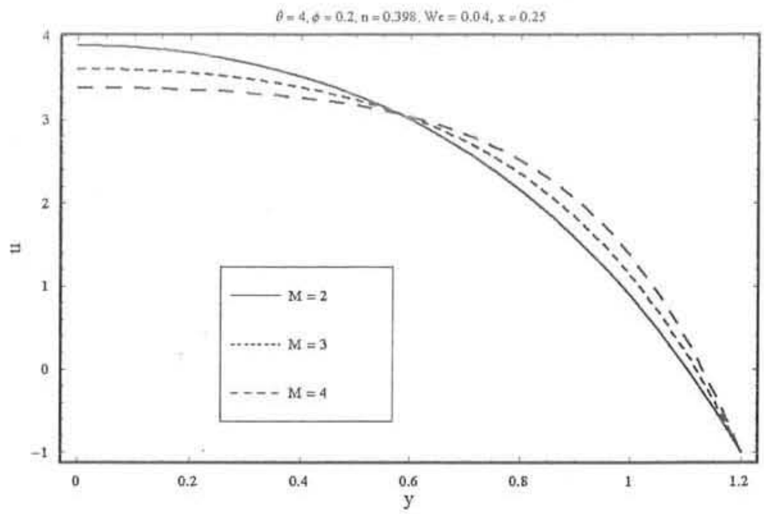

Figure 4.6a: Plot showing velocity u versus y in wider part of the channel for sinusoidal wave.

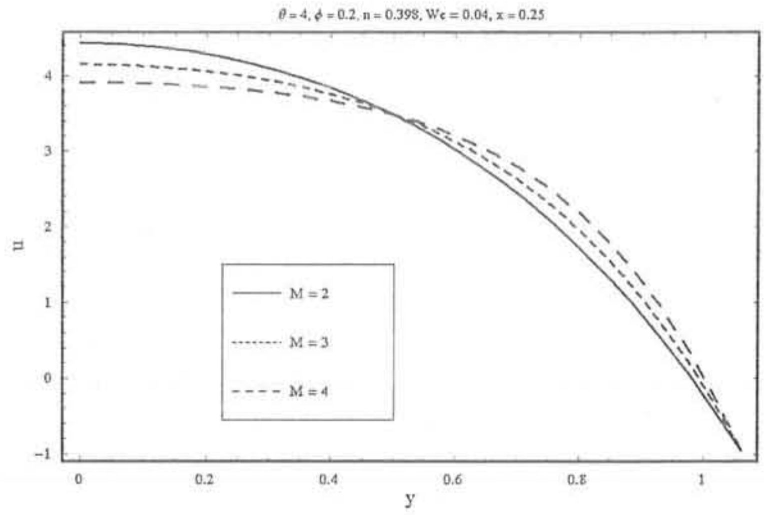

Figure 4.6b: Plot showing velocity u versus y in wider part of the channel for triangular wave.

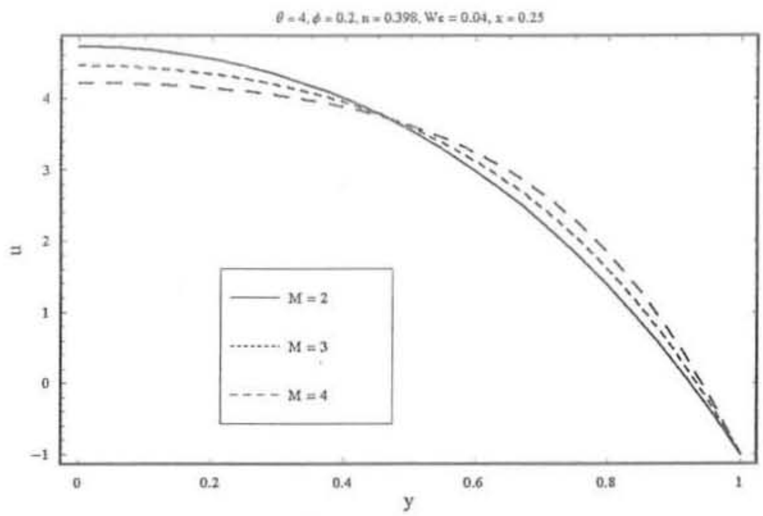

Figure 4.6c: Plot showing velocity u versus y in wider part of the channel for square wave.

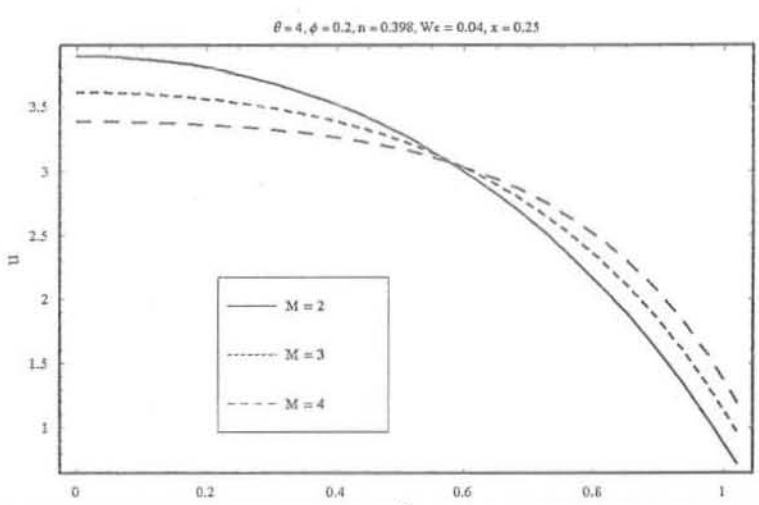

Figure 4.6d: Plot showing velocity u versus y in wider part of the channel for trapezoidal wave.

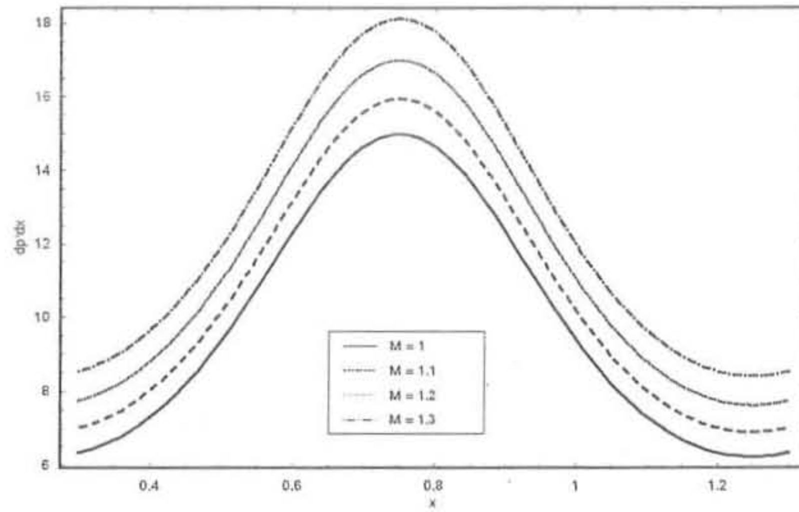

Figure 4.7a: Plot showing dp/dx versus x for sinusoidal wave. Here $\Phi = 0.2$, $\theta = -2$ and $We = 0.1$ $E = -0.4$, and $n = 0.398$.

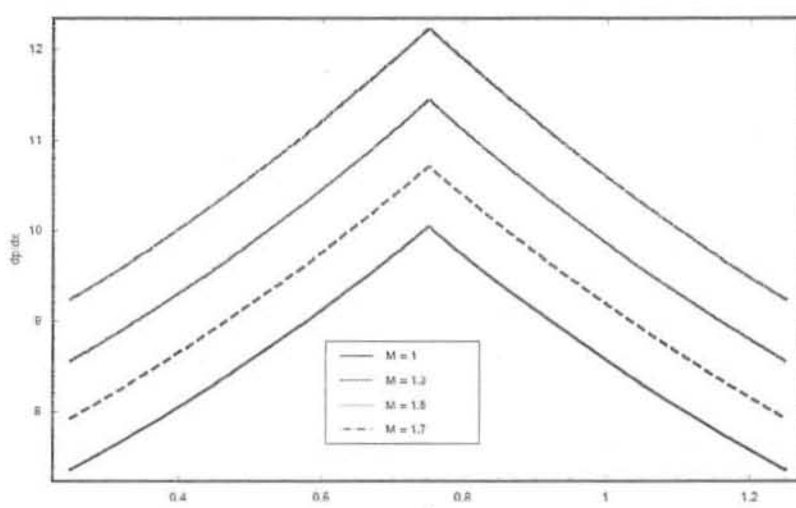

Figure 4.7b: Plot showing dp/dx versus x for triangular wave. Here $\Phi = 0.2$, $\theta = -2$ and $We = 0.1$ $E = -0.4$, and $n = 0.398$.

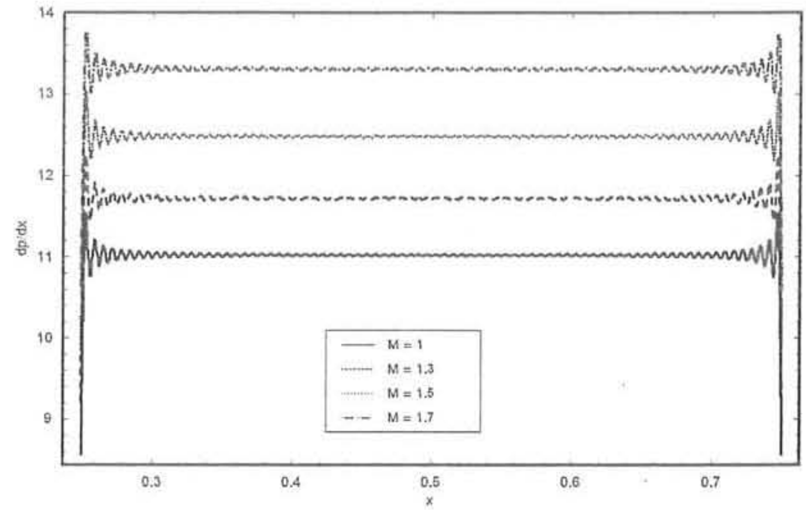

Figure 4.7c: Plot showing dp/dx versus x for square wave. Here $\Phi = 0.2$, $\theta = -2$ and $We = 0.1$ $E = -0.4$, and $n = 0.398$.

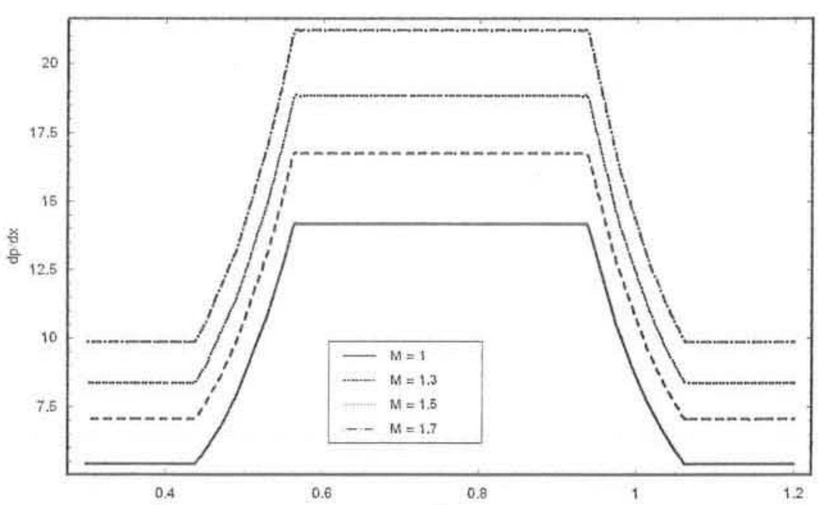

Figure 4.7d: Plot showing dp/dx versus x for trapezoidal wave. Here $\Phi = 0.2$, $\theta = -2$ and $We = 0.1$ $E = -0.4$, and $n = 0.398$.

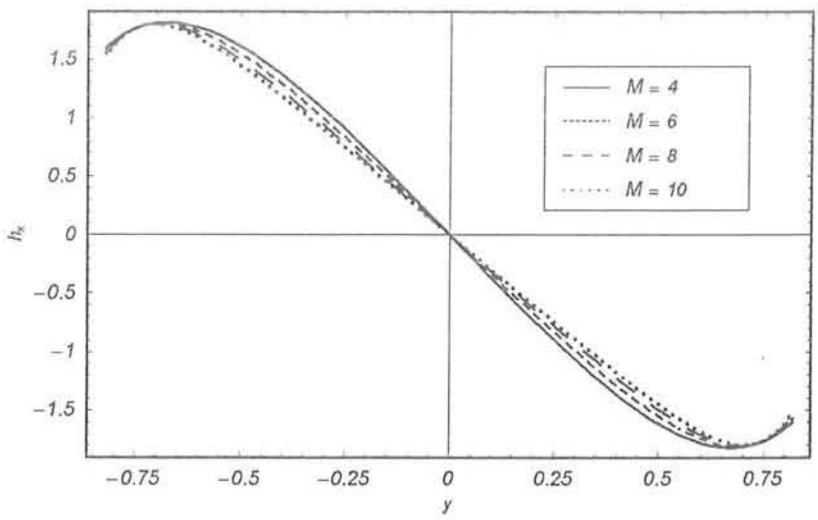

Figure 4.8a: Axial induced magnetic field versus y for sinusoidal wave. Here $E = 0.3$, $n = 0.398$, $\Phi = 0.6$, $R_m = 1$, $We = 0.04$, $x = \frac{\pi}{2}$ and $\theta = 3$.

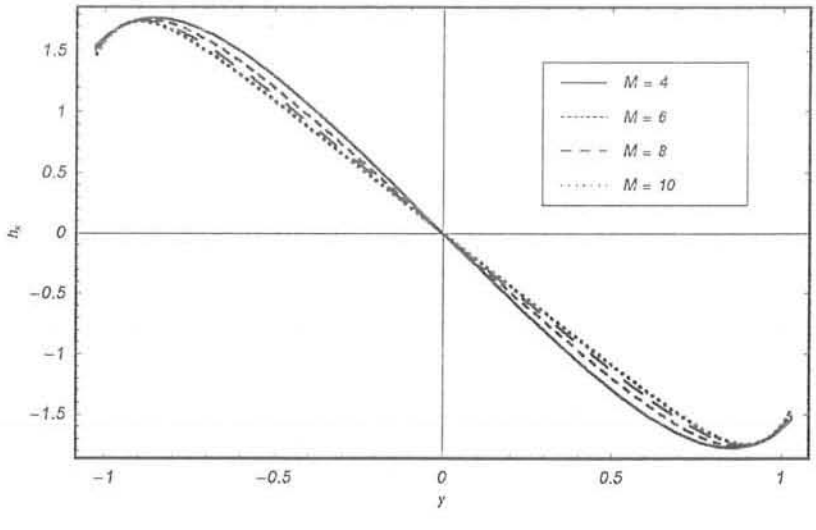

Figure 4.8b: Axial induced magnetic field versus y for triangular wave. Here $E = 0.3$, $n = 0.398$, $\Phi = 0.6$, $R_m = 1$, $We = 0.04$, $x = \frac{\pi}{2}$ and $\theta = 3$.

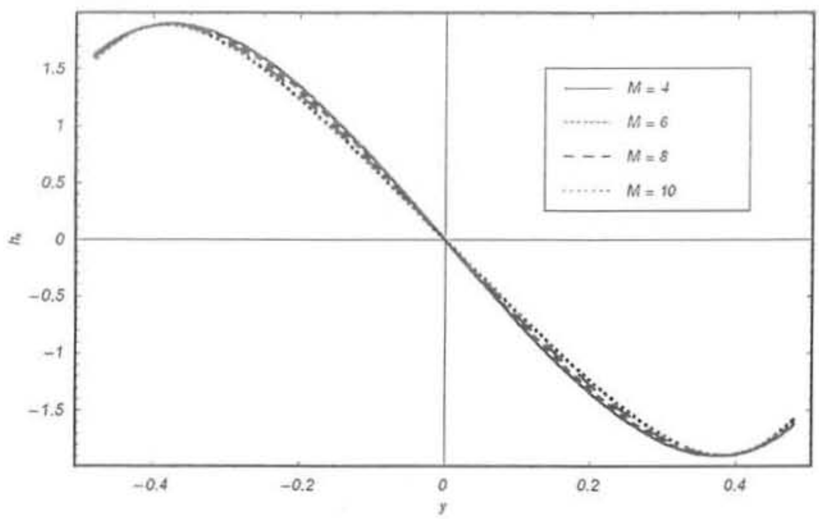

Figure 4.8c: Axial induced magnetic field versus y for square wave. Here $E = 0.3$, $n = 0.398$, $\Phi = 0.6$, $R_m = 1$, $We = 0.04$, $x = \frac{\pi}{2}$ and $\theta = 3$.

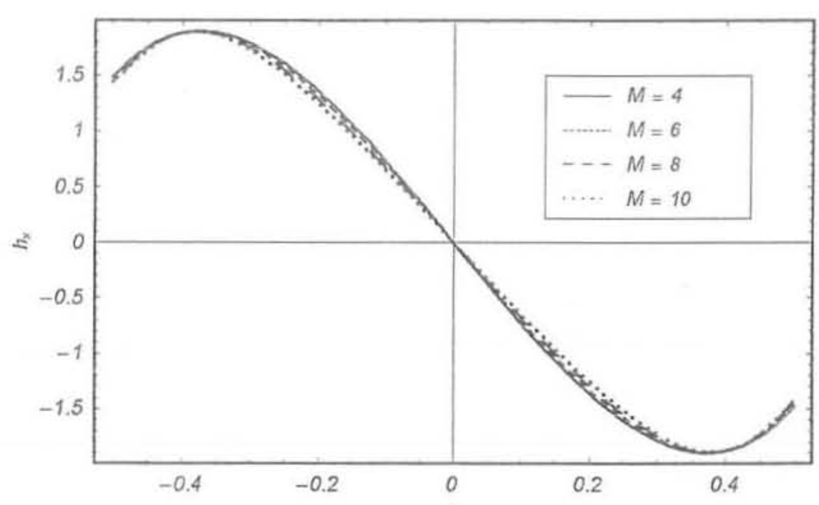

Figure 4.8d: Axial induced magnetic field versus y for trapezoidal wave. Here $E = 0.3$, $n = 0.398$, $\Phi = 0.6$, $R_m = 1$, $We = 0.04$, $x = \frac{\pi}{2}$ and $\theta = 3$.

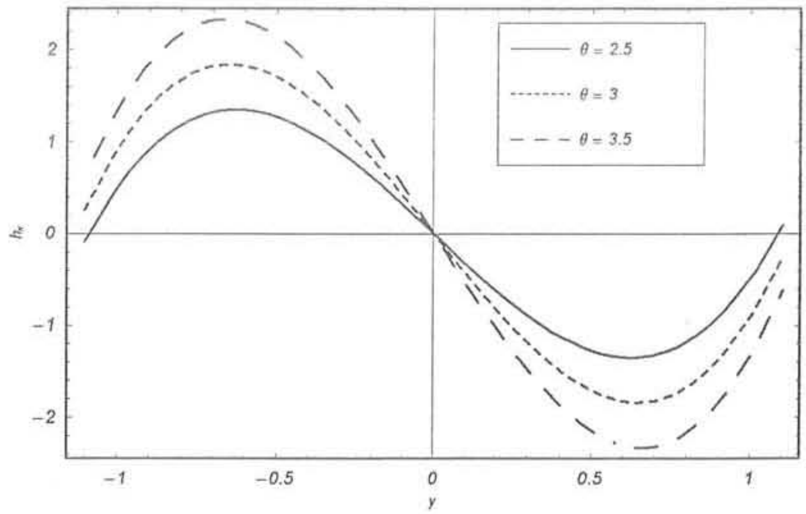

Figure 4.9a: Axial induced magnetic field versus y for sinusoidal wave. Here $E = 0.3$, $n = 0.398$, $\Phi = 0.6$, $R_m = 1$, $We = 0.04$ and $M = 1$.

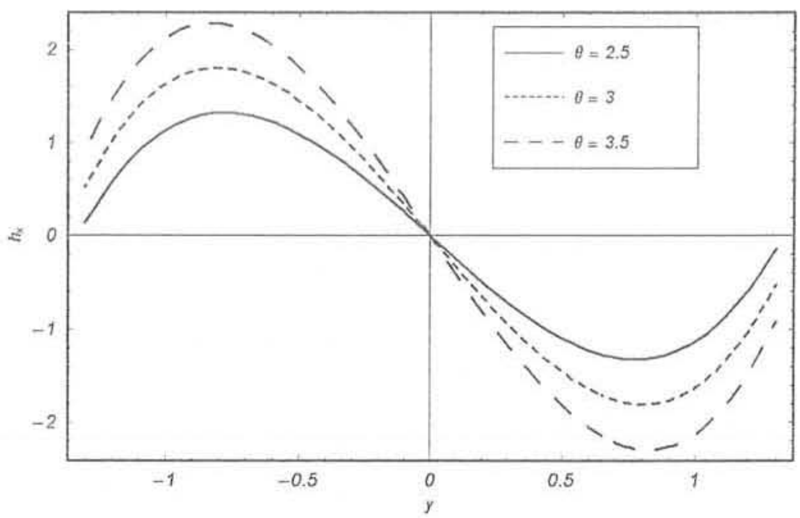

Figure 4.9b: Axial induced magnetic field versus y for triangular wave. Here $E = 0.3$, $n = 0.398$, $\Phi = 0.6$, $R_m = 1$, $We = 0.04$ and $M = 1$.

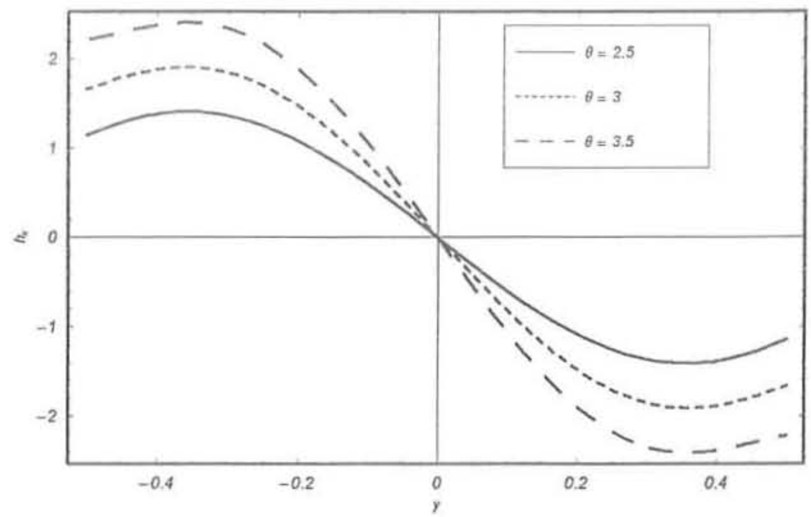

Figure 4.9c: Axial induced magnetic field versus y for square wave. Here $E = 0.3$, $n = 0.398$, $\Phi = 0.6$, $R_m = 1$, $We = 0.04$ and $M = 1$.

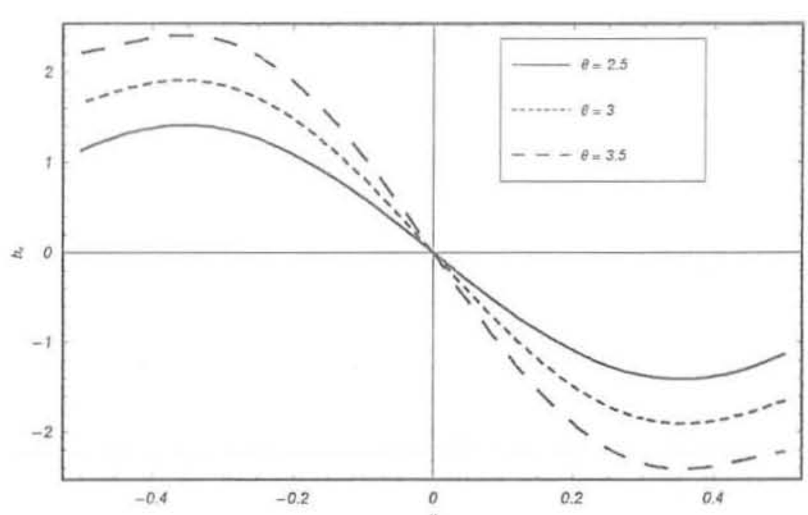

Figure 4.9d: Axial induced magnetic field versus y for trapezoidal wave. Here $E = 0.3$, $n = 0.398$, $\Phi = 0.6$, $R_m = 1$, $We = 0.04$ and $M = 1$.

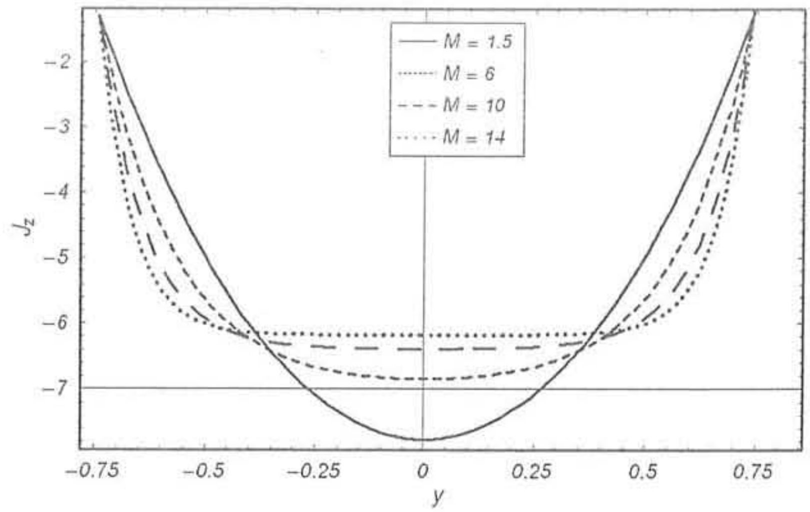

Figure 4.10a: Current density distribution versus y for sinusoidal wave. Here $E = 0.3$, $n = 0.398$, $\Phi = 0.6$, $R_m = 1$, $We = 0.04$, $x = \frac{\pi}{2}$ and $\theta = -3$.

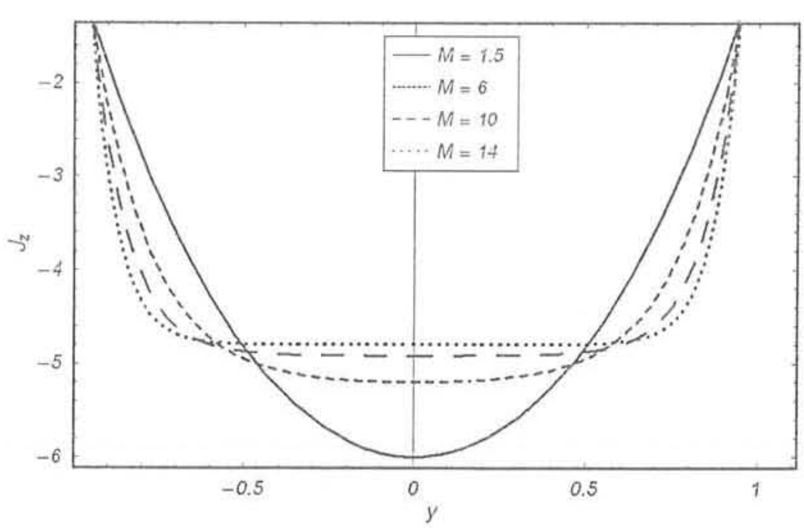

Figure 4.10b: Current density distribution versus y for triangular wave. Here $E = 0.3$, $n = 0.398$, $\Phi = 0.6$, $R_m = 1$, $We = 0.04$, $x = \frac{\pi}{2}$ and $\theta = -3$.

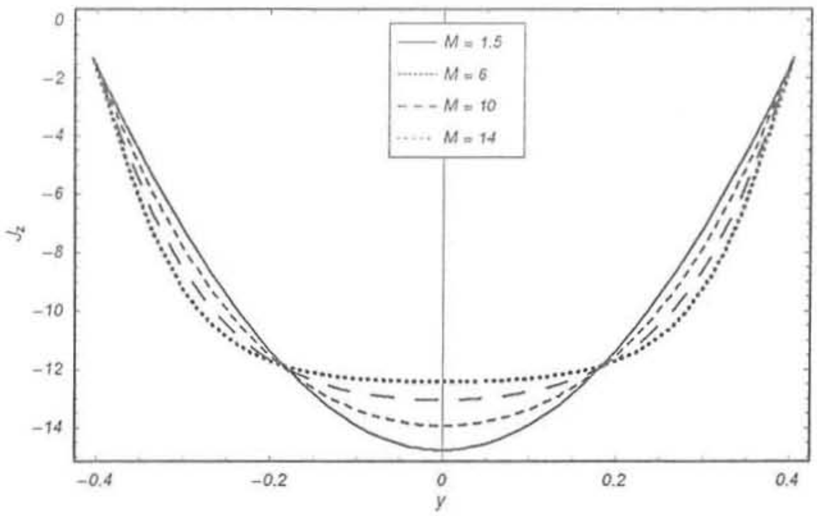

Figure 4.10c: Current density distribution versus y for square wave. Here $E = 0.3$, $n = 0.398$, $\Phi = 0.6$, $R_m = 1$, $We = 0.04$, $x = \frac{\pi}{2}$ and $\theta = -3$.

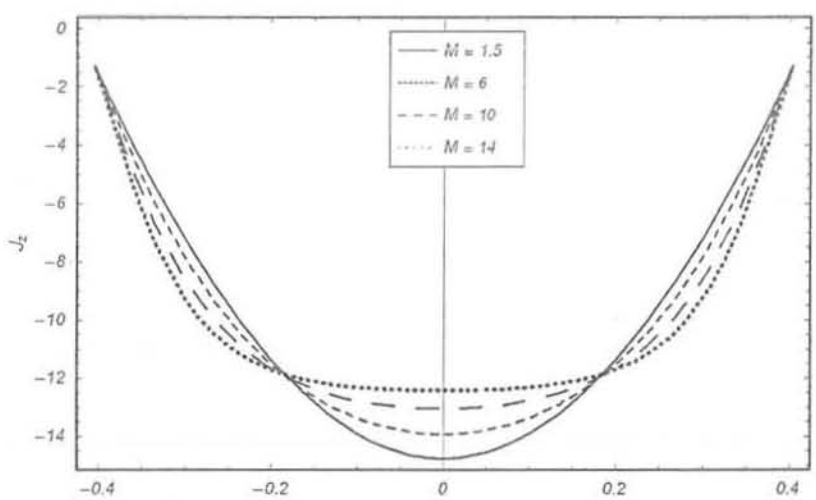

Figure 4.10d: Current density distribution versus y for trapezoidal wave. Here $E = 0.3$, $n = 0.398$, $\Phi = 0.6$, $R_m = 1$, $We = 0.04$, $x = \frac{\pi}{2}$ and $\theta = -3$.

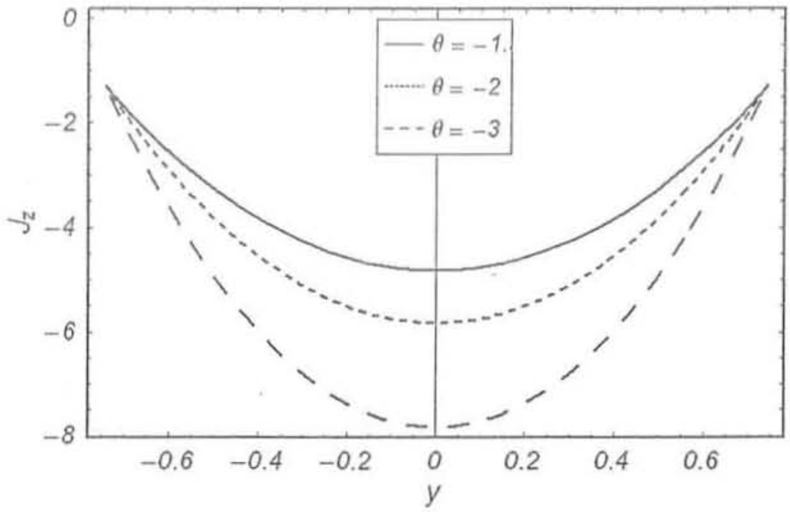

Figure 4.11a: Current density distribution versus y for sinusoidal wave. Here $E = 0.3$, $n = 0.398$, $\Phi = 0.6$, $R_m = 1$, $We = 0.04$, $x = \frac{\pi}{2}$ and $M = 1$.

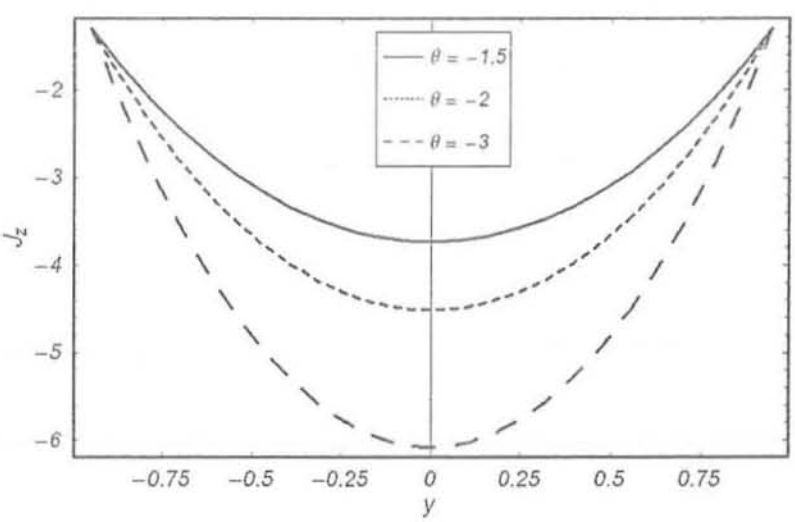

Figure 4.11b: Current density distribution versus y for triangular wave. Here $E = 0.3$, $n = 0.398$, $\Phi = 0.6$, $R_m = 1$, $We = 0.04$, $x = \frac{\pi}{2}$ and $M = 1$.

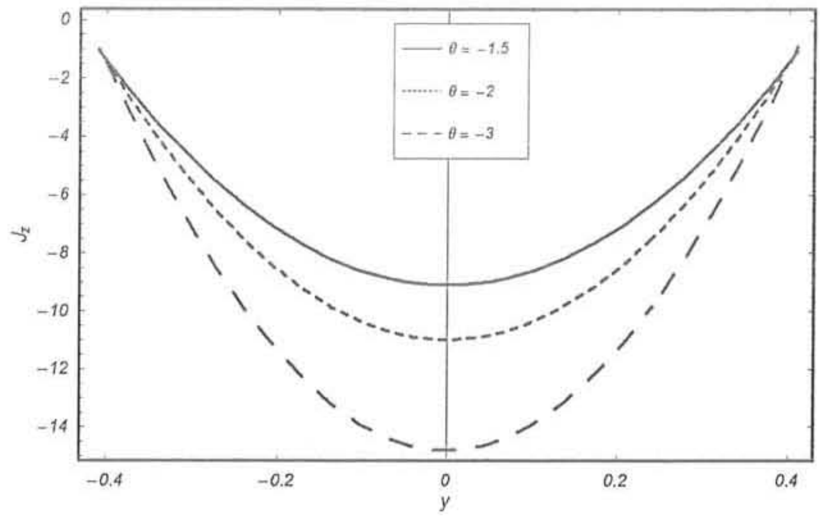

Figure 4.11c: Current density distribution versus y for square wave. Here $E = 0.3$, $n = 0.398$, $\Phi = 0.6$, $R_m = 1$, $We = 0.04$, $x = \frac{\pi}{2}$ and $M = 1$.

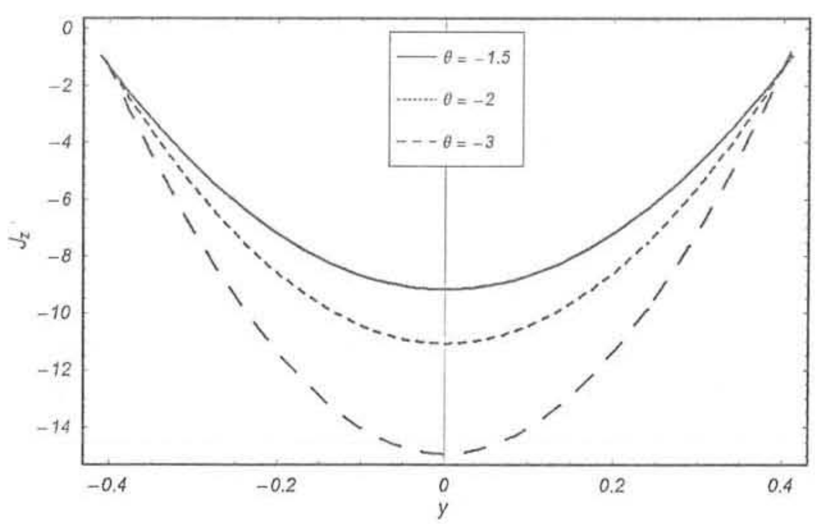

Figure 4.11d: Current density distribution versus y for trapezoidal wave. Here $E = 0.3$, $n = 0.398$, $\Phi = 0.6$, $R_m = 1$, $We = 0.04$, $x = \frac{\pi}{2}$ and $M = 1$.

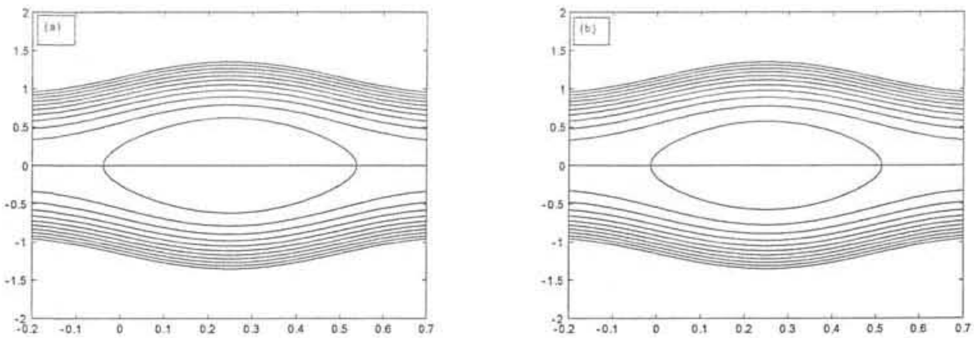

Figure 4.12: Streamlines (sinusoidal wave) for $M = 0.2$, (panel(a)), $M = 1.1$ (panel (b)). The other parameters are $\Phi = 0.2$, $n = 0.398$, $We = 0.4$ and $\theta = 0.67$.

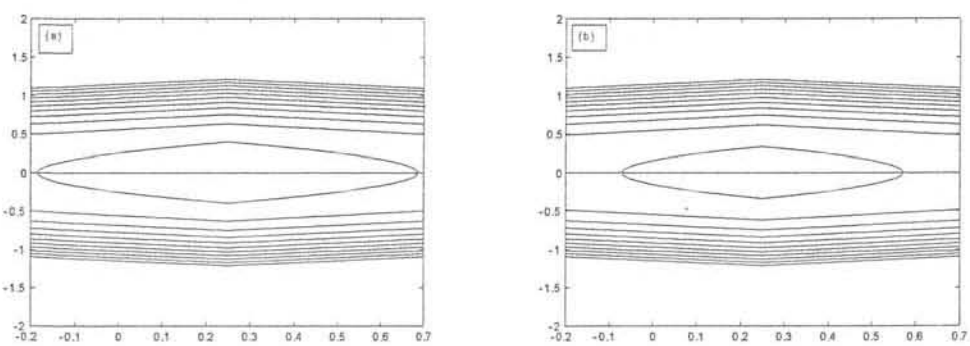

Figure 4.13: Streamlines (triangular wave) for $M = 0.2$, (panel(a)), $M = 1.1$ (panel (b)). The other parameters are $\Phi = 0.2$, $n = 0.398$, $We = 0.4$ and $\theta = 0.67$.

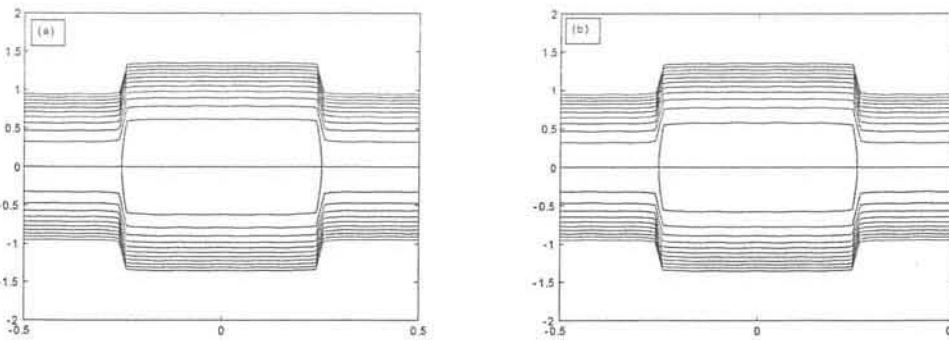

Figure 4.14: Streamlines (square wave) for $M = 0.2$, (panel(a)), $M = 1.1$ (panel (b)). The other parameters are $\Phi = 0.2$, $n = 0.398$, $We = 0.4$ and $\theta = 0.67$.

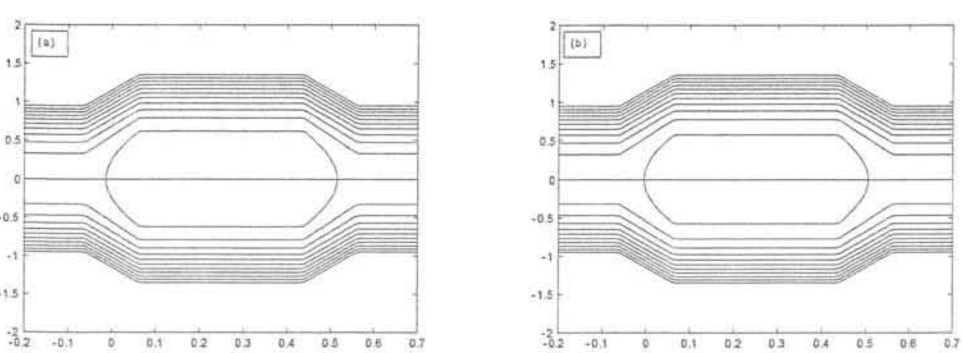

Figure 4.15: Streamlines (trapezoidal wave) for $M = 0.2$, (panel(a)), $M = 1.1$ (panel (b)). The other parameters are $\Phi = 0.4$, $n = 0.398$, $We = 0.4$ and $\theta = 0.67$.

4.3 Results and discussion

To discuss qualitatively the effects of various parameters of interest on flow quantities such as pressure rise per wavelength (Δp_λ), longitudinal velocity (u), pressure gradient (dp/dx), the axial induced magnetic field (h_x), the current density distribution (J_z) and stream function (Ψ), we have prepared Figures 4.1 – 4.15. The effects of M on Δp_λ for different wave forms have been illustrated in the Figures $4.1 - 4.4$.

From Figures $4.1 - 4.4$, we have concluded that the peristalsis has to work against a greater pressure rise for magnetohydrodynamic fluid as compared to hydrodynamic fluid. The peristalsis pumping rate and free pumping rate is enhanced for MHD fluid in comparison to hydrodynamic fluid. In augmented pumping region the pumping rate increases by increasing M. However, for suitably chosen ($\Delta p_\lambda < 0$) the pumping rate decreases for large values of M. A comparison of Figures $4.1 - 4.4$ reveals that the pumping rate of square wave is maximum (for a fixed $\Delta p_\lambda > 0$) and minimum for a triangular wave as is the pumping performance concerned.

Figures 4.5 and 4.6 lead to the following conclusions about the effects of M on longitudinal velocity u :

- The longitudinal velocity near the centre of the channel wall decreases by increasing M. An opposite behaviour is seen near the channel wall.
- A comparison of these figures further reveals that at the channel centre, the longitudinal velocity is maximum in the cases of sinusoidal and trapezoidal waves.
- The behaviour of velocity in the wider part of the channel is quite similar to that in the narrow part.

  Figures $4.7(a - d)$ have been sketched to see the variation of dp/dx versus x for different values of M. We can observe that:

- Magnitude of dp/dx for large values of M both in the wider and narrow part of the channel. Also this result is valid for all the considered wave forms.
- The increase in dp/dx for M is maximum for trapezoidal wave and minimum for triangular and square waves.

The conclusion regarding an axial induced magnetic field (h_x) with y over a cross section $x = \pi/2$ for different wave forms can be seen through Figures 4.8 and 4.9. These are:

- The shape of the curves of h_x is independent of the wave shape.

- Axial induced magnetic field (h_x) decreases by increasing M and θ. This decrease is maximum for the sinusoidal and triangular waves and minimum for square and trapezoidal waves.

- Axial induced magnetic field (h_x) is symmetric about the origin.

- Axial induced magnetic field in the half region ($y \geqslant 0$) is in one direction and it is in opposite direction in the other half region ($y \leq 0$).

- The axial induced magnetic field is zero at $y = 0$.

The profiles of the current density (J_z) over a cross section $x = \pi/2$ are shown in the Figures 4.10 and 4.11 for different values of M and θ. These plots provide the following information:

- The magnitude of the current density increases near the channel walls by increasing M. However, it decreases near and at the centre of the channel. This result holds for all wave forms.

- Contrary to the above result the magnitude of J_z increases by increasing θ for all the considered wave forms.

- The magnitude of the current density is maximum for square and trapezoidal waves and minimum for sinusoidal and triangular waves.

The phenomenon of trapping for different values of M is discussed through Figures 4.12 – 4.15. These figures show that:

- The volume of the trapped bolus decreases in going from hydrodynamic to magnetohydrodynamics (MHD) fluid.

- For large values of M, the bolus disappears.

4.4 Concluding remarks

A mathematical model under long wavelength and low Reynolds number approximations is presented to study the effects of an induced magnetic field on the peristaltic transport of Carreau fluid in a symmetric channel. Closed form expressions of stream function, longitudinal velocity, axial induced magnetic field and current density distribution are developed using perturbation method. A comparative study is made for different wave forms through graphs. It is concluded that the decrease in an induced magnetic field is maximum for sinusoidal and triangular waves and minimum for square and trapezoidal waves. However J_z is maximum for the square and trapezoidal waves and minimum for sinusoidal and triangular waves. We have noticed from these figures that the lower trapping limit for triangular wave is greater when compared with the other wave forms.

Chapter 5

Influence of induced magnetic field and heat transfer on peristaltic transport of a Carreau fluid

The problem of peristaltic flow of a non-Newtonian fluid in an asymmetric channel with heat transfer is investigated in this chapter. The non-Newtonian fluid is characterized by the constitutive equations of a Carreau fluid. The mathematical modelling is based upon the consideration of an induced magnetic field. The series solutions for velocity and temperature are established and their associated characteristics have been analyzed. Pumping and trapping mechanisms have been discussed in detail.

5.1 Development of the mathematical problem

Consider the MHD flow of an incompressible Carreau fluid in an asymmetric channel of width $d_1 + d_2$. The fluid is magnetohydrodynamic and the channel walls are insulating. A uniform magnetic field B_0 is applied in the $y-$direction. The induced magnetic field effects are retained. The sinusoidal waves traveling down its walls may be expressed as

$$
\begin{aligned}
Y &= h_1'(X,Y,t) = d_1 + a_1\cos\left[\frac{2\pi}{\lambda}(X-ct)\right]; \quad \text{upper wall},\\
Y &= h_2'(X,Y,t) = -d_2 - b_1\cos\left[\frac{2\pi}{\lambda}(X-ct)+\phi\right]; \quad \text{lower wall},
\end{aligned} \tag{5.1}
$$

with a_1, b_1 are the wave amplitudes, λ the wavelength and $\phi\,(0 \le \phi \le \pi)$ the phase difference. The symmetric channel case with waves out of phase is noticed for $\phi = 0$ and when $\phi = \pi$, it corresponds to the situation when waves are in phase. Further

$$
a_1^2 + b_1^2 + 2a_1 b_1 \cos\phi \le (d_1 + d_2)^2 .
$$

Thus, the equations governing the conservations of mass and momentum along with the induction equation are introduced by Eqs. (4.1)-(4.6). The conservation of energy is

$$
\rho C_p \frac{dT}{dt} = \kappa \nabla^2 T + \tau_1 . \mathrm{L}, \tag{5.2}
$$

where τ_1 is the Cauchy stress tensor, $\mathrm{L} = grad\mathrm{V}$, ρ the density, C_p the specific heat at a constant volume, κ the thermal conductivity and T the temperature.

The present flow in the laboratory frame need to be solved subject to the equations (4.7), (4.8) and the following equation

$$
\begin{aligned}
\rho C_p \left(\frac{\partial}{\partial t'} + U'\frac{\partial}{\partial X'} + V'\frac{\partial}{\partial Y'}\right) T &= \kappa \left(\frac{\partial^2}{\partial X'^2} + \frac{\partial^2}{\partial Y'^2}\right) T + \tau'_{X'X'}\left(\frac{\partial U'}{\partial X'}\right)\\
&\quad + \tau'_{X'Y'}\left(\frac{\partial V'}{\partial X'} + \frac{\partial U'}{\partial Y'}\right) + \tau'_{X'Y'}\left(\frac{\partial V'}{\partial Y'}\right).
\end{aligned} \tag{5.3}
$$

The above equation in the wave frame is given by

$$
\begin{aligned}
\rho C_p \left(u'\frac{\partial}{\partial x'} + v'\frac{\partial}{\partial y'}\right) T &= \kappa \left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2}\right) T + \tau'_{xx}\left(\frac{\partial u'}{\partial x'}\right) + \tau'_{xy}\left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'}\right)\\
&\quad + \tau'_{yy}\left(\frac{\partial v'}{\partial y'}\right).
\end{aligned} \tag{5.4}
$$

Introducing the velocity components by usual definitions

$$u = \frac{\partial \Psi}{\partial y}, \quad v = -\delta \frac{\partial \Psi}{\partial x}, \tag{5.5}$$

as well as transformations

$$d = \frac{d_2}{d_1}, \quad a = \frac{a_1}{d_1}, \quad b = \frac{a_2}{d_1}, \quad \theta' = \frac{T - T_0}{T_1 - T_0}, \quad \Pr = \frac{\rho \nu C_p}{\kappa}, \quad E = \frac{c^2}{C_p (T_1 - T_0)}, \tag{5.6}$$

the governing equations reduce to

$$\begin{aligned} \operatorname{Re}\delta \left\{ \left(\frac{\partial \Psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial}{\partial y} \right) \frac{\partial \Psi}{\partial y} \right\} &= -\frac{\partial p_m}{\partial x} + 2\delta \frac{\partial}{\partial x}\left\{ \left(1 + \frac{(n-1)}{2} We^2 \dot{\gamma}^2 \right) \frac{\partial^2 \Psi}{\partial x \partial y} \right\} \\ &+ \frac{\partial}{\partial y}\left\{ \left(1 + \frac{(n-1)}{2} We^2 \dot{\gamma}^2 \right) \left(\frac{\partial^2 \Psi}{\partial y^2} - \delta^2 \frac{\partial^2 \Psi}{\partial x^2} \right) \right\} \\ &+ \operatorname{Re}\delta S_t^2 \left(\frac{\partial \Phi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \Phi}{\partial x}\frac{\partial}{\partial y} \right) \frac{\partial \Phi}{\partial y} + \operatorname{Re} S_t^2 \frac{\partial^2 \Phi}{\partial y^2}, \end{aligned} \tag{5.7}$$

$$\begin{aligned} -\operatorname{Re}\delta^3 \left\{ \left(\frac{\partial \Psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial}{\partial y} \right) \frac{\partial \Psi}{\partial x} \right\} &= -\frac{\partial p_m}{\partial y} + \delta^2 \frac{\partial}{\partial x}\left\{ \begin{array}{c} \left(1 + \frac{(n-1)}{2} We^2 \dot{\gamma}^2 \right) \\ \left(\frac{\partial^2 \Psi}{\partial y^2} - \delta^2 \frac{\partial^2 \Psi}{\partial x^2} \right) \end{array} \right\} \\ &- 2\delta^2 \frac{\partial}{\partial x}\left\{ \left(1 + \frac{(n-1)}{2} We^2 \dot{\gamma}^2 \right) \frac{\partial^2 \Psi}{\partial x \partial y} \right\} \\ &- \operatorname{Re}\delta^3 S_t^2 \left(\frac{\partial \Phi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \Phi}{\partial x}\frac{\partial}{\partial y} \right) \frac{\partial \Phi}{\partial x} \\ &- \operatorname{Re}\delta^2 S_t^2 \frac{\partial^2 \Phi}{\partial x \partial y}, \end{aligned} \tag{5.8}$$

$$\frac{\partial \Psi}{\partial y} - \delta \left(\frac{\partial \Psi}{\partial y}\frac{\partial \Phi}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial \Phi}{\partial y} \right) + \frac{1}{R_m} \nabla^2 \Phi = E, \tag{5.9}$$

$$\frac{\partial^2 \theta}{\partial y^2} + Br \left\{ 1 + \frac{(n-1)}{2} We^2 \dot{\gamma}^2 \right\} \left(\frac{\partial^2 \Psi}{\partial y^2} - \delta^2 \frac{\partial^2 \Psi}{\partial x^2} \right) \frac{\partial^2 \Psi}{\partial y^2} = 0 \tag{5.10}$$

$$\dot{\gamma} = \left[2\delta^2 \left(\frac{\partial^2 \Psi}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 \Psi}{\partial y^2} - \delta^2 \frac{\partial^2 \Psi}{\partial x^2} \right)^2 + 2\delta^2 \left(\frac{\partial^2 \Psi}{\partial x \partial y} \right)^2 \right]^{\frac{1}{2}}, \tag{5.11}$$

where incompressibility condition is automatically satisfied and

$$u=\frac{\partial\Psi}{\partial y},\quad v=-\delta\frac{\partial\Psi}{\partial x},\quad h_x=\frac{\partial\Phi}{\partial y},\quad h_y=-\delta\frac{\partial\Phi}{\partial x}$$
$$p_m=p+\frac{1}{2}\operatorname{Re}\delta\frac{\mu_e\left(H'^{+}\right)^2}{\rho c^2},\quad Br=E\Pr.$$

$$\nabla^2=\delta^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},$$

in which δ is a wave number, We the Wessinberg number, S_t the Strommer's number, Re the Reynolds number, R_m the magnetic Reynolds number, p_m the magnetic pressure, M the Hartman number, Pr the Prandtl number, E the Eckert number and Br the Brinkman number.

Due to long wavelength approximation, we have

$$\frac{\partial p}{\partial x}=\frac{\partial}{\partial y}\left[\left\{1+\frac{(n-1)}{2}We^2\left(\frac{\partial^2\Psi}{\partial y^2}\right)^2\right\}\frac{\partial^2\Psi}{\partial y^2}\right]+M^2\left(\epsilon-\frac{\partial\Psi}{\partial y}\right), \tag{5.12}$$

$$\frac{\partial p}{\partial y}=0, \tag{5.13}$$

$$\frac{\partial\Psi}{\partial y}+\frac{1}{R_m}\frac{\partial^2\Phi}{\partial y^2}=\epsilon, \tag{5.14}$$

$$\frac{\partial^2\theta}{\partial y^2}+Br\left[\left\{1+\frac{(n-1)}{2}We^2\left(\frac{\partial^2\Psi}{\partial y^2}\right)^2\right\}\left(\frac{\partial^2\Psi}{\partial y^2}\right)^2\right]=0, \tag{5.15}$$

where Eq. (5.13) shows that $p\neq p(y)$ and hence $p=p(x)$.

Cross differentiation of Eqs. (5.12) and (5.13) yield

$$\frac{\partial^2}{\partial y^2}\left[\left\{1+\frac{(n-1)}{2}We^2\left(\frac{\partial^2\Psi}{\partial y^2}\right)^2\right\}\frac{\partial^2\Psi}{\partial y^2}\right]-M^2\frac{\partial^2\Psi}{\partial y^2}=0. \tag{5.16}$$

The governing equations here need to be solved subject to the following boundary conditions on the stream function, the magnetic force function and the temperature

$$\begin{aligned}&\Psi=\frac{F}{2},\quad \frac{\partial\Psi}{\partial y}=-1,\quad \Phi=0,\quad \theta'=0\ \text{ at } y=h_1=1+a\cos 2\pi x,\\ &\Psi=-\frac{F}{2},\quad \frac{\partial\Psi}{\partial y}=-1,\quad \Phi=0,\quad \theta'=-1\ \text{ at } y=h_2=-d-b\cos\left(2\pi x+\phi\right),\end{aligned} \tag{5.17}$$

where the dimensionless mean flows in laboratory (θ) and wave (F) frames are related by the following expressions

$$\theta = F + d + 1, \tag{5.18}$$

$$F = \int_{h_2(x)}^{h_1(x)} \frac{\partial \Psi}{\partial y} dy = \Psi\left(h_1(x)\right) - \Psi\left(h_2(x)\right) \tag{5.19}$$

and the dimensionless pressure rise per wavelength is

$$\Delta p_\lambda = \int_0^{2\pi} \left(\frac{dp}{dx}\right)\bigg|_{y=0} dx. \tag{5.20}$$

5.2 Perturbation solution

For small Weissenberg number

$$\Psi = \Psi_0 + We^2\Psi_1 + O\left(We^4\right), \tag{5.21}$$

$$\Phi = \Phi_0 + We^2\Phi_1 + O\left(We^4\right), \tag{5.22}$$

$$F = F_0 + We^2 F_1 + O\left(We^4\right), \tag{5.23}$$

$$p = p_0 + We^2 p_1 + O\left(We^4\right). \tag{5.24}$$

Upon making use of above equations into Eqs. (5.12) and (5.14)−(5.17) and then equating the coefficients of like powers of We^2, the following systems can be determined:

5.2.1 O(We^0) System

$$\frac{\partial p_0}{\partial x} = \frac{\partial^3 \Psi_0}{\partial y^3} + M^2\left(E - \frac{\partial \Psi_0}{\partial y}\right),$$

$$\frac{\partial^4 \Psi_0}{\partial y^4} - M^2 \frac{\partial^2 \Psi_0}{\partial y^2} = 0,$$

$$\frac{\partial^2 \Phi_0}{\partial y^2} = R_m\left(E - \frac{\partial \Psi_0}{\partial y}\right),$$

$$\frac{\partial^2\theta_0}{\partial y^2}+Br\left\{\left(\frac{\partial^2\Psi_0}{\partial y^2}\right)^2\right\}=0,$$

$$\begin{aligned}\Psi_0 &= \frac{F_0}{2},\;\; \frac{\partial\Psi_0}{\partial y}=-1,\;\; \Phi_0=0,\;\; \theta_0^{'}=0 \;\;\text{at}\;\; y=h_1,\\ \Psi_0 &= -\frac{F_0}{2},\;\; \frac{\partial\Psi_0}{\partial y}=-1,\;\; \Phi_0=0,\;\; \theta_0^{'}=-1 \;\;\text{at}\;\; y=h_2.\end{aligned}\qquad (5.25)$$

5.2.2 O(We^2) System

$$\frac{\partial p_1}{\partial x}=\frac{\partial^3\Psi_1}{\partial y^3}-M^2\frac{\partial\Psi_1}{\partial y}+\left(\frac{n-1}{2}\right)\frac{\partial}{\partial y}\left\{\left(\frac{\partial^2\Psi_0}{\partial y^2}\right)^3\right\},$$

$$\frac{\partial^4\Psi_1}{\partial y^4}-M^2\frac{\partial^2\Psi_1}{\partial y^2}+\left(\frac{n-1}{2}\right)\frac{\partial^2}{\partial y^2}\left\{\left(\frac{\partial^2\Psi_0}{\partial y^2}\right)^3\right\}=0,$$

$$\frac{\partial^2\Phi_1}{\partial y^2}=-R_m\left(\frac{\partial\Psi_1}{\partial y}\right),$$

$$\frac{\partial^2\theta_1}{\partial y^2}+Br\left\{\left(\frac{\partial^2\Psi_1}{\partial y^2}\right)^2+\frac{n-1}{2}\left(\frac{\partial^2\Psi_0}{\partial y^2}\right)^4\right\}=0,$$

$$\begin{aligned}\Psi_1 &= \frac{F_1}{2},\;\; \frac{\partial\Psi_1}{\partial y}=0,\;\; \Phi_1=0,\;\; \theta_1^{'}=0 \;\;\text{at}\;\; y=h_1,\\ \Psi_1 &= -\frac{F_1}{2},\;\; \frac{\partial\Psi_1}{\partial y}=0,\;\; \Phi_1=0,\;\; \theta_1^{'}=0 \;\;\text{at}\;\; y=h_2.\end{aligned}\qquad (5.26)$$

The above systems admit the following expressions

$$\begin{aligned}\Psi \;=\; & \frac{1}{128L_1^3}(M^4(-1+n)We^2(h_1-h_2)^3(2(-\cosh(M(3y-h_1-2h_2))+\cosh(M(y+h_1-2h_2))\\ &+\cosh(M(3y-2h_1-h_2))-\cosh(M(y-2h_1+h_2))+2My(-6\sinh(M(y-h_1))+6\sinh(M\\ &(y-h_2))-8\sinh(M(h_1-h_2))+\sinh(2M(h_1-h_2))))+M(12M(-1+\cosh(M(y+h_2)))h_1^2\\ &-(24My-12My(\cosh(M(y+h_1))+\cosh(M(y+h_2)))-12\sinh(M(y-h_1))+\sinh(M\\ &(3y-h_1-2h_2))-3\sinh(M(y+h_1-2h_2))+12\sinh(M(y-h_2))+\sinh(M(3y-2h_1-h_2))\\ &-16\sinh(M(h_1-h_2))+2\sinh(2M(h_1-h_2))-3\sinh(M(y-2h_1+h_2))+12M(-1+\cosh(M\end{aligned}$$

$(y+h_1))h_2)+h_1(24My-12My(\cosh(M(y+h_1))+\cosh(M(y+h_2)))+12\sinh(M(y-h_1))$
$+\sinh(M(3y-h_1-2h_2))-3\sinh(M(y+h_1-2h_2))-12\sinh(M(y-h_2))+\sinh(M(3y-2h_1$
$-h_2))+12\sinh(M(h_1-h_2))-2\sinh(2M(h_1-h_2))-3\sinh(M(y-2h_1-h_2))+12M(\cosh(M$
$(y-h_1))-\cosh(M(y+h_2))h_2))))+\frac{1}{L_3L_4}(2\cosh(M(y+h_1))(\cosh(My)-\sinh(My))\sinh(\frac{M}{2}$
$(y-h_2))(\cosh(\frac{M}{2}(2y+h_1+h_2))+\sinh(\frac{M}{2}(2y+h_1+h_2)))+\frac{h_1F}{4}(\cosh(My)-\sinh(My))(2$
$(\cosh(2My)-My\cosh(M(y+h_1))-My\cosh(M(y+h_2))-\cosh(M(h_1+h_2))+\sinh(2My)$
$-My\sinh(M(y+h_1))-My\sinh(M(y+h_2))-\sinh(M(h_1+h_2))+M(\cosh(M(y+h_1))$
$+\sinh(M(y+h_1))+\sinh(M(y+h_2)))h_1+\sinh(M(y+h_2))h_2)+(-\cosh(My)+\sinh(My))(y$
$(\cosh(M(y+h_1))-\cosh(M(y+h_2))+\sinh(M(y+h_1))-\sinh(M(y+h_2))+2\cosh(\frac{M}{2}$
$(y-h_2))\sinh(\frac{M}{2}(y-h_1))(\cosh(\frac{M}{2}(2y+h_1+h_2))+\sinh(\frac{M}{2}(2y+h_1+h_2)))h_2)),\qquad(5.27)$

$$\begin{aligned}\frac{dp}{dx} &= (\frac{M^2}{L_5}(-2(1+\epsilon)\sinh(\frac{M}{2}(h_1+h_2))-\cosh(\frac{M}{2}(h_1+h_2))(F-\epsilon h_1+\epsilon h_2)))+\frac{1}{128L_1^3}(We^2\\
&(192M^7(-1+n)\cosh(\frac{M}{2}(h_1+h_2))(\sinh(\frac{M}{2}(h_1+h_2)))^2(F+h_1-h_2)^3+\frac{1}{L_1}(M^7(-1+n)\\
&(h_1-h_2)^3(-24\sinh(Mh_1)+32\sinh(M(h_1-h_2))-4\sinh(2M(h_1-h_2))+24\sinh(Mh_2)\\
&-2\sinh(M(h_1-2h_2))-2\sinh(M(2h_1-h_2))+6\sinh(M(2h_1+h_2))-6\sinh(M(h_1+2h_2\\
&))-3M(-4M\sinh(Mh_2)h_1^2+h_2(-8+8\cosh(Mh_1)+\cosh(M(h_1-2h_2))+\cosh(M(2h_1-\\
&h_2))-\cosh(M(2h_1+h_2))-\cosh(M(h_1+2h_2))+4M\sinh(Mh_1)h_2)+h_1(8-8\cosh(Mh_2)\\
&-\cosh(M(h_1-2h_2))-\cosh(M(2h_1-h_2))+\cosh(M(2h_1+h_2))+\cosh(M(h_1+2h_2))-\\
&4M\sinh(Mh_1)h_2+4M\sinh(Mh_2)h_2))))+\frac{1}{L_1}(M^7(-1+n)(h_1-h_2)^3(-72\sinh(Mh_1)-72\\
&\sinh(Mh_2)+2\sinh(M(h_1-2h_2))+2\sinh(M(2h_1-h_2))-54\sinh(M(2h_1+h_2))+54\sinh\\
&(M(h_1+2h_2))+3M(-4M\sinh(Mh_2)h_1^2+h_2(16\cosh(Mh_1)+8\cosh(Mh_2)+\cosh(M(h_1-\end{aligned}$$

$$2h_2)) + \cosh(M(2h_1 - h_2)) - 9\cosh(M(2h_1 + h_2)) - 9\cosh(M(h_1 + 2h_2)) + 4M$$
$$\sinh(Mh_1)h_2) + h_1(-8\cosh(Mh_1) - 16\cosh(Mh_2) - \cosh(M(h_1 - 2h_2)) - \cosh(M$$
$$(2h_1 - h_2)) + 9\cosh(M(2h_1 + h_2)) + 9\cosh(M(h_1 + 2h_2)) - 4\sinh(Mh_1)h_2 + 4M$$
$$\sinh(Mh_2)h_2)))))). \tag{5.28}$$

The results of magnetic force function Φ and an axial induced magnetic field h_x are

$$\Phi = \frac{1}{384L_1^4M^2}((64L_1^3M(F(6 + My(12 + My(3 + 2My))) \cosh(\frac{M}{2}(h_1 - h_2)) - 6F(\cosh(\frac{M}{2}(2y - h_1$$
$$- h_2)) + 2\sinh(\frac{M}{2}(2y - h_1 - h_2))) + 6(My^2 + F(-2 + My)) \sinh(\frac{M}{2}(h_1 - h_2)) + 64L_1^3M(6$$
$$\cosh(\frac{M}{2}(2y - h_1 - h_2))h_2 - 6M(F + y)\sinh(\frac{M}{2}(h_1 - h_2))h_2 + \cosh(\frac{M}{2}(h_1 - h_2))h_2(-3(2$$
$$+ FM(4 + M^2y^2)) + FM^2h_2(-3 + Mh_2)) + 3M\epsilon(y^2 - h_2^2)L_1) + (2(36My\cosh(M(y - h_1))$$
$$- 36My\cosh(M(y - h_2)) - 36\sinh(M(y - h_1)) + \sinh(M(3y - h_1 - 2h_2)) - \sinh(M(y + h_1$$
$$- 2h_2)) + 36\sinh(M(y - h_2)) - \sinh(M(3y - 2h_1 - h_2)) + 36\sinh(M(y - h_2)) - \sinh(M(3y$$
$$- 2h_1 - h_2)) + 8(-4 + 3M^2y^2)\sinh(M(h_1 - h_2)) + (2 - 3M^2y^2)\sinh(2M(h_1 - h_2)) + 3\sinh(M$$
$$(y - 2h_1 + h_2))) + Mh_2(\cosh(M(3y - h_1 - 2h_2)) - 9\cosh(M(y + h_1 - 2h_2)) + 72\cosh(M$$
$$(y - h_2)) - 64\cosh(M(h_1 - h_2)) + \cosh(2M(h_1 - h_2)) - 9\cosh(M(y - 2h_1 + h_2)) + 6My$$
$$(-6(-My + \sinh(M(y - h_1)) + \sinh(M(y - h_2))) - 8\sinh(M(h_1 - h_2))(-2 - My(1 + My)$$
$$+ Mh_2(1 + Mh_2)) + 2M\epsilon(-y + h_2)L_1) - (72\cosh(M(y - h_1)) + \cosh(M(3y - h_1 - 2h_2))$$
$$- 9\cosh(M(y + h_1 - 2h_2)) + \cosh(M(3y - 2h_1 - h_2)) - 64\cosh(M(h_1 - h_2)) + 6\cosh(2M$$
$$(h_1 - h_2)) + 6Mh_2(6\sinh(M(y - h_1)) - 6\sinh(M(y - h_2)) - 8\sinh(M(h_1 - h_2)) - 6Mh_2))$$
$$L_2) + 12M^2h_1^2(-3\sinh(M(y - h_2))L_2 + M(y - h_2)(64L_1^3\sinh(\frac{M}{2}(h_1 - h_2)) + 3L_2)))R_m), \tag{5.29}$$

$$
\begin{aligned}
h_x &= \frac{1}{128L_1^4M}((128L_1^3M(F(2+My(1+My))\cosh(\frac{M}{2}(h_1-h_2))-F(2\cosh(\frac{M}{2}(2y-h_1-h_2))\\
&-h_2))h_1-\sinh(\frac{M}{2}(2y-h_1-h_2))h_1+\sinh(\frac{M}{2}(2y-h_1-h_2))h_2-\sinh(\frac{M}{2}(h_1-h_2))(h_1\\
&(1+2My-2Mh_1)+h_2)-\cosh(\frac{M}{2}(h_1-h_2))((-2+FM(1+My))h_1+FM^2(y-h_1)h_2)\\
&+y\epsilon L_1-h_1\epsilon L_1)+2(\cosh(M(3y-h_1-2h_2))-\cosh(M(y+h_1-2h_2))-\cosh(M(3y\\
&-2h_1-h_2))+\cosh(M(y-2h_1+h_2))L_2+M(-4y(-6\sinh(M(y-h_1))+\sinh(M(y\\
&-h_2))-8\sinh(M(h_1-h_2))+\sinh(2M(h_1-h_2)))-h_1(24My-12My(\cosh(M(y-h_1))\\
&+\cosh(M(y-h_2)))+12\sinh(M(y-h_1))+\sinh(M(3y-h_1-2h_2))-3\sinh(M(y+h_1\\
&-2h_2))-3\sinh(M(y+h_1-2h_2))+12\sinh(M(y-h_2))+\sinh(M(3y-2h_1-h_2))-16\\
&\sinh(M(h_1-h_2))+2\sinh(2M(h_1-h_2))-3\sinh(M(y-2h_1-h_2))+12M(-\cosh(M\\
&(y-h_2))h_1)h_2+12M(-1+\cosh(M(y-h_1)))h_2^2)L_2)R_m), \qquad (5.30)
\end{aligned}
$$

and temperature is

$$
\begin{aligned}
\theta' &= (L_8-(\frac{1}{262144L_1M^2L_6^4}(((BrWe^2(16384F^4L_1^6L_2^2M^8L_9-65536F^3F_1^6L_2^2M^8L_{10}+9830F^2L_1^6L_2^2\\
&M^8L_{11}-65536FL_1^6L_2^2M^8L_{12}+16384L_1^6L_2^2M^8L_{13}-393216L_1^6L_2^2M^{10}L_{14}+(4L_{15}+Mh_2(4\\
&L_{16}-L_{17})+L_{18}+4(L_{19}+L_{20})))L_6^4L_7^2)-32M^4h_1^4(512L_1^6L_2^2M^4L_{21}+3L_{22}L_6^4L_7^2)+4M^3h_1^3\\
&(16384L_1^6L_2^2M^5L_{23}L_{24}+L_{25}+L_{26}-L_{27})L_6^4L_7^2-3M^2h_1^2(32768L_1^6L_2^2M^6L_{28}L_{29})+L_{30}+4\\
&Mh_2L_{31})L_6^4L_7^2-2Mh_1(327268F^3L_1^6L_2^2L_{32}+98304F^2L_1^6L_2^2L_{32}L_{33}+98304FL_1^6L_2^2L_{32}L_{34}\\
&+32768L_1^6L_2^2L_{32}L_{35}+19660L_1^6L_2^2L_{36}-L_{37}+L_{38}-L_{39}+L_{40}+6Mh_2L_{41}))), \qquad (5.31)
\end{aligned}
$$

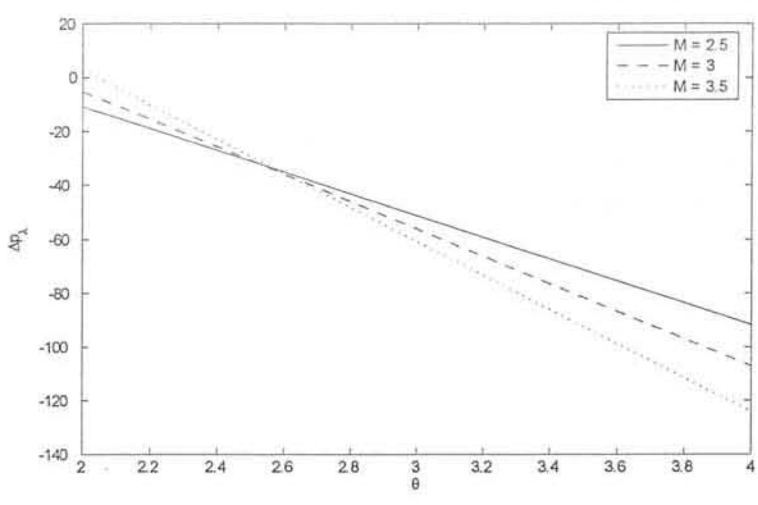

Figure 5.1: Plot showing Δp_λ versus flow rate θ. Here $n = 0.398$, $We = 0.01$, $a = 0.6$, $\epsilon = 0.4$, $\phi = \frac{\pi}{6}$, $b = 0.4$ and $d = 1.1$.

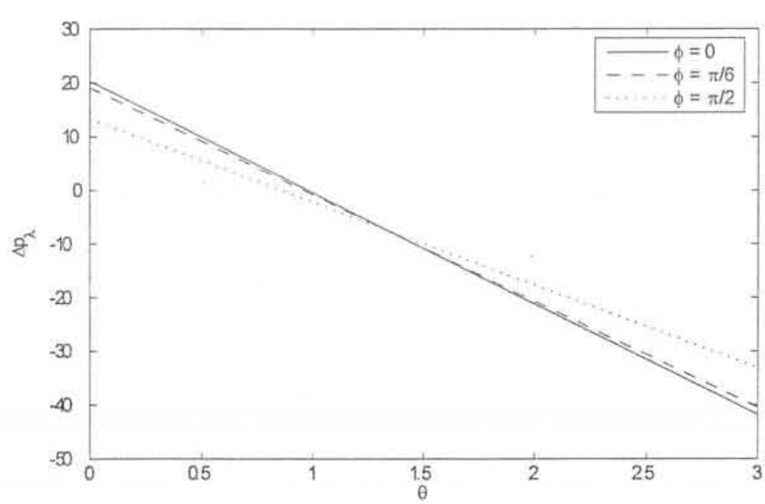

Figure 5.2: Plot showing Δp_λ versus flow rate θ. Here $n = 0.398$, $We = 0.01$, $M = 1$, $\epsilon = 0.4$, $a = 0.6$, $b = 0.4$ and $d = 1.1$.

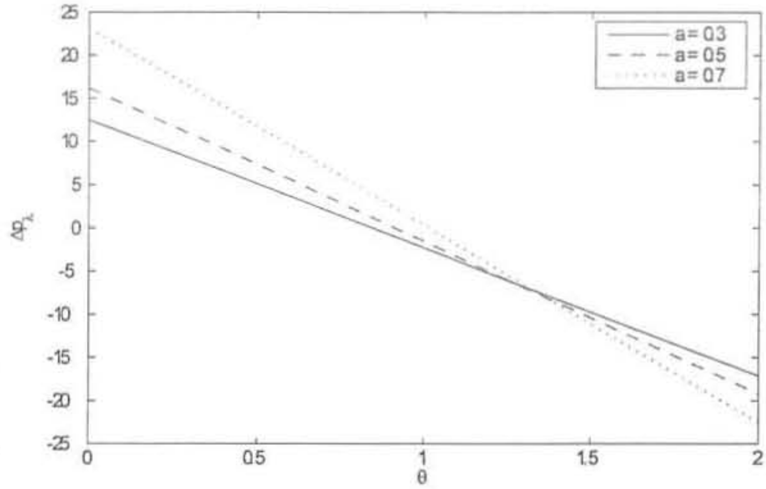

Figure 5.3: Plot showing Δp_λ versus flow rate θ. Here $n = 0.398$, $We = 0.01$, $M = 1$, $\epsilon = 0.4$, $\phi = \frac{\pi}{6}$, $b = 0.4$ and $d = 1.1$.

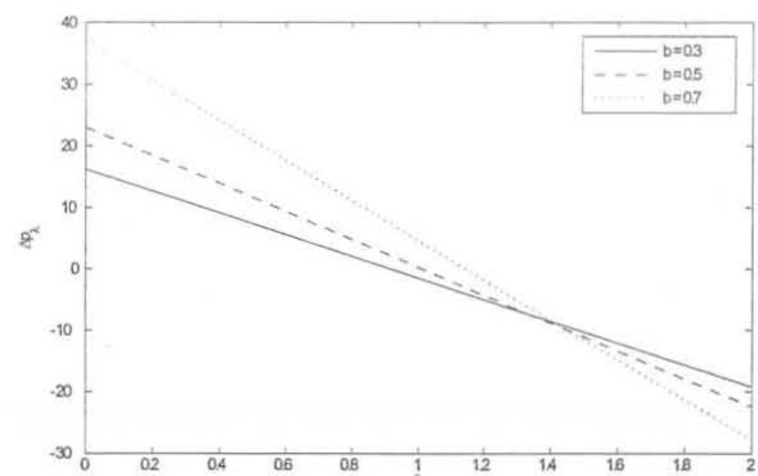

Figure 5.4: Plot showing Δp_λ versus flow rate θ. Here $n = 0.398$, $We = 0.01$, $M = 1$, $\epsilon = 0.4$, $\phi = \frac{\pi}{6}$, $a = 0.6$ and $d = 1.1$.

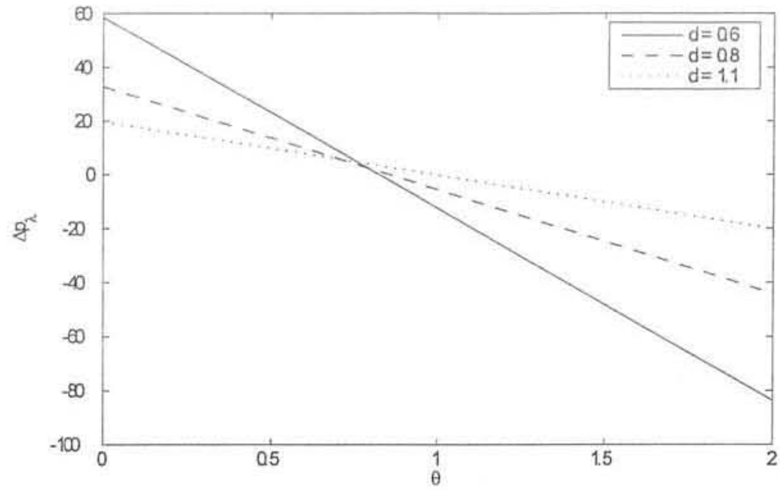

Figure 5.5: Plot showing Δp_λ versus flow rate θ. Here $n = 0.398$, $We = 0.01$, $M = 1$, $\epsilon = 0.4$, $\phi = \frac{\pi}{6}$, $a = 0.6$ and $b = 0.4$.

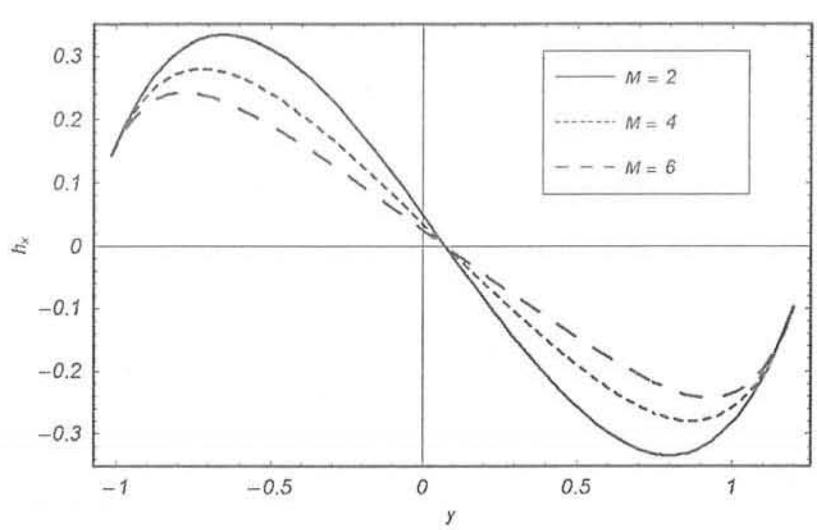

Figure 5.6: Axial induced magnetic field versus y for different values of M. Here $n = 0.398$, $We = 0.01$, $a = 0.6$, $b = 0.3$, $d = 1.1$, $\epsilon = 0.3$, $\theta = 3$, $\phi = \frac{\pi}{6}$ and $x = 0.2$.

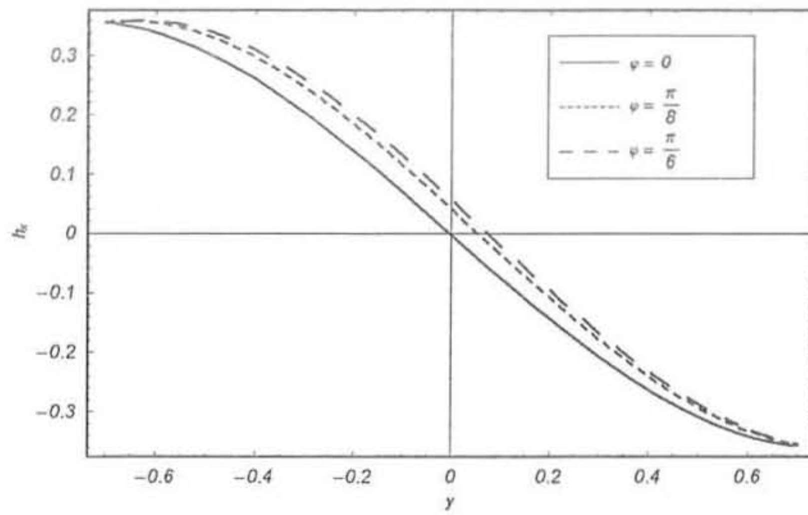

Figure 5.7: Axial induced magnetic field versus y for different values of ϕ. Here $n = 0.398$, $We = 0.01$, $a = 0.6$, $b = 0.3$, $d = 1.1$, $\epsilon = 0.3$, $\theta = 3$, $M = 1$ and $x = 0.2$.

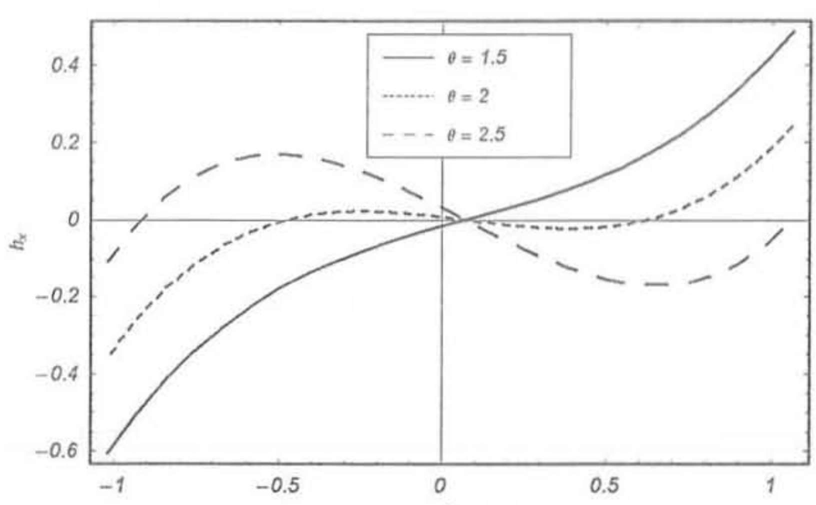

Figure 5.8: Axial induced magnetic field versus y for different values of θ. Here $n = 0.398$, $We = 0.01$, $a = 0.6$, $b = 0.3$, $d = 1.1$, $\epsilon = 0.3$, $M = 1$, $\phi = \frac{\pi}{6}$ and $x = 0.2$.

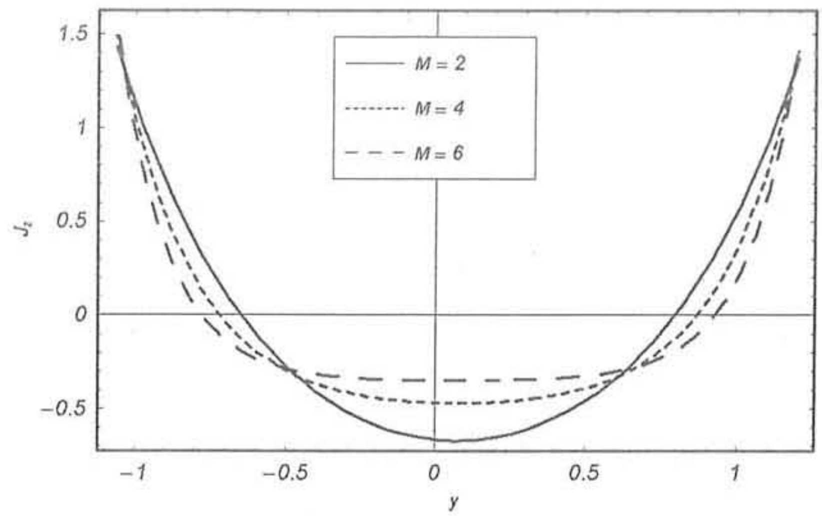

Figure 5.9: Current density distribution versus y for different values of M. Here $n = 0.398$, $We = 0.01$, $a = 0.6$, $b = 0.3$, $d = 1.1$, $\epsilon = 0.3$, $\theta = 3$, $\phi = \frac{\pi}{6}$ and $x = 0.2$.

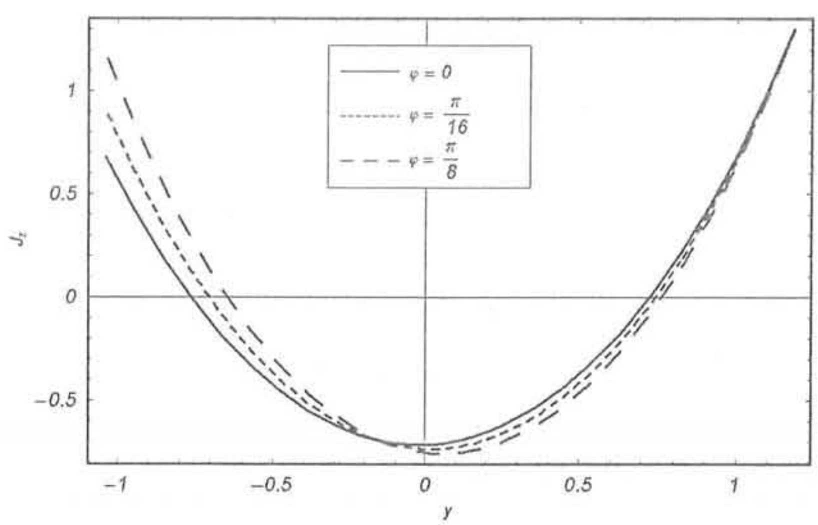

Figure 5.10: Current density distribution versus y for different values of ϕ. Here $n = 0.398$, $We = 0.01$, $a = 0.6$, $b = 0.3$, $d = 1.1$, $\epsilon = 0.3$, $\theta = 3$, $M = 1$ and $x = 0.2$.

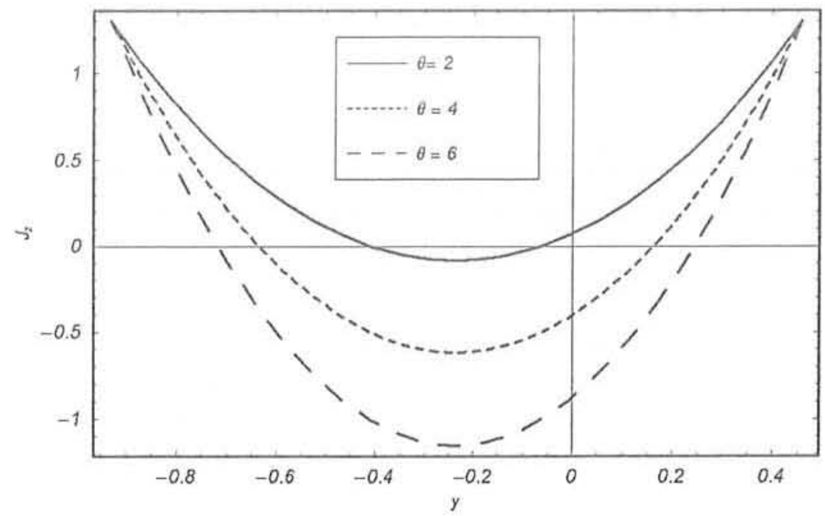

Figure 5.11: Current density distribution versus y for different values of θ. Here $n = 0.398$, $We = 0.01$, $a = 0.6$, $b = 0.3$, $d = 1.1$, $\epsilon = 0.3$, $M = 1$, $\phi = \frac{\pi}{6}$ and $x = 0.2$.

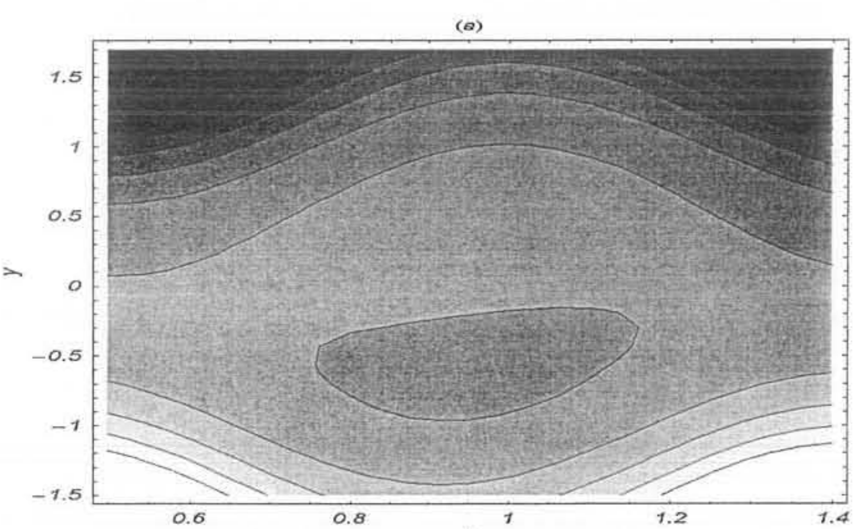

Figure 5.12a: Streamlines for $M = 0.2$. The other parameters are $a = b = 0.4$, $d = 1.1$, $n = 0.498$, $We = 0.03$, $\phi = \frac{\pi}{6}$ and $\theta = 1.5$.

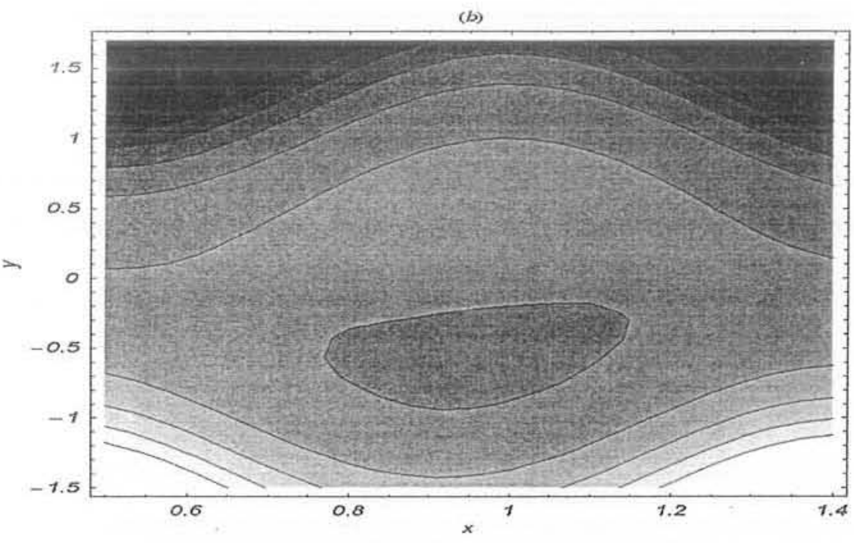

Figure 5.12b: Streamlines for $M = 0.8$. The other parameters are $a = b = 0.4$, $d = 1.1$, $n = 0.498$, $We = 0.03$, $\phi = \frac{\pi}{6}$ and $\theta = 1.5$.

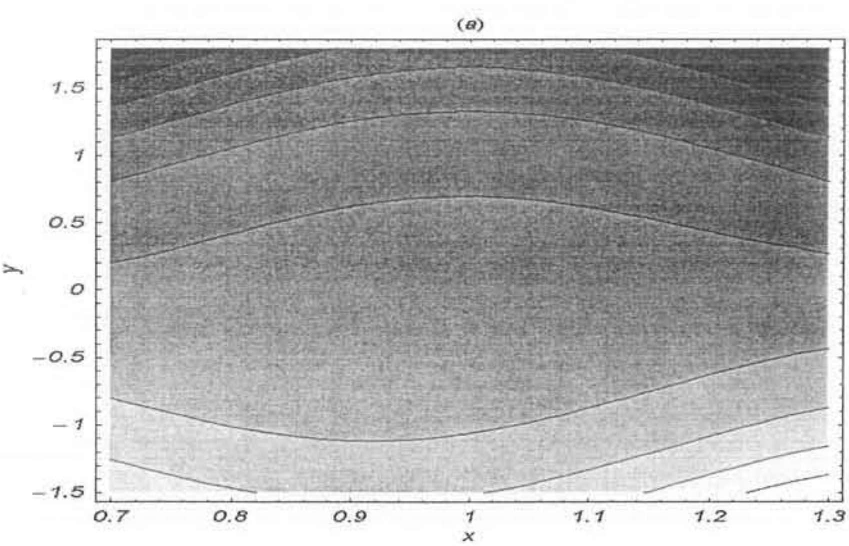

Figure 5.13a: Streamlines for $\theta = 1$. The other parameters are $a = b = 0.4$, $d = 1.1$, $n = 0.498$, $We = 0.03$, $\phi = \frac{\pi}{6}$ and $M = 0.2$.

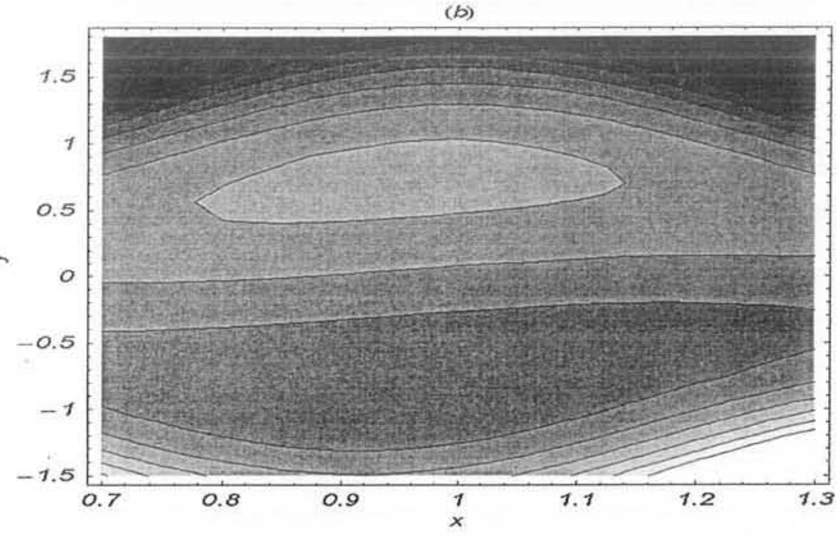

Figure 5.13b: Streamlines for $\theta = 2$. The other parameters are $a = b = 0.4$, $d = 1.1$, $n = 0.498$, $We = 0.03$, $\phi = \frac{\pi}{6}$ and $M = 0.2$.

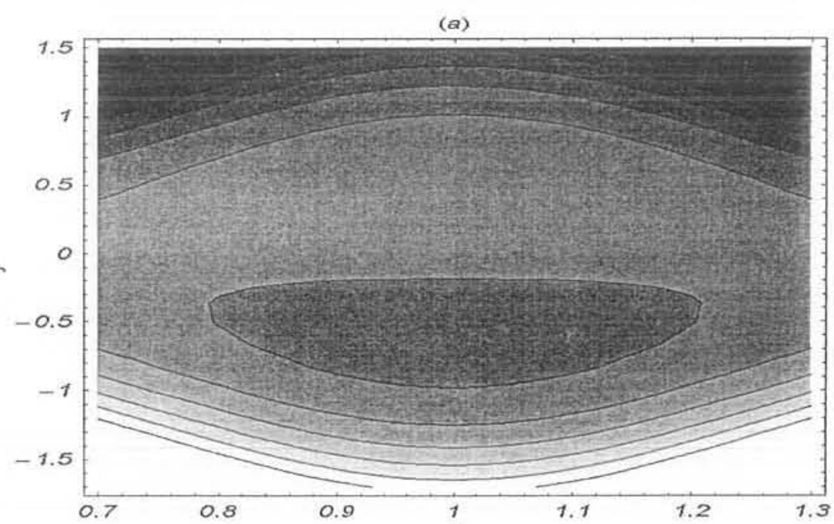

Figure 5.14a: Streamlines for $\phi = 0$. The other parameters are $a = b = 0.4$, $d = 1.1$, $n = 0.498$, $We = 0.03$, $\theta = 1.5$ and $M = 0.2$.

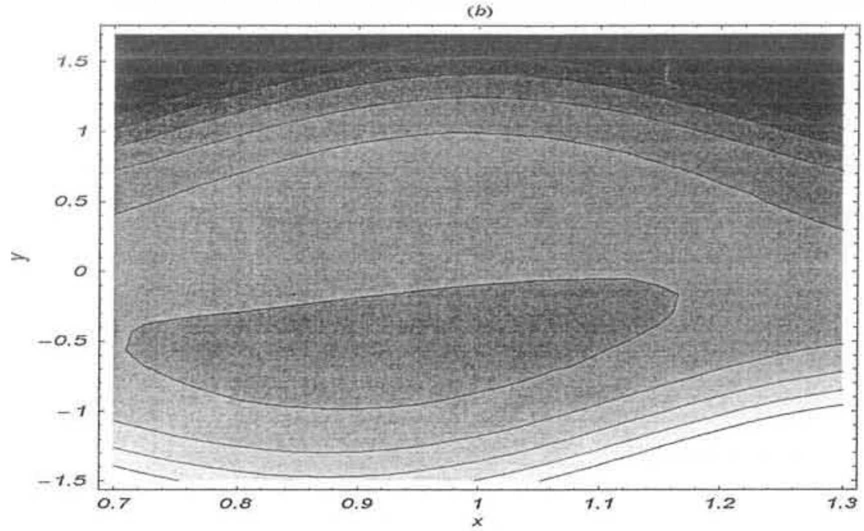

Figure 5.14b: Streamlines for $\phi = \frac{\pi}{4}$. The other parameters are $a = b = 0.4$, $d = 1.1$, $n = 0.498$, $We = 0.03$, $\theta = 1.5$ and $M = 0.2$.

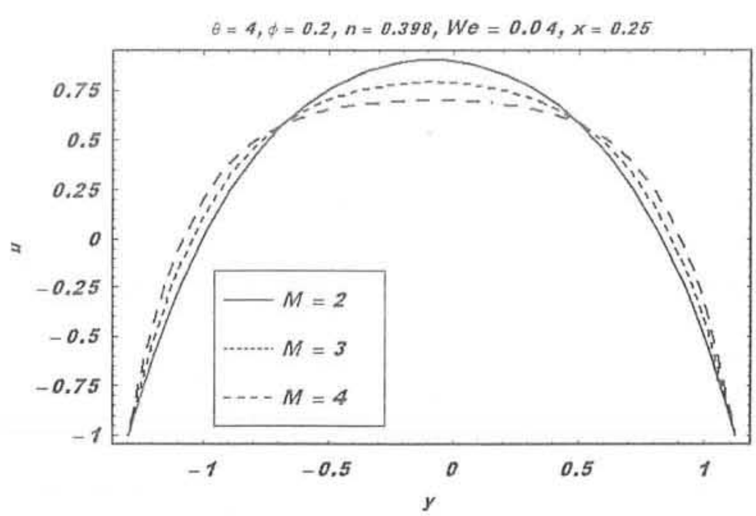

Figure 5.15a: Velocity distribution versus y for different values of M.

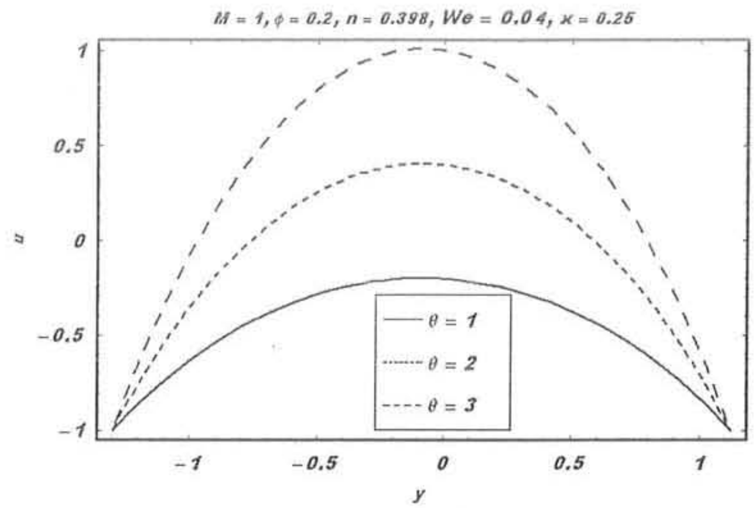

Figure 5.15b: Velocity distribution versus y for different values of θ.

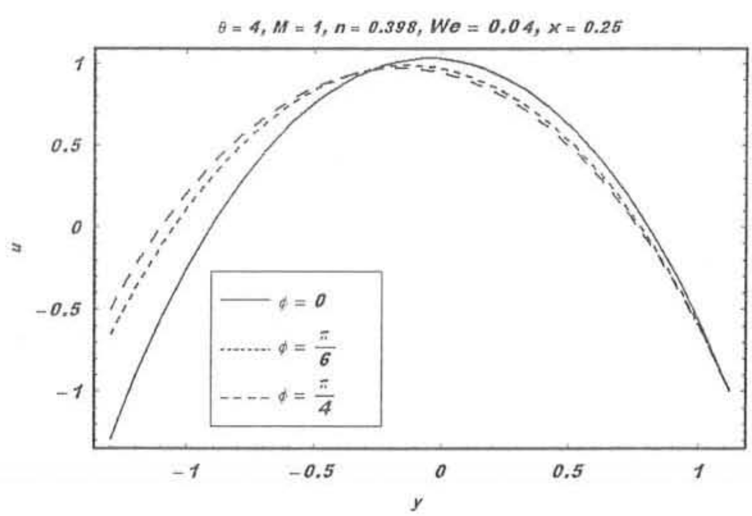

Figure 5.15c: Velocity distribution versus y for different values of ϕ.

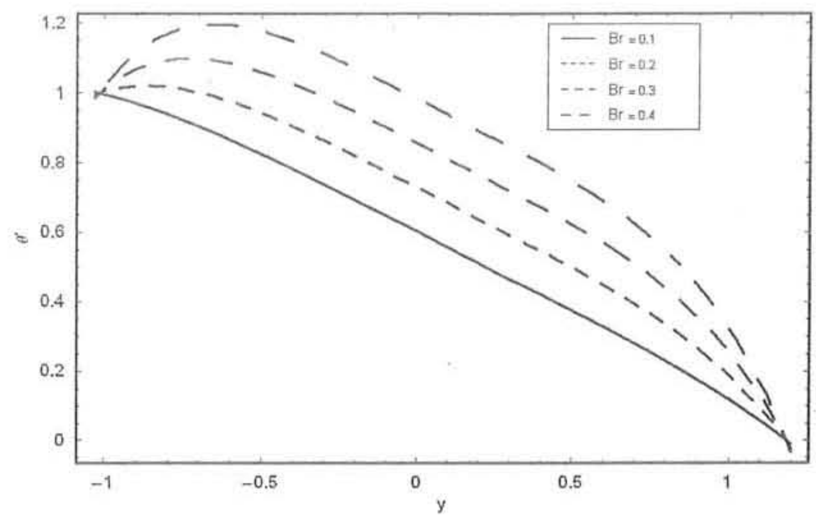

Figure 5.16a: Temperature distribution versus y for different values of Br. Here $n = 0.398$, $We = 0.01$, $a = 0.6$, $b = 0.3$, $d = 1.1$, $\epsilon = 0.3$, $M = 1$, $\phi = 0.6$, $\theta = 2$ and $x = 0.2$.

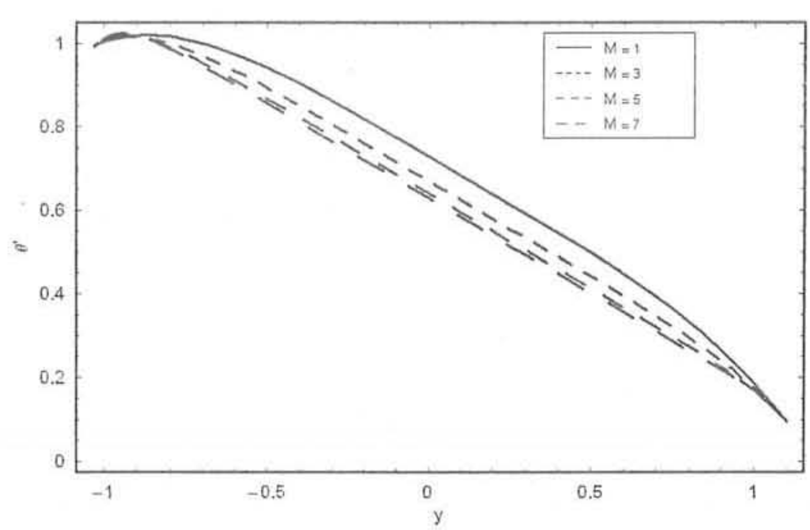

Figure 5.16b: Temperature distribution versus y for different values of M. Here $n = 0.398$, $We = 0.01$, $a = 0.6$, $b = 0.3$, $d = 1.1$, $\epsilon = 0.3$, $Br = 0.3$, $\phi = 0.6$, $\theta = 2$ and $x = 0.2$.

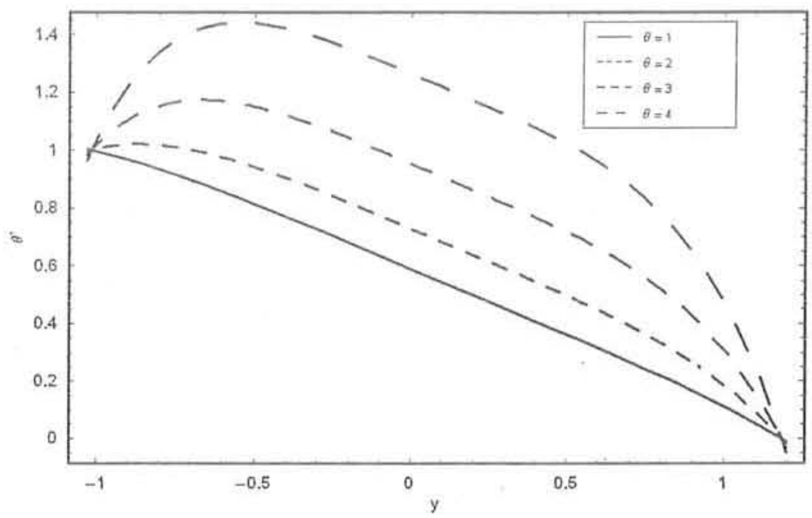

Figure 5.16c: Temperature distribution versus y for different values of θ. Here $n = 0.398$, $We = 0.01$, $a = 0.6$, $b = 0.3$, $d = 1.1$, $\epsilon = 0.3$, $Br = 0.3$, $\phi = 0.6$, $M = 1$ and $x = 0.2$.

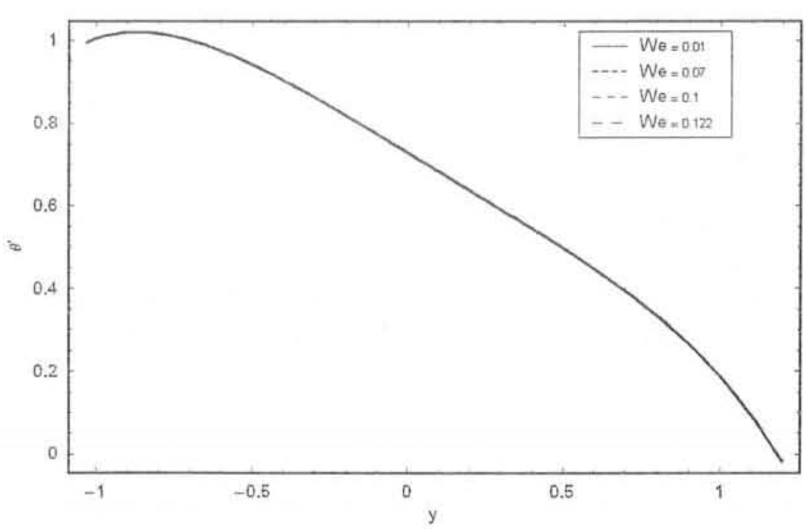

Figure 5.16d: Temperature distribution versus y for different values of We. Here $n = 0.398$, $a = 0.6$, $b = 0.3$, $d = 1.1$, $\epsilon = 0.3$, $M = 1$, $Br = 0.3$, $\phi = 0.6$, $\theta = 2$ and $x = 0.2$.

where

$$
\begin{aligned}
L_1 &= 2\sinh(\frac{M}{2}(h_1-h_2)) + M\cosh(\frac{M}{2}(h_1-h_2))(-h_1+h_2),\\
L_2 &= M^5(-1+n)(h_1-h_2)^3,\\
L_3 &= \cosh(\frac{M}{2}(h_1+h_2)) + \sinh(\frac{M}{2}(h_1+h_2)),\\
L_4 &= 2\sinh(\frac{M}{2}(h_1-h_2)) - M\cosh(\frac{M}{2}(h_1+h_2))h_1 + M\cosh(\frac{M}{2}(h_1-h_2))h_2,\\
L_5 &= -2\sinh(\frac{M}{2}(h_1-h_2)) + M\cosh(\frac{M}{2}(h_1-h_2))(h_1-h_2),\\
L_6 &= 4\sinh(\frac{M}{2}(h_1-h_2)) + 2M\cosh(\frac{M}{2}(h_1-h_2))(-h_1+h_2),\\
L_7 &= M^6(-1+n)(h_1-h_2)^3,\\
L_8 &= \frac{1}{2(h_1-h_2)L_6^2}(2BrM^4h_1^4(-y+h_2) + BrM^2h_2(-F+h_2)^2(-2M^2y^2 - \cosh(M(h_1-h_2))\\
&\quad + \cosh(2My - M(h_1-h_2)) + 2M^2yh_2 + BrM^2h_1(-F+h_2)((-2M^2y^2 - \cosh(M(h_1\\
&\quad -h_2)) + \cosh(2My - M(h_1-h_2))F + 3(2M^2y^2 + \cosh(M(h_1-h_2)) - \cosh(2My\\
&\quad -M(h_1-h_2))h_2 + 2M^2(F-2y)h_2^2 - 2M^2h_2^3 + BrM^2h_1^2(-2F(M^2(F-2y)y - \cosh(M\\
&\quad (h_1-h_2)) + \cosh(2My - M(h_1-h_2))) + (2M^2(F-y)(F+3y) - 3\cosh(M(h_1-h_2))\\
&\quad + \cosh(2My - M(h_1+h_2)) - 8M^2Fh_2^2 + 6M^2h_2^3) + BrM^2h_1^3(2M^2y(-F+y) - \cosh(2My\\
&\quad -M(h_1-h_2)) + 2M^2(2(F+y) - 3h_2)) + 2(-y+h_1)L_6^2),\\
L_9 &= (-1+n)(24M^2y^2 + 16\cosh(M(h_1-h_2)) - \cosh(2M(h_1-h_2)) + \cosh(4My - 2M(h_1\\
&\quad -h_2)) - 16\cosh(2My - M(h_1+h_2)),\\
L_{10} &= (-1+n)(6M^2y(F+4y) + 16\cosh(M(h_1-h_2)) - \cosh(2M(h_1-h_2)) + \cosh(4My - 2M\\
&\quad (h_1+h_2)) - 16\cosh(2My - M(h_1+h_2))h_2,\\
L_{11} &= (-1+n)(8M^2y(2F+3y) + 16\cosh(M(h_1-h_2)) - 16\cosh(2M(h_1-h_2)) + \cosh(4My\\
&\quad -2M(h_1+h_2)) - 16\cosh(2My - M(h_1+h_2))h_2^2,\\
L_{12} &= (-1+n)(12M^2y(3F+2y) + 16\cosh(M(h_1-h_2)) - \cosh(2M(h_1-h_2)) + \cosh(4My\\
&\quad -2M(h_1+h_2)) - 16\cosh(2My - M(h_1+h_2))h_2^3,
\end{aligned}
$$

$$
\begin{aligned}
L_{13} &= (-1+n)(24M^2y(4F+y)+16\cosh(M(h_1-h_2))-M^2y^2)+\cosh(4My-2M-2M(h_1+h_2)) \\
&\quad -16\cosh(2My-M(h_1+h_2))h_2^4, \\
L_{14} &= (-1+n)yh_2^5, \\
L_{15} &= (-1635+5264M^2y^2-192M^4y^4+12(137+24M^2y^2)\cosh(2M(y-h_2))-9\cosh(4M(y \\
&\quad -h_1))-153\cosh(M(4y-h_1-3h_2))+84\cosh(M(2y+h_1-3h_2))+18\cosh(2M(3y \\
&\quad -h_1-2h_2))+2\cosh(2M(y+h_1-2h_2))+12(137+24M^2y^2)\cosh(2M(y-h_2))-9 \\
&\quad \cosh(4M(y-h_2))-153\cosh(M(4y-3h_1-h_2))+18\cosh(2M(3y-2h_1-h_2)) \\
&\quad -3460\cosh(M(2y-h_1-h_2))-576M^2y^2\cosh(M(2y-h_1-h_2))+324\cosh(2M(2y \\
&\quad -2h_1-h_2))-36\cosh(3M(2y-h_1-h_2))+(3529-4872M^2y^2+192M^4y^4)\cosh(M \\
&\quad (h_1-h_2))-4(497+96M^2y^2)\cosh(2M(h_1-h_2))+(105-8M^2y^2)\cosh(3M(h_1 \\
&\quad -h_2))-11\cosh(4M(h_1-h_2))+84\cosh(M(2y-3h_1+h_2))+2\cosh(2M(y-2h_1 \\
&\quad +h_2))+96My\sinh(2M(y-h_1))-108My\sinh(M(4y-h_1-3h_2))+48\sinh(M(2y \\
&\quad +h_1-3h_2))+96My\sinh(2M(y-h_2))-108\sinh(M(4y-3h_1-h_2))-288My\sinh(M \\
&\quad (2y-h_1-h_2))+216My\sinh(2M(2y-h_1-h_2))+48My\sinh(M(2y-3h_1+h_2)), \\
L_{16} &= (-1692\sinh(2M(y-h_1))+18\sinh(4M(y-h_1))-99\sinh(M(4y-h_1-3h_2))+144 \\
&\quad \sinh(M(2y+h_1-3h_2))+6\sinh(2M(y+h_1-2h_2))+1594\sinh(2M(y-h_2))-18 \\
&\quad \sinh(4M(y-h_2))+207\sinh(M(4y-3h_1-h_2))+144\sinh(M(2y-h_1-h_2))-108 \\
&\quad \sinh(2M(2y-h_1-h_2))-534\sinh(2M(h_1-h_2))+4My(-96\cosh(2M(y-h_1))-27 \\
&\quad \cosh(M(4y-3h_1-3h_2))+24\cosh(M(2y+h_1-3h_2))-48\cosh(2M(y-h_2))+27 \\
&\quad \cosh(M(4y-3h_1-h_2))+2(72\cosh(M(2y-h_1-h_2))+(609-48M^2y^2)\cosh(M \\
&\quad (h_1-h_2))+48\cosh(2M(h_1-h_2))+\cosh(3M(h_1-h_2))-12\cosh(M(2y-3h_1 \\
&\quad +h_2))+6My(-6\sinh(2M(y-h_1))+6\sinh(2M(y-h_2))-(102-4M^2y^2+32\cosh(M \\
&\quad (h_1-h_2))+\cosh(2M(h_1-h_2)))\sinh(M(h_1-h_2))))), \\
L_{17} &= 1476\sinh(2M(h_1-h_2))-45\sinh(3M(h_1-h_2))-2(2632My-192M^3y^3+\sinh(4M \\
&\quad (h_1-h_2))+96\sinh(2M(y-2h_1+h_2))+9\sinh(6My-2M(2h_1+h_2))-9\sinh(6My \\
&\quad -2M(h_1+2h_2))),
\end{aligned}
$$

$$
\begin{aligned}
L_{18} &= 9\cosh(4M(y-h_1)) - 27\cosh(M(4y-h_1-3h_2)) + 36\cosh(M(2y-h_1-3h_2)) + 6 \\
&\cosh(2M(y+h_1-2h_2)) + 4(253+24M^2y^2)\cosh(2M(y-h_2)) - 9\cosh(4M(y-h_2)) \\
&-171\cosh(M(4y-3h_1-h_2)) - 36\cosh(2M(2y-h_1-h_2)) + 12\cosh(3M(2y-h_1 \\
&-h_2)) - 293\cosh(M(h_1-h_2)) - 988\cosh(2M(h_1-h_2)) - 21\cosh(3M(h_1-h_2)) \\
&+6\cosh(2M(y-2h_1+h_2)) + 6\cosh(2M(y-2h_1+h_2)) + 288\cosh(2My-M(h_1 \\
&+h_2)) + 6\cosh(6My-2M(2h_1+h_2)) + 6\cosh(6My-2M(h_1+2h_2)), \\
L_{19} &= ((3+48M^2y^2)\cosh(M(2y-h_1-h_2)) + My(168\sinh(2M(y-h_1)) - 9\sinh(M(4y \\
&-h_1-3h_2)) + 12\sinh(M(2y+h_1-3h_2)) + 1624\sinh(M(h_1-h_2)) + 2My((-403 \\
&+8M^2y^2)\cosh(M(h_1-h_2)) - 58\cosh(2M(h_1-h_2)) - 3\cosh(3M(h_1-h_2)) - 64 \\
&My\sinh(M(h_1-h_2)) + 256\sinh(2M(h_1-h_2)) + 8\sinh(3M(h_1-h_2)) - 3(8\sinh(2M \\
&(y-h_2)) + 3\sinh(M(4y-3h_1-h_2)) + 24\sinh(M(2y-h_1-h_2)) + 6\sinh(2M(2y \\
&-h_1-h_2)) - 4\sinh(M(2y-3h_1+h_2))))), \\
L_{20} &= 4Mh_2(92My - 32M^3y^3 - 48My\cosh(2M(y-h_1)) - 48My\cosh(M(2y-h_1-h_2)) \\
&112My\cosh(2M(h_1-h_2)) + 6My\cosh(3M(h_1-h_2)) - 96\sinh(2M(y-h_1)) + 24 \\
&\sinh(2M(2y-h_1-h_2)) - 12\sinh(M(2y-3h_1+h_2)) - 24Mh_2 + 48M^3y^2h_2 + 24 \\
&M\cosh(2M(y-h_1))h_2 - 32M^3y^2h_2^2 + 3\sinh(2M(h_1-h_2))(-11-8M^2y^2+M^2yh_2) \\
&+\sinh(M(h_1-h_2))(-57-72M^2y^2+136M^2yh_2) + 2My\cosh(M(h_1-h_2))(435 \\
&-16M^2y^2+8M^2h_2^2)), \\
L_{21} &= (-1+n)(24M^2(4F+y)y - 16\cosh(M(h_1-h_2)) + \cosh(2M(h_1-h_2)) - \cosh(4My \\
&-2M(h_1+h_2)) + 16\cosh(2My-M(h_1+h_2)) + 24M^2h_2(-4F-3y+4h_2)), \\
L_{22} &= (3-6M^2y^2 - 3\cosh(2M(y-h_2)) + 17M\sinh(M(h_1-h_2))(y-h_2) + 3M\sinh(2M \\
&(h_1-h_2))(y-h_2) - 8M^2yh_2 + 18M^2yh_2 + 10M^2\cosh(M(h_1-h_2))h_2(y-h_2) + 14 \\
&M^2h_2^2), \\
L_{23} &= (-1+n)(F-h_2), \\
L_{24} &= (12M^2y(-3F+2y) + 16\cosh(M(h_1-h_2)) - \cosh(2M(h_1-h_2)) + \cosh(4My \\
&-2M(h_1+h_2)) - 16\cosh(2My-M(h_1+h_2)) + 12M^2(3F+y-3h^2)h_2),
\end{aligned}
$$

$$
\begin{aligned}
L_{25} &= 3(92My - 32M^3y^3 - +24\sinh(2M(y-h_1)) + 9\sinh(M(4y-h_1-3h_2))) - 96 \\
&\quad \sinh(2M(y-h_2)) + 36\sinh(M(2y-h_1-h_2)) + 9\sinh(2M(2y-h_1-h_2)) + 112 \\
&\quad M\cosh(2M(h_1-h_2))(y-h_2) + 6M\cosh(3M(h_1-h_2))(y-h_2) + 4Mh_2 - 96M^3y^2 \\
&\quad h_2 - 48M\cosh(2M(y-h_2)) + 48M\cosh(M(2y-h_1-h_2))h_2 - 32M^3y^2h_2^2 + 160M^3 \\
&\quad h_2^3 + \sinh(M(h_1-h_2))(57 + 72M^2y^2 + 8M^2(50y - 59h_2)h_2), \\
L_{26} &= 3\sinh(2M(h_1-h_2))(11 + 8M^2y^2 + 8M^2(2y-3h_2)h_2), \\
L_{27} &= 2M\cosh(M(h_1-h_2))(-y+h_2)(435 - 16M^2y^2 + 32M^2h_2(y+2h_2)), \\
L_{28} &= (-1+n)(F-h_2)^2, \\
L_{29} &= (8M^2(2F-3y)y - 16\cosh(M(h_1-h_2)) + \cosh(2M(h_1-h_2)) - \cosh(4My - 2M \\
&\quad (h_1+h_2)) + 16\cosh(2My - M(h_1+h_2)) + 8M^2h_2(-2F + y + 2h_2)), \\
L_{30} &= 1131 + 624M^2y^2 - 64M^4y^4 - 4(253 + 24M^2y^2)\cosh(2M(y-h_1)) + 9\cosh(4M(y \\
&\quad -h_1)) + 171\cosh(M(4y-h_1-3h_2)) - 164\cosh(M(2y+h_1-3h_2)) - 6\cosh(2M \\
&\quad (y+h_1-2h_2)) - 12(95 + 8M^2y^2)\cosh(2M(y-h_2)) + 9\cosh(4M(y-h_2)) + 27 \\
&\quad \cosh(M(4y-3h_1-h_2)) + 36\cosh(2M(2y-h_1-h_2)) - 12\cosh(3M(2y-h_1 \\
&\quad -h_2)) + 296\cosh(M(h_1-h_2)) + 988\cosh(2M(h_1-h_2)) + 21\cosh(3M(h_1-h_2)) \\
&\quad -6\cosh(6My - 2M(2h_1+h_2)) - 6\cosh(6My - 2M(h_1+2h_2)) + 4(-3(-1+M \\
&\quad +18M^2y^2)\cosh(M(2y-h_1-h_2)) + My(24\sinh(2M(y-h_1)) + 9\sinh(M(4y-h_1 \\
&\quad -3h_2)) - 12\sinh(M(2y+h_1-3h_2)) - 168\sinh(2M(y-h_2)) + 9\cosh(M(4y-3 \\
&\quad h_1-h_2)) + 72\sinh(M(2y-h_1-h_2)) + 18\sinh(2M(2y-h_1-h_2)) + 1624\sinh(M \\
&\quad (h_1-h_2)) + 2My((403 - 8M^2y^2)\cosh(M(h_1-h_2)) + 56\cosh(2M(h_1-h_2)) + 3 \\
&\quad \cosh(3M(h_1-h_2)) - 64My\sinh(2M(h_1-h_2))) + 256\sinh(2M(h_1-h_2)) + 8 \\
&\quad \sinh(3M(h_1-h_2)) - 12\sinh(M(2y-3h_1+h_2)))), \\
L_{31} &= (-36My - 32M^3y^3 + 48My\cosh(2M(y-h_1)) - 96My\cosh(2M(y-h_2)) - 48My \\
&\quad \cosh(M(2y-h_1-h_2)) + 2My(499 - 16M^2y^2)\cosh(M(h_1-h_2)) + 112My\cosh(2 \\
&\quad M(h_1-h_2)) + 6My\cosh(3M(h_1-h_2)) + 48\sinh(2M(y-h_1)) + 18\sinh(M(4y
\end{aligned}
$$

$-3h_1 - h_2)) + 36\sinh(M(2y - h_1 - h_2)) + 9\sinh(2M(2y - h_1 - h_2)) + (-1453$

$+600M^2y^2)\sinh(M(h_1 - h_2)) + (157 + 72M^2y^2)\sinh(2M(h_1 - h_2)) - 8\sinh(3M$

$(h_1 - h_2)) + 12\sinh(M(2y - 3h_1 + h_2)) + 4Mh_2(-6(-1 + 4M^2y^2 + \cosh(2M(y$

$-h_1)) + \cosh(2M(y - h_2)) - 4\cosh(M(2y - h_1 - h_2)) + (-451 + 48M^2y^2)\cosh($

$M(h_1 - h_2)) - 56\cosh(2M(h_1 - h_2)) - 3\cosh(3M(h_1 - h_2)) + 2Mh_2(-59\sinh(M$

$(h_1 - h_2))) - 9\sinh(2M(h_1 - h_2))) - 2M\cosh(M(h_1 - h_2))(4y + 5h_2) + 2M(2y$

$+7h_2)))),$

$$
\begin{aligned}
L_{32} &= (-2M^2(F^2 - 12y^2) + 16\cosh(M(h_1 - h_2)) + \cosh(2M(h_1 - h_2)) - \cosh(4My - 2\\
&\quad M(h_1 + h_2)) + 16\cosh(2My - M(h_1 + h_2))),\\
L_{33} &= (-2M^2(F^2 - 12y^2) + 16\cosh(M(h_1 - h_2)) - \cosh(2M(h_1 - h_2)) + \cosh(4My - 2\\
&\quad M(h_1 - h_2)) - 16\cosh(2My - M(h_1 + h_2)))h_2,\\
L_{34} &= (4M^2(2F^2 - 3Fy - 6y^2) - 16\cosh(M(h_1 - h_2)) + \cosh(2M(h_1 - h_2)) - \cosh(4My\\
&\quad -2M(h_1 + h_2)) + 16\cosh(2My - M(h_1 + h_2)))h_2^2,\\
L_{35} &= 12M^2(-3F^2 + 4Fy + 2y^2) + 16\cosh(2My - M(h_1 + h_2))h_2^3,\\
L_{36} &= M^9(-1 + n)(4F - 3y)h_2^2,\\
L_{37} &= 96608L_1^6L_2^2M^9(-1 + n)h_2^5,\\
L_{38} &= (2(-1596\sinh(2M(y - h_1)) + 18\sinh(4M(y - h_1)) + 207\sinh(M(4y - h_1 - 3h_2))\\
&\quad +192\sinh(M(2y + h_1 - 3h_2)) + 6\sinh(2M(y + h_1 - 2h_2)) + 1692\sinh(2M(y\\
&\quad -h_2)) - 18\sinh(4M(y - h_2)) + 99\sinh(M(y - 3h_1 - h_2)) - 144\sinh(M(2y - h_1\\
&\quad -h_2)) + 108\sinh(2M(2y - h_1 - h_2)) - 543\sinh(M(h_1 - h_2)) + 4My(48\cosh(2\\
&\quad M(y - h_1)) - 27\cosh(M(4y - h_1 - 3h_2)) + 24\cosh(M(2y + h_1 - 3h_2)) + 96\cosh(2\\
&\quad M(y - h_2)) + 27\cosh(M(4y - 3h_1 - h_2)) - (72\cosh(M(2y - h_1 - h_2)) + (609 - 48\\
&\quad M^2y^2)\cosh(M(h_1 - h_2)) + 48\cosh(2M(h_1 - h_2)) + \cosh(3M(h_1 - h_2)) + 12\cosh(M\\
&\quad (2y - 3h_1 + h_2)) + 6My(6\sinh(2M(y - h_1)) - 6\sinh(2M(y - h_2)) + (102 - 4M^2y^2\\
&\quad +32\cosh(M(h_1 - h_2)) + \cosh(2M(h_1 - h_2)))\sinh(M(h_1 - h_2))))),
\end{aligned}
$$

$$
\begin{aligned}
L_{39} &= 1476\sinh(2M(h_1-h_2)) - 45\sinh(3M(h_1-h_2)) - 2(-2632My + 192M^3y^3 + 3\sinh(4\\
&\quad M(h_1-h_2)) + 72\sinh(M(2y-3h_1+h_2)) + 3\sinh(2M(y-2h_1+h_2)) + 9\sinh(6My\\
&\quad -2M(2h_1+h_2))),\\
L_{40} &= Mh_2(-13921 + 48M^2y^2(9+4M^2y^2) + 12(221+24M^2y^2)\cosh(2M(y-h_1)) - 27\\
&\quad \cosh(4M(y-h_1)) - 297\cosh(M(4y+h_1-3h_2)) + 300\cosh(M(2y+h_1-3h_2))\\
&\quad +18\cosh(2M(3y-h_1-2h_2)) + 18\cosh(2M(y+h_1-2h_2)) - 27\cosh(4M(y\\
&\quad -h_2)) - 297\cosh(M(4y-3h_1-h_2)) + 18\cosh(2M(3y-2h_1-h_2)) + 2052\\
&\quad \cosh(M(2y-h_1-h_2)) - 108\cosh(2M(2y-h_1-h_2)) + 36\cosh(3M(2y-h_1\\
&\quad -h_2)) + 8865\cosh(M(h_1-h_2)) - 2196\cosh(2M(h_1-h_2)) - 47\cosh(3M(h_1\\
&\quad -h_2)) - 9\cosh(4M(h_1-h_2)) + 300\cosh(M(2y-3h_1+h_2)) + 18\cosh(2M(y\\
&\quad -2h_1+h_2)) + 12(221+24M^2y^2)\cosh(2M(y-h_2)) - 56\cosh(2M(h_1-h_2)) - 3\\
&\quad \cosh(3M(h_1-h_2))) + 3(24\sinh(2M(y-h_1)) - 3\sinh(M(4y-h_1-3h_2)) + 4\\
&\quad \sinh(M(2y+h_1-3h_2)) + 24\sinh(2M(y-h_2)) - 3(\sinh(M(4y-3h_1-h_2)) + 8\\
&\quad \sinh(M(2y-h_1-h_2)) + 2\sinh(2M(2y-h_1-h_2)) + 4\sinh(M(2y-3h_1+h_2))))),\\
L_{41} &= ((-36My - 32M^3y^3 - 96My\cosh(2M(y-h_1)) + 48My\cosh(2M(y-h_2)) - 48My\\
&\quad \cosh(M(2y-h_1-h_2)) + 2My(499-16M^2y^2)\cosh(M(h_1-h_2)) + 122M\cosh(2\\
&\quad M(h_1-h_2)) + 6My\cosh(3M(h_1-h_2)) - 120\sinh(2M(y-h_1)) - 9\sinh(M(4y-h_1\\
&\quad -3h_2)) + 12\sinh(M(2y+h_1-3h_2)) + 48\sinh(2M(y-h_2)) + 18\sinh(M(4y-3h_1\\
&\quad -h_2)) + 36\sinh(M(2y-h_1-h_2)) + 9\sinh(2M(2y-h_1-h_2)) + (1453-600M^2\\
&\quad y^2)\sinh(M(h_1-h_2)) + (157-72M^2y^2)\sinh(2M(h_1-h_2)) + 8\sinh(3M(h_1-h_2))\\
&\quad -24\sinh(M(2y-3h_1+h_2)) + 2Mh_2(56\cosh(2M(h_1-h_2)) + 3\cosh(3M(h_1-h_2))\\
&\quad -2(1-24M^2y^2 - 12\cosh(2M(y-h_1)) + 12\cosh(M(2y-h_1-h_2)) + 8M^2h_2(2y\\
&\quad +h_2)) + \cosh(M(h_1-h_2))(435-48M^2y^2+8M^2h_2(5y-h_2))))L_6^4L_7^2).
\end{aligned}
$$

5.3 Discussion

The main aim of this section is to study the behaviour of involved key parameters on pressure rise per wavelength (Δp_λ). Therefore we have prepared Figures 5.1 − 5.5. The variation of Hartman number (M) on Δp_λ is displayed in Figure 5.1. Here pumping rate increases with the decrease of M. In Figure 5.2, we have discussed the effects of phase difference (ϕ). It is noticed that in both pumping ($\Delta p_\lambda > 0$) and free pumping ($\Delta p_\lambda = 0$) regions, the pumping decreases with an increase of ϕ. However in copumping region ($\Delta p_\lambda < 0$) this situation is quite opposite. The pumping increases for large values of ϕ. It is also revealed that pumping rate increases with the decrease of ϕ. As we can see from the Figures 5.3 and 5.4 that pumping rate increases by decreasing the wave amplitudes a and b. To observe the effects of channel width d on Δp_λ we have presented Figure 5.5. It shows that pumping decreases when d increases in pumping and free pumping regions. A quite reverse situation is observed in the copumping region where Δp_λ increases by increasing the channel width.

Figures 5.6 − 5.11 describe the influence of Hartman number (M), phase difference (ϕ) and volume flow rate (θ) on the axial induced magnetic field and the current density distribution. In Figures 5.6 − 5.8, it is focused that an axial induced magnetic field h_x decreases when M, ϕ and θ are increased. Moreover h_x is not symmetric about the origin because the phase difference is taken into account. The axial induced magnetic field is in one direction in some part of the region whereas in the other part it is in opposite direction. The current density distribution with y can be seen in the Figures 5.9 − 5.11. Figure 5.9 depicts that the current density distribution increases near the channel walls when M increases and decreases near the centre of the channel. In Figure 5.10, we found that J_z decreases with an increase of ϕ whereas in the other region, the behaviour is reverse. In Figure 5.11, it is found (contrary to Figure 5.9) that the current density distribution is an increasing function of volume flow rate θ.

Figures 5.12 − 5.14 are for the trapping. Figures 5.12(a, b) show the streamlines for M. These depicts that size of the trapped bolus decreases with an increase in Hartman number M when $\phi = \frac{\pi}{6}$. Figures 5.13(a, b) provide the effects of volume flow rate on trapping. The volume flow rate always effects the trapping process. These figures clearly indicate that the size of bolus increases for large values of θ. Figures 5.14(a, b) compare the trapping phenomenon between symmetric and asymmetric channels. This figure shows that the size of the trapped bolus increases by increasing phase difference ϕ. It is also observed that trapped bolus moves towards left when large values of ϕ are taken into account.

Figures 5.15(a, b, c) show the variation of velocity (u) versus y for the different parameters. The effect of M on the longitudinal velocity (u) at cross section $x = 0.25$ is depicted in Figure 5.15a. Here an increase in M causes a decrease in the magnitude of u at the boundaries and at the center of the channel. Clearly u increases when there is an increase in M. In Figure 5.15b, the effect of θ on u is captured. It is noticed that with an increase in θ, the velocity increases. Figure 5.15c is just displayed to see the variation of u for the different values of ϕ. It is observed that u increases when ϕ decreases and such behaviour is quite opposite in the other region. In Figures 5.16(a, b, c, d), we have plotted the temperature distribution (θ') versus y for the different values of parameters. In Figures 5.16(a, b), θ' increases for large values of Br and for small values of M. Similarly θ' increases with an increase in θ (Figure 5.16c). Figure 5.16d indicates that in the present situation We has no effect on θ'. It is noted from Figure 5.16 that the shape of temperature profile is almost parabolic.

Chapter 6

Peristaltic motion of a Carreau fluid in an asymmetric channel with partial slip and heat transfer characteristics

The simultaneous effects of heat transfer and partial slip on the peristaltic motion in an asymmetric channel are studied in this chapter. Mathematical formulation is based upon the constitutive equations of a Carreau fluid. Slip condition in terms of shear stress is considered. The solution to the resulting problem is presented by employing long wavelength approximation. Results of stream function, pressure gradient and temperature are developed. Pumping, trapping and frictional forces are discussed in detail.

6.1 Mathematical formulation

We note that the physical model of the present problem is similar to that of the previous chapter. We consider the present flow situation for a hydrodynamic incompressible fluid in an asymmetric channel. The flow analysis is taken by taking slip effects on the walls of channel.

The governing two dimensional equations in usual notations can be put into the forms

$$\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}=0, \tag{6.1}$$

$$\rho\left(u'\frac{\partial}{\partial x'}+v'\frac{\partial}{\partial y'}\right)u'=-\frac{\partial p'}{\partial x'}-\frac{\partial S'_{x'x'}}{\partial x'}-\frac{\partial S'_{x'y'}}{\partial y'}, \tag{6.2}$$

$$\rho\left(u'\frac{\partial}{\partial x'}+v'\frac{\partial}{\partial y'}\right)v'=-\frac{\partial p'}{\partial y'}-\frac{\partial S'_{x'y'}}{\partial x'}-\frac{\partial S'_{y'\overline{y}}}{\partial y'}. \tag{6.3}$$

By performing the transformations

$$\begin{aligned}
&x=\frac{2\pi x'}{\lambda},\quad y=\frac{y'}{d_1},\quad u=\frac{u'}{c},\quad v=\frac{v'}{c},\quad t=\frac{2\pi t'c}{\lambda},\quad p=\frac{2\pi d_1^2 p'}{c\lambda\mu},\quad \mathrm{Re}=\frac{\rho ca}{\mu},\\
&S=\frac{d_1 S'}{\mu c},\quad h_1=\frac{h_1'}{d_1},\quad h_2=\frac{h_2'}{d_1},\quad \Psi=\frac{\Psi'}{cd_1},\quad \delta=\frac{2\pi d_1}{\lambda},d=\frac{d_2}{d_1},\quad a=\frac{a_1}{d_1},\\
&b=\frac{a_2}{d_1},\quad \theta'=\frac{T-T_0}{T_1-T_0},\quad \mathrm{Pr}=\frac{\rho\nu C_p}{\kappa},\quad E=\frac{c^2}{C_p\left(T_1-T_0\right)},\quad u=\frac{\partial\Psi}{\partial y},\\
&v=-\delta\frac{\partial\Psi}{\partial x},\quad Br=E\,\mathrm{Pr},
\end{aligned} \tag{6.4}$$

with the stream function $\Psi(x,y)$, we find that Eq. (6.1) is identically satisfied and Eqs. (6.2) and (6.3) can be written as

$$\delta\,\mathrm{Re}\left[\left(\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial\Psi}{\partial y}\right]=-\frac{\partial p}{\partial x}-\delta\frac{\partial S_{xx}}{\partial x}-\frac{\partial S_{xy}}{\partial y}, \tag{6.5}$$

$$-\delta^3\,\mathrm{Re}\left[\left(\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial\Psi}{\partial x}\right]=-\frac{\partial p}{\partial y}-\delta^2\frac{\partial S_{xy}}{\partial x}-\delta\frac{\partial S_{yy}}{\partial y}, \tag{6.6}$$

$$\frac{\partial^2\theta'}{\partial y^2}+Br\left\{1+\frac{(n-1)}{2}We^2\dot{\gamma}^2\right\}\left(\frac{\partial^2\Psi}{\partial y^2}-\delta^2\frac{\partial^2\Psi}{\partial x^2}\right)\frac{\partial^2\Psi}{\partial y^2}=0, \tag{6.7}$$

where for the convenience of readers we present

$$
\begin{aligned}
S_{xx} &= -2\left[1+\frac{n-1}{2}We^2\dot{\gamma}^2\right]\frac{\partial^2\Psi}{\partial x\partial y}, && (6.8)\\
S_{xy} &= -\left[1+\frac{n-1}{2}We^2\dot{\gamma}^2\right]\left(\frac{\partial^2\Psi}{\partial y^2}-\delta^2\frac{\partial^2\Psi}{\partial x^2}\right), && (6.9)\\
S_{yy} &= 2\delta\left[1+\frac{n-1}{2}We^2\dot{\gamma}^2\right]\frac{\partial^2\Psi}{\partial x\partial y}, && (6.10)\\
\dot{\gamma} &= \left[2\delta^2\left(\frac{\partial^2\Psi}{\partial x\partial y}\right)^2+\left(\frac{\partial^2\Psi}{\partial y^2}-\delta^2\frac{\partial^2\Psi}{\partial x^2}\right)^2+2\delta^2\left(\frac{\partial^2\Psi}{\partial x\partial y}\right)^2\right]^{\frac{1}{2}}, && (6.11)
\end{aligned}
$$

in which We is the Weissenberg number, Br the Brinkman number, Pr the Prandtl number, Ec the Eckert number, Re the Reynolds number and δ the wave number. Under the assumptions of long wavelength and low Reynolds number, the Eqs. (6.5)-(6.7) after using Eq. (6.9) become

$$
\frac{\partial p}{\partial x}=\frac{\partial}{\partial y}\left[\left\{1+\frac{(n-1)}{2}We^2\left(\frac{\partial^2\Psi}{\partial y^2}\right)^2\right\}\frac{\partial^2\Psi}{\partial y^2}\right], \tag{6.12}
$$

$$
\frac{\partial p}{\partial y}=0, \tag{6.13}
$$

$$
\frac{\partial^2\theta'}{\partial y^2}+Br\left[\left\{1+\frac{(n-1)}{2}We^2\left(\frac{\partial^2\Psi}{\partial y^2}\right)^2\right\}\left(\frac{\partial^2\Psi}{\partial y^2}\right)^2\right]=0, \tag{6.14}
$$

where Eq. (6.13) shows that $p\neq p(y)$ and hence $p=p(x)$.

From Eqs. (6.13) and (6.14), we get

$$
\frac{\partial^2}{\partial y^2}\left[\left\{1+\frac{(n-1)}{2}We^2\left(\frac{\partial^2\Psi}{\partial y^2}\right)^2\right\}\frac{\partial^2\Psi}{\partial y^2}\right]=0. \tag{6.15}
$$

The slip boundary conditions in the laboratory frame can be defined as

$$
\begin{aligned}
\overline{U}(\overline{X},\overline{H}_1,\overline{t}) &= -\frac{\chi}{\mu}\overline{S}_{\overline{XY}},\\
\overline{U}(\overline{X},\overline{H}_2,\overline{t}) &= \frac{\chi}{\mu}\overline{S}_{\overline{XY}}
\end{aligned} \tag{6.16}
$$

The dimensionless boundary conditions are

$$\Psi = \frac{F}{2}, \frac{\partial \Psi}{\partial y} = -\beta \left\{ \left(\frac{\partial^2 \Psi}{\partial y^2} \right) + \frac{(n-1)}{2} We^2 \left(\frac{\partial^2 \Psi}{\partial y^2} \right)^3 \right\} - 1, \, \theta' = 0 \text{ at } y = h_1,$$

$$\Psi = -\frac{F}{2}, \frac{\partial \Psi}{\partial y} = \beta \left\{ \left(\frac{\partial^2 \Psi}{\partial y^2} \right) + \frac{(n-1)}{2} We^2 \left(\frac{\partial^2 \Psi}{\partial y^2} \right)^3 \right\} - 1, \, \theta' = 1 \text{ at } y = h_2, \tag{6.17}$$

$$\begin{aligned} h_1 &= 1 + a \cos 2\pi x, \\ h_2 &= -d - b \cos(2\pi x + \phi), \end{aligned}$$

where $\beta = (\frac{L}{d_1})$ is the dimensionless slip parameter, L is dimensional slip parameter and h_1 and h_2 are the dimensional form of the peristaltic walls. The dimensionless mean flow rates in laboratory (θ) and wave (F) frames are

$$\theta = F + d + 1, \tag{6.18}$$

$$F = \int_{h_2(x)}^{h_1(x)} \frac{\partial \Psi}{\partial y} dy = \Psi(h_1(x)) - \Psi(h_2(x)), \tag{6.19}$$

and dimensionless pressure rise per wavelength is

$$\Delta p_\lambda = \int_0^{2\pi} \left(\frac{dp}{dx} \right) dx. \tag{6.20}$$

6.2 Perturbation solution

Now we seek the perturbation solution. For that one has

$$\Psi = \Psi_0 + We^2 \Psi_1 + O(We^4), \tag{6.21}$$

$$F = F_0 + We^2 F_1 + O(We^4), \tag{6.22}$$

$$p = p_0 + We^2 p_1 + O(We^4), \tag{6.23}$$

$$\theta = \theta_0 + We^2 \theta_1 + O(We^4). \tag{6.24}$$

Making use of the above equations into Eqs. (6.12)-(6.15) and (6.17) and then equating the coefficients of We^2 we obtain

6.2.1 System of order We^0

$$\frac{\partial^4\Psi_0}{\partial y^4}=0, \tag{6.25}$$

$$\frac{\partial p_0}{\partial x}=\frac{\partial^3\Psi_0}{\partial y^3}, \tag{6.26}$$

$$\frac{\partial p_0}{\partial y}=0, \tag{6.27}$$

$$\frac{\partial^2\theta_0'}{\partial y^2}=Br\left(\frac{\partial^2\Psi_0}{\partial y^2}\right)^2, \tag{6.28}$$

$$\begin{aligned}\Psi_0 &= \frac{F_0}{2}, \quad \frac{\partial\Psi_0}{\partial y}=-\beta\frac{\partial^2\Psi_0}{\partial y^2}-1 \quad \theta_0'=0 \quad \text{on} \quad y=h_1(x),\\ \Psi_0 &= -\frac{F_0}{2}, \quad \frac{\partial\Psi_0}{\partial y}=+\beta\frac{\partial^2\Psi_0}{\partial y^2}-1 \quad \theta_0'=1 \quad \text{on} \quad y=h_2(x).\end{aligned} \tag{6.29}$$

6.2.2 System of order We^2

$$\frac{\partial^4\Psi_1}{\partial y^4}+\left(\frac{n-1}{2}\right)\frac{\partial^2}{\partial y^2}\left[\left(\frac{\partial^2\Psi_0}{\partial y^2}\right)^3\right]=0, \tag{6.30}$$

$$\frac{\partial p_1}{\partial x}=\frac{\partial^3\Psi_1}{\partial y^3}+\left(\frac{n-1}{2}\right)\frac{\partial}{\partial y}\left[\left(\frac{\partial^2\Psi_0}{\partial y^2}\right)^3\right], \tag{6.31}$$

$$\frac{\partial p_1}{\partial y}=0, \tag{6.32}$$

$$\frac{\partial^2\theta_1'}{\partial y^2}=Br\left\{\left(\frac{\partial^2\Psi_1}{\partial y^2}\right)^2+\left(\frac{n-1}{2}\right)\left[\frac{\partial^2\Psi_0}{\partial y^2}\right]^4\right\} \tag{6.33}$$

$$\begin{aligned}\Psi_1 &= \frac{F_1}{2}, \ \frac{\partial\Psi_1}{\partial y}=-\beta\left\{\frac{\partial^2\Psi_1}{\partial y^2}+\left(\frac{n-1}{2}\right)\left(\frac{\partial^2\Psi_0}{\partial y^2}\right)^3\right\}, \ \theta_1'=0 \quad \text{on } y=h_1(x),\\ \Psi_1 &= -\frac{F_1}{2}, \ \frac{\partial\Psi_1}{\partial y}=\beta\left\{\frac{\partial^2\Psi_1}{\partial y^2}+\left(\frac{n-1}{2}\right)\left(\frac{\partial^2\Psi_0}{\partial y^2}\right)^3\right\} \ \theta_1'=0=0 \text{ on } y=h_2(x).\end{aligned} \tag{6.34}$$

6.2.3 Solution for system of order We^0

Solving Eq. (6.25)-(6.28) and then using the boundary conditions (6.29) we have

$$\begin{aligned}\Psi_0 &= \frac{1}{(2(h_1-h_2)^2L_6}(-((2y-h_1-h_2)(2(y-h_1)(y-h_2)(h_1-h_2)+F_0(2y^2-h_1^2\\&-2h_1(y+3\beta-2h_2)-h_2(2y-6\beta+h_2))))),\end{aligned} \quad (6.35)$$

$$\frac{dp_0}{dx}=(\frac{1}{(2(h_1-h_2)^2L_6}(3(4F_0+4(h_1-h_2))), \quad (6.36)$$

$$\begin{aligned}\theta_0' &= \frac{1}{(2(h_1-h_2)^2L_6}((y-h_1)(BrF_0^2(y-h_2)(2y^2-2yh_1+h_1^2-2yh_2+h_2^2)+12BrF_0\\&(y-h_2)(h_1-h_2)(2y^2-2yh_1+h_1^2-2yh_2+h_2^2)+(h_1-h_2)^2(12Bry^3+h_1(-12Br\\&y^2+36\beta^2+h_1(6Bry+12\beta+h_1))-3(8Bry^2+12\beta^2+h_1(-4Bry+8\beta+(1+2\\&Br)h_1))h_2+3(6Bry+4\beta+h_1)h_2^2-(1+6Br)h_2^3))).\end{aligned} \quad (6.37)$$

6.2.4 Solution for system of order We^2

By using the zeroth-order solution (6.35) and (6.37) into the first order system and then solving the resulting system along with the corresponding boundary conditions, we find that

$$
\begin{aligned}
\Psi_1 &= \frac{1}{(2(h_1-h_2)^6L_6^4}((2y-h_1-h_2)(-216(-1+n)F_0^3(y-h_1)(y-h_2)(-6y^2\beta+h_1(-y(y-6\\
&\beta)+(y+2\beta)h_1)+(y(y+6\beta)-h_1(10\beta+h_1))h_2+(-y+2\beta+h_1)h_2^2)+648(-1+n)F_0^2\\
&(-y+h_1)(y-h_2)(h_1-h_2)(-6y^2\beta+h_1(-y(y-6\beta)+(y+2\beta)h_1)+(y(y+6\beta)-h_1(10\\
&\beta+h_1))h_2+(-y+2\beta+h_1)h_2^2)+648(-1+n)F_0(-y+h_1)(y-h_2)(h_1-h_2)^2(-6y^2\beta+\\
&h_1(-y(y-6\beta)+(y+2\beta)h_1)+(y(y+6\beta)-h_1(10\beta+h_1))h_2+(-y+2\beta+h_1)h_2^2)+(h_1\\
&-h_2)^3(1296(-1+n)y^4\beta+5F_1h_1^6+10F_1h_1^5(y+12\beta-4h_2)-5F_1h_1^4(2y^2-18y\beta-10\\
&8\beta^2)+6(y+26\beta)h_2-23h_2^3)+h_2(216(-1+n)y(-y^2(y+12\beta)+2y(y+2\beta)h_2(y-2\beta)\\
&h_2^2)+5F_1(6\beta-h_2)^3(2y^2-h_2(2y-6\beta+h_2)))+4h_1^3(54(-1+n)(y-h_2)(y+2\beta-h_2)\\
&+5F_1(9\beta(-y^2+6y\beta+24\beta^2)+h_2(2(y^2-9y\beta-135\beta^2)+(y+87\beta-8h^2)))+2h_1(10\\
&8(-1+n)(y^3(y-12\beta)+28y^2\beta h_2-y(3y+14\beta)h_2^2+2(y-\beta)h_2^3)-5F_1(-6\beta+h_2)^2(6y^2\\
&\beta+h_2(-4(y^2-9\beta^2)+h_2(3y-30\beta+4h_2))))+h_1^2(-216(-1+n)(y-h_2)(2y(y-2\beta)-\\
&h_2(y-10\beta+h_2))+5F_1(6\beta-h_2)(36\beta)(-y^2+2y\beta+6\beta^2)-h_2(-12(y^2-2y\beta-39\beta^2)\\
&+h_2(4y-210\beta+23h_2))))))), \qquad (6.38)
\end{aligned}
$$

$$
\frac{dp_1}{dx}=-\frac{1}{2(h_1-h_2)^6L_6^3}\left[648(-1+n)(2y-h_1-h_2)^2(F_0+h_1-h_2)^3\right], \qquad (6.39)
$$

$$
\begin{aligned}
\theta_1' &= -\frac{1}{175(h_1-h_2)^{12}L_6^8}(Br(n-1)(y-h_1(y-h_2)(F+h_1-h_2)^4(L_{32}+L_{35}\\
&+777600F^2(n-1)y^6\beta^2+12BrF_1(y-h_2)(h_1-h_2)(2y^2-2yh_1+h_1^2\\
&-2yh_2+h_2^2)+2(L_{31}+L_{33})h_1^2+4h_1^3L_{39}+105h_1^{12}-420h_1^{11}(y-6\beta+2\\
&h_2)+2h_1^{10}L_{34}+h_1^8L_{12}-8h_1^7L_{16}-8h_1^5L_{17}+h_1^4L_{18}+4h_1(L_{36}64800F(n\\
&-1)y^5\beta(F(y-9\beta)+6y\beta)-2160(n-1)L_{37}-216y^2((n-1)(F-10y)y^2\\
&(31F+10y)+L_{38}-4(27F^2(n-1)+L_{19})h_2^7+4(81F(n-1)+490y^3+\\
&8190y^2\beta-27y(-23+23n-910\beta^2)+810\beta(-2+2n+35\beta^2)\\
&h_2^8-L_{20}+210h_2^{11}+L_{23}+L_{25}+L_{28}))), \qquad (6.40)
\end{aligned}
$$

Perturbation solutions up to second order for Ψ, dp/dx and θ' may be composed as

$$
\begin{aligned}
\Psi &= \frac{1}{10(h_1-h_2)^6(6\beta+h_1-h_2)^4}((2y-h_1-h_2)(-10(y-h_1)(y-h_2)(h_1-h_2)^5 \\
&+(6\beta+h_1-h_2)^3+We^2(-216F^3(-1+n)(y-h_1)(y-h_2)(-6y^2\beta+h_1(-y(y-6\beta) \\
&+(y+2\beta)h_1)+(y(y+6\beta)-h_1(10\beta+h_1)h_2+(-y+2\beta+h_1)h_2^2)+648F^2(-1+n) \\
&(-y+h_1)(y-h_2)(h_1-h_2)(-6y^2\beta+h_1(-y(y-6\beta)+(y+2\beta)h_1)+(y(y+6\beta)-h_1 \\
&(10\beta+h_1))h_2+(-y+2\beta+h_1)h_2^2)+648F(-1+n)(-y+h_1)(y-h_2)(h_1-h_2)^2 \\
&(-6y^2\beta+h_1(-y(y-6\beta)+(y+2\beta)h_1)+(y(y(y+6\beta)-h_1(10\beta+h_1))h_2+(-y+2\beta \\
&+h_1)h_2^2)+(h_1-h_2)^3(1296(-1+n)y^4\beta+216h_1^3(-1+n)(y-h_1)(y+2\beta-h_2) \\
&+216(-1+n)yh_2(-y^2(y+12\beta)+2y(y+2\beta)h_2-(y-2\beta)h_2^2+216(-1+n)yh_1(y^3 \\
&(y-12\beta)+28y^2\beta h_2-y(3y+14\beta)h_2^2+(y-\beta)h_2^3)-216(-1+n)h_1^2(2y(y-2\beta)-h_2 \\
&(y-10\beta-h_2)))))),
\end{aligned}
\tag{6.41}
$$

$$
\begin{aligned}
\frac{dp}{dx} &= \frac{1}{5(h_1-h_2)^4(6\beta+h_1-h_2)^4}(12(540F^2(-1+n)We^2\beta+(54F(-1+n)We^2(F+20\beta) \\
&+(108(F(-1+n)We^2+5(-1+n)We^2\beta+10\beta^3)+(54(-1+n)We^2+540\beta^2+5(h_1 \\
&-h_2)(18\beta+h_1-h_2))(h_1-h_2))(h_1-h_2))(h_1-h_2))(F+h_1-h_2)),
\end{aligned}
\tag{6.42}
$$

$$\theta' = \frac{1}{175(h_1-h_2)^{12}L_6^8}((108Br(n-1)We^2(y-h_1)(y-h_2)(F+h_1-h_2)^4(L_{43}-4h_1^3(L_{42}$$
$$+216(n-1)y^3(-300y^2(y-9\beta)\beta-10Fy(5y^2-180y\beta+552\beta^2)+F^2(75y^2-976y\beta$$
$$+1020\beta^2))+L_{44}+2h_1^2(L_{41}+2160(n-1)y^4(120Fy(y-9\beta)\beta+180y^2\beta^2+F^2(5y^2$$
$$-180y\beta+552\beta^2))+L_{45}-864y(-F(n-1)y(256y^2+2096y\beta-4620\beta^2)-F^2(n$$
$$-1)+20y^2(21y^2(n-1-10\beta^2)-2y\beta(187-187n+280\beta^2)))h_2^4-72(3F^2(n-1)$$
$$+2y(980y^3\beta+y^2(981-981n+7560\beta^2)))h_2^5+6L_{22}+1645h_2^{10})+h_1^8L_{12}+4320$$
$$(n-1)y^4h_2^2L_{13}+432h_2^6(4F(n-1)y+L_{24}+4y^2L_{14})+4h_1^5L_{15}-8h_1^7L_{16}-8h_1^5L_{17}$$
$$+h_1^4L_{18}-48h_2^7L_{21}-L_{29}-L_{30}-12(9F^2(n-1)L_{26}+18FL_8-4(27F^2(n-1)$$
$$+2(140y^4+5040y^3\beta+270y\beta(17-17n)+140\beta^2)))h_2^7+4(81F(n-1)+490y^3\beta$$
$$+810(-2+2n+35\beta^2))h_2^8-2(108n+805y^2+67y\beta+54(-2+315\beta^2))h_2^9+35(19$$
$$y+126\beta)h_2^{10}-210h_2^{11}+L_{23}+L_{25}+L_{28})))+L_{40}), \tag{6.43}$$

where

$$L_1 = \frac{684(n-1)(F+h_1-h_2^4)}{(h_1-h_2)^8(6\beta+h_1-h_2)^4},$$
$$L_2 = (108F^3(n-1)), \qquad L_3 = (324F^2(n-1)(h_1-h_2))$$
$$L_4 = (324F^2(n-1)(h_1-h_2)^2), \quad L_5 = (60\beta h_1 + 7h_1^2),$$
$$L_6 = (6\beta+h_1+h_2), \quad L_7 = \left(60h_2 - 7h_2^2\right)$$

$$L_8 = (324F(n-1)(y-2\beta)), \quad L_9 = ((-F^2+2Fy+2y^2),$$
$$L_{10} = (F^2+2Fy+4y^2), \quad L_{11} = (2BrF+3Bry2+\beta),$$

$$L_{12} = (8(27F^2(n-1)-L_8+2(35y^2-1680y^3\beta++405\beta^2(4n-4+21\beta^2)+135y^2$$
$$-162y(\beta-n\beta+140\beta^3)))-16(81F(n-1)-490y^3+8190y^2\beta+27y(-23$$
$$+n-910\beta^2)+810\beta(-2+n+35\beta^2))h_2+12(-738+1435y^2-9240y\beta)h_2$$
$$+700(13y-210\beta)h_2^3+17225h_2^4),$$

$$
\begin{aligned}
L_{13} &= (180y^2\beta^2 + 120Fy\beta(y+9\beta) + F^2(5y^2 + 180y\beta + 552\beta^2))), \\
L_{14} &= (15\beta^2(n-1+49\beta^2) + y^2(-43+n+70\beta^2) + 2y\beta(-113+113n+280\beta^2))),
\end{aligned}
$$

$$
\begin{aligned}
L_{15} &= (108(-4F(n-1)y(47y^2 - 73y\beta - 30\beta^2) + F^2(n-1)(25y^2 + 6y\beta + 60\beta^2) \\
&\quad +4y^2(15\beta^2(n-1+49\beta^2) + y^2(43n - 43 + 70\beta^2) - 2y\beta - 2y\beta(-113n \\
&\quad +113 + 280\beta^2))) - 12(9F^2 - 18F(n-1)(107y^2 - 334y\beta - 480\beta^2) + 10y \\
&\quad (196y^3\beta + 9y\beta(169 - 169n + 700\beta^2)))h_2 + 8(243F^2(n-1) + y^2(3267(1 \\
&\quad -n) + 49140\beta^2))h_2^2 - 8(1053F - 81y(23(n-1) + 140\beta^2) + 135\beta(79n(n \\
&\quad -1) + 560\beta^2))h_2^3 + 490(y - 186\beta)h_2^5 + 7595h_2^6),
\end{aligned}
$$

$$
\begin{aligned}
L_{16} &= (6(27F^2(n-1)(y-2\beta) - 18F(n-1)(25y^2 + 6y\beta + 60\beta^2) + 2y(-140y^3 \\
&\quad -9y\beta(-73(1-n) + 980\beta^2))) + 2(27F^2(n-1) + 2(140y^4 - 5040y^3\beta \\
&\quad -27y^2 - 270y\beta(17(1-n) + 140\beta^2)))h_2 - 12(-261 - 1120y\beta + 9450\beta^2 \\
&\quad +245y^2)h_2^3 + 3290h_2^5),
\end{aligned}
$$

$$
\begin{aligned}
L_{17} &= (180y(F^2(n-1)(47y^2 - 73y\beta - 30\beta^2) - 4F(n-1)y(43y^2 - 226y\beta + 15 \\
&\quad \beta^2) + y^2(75(n-1)y^2 + 60\beta^2(17n - 17 + 28\beta^2) - 16y\beta(16n - 16n + 35 \\
&\quad \beta^2))) - 108(2F^2(n-1)(33y^2 - 82y\beta - 105\beta^2) - 4Fy^2(-180\beta^2(16n \\
&\quad -16 + 21\beta^2) + 8y\beta(149 - 149n + 560\beta^2)))h_2 + 18(3F^2 - 2y(980y^3\beta \\
&\quad +27y\beta(83 - 83n + 420\beta^2)))h_2^2 + 2(567F^2(n-1) - 2y(980y^3\beta + 27y \\
&\quad \beta(83 - 83n + 420\beta^2)))h_2^2 + 189\beta(49 - 49n + 590\beta^2))h_2^4 + 42(35y^2 \\
&\quad +72n - 72)h_2^5 + 3290h_2^7),
\end{aligned}
$$

$$L_{18} = (864y^2(2F^2(n-1)(43y^2-226y\beta+15\beta^2)-2F(n-1)y(75y^2-976y\beta
+1020\beta^2)+5y^2(5(n-1)y^2-180(n-1)y\beta+24\beta^2(-23+23n+7\beta^2
+y\beta)))+864y(-2F+5y^2(45(n-1)y^2+36\beta^2(33-33n+28\beta^2)-4
y\beta(-19+19n+140\beta^2)))h_2-432(-4F(n-1)y-20y^2(21y^2(n-1-10
\beta^2)-9\beta^2(19(n-1)+21\beta^2)+2y\beta(187-187n+280\beta^2)))h_2^2+48(63
F^2(n-1)(11y-18\beta)-10y(980y^3\beta+9y^2(53n-53-280\beta^2)+9y\beta
(-499+499n+140\beta^2)))+10(245y^4+1701\beta^2(4-4n+11\beta^2)+27
y^2(61n-61+70\beta^2)))h_2^4-32(756F(n-1)+189\beta(-49+49n+590
\beta^2))h_2^5+24(-1206+735y^2+62370\beta^2)h_2^6-280(23y+954\beta)h_2^7
+17255h_2^8),$$

$$L_{19} = 48(27F^2(n-1)(25y^2-6y\beta+60\beta^2)+2y(140y^3\beta+270\beta^2(1-n+21
\beta^2)+9y^2(47n-47+280\beta^2)+9y\beta(73n-73+980\beta^2)))h_2^7+8(27F^2
(n-1)+405\beta^2(4n-4+21\beta^2)+135y^2(5n-5+98\beta^2)+162y(\beta(n
-1+140\beta^2))))h_2^8-16(27F(n-1)+70y^3+14070y^2\beta+81y(n-1
+70\beta^2))h_2^9+2h_1^{10}(2(-54+54n+245y^2-2520y\beta+5670\beta^2)+1645
h_2^2)+4h_1^9(4(27F(n-1)-70y^3+1470y^2\beta+162\beta(n-1+35\beta^2)-81
y(n-1+70\beta^2))-35(49y-438\beta)h_2^2-2170h_2^3),$$

$$
\begin{aligned}
L_{20} &= 48(27F^2(n-1)(y+2\beta)+18F(n-1)(25y^2-6y\beta+60\beta^2)+2y(140y^3\\
&\quad \beta+270\beta^2(1-n+21\beta^2)+9y\beta(-73+73n+980\beta^2)))h_2^7+8(27F^2(n\\
&\quad -1)+2(34y^4+1680y^3\beta+405\beta^2(4n-4+21\beta^2)+135y^2(-5+5n\\
&\quad +98\beta^2+162y(\beta-n\beta+140\beta^3)))h_2^8-16(27F(n-1)+81y(n-1+70\\
&\quad \beta^2))h_2^9+2h_1^{10}(2(-54+54n+245y^2-2520y\beta+5670\beta^2)+70(-126\beta\\
&\quad +19y)h_2+1645h_2^2)+4h_1^9(4(27F(n-1)-70y^3-81y(n-1+70\beta^2))\\
&\quad -2(108n-6720y\beta+54(-2+315\beta^2))h_2-35(49y-438\beta)h_2^2-2170\\
&\quad h_2^3),
\end{aligned}
$$

$$
\begin{aligned}
L_{21} &= (27F^2(n-1)(y+2\beta)+18F(n-1)(25y^2-6y\beta+60\beta^2)+2y(140y^3\beta\\
&\quad +270\beta^2(1-n+21\beta^2)+9y^2(47n-47+280\beta^2)+9y\beta(-73+73n+\\
&\quad 980\beta^2))),\\
L_{22} &= (2160(n-1)y^4(120Fy^2\beta+180y^2\beta^2+F^2(5y^2-798\beta^2))h_2),\\
L_{23} &= (648(n-1)y^3(300y^2\beta(y+3\beta)+F^2(25y^2-524y\beta-2660\beta^2)+10Fy\\
&\quad (5y^2+60y\beta-348\beta^2))h_2^2),\\
L_{24} &= 216y^2(4F(n-1)y(75y^2-298y\beta-3480\beta^2)-F^2(n-1)(31y^2+2096\\
&\quad y\beta+2940\beta^2)+20y^2(5(n-1)y^2+90(n-1)y\beta+3\beta^2(41-41n+56\\
&\quad \beta^2)))h_2^3,\\
L_{25} &= (216(2F^2(n-1)(33y^2+82y\beta-105\beta^2)+4F(n-1)y(31y^2+459y\beta\\
&\quad +120\beta^2)-y^2(180\beta^2(16-16n+21\beta^2)+3y^2(-47+47n+280\beta^2)\\
&\quad +8y\beta(149+560\beta^2)))),
\end{aligned}
$$

$$
\begin{aligned}
L_{26} &= (4(27F^2(n-1)-108F(n-1)(13y-27\beta)+2(140y^4+5040y^3\beta-27\\
&\quad y^2(-41+41n+1120\beta^2)+2430\beta^2(-3+3n+7\beta^2)+270y\beta(17-17n\\
&\quad +140\beta^2)))h_2^7),
\end{aligned}
$$

$$
\begin{aligned}
L_{27} &= (2160(n-1)y^4(120Fy^2\beta+180y^2\beta^2+F^2(5y^2-798\beta^2))h_2),\\
L_{28} &= (216y(2F(n-1)y(55y^2-1644y\beta-2910\beta^2)-F^2(n-1)(109y^2+991\\
&\quad y\beta+210\beta^2)+5y^2(45+4y\beta(-19+19n+140\beta^2)))h_2^4),\\
L_{29} &= (16(513F(n-1)+5(140y^3+1764y^2\beta-27y(n-1-140\beta^2)+27\beta\\
&\quad (-41+41n+280\beta^2)))h_2^7),
\end{aligned}
$$

$$
\begin{aligned}
L_{30} &= (72(3F^2(n-1)(45y+308\beta)-18F(n-1)(45y^2-278y\beta-560\beta^2)+2\\
&\quad y(980y^3\beta+270\beta^2(23n-23+7\beta^2)+27y\beta(83-83n+420\beta^2)+y^2(91\\
&\quad -981n+7560\beta^2)))h_2^5),
\end{aligned}
$$

$$
\begin{aligned}
L_{31} &= 1296(n-1)y^3(-300y^2(y-3\beta)\beta+F^2(25y^2+524y\beta-2660\beta^2)+10Fy\\
&\quad (-5y^2+60y\beta+384\beta^2))h_2(2592y^2((n-1)y^2(-39F^2+25y^2)-1048F\\
&\quad (n-1)y^2\beta+60(n-1)(21F^2-29y^2)\beta^2+840y^2\beta^4)+h_2(864y((n-1)\\
&\quad y^2(78F^2-75y^2)-2800y^3\beta^3-1680y^2\beta^4)+h_2(216(F^2+20y^2(-374(n\\
&\quad +1)y\beta-21y^2(n-1-10\beta)))+h_2(72(18F(n-1)+2y(-980y^3\beta+9y^2\\
&\quad (109(n-1)-840\beta^2)-27y\beta(83-83n+420\beta^2)))+h_2(16(243F^2+y^2\\
&\quad (-3267+49140\beta^2))+h_2(-16(513F(n-1)+5(140y^3-27y(n-1-14\\
&\quad \beta^2)27\beta(41n-41+280\beta^2)))+h_2(6(-738+35(41y^2+990\beta^2))+35h_2\\
&\quad (-98y+47h_2))))))))),
\end{aligned}
$$

$$L_{32} = h_2(-259200F(n-1)y^5\beta(6y\beta+F(y+9\beta))+h_2(-864(n-1)y^3(300y^2\beta$$
$$(y+9\beta)+10Fy(5y^2+180y\beta+552\beta^2)+F^2(75y^2+976y\beta+1020\beta^2))$$
$$h_2+864y^2((n-1)y^2L_1+4(n-1)yL_2+L_3+840y^2\beta^4)h_2^2-864y^3((n$$
$$-1)y^2(47F^2+172Fy+75y^2)+L_4)h_2^3+8(27F^2(n-1)+324F(n-1)$$
$$(y+2\beta)+2(35y^4+1680y^3\beta+405\beta^2(-4+4n+21\beta^2)+L_5))h_2^6+L_7$$
$$-420(y+6\beta)h_2^9+105h_2^{10}+4320(n-1)y^4L_{13}-432h_2^4(F(n-1)(y^2(25$$
$$F+188y)-2(3F-146y)y\beta+60(F-2y)\beta^2)+4y^2L_{14})-48h_2^5L_{21}))),$$

$$L_{33} = (2160(n-1)y^4(120Fy(y-9\beta)\beta+180y^2\beta^2+F^2(5y^2-180y\beta+552\beta^2$$
$$))),$$

$$L_{34}=(2(-54+54n+35(7y^2-72y\beta+162\beta^2))+35h_2(38y-252\beta+47h_2)),$$

$$L_{35}=(10Fy(5y^2-180y\beta+552\beta^2)+F^2(75y^2-976y\beta+1020\beta^2)),$$

$$L_{36}=(F^2(n-1)(78y^2+584y\beta-1260\beta^2)),$$

$$L_{37} = (120Fy^2\beta+180y^2\beta^2+F^2(5y^2-798\beta^2))h_2-216y^2((n-1)(F-10y)y^2$$
$$(31F+10y)),$$

$$L_{38} = 8(n-1)y(-262F^2-149Fy+225y^2)\beta-60(n-1)(49F^2+232Fy+41y^2)$$
$$\beta^2+3360y^2\beta^4)h_2^3,$$

$$L_{39} = (54y^2(41-41n-1120\beta^2)+4860\beta^2(-3+3n+7\beta^2)+540y\beta(17-17n$$
$$+140\beta^2)),$$

$$L_{40} = (\frac{-1}{(h_1-h_2)^4L_6^2}((y-h_1)(12BrF^2y^3+h_1(-12BrF(F-2y)y^2+h_1(6Bry(F^2 -4Fy+2y^2)+h_1(12Br+h_1(6Bry+12\beta+h_1))))+2(3Bry(F^2-4Fy+2 y^2)+h_1(12Br(F-y)y+h_1(6Bry+12\beta+h_1))))+2(3Bry(3F^2+8Fy+2 y^2)+h_1(6Bry(F+3y)+54\beta^2+6(BrF+6\beta)h_1+(5+6Br)h_1^2))h_2^2-(1+6 \beta)h_2^5-h_2(24BrFy^2(F+y)+h_1(h_1(6BrF(F-2y)+108\beta^2+12(Br+4\beta)h_1 +h_1^2)))+6L_{11})),$$

$$L_{41} = (((-300y^2(y-9\beta)\beta-L_{35}+216y^2(-4F(n-1)y(75y^2+289y\beta)+20y^2(5y^2 +3\beta^2(41-41n+56\beta^2)))h_2+432y(F(n-1)+(265y^2-2096\beta^2))+432 y^2(77F^2-5(y^2(-53+53n-280\beta^2)+84\beta^2(11-11n+\beta^2)))h_2^3),$$

$$L_{42} = (16(513F(n-1)+5(140y^3+1764y^2\beta-27y(-1+n-140\beta^2)+27\beta(-41 +41n+280\beta^2)))),$$

$$L_{43} = (16(243F^2(n-1)+490y^4+11760y^3\beta+810\beta^2(-36+36n+49\beta^2) (3267-3267n+49140\beta^2))h_2^6-L_{42}h_2^7)),$$

$$L_{44} = 4(567F^2+2(980y^4+54y\beta(-499+499n+140\beta^2)+135y^2(-17+17n +244\beta^2)))h_2^5,$$

$$L_{45} = 1296(n-1)y^3(-300y^2+10Fy(-5y^2+60y\beta+384\beta^2))h_2+2592y^2(-1048 F(n-1)y^2\beta-3F^2(n-1)(13y^2-420\beta^2)+5y^2(5(n-1)y^2+12\beta^2(29-29 n+14\beta^2)))h_2^2.$$

6.3 Discussion

The main concern in this section is to study the phenomena of pumping, trapping and frictional forces. First the pumping characteristic against pressure rise per wavelength Δp_λ is discussed. Δp_λ is denoted by p_0 at zero volume flow rate (θ). When $\Delta p_\lambda > p_0$, then we have negative flux. The regions $\Delta p_\lambda = 0$, $\Delta p_\lambda > 0$ and $\Delta p_\lambda < 0$ correspond to free pumping, peristaltic pumping and copumping regions respectively. For the analysis of pumping phenomenon, numerical integration is performed and results are plotted. The effects of slip parameter (β) on Δp_λ is shown in Figure 6.1. This figure concludes that peristaltic pumping rate increases by increasing β and ϕ. The results of no-slip condition can be deduced when $\beta = 0$.

Our next interest is to examine the trapping phenomenon. In the wave frame, the streamlines are plotted for an asymmetric channel. To see the effects of physical parameters of interest, we have shown the Figures 6.2 and 6.3. Figures 6.2 (a,b) describe the variations of slip parameter β on trapping. This Fig. indicates that with an increase in β, the upper bolus is going to expand and the lower one is going to squeeze. The dimensionless flow rate θ is displayed in Figures 6.3 (a, b). Here size of the upper bolus increases and of the lower bolus decreases when θ is increased.

Temperature (θ') for different Brinkman number (Br), phase difference (ϕ) and volume flow rate (θ) is sketched in the Figures (6.4-6.6). Figure 6.4 depicts the effects of Br on temperature. In this figure, temperature increases when Br increases. The temperature profile is almost parabolic. Figure 6.5 describes the influence of ϕ on temperature. Clearly temperature decreases when ϕ increases. It is noted that temperature increases much in symmetric channel. The effects of θ on temperature (θ') is displayed in Figure 6.6. Clearly temperature increases with an increase in θ and temperature profile looks parabolic.

Figures 6.7 and 6.8 explain the frictional forces F_λ for various values of slip parameter β at the channel walls. The frictional forces versus mean flow rate θ at the upper wall of the channel can be presented in Figure 6.7. In this figure, there is critical value of θ $(0 < \theta < 0.5)$ for which F_λ resists the flow along the channel wall. Above (below) this critical point F_λ decreases (increases) for large values of β. Figure 6.8 is plotted for the variation of β on the lower wall of the channel and its behaviour is quite similar to that in Figure 6.7.

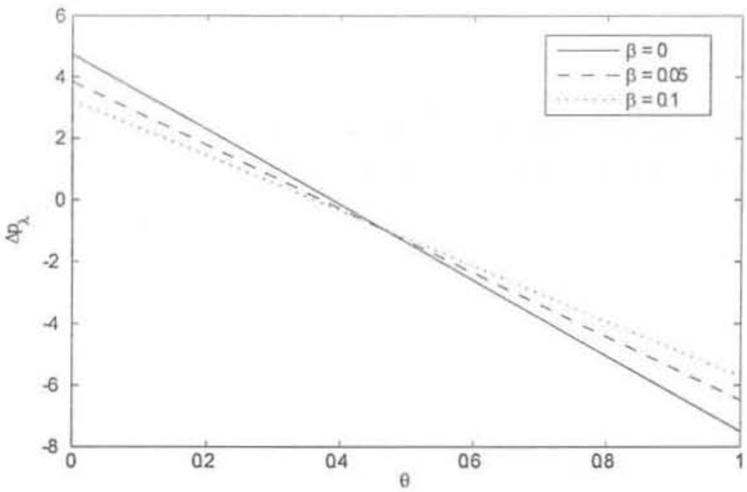

Figure 6.1: Plot showing Δp_λ versus flow rate θ. Here $n = 0.398$, $We = 0.01$, $\phi = \frac{\pi}{6}$, $a = b = 0.4$ and $d = 1.1$.

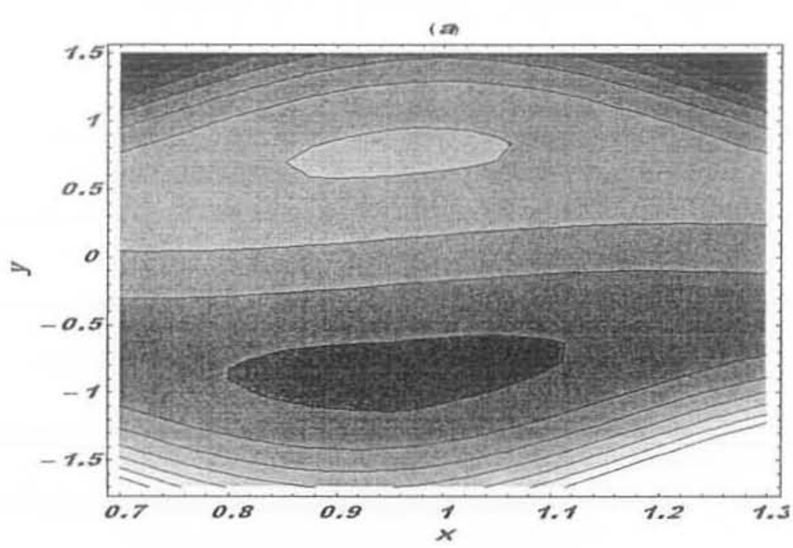

Figure 6.2a: Streamlines for $\beta = 0.03$. The other parameters are $n = 0.398$, $We = 0.03$, $\theta = 1.5$, $\phi = \frac{\pi}{6}$, $a = b = 0.4$ and $d = 1.1$.

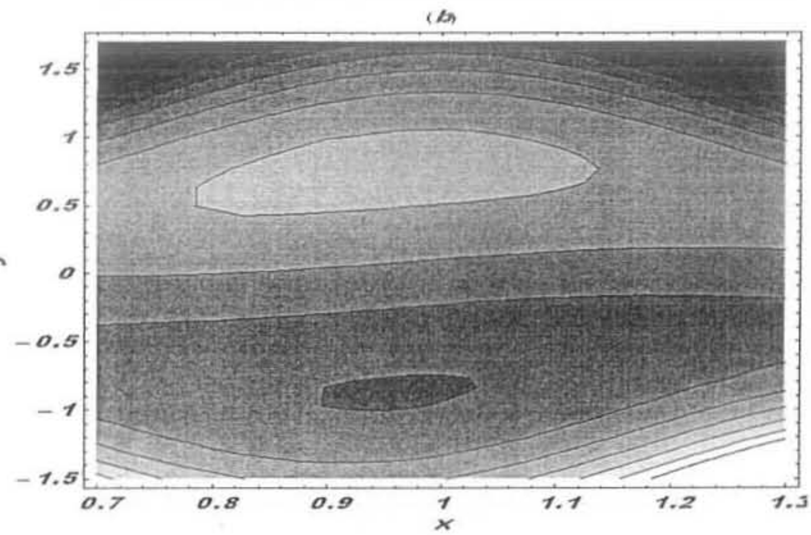

Figure 6.2b: Streamlines for $\beta = 0.05$. The other parameters are $n = 0.398$, $We = 0.03$, $\theta = 1.5$, $\phi = \frac{\pi}{6}$, $a = b = 0.4$ and $d = 1.1$.

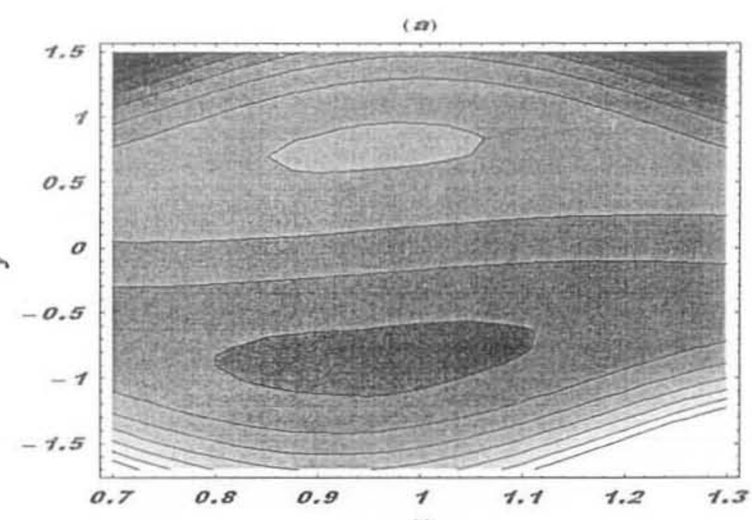

Figure 6.3a: Streamlines for $\theta = 1$. The other parameters are $n = 0.398$, $We = 0.03$, $\beta = 0.03$, $\phi = \frac{\pi}{6}$, $a = b = 0.4$ and $d = 1.1$.

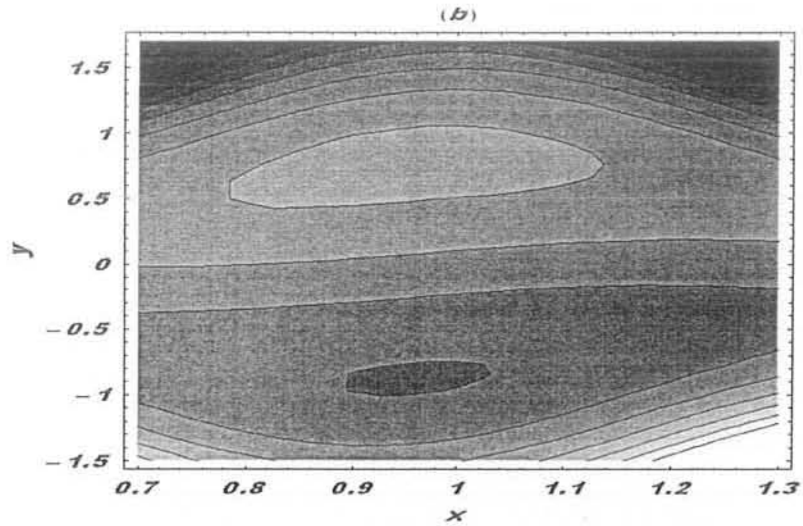

Figure 6.3b: Streamlines for $\theta = 2$. The other parameters are $n = 0.398$, $We = 0.03$, $\beta = 0.03$, $\phi = \frac{\pi}{6}$, $a = b = 0.4$ and $d = 1.1$.

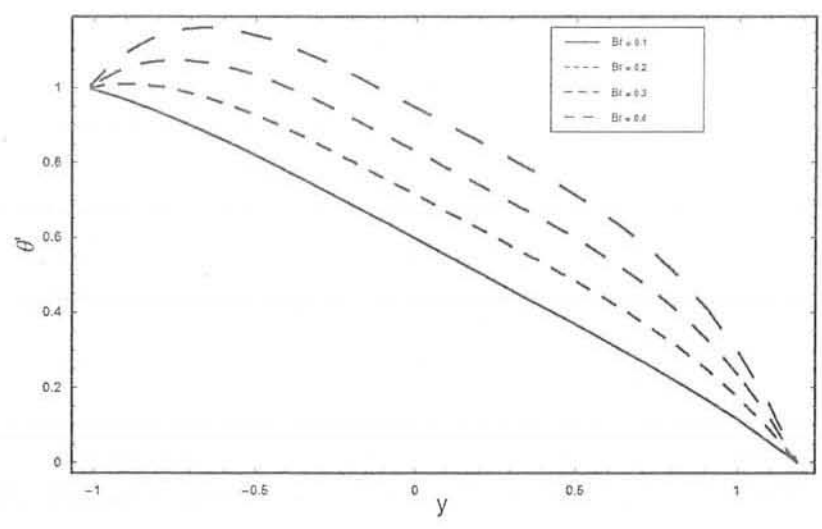

Figure 6.4: Temperature distribution versus y for different values of Br. Here $n = 0.398$, $We = 0.01$, $a = 0.6$, $b = 0.3$, $d = 1.1$, $\theta = 2$, $\phi = 0.6$, and $x = 0.1$.

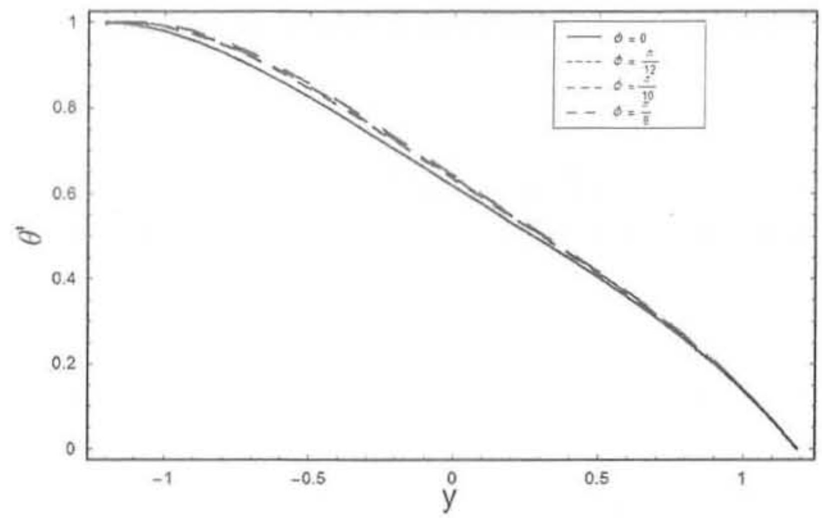

Figure 6.5: Temperature distribution versus y for different values of ϕ. Here $n = 0.398$, $We = 0.01$, $a = 0.6$, $b = 0.3$, $d = 1.1$, $\theta = 2$, $Br = 0.2$, and $x = 0.1$.

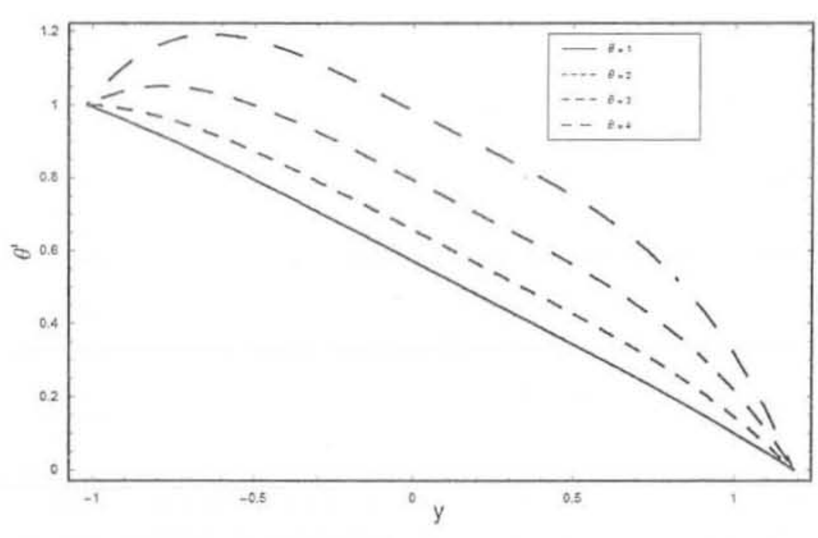

Figure 6.6: Temperature distribution versus y for different values of θ. Here $n = 0.398$, $We = 0.01$, $a = 0.6$, $b = 0.3$, $d = 1.1$, $\phi = 0.6$, $Br = 0.2$, and $x = 0.1$.

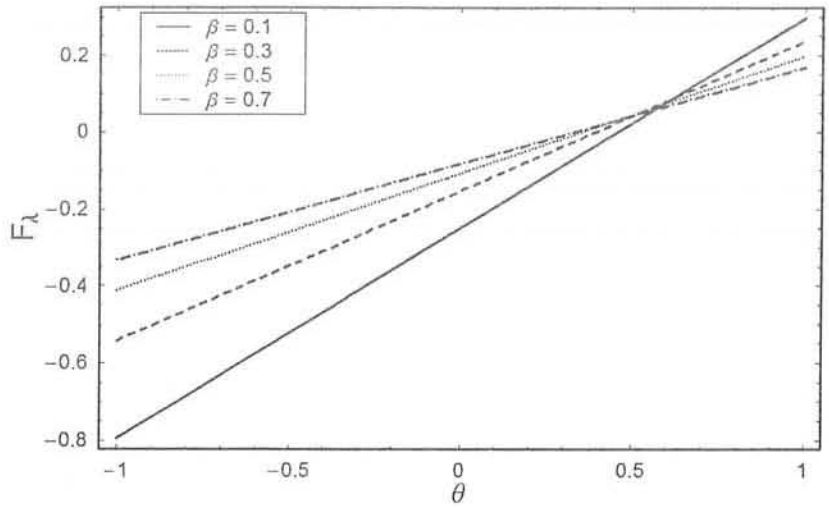

Figure 6.7: Effect of β on variation of F_λ with θ at the upper wall of the channel. Here $n = 0.398$, $We = 0.01$, $a = 0.7$, $b = 1.2$, $d = 2$, and $\phi = \frac{3\pi}{2}$.

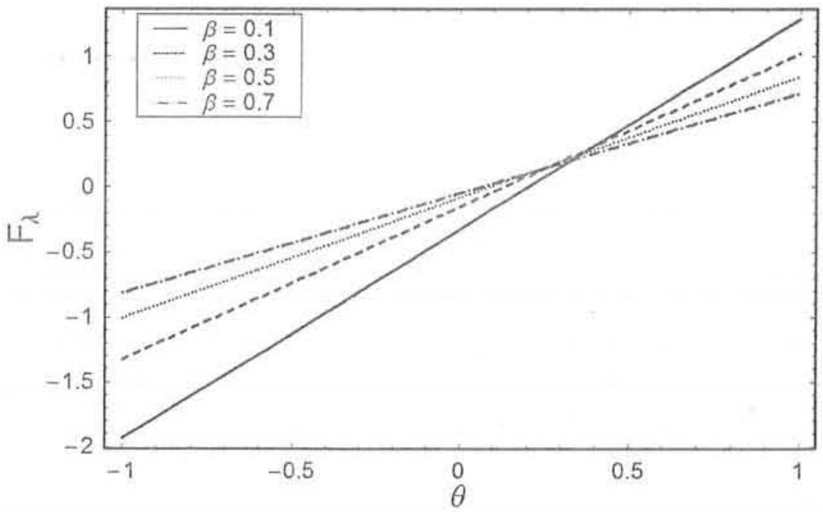

Figure 6.8: Effect of β on variation of F_λ with θ at the lower wall of the channel. Here $n = 0.398$, $We = 0.01$, $a = 0.7$, $b = 1.2$, $d = 2$, and $\phi = \frac{3\pi}{2}$.

Chapter 7

Peristaltic flow of a second order fluid in the presence of an induced magnetic field

This chapter studies the effect of an induced magnetic field on peristaltic flow of an incompressible conducting second order fluid in a symmetric and planar channel. The flow analysis is discussed by taking into account the small wave number. The series solutions have been established for the pressure gradient, stream function, magnetic force function, axial induced magnetic field and current distribution across the channel. The effects of important parameters on the pressure rise per wavelength have been investigated by means of numerical integration. We further study the effect of embedded flow parameters on the magnetic force function, the axial induced magnetic field and the current distribution across the channel. The phenomena of trapping and pumping are also examined.

7.1 Basic equations

We consider the magnetohydrodynamic flow of an incompressible second order fluid in a two dimensional channel of uniform thickness. We assume that the sinusoidal waves are travelling down its wall. A wave of small amplitude b propagates along its walls with constant speed c.

Its instantaneous height at any axial station $X^{'}$ is

$$Y' = H\left(\frac{X' - ct'}{\lambda}\right). \tag{7.1}$$

It is further noticed that flow in laboratory (X', Y') and wave (x', y') frames are treated unsteady and steady, respectively. The transformations between the two frames are

$$x' = X' - ct', \quad y' = Y',$$
$$u'(x', y') = U' - c, \quad v'(x', y') = V', \tag{7.2}$$

in which (U', V') and (u', v') are the respective velocities in the laboratory and wave frames.

The system is stressed by an external transverse uniform magnetic field of strength $H_0^{'}$ which gives rise to an induced magnetic field $H^{'}(h_{X'}^{'}(X^{'}, Y^{'}, t^{'}),\ h_{Y'}^{'}(X^{'}, Y^{'}, t^{'}),\ 0)$ and the total magnetic field therefore is $H^{'+}(h_{X'}^{'}(X^{'}, Y^{'}, t^{'}),\ H_0^{'} + h_{Y'}^{'}(X^{'}, Y^{'}, t^{'}),\ 0)$. Moreover, the walls of the channel are non conductive.

The Cauchy stress tensor (T) in second order fluid is

$$\mathbf{T} = -\overline{p}\mathbf{I} + \mu\overline{\mathbf{A}}_1 + \alpha_1\overline{\mathbf{A}}_2 + \alpha_2\overline{\mathbf{A}}_1^2 \tag{7.3}$$

with the Rivlin-Ericksen tensors in the forms

$$\overline{\mathbf{A}}_1 = (\operatorname{grad}\overline{\mathbf{V}}) + (\operatorname{grad}\overline{\mathbf{V}})^T, \tag{7.4}$$

$$\overline{\mathbf{A}}_2 = \frac{d\overline{\mathbf{A}}_1}{dt} + \overline{\mathbf{A}}_1(\operatorname{grad}\overline{\mathbf{V}}) + (\operatorname{grad}\overline{\mathbf{V}})^T\overline{\mathbf{A}}_1. \tag{7.5}$$

The fundamental equations of magnetohydrodynamic fluid are:

(i) Maxwell's equations

$$\nabla \cdot \mathbf{H}^{'} = 0, \quad \nabla \cdot \mathbf{E}^{'} = 0, \tag{7.6}$$

$$\nabla \wedge \mathbf{H}^{'} = \mathbf{J}^{'}, \quad \mathbf{J}^{'} = \sigma\left\{\mathbf{E}^{'} + \mu_e\left(\mathbf{V}^{'} \wedge \mathbf{H}^{'+}\right)\right\}, \tag{7.7}$$

$$\nabla \wedge \mathbf{E}^{'} = -\mu_e \frac{\partial \mathbf{H}^{'}}{\partial t^{'}}. \tag{7.8}$$

(*ii*) The continuity equation

$$\nabla \cdot \overline{\mathbf{V'}} = 0. \tag{7.9}$$

(*iii*) The equation of motion is

$$\rho \left[\frac{\partial}{\partial t'} + (\mathbf{V}' \cdot \nabla) \right] \overline{\mathbf{V'}} = -\nabla p' + \operatorname{div} \overline{\tau'} - \mu_e \left\{ (\mathbf{H}^{'+} \cdot \nabla) - \frac{1}{2} \left(H^{'+} \right)^2 \nabla \right\}, \tag{7.10}$$

where p' is the fluid pressure, ρ the fluid density, μ the coefficient of viscosity, α_1 and α_2 are the material constants, $\mathbf{J}'$ the current density, μ_e the magnetic permeability, σ the electrical conductivity, $\overline{\tau'}$ an extra stress tensor, $\mathbf{E}'$ an electric field and the velocity $\overline{\mathbf{V'}}$ is given by

$$\overline{\mathbf{V'}} = \left(u', v', 0 \right). \tag{7.11}$$

From equations (7.6)-(7.8), we obtain the following induction equation

$$\frac{\partial \mathbf{H}^{'+}}{\partial t'} = \nabla \Lambda \left\{ \mathbf{V}' \Lambda \mathbf{H}^{'+} \right\} + \frac{1}{\zeta} \nabla^2 \mathbf{H}^{'+}, \tag{7.12}$$

where $\zeta = \frac{1}{\sigma \mu_e}$ indicates the magnetic diffusivity.

In view of the following dimensionless variables

$$\begin{aligned}
x &= \frac{2\pi x'}{\lambda}, \quad y = \frac{y'}{a}, \quad u = \frac{u'}{c}, \quad v = \frac{v'}{c}, \quad t = \frac{2\pi t' c}{\lambda}, \quad p = \frac{2\pi a^2 p'}{c \lambda \mu}, \\
\tau &= \frac{a \tau'}{\mu c}, \quad h = \frac{h'}{a}, \quad \Psi = \frac{\Psi'}{ca}, \quad \Phi = \frac{\Phi'}{H_0 a}, \quad \delta = \frac{a}{\lambda}, \quad \mathrm{Re} = \frac{\rho c a}{\mu}, \\
R_m &= \sigma \mu_e a c, \quad S_t = \frac{H_0}{c} \sqrt{\frac{\mu_e}{\rho}}, \quad \omega = \frac{a \overline{\omega}}{c}, \; u = \frac{\partial \Psi}{\partial y}, \quad v = -\delta \frac{\partial \Psi}{\partial x}, \\
h_x &= \frac{\partial \Phi}{\partial y}, \quad h_y = -\delta \frac{\partial \Phi}{\partial x}, \quad \mathrm{Re}\, R_m S_t^2 = M^2, \quad p_m = p + \frac{1}{2} \mathrm{Re}\, \delta \frac{\mu_e \left(H^{'+} \right)^2}{\rho c^2}, \\
\lambda_1 &= \frac{\alpha_1 c}{\mu a}, \; \lambda_2 = \frac{\alpha_2 c}{\mu a}
\end{aligned} \tag{7.13}$$

equations (7.3)-(7.10) admit the following forms

$$\mathrm{Re}\,\delta\left\{\left(\Psi_y\frac{\partial}{\partial x}-\Psi_x\frac{\partial}{\partial y}\right)\Psi_y\right\}=-\frac{\partial p_m}{\partial x}+\delta\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}$$
$$+\,\mathrm{Re}\,\delta S_t^2\left\{\left(\Phi_y\frac{\partial}{\partial x}-\Phi_x\frac{\partial}{\partial y}\right)\Phi_y\right\}+\mathrm{Re}\,S_t^2\Phi_{yy},\qquad(7.14)$$

$$\mathrm{Re}\,\delta^3\left\{\left(\Psi_y\frac{\partial}{\partial x}-\Psi_x\frac{\partial}{\partial y}\right)\Psi_x\right\}=-\frac{\partial p_m}{\partial y}+\delta^2\frac{\partial\tau_{xy}}{\partial x}+\delta\frac{\partial\tau_{yy}}{\partial x}$$
$$-\,\mathrm{Re}\,\delta^3 S_t^2\left\{\left(\Phi_y\frac{\partial}{\partial x}-\Phi_x\frac{\partial}{\partial y}\right)\Phi_x\right\}-\mathrm{Re}\,\delta^2 S_t^2\Phi_{xy},\qquad(7.15)$$

$$\Psi_y-\delta\left(\Psi_y\Phi_x-\Psi_x\Phi_y\right)+\frac{1}{R_m}\nabla^2\Phi=E,\qquad(7.16)$$

where δ is the wave number and

$$\begin{aligned}\tau_{xx} &= \delta\frac{\partial^2\Psi}{\partial x\partial y}+\lambda_1\{2\delta^2\frac{\partial^3\Psi}{\partial x^2\partial y}+2\delta^2\frac{\partial^3\Psi}{\partial x\partial y^2}+4\delta^2\left(\frac{\partial^2\Psi}{\partial x\partial y}\right)^2+2\delta^4\left(\frac{\partial^2\Psi}{\partial x^2}\right)^2\\ &\quad -\delta^2\frac{\partial^2\Psi}{\partial x^2}\frac{\partial^2\Psi}{\partial y^2}\}+\lambda_2\{4\delta^4\left(\frac{\partial^2\Psi}{\partial x\partial y}\right)^2+4\left(\frac{\partial^2\Psi}{\partial y^2}\right)^2+\delta^4\left(\frac{\partial^2\Psi}{\partial x^2}\right)^2\\ &\quad -\delta\frac{\partial^2\Psi}{\partial y^2}\frac{\partial^2\Psi}{\partial x^2}\},\end{aligned}$$

$$\begin{aligned}\tau_{xy} &= \left(\frac{\partial^2\Psi}{\partial y^2}-\delta^2\frac{\partial^2\Psi}{\partial x^2}\right)+\lambda_1\{\delta\frac{\partial^3\Psi}{\partial x\partial y^2}-\frac{\partial^3\Psi}{\partial y^3}-\delta^3\frac{\partial^3\Psi}{\partial x^3}-\delta^2\frac{\partial^3\Psi}{\partial x^2\partial y}\\ &\quad +\delta\frac{\partial^2\Psi}{\partial x\partial y}\frac{\partial^2\Psi}{\partial y^2}+2\delta^3\frac{\partial^2\Psi}{\partial x^2}\frac{\partial^2\Psi}{\partial x\partial y}\},\end{aligned}$$

$$\begin{aligned}\tau_{yy} &= -2\delta\frac{\partial^2\Psi}{\partial x\partial y}+\lambda_1\{-2\delta^2\frac{\partial^3\Psi}{\partial x^2\partial y}-2\delta\frac{\partial^3\Psi}{\partial x\partial y^2}+4\delta^2\left(\frac{\partial^2\Psi}{\partial x\partial y}\right)^2+2\left(\frac{\partial^2\Psi}{\partial y^2}\right)^2\\ &\quad -2\delta^2\frac{\partial^2\Psi}{\partial x^2}\frac{\partial^2\Psi}{\partial y^2}\}+\lambda_2\{4\delta^2\left(\frac{\partial^2\Psi}{\partial x\partial y}\right)^2+\left(\frac{\partial^2\Psi}{\partial y^2}\right)^2-\delta^4\left(\frac{\partial^2\Psi}{\partial x^2}\right)^2\\ &\quad -\delta\frac{\partial^2\Psi}{\partial x^2}\frac{\partial^2\Psi}{\partial y^2}\}.\end{aligned}$$

Eliminating the pressure from equations (7.14) and (7.15), we have

$$\operatorname{Re}\delta\left\{\left(\Psi_y\frac{\partial}{\partial x}-\Psi_x\frac{\partial}{\partial y}\right)\nabla^2\Psi\right\}=\nabla^4\Psi+\lambda_1\delta\left\{\left(\Psi_y\frac{\partial}{\partial x}-\Psi_x\frac{\partial}{\partial y}\right)\nabla^4\Psi\right\}$$
$$+\operatorname{Re}S_t^2\delta\left\{\left(\Phi_y\frac{\partial}{\partial x}-\phi_x\frac{\partial}{\partial y}\right)\nabla^2\Phi\right\}+\operatorname{Re}S_t^2\nabla^2\Phi_y, \quad (7.17)$$

where

$$\nabla^2=\delta^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},\ \nabla^4=\nabla^2\nabla^2. \quad (7.18)$$

In a wave frame, the dimensionless boundary conditions and pressure rise per wavelength Δp_λ are

$$\Psi=0,\quad \frac{\partial^2\Psi}{\partial y^2}=0,\quad \frac{\partial\Phi}{\partial y}=0\ \text{ at } y=0,$$
$$\Psi=F,\quad \frac{\partial\Psi}{\partial y}=-1,\quad \Phi=0\ \text{ at } y=h, \quad (7.19)$$

$$\Delta p_\lambda=\int_0^{2\pi}\left(\frac{dp}{dx}\right)dx. \quad (7.20)$$

The dimensionless mean flows in the laboratory (θ) and wave (F) frames are related by the following expressions

$$\theta=F+1, \quad (7.21)$$
$$F=\int_0^h\frac{\partial\Psi}{\partial y}dy, \quad (7.22)$$

in which

$$h(x)=1+\ \phi\sin 2\pi x, \quad (7.23)$$

and $\phi(=b/a)$ is the amplitude ratio.

7.2 Perturbation solution

In order to proceed for the series solutions, we express the flow quantities in terms of small wave number as

$$\Psi = \Psi_0 + \delta\Psi_1 + O(\delta)^2, \tag{7.24}$$

$$F = F_0 + \delta F_1 + O(\delta)^2, \tag{7.25}$$

$$p = p_0 + \delta p_1 + O(\delta)^2, \tag{7.26}$$

$$\Phi = \Phi_0 + \delta\Phi_1 + O(\delta)^2. \tag{7.27}$$

Invoking equations (7.24)-(7.27) into equations (7.14)-(7.16), (7.18) and (7.19) and then comparing the coefficients of like powers of δ, we get

7.2.1 Zeroth order system

$$\frac{\partial^4\Psi_0}{\partial y^4} - M^2\frac{\partial^2\Psi_0}{\partial y^2} = 0, \tag{7.28}$$

$$\frac{\partial^2\Phi_0}{\partial y^2} = R_m(E - \frac{\partial\Psi_0}{\partial y}), \tag{7.29}$$

$$\frac{\partial p_0}{\partial x} = \frac{\partial^3\Psi_0}{\partial y^3} + M^2(E - \frac{\partial\Psi_0}{\partial y}), \tag{7.30}$$

$$\frac{\partial p_0}{\partial y} = 0, \tag{7.31}$$

$$\Psi_0 = 0, \quad \frac{\partial^2\Psi_0}{\partial y^2} = 0, \quad \frac{\partial\Phi_0}{\partial y} = 0, \quad \text{at} \quad y = 0,$$
$$\frac{\partial\Psi_0}{\partial y} = -1, \quad \Psi_0 = F_0, \quad \Phi_0 = 0, \quad \text{at} \quad y = h, \tag{7.32}$$

where equation (7.31) shows that $p_0 \neq p_0(y)$.

7.2.2 First order system

$$\mathrm{Re}\left\{\left(\frac{\partial\Psi_0}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\Psi_0}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial^2\Psi_0}{\partial y^2}\right\}=\frac{\partial^4\Psi_1}{\partial y^4}+\lambda_1\left\{\left(\frac{\partial\Psi_0}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\Psi_0}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial^4\Psi_0}{\partial y^4}\right\}$$
$$+\mathrm{Re}\,S_t^2\left\{\left(\frac{\partial\Phi_0}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\Phi_0}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial^2\Phi_0}{\partial y^2}\right\}+\mathrm{Re}\,S_t^2\frac{\partial^3\Phi_1}{\partial y^3},\tag{7.33}$$

$$\frac{\partial^2\Phi_1}{\partial y^2}=R_m\left\{\left(\frac{\partial\Psi_0}{\partial y}\frac{\partial\Phi_0}{\partial x}-\frac{\partial\Psi_0}{\partial x}\frac{\partial\Phi_0}{\partial y}\right)-\frac{\partial\Psi_1}{\partial y}\right\},\tag{7.34}$$

$$\frac{\partial p_1}{\partial x}=-\mathrm{Re}\left\{\left(\frac{\partial\Psi_0}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\Psi_0}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial\Psi_0}{\partial y}\right\}+\frac{\partial^3\Psi_1}{\partial y^3}-\lambda_1\frac{\partial\Psi_0}{\partial x}\frac{\partial^4\Psi_0}{\partial y^4}+\frac{\partial}{\partial x}\left\{\begin{array}{c}\frac{\partial\Psi_0}{\partial y}\frac{\partial^3\Psi_0}{\partial y^3}+\\ \frac{1}{2}(2\lambda_1+\lambda_2)(\frac{\partial^2\Psi_0}{\partial y^2})^2\end{array}\right\},\tag{7.35}$$

$$\frac{\partial p_1}{\partial y}=2(2\lambda_1+\lambda_2)\frac{\partial^2\Psi_0}{\partial y^2}\frac{\partial^3\Psi_0}{\partial y^3},\tag{7.36}$$

$$\Psi_1=0,\quad \frac{\partial^2\Psi_1}{\partial y^2}=0,\quad \frac{\partial\Phi_1}{\partial y}=0,\quad \text{at}\ \ y=0,\tag{7.37}$$

$$\frac{\partial\Psi_1}{\partial y}=0,\quad \Psi_1=F_1,\quad \Phi_1=0,\quad \text{at}\quad y=h.\tag{7.38}$$

7.2.3 Solution of zeroth order system

From equations (7.28)-(7.30) and (7.32), we obtain stream function (Ψ_0), magnetic force function (Φ_0) and axial pressure gradient as

$$\Psi_0=\left(\frac{F_0M\cosh(Mh)+\sinh(Mh)}{Mh\cosh(Mh)-\sinh(Mh)}\right)y-\left(\frac{(F_0+h)\sinh(My)}{Mh\cosh(Mh)-\sinh(Mh)}\right),\tag{7.39}$$

$$\begin{aligned}\Phi_0 &= \frac{1}{2M(Mh\cosh(Mh)-\sinh(Mh))}((\Phi_{10}\cosh(Mh)+2(F+h)\cosh(My)\\ &\quad +M(h-y)(h+y)(1+E)\sinh(Mh)+\Phi_{20})R_m),\end{aligned}\tag{7.40}$$

$$\frac{\partial p_0}{\partial x}=-\left(\left(\frac{F_0M^3\cosh(Mh)+M^2\sinh(Mh)}{Mh\cosh(Mh)-\sinh(Mh)}\right)-M^2E\right),\tag{7.41}$$

$$\begin{aligned}\Phi_{10} &= F(-2+M^2(h-y)(h+y)+h(-2+M^2(-h^2+y^2)E)),\\ \Phi_{20} &= \delta((2+M^2(-h^2+y^2))\cosh(Mh)-2\cosh(My))F_1.\end{aligned}$$

7.2.4 Solution of first order system

Expressions of stream function (Ψ_1), magnetic force function (Φ_1) and axial and transverse pressure gradients at this order are

$$
\begin{aligned}
\Psi_1 = & \frac{1}{24Q_1}[(\cosh(2M(h+y)) - \sinh(2M(h+y)))(M^2(1+M)(1+hM)p_5(\cosh(Mh) \\
& + \sinh(Mh)) + M^2(1+M)(-1+hM)p_5(\cosh(4Mh) + \sinh(4Mh)) + M^3(1+M) \\
& p_5 y(\cosh(2My) + \sinh(2My)) - M^2(1+M)(1+2hM)p_5(\cosh(M(h+y)) + \\
& \sinh(M(h+y))) + (-144p_8 + 24F_1M^7y + 6M^2(p_5 + 8hp_7 - 4p_9 - 8p_7y) + 3M^3 \\
& 2h(p_5 - 4p_9) - 3p_5y + 8p_9y) + 4hM^5p_3y(h^2 - y^2) + M^4y(12h^2p_3 - 3p_5 - 4p_3 \\
& y^2) + 48M(p_7 + 3p_8(-h+y)))(\cosh(2M(h+y)) + \sinh(2M(h+y))) + (-24F_1 \\
& M^6 + 8h^3M^4p_8 + 6(8Mp_7 - 24p_8 + M^2(p_5 - 4p_9)))(\cosh(3M(h+y)) + \sinh 3 \\
& M(h+y)))(-1+M)M^2(-1+hM)p_5(\cosh(4M(h+y)) - \sinh(4M(h+y))) + M \\
& (h^3M^2(-3Mp_2 + 11p_4 + 3Mp_7 - 11p_8) + 2h^4M^3(p_4 - p_8) - 6h^2M(-6p_4 + 2M \\
& (p_2 - p_7) + 6p_8 + M^2(p_1 + p_9)) + y(51(-p_4 + p_8) + 15M(p_2 - p_7 - p_4y + p_8y) \\
& + M^2(6p_1 + 6p_9 + y(3p_2 - 3p_7 - 2p_4y + 2p_8y))) + hMy(51(-p_4 + p_8) + 15M \\
& (p_2 - p_7 - p_4y + p_8y) + M^2(6p_1 + 6p_9 + y(3p_2 - 3p_7 - 2p_4y + 2p_8y))))(\cosh(\\
& M(2h+y)) + \sinh(M(2h+y))) + (144p_8 - 48M(p_7 + 3p_8(h-y)) + 24F_1M^7y \\
& -6M^2(p_5 - 8hp_7 - 4p_9 + 8p_7y) + 3M^3(2h(p_5 - 4p_9) - 3p_5y + 8p_9y) + 4hM^5p_3 \\
& y(h^2 - y^2) + M^4(-12h^2p_3y + 3p_5y + 4p_3y^3))(\cosh(2M(2h+y)) + \sinh(2M(2h \\
& +y))) + 2(12F_1M^6 - 4h^3M^4p_3 - 3M^2p_5 - 24Mp_7 + 72p_8 + 12M^2p_9)(\cosh(M \\
& (3h+y)) + \sinh(M(3h+y))) - (-1+M)M^3p_5y(\cosh(2M(3h+y)) + \sinh(2M \\
& (3h+y))) - 6h^2M^2(2M(p_2 + p_7) - 6(p_4 + p_8) + M^2(p_1 - p_9)) + M(-48p_7 + 51 \\
& (-p_4 + p_8)y) + M^3y(-6p_1 + 6p_9 + y(3p_2 - 3p_7 - 2p_4y + 2p_8y)) + M^2(-8p_5 + 3(8 \\
& p_9 + 5y(p_2 - p_7 - p_4y + p_8y))) + hM(-144p_8 + M(48p_7 + 51(p_4 - p_8)y) + M^3y(
\end{aligned}
$$

$$-6p_1 - 6p_9 + y(-3p_2 + 3p_7 + 2p_4y - 2p_8y)) + M^2(8p_5 - 3(8p_9 + 5y(p_2 - p_7 - p_4y + p_8y)))))(\cosh(M(4h+y)) + \sinh(M(4h+y))) - (-1+M)M^2(-1+2hM)p_5 (\cosh(M(5h+y)) + \sinh(M(5h+y))) - 3M(17(-p_4+p_8) + 5M(p_2 - 2hp_4 - p_7 + 2hp_8) + 2M^2(p_1 + h(p_2 - p_7) + h^2(-p_4+p_8) + p_9))y(\cosh(M(h+2y)) + \sinh(M (h+2y))) - (-1+M)M^2(1+hM)p_5(\cosh(2M(h+2y)) + \sinh(2M(h+2y))) + 2M(-12hM(M^2p_1 + 2Mp_2 - 6p_4) + 4h^3M^3p_4 + 33Mp_7 + 6h^2M^2(Mp_7 - 4p_8) - 93p_8 + 2M^2(4p_5 - 9p_9))y(\cosh(M(3h+2y)) + \sinh(M(3h+2y))) + 3M(5Mp_2 + 10hMp_4 + 5Mp_7 + 10hMp_8 - 17(p_4+p_8) + 2M^2(p_1 - h(p_2 + p_7 + h(p_4+p_8)) - p_9))y(\cosh(M(5h+2y)) + \sinh(M(5h+2y))) + M^2(1+M)(1+2hM)p5(\cosh(M(h+3y)) + \sinh(M(h+3y))) - (h^3M^3(-3Mp2 + 11p4 + 3Mp7 - 11p8) + 2h^4 M^4(p4 - p8) - 144p8 - 6h^2M^2(-6p4 + 2M(p2 - p7) + 6p8 + M^2(p1 + p9) + M (48p7 + 51(p4+p8)y) + M^3y(-6p1 + 6p9 + y(3p2 + 3p7 + 2(p4+p8)y)) + M^2 (8p5 - 3(8p9 + 5y(p2 + p7 + (p4+p8))) + hM(-144p8 + M(48p7 + 51(p4 + p8)y) + M^3y(-6p1 + 6p9 + y(3p2 + 3p7 + 2(p4+p8)y)) + M^2(8p5 - 3(8p9 + 5 y(p2 + p7 + (p4+p8)y)))))(\cosh(M(2h+3y)) + \sinh(M(2h+3y))) + M(2h^4M^3 (p_4+p_8) + h^3M^2(3M(p_2+p_7) - 11(p_4+p_8)) - 6h^2M(2M(p_2+p_7) - 6(p_4+p_8) + M^2(p_1 - p_9) + y(51(p_4+p_8) - 15M(p_2 + p_7 + (p_4+p_8)y) + M^2(-6p_1 + 6p_9 + y(3p_2 + 3p_7 + 2(p_4+p_8)y))) + hMy(51(p_4+p_8) - 15M(p_2 + p_7 + (p_4+p_8)y) + M^2(-6p_1 + 6p_9 + y(3p_2 + 3p_7 + 2(p_4+p_8)y))))(\cosh(M(4h+3y)) + \sinh(M(4h + 3y)))(-1+M)M^2(-1+2hM)p_5(\cosh(M(5h+3y)) + \sinh(M(5h+3y))), \tag{7.42}$$

$$\begin{aligned}\Phi_1 = \frac{1}{24M^4}[&\cosh(Mh)(6(-6p_8 + p_9 + M(2p_{13} + 8h^5M^6p_4 + 12h^4M^5(Mp_7 - 4p_8) \\ &-M(p_{10} + 4(-237p_8 + M(8Mp_5 + 69p_7 - 30Mp_9))) - 2M^3(-93p_8 + M(8Mp_5 \\ &+33p_7 - 18Mp_9)) + y^2 + h(-Mp_{13} + 4p_8 + 24M^4(M^2p_1 + 2Mp_2 - 6p_4)y^2)\end{aligned}$$

$$-8h^3M^4(-22p_4+M(3Mp_1+6p_2+Mp_4y^2))-h^2M(p_8+2M^2(-165p_8+M$$
$$(-8Mp_5-45p_7+18Mp_9+6M(Mp_7-4p_8)y^2)))))+M^2(-3456p_8+288F_1M^6$$
$$(-2+M^2(h-y)(h+y)+M(h^4Mp_{11}+24h^5M^5p_3+6h^2(p_5+96M^2p_7-M$$
$$(p_{12}+288p_8)+6M^3(p_5-8p_9))+36(5Mp_5+32p_7-16Mp_9)+6(-p_5+M(p_{12}$$
$$+6(-48p_8+M(3Mp_5+16p_7-8Mp_9))))y^2-Mp_{11}y^4-48h^3M^3p_3(-4+M^2y^2)$$
$$+12hM(-96Mp_7y+288p_8y+2M^4p_3y^4+3M^2(p_5-4p_5y+16p_9y))))\cosh(Mh)$$
$$-6(6p_8-p_9+M(-2p_{13}+12h^3M^4(Mp_2-9p_4)+12h^4M^5p_8+h(4p_8+M(-p_{13}$$
$$+24M(2M^2p_1+7Mp_2-31p_4)-12M^3(Mp_2-5p_4)y^2))+h^2(-78M^4p_7+Mp_8$$
$$+414M^3p_8+12M^5(p_9-p_8y^2))+M(p_{10}+372p_8+6M(-14p_7+4Mp_9+M$$
$$(5Mp_7-17p_8-2M^2p_9)y^2))))M(p_{10}+372p_8+6M(-14p_7+4Mp_9+M(5Mp_7$$
$$-17p_8-2M^2p_9)y^2))))\cosh(2hM)+12M^4p_5(1+M^2(h(3+h)-y^2))\cosh(3hM)$$
$$+2M^5p_5\cosh(2My)(hM\cosh(Mh)-\sinh(Mh))-M^3(h^4M(p_{11}+120M^3p_3)+36$$
$$M^2p_5+36h(-96p_8+M(5Mp_5+32p_7-16Mp_9))-144(8Mp_7-24p_8+M^2(p_5$$
$$-4p_9))y+6(Mp_{12}-p_5+6M^4p_5)y^2+M(p_{11}+24M^3p_3)y^4-6h^2(Mp_{12}(7p_2+$$
$$62hp_8)+M(p_{10}-hp_{13}-372p_4+h^2p_8)-12M^5(-p_1+h(hp_4+p_7))-p_5+6$$
$$M^4(p_5+4p_3y^2)))\sinh(Mh)+6(6p_8-p_9+M(-2p_{13}+4hp_8+12M^2(h-y)(h+$$
$$y)+6M^3(4p_1-69h^2p_4-28hp_7+17p_4y^2)+6M^2(h(13hp_2+18h^2p_8+8p_9)-$$
$$5(p_2+2hp_8)y^2)))\sinh(2Mh)+12\cosh(My)4M^2(12F_1M^6+72p_8-M(4h^3M^3p_3$$
$$+3Mp_5+24p_7-12Mp_9))+(6p_8-p_9+M(-2p_{13}+M(p_{10}+4h^3M^3(-3Mp_2+$$
$$11p_4)-8h^4M^4p_8+16M(6p_7+M(p_5-3p_9))-24h^2(-2Mp_7+6p_8+M^2p_9)$$
$$+p_8(-288+y^2)+4hM(-93p_4+M(21p_2+93p_8y+M^2y(6p_9+3p_2y+2p_8y^2)+$$
$$3M(2p_1-7y(p_7+p_4y)))))))\cosh(Mh)-4M^4(1+2hM^2)p_5\cosh(2Mh)-(6p_8-$$
$$p_9+M(-2p_{13}+M(p_{10}-372p_4+4h^2M^4(-6p_1+h(2hp_4+3p_7))+p_8y^2+12M$$
$$(7p_2-24hp_8+31p_8y)+12M^2(2p_1+12h^2p_4+8hp_7-7y(p_7+p_4y))-4M^3(12$$
$$h^2p_2+11h^3p_8-4h(p_5-3p_9)-y(6p_9+3p_2y+2p_8y^2)))))\sinh(Mh)+4(1+2h)$$
$$M^5p_5\sinh(2Mh)-12M^5p_5(1+h(3+hM^2)-M^2y^2)\sinh(3Mh)+12M(((Mp_{13}-4$$
$$p_8)y+4hM^2(-165p_8+M(45p_7+93p_4y+M^2y(-6p_1+3p_7y+2p_4y^2)+M(4p_5$$

$$-3(6p_9+7y(p_2+p_8)))))) \cosh(Mh)+(4p_8y+M(660p_8-p_{13}y)-12M^2(15p_7+$$
$$31p_4y)-4M^4y(-6p_1+3p_7y+2p_4y^2)+M^3(-16p_5+12(6p_9+7y(p_2+p_8y))))$$
$$\sinh(Mh)\sinh(My)-24M^4p_5(Mh\cosh(Mh)-\sinh(Mh))R_m], \tag{7.43}$$

$$\frac{\partial p_1}{\partial x} = A_3-\frac{p3}{M^2}-A_4R_e+A_5R_e+(A_2+\frac{A_1}{M^4})R_eS_t+\frac{MR_eR_m}{24Q_1}\{-((\cosh(2Mh)$$
$$-\sinh(2Mh))(2M^2(1+M)(1+2hM)p_5(\cosh(Mh)+\sinh(Mh))-3(2M^2p_1$$
$$+5Mp_2+2hM^2p_2-17p_4-10hMp_4-2h^2M^2p_4-5Mp_7-2hM^2p_7+17p_8$$
$$+10hMp_8+2h^2M^2p_8+2M^2p_9)(\cosh Mh+\sinh(Mh))-2(-1+M)M^2(1$$
$$+hM)p_5(\cosh(2Mh)+\sinh(2Mh))-2M^2(1+M)(1+hM)p_5(\cosh(2Mh)$$
$$+\sinh(2Mh))+(12h^2M^3p_3+4h^3M^4p_3-9M^2p_5-3M^3p_9-48Mp_7+144p_8$$
$$+24M^2p_9)(\cosh(2Mh)+\sinh(2Mh))+(6M^2p_1+6hM^3p_1+6h^2M^4p_1+15M$$
$$p_2+15hM^2p_2+12h^2M^3p_2+3h^3M^4p_2-51p_4-51hMp_4-36h^2M^2p_4-11h^3$$
$$M^3p_4-2h^4M^4p_4-15Np_5-15hM^2p_7-12h^2M^3p_7-3h^3M^4p_7+51p_7(1+h)$$
$$+h^2M^2p_8(36+11hM+2h^2M^2)+6M^2p_9(1+hM+6h^2M^2)(\cosh(2Mh)+\sinh$$
$$(2Mh)+\sinh(2Mh))(6M^2p_1(1+hM+h^2M^2)+Mp_2(15+15hM+12h^2M^2+3$$
$$h^3M^3)+p_4(-51-51hM-36h^2M^2-11h^3M^3-2h^4M^4)-8M^2p_5(1+hM)-M$$
$$p_7(33hM+12h^2M^2+3h^3M^3)+p_8(93+93hM+36h^2M^2+11h^3M^3+2h^4M^4)$$
$$+M^2p_9(18M^2+18hM+6h^2M^2))(\cosh(2Mh)+\sinh(2Mh))+2(8h^3M^4p_3$$
$$+6M^2p_5+487Mp_7-144p_8-24M^2p_9)(\cosh(3Mh)+\sinh(3Mh))+2$$
$$(-12hM^3p_1-24hM^2p_2+72hMp_4+4h^3M^3p_4+8M^2p_5+33Mp_7+6h^2M^3p_7$$
$$-93p_8-24h^2M^2p_8-18M^2p_9)(\cosh(3Mh)+\sinh(3Mh))-2(-1+M)M^2$$
$$(-1+hM)p_5(\cosh(4Mh)+\sinh(4Mh))-(1+M)M^2(-1+hM)p_5(\cosh(4Mh)$$
$$+\sinh(4Mh))+(-12h^2M^3p_3+4h^3M^4p_3-9M^2p_5+3M^3p_5-48Mp_7+144p_8$$
$$+24M^2p_9)(\cosh(4Mh)+\sinh(4Mh))+(-6M^2p_1+6hM^3p_1-6h^2M^4p_1$$
$$-15Mp_2+15hM^2p_2-12h^2M^3p_2+3h^3M^4p_2+51p_4-51hMp_4+36h^2M^2p_4$$

$-11h^3M^3p_4 + 2h^4M^4p_4 - 8M^2p_5 + 8hM^3p_5 - 33Mp_7 + 33hM^2p_7 - 12h^2$
$M^3p_7 + 3h^3M^4p_7 + 93p_8 - 93hMp_8 + 36h^2M^2p_8 - 11h^3M^3p_8 + 2h^4M^4p_8 + 18$
$M^2p_9 - 18hM^3p_9 + 6h^2M^4p_9)(\cosh(4Mh) + \sinh(4Mh)) + (-6M^2p_1 + 6hM^3p_1$
$6h^2M^4p_1 - 15Mp_2 + 15hM^2p_2 - 12h^2M^3p_2 + 3h^3M^4p_2 + 51p_4 - 51hMp_4$
$+36h^2M^2p_4 - 11h^3M^3p_4 + 2h^4M^4p_4 - 15Mp_7 + 15hM^2 + 15hM^2p_7 + 15hM^2$
$p_7 - 12h^2M^3p_7 + 3h^3M^4p_7 + 51p_8 - 51hMp_8 + 36h^2M^2p_8 - 11h^3M^3p_8 + 2h^4M^4$
$p_4 + -15Mp_7 + 15hM^2p_7 - 12h^2M^3p_7 + 3h^3M^4p_7 + 51p_8 - 51hMp_8 + 36h^2M^2p_8$
$-11h^3M^3p_8 + 2h^4M^4p_8 + 6M^2p_9 - 6hM^3p_9 + 6h^2M^4p_9)(\cosh(4Mh) + \sinh(4Mh))$
$+2(-1+M)M^2(-1+2hM)p_5(\cosh(5Mh) + \sinh(5Mh)) + 3(5Mp_2 + 10hMp_4 + 5M$
$p_7 + 10hMp_8 - 17(p_4 + p_8) + 2M^2(p_1 - h(p_2 + hp_4 + p_7 + hp_8) - p_9))(\cosh(5Mh)$
$+\sinh(5Mh)) - (-1+M)M^2p_5(\cosh(6Mh) + \sinh(6Mh)))\} + \frac{1}{L^3M}(k(FM\cosh(Mh)$
$-\sinh(Mh))(-2L\cosh(Mh) + 2LFhM^2\cosh(Mh) + Lh^2M^2\cosh(Mh) - 2Lh^2M^2E\cosh$
$(Mh) - 2LFM\sinh(Mh) + LFh^2M^3\sinh(Mh) + 2LhME\sinh(Mh) - Lh^3M^3E\sinh(Mh) +$
$2FhM^2\cosh(Mh)\sinh(Mh) + 2h^2M^2\cosh(Mh)\sinh(Mh) - Fh^3M^4\cosh(Mh)\sinh(Mh) +$
$2h^2M^2\cosh(Mh)\sinh(Mh) - Fh^3M^4\cosh(Mh)\sinh(Mh) + h^4M^4\cosh(Mh)\sinh(Mh) -$
$h^3M^3(\sinh(Mh))^2(1+E))R_m)S_t^2) + p_6],$ (7.44)

$$\frac{\partial p_1}{\partial y} = \frac{1}{L^2}[2M^5\cosh(My)\sinh(My)(h+F_0)^3(2\lambda_1+\lambda_2)], \quad (7.45)$$

where

$$Q_1 = M^6(1 + hM + (-1+hM)(\cosh(2Mh) + \sinh(2Mh))),$$

$$\begin{aligned} L &= (hM\cosh(Mh) - \sinh(Mh)), \\ p_1 &= -\frac{1}{2L^3}[((kM^2(4M^2\lambda_1(FM\cosh(Mh) + \sinh(Mh))(-hM\cosh(Mh) + (1+h(F+h) \\ &\quad M^2)\sinh(Mh)) - 4(FM\cosh(Mh) + \sinh(Mh))(-hM\cosh(Mh) + (1+h(F+h) \\ &\quad M^2)\sinh(Mh))R_e + (-h(F+h)M^2(2+h(3F+2h)M^2) + 2(-1-h(2F+h)M^2 \end{aligned}$$

$$
\begin{aligned}
& +h^3(F+h)M^4)E + (h(F+h)M^2(2-FhM^2) + 2(1+h(2F+3h)M^2+h^3(F \\
& +h)M^4)E)\cosh(2Mh) - hM(4E-(F+h)M^2(2F-h-6hE))\sinh(2Mh)) \\
& R_m))], \\
p_2 &= \frac{1}{L^3}[(kM^3((F+h)M^3\lambda_1((-F-2h)M + (FM\cosh(Mh)+\sinh(Mh)) + (-F-h) \\
& M((-2F-2h)M + (FM\cosh(Mh)+\sinh(Mh))R_e + (2+M^2(F^2+4h^2+Fh(7 \\
& -2E)) - 2E - (2-2E+M^2(F^2+Fh+2h^2-2h(F+2h)E))\cosh(2Mh) - M \\
& (-h+F(3+2h(F+h)M^2)+2h(2+h(F+h)M^2)E)\sinh(2Mh))R_m))], \\
p_3 &= \frac{1}{L^3}[(kM^3(M(2F-3hE)\cosh(Mh)+(2+3E)\sinh(Mh))], \\
p_4 &= \frac{(F+h)M^4R_m}{2L}, \\
p_5 &= -\frac{1}{L^3}[2(F+h)kM^2(-hM\cosh(Mh)(1+h(F+h)M^2)\sinh(Mh))R_m], \\
p_6 &= \frac{F_1M^3\cosh(Mh)}{L}, \\
p_7 &= -\frac{1}{L^3}[(F+h)kM^3((-F-2h)M+FM\cosh(Mh)+\sinh(Mh))R_m], \\
p_8 &= -\frac{1}{2L^3}[(F+h)kM^2((-F-2h)M+FM\cosh(2Mh)+\sinh(2Mh))R_m], \\
p_9 &= -\frac{1}{2L^3}[(F+h)kM^2R_m(2M(F+2h-hE+h^3M^3E)+2M(-F+hE+h^3M^2 \\
& \cosh(2Mh) - 2(1+2h^2M^2E)\sinh(2Mh) - hM(2+h(3F+2h)M^2+(-2+FhM^2) \\
& +FhM^2)\cosh(2Mh) + (-2F+h)M\sinh(2Mh))R_m))], \\
p_{10} &= -\frac{1}{ML^3}[2k(FM\cosh(Mh)+\sinh(Mh))(-hM\cosh(Mh)+(1+h(F+h)M^2) \\
& \sinh(Mh))R_m], \\
p_{11} &= \frac{1}{2L^3}[kM(M(F-2hE)\cosh(Mh)+(1+2E)\sinh(Mh))((-F-2h)M+FM \\
& \cosh(2Mh)+\sinh(2Mh))R_m], \\
p_{12} &= \frac{1}{2L^3M}[k(FM\cosh(Mh)+\sinh(Mh))(M(F(-2+3h^2M^2)-2h(-1+hM) \\
& (1+hM)(-1+E)+M(F(2+h^2M^2)-2h(1+E+h^2M^2E))\cosh(2Mh) \\
& +(2+hM^2(-2F+h+4hE))\sinh(2Mh)R_m], \\
p_{13} &= \frac{1}{L^3}[(k(-1+E-M^2(F^2+3h^2+Fh(5+E))+(1+E+M^2(F^2+Fh+h^2
\end{aligned}
$$

$$+ h(F+2h)E))\cosh(2Mh) + M(2+h(F+h)M^2)(F-hE)\sinh(2Mh),$$

$$A_1 = \frac{1}{L^3}[2(F+h)kM^4(-hM\cosh(Mh) + (1+h(F+h)M^2)\sinh(Mh))],$$

$$A_2 = -\frac{1}{2L^3}(k(-LE + FM\cosh(Mh) + \sinh(Mh))(-2L(-2+h(2F+h)M^2\cosh(Mh)$$

$$+ M(-h^3M^2 + 4hL^2E + 4FL\sinh(Mh) + hM(hM(h\cosh(2Mh) - 2FL\sinh(Mh))$$

$$+ (-2(F+h) + Fh^2M^2)\sinh(2Mh))))),$$

$$A_3 = -\frac{1}{L^3}[(F+h)kM((-F-2h)M + FM\cosh(2Mh) + \sinh(2Mh))(3\lambda_1 + 2\lambda_2)],$$

$$A_4 = \frac{1}{2L^3}(kM((1+(F^2+8Fh+4h^2)M^2)\cosh(Mh) - (1+F^2M^2)\cosh(3Mh) - 4M$$

$$(F + h(F+h)^2M^2 + F\cosh(2Mh))\sinh(Mh))],$$

$$A_5 = -\frac{1}{L^3}[(kM(-1-(F^2+5Fh+3h^2)M^2 + (1+(F^2+Fh+h^2)M^2)\cosh(2Mh)$$

$$+ FM(2+h(F+h)M^2)\sinh(2Mh)))],$$

$$A_6 = -\frac{1}{L^3}[(M^3k(F+h)(-(F+2h)M + FM\cosh(2Mh) + \sinh(2Mh)))].$$

The stream and magnetic force functions upto order $O(\delta)$ are:

$$\begin{aligned}\Psi \;=\; & \frac{1}{Q_1}[\frac{48FMy\cosh(Mh)}{(\cosh(Mh)-\sinh(Mh))} + (2(-24M^2y + 6M^2p_5\delta + 48Mp_7\delta + 48hM^2p_7\delta \\ & -144p_8\delta - 144hMp_8\delta - 24M^2p_9\delta - 24hM^3p_9\delta + 12h^2M^4p_3y\delta + 4h^3M^5p_3 \\ & y\delta - 9M^3p_5y\delta - 3M^4p_5y\delta - 48M^2p_7y\delta + 144Mp_8y\delta + 24M^3p_9y\delta - 4M^4p_3 \\ & y^3\delta - 4hM^5p_3y^3\delta - 3M(2M^2p_1 + 5Mp_2 + 2hM^2p_2 - 17p_4 - 10hMp_4 - 2h^2 \\ & M^2p_4 - 5Mp_7 - 2hM^2p_7 + 17p_8 + 10hMp_8 + 2h^2M^2p_8 + 2M^2p_9)y\delta(\cosh(Mh) \\ & -\sinh(Mh)) + 2M(-12hM^3p_1 - 24hM^2p_2 + 72hMp_4 + 4h^3M^3p_4 + 8M^2p_5 \\ & +33Mp_7 + 6h^2M^3p_7 - 93p_8 - 24h^2M^2p_8 - 18M^2p_9)y\delta(\cosh(Mh) + \sinh(Mh)) \\ & M^3(1+M)p_5y\delta(\cosh(2Mh) - \sinh(2Mh))(24M^6y - 6M^2p_5\delta + 6hM^3p_5\delta - 48 \\ & Mp_7\delta + 48hM^2p_7\delta + 144p_8\delta - 144hMp_8\delta + 24M^2p_9\delta - 24hM^3p_9\delta - 12h^2M^4 \\ & p_3y\delta + 4h^3M^5p_3y\delta - 9M^3p_5y\delta + 3M^4p_5y\delta - 48M^2p_7y\delta + 144Mp_8y\delta + 24M^3 \\ & p_9y\delta + 4M^4p_3y^3\delta - 4hM^5p_3y^3\delta)(\cosh(2Mh) + \sinh(2Mh)) + 3M(5Mp_2 + 10hM\end{aligned}$$

$$p_4+5Mp_7+10hMp_8-17(p_4+p_8)+2M^2(p_1-h(p_2+hp_4+p_7+hp_8)-p_9))y\delta($$
$$\cosh(3Mh)+\sinh(3Mh))-(-1+M)M^3p_5y\delta(\cosh(4Mh)+\sinh(4Mh))2(12FM^6$$
$$+12hM^6-4h^3M^4p_3\delta-3M^2p_5\delta-24Mp_7\delta+72p_8\delta+12M^2p_9\delta(\cosh(M(h-y))$$
$$+\sinh(M(h-y)))+M^2(1+M)(-1+hM)p_5\delta(\cosh(M(h-y))+\sinh(M(h-y)))$$
$$-(1+M)M^2(1+2hMp_5\delta(\cosh(M(3h-y))+\sinh(M(3h-y))))M(-6h^2M^3p_1-$$
$$12h^2M^2p_2-3h^3M^3p_2+36h^2Mp_4+11h^3M^2p_4+2h^4M^3p_4+12h^2M^2p_7+3h^3M^3$$
$$p_7-36h^2Mp_8-11h^3M^2p_8-2h^4M^3p_8-6h^2M^3p_9+3(1+hM)(2M^2p_1+5Mp_2$$
$$-17p_4-5Mp_7+17p_8+2M^2p_9)y+3M(1+hM)(Mp_2-5p_4-Mp_7+5p_8)y^2-$$
$$2M^2(1+hM)(p_4-p_8)y^3)\delta(\cosh(My)-\sinh(My))(6h^2M^4p_1+12h^2M^3p_2+3h^3M^4$$
$$p_2-36h^2M^2p_4-11h^3M^3p_4-2h^4M^4p_4-8M^2p_5(1+M)-48Mp_7(1+M)-3h^2$$
$$M^3p_7(4+hM)+144p_8(1+hM)+h^2M^2p_8(36+11hM+2h^2M^2)+24M^2p_9(1+h$$
$$M)+6h^2M^4p_9+6M^3p_1y(1+hM)+15M^2p_2y(1+hM)-51Mp_4y(1+hM)+15M^2$$
$$p_7y(1+hM)-51Mp_8y(1+hM)-6M^3p_9y(1+hM)+3M^3p_2y^2(1+hM)+15M^2p_4$$
$$y^2(1+hM)-3M^3p_7y^2(1+hM)+15M^2p_8y(1+hM)-2M^3p_4y^3(1+hM)-2M^3p_8$$
$$y^3(1+hM))\delta(\cosh(My)+\sinh(My))+M^2(1+hM)p_5\delta(\cosh(2My)+\sinh(2My))+$$
$$M^2(1+M)(1+2hM)p_5\delta(\cosh(M(-h+y))+\sinh(M(-h+y)))-M^2(1+M)(1+2$$
$$hM)p_5\delta(\cosh(M(h+y))+\sinh(M(h+y)))-2(12M^6F+12hM^6h-4h^3M^4p_3\delta-$$
$$3M^2p_5\delta-24Mp_7\delta+72p_8\delta+12M^2p_9\delta)(\cosh(M(h+y))+\sinh(M(h+y)))+M^2$$
$$(1+M)(-1+hM)\ p_5\delta(\cosh(2M(h+y))+\sinh(2M(h+y)))M(-6h^2M^3p_1-12h^2$$
$$M^2p_2+12h^3M^3p_2+36Mh^2p_4-11h^3M^2p_4+2h^4M^3p_4-3h^2M^2p_7(4-hM)+h^2$$
$$Mp_8(36-11hM+2h^2M^2)+6h^2M^3p_9+(-1+hM)(2M^2p_1+5Mp_2-17p_4+5M$$
$$p_7-17p_8-2M^2p_9)y-3M(-1+hM)\ (Mp_2-5p_4+Mp_7-5p_8)y^2-2M^2(-1+h$$
$$M)(p_4+p_8)y^3)\delta(\cosh(M(2h+y))+\sinh(M(2h+y)))+(-1+M)M^2(-1+2hM)p_5$$
$$\delta(\cosh(M(3h+y))+\sinh(M(3h+y)))(-6h^2M^4p_1-3h^2M^3p_2(4-hM)+h^2M^2p_4$$
$$(36-11hM+2h^2M^2)-8M^2p_5(1-hM)-48Mp_7(1-hM)-3h^2M^3p_7(4-hM)$$
$$+144p_8(1+hM)+h^2M^2p_8(36-11hM+2h^2M^2)+24M^2p_9(1-hM)+6h^2M^4p_9$$

$$+ 6M^3p_1y(1-hM) + 15M^2p_2y(1-hM) - 51Mp_4y(1-hM) - 15M^2p_7y(1-hM)$$
$$+ 51Mp_8y(1-hM) + 6M^3p_9y(1-hM) + 3M^3p_2y^2(1-hM) - 15M^2p_4y^2(1+hM)$$
$$- 3M^3p_7y^2(1-hM) + 15M^2p_8y^2(1-hM) - 2M^3p_4y^3(1-hM) + 2M^3p_8y^3(1-h$$
$$M)\delta(\cosh(M(2h-y)) + \sinh(M(2h-y))], \tag{7.46}$$

$$\Phi = -\frac{1}{24M^5L}[(12M^4((F(-2+M^2(h-y)(h+y)) + h(-2+M^2(-h^2+y^2)E))$$
$$\cosh(Mh) + 2(F+h)\cosh(My) + M(h-y)(h+y)(1+E)\sinh(Mh) + \delta$$
$$((93p_8 + M(12h(M^2p_1 + 2Mp_2 - 6p_4) - 4h^3M^2p_4 - 8Mp_5 - 33p_7 - 6h^2$$
$$M(Mp_7 - 4p_8) + 81Mp_9))\delta - M^2p_5\delta\cosh(3Mh) + \delta\cosh(2Mh)(51p_8 +$$
$$3M(-5p_7 + 2h(-5p_4 + M(p_2 + hp_8)) + 2Mp_9) - 2M^2(1+2hM^2)p_5$$
$$\cosh(My) - 24M^5\sinh(Mh) + 12h^2M^3p_3\delta\sinh(Mh) - 3M^3p_5\delta\sinh(Mh)$$
$$12M^3p_3y^2\delta\sinh(Mh) - 24M^5E\sinh(Mh) - 4M^3p_5\delta\cosh(2My)\sinh(Mh)$$
$$-(6M^2p_1 + 15Mp_2 - 51p_4 - 6h^2M^2p_4 - 6hM^2p_7 - 30hMp_8)\delta$$
$$\sinh(2Mh) + 2\cosh(My)(12(F+h)M^6 - (-72p_8 + M(4h^3M^3p_3 + 3Mp_5$$
$$+24p_7 - 12Mp_9))\delta + (-51p_4 + h^2M^4(6p_1 - h(2hp_4 + 3p_7)) + 3M(5p_2$$
$$+24hp_8 - 7p_8y) + 3M^2(2p_1 - 12h^2p_4 - 8hp_7 + 3y(p_7 + p_4y)) + M^3(12$$
$$h^2p_2 + 11h^3p_8 - 4h(p_5 - 3p_9) - y(6p_9 + 3p_2y + 2p_8y^2)))\delta\sinh(Mh) +$$
$$(1+2h)M^3p_5\delta\sinh(2Mh) + M^3p_5\delta\sinh(3Mh) - 2(-21p_8 + M(9p_7$$
$$+21p_4y + M^2y(-6p_1 + y(3p_7 + 2p_4y)) + M(4p_5 - 3(2p_9 + 3y(p_2$$
$$+p_8y)))))\delta\sinh(Mh)\sinh(My) + \cosh(Mh)(-24FM^6 - (144p_8 + M(-9p_5$$
$$+24p_9 + 4hM^2p_3(h^2 - 3y^2))))\delta + 24hM^6E + 2\delta((-72p_8 + M(h^2M^2$$
$$(-3Mp_2 + 11p_4) + 24p_7 - 2h^4M^3p_8 + 4M(p_5 - 3p_9) - 6h^2M(-2Mp_7$$
$$+6p_8 + M^2p_9) + h(51p_4 + M(-15p_2 + 21p_8y - 3M(2p_1 + 3y(p_7 +$$
$$p_4y)) + 21M^2y(6p_9 + y(3p_2 + 2p_8y)))))) \cosh(My) + 12hM(2M^3p_5$$
$$\cosh(2My) + (-21p_8 + M(9p_7 + 21p_4yM^2y(-6p_1 + 3p_7y + 2p_4y^2) + M$$

$$(4p_5 - 3(2p_9 + 3y(p_2 + p_8y)))) - 4M^2p_5\cosh(My)\sinh(My)))) + 4M^2$$
$$p_5\delta\sinh(Mh)\sinh(2My)))R_m]. \qquad (7.47)$$

Expression of an axial induced magnetic field is

$$\begin{aligned} h_x &= \frac{1}{12M^4L}[(M(3p_4y + M^2y(-6p_1 + y(3p_7 + 2p_4y)) + M(4p_5 - 3y(p_2 + \\ &\quad p_8y)))\delta\cosh(My)L - 4M^2p_5\delta\cosh(2My)L + 12M^2y(M(hp_3\delta + M^2 \\ &\quad (-F + hE)\cosh(Mh) - (p_3\delta + M^2(1+E))\sinh(Mh) + (12(F+h)M^4 \\ &\quad +12(F+h)M^6\delta - (4h^3M^4p_3 + 3(8Mp_7 - 24p_8 + M^2(p_5 - 4p_9)))\delta \\ &\quad +\delta((-72p_8 + M(h^3M^2(-3Mp_2 + 11p_4) + 24p_7 - 2h^4M^3p_8 + 4M \\ &\quad (p_5 - 3p_9) - 6h^2M(-2Mp_7 + 6p_8 + M^2p_9) + h(72p_4 + M(-24p_2 \\ &\quad +3p_8y + M^2y(6p_9 + 3p_2y + 2p_8y^2) - 3M(4p_1 + y(p_7 + p_4y)))))) \\ &\quad \cosh(Mh) - M^2(1 + 2hM^2)p_5\cosh(2Mh) + (-72p_4 + M(24p_2 \\ &\quad +h^2M^3(6p_1 - 2h^2p_4 - 3hp_7) + 72hp_8 - 3p_8y + 3M(4p_1 - 4h \\ &\quad (3hp_4 + 2p_7) + p_7y + p_4y^2) + M^2(h(12hp_2 - 4p_5 + 11h^2p_8 + 12p_9) \\ &\quad -6p_9y - 3p_2y^2 - 2p_8y^3))) \sinh(Mh) + (1+2h)M^3p_5\sinh(2Mh))) \\ &\quad \sinh(My) + 4M^3p_5\delta L\sinh(2My))R_{m]}. \end{aligned} \qquad (7.48)$$

The current density distribution is given by

$$J_z = -\frac{\partial^2\Phi}{\partial y^2}. \qquad (7.49)$$

7.3 Analysis

Our special concern here is to discuss the influence of various parameters of interest on the pressure rise per wave length (Δp_λ), the magnetic force function (Φ), the axial induced magnetic field (h_x), the current density distribution (J_z) and trapping. For this purpose, we have prepared the Figures (7.1-7.8).

The expression for pressure rise per wavelength is computed numerically (due to the complexity of the integral) and few observations have been extracted by graphs. The variation of pressure rise per wavelength Δp_λ versus dimensionless flow rate θ for different values of parameters λ_1, λ_2, Hartman number M and wave number δ (i.e. our perturbation parameter) can be seen from Figures 7.1-7.4. Figures 7.1 and 7.2 show that for the pumping region ($\Delta p_\lambda > 0$), the increase in the values of λ_1 and λ_2 gives better pumping performance. For free pumping region ($\Delta p_\lambda = 0$), there is no difference between the pumping characteristics of the Newtonian and second order fluids. However in copumping region ($\Delta p_\lambda < 0$) the behaviour is quite opposite when compared to the pumping region. There is decrease in pumping when we increase λ_1 and λ_2 in copumping region. Figure 7.3 reveals that the pumping increases with an increase in perturbation parameter δ both in pumping and free pumping regions whereas pumping decreases with the increase of δ in copumping region. Figure 7.4 is plotted to see the effect of Hartman number M on Δp_λ. It is observed that Δp_λ increases for large values of M both in pumping and copumping regions. For an appropriate value of volume flow rate θ, the pressure rise per wavelength (Δp_λ) increases for small values of M in the copumping regions.

Figures 7.5-7.7 discuss the variations of magnetic force function Φ, axial induced magnetic field h_x and current density distribution J_z versus y for the different values of viscoelasticity λ_1, perturbation parameter δ and Hartman number M. Figures 7.5(a-c) show that the magnetic force function Φ increases as λ_1 and M increase but decreases for large values of δ. In Figures 7.6(a-c), it is noted that the axial induced magnetic field h_x is symmetric about the channel and decreases with an increase in λ_1 and M. It also decreases with the decrease of δ. Moreover, in the half region h_x is one direction while in the other half region, it is in opposite direction and is equal to zero at $y = 0$. The Figures 7.7(a-c) are plotted to show that the current density distribution J_z increases with the increase in λ_1 and M. Moreover J_z increases with a decrease in δ.

We now discuss trapping. The formation of an internally circulating bolus of fluid by closed stream lines is called trapping and this trapped bolus is pushed ahead along with peristaltic wave. The effects of Hartman number on trapping is illustrated in the Figures 7.8(a-c). The Figures 7.8(a-c) reflect that the size of the trapped bolus decreases with the increase of λ_1. In these figures, we have observed that the size of the trapped bolus decreases with the increase of M and vanishes ultimately when large values of M are taken into account.

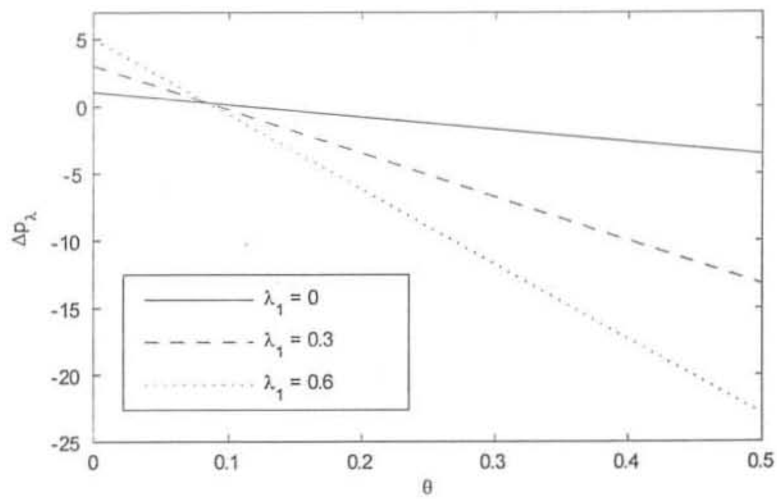

Figure 7.1: Plot showing Δp_λ versus flow rate θ when $\phi = 0.4$, $\delta = 0.6$, $M = 0.5$, $E = 0.8$, $R_m = 1$, $R_e = 1$, $S_t = 1$ and $\lambda_2 = 0.8$.

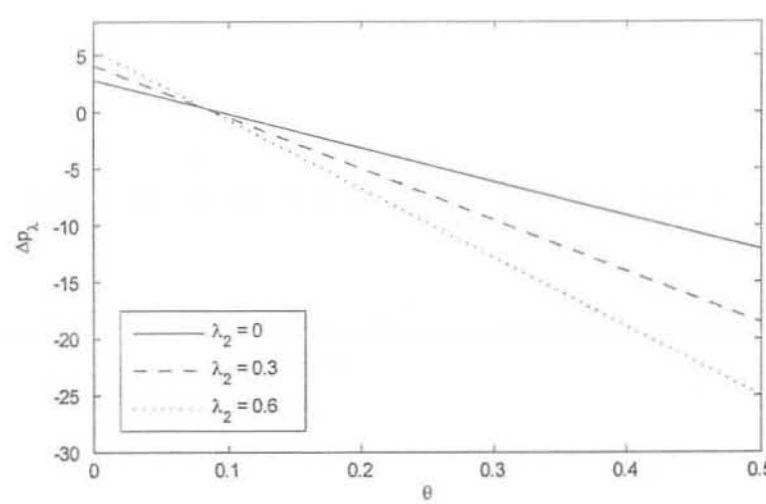

Figure 7.2: Plot showing Δp_λ versus flow rate θ when $\phi = 0.4$, $\delta = 0.6$, $M = 0.5$, $E = 0.8$,

$R_m = 1$, $R_e = 1$, $S_t = 1$ and $\lambda_1 = 0.8$.

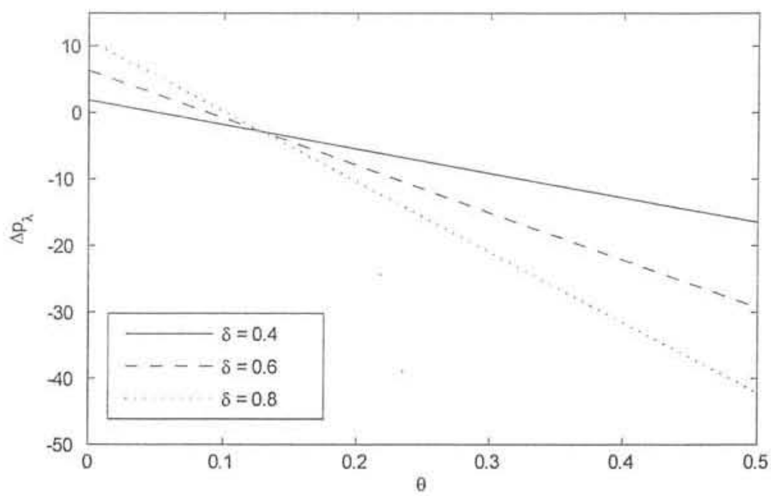

Figure 7.3: Plot showing Δp_λ versus flow rate θ when $\phi = 0.4$, $M = 0.5$, $E = 0.8$, $R_m = 1$, $R_e = 1$, $S_t = 1$, $\lambda_1 = 0.8$, and $\lambda_2 = 0.8$.

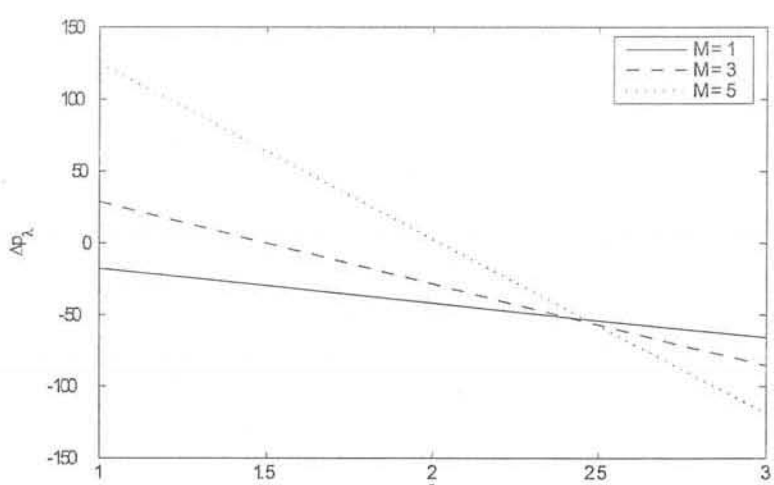

Figure 7.4: Plot showing Δp_λ versus flow rate θ when $\phi = 0.4$, $\delta = 0.6$, $E = 0.8$, $R_m = 1$,

$R_e = 1$, $S_t = 1$, $\lambda_1 = 0.8$ and $\lambda_2 = 0.8$.

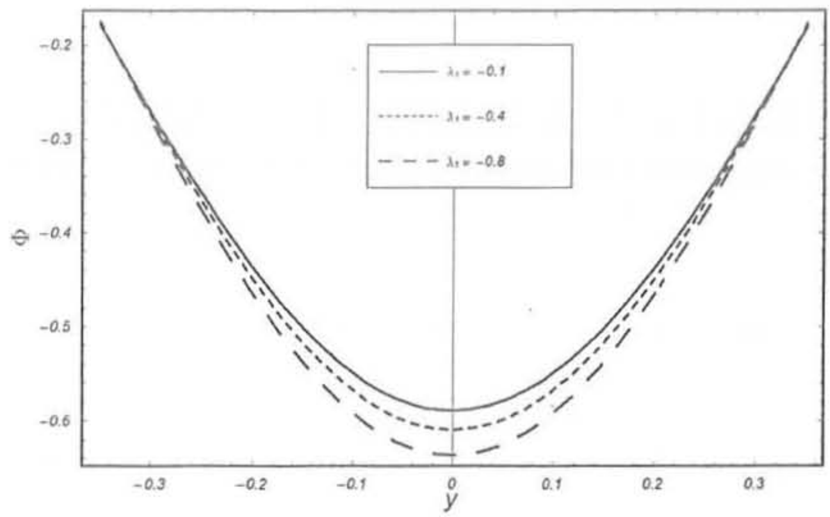

Figure 7.5a: Magnetic force function versus y when $\theta = 2$, $R_m = 1$, $R_e = 1$, $M = 1$, $E = 0.8$, $\phi = 0.6$, $\delta = 0.08$ and $x = \frac{\pi}{2}$.

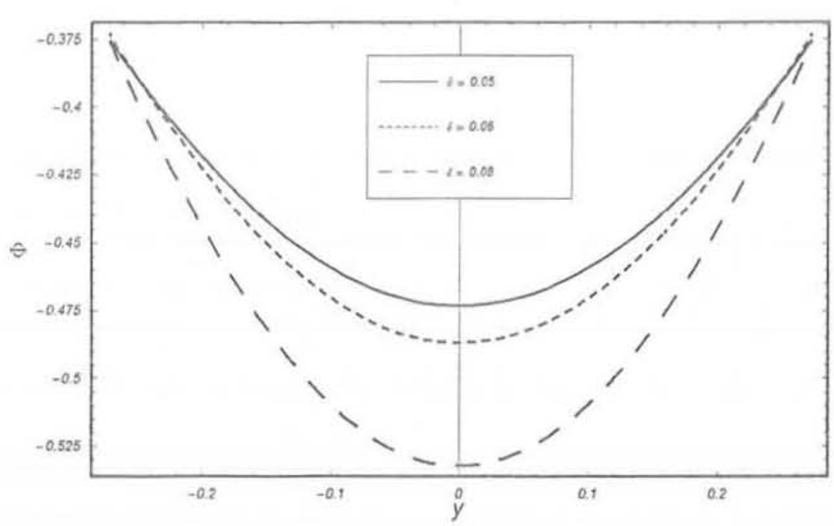

Figure 7.5b: Magnetic force function versus y when $\theta = 2$, $R_m = 1$, $R_e = 1$, $M = 1$, $E = 0.8$, $\phi = 0.6$, $\lambda_1 = -0.8$ and $x = \frac{\pi}{2}$.

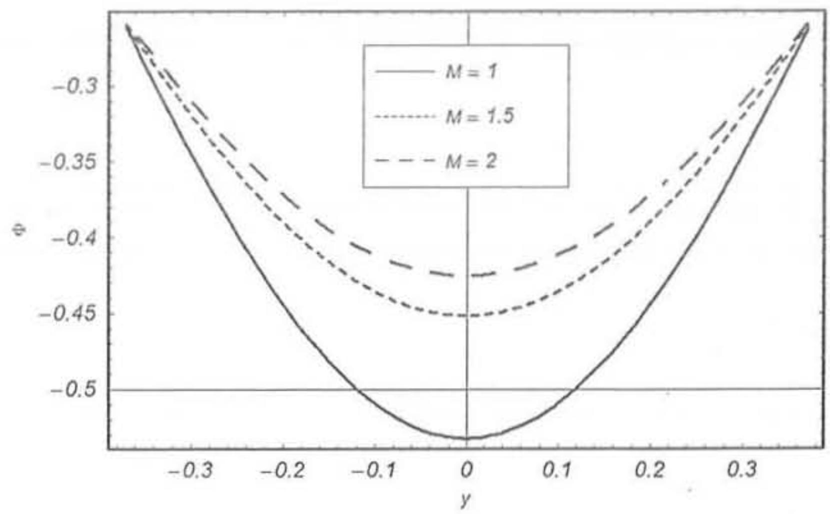

Figure 7.5c: Magnetic force function versus y when $\theta = 2$, $R_m = 1$, $R_e = 1$, $E = 0.8$, $\phi = 0.6$, $\delta = 0.08$, $\lambda_1 = -0.8$ and $x = \frac{\pi}{2}$.

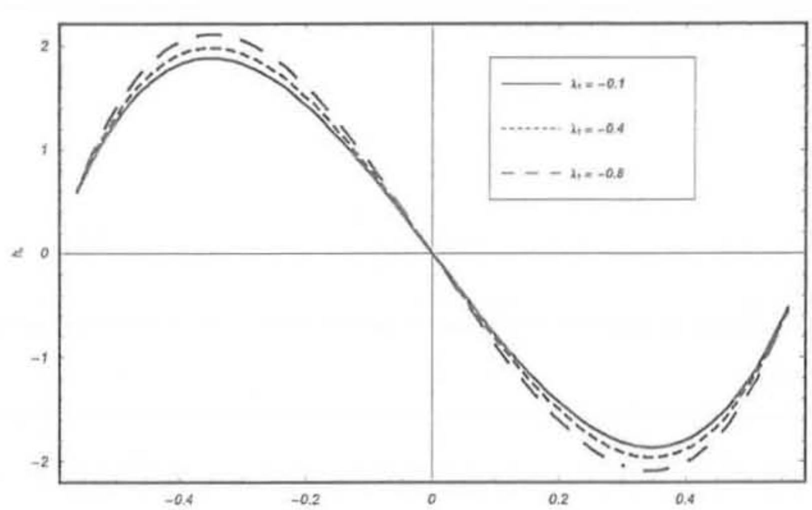

Figure 7.6a: Axial induced magnetic field h_x versus y when $\theta = 2$, $R_m = 1$, $R_e = 1$, $M = 1$, $E = 0.8$, $\phi = 0.6$, $\delta = 0.08$ and $x = \frac{\pi}{2}$.

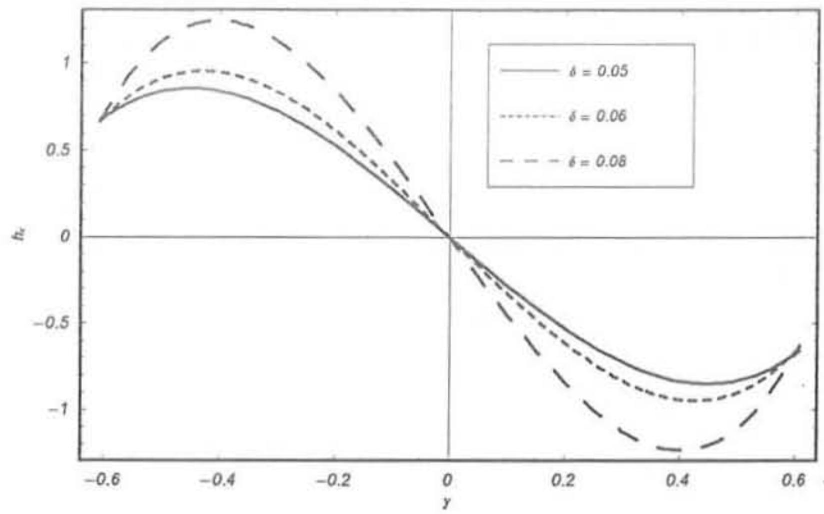

Figure 7.6b: Axial induced magnetic field h_x versus y when $\theta = 2$, $R_m = 1$, $R_e = 1$, $M = 1$, $E = 0.8$, $\phi = 0.6$, $\lambda_1 = -0.8$ and $x = \frac{\pi}{2}$.

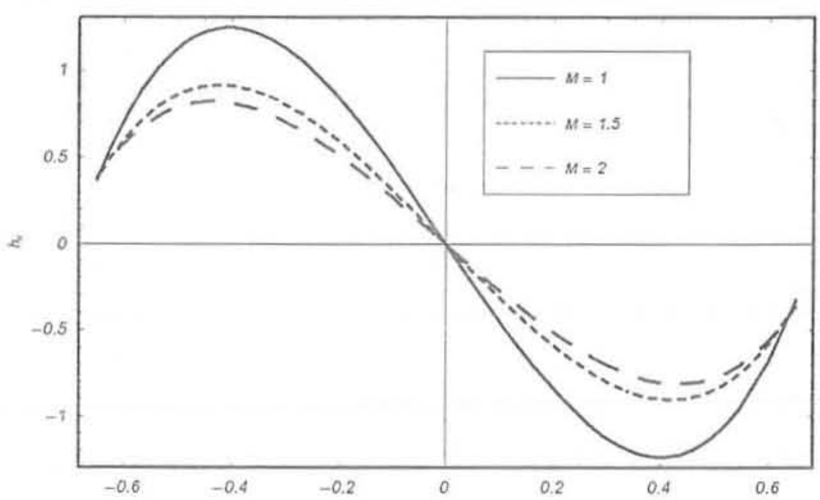

Figure 7.6c: Axial induced magnetic field h_x versus y when $\theta = 2$, $R_m = 1$, $R_e = 1$, $\delta = 0.08$, $E = 0.8$, $\phi = 0.6$, $\lambda_1 = -0.8$ and $x = \frac{\pi}{2}$.

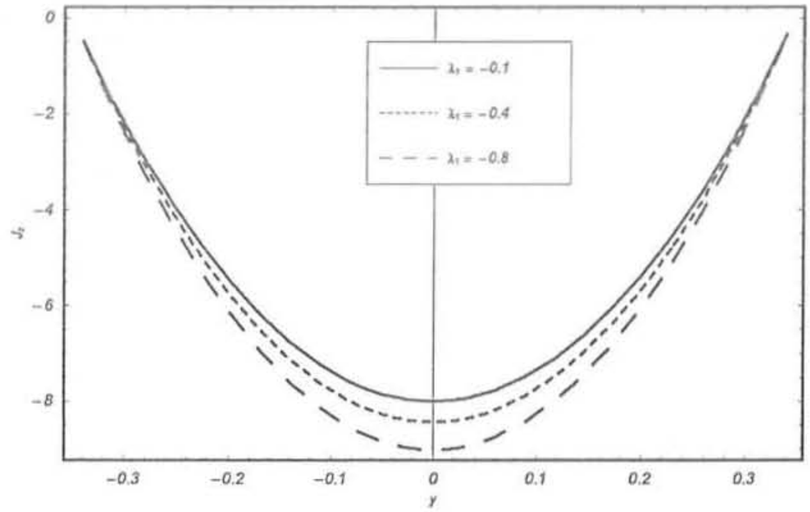

Figure 7.7a: Current density distribution versus y when $\theta = 2$, $R_m = 1$, $R_e = 1$, $M = 1$, $E = 0.8$, $\phi = 0.6$, $\delta = 0.08$ and $x = \frac{\pi}{2}$.

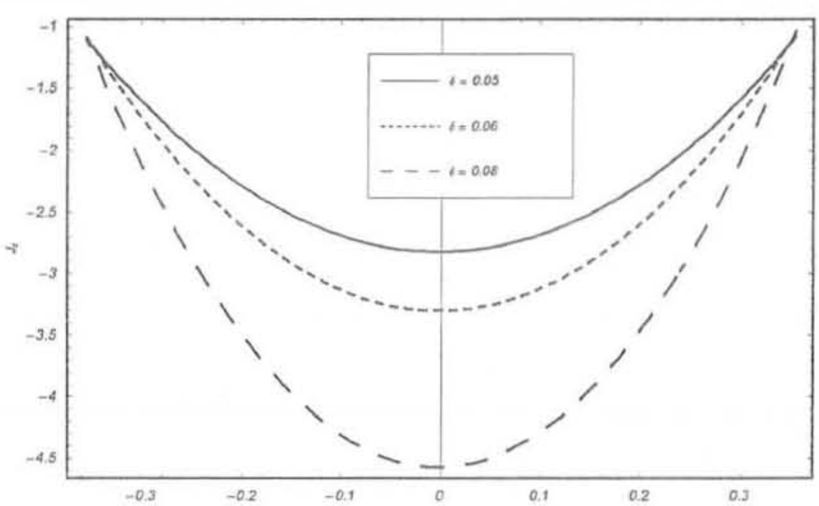

Figure 7.7b: Current density distribution versus y when $\theta = 2$, $R_m = 1$, $R_e = 1$, $M = 1$, $E = 0.8$, $\phi = 0.6$, $\lambda_1 = -0.8$ and $x = \frac{\pi}{2}$.

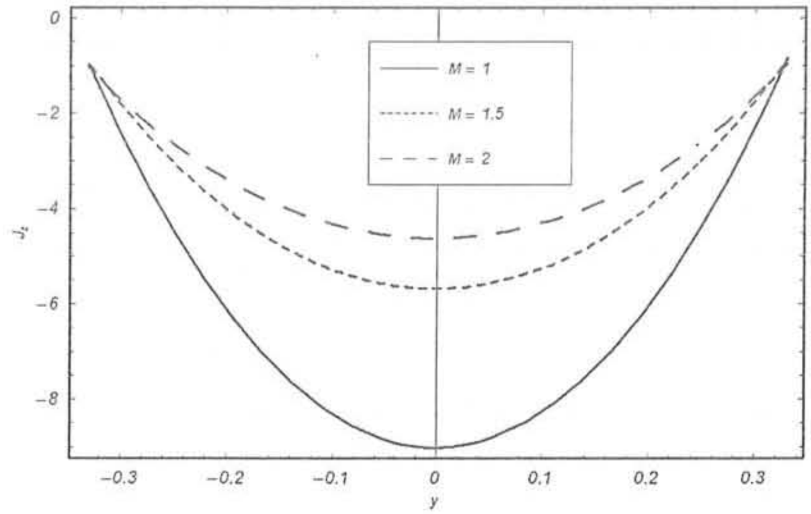

Figure 7.7c: Current density distribution versus y when $\theta = 2$, $R_m = 1$, $R_e = 1$, $\delta = 0.08$, $E = 0.8$, $\phi = 0.6$, $\lambda_1 = -0.8$ and $x = \frac{\pi}{2}$.

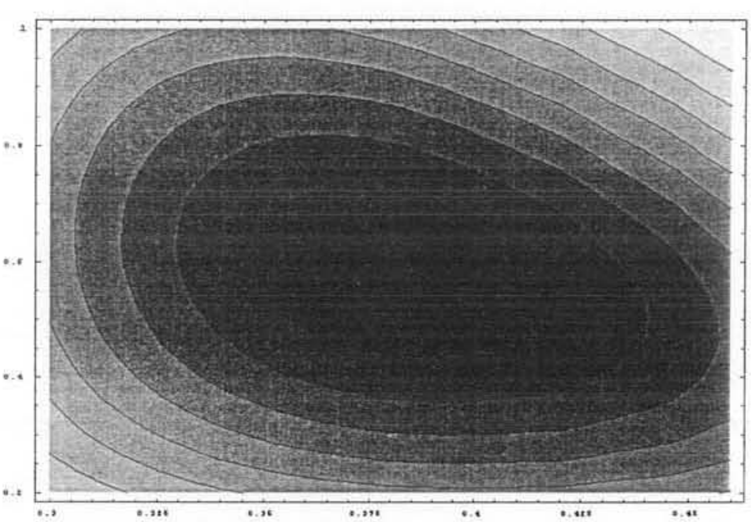

Figure 7.8a: Streamlines for $M = 0.2$ when $\phi = 0.6$, $R_e = 1$, $R_m = 0.5$, $\delta = 0.09$, $\theta = 1$, $E = 0.8$ and $\lambda_1 = 0.9$.

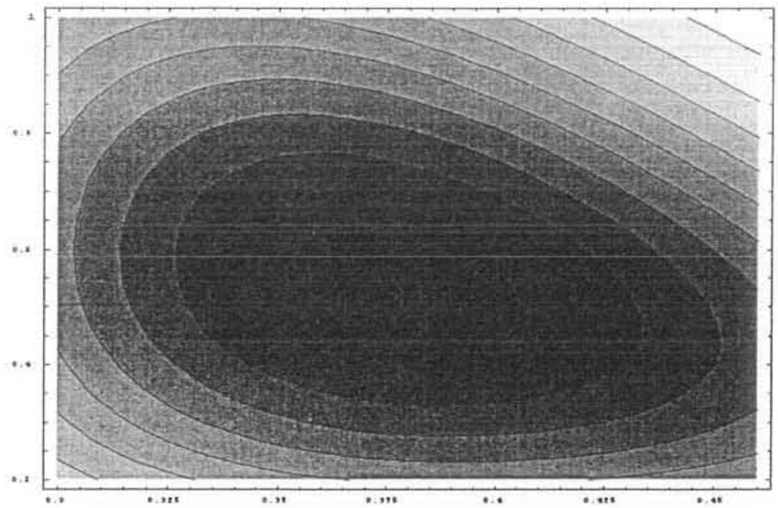

Figure 7.8b: Streamlines for $M = 0.6$ when $\phi = 0.6$, $R_e = 1$, $R_m = 0.5$, $\delta = 0.09$, $\theta = 1$, $E = 0.8$ and $\lambda_1 = 0.9$.

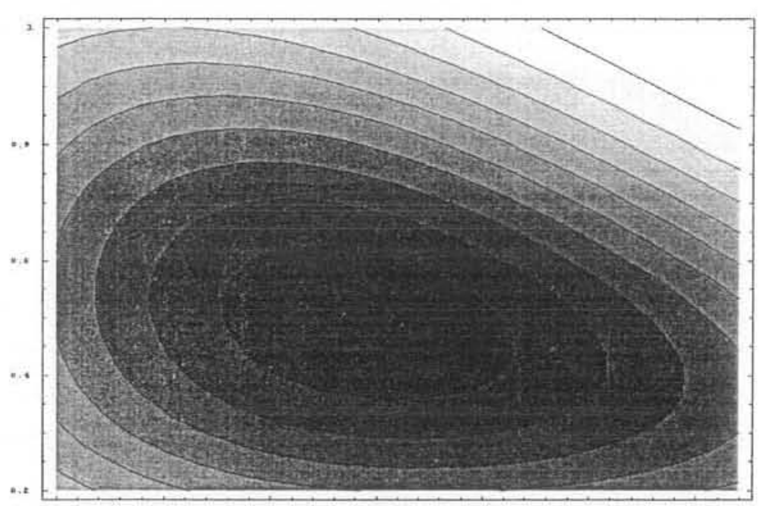

Figure 7.8c: Streamlines for $M = 1$ when $\phi = 0.6$, $R_e = 1$, $R_m = 0.5$, $\delta = 0.09$, $\theta = 1$, $E = 0.8$ and $\lambda_1 = 0.9$.

Chapter 8

Simultaneous effects of induced magnetic field and heat and mass transfer on the peristaltic motion of second order fluid in a channel

This chapter mainly investigates the effects of heat and mass transfer on the peristaltic flow of magnetohydrodynamic (MHD) second order fluid in a channel. The induced magnetic field is taken into account. Problem formulation in a wave frame of reference is presented. The analysis of governing nonlinear problem is carried out under the assumption of small wave number. Explicit expressions of the pressure gradient, the stream function, the magnetic force function, the axial induced magnetic field, the current density distribution, the temperature and concentration distribution are derived. The effects of embedded parameters into the mathematical problem are also highlighted.

8.1 Problem formulation

We consider the peristaltic flow of an incompressible second order and electrically conducting fluid in the presence of a uniform magnetic field. In addition heat and mass transfer effects are considered. Mathematical analysis is performed in the presence of an induced magnetic field. The second order fluid is considered in a symmetric channel of uniform thickness and non-conducting walls. A wave of amplitude b propagates along the channel walls with constant speed c. The geometry of the wall is

$$h'(X',t') = a + b\sin\frac{2\pi}{\lambda}(X'-ct'), \tag{8.1}$$

in which a is the half width of channel, b the wave amplitude, λ the wavelength, t' the time and c the wave speed in the X'-direction. According to the Cartesian coordinate system, the X'-axis is chosen parallel to the channel walls and Y'-axis is taken normal to it. The fluid properties are taken to be constant. The physical properties ρ, μ, α_1, α_2, μ_e, σ, D, K_T, T_m, etc are taken constant throughout the fluid. The system is stressed by an external transverse uniform magnetic field of strength $H_0^{'}$ which gives rise to an induced magnetic field $H^{'}(h_{X'}^{'}(X^{'},Y^{'},t^{'}),\ h_{Y'}^{'}(X^{'},Y^{'},t^{'}),\ 0)$ and the total magnetic field therefore is $H^{'+}(h_{X'}^{'}(X^{'},Y^{'},t^{'}),\ H_0^{'}+h_{Y'}^{'}(X^{'},Y^{'},t^{'}),\ 0)$. The governing equations of magneto second order fluid are:

(i) Maxwell's equations

$$\nabla\cdot\mathrm{H}^{'}=0,\quad \nabla\cdot\mathrm{E}=0, \tag{8.2}$$

$$\nabla\wedge\mathrm{H}^{'}=\mathrm{J}^{'},\quad \mathrm{J}^{'}=\sigma\left\{\mathrm{E}+\mu_e\left(\mathrm{V}^{'}\wedge\mathrm{H}^{'+}\right)\right\}, \tag{8.3}$$

$$\nabla\wedge\mathrm{E}=-\mu_e\frac{\partial\mathrm{H}^{'}}{\partial t^{'}}. \tag{8.4}$$

(ii) The induction equation

$$\frac{\partial\mathrm{H}^{'+}}{\partial t^{'}}=\nabla\wedge\left\{\mathrm{V}'\wedge\mathrm{H}^{'+}\right\}+\frac{1}{\zeta}\nabla^2\mathrm{H}^{'+}. \tag{8.5}$$

(iii) The continuity equation

$$\nabla\cdot\overline{\mathrm{V}'}=0. \tag{8.6}$$

(iv) The equation of motion

$$\rho\left[\frac{\partial}{\partial t'}+(\mathbf{V}'\cdot\nabla)\right]\overline{\mathbf{V}'}=-\nabla p'+\operatorname{div}\overline{\tau'}-\mu_e\left\{(\mathbf{H}'^{+}\cdot\nabla)-\frac{1}{2}\left(H'^{+}\right)^2\nabla\right\}. \tag{8.7}$$

(v) The energy equation

$$\rho C_p\frac{dT}{dt}=\kappa\nabla^2T+trac\,(\mathbf{T.L}). \tag{8.8}$$

(vi) The concentration equation

$$\frac{dC}{dt}=D\nabla^2C+\frac{DK_T}{T_m}\nabla^2T, \tag{8.9}$$

where p' is the fluid pressure, ρ the density, $\mathbf{J}$ the current density, μ_e magnetic permeability, σ the electrical conductivity, $\mathbf{E}$ the electric field, $\varsigma=\sigma\mu_e$ the magnetic diffusivity, C_p specific heat, T the temperature, D the coefficient of mass diffusivity, T_m the mean temperature, K_T the thermal diffusion ratio, C the concentration, κ the thermal conductivity, $\overline{\mathbf{V}'}$ the velocity and extra stress $\overline{\tau'}$ in a second order fluid is defined by equations (7.3)-(7.5).

Using the transformations (7.2) and the velocity field (7.11), the resulting two-dimensional equations in the present case are

$$\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}=0, \tag{8.10}$$

$$\begin{aligned}\rho\left(u'\frac{\partial}{\partial x'}+v'\frac{\partial}{\partial y'}\right)u'+\frac{\partial p'}{\partial x'} &= \frac{\partial\tau'_{x'x'}}{\partial x'}+\frac{\partial\tau'_{x'y'}}{\partial y'}-\frac{\mu_e}{2}\left(\frac{\partial H^{+^2}}{\partial x'}\right)\\ &\quad+\mu_e\left(h'_{x'}\frac{\partial h'_{x'}}{\partial x'}+h'_{y'}\frac{\partial h'_{x'}}{\partial y'}+H_0\frac{h'_{x'}}{\partial y'}\right),\end{aligned} \tag{8.11}$$

$$\begin{aligned}\rho\left(u'\frac{\partial}{\partial x'}+v'\frac{\partial}{\partial y'}\right)v'+\frac{\partial p'}{\partial y'} &= \frac{\partial\tau'_{x'y'}}{\partial x'}+\frac{\partial\tau'_{y'y'}}{\partial y'}-\frac{\mu_e}{2}\left(\frac{\partial H^{+^2}}{\partial y'}\right)\\ &\quad+\mu_e\left(h'_{x'}\frac{\partial h'_{y'}}{\partial x'}+h'_{y'}\frac{\partial h'_{y'}}{\partial y'}+H_0\frac{\partial h'_{y'}}{\partial y'}\right),\end{aligned} \tag{8.12}$$

$$\rho C_p\left(u'\frac{\partial}{\partial x'}+v'\frac{\partial}{\partial y'}\right)\bar{T} = \kappa\left(\frac{\partial^2 T}{\partial x'^2}+\frac{\partial^2 T}{\partial y'^2}\right)+\tau'_{x'x'}\frac{\partial u'}{\partial x'} +\tau'_{x'y'}\left(\frac{\partial v'}{\partial x'}+\frac{\partial u'}{\partial y'}\right)+\tau'_{y'y'}\frac{\partial v'}{\partial y'}, \quad (8.13)$$

$$\left(u'\frac{\partial}{\partial x'}+v'\frac{\partial}{\partial y'}\right)C = D\left[\frac{\partial^2 C}{\partial x'^2}+\frac{\partial^2 C}{\partial y'^2}\right]+\frac{DK_T}{T_m}\left(\frac{\partial^2 T}{\partial x'^2}+\frac{\partial^2 T}{\partial y'^2}\right). \quad (8.14)$$

The non-dimensional variables may be posited in the form

$$x=\frac{2\pi x'}{\lambda},\ y=\frac{y'}{a},\ u=\frac{u'}{c},\ v=\frac{v'}{c},\ t=\frac{2\pi t'c}{\lambda},\ p=\frac{2\pi a^2 p'}{c\lambda\mu},\ \tau=\frac{a\tau'}{\mu c},$$
$$h=\frac{h'}{a},\ \Psi=\frac{\Psi'}{ca},\ \Phi=\frac{\Phi'}{H_0 a},\ \delta=\frac{a}{\lambda},\ \mathrm{Re}=\frac{\rho ca}{\mu},\ R_m=\sigma\mu_e ac,\ \lambda_1=\frac{\alpha_1 c}{\mu a},$$
$$\lambda_2=\frac{\alpha_2 c}{\mu a},\ u=\frac{\partial\Psi}{\partial y},\ v=-\delta\frac{\partial\Psi}{\partial x},\ h_x=\frac{\partial\Phi}{\partial y},\ h_y=-\delta\frac{\partial\Phi}{\partial x},\ \mathrm{Re}\,R_m S_t^2=M^2,$$
$$S_t=\frac{H_0}{c}\sqrt{\frac{\mu_e}{\rho}},\ p_m=p+\frac{1}{2}\mathrm{Re}\,\delta\frac{\mu_e\left(H'^{+}\right)^2}{\rho c^2},\ E=\frac{-E}{cH_0\mu_e},\ E_1=\frac{c^2}{C_p\left(T_1-T_0\right)},$$
$$\mathrm{Pr}=\frac{\mu C_p}{\kappa},\ \theta'=\frac{T-T_0}{T_1-T_0},\ \varphi=\frac{C-C_0}{C_1-C_0},\ Sc=\frac{\mu}{\rho D},\ Sr=\frac{\rho T_0 DK_T}{\mu T_m C_0},\ Br=E_1 P_r, \quad (8.15)$$

where E_1, P_r, Sc, Sr, δ, Re, R_m, S and M are respectively the Eckert, Prandtl, Schmidt, Soret, wave, Reynolds, magnetic Reynolds, Strommer's and Hartman numbers. Here p_m is the total pressure which is sum of ordinary and magnetic pressures, Ψ the stream function and Φ the magnetic force function. Also T_0, C_0 and T_1, C_1 are temperatures and concentrations at $y=0$ and $y=h$ respectively.

The non-dimensional forms of Eqs. (8.11)-(8.14) are

$$\mathrm{Re}\,\delta\left\{\left(\Psi_y\frac{\partial}{\partial x}-\Psi_x\frac{\partial}{\partial y}\right)\Psi_y\right\}=-\frac{\partial p_m}{\partial x}+\delta\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{xy}}{\partial y} +\mathrm{Re}\,\delta S_t^2\left\{\left(\Phi_y\frac{\partial}{\partial x}-\Phi_x\frac{\partial}{\partial y}\right)\Phi_y\right\}+\mathrm{Re}\,S_t^2\Phi_{yy}, \quad (8.16)$$

$$\mathrm{Re}\,\delta^3\left\{\left(\Psi_y\frac{\partial}{\partial x}-\Psi_x\frac{\partial}{\partial y}\right)\Psi_x\right\}=-\frac{\partial p_m}{\partial y}+\delta^2\frac{\partial\tau_{xy}}{\partial x}+\delta\frac{\partial\tau_{yy}}{\partial x}$$
$$-\mathrm{Re}\,\delta^3S_t^2\left\{\left(\Phi_y\frac{\partial}{\partial x}-\Phi_x\frac{\partial}{\partial y}\right)\Phi_x\right\}-\mathrm{Re}\,\delta^2S_t^2\Phi_{xy}, \qquad (8.17)$$

$$\Psi_y-\delta\left(\Psi_y\Phi_x-\Psi_x\Phi_y\right)+\frac{1}{R_m}\nabla^2\Phi=E, \qquad (8.18)$$

$$\delta\,\mathrm{Pr}\,\mathrm{Re}\left[\frac{\partial\Psi}{\partial y}\frac{\partial\theta'}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial\theta'}{\partial y}\right]=\left(\delta^2\frac{\partial^2\theta'}{\partial x^2}+\frac{\partial^2\theta'}{\partial y^2}\right)+Br[\delta\tau_{xx}\frac{\partial^2\Psi}{\partial x\partial y}$$
$$+\tau_{xy}\left(\frac{\partial^2\Psi}{\partial y^2}-\delta\frac{\partial^2\Psi}{\partial x^2}\right)+\delta\tau_{yy}\frac{\partial^2\Psi}{\partial y\partial x}], \qquad (8.19)$$

$$\delta\,\mathrm{Re}\left[\frac{\partial\Psi}{\partial y}\frac{\partial\varphi}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial\varphi}{\partial y}\right] = \frac{1}{Sc}\left[\delta^2\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\right]$$
$$+Sr\left[\delta^2\frac{\partial^2\theta'}{\partial x^2}+\frac{\partial^2\theta'}{\partial y^2}\right], \qquad (8.20)$$

where Eq. (8.10) is automatically satisfied and the values of τ_{xx}, τ_{xy} and τ_{yy} are given in the previous chapter.

Elimination of pressure between Eqs. (8.16) and (8.17) gives

$$\mathrm{Re}\,\delta\left\{\left(\Psi_y\frac{\partial}{\partial x}-\Psi_x\frac{\partial}{\partial y}\right)\nabla^2\Psi\right\} = \nabla^4\Psi+\lambda_1\delta\left\{\left(\Psi_y\frac{\partial}{\partial x}-\Psi_x\frac{\partial}{\partial y}\right)\nabla^4\Psi\right\}$$
$$+\mathrm{Re}\,S_t^2\delta\left\{\left(\Phi_y\frac{\partial}{\partial x}-\phi_x\frac{\partial}{\partial y}\right)\nabla^2\Phi\right\}$$
$$+\mathrm{Re}\,S_t^2\nabla^2\Phi_y, \qquad (8.21)$$

with

$$\nabla^2=\delta^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},\ \nabla^4=\nabla^2\nabla^2.$$

The non-dimensional boundary conditions and pressure rise per wavelength Δp_λ in wave frame are given as

$$\Psi = 0, \quad \frac{\partial^2 \Psi}{\partial y^2} = 0, \quad \frac{\partial \Phi}{\partial y} = 0, \quad \frac{\partial \theta'}{\partial y} = 0, \quad \frac{\partial \varphi}{\partial y} = 0 \text{ at } y = 0,$$

$$\Psi = F, \quad \frac{\partial \Psi}{\partial y} = -1, \quad \Phi = 0, \quad \theta' = 0, \quad \varphi = 0 \text{ at } y = h, \tag{8.22}$$

$$\Delta p_\lambda = \int_0^{2\pi} \left(\frac{dp}{dx}\right) dx. \tag{8.23}$$

The dimensionless mean flows in laboratory (θ) and wave (F) frames are related through the expressions given below

$$\theta = F + 1, \tag{8.24}$$

$$F = \int_0^h \frac{\partial \Psi}{\partial y} dy, \tag{8.25}$$

whence

$$h(x) = 1 + \phi \sin 2\pi x \tag{8.26}$$

and $\phi (= b/a)$ is the amplitude ratio.

8.2 Perturbation solution

For series solution, we expand the flow quantities in terms of small wave number (δ) as:

$$\Psi = \Psi_0 + \delta \Psi_1 + O(\delta)^2, \tag{8.27}$$

$$F = F_0 + \delta F_1 + O(\delta)^2, \tag{8.28}$$

$$p = p_0 + \delta p_1 + O(\delta)^2, \tag{8.29}$$

$$\Phi = \Phi_0 + \delta \Phi_1 + O(\delta)^2. \tag{8.30}$$

Substituting Eqs. (8.27)-(8.30) into Eqs.(8.16)-(8.22) and then comparing the coefficients of like powers of δ, we have the following systems

8.2.1 Zeroth order system

$$\frac{\partial^4\Psi_0}{\partial y^4} - M^2\frac{\partial^2\Psi_0}{\partial y^2} = 0,$$

$$\frac{\partial^2\Phi_0}{\partial y^2} = R_m(E - \frac{\partial\Psi_0}{\partial y}),$$

$$\frac{\partial p_0}{\partial x} = \frac{\partial^3\Psi_0}{\partial y^3} + M^2(E - \frac{\partial\Psi_0}{\partial y}),$$

$$\frac{\partial p_0}{\partial y} = 0,$$

$$\frac{\partial^2\theta_0'}{\partial y^2} + Br\left\{\left(\frac{\partial^2\Psi_0}{\partial y^2}\right)^2\right\} = 0$$

$$\frac{1}{Sc}\frac{\partial\varphi_0}{\partial y} + Sr\frac{\partial^2\theta_0'}{\partial y^2} = 0,$$

$$\Psi_0 = 0, \quad \frac{\partial^2\Psi_0}{\partial y^2} = 0, \quad \frac{\partial\Phi_0}{\partial y} = 0, \quad \frac{\partial\theta_0'}{\partial y} = 0, \quad \frac{\partial\varphi_0}{\partial y} = 0, \text{ at } y = 0,$$
$$\frac{\partial\Psi_0}{\partial y} = -1, \quad \Psi_0 = F_0, \quad \Phi_0 = 0, \quad \theta_0' = 0 \quad \varphi_0 = 0 \text{ at } y = h.$$

8.2.2 First order system

$$\mathrm{Re}\left\{\left(\frac{\partial\Psi_0}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\Psi_0}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial^2\Psi_0}{\partial y^2}\right\} = \frac{\partial^4\Psi_1}{\partial y^4} + \lambda_1\left\{\left(\frac{\partial\Psi_0}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\Psi_0}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial^4\Psi_0}{\partial y^4}\right\}$$
$$+\,\mathrm{Re}\,S_t^2\left\{\left(\frac{\partial\Phi_0}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\Phi_0}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial^2\Phi_0}{\partial y^2}\right\} + \mathrm{Re}\,S_t^2\frac{\partial^3\Phi_1}{\partial y^3},$$

$$\frac{\partial^2\Phi_1}{\partial y^2} = R_m\left\{\left(\frac{\partial\Psi_0}{\partial y}\frac{\partial\Phi_0}{\partial x} - \frac{\partial\Psi_0}{\partial x}\frac{\partial\Phi_0}{\partial y}\right) - \frac{\partial\Psi_1}{\partial y}\right\},$$

$$\frac{\partial p_1}{\partial x} = -\mathrm{Re}\left\{\left(\frac{\partial\Psi_0}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\Psi_0}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial\Psi_0}{\partial y}\right\} + \frac{\partial^3\Psi_1}{\partial y^3} - \lambda_1\frac{\partial\Psi_0}{\partial x}\frac{\partial^4\Psi_0}{\partial y^4} + \frac{\partial}{\partial x}\left\{\begin{array}{c}\frac{\partial\Psi_0}{\partial y}\frac{\partial^3\Psi_0}{\partial y^3} + \\ \frac{1}{2}(2\lambda_1 + \lambda_2)(\frac{\partial^2\Psi_0}{\partial y^2})^2\end{array}\right\},$$

$$\frac{\partial p_1}{\partial y} = 2(2\lambda_1 + \lambda_2)\frac{\partial^2\Psi_0}{\partial y^2}\frac{\partial^3\Psi_0}{\partial y^3},$$

$$\mathrm{Re}\left(\frac{\partial\Psi_0}{\partial y}\frac{\partial\theta_0'}{\partial x}-\frac{\partial\Psi_0}{\partial x}\frac{\partial\theta_0'}{\partial y}\right)=\frac{\partial^2\theta_1'}{\partial y^2}+Br\left[\begin{array}{c}\left\{2(\frac{3}{2}\lambda_2-\lambda_1)\left(\frac{\partial^2\Psi_0}{\partial y^2}\right)^2\right\}\frac{\partial^2\Psi_0}{\partial x\partial y}\\+\left\{\frac{\partial^2\Psi_1}{\partial y^2}+\lambda_1\left(\frac{\partial^3\Psi_0}{\partial x\partial y^2}-\frac{\partial^3\Psi_1}{\partial y^3}+\frac{\partial^2\Psi_0}{\partial x\partial y}\frac{\partial^2\Psi_0}{\partial y^2}\right)\right\}\frac{\partial^2\Psi_0}{\partial y^2}\end{array}\right]$$

$$\mathrm{Re}\left[\frac{\partial\Psi_0}{\partial y}\frac{\partial\varphi_0}{\partial x}-\frac{\partial\Psi_0}{\partial x}\frac{\partial\varphi_0}{\partial y}\right]=\frac{1}{Sc}\left(\frac{\partial^2\varphi_1}{\partial y^2}\right)+Sr\left(\frac{\partial^2\theta_1'}{\partial y^2}\right)$$

$$\Psi_1=0,\quad \frac{\partial^2\Psi_1}{\partial y^2}=0,\quad \frac{\partial\Phi_1}{\partial y}=0,\quad \frac{\partial\theta_1'}{\partial y}=0,\quad \frac{\partial\varphi_1}{\partial y}=0,\ \text{ at }\ y=0,$$
$$\frac{\partial\Psi_1}{\partial y}=0,\quad \Psi_1=F_1,\quad \Phi_1=0,\quad \theta_1'=0,\quad \varphi_1=0\ \text{ at }\ y=h.$$

The above systems give the following solutions

$$\begin{aligned}\frac{dp}{dx} &= (A_{14}+(\delta(\frac{1}{A_1^3}(2A_2A_{15}M(A_1(-1+\cosh(hM)+A_{16})\,\mathrm{Re})\\&+\frac{1}{12A_1M^3}(A_9+A_{17}+M(\frac{A_3}{A_1^3}+3A_{18})+A_4\sinh(hM)\\&+\cosh(hM)(A_{13}+M(A_5-h(\frac{75(F+h)M^4R_m}{2A_1}-\frac{A_6}{A_1^3})\\&-4M(6A_{13}+A_7)))+A_{19}-\frac{1}{M^2}(e^{hM}\,\mathrm{Re}\,R_mA_{20}+M\\&A_{21})12hA_8))+\sinh(hM)(-\frac{MA_{10}}{2A_1^3}+A_{11}+A_{22}-12M^2\\&A_{23})+4h^2M^4A_{12}+2M^2A_{24}\cosh(2hM)+\cosh(hM)(\\&A_{25}+24h^2M^4A_{26}-4hM^3A_{27}-M^2A_{28}+MA_{29})S_t^2)+\\&A_{30})),\end{aligned}\tag{8.31}$$

$$\begin{aligned}\Psi &= (\frac{-L_5}{L_4}+\delta(-\frac{L_2}{ML_1^3}+\frac{1}{3L_1^3}(e^{3hM}kMy^2L_6L_7-\frac{2e^{3hM}kL_3R_m}{M^3L_1^3}-\frac{L_8}{3ML_1^3}\\&-\frac{1}{3ML_1^3}(e^{M(3h+2y)}kL_9+\frac{1}{12M^4L_1^3}(L_{10}+L_{14})+\frac{L_{20}}{6M^3L_1^4}+\frac{1}{24M^4L_1^4}\\&(e^{My}(kML_{21}+L_{22}+L_{23}+L_{24}+L_{25}+L_{26}+\frac{1}{24M^4L_1^4}L_{38})))))),\end{aligned}\tag{8.32}$$

$$\Phi = -\frac{1}{24M^5A_1}[(12M^4((F(-2+M^2(h-y)(h+y))+h(-2+M^2(-h^2+y^2)E))$$
$$\cosh(Mh)+2(F+h)\cosh(My)+M(h-y)(h+y)(1+E)\sinh(Mh)+\delta$$
$$((93p_8+M(12h(M^2p_1+2Mp_2-6p_4)-4h^3M^2p_4-8Mp_5-33p_7-6h^2$$
$$M(Mp_7-4p_8)+81Mp_9))\delta - M^2p_5\delta\cosh(3Mh)+\delta\cosh(2Mh)(51p_8+$$
$$3M(-5p_7+2h(-5p_4+M(p_2+hp_8))+2Mp_9)-2M^2(1+2hM^2)p_5$$
$$\cosh(My)-24M^5\sinh(Mh)+12h^2M^3p_3\delta\sinh(Mh)-3M^3p_5\delta\sinh(Mh)$$
$$12M^3p_3y^2\delta\sinh(Mh)-24M^5E\sinh(Mh)-4M^3p_5\delta\cosh(2My)\sinh(Mh)$$
$$-(6M^2p_1+15Mp_2-51p_4-6h^2M^2p_4-6hM^2p_7-30hMp_8)\delta\sinh(2Mh)$$
$$+2\cosh(My)(12(F+h)M^6-(-72p_8+M(4h^3M^3p_3+3Mp_5+24p_7-12$$
$$Mp_9))\delta+(-51p_4+h^2M^4(6p_1-h(2hp_4+3p_7))+3M(5p_2+24hp_8$$
$$-7p_8y)+3M^2(2p_1-12h^2p_4-8hp_7+3y(p_7+p_4y))+M^3(12h^2p_2$$
$$+11h^3p_8-4h(p_5-3p_9)-y(6p_9+3p_2y+2p_8y^2)))\delta\sinh(Mh)+(1$$
$$+2h)M^3p_5\delta\sinh(2Mh)+M^3p_5\delta\sinh(3Mh)-2(-21p_8+M(9p_7+21$$
$$p_4y+M^2y(-6p_1+y(3p_7+2p_4y))+M(4p_5-3(2p_9+3y(p_2+p_8y)))$$
$$))\delta\sinh(Mh)\sinh(My)+\cosh(Mh)(-24FM^6-(144p_8+M(-9p_5+24$$
$$p_9+4hM^2p_3(h^2-3y^2))))\delta+24hM^6E+2\delta((-72p_8+M(h^2M^2(-3Mp_2$$
$$+11p_4)+24p_7-2h^4M^3p_8+4M(p_5-3p_9)-6h^2M(-2Mp_7+6p_8+$$
$$M^2p_9)+h(51p_4+M(-15p_2+21p_8y-3M(2p_1+3y(p_7+p_4y))+M^2$$
$$y(6p_9+y(3p_2+2p_8y))))))\cosh(My)+hM(2M^3p_5\cosh(2My)+(-21$$
$$p_8+M(9p_7+21p_4yM^2y(-6p_1+3p_7y+2p_4y^2)+M(4p_5-3(2p_9+3y$$
$$(p_2+p_8y))))-4M^2p_5\cosh(My)\sinh(My))))+4M^2p_5\delta\sinh(Mh)$$
$$\sinh(2My)))R_m]. \tag{8.33}$$

The axial induced magnetic field and the distribution of current density are given by

$$h_x = -\frac{\partial\Phi}{\partial y}, \tag{8.34}$$

$$J_z = -\frac{\partial^2 \Phi}{\partial y^2}. \tag{8.35}$$

The temperature and concentration distributions are

$$\begin{aligned}
\theta' \;=\; & \frac{L_{110}}{4L_{39}^2} - \frac{1}{2160L_{39}^5M^2}(Br(F+h)k\delta(\cosh(3My)-\sinh(3My))(M(10ML_{143}+9(-240\\
& L_{72}(\cosh(3M(h+y))-\sinh(3M(h+y)))-5L_{97}(\cosh(M(2h+y))-\sinh(M(2h\\
& +y)))-2L_{98}(\cosh(3M(2h+y))-\sinh(3M(2h+y)))+120L_{96}(\cosh(3M(3h+y))\\
& -\sinh(3M(3h+y)))-5L_{101}(\cosh(M(4h+y))-\sinh(M(4h+y)))-5L_{106}(\cosh(M\\
& (6h+y))-\sinh(M(6h+y)))+k_1(\cosh(M(9h+y))-\sinh(M(9h+y)))-k_{10}(\cosh\\
& (M(h+2y))-\sinh(M(h+2y)))-960L_{109}(\cosh(M(3h+2y))-\sinh(M(3h+2y)))\\
& -k_{11}(\cosh(M(5h+2y))-\sinh(M(5h+2y)))-k_{10}(\cosh(M(9h+2y))-\sinh(M(9h\\
& +2y)))+L_{99}(\cosh(M(2h+3y))-\sinh(M(2h+3y)))-2L_{109}(\cosh(M(4h+3y))\\
& -\sinh(M(4h+3y)))+240L_{103}(\cosh(M(5h+3y))-\sinh(M(5h+3y)))+240L_{104}\\
& (\cosh(M(7h+3y))-\sinh(M(7h+3y)))+L_{74}(\cosh(M(8h+3y))-\sinh(M(8h+3\\
& y)))-k_{10}(\cosh(M(h+4y))-\sinh(M(h+4y)))+960L_{108}(\cosh(M(7h+4y))-\sinh\\
& (M(7h+4y)))+480k_5(1+FM)(-1+hM)(\cosh(M(9h+4y))-\sinh(M(9h+4y)))\\
& -120(-1+FM)(2+3hM+h^2M^2)(hM(-1+FM)-2E+2h^2M^2E)(\cosh(M(h+5y\\
&))-\sinh(M(h+5y)))-5L_{107}(\cosh(M(2h+5y))-\sinh(M(2h+5y)))-k_9(\cosh(M\\
& (3h+5y))-\sinh(M(3h+5y)))-5L_{100}(\cosh(M(4h+5y))-\sinh(M(4h+5y)))-5\\
& L_{90}(\cosh(M(6h+5y))-\sinh(M(6h+5y)))-k_9(\cosh(M(7h+5y))-\sinh(M(7h+5\\
& y)))-5L_{89}(\cosh(M(8h+5y))-\sinh(M(8h+5y)))-120L_{113}(\cosh(M(h+3y))-\\
& \sinh(M(h+3y)))-960L_{116}+L_{120}+120L_{122}+15L_{127}-5L_{140}+L_{141}+120(L_{143}\\
& +L_{144}+L_{145})+L_{150})R_m)-5\,\mathrm{Re}(2M^2(L_{115}+L_{122}-2M\phi\sin(2\pi x)L_{125})+9L_{112}R_m^2\\
& S_t^2))),
\end{aligned} \tag{8.36}$$

$$\varphi = \frac{k_{36}}{4L_{39}^2} + \frac{1}{2160L_{39}^5 M^2}(Br(F+h)kSc\delta(\cos(3My) - \sinh(3My))(-5k_{34}Sr\,\mathrm{Re} + M(9k_{52}$$
$$SrR_m + 10M(18k_{54}\,\mathrm{Re}(\Pr Sr + ScSr) + MSr(k_{55} + k_{56} + k_{57} - L_{58} + k_{59} + k_{60} + L_{91}$$
$$(\cosh(M(10h+3y)) - \sinh(M(10h+3y))) - 9L_{144}(\cosh(M(4h+5y)) - \sinh(M(4h$$
$$+5y)))\lambda_1)))), \tag{8.37}$$

where

$$A_1 = (hM\cosh(hM) - \sinh(hM)),$$
$$A_2 = (M^2\delta\cosh(2hM)(8(F+h)kM^2(1+2hM^2)\cosh(My) - hM\cosh(hM) + (1+h(F+h)$$
$$M^2\sinh(hM)R_m - 51(F+h)k(-(F+2h)M + FM\cosh(2hM) + \sinh(2hM))R_m + 6M^2($$
$$5(F+h)k(-(F+2h)M + FM\cosh(2hM) + \sinh(2hM))R_m + (F+h)kR_m(2M(F+h(2$$
$$+(-1+h^2M^2)E)) + 2M(-F+hE)\cosh(2hM) - hM(2+h(3F+2h)M^2 + (-2+2h^2$$
$$M^2E) + (-2F+h)M\sinh(2hM)R_m) + h(-5(F+h)M(-hM\cosh(hM) + \sinh(hM))^2R_m$$
$$+2kM((F+h)M^3\lambda_1(-(F+2h)M + FM\cosh(2hM) + \sinh(2hM)) - (F+h)M(-(F$$
$$+2h)M + FM\cosh(2hM) + \sinh(2hM))\,\mathrm{Re} + (2 + M^2(F^2 + 4h^2 + Fh(7-2E)) - 2$$
$$E - (2 - 2E + M^2(F^2 + Fh + 2h^2 - 2h(F+2h)E))\cosh(2hM) - M(-h + F(3+2h($$
$$F+h)M^2) + 2h(2 + h(F+h)M^2)E)\sinh(2hM)R_m))))),$$
$$A_3 = 4h^3kM^6(M(2F-3hE)\cosh(hM) + (2+3E)\sinh(hM))(-F+2h)M + FM\cosh(2hM)$$
$$+\sinh(2hM))R_m,$$

$$A_4 = M(3M(A_{30} - 4A_{34}h^2M) + A_{33} + \frac{1}{2A_1^3}(-6A_{29} + 2A_{30}h^2M^3 + h(F+h)kM^2R_m((F+$$
$$2h)M(-72 + M^2(-24 + h^2(-11 + 6M^2))) - 24hM^3(-1 + h^2M^2)E + 16hM^3\cosh(h$$
$$M) + M(F(72 + M^2(24 + h^2(11 - 6M^2))) - 24hM^2(1 + h^2M^2)E)\cosh(2hM) - 16$$
$$M^2(1 + h(F+h)M^2)\sinh(hM) + (72 + M^2(24 + h^2(11 + 6M^2(-1+8E))))\sinh(2h$$
$$M) + 12hM^3(2 + hM^3(2 + hM^2 + (-2 + FhM^2)\cosh(2hM) + (-2F+h)M\sinh(2hM$$
$$))R_m))),$$

$$
\begin{aligned}
A_5 &= h^2M^2(\frac{11h(F+h)M^4R_m}{2A_1}+\frac{1}{A_1^3}(A_{31}h+(F+h)kMR_m((F+2h)M(18+(-6+h^2)M^2\\
&+6hM^3(-1+h^2M^2)E+M(F(-18-(-6+h^2)M^2)+6hM^2(1+h^2M^2)E)\cosh(2hM\\
&)-(18+M^2(-6+h^2+12h^2M^2E))\sinh(2hM)-3hM^3(2+h(3F+2h)M^2+(-2+F\\
&hM^2)\cosh(2hM)+(-2F+h)M\sinh(2hM))R_m))),\\
A_6 &= 9kM^4(3(F+h)M(-(F+2h)M+FM\cosh(2hM)+\sinh(2hM))(M^2\lambda_1-\mathrm{Re})+3(2+M^2\\
&(F^2+4h^2+Fh(7-2E))-2E-(2-2E+M^2(F^2+Fh+2h^2(F+2h)E))\cosh(2hM)\\
&-M(-h+F(3+2h(F+h)M^2)+2h(2+h(F+h)M^2)E)\sinh(2hM))R_m+\frac{1}{A_1^3}(4(FM\\
&\cosh(hM)+\sinh(hM))(-hM\cosh(hM)+(1+h(F+h)M^2)\sinh(hM))(M^2\lambda_1-\mathrm{Re})\\
&+(-h(F+h)M^2(2+h(3F+2h)M^2)+2(-1-h(2F+h)M^2+h^3(F+h)M^4)E+(h\\
&(F+h)M^2(2-FhM^2)+2(1+h(2F+3h)M^2+h^3(F+h)M^4)E)\cosh(2hM)-hM\\
&(4E-(F+h)M^2(2F-h-6hE))\sinh(2hM))R_m)),\\
A_7 &= \frac{1}{2A_1^3}((((F+h)kM^2R_m(-6M(F+2h-hE+h^3M^2E)+4hM\cosh(hM)-6M(-F+hE\\
&+h^3M^2E)\cosh(2hM)-4(1+h(F+h)M^2\sinh(hM)+6(1+2h^2M^2E)\sinh(2hM)\\
&+3hM(2+h(3F+2h)M^2+(-2+FhM^2)\cosh(2hM)+(-2F+h)M\sinh(2hM))R_m)),\\
A_8 &= \frac{M^4}{2}(\frac{-6(F+h)R_m}{A_1}+\frac{1}{A_1^3}(k(2(-1+2F^2+4Fh+3h^2)M^2+(1+(-2F^2-2Fh+h^2)\\
&M^2)\cosh(2hM)+M(-F-3h+Fh(F+h)M^2)\sinh(2hM))(-M^2\lambda_1+\mathrm{Re})+(8-6E+\\
&M^2(F^2(4+3h^2M^2)+Fh(30+h^2M^2(5-2E)-4E)+2h^2(9-h^2M^2(-1+E)+E))\\
&+(-8+6E+M^2(F^2(-4+h^2M^2)+Fh(-6+h^2M^2(1-2E)+4E)-2h^2(5-5E+h\\
&M^2E)))\cosh(2hM)-M(10F^2hM^2+h(-4+12E+h^2M^2(-1+2E))+F(12+h^2M^2\\
&(9+2E)))\sinh(2hM))R_m))),\\
A_9 &= \frac{1}{A_1^3}(2(F+h)kM^2(-M^2(1+2hM^2)\cosh(2hM)(-hM\cosh(hM)+(1+h(F+h)M^2)\sinh\\
&(hM))-8M^3(-hM\cosh(hM)+\sinh(hM)(-hM\cosh(hM)+(1+h(F+h)M^2)\sinh(2hM\\
&))+18(-(F+2h)M+FM\cosh(2hM)+\sinh(2hM)))R_m),\\
A_{10} &= 9(F+h)kM^3R_m(2M(F+2h-hE+h^3M^2E)+2M(-F+hE+h^3M^2E)\cosh(2hM)-2\\
&(1+2h^2M^2E)\sinh(2hM)-hM(2+h(3F+2h)M^2+(-2+FhM^2)\cosh(2hM)+(-2F+
\end{aligned}
$$

$$
\begin{aligned}
& h)M\sinh(2hM))R_m), \\
A_{11} &= \frac{1}{2A_1^3}(kM(4(F+h)M^2(-hM\cosh(hM)+(1+h(F+h)M^2)\sinh+h)M^6(-(hM))-32(F \\
& hM\cosh(hM)+(1+h(F+h)M^2)\sinh(hM)+(FM\cosh(hM)+\sinh(hM))(M(F(-2+3h^2 \\
& M^2)-2h(-1+h^2M^2)(-1+E))+M(F(2+h^2M^2)-2h(1+E+h^2M^2E))\cosh(2hM) \\
& +(2+hM^2(-2F+h+4hE))\sinh(2hM))+21M^6(-(F+h)(-hM\cosh(hM)+(1+h(F \\
& +h)M^2)\sinh(hM))-2h^2M(M(2F+3hE)\cosh(hM)+(2+3E)\sinh(hM)(-(F+2h)M \\
& +FM\cosh(2hM)+\sinh(2hM))))R_m), \\
A_{12} &= \frac{1}{2A_1^3}(kM(4(F+h)M^2(-hM\cosh(hM)+(1+h(F+h)M^2)\sinh(hM))-32(F+h)M^6(-h \\
& M\cosh(hM)+(1+h(F+h)M^2M\sinh(hM))+(FM\cosh(hM)+\sinh(hM))(M(F(-2+3 \\
& h^2M^2)-2h(-1+h^2M^2)(-1+E))+M(F(2+h^2M^2)-2h(1+E+h^2M^2E))\cosh(2hM) \\
& +(2+hM^2(2F+h+4hE))\sinh(2hM)+24M^6((-F-h)(-hM\cosh(hM)+(1+h(F+h) \\
& M^2)\sinh(hM)-2h^2M(M(2F-3hE)\cosh(hM)+(2+3E)\sinh(hM))-(-(F+2h)M+2 \\
& FM\cosh(2hM)+\sinh(2hM))))R_m), \\
A_{13} &= -\frac{1}{A_1^3}(36(F+h)kM^2((-F-2h)M+FM\cosh(2hM)+\sinh(2hM))R_m), \\
A_{14} &= M^2E+\frac{1}{A_1}(M^2(FM\cosh(hM)+\sinh(hM))), \\
A_{15} &= ((-F-h)M+FM\cosh(2hM)+\sinh(hM)), \\
A_{16} &= (M(A_1F+h(F+h)M-FhM\cosh(hM))\sinh(hM)-hM(\sinh(hM))^2), \\
A_{17} &= \frac{1}{A_1^3(12kM^4}(-kM\cosh(hM)+\sinh(hM))(M(2F-3hE)\cosh(hM)+(2+3E)\sinh(hM))((- \\
& F-2h)M+FM\cosh(2hM)+\sinh(2hM))R_m), \\
A_{18} &= (-\frac{1}{A_1^3}(8(F+h)kM^3((-F-2h)M+FM\cosh(2hM)+\sinh(2hM))R_m)+M(-\frac{A_{34}}{A_1^3})), \\
A_{19} &= \frac{1}{A_1^3M}(A_2((-F-h)M+(F-hE)\cosh(hM)+(1+E)\sinh(hM))(2A_1+A_1(-2+hM^2(2F \\
& +h-2hE))\cosh(hM)+M(-2h(F+h)M+A_1(-2+h^2M^2)(F-hE)+hM(F(2-h^2M^2) \\
& +h(2+h^2M^2E))\cosh(hM))\sinh(hM)-h^3M^3(1+E)(\sinh(hM))^2)\operatorname{Re}R_m^2S_t^2), \\
A_{20} &= (-\frac{1}{A_1^3}(4(F+h)kM^6\cosh(3hM)-(hM\cosh(hM)+(1+h(F+h)M^2)\sinh(hM))R_m)+A_{35}
\end{aligned}
$$

$$+2M^2(\frac{1}{2A_1^3}(93(F+h)kM^2(FM\cosh(2hM)+\sinh(2hM)+2M^2(\frac{1}{2A_1^3}(93(F+h)kM^2((-F$$
$$-2h)M+FM\cosh(2hM)+\sinh(2hM))R_m)+\frac{1}{A_1^3}(8h^3kM^7(M(2F-3hE)\cosh(hM)+(2$$
$$+3E)\sinh(hM))((-F-2h)M+FM\cosh(2hM)+\sinh(2hM))R_m)+(\frac{1}{A_1^3}(12(F+h)kM^2$$
$$((-F-2h)M+FM\cosh(2hM)+\sinh(2hM))R_m)-\frac{1}{A_1^3}(8(F+h)kM^4((-F-2h)M+FM$$
$$\cosh(2hM)+\sinh(2hM))R_m)+M^2(\frac{1}{A_1^3}(2(F+h)kM^2(-hM\cosh(hM)+(1+h(F+h)M^2)$$
$$\sinh(hM))R_m)-\frac{1}{A_1^3}(2(F+h)kM^2R_m(2M(F+2h-hE+h^3M^2E)+2M(-F+hE+h^3M^2E)$$
$$\cosh(2hM)-2(1+2h^2M^2E)\sinh(2hM)-hM(2+h(3F+2h)M^2+(-2+FhM^2)\cosh(2hM)$$
$$+(-2F+h)\sinh(2hM))R_m)))),$$

$$\begin{aligned}
A_{21} &= (\frac{4h^3(F+h)M^6R_m}{2A_1}-\frac{1}{A_1^3}(16(F+h)kM^3(-hM\cosh(hM)+(1+h(F+h)M^2\sinh(hM))R_m) \\
&\quad -\frac{1}{A_1^3}(33(F+h)kM^3((-F-2h)M+FM\cosh(2hM)+\sinh(2hM))R_m-\frac{A_{36}}{A_1^3}+6h^2M(\frac{1}{A_1^3} \\
&\quad (2(F+h)kM^2((-F-2h)M+FM\cosh(2hM)+\sinh(2hM))R_m)-\frac{1}{A_1^3}((F+h)kM^4 \\
&\quad ((-F-2h)M+FM\cosh(2hM)+\sinh(2hM))R_m), \\
A_{22} &= (\frac{204(F+h)M^4R_m}{2A_1}-\frac{1}{MA_1^3}(2k(FM\cosh(hM)+\sinh(hM))(-hM\cosh(hM)+(1+h(F+h \\
&\quad)M^2)\sinh(hM))R_m)-12M(5A_{32}-\frac{1}{A_1^3}(12h(F+h)kM^2((-F-2h)M+FM\cosh(2hM)+ \\
&\quad \sinh(2hM))R_m)), \\
A_{23} &= (-\frac{12h^2(F+h)M^4R_m}{2A_1}+\frac{1}{A_1^3}(8h(F+h)kM^3((-F-2h)M+FM\cosh(2hM)+\sinh(2hM)) \\
&\quad R_m)-\frac{1}{A_1^3}(kM^2(4M^2\lambda_1(FM\cosh(hM)+\sinh(hM))(-hM\cosh(hM)+(1+h(F+h)M^2) \\
&\quad \sinh(hM))-4(FM\cosh(hM)+\sinh(hM))(-hM\cosh(hM)+(1+h(F+h)M^2)\sinh(hM)) \\
&\quad \mathrm{Re}+(-h(F+h)M^2(2+h(3F+2h)M^2)+2(-1-h(2F+h)M^2+h^3(F+h)M^4)E+(h \\
&\quad (F+h)M^2(2-FhM^2)+2(1+h(2F+3h)M^2+h^3(F+h)M^4)E)\cosh(2hM)-hM(4E- \\
&\quad (F+h)M^2(2F-h-6hE))\sinh(2hM)))), \\
A_{24} &= (-\frac{1}{A_1^3}(4(F+h)kM^4(1+2hM^2)(-hM\cosh(hM)+(1+h(F+h)M^2)\sinh(hM))R_m) \\
&\quad -3(-\frac{1}{2A_1^3}(17(F+h)kM^2((-F-2h)M+FM\cosh(2hM)+\sinh(2hM))R_m)+M \\
&\quad (\frac{1}{A_1^3}(5(F+h)kM^3((-F-2h)M+FM\cosh(2hM)+\sinh(2hM))R_m)+\frac{1}{A_1^3}((F+h)kM^3
\end{aligned}$$

$R_m(2M(F+2h-hE+h^3M^2E)+2M(-F+hE+h^3M^2E)\cosh(2hM)-2(1+2h^2M^2E)$
$\sinh(2hM)-hM(2+h(3F+2h)M^2+(-2+FhM^2)\cosh(2hM)+(-2F+h)M\sinh(2hM))$
$R_m))+2h(\frac{1}{2A_1}(5(F+h)M^4R_m)+M(A_{32}-\frac{1}{2A_1^3}(h(F+h)kM^2((-F-2h)M+FM$
$\cosh(2hM)+\sinh(2hM))R_m)))))),$

$$
\begin{aligned}
A_{25} &= \frac{-1}{A_1^3}(h^4(F+h)kM^8(FM\cosh(2hM)+\sinh(2hM))R_m)+4h^3M^5(\frac{A_{31}}{A_1^3}-\frac{11(F+h)M^4R_m}{2A_1}) \\
&+\frac{1}{2A_1^3}((F+h)kM^2R_m(2M(F+2h-hE+h^3M^2E)+2M(-F-hE+h^3M^2E) \\
&(\cosh(2hM)-2(1+2h^2M^2E)\sinh(2hM)-hM(2+h(3F+2h)M^2+(-2+FhM^2) \\
&(\cosh(2hM)+(-2F+h)M\sinh(2hM))R_m)), \\
A_{26} &= (\frac{-1}{A_1^3}(3(F+h)kM^2((-F-2h)M+FM(\cosh(2hM)+\sinh(2hM))R_m)+\frac{1}{A_1^3}(2(F+h)k \\
&M^4((-F-2h)M+FM(\cosh(2hM)+\sinh(2hM))R_m)+\frac{1}{2A_1^3}((F+h)kM^4R_m(2M(F+ \\
&2h-hE+h^3M^2E)+2M(-F+hE+h^3M^2E)\cosh(2hM)-2(1+2h^2M^2E)\sinh(2hM) \\
&-hM(2+h(3F+2h)M^2+(-2+FhM^2)(\cosh(2hM)+(-2F+h)M\sinh(2hM))R_m))), \\
A_{27} &= (\frac{51(F+h)M^4R_m}{2A_1}+M(-15A_{32}+\frac{1}{A_1^3}(3kM^3(4M^2\lambda_1(FM(\cosh(hM)+\sinh(hM))(-hM \\
&\cosh(hM)+(1+h(F+h)M^2)\sinh(2hM))-4(FM\cosh(hM)+\sinh(hM))(-hM\cosh(h \\
&M)+(1+h(F+h)M^2)\sinh(hM))\,\mathrm{Re}+(-h(F+h)M^2(3F+2h)M^2)+2(-1-h(2F+ \\
&h)M^2+h^3(F+h)M^4)E+(h(F+h)M^2(2-FhM^2)+2(1+h(2F+3h)M^2+h^3(F+h \\
&)M^4)E\cosh(2hM)-hM(4E-(F+h)M^2(2F-h-6hE))\sinh(2hM))R_m)))), \\
A_{28} &= (\frac{-1}{MA_1^3}(2k(FM\cosh(hM)+\sinh(hM))(-hM\cosh(hM)+(1+h(F+h)M^2)\sinh(hM))R_m) \\
&+\frac{1}{A_1^3}(144(F+h)kM^2((-F-2h)M+FM\cosh(2hM)+\sinh(2hM))R_m)+16M(\frac{-1}{A_1^3}(6(\\
&F+h)kM^3((-F-2h)M+FM\cosh(2hM)+\sinh(2hM))R_m)+M(\frac{-1}{A_1^3}(2(F+h)kM^2(-h \\
&M\cosh(hM)+(1+h(F+h)M^2)\sinh(hM))R_m)-\frac{1}{2A_1^3}(3(F+h)kM^2R_m(2M(F+2h-h \\
&E+h^3M^2E)+2M(-F+hE+h^3M^2E)\cosh(2hM)-(1+2h^2M^2E)\sinh(2hM) \\
&-hM(2+h(3F+2h)M^2+(-2+FhM^2)\cosh(2hM)+(-2F+h)M\sinh(2hM)))))), \\
A_{29} &= (\frac{-1}{A_1^3}(2(F+h)kM^2(-hM\cosh(hM)+(1+h(F+h)M^2)\sinh(hM))R_m)+\frac{1}{A_1^3}(16h(F+h) \\
&kM^7(-hM\cosh(hM)+(1+h(F+h)M^2)\sinh(hM))R_m)+\frac{1}{A_1^3}(16h(F+h)kM^7(-hM\cosh
\end{aligned}
$$

$$
\begin{aligned}
& (hM) + (1 + h(F+h)M^2)\sinh(hM))R_m) + M(\frac{-1}{A_1^3}(144(F+h)kM^2((-F-2h)M + FM \\
& \cosh(2hM) + \sinh(2hM))R_m) - \frac{1}{2MA_1^3}(k(FM\cosh(hM) + \sinh(hM))(M(F(-2+3h^2M^2) \\
& -2h(-1+hM)(1+hM)(-1+E)) + M(F(2+h^2M^2) - 2h(1+E+h^2M^2E))\cosh(2h \\
& M) + (2 + hM^2(-2F+h+4hE))\sinh(2hM))R_m) + 2M(\cosh(hM) + \sinh(hM)(48(F+h \\
&)kM^3((-F-2h)M + FM\cosh(2hM) + \sinh(2hM))R_m) + M(\frac{1}{A_1^3}(18(F+h)kM^2(-hM \\
& \cosh(hM) + (1+h(F+h)M^2)\sinh(hM))R_m) + \frac{1}{A_1^3}(4kh^3M^5(M(\cosh(hM) + (2+3E) \\
& \sinh(hM))(FM\cosh(2hM) + \sinh(2hM))R_m) + \frac{1}{A_1^3}(12(F+h)kM^2R_m(2M(F+2h-hE \\
& +h^3M^2E) + 2M(-F+hE+h^3M^2E)\cosh(2hM) - (1+2h^2M^2E)\sinh(2hM) + (-2+F \\
& hM^2)\cosh(2hM) + (-2F+h)\sinh(2hM))R_m))))))), \\
A_{30} &= \frac{1}{A_1^3}(A_2M^3(4(F+h)M(-A_1 + h(F+h)M^2\sinh(hM)) + 2(h(F+h)M^2 + A_1(2 \\
& F+h)M\cosh(hM) + A_1\sinh(hM) - (F+h)M^2(h\cosh(2hM) - FA_1\sinh(hM) + FhM \\
& \sinh(2hM)))\lambda_1)), \\
A_{31} &= 3kM^4((F+h)M^3\lambda_1((-F-2h)M + FM\cosh(2hM) + \sinh(2hM)) + (-F-h)M((-F \\
& -2h)M + FM\cosh(2hM) + \sinh(2hM))\,\mathrm{Re} + (2 + M^2(F^2+4h^2+Fh(7-2E)) - 2 \\
& E - (2 - 2E + M^2(F^2+Fh+2h^2-2h(F+2h)E))\cosh(2hM) - M(-h+F(3+2h(\\
& F+h)M^2) + 2h(2+h(F+h)M^2)E)\sinh(2hM))R_m)), \\
A_{32} &= \frac{1}{A_1^3}(kM^3((F+h)M^3\lambda_1((-F-2h)M + FM\cosh(2hM) + \sinh(2hM)) + (-F-h)M((\\
& -F-2h)M + FM\cosh(2hM) + \sinh(2hM))\,\mathrm{Re} + (2 + M^2(F^2+4h^2+Fh(7-2E)) \\
& -2E - (2 - 2E + M^2(F^2+Fh+2h^2-2h(F+2h)E))\cosh(2hM) - M(-h+F(3+ \\
& 2h(F+h)M^2) + 2h(2+h(F+h)M^2)E)\sinh(2hM))R_m)), \\
A_{33} &= \frac{(F+h)M^3(75+2h^4M^4)R_m}{2A_1}, \\
A_{34} &= 2(F+h)kM^2R_m(-hM\cosh(hM) + \sinh(hM)) - \frac{1}{A_1^3}(2M(F+h)) + 2M(-F+hE)\cosh \\
& (2hM) - 2(1+2h^2M^2E)\sinh(2hM) - hM(2 + M^2\cosh(2hM) + (-2F+h)\sinh(2hM)M \\
&)R_m)), \\
A_{35} &= 2M^3\sinh(2hM)(15A_{32} - \frac{1}{2A_1}(3(F+h)M^3(17+2h^2M^2)R_m) + \frac{1}{A_1^3}(kM^2(12M(FM\cosh
\end{aligned}
$$

$$
\begin{aligned}
& (hM) + \sinh(hM)) + (3h(F+h)M(5(F+2h) - 2(-1+F+2h)M^2 + h(3F+2h)M^4) - 6 \\
& M(-1 + h^3(F+h)M^4)E - 3M(h(F+h)(5F - 2(-1+F)M^2) + 4(F+h)(1+2h)M^2 \sinh(\\
& Mh) + 3h(-5h + F(-5+2M^2) + M^2(4E + h(2 + hM^2(1+6E)))) \sinh(2hM)R_m))), \\
A_{36} &= 9(F+h)kM^3R_m(2M(F+2h) + 2M(hE - F + h^3M^2E)\cosh(2hM) - 2(1+2h^2M^2E)\sinh(\\
& 2hM) - hM(2 + h + (3F+2h)M^2\cosh(2hM) + (-2F+h)M\sinh(2hM))R_m), \\
p_1 &= -\frac{1}{2A_1^3}[(kM^2(4M^2\lambda_1(FM\cosh(Mh) + \sinh(Mh))(-hM\cosh(Mh) + (1 + h(F+h) \\
& M^2)\sinh(Mh)) - 4(FM\cosh(Mh) + \sinh(Mh))(-hM\cosh(Mh) + (1+h(F+h) \\
& M^2)\sinh(Mh))R_e + (-h(F+h)M^2(2 + h(3F+2h)M^2) + 2(-1 - h(2F+h)M^2 \\
& + h^3(F+h)M^4)E + (h(F+h)M^2(2 - FhM^2) + 2(1 + h(2F+3h)M^2 + h^3(F \\
& + h)M^4)E)\cosh(2Mh) - hM(4E - (F+h)M^2(2F - h - 6hE))\sinh(2Mh)) \\
& R_m))], \\
p_2 &= \frac{1}{A_1^3}[(kM^3((F+h)M^3\lambda_1((-F-2h)M + (FM\cosh(Mh) + \sinh(Mh)) + (-F-h) \\
& M((-2F-2h)M + (FM\cosh(Mh) + \sinh(Mh))R_e + (2 + M^2(F^2 + 4h^2 + Fh(7 \\
& -2E)) - 2E - (2 - 2E + M^2(F^2 + Fh + 2h^2 - 2h(F+2h)E))\cosh(2Mh) - M \\
& (-h + F(3 + 2h(F+h)M^2) + 2h(2 + h(F+h)M^2)E)\sinh(2Mh))R_m))], \\
p_3 &= \frac{1}{A_1^3}[(kM^3(M(2F - 3hE)\cosh(Mh) + (2+3E)\sinh(Mh))], \\
p_4 &= \frac{(F+h)M^4R_m}{2A_1^3}, \\
p_5 &= -\frac{1}{A_1^3}[2(F+h)kM^2(-hM\cosh(Mh)(1 + h(F+h)M^2)\sinh(Mh))R_m], \\
p_6 &= \frac{F_1M^3\cosh(Mh)}{A_1}, \\
p_7 &= -\frac{1}{A_1^3}[(F+h)kM^3((-F-2h)M + FM\cosh(Mh) + \sinh(Mh))R_m], \\
p_8 &= -\frac{1}{2A_1^3}[(F+h)kM^2((-F-2h)M + FM\cosh(2Mh) + \sinh(2Mh))R_m], \\
p_9 &= -\frac{1}{2A_1^3}[(F+h)kM^2R_m(2M(F + 2h - hE + h^3M^3E) + 2M(-F + hE + h^3M^2 \\
& \cosh(2Mh) - 2(1 + 2h^2M^2E)\sinh(2Mh) - hM(2 + h(3F+2h)M^2 + (-2 + FhM^2) \\
& + FhM^2)\cosh(2Mh) + (-2F+h)M\sinh(2Mh))R_m))],
\end{aligned}
$$

$$L_1 = (2160h^3 + M^2(1 + hM + (-1 + hM)(\cosh(2hM) + \sinh(2hM))^5),$$

$$L_2 = e^{3hM}ky^2((-4E + hM^2(7F^2hM^2 + h(-9 + 20E) + 2F(-2 + E + h^2M^2(2 + 3 E)))) \cosh(hM) + hM^2(h + F(-4 + FhM^2)) + 2(-2 + FhM^2)(-1 + h^2M^2)E) \cosh(3Mh) - 4M(-F^2hM^2 + 2F(-1 + h^2M^2(-3 + E) + E) - h(3 - 7E + h^2 M^2(3 + E)) + (h + F(F - h)hM^2 + (2F - h)(-1 + hM)E) \cosh(3hM) - 4M (-F^2hM^2 + 2F(-1 + h^2M^2(-3 + E) + E) - h(3 - 7E + h^2M^2(3 + E)) + (h + F(F - h)h M^2 + (2F - h)(-1 + h^2M^2)E) \cosh(2Mh)) \sinh(2Mh))R_m),$$

$$L_3 = (2(-6 + 4E + M^2(3F^2(-4 + h^2M^2) + 2h^2(-14 - E + h^2M^2(1 + E) + Fh(2 - (23 + E) + (5 + 2E)h^2M^2))) + (-4E + hM^2(7F^2hM^2 + h(-9 + 20E) + 2F(-2 + E + h^2M^2(2 + 3E)))) \cosh(hM) + \cosh(2hM) + (hM^2(h + F(-4 + FhM^2)) + 2(-2 + FhM^2)(-1 + h^2M^2)E) \cosh (3hM) + 2M(3F^2hM^2 + F(4 + h^2M^2(11 - 2E) - 6E) + h(7 - 13E + h^2M^2(6 + E))) \sinh(h M) + 2M(4F^2hM^2 + F(18 + 5h^2M^2) + (6 + h^2M^2 + 8E)) \sinh(2hM) + 2hM(h(-1 + F(-F + h)M^2) + (-F + h)(-1 + h^2M^2)E) \sinh(3hM)),$$

$$L_4 = (1 + hM + (-1 + hM) \cosh(2Mh) + (-1 + hM) \sinh(2hM)),$$

$$L_5 = ((\cosh(My) - \sinh(My))(-F \cosh(Mh) - h \cosh(Mh) + y \cosh(My) - FMy \cosh(My) - y \cosh (2Mh + My) - F(1 - My) \cosh(2Mh + My) + (F + h) \cosh(Mh + 2My) - (F + h) \sinh(Mh)+ y(1 - FM) \sinh(My) - y(1 + FM) \sinh(2Mh + My) + (F + h) \sinh(Mh + 2My))),$$

$$L_6 = (M(F + 2hE) \cosh(hM) + (1 - 2E) \sinh(hM)),$$

$$L_7 = (-(F + 2h)M + FM \cosh(2hM) + \sinh(2hM))R_m),$$

$$L_8 = (e^{M(3h-2y)}k(-hM \cosh(hM) + (1 + h(F + h)M^2 \sinh(hM))(2(F + h)M^2\lambda_1)),$$

$$L_9 = (-hM \cosh(hM) + (1 + h(F + h)M^2) \sinh(hM)),$$

$$L_{10} = (e^{3hM}k(\cosh(My) - \sinh(My))(ML_4R_m + R_e(6M^2(-10 + M(-F^2M(17 + 2My(5 + My)) - FhM(71 + 2My(19 + 3My)) - 4(y + h^2M(11 + My(6 + My)))) + (10 + M(4y + 2h^2M(5 + My)))) \cosh(2Mh) + M(7h + F(27 + 10h(F + h)M^2) + 2M(7F + 3h + 2Fh(F + h)M^2)y + 2 (F + h)M^2y^2) \sinh(2hM)) + (102(-1 + E) + M(F^2M(-147 + 18h^2M^2(5 + 2My) - 2My(51+$$

$$
\begin{aligned}
& \sinh(2hM)) + (102(-1+E) + M(F^2M(-147 + 18h^2M^2(5+2My) - 2My(51 + My(51 \\
& +2My))) + 4(3y(5+My)(-1+E) + 3h^4M^3(5+2My)(1+E) - h^2M(84 + 15E + 2My \\
& (My(9+My) + 3(10+E)))) + 3FhM(-195 + 14E + 2M(h^2M(5+2My)(5+2E) + y \\
& (-67 + 6E + My(-19 - 2My(-19 - 2My + 2E))))) + (F^2M^2(147 + 6h^2M^2(5+2My) \\
& +2My(51 + My(15+2My))) + FhM^2(87 + 2My(39 + My(15 + 2My - 6E) - 18E) \\
& -42E + 6h^2M^2(5+2My)(1+2E)) + 6((-17 - 2My(5+My))(-1+E) + 2h^4M^4(5 \\
& +2My)E + h^2M^2(7 - 24E + 2My(3 + My - 2(4+My)E)))) \cosh(2hM) + M(6F^2hM^2 \\
& (7 + 2My(3+My)) + h(45 + 204E + 2My(21 + 60E + My(9 + 2My + 12E)) + 6h^2M^2 \\
& (5 - 3E + 2My(1 + E + MyE))) + F(249 + 2My(81 + My(21+My)) + 6h^2M^2(-3(-4 \\
& +E) + 2My(4 + E + My(1+E)))))) \sinh(2hM) R_m^2 S_t^2) + 6M^4(-10 + M(-4y + M(4h^2 \\
& (6 + My(4+My)) + F^2(17 + 2My(5+My)) + Fh(31 + 2My(11 + 3My))))) - (-2 \\
& (1 + h^2M^2)(5+2My) + F^2M^2(17 + 2My(5+My)) + FhM^2(17 + 2My(5+My))) \\
& \cosh(2hM) + M(-27h + F(-7 + 10h(F+h)M^2 + 2M(-3F - 7h + 2Fh(F+h)M^2)y \\
& -2(F+h)M^2y^2) \sinh(2hM)\lambda_1)),
\end{aligned}
$$

$$
\begin{aligned}
L_{11} &= (51 + My(-15 + 2My - 12E) + 60E), \\
L_{12} &= (-4y + M(17F^2 + 71Fh + 44h^2 - 2(F+3h)(F+4h)My + 2(F+h)(F+2h)M^2y^2)), \\
L_{13} &= 3FMh(195 - 14E + 2M(h^2M(-5+2My)(5+2E) + y(-67 + My(19 - 2My - 2E + 6 \\
&\quad E))), \\
L_{14} &= (e^{3Mh}k(\cosh(My) + \sinh(My))(M(6(-27+17E) + M(F^2M(-309 + 18h^2M^2(5-2My) \\
&\quad +2My) + 2My(93 + My(-21+My))) + 3FMh(-397 + 14E - 2h^2M^2(-5+2My)(5 \\
&\quad +2E) + 2My(117 - 6E + My(-25 + 2My + 2E))) + 4(3y(7 + My(-1+E) - 5E) - \\
&\quad 3h^4M^3(-5 + 2My(1+E) + h^2M(-15(12+E) + 2My(My(12+My) + 3(18+E))))) \\
&\quad +(162 - 102E + M(F^2M(309 - 2M(3h^2M(-5+2My) + y(93 + My(-21 + 2My + 6 \\
&\quad E))) + 6(-2h^4M^3(-5+2My)E + 2y(-7 + My + 5E - MyE) + h^2M(17 - 24E + 2M
\end{aligned}
$$

$$y(-5+My+8E-2MyE))))) \cosh(2Mh)+M(6F^2M^2h(17+2My(-5+My))+h(-$$
$$2MyL_{13}+3(49+68E)+6h^2M^2(5-3E+2My(-1-E+MyE)))+F(471-2My$$
$$(135+My(-27+2My)+6h^2M^2(22-3E+2My(-6-E+My(1+E)))))) \sinh(2Mh)$$
$$R_m+R_e(6M^2(10+ML_{12}-(10+M(-4y+2h^2M(5-2My)+F^2M(17+2My(-5$$
$$+My))+FhM(17+2My(-5+My)))) \cosh(2hM)+MM(-7h+F(-27-10h(F+h)$$
$$M^2)+2M(7F+3h+2Fh(F+h)M^2)y-2(F+h)M^2y^2) \sinh(2hM)+(-102(-1+E)$$
$$+M(F^2M(147+2M(9h^2M(-5+2My)+y(-51+My(15-2My))))+L_{13}+4(-3y$$
$$(-5+My)(-1+E)+3h^4M^3(-5+2My)2E)+6E)))+4(-3y(-5+My)(-1+E)+$$
$$3h^4M^3(-5+2My)(1+E)+h^2M(84+15E-2My(My(-9+My)+3(10+E)))))+$$
$$(F^2M^2(-147+6h^2M^2(-5+2My)+2My(51+My(-15+2My)))+FhM^2(-87+42$$
$$E+2My(39-18E+My(-15+2My+6E)))+6((17+2My(-5+My))(-1+E)+$$
$$2h^4M^4+h^2M^2(-7+24E+2My(3-8E+My(-1+2E))))) \cosh(2Mh)+M(-6F^2h$$
$$M^2(7+2My(-3+My))+h(-3(15+68E)+2M(y(21+My(-9+2My-12E)+60$$
$$E)+3h^2M(-5+3E+2My)))+F(-249+2M(81y+M(y^2+h^2(9(-4+E)+6My))))$$
$$)) \sinh(2hM)R_m^2S_t^2)+6M^4(10+M(FhM+F^2M-4(y+h^2M(6+My(-4+My))))$$
$$+(2(1+h^2M^2)+FhM^2(17+2My(-5+My))) \cosh(2hM)+M(27h+2My+2(F+h)$$
$$M^2y^2 \sinh(2hM)\lambda_1)),$$

$$L_{15} = (24(8E+M^2(12F^2+Fh(14+h^2M^2)+h^2(3-2E+2h^2M^2E))) \cosh(3Mh)-42 \cosh(4Mh)),$$

$$L_{16} = M(4F^2hM^2(3+5h^2M^2)+F(135+18h^2M^2(-4+E)-48E+2h^4M^4(-5+19E)),$$

$$L_{17} = (1+7F-6h-h(12F^2+h(-1+41h)+F(-1+61h))M^2),$$

$$L_{18} = (-1+7h+h(9F^2+(-1+h)h+F(-1+4h))M^2) \cosh(2hM),$$

$$L_{19} = (1-21F-10h-(-1+h)h(F+h)M^2) \cosh(4hM)+8\phi \sinh(2\pi x)(\sinh(hM))^3(-hM \cosh(hM)+(1+h(F+h)M^2) \sinh(hM)+2M(15F^2+h+93Fh+47h^2+12Fh^2(F+h)M^2 \sinh(2hM)-(15F^2+h+9Fh-h^2)M \sinh(4hM)),$$

$$
\begin{aligned}
L_{20} = {} & (e^{4hM}y(kM(162-150E+M^2(F^2(297+96h^2M^2+43h^4M^4)+h^2(-24(-39+E)+h^2\\
& M^2(203+6E+8h^2M^2E)+Fh(1419-162E+2h^2M^2(178-54E+h^2M^2(14+19E\\
&))))-24(6-8E+M^2(12F^2+F(78h-7h^3M^2)+h^2(43+30E-2h^2M^2(2+3E))))\\
& \cosh(Mh)+4(6(-5+3E)+M^2(3F^2(-17-11h^2M^2+4h^4M^4)+h^2(-3+180E+2h^2\\
& M^2(-29+10E+h^2M^2E))+Fh(6(8+9E)+h^2M^2(-97+14E+h^2M^2(7+12E)))))\\
& \cosh(2hM)+144\cosh(3Mh)+L_{15}+(78E+M^2(F^2(-93+36h^2M^2+5h^4M^4)+h^2(36\\
& +h^2M^2(5-38E)+72E)+Fh(-75-54E+2h^2M^2(-20+26E+5h^2M^2E))))\cosh(4\\
& hM)-24M(2F^2hM^2+h(8-34E+h^2M^2(11+2E))-F(h^2M^2(-17+4E)+2(5+6\\
& E)))\sinh(hM)-2M(F^2(84hM^2+40h^3M^4)+F(249+48E+2h^2M^2(57+39E+7h^2\\
& M^2(-2+5E)))+h(33+240E+2h^2M^2(-13+107E+h^2M^2(-19+11E))))\sinh(2h\\
& M)+24M(6F^2hM^2+h(8+h^2M^2(1-2E)+10E)+F(18-4E+h^2M^2(3+4E)))\sinh\\
& (3hM)-L_{16}+h(9(7+16E)-2h^2M^2(-10+(11+5h^2M^2)E)))\sinh(4hM))R_m--kR_e\\
& (2M^2(3ML_{17}+12\sinh(2hM))+M(4L_{18}+L_{19})+(M(-63h+144hE+32h^5M^4(1+2\\
& E)+F(81+6h^2M^2(-82+E)+4h^4M^4(28+13E)))+4hM(-18(-1+E)+M^2(F^2(21\\
& +8h^2M^2)+4h^2(-9+(-6+4h^2M^2)E)+3Fh(-15+2E+4h^2M^2(1+E))))\cosh(2h\\
& M)-M(4hF^2M^2+F(81+4Eh^4M^4)+h(9+72E+2h^2M^2(2+7E)))\cosh(4hM)+2\\
& (-30(-1+E)+M^2(51F^2+12h^2(11+2h^2M^2(1-2E)+10E)))\sinh(2hM)+(30(-1\\
& +E)+M^2(3F^2(-17+4h^2M^2)+Fh(-1+2E)))\sinh(4hM)R_m^2S_t^2)+2kM^4(3M+4M\\
& \cosh(2hM)+2(-6-12Fh^2(F+h)M^4)\sinh(2hM)+(6+(-15F^2-7Fh+14h^2)M^2\\
&)\sinh(4hM))\lambda_1)),\\
L_{21} = {} & (24(-1+FM)(1+hM)(2+hM)(-2E+hM(-1+FM+2hME))+24e^{6hM}(F^2M^2\\
& (-24+hM(-5+hM)(2+hM))+12(-1+E)-hM(14(1+E)+hM(9+hM\\
& +2(-3+hM(1+hM))E))+2FM(2(-9+E)+hM(3(-4+E)+hM(-6+(-3+hM\\
& (-3+hM))E)))),
\end{aligned}
$$

$$
\begin{aligned}
L_{22} &= 24e^{2hM}(4(-3+E)+M(F^2M(-24+hM(2+hM(-1+7hM)))+4F(7+hM(-8+hM\\
&\quad(-4+hM(-5+hM))))+6F(1+hM)^2-h(2-38E+hM(31-58E+hM)))),\\
L_{23} &= 24e^{4hM}(24-20E+M(F^2M(48+hM(6+hM(1+7hM)))+h(14-30E+hM(117\\
&\quad+2E+hM(3+hM(4-6E)+22E)))+2F(4-6E+hM(94-9E+hM(9-E+h\\
&\quad M(3-3E+hM(2+3E))))))),\\
L_{24} &= e^{5hM}(6+M(F^2M(-351+hM(831+2hM))+F(3(67+96E)+hM(-6(277+3E)\\
&\quad+hM(2505-168E+2hM(-328+39E))))+h(255(1+2E)+hM(-3+2hM(768\\
&\quad-153E+hM(-283+2hM(59-36E+hM(8+33E)))))))),\\
L_{25} &= e^{7hM}(162-198E+M(F^2M(309+hM(-159+2hM(24+5hM(-4+hM))))+h(33+354\\
&\quad E+hM(3-252E+2hM(9(-2+9E)+hM(1+2hM+3(-9+2hM(-2+hM))E))))\\
&\quad+F(471-96E+hM(6(-21+E)+hM(51+168E+2hM(-38-9E+2hM(3-21E\\
&\quad+hM(1+6E)))))))),\\
L_{26} &= e^{hM}(126-186E+M(F^2M(267+hM(129+2hM(24+hM(14+5hM)))0+h(-63-318\\
&\quad E+hM(-3(1+76E)+2hM(18-99E+hM(7-57E+2hM(-1+3hME)))))+F(-393\\
&\quad+96E+hM(-6(11+E)+hM(-9(5+24E)+2hM(-32-39E+2hM(-6+15E+h(M\\
&\quad+6ME)))))))))R_m+2e^{4hM}kR_e(2M^2(-M(2+81F-11h-h(21F^2+6h(-1+4h)-F(2\\
&\quad+13h))M^2+6Fh^3(F+h)M^4)\cosh(3hM)+2\sinh(hM)(30+(51F^2+294Fh+h(2+16\\
&\quad 3h))M^2+6h^2(9F^2+22Fh+11h^2)M^4+(-30-3(17F^2-10Fh+h(-2+9h))M^2+2h^2\\
&\quad(3F^2+(2-5h)h+F(2+4h))M^4)\cosh(2hM)+2M(1+h(F+h)M^2)\phi\sinh(2\pi x)(2hM\\
&\quad\cosh(2hM)-\sinh(2hM))+M\cosh(hM)(2+81F-11h-h(F(-2+21F)+(2+311F)h\\
&\quad+220h^2)M^2+6(F-4h)h^3(F+h)M^4+4hM\phi\sinh(2\pi x)(-2hM\cosh(2hM)+\sinh(2hM))\\
&\quad)+(M(7F^2hM^2(-15+28h^2M^2)+F(294+3h^2M^2(-297+8E)+4h^4M^4(37+44E))+\\
&\quad h(-57+306E+4h^{40}M^4(9+23E)-2h^2M^2(302+79E)))\cosh(hM)+M(F^2M^2h(105+\\
&\quad 44h^2M^2)+h(57+2h^2M^2(8-125E)-306E+4h^4M^4(-1+9E))+F(-294-3h^2M^2(\\
&\quad 35+8E)+4h^4M^4(3+16E)))\cosh(3hM)-2(102(-1+E)+M^2(F^2(-147-24h^2M^2+
\end{aligned}
$$

$$
\begin{aligned}
& 20h^4M^4) + 2Fh(-417 + 21E + 6h^4M^4(3 + E) + h^2M^2(-122 + 43E)) + h^2(4h^4M^4(4 + 3\\
& E) - 2h^2M^2(58 + 9E) - 3(143 + 72E))) + (-102(-1 + E) + M^2(F^2(147 + 60h^2M^2 + 4h^4\\
& M^4) + 2Fh(2h^4M^4(1 + 3E) + 5h^2M^2 - 3) + h^2(45 - 396E + 12h^4M^4E))) \cosh(2hM))\\
& \sinh(hM))R_m^2S_t^2) - 4e^{4hM}(k(120(\sinh(hM))^3 + M((17F + 107h - h(85F^2 + 153Fh)M^2 - 6\\
& h^3(F + h)M^4) + (-17F - 107h + 18Fh^3(F + h)M^4) \cosh(3hM) + M(153F^2 + 262Fh + 16\\
& 3h^2 + 2h^2M^2) \sinh(hM) - M(51F^2 + 42Fh - 111h^2 + 2h^2M^2) \sinh(3hM)))\lambda_1))),\\
L_{27} &= \sinh(2\pi x)(-h \cosh(hM) + (1 + h(F + h)M^2) \sinh(hM))(2hM \cosh(2hM) - \sinh(2hM)),\\
L_{28} &= (120(\sinh(hM))^3 + M((-2 - 81F + 11h + h(F(-2 + 21F) + (2 + 311F)h + 220h^2)M^2 -\\
& 6(F - 4h)h^3(F + h)M^4) \cosh(hM) + (2 + 81F - 11h - h(21F^2 + 6h(-1 + 4h) - F(2 + 1\\
& 3h))M^2 + 6Fh^3(F + h)M^4) \cosh(3hM) - M(153F^2 - 2h + 558Fh + 353h^2 + 2h^2(51F^2\\
& +2F(-1 + 64h) + h(-2 + 71h))M^2) \sinh(hM)),\\
L_{29} &= (-51F^2 + 30Fh + 3(2 - 9h)h + 2h^2(3F^2 + (2 - 5h)h + F(2 + 4h))M^2),\\
L_{30} &= (-M(7F^2hM^2(-15 + 28h^2M^2) + F(249 + 3h^2M^2(-297 + 8E) + 4h^4M^4(37 + 44E)) + h\\
& (-57 + 306E + 4h^4M^4(9 + 23E) - 2h^2M^2(302 + 79E))) \cosh(hM) + M(-F^2hM^2(105 +\\
& 44h^2M^2) + F(249 + 3h^2M^2(35 + 8E) - 4h^4M^4(3 + 16E)) + h(-57 + 4h^4M^4(1 - 9E) +\\
& 306E + 2h^2M^2(-8 + 125E))) \cosh(3hM) + (306(-1 + E) + M^2(9F^2(-49 - 12h^2M^2 + 4\\
& h^4M^4) + h^2(-903 - 36E - 2h^2M^2(108 + E) + 4h^4M^4(8 + 3E)) + 2Fh(-753 + 63E + 2\\
& h^4M^4(17 + 3E) + h^2M^2(-214 + 51E)))) \sinh(hM) + (-102(-1 + E) + M^2(F^2(147 + 60\\
& h^2M^2 + 4h^4M^4) + 2Fh(2h^4M^4(1 + 3E) + 5h^2M^2(-6 + 7E) - 3(27 + 7E)) + h^2(45 - 3\\
& 96E + 123h^4M^4E - 2h^2M^2(8 + 17E))) \sinh(3hM)R_m^2S_t^2),\\
L_{31} &= (-309F^2M - h(-1 + FM)(33 + FM(159 + 2hM(24 + 5hM(4 + hM))) + hM(-3 + 2hM(\\
& 81 + hM(-1 + 2hM)))) - 6(F(1 + hM)(16 + hM(-15 + hM(-13 + 2hM(5 + 2hM)))) +\\
& h(-59 + hM(-42 + hM(-27 + hM(-9 + 2hM(2 + hM))))))E,\\
L_{32} &= 24e^{7hM}(1 + FM)(-2 + hM)(-1 + hM)(-2E + hM(1 + FM + 2hME)) + 24e^{hM}(-12(-1 +\\
& E) + M(F^2M(24 + hM(-2 + hM)(5 + hM)) + h(-14(1 + E) + hM(9 - 6E + hM(-1 + 2
\end{aligned}
$$

$$
\begin{aligned}
& (-1+hM)E)))+2F(2(-9+E)+hM(-3(-4+E)+hM(-6+(-3+hM(3+hM)E))))), \\
L_{33} &= 24e^{5hM}(-4(-3+E)+M(F^2M(24+hM(2+hM(1+7hM)))+4F(7+hM(8+hM(-4+ \\
& hM(5+hM))))+6F(-1+hM)^2(2+hM(1+hM))E+h(-2+38E+hM(31-58E+h \\
& M(-19+18E+6hM(2+E))))))+24e^{3hM}(4(-6+5E)+M(F^2M(-48+hM(6+hM(7 \\
& hM-1)))+h(14-30E+hM(-177-2E+hM(3+22E+2hM(-2+3E))))+2F(4- \\
& 6E+hM(-94+9E+hM(9-E+hM(3(-1+E)+hM(2+3E))))))), \\
L_{34} &= -e^{4hM}(-90+174E+M(F^2M(-225+hM(801+2hM(-36+hM(72+91hM))))+F(9(\\
& 31+32E)+hM(6(-203+3E)+hM(2199-216E+2hM(-370+9E+2hM(119-33E \\
& +7hM(7+6E))))))+h(225+546E+hM(-855-492E+2hM(684-195E+hM(-30 \\
& 1+123E+2hM(71-81E+hM(8+33E)))))))), \\
L_{35} &= e^{2hM}(198-210E+M(F^2M(351+hM(831+2hM(12+hM(30-91hM))))+F(3(67+ \\
& 96E)+hM(6(277+3E)-hM(3(-835+56E)+2hM(-328+39E+2hM(-86+51E+ \\
& 7hM(7+6E))))))+h(255(1+2E)+hM(1065+546E+2hM(768-153E-hM(-283 \\
& +153+2hM(-59+36E+hM(8+33E)))))))), \\
L_{36} &= e^{6hM}(126-186E+M(F^2M(267+hM(-129+2hM(24+hM(-14+5hM))))+h(63+ \\
& 318E+hM(-3(1+76E)+2hM(-18+99E+hM(7-57E+2hM(1+3hME)))))+F \\
& (393+hM(-6+hM(9+2hM(-32+2hM(6-15E))))))), \\
L_{37} &= e^{3hM}(-M(-17F-107h+h(85F^2+153Fh+134h^2)M^2+6h^3(F+h)(3F+4h)M^4 \\
& \cosh(hM)+120(\sinh(hM))^3+M((-17F-107h+h(85F^2+85Fh-54h^2)M^2+18 \\
& Fh^3(F+h)M^4\cosh(3hM)+M(153F^2+262Fh+163h^2+2h^2(25F^2+56Fh+37 \\
& h^2)M^2)\sinh(hM)-M(51F^2+262Fh+163h^2+2h^2(25F^2+56Fh+37h^2)M^2)\sinh \\
& (hM)-M(51F^2+42Fh-111h^2+2h^2M^2)L_{43}\sinh(3hM)))\lambda_1, \\
L_{38} &= (k(2+e^{3hM}R_e(-8M^3\phi L_{27}+2M^2L_{28}-ML_{29}\sinh(hM)))+L_{30}+M((3(-54+157F \\
& M+66E)+ML_{31})+L_{32}+L_{33}+L_{34}+L_{35}+L_{36})R_m+4M^3L_{37}),
\end{aligned}
$$

$$
\begin{aligned}
L_{39} &= (1+hM+(-1+hM)(\cos(2hM)+\sinh(2hM)),\\
L_{41} &= (1+hM+FhM^2+h^2M^2),\\
L_{42} &= (1-hM+FhM^2+h^2M^2),\\
L_{43} &= (31F^2+32Fh-5h^2),\\
L_{44} &= (36+M(h(24+M(-41+h(24+M(49+2h(6+M(16+hM(37+12hM))))))+6M\\
&\quad (-6+h(-4+M(-9+h(-4+M(21+2h(-1+9M))))))y-6hM^3(-1+hM(5+4hM\\
&\quad y^2-36(1+y)+F^2M^2(63+h(-12+M(37+2h(6+M(34+9hM))))-6M(-15+h\\
&\quad (-2+M(-7+2h(1+M))))y-6M^2(3+hM)y^2)+F(-12+M(-5+h(60+M(112\\
&\quad +h(36+M(57+2h(6+M(65+21hM)))))))+12y-6M(3+F(-12+M(-5+h(60\\
&\quad +M(112+h(36+M(57+2h(6+M(65+21hM))))))+12y-6M(3+h(10+M(-30\\
&\quad +h(6+M(-19+2h(1+M))))))y-6M^2(-1+hM(8+5hM))y^2)))),\\
L_{45} &= (-12+M(hM(-41+h(-24+M(45+2h(6+M(-18+5hM))))-6M(-2+h(4+M\\
&\quad (9+h(-4+M(-9+2h(1+M))))))y-6hM^3(-1+hM)y^2+12(1+2h+y)+F^2M^2\\
&\quad (-21+h(-12+M(37+2h(6+h(-22+9hM))))-6M(-1+hM)(-5+2h(1+M))\\
&\quad y-6M^2(-1+hM)y^2)+F(-12+M(-5+18h^4M^4-2h^3M^2(-6+26M+6M(1+M)\\
&\quad y)+6y(2+M(-3+My))-12h(-1+M(2+y+My))+h^2M(-12+M(37+6y(2\\
&\quad +M(7-My)))))))),\\
L_{46} &= (12+M(h(24-M(41+h(-24+M(45+2h(-6+M(18+5hM))))))-6M(2+h(4\\
&\quad +M(9+h(4+M(9+2h(1+M))))))y+6hM^3(1+hM)y^2-12(1+y)+F^2M^2(21\\
&\quad +h(-12+M(37+2h(-6+M(22+9hM))))+6M(1+hM)(5+2h(1+M))y-6\\
&\quad M^2(1+hM)y^2)+F(-12+M(-5+18h^4M^4+2h^3M^2(-6+23M+6M(1+M)y)\\
&\quad +6y(2+M(-3+My))+12h(-1+M(2+y+My))+h^2M(-12+M(37+6y(2+M\\
&\quad (7-My)))))))),\\
L_{47} &= (-12+6Fh(F+h)M^5(h-y)^2+MM(2+45F+7h+12)+M^3(h(-21F^2-6(-1+h)
\end{aligned}
$$

$$
\begin{aligned}
& h+F(2+13h))+6(3F^2-2Fh+h^2)y-6(F+h)y^2)+M^2(-33F^2+6F(2h-5y) \\
& +3h+2y))-2M^4(Fh^2(-2+5h-3y)+3F^2(h-y)(2h+y)+h^2(h(-2+5h)-6 \\
& hy+3y^2))), \\
L_{48} &= (-108+(80-134h)M+M^2(-297F^2-576Fh+2(89-683h)h+648y^2+M^3(-h(81 \\
& F^2+4h(41+94h)+F(-80+1619h))+18(1+36F-h)y^2)-9h(F+h)M^7(h-y) \\
& (h+y)((7F-20h)h^2(F+4h)y^2)-9M^6(h-y)(h+y)(h^2(93F^2+h(-4+137h))-(F \\
& +h)(3F+5h)y^2)+M^4(-h^2(1485F^2+F(-98+5363h)))-(F+h)(3F+5h)y^2)M^4 \\
& (-h^2(1485F^2+2h(-58+1759h)+F(-98+5363h))+18(72F^2+252Fh+h(-1+163 \\
& h))y^2)+9M^5(h^3(18F^2+h(2+203h)+F(-2+273h))-2h(F(-1+12F)+h+139 \\
& Fh+101h^2)y^2-(F+h)y^4)), \\
L_{49} &= (36-M(2+45F+7h+36y)+M^3(h(21F^2+2h(1+65h)+F(-2+167h))-6(9F^2 \\
& +34Fh+19h^2)y+6(F+h)y^2)+2M^4(h^2(42F^2+h(-2+71h)+F(-2+119h))-3 \\
& h(F^2+21Fh+14h^2)y-3(F+h)(3F+5h)y^2)-6h(F+h)M^5(h+y)(4h(-h+y)+F \\
& (h+y))+M^2(99F^2+h(-2+227h-6y)+6F(54h+5y))), \\
L_{50} &= (12+M(2+45F+7h-12y)+6Fh(F+h)M^5(h+y)^2+3M^2(11F^2+h(-2+3h+2y) \\
& -2F(2h+5y))+M^3(-3F^2(7h+6y)+F(h(2+13h)+12hy-6y^2)-6h((-1+h)h \\
& +hy+y^2))+2M^4(3F^2(2h-y)(h+y)+Fh^2(-2+5h+3y)+h^2(h(-2+5h)+6hy \\
& +3y^2))), \\
L_{51} &= (-36-M(2+45F-7h-36y)+M^3(h(21F^2+2h(1+65h)+F(-2+167h))+6(9F^2 \\
& +34Fh+19h^2)y+6(F+h)y^2+2M^4(-h^2(42F^2+h(-2+71h)+F(-2+119h)-3h \\
& (F^2+21Fh+14h^2)y+3(F+h)(3F+5h)y^2)+6h(F+h)M^5(h-y)(F(-h+y)+4h \\
& (h+y))-M^2(99F^2+6F(54h-5y)+h(-2+227h+6y))), \\
L_{52} &= (324+M(334-739h+9M(h(62+h(253+2M(29+h(-17+2M(1+h(17+M(-6 \\
& +hM)))))))-8(12+M(-1+h(1+M(-3+h(9+M(-3+h(9+(-2+5h)M))))) \\
&))y^2-4hM^3(1+hM)y^4)+F(-405+hM(3186+M(334+h(-3223+36M(7-h(16
\end{aligned}
$$

$$+M(2+h(15+M(4+h(2+7hM)))))))))+72M^2(36+hM(12+M(1+h(5+M$$

$$(2+h+3h^2M)))))y^2+36M^4(-1+h^2M^2)y^4)+9F^2M(99+M(-h(33+4hM(-63+h$$

$$M(-18+hM(1+7hM))))+24M(-8-4hM+h^3M^3)y^2+4M^3(1+hM)y^4)))),$$

$$\begin{aligned}
L_{53} &= (108+(80-134h)M+M^3(-h(81F^2+4h(41+94h)+F(-80+1691h))+18(1+36\\
&\quad F-h)y^2)-9h(h-y)(h+y)+9M^6(h-y)(h+y)(h^2(93F^2+h(-4+137h)+F(-4\\
&\quad +242h))-(F+h)(3F+5h)y^2)+M^4(h^2(1485F^2+2h(-58+1759h)+F(-98+5363\\
&\quad h))-18(72F^2+252Fh)y^2)+9M^5(h^3(18F^2+h(2+203h)+F(-2+273h))-2h(F(\\
&\quad -1+12F)+h+139Fh+101h^2)y^2-(F+h)y^4)),\\
L_{54} &= (-12+M(2+45F+7h-12y)+6Fh(F+h)M^5(h+y)^2-M^3(h(21F^2+6(-1+h)h\\
&\quad -F(2+13h))+6(3F^2-2Fh+h^2)y+6(F+h)y^2)+M^2(-33F^2-3h(-2+3h+2y)\\
&\quad +6F(2h+5y))-2M^4(3F^2(2h-y)(h+y)+F^2h(-2+5h+3y)+h^2(h(-2+5h)\\
&\quad +6hy))\\
L_{55} &= (-36-M(2+45F+7h+36y)+M^3(h(21F^2+2h(1+65h)+F(-2+167h))-6(9F^2\\
&\quad +34Fh+19h^2)y)+2M^4(-h^2(42F^2+h(-2+71h))+3h(F^2+21Fh+14h^2)y+3(F\\
&\quad +h)y^2)-M^2(99F^2+h(-2+227h-6y)+6F(54h+5y))),\\
L_{56} &= (324+(-324+405F+739h)M+9M^2((F^2+354Fh+h(62+253h)-96y^2)+36Fh(F\\
&\quad +h)M^7(h-y)(h+y)(7h^2+y^2)+M^3(h(297F^2+18h(-29+17h)+F(-334+3223h))\\
&\quad +72(-1-36F+h)y^2)-36M^2(h-y)(h+y)(h^2(F^2+(F-h))(F+h)y^2)+36M^4(h^2\\
&\quad (7F(1+9F)+h-16Fh+17h^2)-6(8F^2-4Fh+h(-1+3h))y^2)+36M^5(h^3(-18F^2+\\
&\quad F(2+15h))+2h(F(-1+12F)-(3+5F)h+9h^2)y^2+(F+h)y^4)),\\
L_{57} &= (108+M(14+27F^2M(11+hM)+F(-405+hM(162+M(14+13h+4(-1+h)hM)))\\
&\quad +h(-203+M(10+h(-37+2(-1+h)M(-5+2hM))))),\\
L_{58} &= (108+M(14+27F^2M(11+hM)+F(-405+hM(162+M(14+13h+4(-1+h)hM)))\\
&\quad +h(-203+M(10+h(-37+2(-1+h)M(-5+2hM)))))),
\end{aligned}$$

$$L_{59} = (12 + 2M + 45FM + 7hM + 33F^2M^2 - 6hM^2 - 12FhM^2 + 9h^2M^2 + 2FM^3h - 21F^2hM^3 + 6h^2M^3 + 13Fh^2M^3 - 6h^3M^3 - 4Fh^2M^4 + 12F^2h^2M^4 - 12FhM^3y + 6h^2M^3y - 6F^2h M^4 - 6Fh^2M^4y - 12h^3M^4y - 12F^2h^2M^5y - 12Fh^3M^5y - 6FM^3y^2 - 6hM^3y^2 - 6F^2M^4 y^2 + 6h^2M^4y^2 + 6F^2hM^5 + 6Fh^2M^5y^2),$$

$$L_{60} = (40 + M(h(89 + FM(40 + hM(49 + 9hM(-1 + 2hM))) + hM(-82 + hM(58 + 9hM(1 + 2 hM)))) - 9M(-1 + 2hM)(1 + hM(1 + (F + h)M))y^2)),$$

$$L_{61} = (-126(-1 + E) + M(75F^2M + 48M(6 + FM(5 + 9FM))y^2 - 3M^3(2 + FM(1 + FM))y^4 + 6 (32F - 48My^2 + M^3y^4)E + 2h^8M^7(16 + 7E) + h(-6(83 + FM(-88 + 19FM)) - 48My^2 + M^3y^4)E + 2h^8M^7(16 + 7E) + h(-6(83 + FM(-88 + 19FM)) - 48M^2y^2 - M^4(-3 + F M(18 + FM))y^4 + 6(115 - 9FM - 24M^2(-2 + FM)y^2 + M^4(-1 + FM)y^4)E) + 2h^7M^6(-9 - 29E + 7FM(5 + E)) + h^2M(-213 + 612E + M(912My^2 - 9M^3y^4 - 2F^2M(195 - 72 M^2y^2 + M^4y^4) + F(-1497 + M^4y^4(11 - 4E) + 48M^2y^2(-18 + E) + 6E))) + h^6M^5(-209 -20E + M(38F^2M - 4My^2(8 + 3E) - F(93 + 136E))) + h^5M^4(609 + 148E + M(-153F^2 M + 2M^2y^2(17 + 3E))) + h^4M^3(-978 + 164E + M(9F^2M(13 + 4M^2y^2) + 2My^2(117 + 16 E - M^2y^2E) + F(893 - 6E + 2M^2y^2(41 + 70E)))) + h^3M^2(-3(101 + 358E) + 2M(F^2M(65 + 77M^2y^2) + My^2(-302 - 58E + M^2y^2(3 + E)) - F(720 - 118E + M^2y^2(-214 + 36E +M^2y^2(1 + E))))))),$$

$$L_{62} = 198(-1 + E) + M(F^2M(-351 + M(h(-75 + 2hM(-99 + hM(35 + 18hM))) - 12(-1 +hM)(9 + hM(11 + 7hM))y + 6M(9 + hM(1 + 2hM))y^2 + 4M^2(3 + hM)y^3)) - 36 y(3 + My)(-1 + E) + 4h^6M^5(8 + 3E) + h(3(-5 + 66E) - 2My(-18 + 54E + My(-3 +2My + 18E))) + F(183 + M(-2y(36 + My(15 + 2My)) + 4h^5M^4(17 + 3E) + h^2 M(-651 + 2My(138 + My(63 + 10My - 12E)) + 60E) + 2h(-597 + 2My(99 + M y(51 + 8My - 9E) - 9E) + 63E) - 2h^4M^3(25 - 34E + 6My(11 + 6E)) + 2h^3M^2(-286 + 15E + 6My(-7 - 3E + My(1 + E)))))),$$

$$
\begin{aligned}
L_{63} &= -582(-1+E)+M(-1437h-h^2M(1011+4hM(65+hM(-50+hM(-17+hM(13\\
&\quad +6hM)))))+8M(-48+hM(-24+hM(-24+hM(-14+hM(5+2hM)))))y^2+F^2\\
&\quad M(735+M(h(159+4hM(222+hM(86+hM(55+3hM(13+2hM)))))-8M(3\\
&\quad +2hM)(24+h^2M^2(8+hM))y^2-4M^3(1+hM)(-1+2hM)y^4))+2(h(819+hM\\
&\quad (2148+hM(1045+hM(-397+4hM(-126+hM(-1+hM)(22+7hM))))))-8\\
&\quad M(1+hM)(-24+hM(-48+hM(-48+hM(-19+3hM(1+hM)))))y^2-4M^3(1\\
&\quad +hM)y^4)E+F(-951+960M^2y^2-12M^4y^4+1152E+8h^7M^7(3+7E)+h^2M^2\\
&\quad (1245+744E+192M^2y^2(2+E)-4M^4y^4(1+4E))-4h^4M^4(99+50E+2M^2\\
&\quad y^2(3+26E))+2h^3M^3(3(48+E)-4M^2y^2(-33+20E+M^2y^2(1+E)))))),\\
L_{64} &= 198(-1+E)+M(F^2M(-351+M(h(75+2hM(-99+hM(-35+18hM)))+12(1\\
&\quad +hM)(9+hM(-11+7hM))y+6M(9+hM(-1+2hM))y^2-4M^2(-3+hM)y^3))\\
&\quad -36y(3+My)(-1+E)+4h^6M^5(8+3E)+h^2M(-789+72E+2My(90+My(57\\
&\quad +10My)+36E))+2h^3M^3(-135-37E+6My(-3-3E+MyE))-h^2M(-651\\
&\quad +60E)+2h(-597+2My(99+My(51+8My-9E)-9E)+63E)+2h^4M^3(25\\
&\quad -34E+6My(11+6E))+2h^3M^2(-286+6My(-7-3E+My(1+E)))))),\\
L_{65} &= -66(-1+E)+M(F^2M(117+M(h(-75+2hM(-3+hM)(-7+2hM))+12(-3\\
&\quad +2hM+h^3M^3)y+6M(-1+hM)(3+2hM)y^2+4M^2(-1+hM)y^3))+12y(3\\
&\quad +My)(-1+E)+12h^6M^5E+4h^5M^4(1+6MyE)+2hg^4M^3(1-53E+6My(1\\
&\quad -3E+MyE))-2h^3M^2(17-125E+6My(3E+My(-1+3E)))+h(3(-5+66\\
&\quad E)-2My(-18+54E+My(-3+2My+18E)))+h^2M(39-288E+2My(-6\\
&\quad +60E+My(-3+2My+24E)))+F(183+M(-2y+h^2M(81+2My(-6+My\\
&\quad (3+2My-12E))+60E)+2h^3M^2(-30-E+6My(1-3E+My(1+E)))+6\\
&\quad h(-15-7E+2My(5+E+My(1+E)))))),\\
L_{66} &= (-126(-1+E)+M(75F^2M+48M(6+FM(-5+9FM))y^2-3M^3(2+FM(-1+F\\
&\quad M))y^4+6(-32F-48My^2+M^3y^4)E+2h^8M^7(16+7E)+2h^7M^6(9+29E+7F
\end{aligned}
$$

$$M(5+E)) + h^6 M^5(-209 - 20E + M(38F^2M - 4My^2(8+3E) + F(93+136E)))$$
$$+h(498 - 690E + M(F^2M(114 + 96M^2y^2 + M^4y^4) + 3My^2(16 - 96E + M^2y^2(-1$$
$$+2E)) + 6F(88 + M^4y^4(-3+E) - 9E - 8M^2y^2(-31+3E)))) + h^5M^4(-609 - 1$$
$$48E - M(-153F^2M + 4My^2(3+14E) + 2F(212 - 39E + 2M^2y^2(17+3E)))) + h^2$$
$$M(-213 + 612E + M(912My^2 + F(1497 - 48M^2y^2(-18+E) - 6E + M^4y^4(-11+4$$
$$E)))) + h^3M^2(303 + 1074E - 2M(F^2M + F(720 - 118E + M^2y^2(-241+36E))))$$
$$+F(893 - 6E + 2M^2y^2(41+70E)))))),$$

$$L_{67} = (-6 + M(-3h(-60 + hM(-3 + hM(61 + 2hM(4 + 3hM(-8 + hM(-1+hM)))))) + 6$$
$$M(-2 + hM(-1 + hM(-16 + hM(9 + 2hM(-1+hM)))))y^2 + 6M^3(-1+hM)^2(1+h$$
$$M)y^4 + F^2M(6 + M(h(135 + hM(-243 + 2hM(108 + hM(18 + hM(-14+5hM)))))$$
$$-6M(-1 + hM(4 + hM(5 + 2hM(-2+hM))))y^2 + 2M^3(-1+hM)(3+hM)y^4))$$
$$+6M(1 + hM(9 + hM(3 + hM(13 + 2hM(-6+hM)))))y^2 + 2hM^4(-1+hM)(3+h$$
$$M)y^4)))),$$

$$L_{68} = (450(-1+E) + M(-F^2M(501 + hM(615 + 16hM(12+hM))) + h(-555 + 1122E$$
$$+hM(-183 + 936E + 2hM(-8 + 11(13+hM)E))) + F(951 - 384 + hM(1170 + h$$
$$M(375 - 2(360+hM)E))))),$$

$$L_{69} = (-582(-1+E) + M(1437 + h^2M(-1011 + 4hM(65 + hM(50 + hM(-17 + hM(-13$$
$$+6hM))))) - 8M(48 + hM(-24 + hM(24 + hM(-14 + hM(-5+2hM)))))y^2 - 4M^3$$
$$(2+hM)y^4 + F^2M(735 + M(h(-159 + 4hM(222+hM)) - 8M(-3+2hM) - 4M^3($$
$$-1+hM)(1+2hM)y^4)) + 2(h(-819 + hM(2148 + hM(-1045 + hM(-397 + 4hM($$
$$126 + hM(1+hM)))))) - 4M^3(1 + hM(-1+hM))y^4)E + F(951 + M(12My^2(-80+$$
$$M^2y^2) + h(-8M^4y^4(1+E) + 192M^2y^2(3+E) + 6(403+267E)) + 2h^3M^2(3(48+E$$
$$) - 4M^2y^2(-33 + 20E + M^2y^2(1+E))))))),$$

$$L_{70} = (405(-1+E) + M(F^2M(-501 + hM(615 + 16hM(-12+hM))) + h(555 - 1122E + h$$
$$M(-183 + 936 + 2hM(8 + 11(-13+hM)E))) + F(-951 + hM(1170 - 918E + hM($$

$$-375 + 2hM(16 + (-101 + 8hM)E)))))),$$

$$L_{71} = (-66(1+E) + M(F^2M(117 + M(h(75 + 2hM(3 + hM)(7 + 2hM)) + 4M^2(1 + hM)y^3)) + 12h^6M^5E + h(15 - 198E - 2My(-18 + My(3 + 2My - 18E) + 54E)) + h^2M(39 -288E - 2My(-6 + My(3 + 2My - 24E) + 60E)) + F(-183 + M(-2y(36 + My(-15 +2My)) + 4h^5M^4(1 + 3E) + 2h^3M^2(-30 - E + 6My(-1 + 3E + My(1 + E))) + h^2 M(-81 - 60E + 2My(-6 + My(-3 + 2My + 12E))))))),$$

$$L_{72} = (198(-1+E) + M(F^2M(-351 + M(h(-75 + 2hM(-99 + hM(35 + 18hM))) + 12(-1 +hM)(9 + hM(11 + 7hM))y + 6M(9 + hM(1 + 2hM))y^2 - 4M^2(3 + hM)y^3)) - 36y (-3 + My)(-1 + E) + h^6M^5(8 + 3E) + h(2My(-18 + My(3 + 2My - 18E) + 54E) + 3 (-5 + 66E)) + +2h^4M^3(-135 - 37E + 6My(3 + 3E + MyE)) - 2h^3M^2(251 + 7E + 2 My(9(4 + 3E) + My(4My - 3(7 + E)))) + F(183 + M(2y(36 + My(-15 + 2My)) + 4 h^5M^4(17 + 3E) + 2h^4M^3(-25 + 34E + 6My(11 + 6E)) + 2h^3M^2(-286 + 15E + 6My (7 + 3E + My(1 + E))) - 2h(597 - 63E + 2My(99 - 9E + My(-51 + 8My + 9E))) -h^2M(651 - 60E + 2My(138 + My(-63 + 10My + 12E))))))),$$

$$L_{73} = (198(-1+E) + M(F^2M(-351 + M(h(-75 + 2hM(-99 + hM(35 + 18hM))) + 12(-1 +hM)(9 + hM(11 + 7hM))y + 6M(9 + hM(1 + 2hM))y^2 - 4M^2(3 + hM)y^3)) - 36y (-3 + My)(-1 + E) + 4h^6M^5(8 + 3E) + 4h^5M^4(-9 - 4E + 6My(2 + 3E)) - h^2M(789 -72E + 2My(90 + My(-57 + 10My) + 36E)) + h(2My(-18 + My(3 + 2My - 18E) +54E) + 3(-5 + 66E)) + 2h^4M^3(-135 - 37E + 6My(3 + 3E + MyE)) - 2h^3M^2(251 +7E + 2My(9(4 + 3E) + My(4My - 3(7 + E)))) + F(183 + M(2y(36 + My(-15 + 2M y)) + 4h^5M^4(17 + 3E) + 2h^4M^3(-25 + 34E + 6My(11 + 6E)) + 2h^3M^2(-286 + 15E +6My(7 + 3E + My(1 + E))) - 2h(597 - 63E + 2My(99 - 9E + My(-1 + 8My + 9E))) - h^2M(651 - 60E + 2My(138 + My(-63 + 10My + 12E))))))),$$

$$L_{74} = (8730 - 22530E + M(16005h - 480y + M(h^2(-37155 + 2hM(10500 + hM(-3905 + 4 hM(190 + hM(-21 + hM))))) + 60h(5 + hM(-1 + 2hM(1 + hM)))y + 80(36 + hM(12$$

$$+h^2M^2(-9+hM(-21+hM))))) + 60h(5+hM(-1+2hM(1+hM)))y + 80(36+hM$$
$$(12+h^2M^2(-9+hM(2+hM))))y^2 - 40M(-6+hM(5-hM+2h^3M^3))y^3 - 8hM^4(-1$$
$$+hM)y^5) + F^2M(8415+M(-h(-3045+2hM(3390+hM(230+hM(155+56hM(-6$$
$$+hM))))) + 60(-1+hM)(11+2hM(2+hM))y + 40M(144+hM(-72+hM(24+h$$
$$M(-17+5hM))))y^2 - 40M^2(-7+5hM+2h^3M^3)y^3 - 8M^4(-1+hM)y^5)) - 10(h(12$$
$$hM(-9948+hM(8755+hM(-2931+2hM(343+2hM))))) - 24(-1+hM)^2(1-hM$$
$$+h^3M^3)y - 24M(-2+hM)(-1+hM)(-8+hM(2+hM(3+hM)))y^2 + 8M^2(1+hM$$
$$(1+2hM))y^3)E + F(22995 - 36960E + M(4y(-285+2My(65My+M^3y^3+120(9-$$
$$2E))) + 8h^7M^6(1-40E) + 40h^6M^5(-7+26E) + 4h^5M^4(428+395E)))),$$

$$L_{75} = (198(-1+E) + M(F^2M(-351+M(h(75+2hM(-99+hM(-35+18hM))) - 12(1$$
$$+hM)(9+hM(-11+7hM))y + 6M(9+hM(-1+2hM))y^2 + 4M^2(-3+hM)y^3)) - 3$$
$$y(-3+My)(-1+E) + 4h^6M^5(8+3E) + F(-183+M(-2y(36+My(-15+2My)) + 4$$
$$h^5M^4(17+3E) - 2h^4M^3(-25+34E+6My(11+6E)) + 2h^3M^2(-286+15E+6My$$
$$(7+3E+My(1+E))) - 2h(597-63E+2My(99-9E+My(-51+8My+9E)))$$
$$+h^2M(651-60E+2My(138+My(-63+10My+12E))))))),$$

$$L_{76} = (-66(-1+E) + M(F^2M(117+M(h(-75+2hM(-3+hM)(-7+2hM)) - 12(-3+2$$
$$hM+h^3M^3)y + 6M(-1+hM)(3+2hM)y^2 - 4M^2(-1+hM)y^3)) + 12y(-3+My)(E$$
$$-1) + 12h^6M^5E + +12h^6M^5E + 4h^5M^4(1-6MyE) + h^2M(39-288E-2My(-6$$
$$+My(3+2My-24E)+60E)) + h(2My(-18+My(3+2My-18E)+54E) + 3(-5$$
$$+66E)) + 24h^3M^2(-17+125E+66My(My+3E-3MyE)) + 2h^4M^3(1-53E$$
$$+6My(-1+3E+2MyE)) + F(183+M(2y(36+My(-15+My)) + 4h^5M^4(1$$
$$+3E) + h^2M(81+2My(6+My(3-2My-126E))+60E) - 2h^4M^3(-3+14$$
$$E+6My(1+2E)) + 6h(-15-71E+2My(-5-E+My(1+4E))) + 2h^3M^2(-30$$
$$-E+6My(-1+3E+My(1+E))))))),$$

$$
\begin{aligned}
L_{77} &= (-324 + M(12(27 - 89h + 18y) + F^2M^2(-567 + h(-1500 + M(-653 + 36h(30 \\
&\quad +M(-46 + h(9 + M(82 + h(4 + M(-55 + h(-4 + 17M))))))))) - 216hM(-1 + \\
&\quad hM)y - 72M^2(-24 + hM(38 + hM(-28 + 9hM)))y^2 + 144hM^3(-1 + hM)y^3 + 36 \\
&\quad M^4(-1 + hM)y^4) + M(h(-221 + 9h(288 + M(25 + 4h(-71 + M(-122 + h(31 + M \\
&\quad (116 + h(8 + M(-47 + h(-4 + 9M))))))))))) - 216h(2 + hM(-2 + hM))y - 72M \\
&\quad (-2 + hM)(-6 + hM(20 + hM(-14 + 5hM)))y^2 + 144My^3 + 36hM^4(-1 + hM)y^4) \\
&\quad +F(-1500 + M(-221 + h(1944 + M(-882 + h(-2904 + M + 36h(58 + M(-20 + \\
&\quad h(5 + M(83 + h(4 + M(-58 + h(-4 + 17M))))))))))) - 72M^2(-10 + hM(-1 + h \\
&\quad M))y^2 + 144M^2(-1 + hM)(1 + h^2M^2)y^3 + 36M^4(-1 + h^2M^2)y^4)))), \\
L_{78} &= (-36 - 12M(-3 + F - 2h - 3y) + 6h(F + h)M^6(h + y)(h(3F + 4h) - (F + 4h)y) \\
&\quad +M^4(h(F^2(37 - 12h) + 3F(19 - 4h)h + 32h^2) - 6(F^2(15 - 2h) + 6Fh(5 + h) \\
&\quad +h^2(21 + 2h))y + 6(F + h)y^2) + 2M^5(-h^2(34F^2 + 65Fh + 37h^2) + 3h(7F^2 + F \\
&\quad (19 + 2F)h + 2(9 + F)h^2)y + 3(F + h)(3F + 5h)y^2) + M^2(-41h + 36y - 24h(h + \\
&\quad y) + F(-5 - 60h + 12y)) - M^3(3F^2(21 + 4h) + h(h(49 - 12h - 24y) + 54y) - 2F \\
&\quad (-9y + 2h(-28 + 9h + 15y)))), \\
L_{79} &= (-324 + M(F^2M^2(567 + h(1500 + M(-653 + 36h(30 + M(46 + h(-9 + M(62 + \\
&\quad h(4 + M(55 + h(4 + 17M))))))))))) + 216hM(1 + hM)y - 72M^2(24 + hM(38 + h \\
&\quad M(28 + 9hM)))y^2 - 144hM^3(1 + hM)y^3 + 36M^4(1 + hM)y^4) + M(h(-221 + 9h \\
&\quad (288 + M(-25 + 4h(71 + M(-122 - h(-31 + M(116 + h(8 + M))))))))) + F(1500 \\
&\quad +M(-221 + h(1944 + M(882 + h(2904 + M(-2237 + 36h(58 + M(20 + h(-5 + M \\
&\quad (83 + h(4 + M(58 + h(4 + 17M))))))))))))) + 216y + 36M^4(-1 + h^2M^2)y^4)))), \\
L_{80} &= (-54 + M(F^2M^2(189 + h(930 + M(-175 + h(204 + M(1723 + 9h(6 + M(82 + h(4 \\
&\quad +M(53 + h(-4 + 17M))))))))))) - 54hM(-1 + hM)y + 36hM^3y^3) + M(h(14 + h(-32 \\
&\quad +M(-802 + h(618 + M(-1322 + h(294 + M(1786 + 9h(-8 + M(129 + h(-4 + M(\\
&\quad 69 + 20hM))))))))))) - 54h(2 + hM(2 + hM))y - 18M(-36 + hM(-46 + hM(80 + h
\end{aligned}
$$

$$M(64+hM(37+12hM)))))y^2+36M(3+hM(2+hM(2+hM)))+y^3+9hM^4(-1$$
$$+hM(5+4hM))y^4)+6(191h-9(2+3y))+F(930+M(-40+h(408+M(305+h$$
$$(1872+M(-579+h(876+M(3131+9h(-14+h(-4+37M))))))))+36M^2(-1+h$$
$$M(5+hM(3+hM)))y^3+9M^4(-1+hM(8+5hM))y^4)))),$$

$$\begin{aligned}
L_{81} &= (12+M(hM(-41+h(24+M(45+2h(-6+M(-18+5hM)))))+6M(-2+h(-4+M \\
&\quad (9+h(4+(-9+2h(-1+M))M))))y-6hM^3(-1+hM)y^2+12(1-2h+y)+F^2M^2 \\
&\quad (-21+h(12+M(37+2h(-6+M(-22+9hM))))+6(-5+2h(-1+M))M(-1+h \\
&\quad M)y-6M^2(-1+hM)y^2)+F(12+M(-5+18h^4M^4+12h(-1+2M+(-1+M)My) \\
&\quad +2h^3M^2(-6-23M+6(-1+M)My)+6y(2+M(3+My))+h^2M(12+M(37-6y \\
&\quad (-2+M(7+My)))))))), \\
L_{82} &= (-6+M(3h(-60+hM(3+hM(61+2hM(-4+3hM(-8+hM(1+hM))))))-6M(2 \\
&\quad +hM(-1+hM(16+hM(9+2hM(1+hM)))))y^2-6M^3(-1+hM)y^4+F^2M(6+M \\
&\quad (h(-136+hM(-243+2hM(-108+hM(18+hM(14+5hM)))))-6M(-1+hM(-4 \\
&\quad +hM(5+2hM(2+hM))))y^2+2M^3(-3+hM)y^4))+F(-180+M(h(3+hM(54+h \\
&\quad M(-297+2hM(-201+hM(42+hM(38+5hM))))))-6M(-1+hM(9+hM(-3+h \\
&\quad M(13+2hM))))+(1+hM)y^4)))), \\
L_{83} &= (36+12M(3+F-2h+3y)+M^2(-5F-41h+60Fh+24h^2+12(-3+F-2h)y) \\
&\quad +M^4(h(37F^2+3F(19+4F)h+4(8+3F)h^2)+6(-6F(-5+h)h+(21-2h)h^2 \\
&\quad +F^2(15+2h))y+6(F+h)y^2)+2M^5(-h^2(34F^2+65Fh+37h^2)+3h(-7F^2+2 \\
&\quad (-9+F)h^2)y+3(F+h)(3F+5h)y^2)+6h(F+h)M^6(h-y)(4h(h+y)+F(3h+y)) \\
&\quad +M^3(3F^2(-21+4h)-2F(2h(28+9h-15y)-9y)+h(-h(49+12h)+6(9+4h) \\
&\quad y))), \\
L_{84} &= (-36+12M(-3+F-2h-3y+6h(F+h)M^6(h+y)(h(3F+4h)-(F+4h)y)+M^4 \\
&\quad (h(F^2(37-12h)+3F(19-4h)h+32h^2)-6(F^2(15-2h)+6Fh(5+h)+h^2(21+2
\end{aligned}$$

$$h))y+6(F+h)y^2+2M^5(h^2(34F^2+65Fh+37h^2)-3h(7F^2+F(19+2F)h+2(9+F)h^2)y-3(F+h)(3F+5h)y^2)+M^2(-41h+36y-24h(h+y)+F(-5-60h+12y))+M^3(3F^2(21+4h)-2F(-9y+2h(-28+9h+15y)))),$$

$$L_{85} = (-54+M(6(18-191h+27y)+F^2M^2(-189+h(-930+M(-175+h(204+M(-1723+9h(-6+M(82+h(4+M(-53+h(4+17M)))))))))+54hM+M(h(14+h(732+M(802+h(-618+M(-1322+h(294+M(-1786+9h(8+M(129+h(-4+M(-69+20hM)))))))))))-18M(36+hM(-46+hM(-80+hM(64+hM(-37+12hM)))))y^2+9hM^4(-1+hM(-5+4hM))y^4)+F(-930+M(-40+h(408+M(-305+h(-1872+M(-579+h(876+M(-3131+9h(14+M(163+h^2M(4+37M)-2h(6+55M)))))))))+54y+54hM(5+hM(-3+hM))y-18M^2(-10+hM(-134+hM(78+hM(-59+21hM))))y^2-36M^2(1+hM(5+hM(-3+hM)))y^3+9M^4(-1+hM(-8+5hM))y^4)))),$$

$$L_{86} = (-12+M(-hM(41+h(24+M(45+2h(6+M(18+5hM)))))+6M(2+h(-4+M(9+h(-4+M))))y+F^2M^2(21+h(12+M(37+2h))-6(5+2h(-1+M))-6M^2(1+hM)y^2)+F(12+M(-5+2h^3M^2(6+23M-6(-1+M)My)+h^2M(12+M(37-6y(-2+M(7+My)))))))),$$

$$L_{87} = ((F+h)(14+M(h(-31+2hM(13+hM(-3+hM(-3+hM))))+10(-1+hM)(1+hM(-1+hM))y-2M(-1+hM)(1+hM(-1+hM))y^2+F(-1+2hM(-1+hM)(2+hM-My)(-3+M(h+y)))))),$$

$$L_{88} = (9+M(h(-351+4hM(294+hM(-297+hM(172+hM(-24+5hM(-3+hM))))))-6M(-3+hM(-1+2hM(-4+hM(9+2hM(-3+hM)))))y^2+4M^3(-1+hM)^3y^4+F^2M(21+M(h(-285+2hM(240+hM(-117+2hM(14+5hM(-2+hM))))))),$$

$$L_{89} = -(54h(2+hM(-2+hM))y),$$

$$L_{90} = (6(3+17E)+M(F^2M(-45+M(h(765+2hM+2hM(-39+hM(93+91hM)))-12(24+hM(-1+hM)(5+7hM))y+6M(15+hM(-3+2hM))y^2+4M^2(-3+hM)y^3))+F(375+288E+M(-2y(78+My(-21+2My))+28h^5M^4(7+6E)-2h^4M^3(-271$$

$$+48E+6My(11+6E))-2h(9(33+E)+2My(294-9E+My(-75+8My+9E)))$$
$$+h^2M(1797-204E+2My(324+My(-93+10My+12E))))))),$$

$$\begin{aligned}
L_{91} &= (108+M(-108+F^2M^2(189+h(156+M(-47+4h(-66+(-13+27h)M))))+h \\
&\quad (-276+M(277+h(336+M(49+4h(-57+M(-46+h(15+14M)))))))+F(156 \\
&\quad +M(61+h(-312+M(346+h(360+M(-231+4h(-78+M+27hM))))))))), \\
L_{92} &= (3+M(F(2+hM(13+hM(-8+hM))+M(-1+hM(-8+5hM))y)+F^2M(3+M \\
&\quad (-3y+hM(h+y)))+h(-1+M(-y+h(7-5My+hM(-5+4My)))))), \\
L_{93} &= (-3+M(-F^2M(3+h^2M^2)-h(1+hM(7+5hM))-F2+hM(13+hM(8+hM))) \\
&\quad +(F+h)M(-1+FM(3+hM)+hM(5+4hM))y)), \\
L_{94} &= ((F+h)(-14+M(h(-31+2hM(-13+hM(-3+hM(3+hM))))+10(1+hM)(1 \\
&\quad +hM(1+hM))y-2M(1+hM)(1+hM(1+hM))y^2+F(-1+2hM(1+hM)(3+h \\
&\quad M-My)(-2+M(h+y)))))), \\
L_{95} &= (-9+M(h(-351+4hM(-294+hM(-297+hM(-172+hM(-24+5hM(3+hM)))))) \\
&\quad -6M(3+hM(-1+2hM(4+hM(9+2hM(3+hM)))))y^2+4M^3(1+hM)^3y^4-2hM^5 \\
&\quad (1+hM)^2y^4)+F^2M(-21+M(-h(285+2hM(240+hM(117+2hM(14+5hM(2+h \\
&\quad M)))))+6M(-1+hM(7+2hM(5+2hM(2+hM))))y^2-4M^3(1+hM)^2y^4)))), \\
L_{96} &= (F^2M^2(-24+M(-16y+h(6+M(2h^4M^3-4y(-1+My)-h^2M(13+2My(3+My)) \\
&\quad +h(-7+6My(3+My))))))+12(-1+E)+M(2h^5M^4+8y(-2+7E+MyE)+h^3M^2 \\
&\quad (-5(3+2E)+2My(3+My)(-1+6E))+h^2M(-13+98E+2My(9+My(3-2E)+10 \\
&\quad E))-2h(-1+35E+2My(-1+My+25E+3MyE)))+2FM(2(-9+E)+M(2h^5M^4 \\
&\quad +4y(-4+7E+MyE)+h^3M^2(-7(2+E)+2My(3+My))+h^2M(-10+43E+2My \\
&\quad (9-My(-3+E)+5E))-h(-4+25E+2My(-2+25E+My(2+3E)))))), \\
L_{97} &= (90-162E+M(F^2M(207+M(h(93+2hM(33+hM(11+5hM)))-12(1+hM)(8 \\
&\quad +hM(-3+hM))y+6M(1+hM)(-5+2hM)y^2-4M^2(1+hM)y^3))+F(-297+96E \\
&\quad +M(2y(78+My(21+2My))+4h^5M^4(1+6E)-h^2M(63+2My(12+My(9+2My
\end{aligned}$$

$$-12E)) + 204E) - 6h^4M^3(5 + 2My - 8E + 4MyE) + 2h^3M^2(-5 + 6My(3 - 3E + M$$
$$y(1 + E))) + 6h(-3 + E + 2My(7 + E + My(1 + E))))))),$$

$$
\begin{aligned}
L_{98} \;=\;& (90 + (35 + 87E) + M(F^2M(3285 + M(h(12285 + 2hM(-1965 + hM(4905 + 2hM(-35 \\
& +2hM(146 + 193hM))))) - 30(33 + hM(-7 + 2hM(-3 + 7hM)))y + 20M^2(21 + hM \\
& (-5 + 2hM(-4 + 7hM)))y^3 - 4M^4(-3 + hM)y^5)) + 20M^2(21 + hM(-5 + 2hM(-4 + 7 \\
& hM)))y^3 - 4M^4y^5)) + 20h^7M^6(135 - 44E) + 8h^8M^7(22 + 135E) + h(75(445 - 934 \\
& E) + 2My(75 - 180E + 2My(M^3y^3 + 5My(-5 + 6E) - 120(5 + 11E)))) + 5h^2M(3(\\
& -957 + 4016E) + 2My(-3(59 + 12E) + 2My(480 + M^3y^3 + 264E + My(35 + 12E)))) \\
& +F(16125 - 37440E + M(4h^6M^5(1087 - 370E) + 20h^7M^6(77 + 72E) - 4h^5M^4(1356 \\
& -205E + 70M^2y^2(7 + 6E)) + 2y(-285 + 2My(65My + M^3y^3)) - 10h^3M^2(3128 - 57E \\
& +2My(-6(3 + 2E) + My(-349 + 24E + 2My(7 + 5E)))) + 5h^2M(7491 + 1956 - 2My \\
& (-3(77 + 4E) + 2My(1176 + M^3y^3 - 96E + My(51 + 8E)))))))),
\end{aligned}
$$

$$
\begin{aligned}
L_{99} \;=\;& (8730 - 22530E + M(-16005h + 480y + M(-h^2(37155 + 2hM + 2hM(10500 + hM \\
& (3905 + 4hM(190 + hM(21 + hM))))) - 60h(-5 + hM(-1 + 2hM(-1 + hM)))y - 80 \\
& (-36 + hM(12 + h^2M^2(-9 + hM(-2 + hM))))y^2 + 40M(-6 + hM(-5 - hM + 2h^3M^3 \\
&))y^3 + 8hM^4(1 + hM)y^5) + F^2M(8415 + M(-h(3045 + 2hM(3390 + hM(-230 + hM \\
& (155 + 56hM(6 + hM))))) - 8M^4(1 + hM)y^5)) - 10(h(-1293 + hM(-9948 + hM \\
& (-8755 + hM(-2931 + 2hM(-343 + 2hM(-51 + 2hM(-2 + hM))))))) + 8M^2(-1 \\
& +hM(1 + hM)(-1 + 2hM))y^3)E + F(105 + M(-10h(261 + 8My(9 + My(24 + My \\
& (-5 + E))) - 5463E) - 4y(-285 + 2My(65My + M^3y^3 + 120(-9 + 2E))) - 20h^4 \\
& M^3(-136 - 373E + 2My(-3 - 6E + My(9 + 2My + 4(-9 + My)E))) + 10h^3M^2(\\
& 7(32 + 63E) + 4My(-6 + 12E - My(35 + 24E + 2My(-1 + 5E)))))))),
\end{aligned}
$$

$$
\begin{aligned}
L_{100} \;=\;& (6(-51 + 47E) + M(F^2M(-531 + M(h(-867 + 2hM(-9 + hM(-9 + 91hM))) - 12 \\
& (-24 + hM(1 + hM)(-5 + 7hM))y - 6M(15 + hM(3 + 2hM))y^2 + 4M^2(3 + hM)y^3 \\
&)) + h(-3(89 + 146E) + 2My(12 - 54E + My(3 - 2My + 18E))) + F(-3(35 + 96
\end{aligned}
$$

$$E) + M(-2y(78 + My(-21 + 2My)) + 28h^5M^4(7 + 6E) - 2h^4M^3(139 - 120E)$$
$$-2h^3M^2(316 - 45E + 6My(5 + My(1 + E))) + h^2M(9(-323 + 20E) + 2My(324$$
$$+My(-93 + 10My + 12E))))))),$$

$$L_{101} = (6(3 + 17E) + M(F^2M(-45 + M(h(-765 + 2hM(-39 + hM(-93 + 91hM))) - 12$$
$$(-12 + hM(1 + hM)(-5 + 7hM))y + 6M(15 + hM(3 + 2hM))y^2 + 4M^2(3 + hM)y^3))$$
$$+4h^6M^5(8 + 33E) + 36y(5 + My - (3 + My)E) + 2h^3M^2(-555 + 225E + 2My(My$$
$$(33 + 4My + 3E) + 3(34 + 9E))) + h^2M^2(2My(324 + My(93 + 10My - 12E)) + 3$$
$$(-599 + 68E)) + 2h^3M^3(-382 + 3E + 6My(-5 - 3E + My(1 + E)))))),$$

$$L_{102} = (90(35 - 87) + M(F^2M(3285 + M(h(-12285 + 2hM(-1965 + hM(-4905 + 2hM$$
$$(-35 + 2hM(-146 + 193hM))))) - 30(-33 + hM(-7 + 2hM(3 + 7hM)))y - 20M$$
$$(-144 + hM(-408 + hM(-24 + hM(-51 + 91hM))))y^2 + 20M^2(-21 + hM(-5 + 2$$
$$hM(4 + 7hM)))y^3 - 4M^4(3 + hM)y^5)) + 20y(6 + My(12 + My(-3 + E) - 16E)$$
$$-3E) + 20h^7M^6(-135 + 44E) + 8h^6M^7(22 + 135E) - 4h^6M^5(1261 - 835E + 10$$
$$M^2y^2(8 + 33E)) + -2h(-15(352 + 607E) + 4My(-435 + My(-1800 + 4M^3y^3 + 5$$
$$My(37 + 3E)))) + 10h^3M^2(-3128 + 57E + 2My(-6(3 + 2E) + My(349 - 24E + 2$$
$$My(7 + 5E)))) + 10h^4M^3(-2495 + 84E + 2My(-3(11 + 6E) + My(205 - 84E + 2$$
$$My(11 + 6E)))) + 5h^2M(-3(2497 + 652E) - 2My(-3(77 + 4E) + 2My(M^3y^3 + 24$$
$$(-49 + 4E) + My(51 + 8E)))))))),$$

$$L_{103} = (F^2M^2(-24 + M^2(40h^4M^2 - 52hy - 42h^3M^2y + 96y^2 + h^2(-45 + 2M^2y^2))) + 4(-3$$
$$+E) + 2FM^2(-40y + h(24 + 2M^2y^2(94 - 9E) - 75E) + 84yE - 6h^4M^4y(2 + 3E) +$$
$$6h^5M^4(1 + 4E) - 2h^2M^2y(43 + 5E) + h^3M^2(-112 - 6M^2y^2(-1 + E) + 31E)) + M^2$$
$$(8y^2(6 - 5E) + 4h^6M^4(-2 + 3E) + 4hy(-41 + 93E) + 2h^4M^2(-114 + 2M^2y^2(2 - 3$$
$$E) + 125E) - 2h^3M^2y(9 + 130E) + h^2(85 - 274E + 2M^2y^2(117 + 2E)))),$$

$$L_{104} = (12 - 8E + M(F^2M(24 + M(h(-14 + hM(26 + hM(9 + 2hM(14 + 3hM)))) + 16y$$

$$
\begin{aligned}
& -6hM(-4+hM(-1+4hM))y-2M(24+hM(6+hM(2+3hM)))y^2))+h^3M^2 \\
& (111-250E)+4h^6M^5(3+2E)+2F(2-2E+M(h^3M^2(3-2My(15+5My-9 \\
& E)-12E)+2h^6M^5(1+E)+4h^5M^4(4+3E)-4My(-7+14E+My(8+E)) \\
& +h^2M(34-47E+2My(13+My-3MyE))-h^4M^3(-29+11E+2My(3+6E \\
& +My(1+E)))+2h(9+25E+My(2+25E+2My(-7+3E))))))),
\end{aligned}
$$

$$
\begin{aligned}
L_{105} &= (12-8E+M(F^2M(24+M(h(-14+hM(26+hM(9+2hM(14+3hM))))+16y \\
& -6hM(-4+hM(-1+4hM))y-2M(24+hM(6+hM(2+3hM)))y^2))+h^3M^2 \\
& (111+2My(30+My(9-10E)-56E)-250E)+4h^6M^5(3+2E)+2h^5M^4(9 \\
& +16E)+8y(2-7E+My(-3+2E))-2h^4M^3(9-22E+2My(9+3My+3E \\
& +2MyE))+2h^2M(2+60E+My(-49+158E+4My(-5+8E)))-2h(5-19 \\
& E+2My(-20+34E+My(3+13E)))+2F(2-2E+M(h^3M^2(3-2My(15+5 \\
& My-9E)-12E)+2h^6M^5(1+E)+4h^5M^4(4+3E)-4y(14E-7+My(8+E)) \\
& +2h(9+25E+My(2+25E+2My(-7+3E))))))),
\end{aligned}
$$

$$
\begin{aligned}
L_{106} &= (6(-51+47E)+M(F^2M(-531+M(h(867+2hM(-9+hM(9+91hM)))-12 \\
& (24+hM(-1+hM))y-6M(15+hM(-3+2hM))y^2+4M^2(-3+hM)y^3))+36 \\
& y(-5+My(-1+E)+3E)-h^2M(1401+600E+2My(264+My(87+10My)+ \\
& 36E))+h(267+438E-2My(6(-2+9E)+My(3+2My+18E)))+F(3(35+ \\
& 96E)+M(-2y(78+My(21+2My))+2h(-2My(249+My(75+8My-9E)-9 \\
& E)+9(-127+E))+h^2M(2907+2My(324+My(93+10My-12E))-180E) \\
& -2h^3M^2(316-45E+6My(-5-3E+My(1+E))))))),
\end{aligned}
$$

$$
\begin{aligned}
L_{107} &= (198-222E+M(-3(7h+20y)+M(-h^2(-3+2hM(-27+2hM(1+hM)))+12 \\
& h(-2+h^2M^2(-2+hM))y+6(1+hM)+4hM(1+hM)y^3)+F^2M(369+M(h(195 \\
& +2hM(15+hM(23+5hM)))-12(1+hM)-6M(1+hM)-4M^2(1+hM)y^3)) \\
& +6(h(-71+hM(-56+hM(-33+hM(-7+2hM(3+hM)))))-2M(1+hM(1 \\
& +hM))y^2(1+hM)^2)E+F(-567+M(2y-6h^4M^3(1+2My+4(-4+My)E)-h^2 \\
& M(33+180E+2My(12+My(-9+2My+12E))))))),
\end{aligned}
$$

$$
\begin{aligned}
L_{108} &= (-6+16E+M(F^2M(-6+M(h(-4+hM(-1+4hM))+4y-My^2))+2M^3(3+\\
&\quad E)y(-4+My)(-1+2E)+2h^3M^2(-5+2E(8+My(-4+My))E)+h^2M(23-86\\
&\quad E-2My(-4+My)(-1+5E))+2h(My(-4+My)(-1+2E)+2(-5+9E))+2\\
&\quad F(hM(-1+My)+h^4M^4(1+2E)+hM(-4-15E-My(-4+My)(1+2E))))),\\
L_{109} &= (6-16E+M(-2h(10+My(4+My))+F^2M(6+M(h(-4+hM(1+4hM))+4\\
&\quad y+My^2))+4h(9+My(4+My))E-2h^4M^3(3+E)-y(4+My)(-1+2E)+2h^3\\
&\quad M^2(-5+2(8+My(4+My))E)+h^2M(-23+86E+2My(45+My)(-1+5E))+\\
&\quad 2F(h^2M^2(-1+My(4+My))+h^3M^3(-5+3E)+hM(4+15E+My(4+My)(1\\
&\quad +2E))))),\\
L_{110} &= (2Bre^{2hM}(F+h)^2M^2(2M^2(h-y)(h+y)-\cosh(2hM)+\cosh(2My))),\\
L_{111} &= (9L_{68}(\cosh(M(2h+y))+\sinh(M(2h+y)))-4L_{85}(\cosh(3M(2h+y))+\sinh\\
&\quad (3M(2h+y)))-9L_{83}(\cosh(M(6h+y))+\sinh(M(6h+y)))+9L_{81}(\cosh(M\\
&\quad (8h+y))+\sinh(M(8h+y)))-16L_{41}(F+h)(3+M)(1+hM)(\cosh(3M(2h\\
&\quad +y))+\sinh(3M(2h+y)))+144L_{41}(F+h)(-9+M)(\cosh(M(3h+2y))+\\
&\quad \sinh(M(3h+2y)))-288(F+h)(FhM+1)M(\cosh(M(5h+2y))+\sinh(M(5h\\
&\quad +2y)))-144L_{42}(F+h)(M-9)(hM-1)M(\cosh(M(7h+2y))+\sinh(M(7h+\\
&\quad 2y)))+L_{79}(\cosh(M(2h+3y))+\sinh(M(2h+3y)))+4L_{80}(\cosh(M(4h+3y))\\
&\quad +\sinh(M(4h+3y)))-4608(F+h)M^2(1+FhM^2)(h-y)(\cosh(M(5h+3y))+\\
&\quad \sinh(M(5h+3y)))-2304(F+h)L_{42}M^2(hM-1)(h-y)(\cosh(M(7h+3y))+\\
&\quad \sinh(M(7h+3y)))+L_{77}(\cosh(M(8h+3y))+\sinh(M(8h+3y)))+L_{91}(\cosh\\
&\quad (M(10h+3y))+\sinh(M(10h+3y)))+144(F+h)L_{41}M(9+M)(1+hM)(\cosh\\
&\quad (M(3h+4y))+\sinh(M(3h+4y)))-288M(F+h)(9+M)(1+FhM^2)(\cosh(M\\
&\quad (5h+4y))+\sinh(M(5h+4y)))-144ML_{42}(F+h)(9+M)(hM-1)(\cosh(M(7\\
&\quad h+4y))+\sinh(M(7h+4y)))+9L_{46}(\cosh(M(2h+5y))+\sinh(M(2h+5y)))\\
&\quad -9L_{44}(\cosh(M(4h+5y))+\sinh(M(4h+5y)))-9L_{78}(\cosh(M(6h+5y))+\\
&\quad \sinh(M(6h+5y)))+9L_{45}(\cosh(M(8h+5y))+\sinh(M(8h+5y)))+32M(F
\end{aligned}
$$

$+h)(3+M)(1+FhM^2)(\cosh(M(5h+6y))+\sinh(M(5h+6y)))),$

$$
\begin{aligned}
L_{112} &= (L_{60}e^{M(2h+y)}-4L_{61}e^{3M(2h+y)}+L_{62}e^{M(4h+y)}+L_{63}e^{M(2h+3y)}-L_{64}e^{M(6h+y)}-L_{65}e^{M(8h+y)}\\
&\quad +4L_{66}e^{M(4h+3y)}+L_{68}e^{3My}+L_{69}e^{M(8h+3y)}-L_{70}e^{M(10+3y)}+L_{71}e^{(2h+5y)}\\
&\quad +L_{72}e^{(4h+5y)}-L_{75}e^{(6h+5y)}-L_{76}e^{(8h+5y)}-192(1+FM)(-1+hM)(2+My)(-1+F\\
&\quad M-E+hME)(e^{(9h+4y)}+e^{(9h+2y)}+e^{(h+4y)}+e^{(h+2y)})+384M(-2+My)(5h+F(2+h\\
&\quad (F+4h)M^2)+(-3(F+2h)+h^2(F+4h)M^2)E)(e^{(5h+4y)}-e^{(5h+2y)}+e^{(7h+4y)})),\\
L_{113} &= (F^2M^2(24+hM(6+hM(7-13hM+2h^3M^3))+2M(1+hM)-2hM^3(1+hM)(2\\
&\quad +hM)y^2)-12(-1+E)+M(-16y+h(2+M(h(13-15hM+2h^3M^3)-4y+6hM\\
&\quad (3+hM)y-2M(1+hM)))-2(h(35+hM(49+hM))-2M(-1+hM)(2+hM)y^2\\
&\quad (1+hM)^2)E)+2FM(2(-9+E)+M(16y+2h(-2+M(-h+2y+M(1+hM)(2+\\
&\quad hM)y^2))+(h(25+hM)+2(1+hM)^2(-14+3hM(1+hM))y-2My^2(-1+hM)(1\\
&\quad +hM)^2)E))),\\
L_{114} &= (-16L_{41}(F+h)(-3+h)M(1+hM)(\cosh(3hM)+\sinh(3hM))+32(F+h)(-3+M)\\
&\quad (1+FhM^2)(\cosh(5hM)+\sinh(5hM))+16L_{42}(F+h)(-3+h)M(1+hM)(\cosh(7h\\
&\quad M)+\sinh(7hM))-(-108+(F(-61+312h)-h(277+336h))M^2+h(4(46-15h)\\
&\quad h^2+F^2(47+264h))M^4+4h^2(14h^2+Fh(1+27h))M^5)(\cosh(3My)+\sinh(3My))\\
&\quad +2304L_{41}(F+h)(h-y)(\cosh(3M(h+y))+\sinh(3M(h+y)))+16L_{42}M(3+M)\\
&\quad (-1+hM)(\cosh(M(7h+6y))+\sinh(M(7h+6y)))+L_{111}),\\
L_{115} &= (288L_{41}(-1+h)M(1+hM)(\cosh(M(3h+4y))+\sinh(M(3h+4y)))-576(-1+h)\\
&\quad M(1+FhM^2)(\cosh(M(5h+4y))+\sinh(M(5h+4y)))-288(-1+h)L_{42}M(\cosh(M\\
&\quad (7h+4y))+\sinh(M(7h+4y)))-9L_{54}(\cosh(M(4h+5y))+\sinh(M(4h+5y)))-\\
&\quad 9L_{58}-9L_{55}(\cosh(M(6h+5y))+\sinh(M(6h+5y)))-9L_{59}(\cosh(M(8h+5y))+\\
&\quad \sinh(M(8h+5y)))+64(-1+h)M(1+FhM^2)(\cosh(M(5h+6y))+\sinh(M(5h+6\\
&\quad y)))+32(-1+h)ML_{42}(-1+hM)(\cosh(M(7h+6y))+\sinh(M(7h+6y))),\\
L_{116} &= ((-6-4My-M^2y^2+F^2M^2(-6-4hM-h^2M^2+4h^3M^3-4My-M^2y^2+h^2M^2(23
\end{aligned}
$$

$+8My(1-5E)+2M^2y^2(1+5E)-86E)+(16E+8MyE+2M^2y^2+2hM(4My(-1$
$+2E)+M^2y^2(-1+2E)+2(-5+9E))+2FM(h^2M^2(-1+4My+M^2y^2)+h^4M^4(1$
$+2E)-hM(4+15E+M^2y^2(1+2E)+4My(1+2E))))(\cosh(M(7h+2y))+\sinh$
$(M(7h+2y)))),$

$$L_{117} = (24M(F^2M(-1-4h^3M^3+2h^4M^4-2h^2M^2(5+5y+M^2y^2)+2hM(6+5My+M^2y^2)) +2F(7-5h^4M^4+2h^5M^5+5My+M^2y^2-2h^{33}M^3(4+5My+M^2y^2)-2hM(8+5My+M^2y^2)+h^2M^2(19+15My+3M^2y^2))+h(-6h^4M^4+2h^5M^5+hM(31+20My+4M^2y^2)))(\cosh(M(7h+2y))+\sinh(M(7h+2y))),$$

$$L_{118} = (120(F^2M^2(-24-10hM-3h^2M^2+h^3M^3)+12(-1+E)-2h^4M^4E-14hM(1+E +2FM(2(-9+E)+3hM(-4+E-3h^3M^3E+h^4M^4E-3h^2M^2(2+E)))(\cosh(M(7h+5y))+\sinh(M(7h+5y)))),$$

$$L_{119} = (120(F^2M^2(-24+2hM-h^2M^2+7h^3M^3)+4(-3+E)-6h^4M^4(2+E)+h^3M^3(-19+18E)+2hM(-1+19E)+h^2M^2(-31+58E)+2FM(14+6E+h^4M^4(2+3E)+hM(-16+9E)))(\cosh(M(3h+5y))+\sinh(M(3h+5y)))),$$

$$L_{120} = (960(6-4My+M^2y^2+F^2M^2(6-4hM+h^2M^2+4h^3M^3-4My+M^2y^2)-16E+8MyE-2M^2y^2E+h^2M^2(-23+8My(1-5E)+86E)+2hM(My(4-8E)+M^2y^2(-1+2E))+2FM(h^2M^2(-1-4My+M^2y^2)+(8-4My+M^2y^2)(-1+2E)+h^3M^3(-5+3E)+hM(4+15E+M^2y^2(1+2E)-4My(1+2E))))(\cosh(M(3h+4y))+\sinh(M(3h+4y)))),$$

$$L_{121} = (24(F+h)M(29-12h^2M^2+4h^4M^4+20My+4M^2y^2+FhM^2(23+20My+4M^2y^2))(\cosh(M(5h+2y))+\sinh(M(5h+2y)))-12(F+h)M(1+FM)(-1+hM)(\cosh(M(9h+2y))+\sinh(M(9h+2y)))+L_{95}(\cosh(M(2h+3y))+\sinh(M(2h+3y)))+2L_{82}(\cosh(M(4h+3y))+\sinh(M(4h+3y)))+L_{88}(\cosh(M(8h+3y))+\sinh(M(8h+3y)))+L_{88}-12M(F+h)(-1+FM)(1+hM)(\cosh(M(h+4y))+\sinh(M(h+4y)))-24ML_{94}(\cosh(M(3h+4y))+\sinh(M(3h+4y)))-24(F$$

$+h)M(29-12h^2M^2+4h^4M^4-20My+4M^2y^2)(\cosh(M(5h+4y))+\sinh(M(5h$
$+4y)))+24ML_{87}(\cosh(M(7h+4y))+\sinh(M(7h+4y)))-12M(F+h)(-1+$
$hM)(\cosh(M(9h+4y))+\sinh(M(9h+4y)))+3(1+hM)(1+h^2M^2+hM^2y+$
$FM(-1+hM+My))(\cosh(M(2h+5y))+\sinh(M(2h+5y)))-3L_{93}(\cosh(M(4$
$h+5y))+\sinh(M(4h+5y)))-3L_{92}(\cosh(M(6h+5y))+\sinh(M(6h+5y)))),$

$$L_{122} = ((F^2M^2(-48+6hM-h^2M^2+7h^3M^3)+2hM(7-15E)-h^2M^2(117+2E)+2h^4 M^4(-2+3E)+4(-6+5E)+h^3M^3(3+22E)+2FM(4-h^2M^2-6E+h^4M^4(2+3E)))(\cosh(M(5h+y))+\sinh(M(5h+y)))),$$

$$L_{123} = (-32M(-1+h)L_{41}(\cosh(3hM)+\sinh(3hM))+64M(-1+h)(1+FhM^2)(\cosh(5hM)+\sinh(5hM))+32M(-1+h)(1+FhM^2)(\cosh(7hM)+\sinh(7hM))-L_{57}(\cosh(3My)+\sinh(3My))-9L_{47}(\cosh(M(2h+y))+\sinh(M(2h+y)))-4L_{48}(\cosh(3M(2h+y))+\sinh(3M(2h+2y)))-9L_{49}(\cosh(M(4h+y))+\sinh(M(4h+y)))-9L_{50}(\cosh(M(8h+y))+\sinh(M(8h+y)))+9L_{51}(\cosh(M(6h+y))+\sinh(M(6h+y)))+L_{52}(\cosh(M(2h+3y))+\sinh(M(2h+3y)))-4L_{53}(\cosh(M(4h+3y))+\sinh(M(4h+3y)))-L_{56}(\cosh(M(8h+3y))+\sinh(M(8h+3y)))+(108+(-14+405F+203h)M+(297h^2+162Fh)M^2+4(-1+h)h^2(F+h)M^4)(\cosh(M(10h+3y))+\sinh(M(10h+3y)))),$$

$$L_{124} = (-16L_{41}(1+hM)(\cosh(3hM)+\sinh(3hM))+32(1+FhM^2)(\cosh(5hM)+\sinh(5hM))+16L_{42}(-1+hM)(\cosh(7hM)+\sinh(7hM))-9L_{41}(1+2hM)(\cosh(M(2h+y))+\sinh(M(2h+y)))+4L_{60}(\cosh(3M(2h+y))+\sinh(3M(2h+y)))-9L_{42}(1+2hM)(\cosh(M(6h+y))+\sinh(M(6h+y)))-9L_{42}(-1+2hM)(\cosh(M(8h+y))+\sinh(M(8h+y)))-16L_{41}(\cosh(3M(h+2y))+\sinh(M(h+2y)))-288(1+FhM^2)(\cosh(M(2h+2y))+\sinh(M(5h+2y)))-144L_{42}(-1+hM)(\cosh(M(7h+2y))+\sinh(M(7h+2y)))),$$

$$L_{125} = (((-167 + 72h^5M^5 - 18h^3M^3(1 - 2FM + 4M^2y^2) - hM(179 + 167FM + 108 M^2y^2) - 4(-40 + 18h^5M^5 + 9h^4M^4(-1 + 2FM) - 9M^2y^2 + h^3M^3(58 + 9FM) + h^2M^2(82 + 49FM - 18FM^3y^2) - hM(-89 + 9M^2y^2 + FM(40 + 9M^2y^2 + F M(40 + 9My)))(\cosh(M(4h + 3y)) + \sinh(M(4h + 3y))) + L_{26} + L_{42}(7 + 2hM) (\cosh(M(10h + 3y)) + \sinh(M(10h + 3y))) + 9L_{41}(-1 + 2hM)(\cosh(M(6h + 5 y)) + \sinh(M(6h + 5y))) - 9L_{42}(-1 + 2hM)(\cosh(M(8h + 5y)) + \sinh(M(8h +5y))) + 32(1 + FhM^2)(\cosh(M(5h + 6y)) + \sinh(M(5h + 6y))) + 16L_{42}(-1 +hM)(\cosh(M(7h + 6y)) + \sinh(M(7h + 6y))) + L_{124})),$$

$$L_{126} = ((-167 - 72h^5M^5 - 36h^4M^4(-3 + 2FM) - 36M^2y^2 + 18h^3M^3(1 + 2FM + 4 M^2y^2) + hM(179 - 167FM + 108M^2y^2 - 36FM^3y^2) + 9h^2M^2(-39 - 12M^2 y^2 + 2FM(7 + 4M^2y^2)))(\cosh(M(8h + 3y)) + \sinh(M(8h + 3y)))),$$

$$L_{127} = ((-162 + F^2M^2(-123 - 257hM - 72h^2M^2 + 8h^3M^3 + 2h^4M^4) + 2h^4M^4)) +458E - 4h^5M^5E + 2h^3M^3(4 + 95E) + h^2M^2(-77 + 888E) + hM(-301 +1146E) + FM(285 + h^2M^2(149 - 960E) + 2hM(279 - 607E) - 480E +4h^5M^5E + 4h^4M^4(-1 + 3E) - 2h^3M^3(8 + 109E)))(\cosh(7My) + \sinh (7My))),$$

$$L_{128} = (F^2M(-1 + 4h^3M^3 + 2h^4M^4 - 2h^2M^2(5 + My + M^2y^2) - 2hM(6 + 5My + M^2 y^2) - 2hM(6 + 5My + M^2y^2)) + 2F(-7 + 5h^4M^4 + 2h^5M^5 - 5My + M^2y^2) - 2 hM(8 + 5My + M^2y^2) - h^2M^2(19 + 15My + 3M^2y^2)) + h(6h^4M^4 + 2h^5M^5 - 2 h^3M^3(3 + 5My + M^2y^2) + 2h^2M^2(13 + 10My + 2M^2y^2) - hM(31 + 20My + 4 M^2y^2)))(\cosh(M(3h + 2y)) + \sinh(M(3h + 2y))),$$

$$L_{129} = (3(1 + FM)(-1 + hM)(1 + h^2M^2 - hM^2y + FM(1 + hM - My))(\cosh(M(8h + 5 y)) + \sinh(M(8h + 5y)))),$$

$$L_{130} = (3(-1 + FM)(1 + hM)(1 + h^2M^2 + hM^2y + FM(-1 + hM + My))(\cosh(M(2h$$

$$+5y)) + \sinh(M(2h + 5y)))),$$

$$L_{131} = (24(F + h)M(29 - 12h^2M^2 + 4h^4M^4 - 20My + 4M^2y^2 + FhM^2(23 - 20My + 4M^2y^2))(\cosh(M(5h + 4y)) + \sinh(M(5h + 4y)))),$$

$$L_{132} = (4(F + h)M(29 - 12h^2M^2 + 4h^4M^4 + 20My + 4M^2y^2 + FhM^2(23 + 20My + 4M^2y^2))(\cosh(M(5h + 2y)) + \sinh(M(5h + 2y)))),$$

$$L_{133} = (2L_{82}(\cosh(M(4h + 3y)) + \sinh(M(4h + 3y))) + L_{88}(\cosh(M(8h + 3y)) + \sinh(M(8h + 3y)))),$$

$$L_{134} = (3(1 + FM)(-1 + hM)(-1 - 4hM + 2h^2M^2 - 2M^2y^2 + FM(-5 + 2h^2M^2 - 2M^2 y^2))(\cosh(M(10h + 3y)) + \sinh(M(10h + 3y)))),$$

$$L_{135} = (12(F + h)M(-1 + FM)(1 + hM)(\cosh(M(h + 4y)) + \sinh(M(h + 4y))) - 24M L_{94}(\cosh(M(3h + 4y)) + \sinh(M(3h + 4y)))),$$

$$L_{136} = (24(F + h)M(29 - 12h^2M^2 + 4h^4M^4 - 20My + 4M^2y^2 + FhM^2(23 - 20My + 4 M^2y^2))(\cosh(M(5h + 4y)) + \sinh(M(5h + 4y)))),$$

$$L_{137} = (24ML_{87}(\cosh(M(7h + 4y)) + \sinh(M(7h + 4y))) - 12(F + h)(1 + FM)(hM - 1) M(\cosh(M(9h + 4y)) + \sinh(M(9h + 4y)))),$$

$$L_{138} = (120(24 + F^2M^2(48 + 6hM + h^2M^2 + 7h^3M^3) + 2hM(7 - 15E) + 2h^4M^4(2 - 3E) -20E + h^2M^2(117 + 2E) + h^3M^3(3 + 22E) + 2FM(4 + hM(94 + 9E) - h^2M^2(9 -E) - 3h^3M^3(-1 + E) - 6E))),$$

$$L_{139} = (12h^6M^5E - 4h^5M^4(-1 + 9E + 6MyE) + 12y(5 + My - (3 + My)E) - 12h^4M^3(2$$

$$L_{139} = (12h^6M^5E - 4h^5M^4(-1 + 9E + 6MyE) + 12y(5 + My - (3 + My)E) - 12h^4M^3(2$$

$$E)))),$$

$$L_{140} = (198 + 22E + M(F^2M(369 + M(h(-195 + 2hM(15 + hM(-23 + 5hM))) - 12(-1 +hM)(8 + hM) - 4M^2(-1 + hM)y^3)) + L_{139} + h(21 + 426E + 2My(6 + My)) + F (567 - 96E + M(2y(78 + My(21 + 2My)) + h^2M(33 - 2My(12 + My(9 + 2My))$$

$$+180E)-2h^3M^2(50+3E+6My(3-3E+My(1+E)))-6h(29+E+2My(7+E+My(1+E))))))),$$

$$L_{141} = (960M(F^2hM^2(12+7h^2M^2+My(-4+My))+h(44-102E-My(-4+My)(-5+12E)+h^2M^2(3+(71+8My)E))+2F(10-24E-My(-4+My)(-1+3E)+h^4M^4(2+3E)+h^2M^2(21+5E+My(-4+My)(2+3E))))),$$

$$L_{142} = ((1+FM(-1+hM)(-(1+FM)(-6+M(-2h+h^2M-y(4+My)))-2(-1+hM)(-8+M(-h+h^2M-y(4+My)))E)+2M(F^2M^2h(12+7h^2M^2+My(4+My))+2F(10-24E+h^4M^4(2+3E)+h^2M^2(21+5E+My(4+My)(2+3E)))+h(44-102E+M(y(4+My)(5-12E)+h^2M(3+(70+8My(4+My))E))))),$$

$$L_{143} = (18\,\mathrm{Pr}\,\mathrm{Re}(3(-1+FM)(1+hM)(-1+4hM+2h^2M^2-2M^2y^2+F(5M-2M^3(h^2y^2)))(\cosh(3My)+\sinh(3My))+3(-1+FM)(1+hM)(1+h^2M^2-hM^2y+FM(-1+hM-My))(\cosh(M(2h+y))+\sinh(M(2h+y)))+2L_{67}(\cosh(3M(2h+y))+\sinh(3M(2h+y)))+3(3+h^3M^3(5+4My)+h^2M^2(7+5My)+hM(1-My)+F^2M^2(3+h^2M^2+3My+hyM^2)-3(1+FM)(-1+hM)(1+h^2M^2+hM^2y+FM(1+hM+My))(\cosh(M(8h+y))+\sinh(M(8h+y)))-12(F+h)M(-1+FM)(1+hM)-2(\cosh(M(2h+y))+\sinh(M(2h+y)))-24(F+h)(29-12h^2M^2)(\cosh(M(5h+2y))+\sinh(M(5h+2y)))+L_{95}(\cosh(M(2h+3y))+\sinh(M(2h+3y)))-2L_{82}(\cosh(M(4h+3y))+\sinh(M(4h+3y)))+L_{88}(\cosh(M(8h+3y))+\sinh(M(8h+3y)))-3L_{92}(\cosh(M(6h+5y))+\sinh(M(6h+5y)))-3L_{93}(\cosh(M(4h+5y))+\sinh(M(4h+5y)))-24L_{94}M(\cosh(M(3h+4y))+\sinh(M(3h+4y)))+L_{117}-24ML_{128}+L_{130}+L_{137}+M\lambda_1L_{114}),$$

$$L_{144} = (24-20E+M(F^2M(48+hM(6+hM(1+7hM)))+h(14-30E+hM(117+2E+hM(3+hM(4-6E)+22E)))+2F(4-6E+hM(94-9E+hM(9-E+hM(3-3E+hM(2+3E)))))).$$

$$
\begin{aligned}
L_{145} &= (-4(-3+E)+M(F^2M(24+hM(2+hM(1+7hM)))+4F(7+hM(8+hM(-4\\
&\quad +hM(5+hM))))+6F(-1+hM)^2(2+hM(1+hM))E+h(-2+38E+hM(31\\
&\quad -58E+hM(-19+18E+6hM(2+E)))))),\\
L_{146} &= (960M(F^2M^2h(12+7h^2M^2+My(4+My))+2F(10-24E-My(4+My)(-1\\
&\quad +3E)+h^4M^4(2+3E)+h^2M^2(21+5E+My(4+My)(2+E)))+h(44-102\\
&\quad E+M(y(4+My)(5-12E)+h^2M(3+(70+8My(4+My))E))))),\\
L_{147} &= (12(4(-3+E)+M(F^2M(-24+hM(2+hM(-1+7hM)))+4F(7+hM(-8+h\\
&\quad M(-4+hM(-5+hM))))+6F(1+hM)^2(2+hM(-1+hM))E-h(2-38E+h\\
&\quad M(31-58E+hM(19-18E+6hM(2+E))))))),\\
L_{148} &= (480(-1+FM)(1+hM)((-1+FM)(-6+M(2h+h^2M-y(4+My)))+2(1\\
&\quad +hM)(-8+M(h+h^2M-y(4+My)))E)),\\
L_{149} &= (960M(F^2hM^2(12+7h^2M^2+My(4+My))+2F(10-24E+My(4+My)\\
&\quad (-1+3E)+h^4M^4(2+3E)+h^2M^2(21+5E+My(4+My)(2+E)))+h(44\\
&\quad -102E+M(y(4+My)(5-12E)+h^2M(3+(70+8My(4+My))E))))),\\
L_{150} &= (15(-162+458E+M(F^2M(-123+hM(257+2hM(-36+hM(-4+hM))))\\
&\quad +h(301-1146E+hM(-77+888E+2hM(-4+hM(-7+2hM))E)))+F\\
&\quad (-285+480E+hM(558-1214E+hM(-149+960E+2hM(-8-109E+2\\
&\quad hM(1+(-3+hM)E)))))))),\\
k &= \pi\phi\cos(2\pi x),\\
k_1 &= (120(1+FM)(-2+hM)(-1+hM)(-2E+hM(1+FM+2hME))),\\
k_2 &= (480(-1+FM)(1+hM)((-1+FM)(-6+M(2h-y(4+My)))+2(1+hM)\\
&\quad (-8+M(h+h^2M-y(4+My)))E)),\\
k_3 &= (960M(F^2M^2h(12+7h^2M^2+My(4+My))+2F(10-My(4+My)(-1+3E)\\
&\quad +h^4M^4(2+3E)+h^2M^2(21+5E+My(4+My)(2+E)))+h(44-102E+M(y(4
\end{aligned}
$$

$$
\begin{aligned}
& +My)(5-12E)+h^2M(3+(70+8My(4+My))E))))), \\
k_4 &= (480(1+FM)(-1+hM)((1+FM)(-6+M(-2h+h^2M(3+(70+8My(4+My))E))))), \\
k_5 &= (1+FM)(-6+M(-2h+h^2M+y(4-My)))-2(-1+hM)(8+M(h-h^2M+y(-4 \\
& +My)))E), \\
k_6 &= (120(F^2M^2(-24+hM(-5+hM(2+hM))+12(-1+E)-hM(14(1+E)+hM(9 \\
& +hM+2(-3+hM(1+hM))E))+2FM(2(-9+E)+hM(3(-4+E)+hM(-6+(\\
& -3+hM(-3+hM))E))))), \\
k_7 &= (480(-1+FM)(1+hM)(M(-6+M(2h+h^2M-y))+2(1+2hM)(-8+M(h+h^2 \\
& M-y(4+My)))E)), \\
k_8 &= (960M(F^2M^2h(12+7h^2M^2+My(4+My))+2F(10-24E-My(4+My)(-1+3 \\
& E)+h^4M^4(2+3E)+h^2M^2(21+5E+My(4+My)(2+3E)))+h(44-102E+M \\
& (y(4+My)(5-12E)+h^2M(3+(70+8My(4+My))E))))), \\
k_9 &= (120(4(-3+E)+M(F^2M(-24+hM(2+hM(-1+7hM)))+4F(7+hM(-8+hM \\
& (-4+hM(-5+hM))))+6F(1+hM)^2(2+hM(-1+hM))E-h(2-38E+hM(31 \\
& -58E+hM(19-18E+6hM(2+E))))))), \\
k_{10} &= (480(-1+FM)(1+hM)((-1+FM)(-6+M(2h+h^2M-y(4+My)))+2(1+hM) \\
& (-8+M(h+h^2M-y(4+My)))E)), \\
k_{11} &= (960M(F^2M^2h(12+7h^2M^2+My(4+My))+2F(10-24E-My(4+My)(-1+3 \\
& E)+h^4M^4(2+3E)+h^2M^2(21+5E+My(4+My)(2+3E)))+h(44-102E+M \\
& (y(4+My)(5-12E)+h^2M(3+(70+8My(4+My))E))))), \\
k_{12} &= (108-14M+297F^2M^2+10hM^2+162FhM^2-37h^2M^2-14FhM^3-113Fh^2 \\
& M^3+10h^3M^3-4Fh^2M^4+4h^4M^4), \\
k_{13} &= (-40+89hM-40FhM^2+82h^2M^2+49Fh^2M^3+58h^3M^3+9Fh^3M^3-9M^2y^2 \\
& +9h^2M^4y^2-18Fh^2M^5y^2-18h^3M^5y^2),
\end{aligned}
$$

$$
\begin{aligned}
k_{14} &= (-162 + 285FM - 301hM - 123F^2M^2 + 558FhM^2 - 77h^2M^2 - 257F^2hM^3 + 149 \\
&\quad h^2M^3 + 8h^3M^3 - 72F^2h^2M^4 - 16Fh^3M^4 + 2h^4M^4 + 8F^2h^3M^5 - 4Fh^4M^5 + 2F^2 \\
&\quad h^4M^6 + 458E - 480FME + 1146hME + 1214FhM^2 + 888h^2M^2E - 14h^4M^4E \\
&\quad +12Fh^4M^5E - 4h^5M^5E + 4Fh^5M^6E), \\
k_{15} &= (8E + M(-2hM(2y(20 + 3My) - 4FMy(1 + 7My) + F^2M(7 + 6My(2 + My))) + 8 \\
&\quad My^2(3 + FM(-8 + 6FM - E) - 2E) - 4FE - 2h(-19 + 2FM(-5 + 2My)(-5 + 3M \\
&\quad y) + 2My(-34 + 13My))E + 4h^6M^5(-3 - 2E + FM(1 + E)) + 8y(2 - 7E + FM) \\
&\quad +h^3M^2(111 - 250E + M(3F^2M + 2y(-30 + My(9 - 10E) + 56E) + F(-6 + 4My \\
&\quad (-15 + 9E)))) + 2h^2M(2 + M(F^2M(-13 + My(3 + 2My)) + F(34 - 47E + 2My \\
&\quad (-13 + My - 3MyE)))) + 2h^4M^3(9 - 22E + M(-14F^2M + F(29 - 11E - 2My(-3 \\
&\quad -6E + My(1 + E))))))), \\
k_{16} &= (24 - 20E + M(F^2M(48 + hM(6 + hM(1 + 7hM))) + h(14 - 30E + hM(117 + 2E \\
&\quad +hM(3 + hM(4 - 6E) + 22E))) + 2F(4 - 6E + hM(94 - 9E + hM(3 - 3E + hM \\
&\quad (2 + 3E)))))), \\
k_{17} &= (198 - 222E + M(F^2M(369 + M(h(-195 + 2hM(15 + hM(-23 + 5hM))) - 12(-1 \\
&\quad +hM)(8 + hM(3 + hM))y - 6M(-1 + hM) - 4M^2(-1 + hM)y^3)) + +6h^3M^2(-9 + \\
&\quad 33E + 2My(-2 + 3E + My(3 + 2My + 18E))) + F(567 - 96E + M(4h^5M^4(1 + 6E) \\
&\quad +h^2M(33 - 2My + 180E) + 2h^3M^2(50 + 3E + 6My(3 - 3E + My(1 + E))) - 6h \\
&\quad (29 + E + 2My(7 + E + My(1 + E))))))), \\
k_{18} &= F^2M^2(24 + hM(6 + hM(7 - 13hM + 2h^3M^3)) + 2M(1 + hM)(-8 + 3hM \\
&\quad (2 + hM))y^2) - 12(-1 + E) + M(-16y + h(2 + M(h(13 - 15hM + 2h^3M^3) \\
&\quad -4y + 6hM(3 + hM)y - 2My^2)) - 2(h(35 + hM(49 + hM(5 - 17hM + 2 \\
&\quad h^3M^3))) - 2M(-1 + hM)(2 + hM)y^2(1 + hM)^2)E) + M(16y + 2h(-2 + M
\end{aligned}
$$

$$(-h(5-7hM+h^3M^3)+2y-3hM(3+hM)y^2))+(h(25+hM(43+hM(7-15hM+2h^3M^3)))+2(1+hM)^2(-14+3hM(1+hM))y-2M(-1+hM)(2+hM)(y+hMy)^2)E)),$$

$$k_{19} = (-167+M(h(279+M(-167F-261h+18h(7F+h)M+36h^2(F+3h)M^2-72h^3(F+h)M^3))+36M(-1+2hM)(1+hM(-1+(F+h)M))y^2)),$$

$$k_{20} = (-6+16E+M(-2h(10+My(4+My))+F^2M(-6+M(h(-4+hM(-1+4hM))-4y-My^2))+4h(9+My(4+My))E+2h^3M^2(-5+2(8+My(4+My))E)+h^2M(23-86E)+2F(h^2M^2(-1+My(4+My))+h^3M^3(5-3E)+(8+My(4+My))(-1+2E)+h^4M^4(1+2E)+hM(-4-15E-My(4+My)(1+2E))))),$$

$$k_{21} = ((F+h)(-14+M(h(-31+2hM(-13+hM(-3+hM(3+hM))))-10(1+hM)(1+hM(1+hM))y-2M(1+hM)(1+hM(1+hM))y^2+F(-1+2hM(1+hM)(-2+hM-My)(3+M(h+y)))))),$$

$$k_{22} = (F+h)(14+M(h(-31+2hM(13+hM(-3+hM(-3+hM))))-2M(-1+hM)(1+hM(-1+hM))y^2+F(-1+2hM(-1+hM)(-3+hM-My)(2+M(h+y)))))),$$

$$k_{23} = -32ML_{41}(-1+hM)(\cosh(3hM)+\sinh(3hM))+64(-1+h)M(1+FhM^2)(\cosh(5hM)+\sinh(5hM))+32ML_{42}(-1+hM)(\cosh(7hM)+\sinh(7hM))-L_{57}(\cosh(3My)+\sinh(3My))-9L_{47}(\cosh(M(2h+y))+\sinh(M(2h+y)))-4L_{48}(\cosh(M(2h+y))+\sinh(M(2h+y)))-9L_{49}(\cosh(M(4h+y))+\sinh(M(4h+y)))-9L_{50}(\cosh(M(8h+y))+\sinh(M(8h+y)))-9L_{51}(\cosh(M(6h+y))+\sinh(M(6h+y)))+32(-1+h)L_{41}(\cosh(3M(2h+y))\sinh(3M(2h+y)))-576ML_{41}(-1+h)(1+FhM^2)(\cosh(M(5h+2y))+\sinh(M(5h+2y)))+L_{52}(\cosh(M(2h+3y))+\sinh(M(2h+3y)))+4L_{53}$$

$$
\begin{aligned}
& (\cosh(M(4h+3y)) + \sinh(M(4h+3y))) - L_{56}(\cosh(M(8h+3y)) + \sinh \\
& (M(8h+3y))) + k_{12}(\cosh(M(10h+3y)) + \sinh(M(10h+3y))) - L_{54}(\cosh \\
& (M(2h+5y)) + \sinh(M(2h+5y))) - 9L_{58}(\cosh(M(4h+5y)) + \sinh(M(4h \\
& +5y))) - 9L_{55}(\cosh(M(6h+5y)) + \sinh(M(6h+5y))) - 9L_{59}(\cosh(M(8h \\
& +5y)) + \sinh(M(8h+5y))) + 32ML_{42}(-1+hM)(\cosh(M(7h+6y)) + \sinh \\
& (M(7h+7y))), \\
k_{24} &= (-16(1+hM)L_{41}((\cosh(3hM) + \sinh(3hM)) + (\cosh(3M(h+2y)) + \sinh(3M(h+2y) \\
&))) + 32(1+FhM^2)(\cosh(5hM) + \sinh(5hM)) + 16L_{42}(\cosh(7hM) + \sinh(7hM)) + \\
& L_{60}(\cosh(3M(2h+y)) + \sinh(3M(2h+y)))(-167 - 36M^2y^2 + 18h^3M^3(1 - 2FM + 4 \\
& M^2y^2) - 9h^2M^2(29 + 12M^2y^2 + 2FM(7 + 4M^2y^2)))(\cosh(M(2h+3y)) + \sinh(M(2h \\
& +3y))) + k_{19}(\cosh(M(8h+3y)) + \sinh(M(8h+3y))) - 9L_{42}(1+2hM)(\cosh(M(4h \\
& +5y)) + \sinh(M(4h+5y))) + 9L_{41}(-1+2hM)(\cosh(M(6h+5y)) + \sinh(M(6h+5y \\
&))) - 9L_{42}(-1+2hM)(\cosh(M(8h+5y)) + \sinh(M(8h+5y))) + 32(1+FhM^2)(\cosh \\
& (M(5h+6y)) + \sinh(M(5h+6y))) + 16L_{42}(-1+hM)(\cosh(M(7h+6y)) + \sinh(M(7h \\
& +6y))) \\
k_{25} &= (1 + F^2M^2 + 2hM(-1+E) - E + 2h^3M^3E + h^2M^2(-2+5E) + 2FM(-1 + h^2M^2 + E \\
& +hM(1+E))), \\
k_{26} &= F^2M^2(24 - 10hM + 3h^2M^2 + h^3M^2) - 12(-1+E) + 2h^4M^4E - 14hM(1+E) - h^3M^3 \\
& (1+2E) + 2FM(2(-9+E) - 3hM(-4+E) - 3hM(-4+E) + 3h^3M^3E + h^4M^4E - 3 \\
& h^2M^2(2+E)), \\
k_{27} &= F^2M^2(-48 + 6hM - h^2M^2 + 7h^3M^3) + 2hM(7 - 15E) - h^2M^2(117 + 2E) + 2h^4M^4 \\
& (-2+3E) + 4(-6+5E)h^3M^3(3+22E) + 2FM(4 - h^2M^2(-9+E) + 3h^3M^3(-1+ \\
& E) - 6E + h^4M^4(2+3E) + hM(-94+9E)), \\
k_{28} &= F^2M^2(-48 + 6hM - h^2M^2 + 7h^3M^3) + 2hM(7 - 15E) - h^2M^2(117 + 2E) + 2h^4M^4
\end{aligned}
$$

$$(-2+3E)+4(-6+5E)+h^3M^3(3+22E)+2FM)4+h^4M^4(2+3E)+hM(-94+9E)),$$

$$
\begin{aligned}
k_{29} &= F^2M^2(24+2hM)+h^3M^3(31-58E)-4(-3+E)+h^3M^3(-19+18E)+2hM(-1\\
&\quad +19E)+2FM(14+hM(16-9E)+6E+h^2M^2(-8+3E)+h^4M^4(2+3E)),\\
k_{30} &= (-4(-3+E)+M(F^2M(24+hM(2+hM(1+7hM)))+4F(7+hM(8+hM(-4\\
&\quad +hM(5+hM))))+6F(-1+hM)^2(2+hM(1+hM))E+h(-2+38E+hM(31\\
&\quad -58E+hM(-19+18E+6hM(2+E)))))),\\
k_{31} &= (3+M(h+F(-2+hM(13+hM(8+hM))+M(-1+hM(8+5hM))y)+F^2M(3\\
&\quad +M(3y+hM(h+y)))+hM(-y+h(7+5My+hM(5+4My))))),\\
k_{32} &= (20F+44h+12F^2M^2h+3h^3M^2+4Fh^4M^4-8FMy+20Mhy+F^2M^4y^2h+2\\
&\quad FM^2y^2+5hM^2y^2-48FE-102hE+10Fh^2M^2+70h^3M^2E+6Fh^4M^4E+24F\\
&\quad MyE+48hMyE-12hM^2y^2E+2Fh^2M^4y^2E+8h^3M^4y^2E),\\
k_{33} &= (F^2M^2h(12+7h^2M^2+My(-4+My))+h(44-102E-My(-4+My)+h^2M^2(3\\
&\quad +(70+8My(-4+My))E))+2F(-24E-My(-4+My)(-1+3E)+h^4M^4(2+3\\
&\quad E)+h^2M^2(21+5E+My(-4+My)(2+3E)))),\\
k_{34} &= ((2M^2(k_{33}-2k_{24}M+9(L_{68}(\cosh(3My)+\sinh(3My))+L_{60}(\cosh(M(2h+y))+\\
&\quad \sinh(M(2h+y)))-4L_{61}(\cosh(3M(2h+y))+\sinh(3M(2h+y)))+L_{62}(\cosh(M\\
&\quad (4h+y))+\sinh(M(4h+y)))-L_{64}(\cosh(M(6h+y))+\sinh(M(6h+y)))-L_{65}\\
&\quad (\cosh(M(8h+y))+\sinh(M(8h+y)))+192(-1+FM)(1+hM)(2+My)(\cosh(\\
&\quad M(h+2y))+\sinh(M(h+2y)))-384k_{25}(\cosh(M(3h+2y))+\sinh(M(3h+2y)))\\
&\quad -384k_{26}(\cosh(M(5h+2y))+\sinh(M(5h+2y)))+L_{63}(\cosh(M(2h+3y))+\\
&\quad \sinh(M(2h+3y)))+4L_{66}(\cosh(M(4h+3y))+\sinh(M(4h+3y)))-L_{69}(\cosh(M\\
&\quad (8h+3y))+\sinh(M(8h+3y)))-L_{70}(\cosh(M(10h+3y))+\sinh(M(10h+6y)))\\
&\quad +384k_{25}(\cosh(M(3h+4y))+\sinh(M(3h+4y)))+k_{26}(\cosh(M(5h+4y))+\sinh
\end{aligned}
$$

$(M(5h+4y))) + L_{71}(\cosh(M(2h+5y)) + \sinh(M(2h+5y))) + L_{72}(\cosh(M(4h+$
$5y)) + \sinh(M(4h+5y))) - L_{75}(\cosh(M(6h+5y)) + \sinh(M(6h+5y))) - L_{76}$
$(\cosh(M(8h+5y)) + \sinh(M(8h+5y))))R_m^2 S_t^2),$

$$\begin{aligned}
k_{35} &= (6 - 16E + M(F^2M(6 + M(h(-4 + hM(1 + 4hM)) - 4y + My^2)) - 2h^4M^3(3 + E)\\
&\quad -y(-4 + My)(-1 + 2E) + 2h^3M^2(-5 + 2(8 + My(-4 + My))E + h^2M(-23 + 86E)\\
&\quad +2F(h^2M^2(-1 + My(-4 + My)) + (8 + My(-4 + My))(-1 + 2E) + h^4M^4(1 + 2E)\\
&\quad +h^3M^3(-5 + 3E) + hM(4 + 15E + My(-4 + My)(1 + 2E))))),\\
k_{36} &= (2Br(F + h)^2M^2ScSr(2M^2(h - y)(h + y) - \cosh(2hM) + \cosh(2My))),\\
k_{37} &= (-162 + 458E + M(F^2M(-123 + hM(257 + 2hM(-36 + hM(-4 + hM)))) + h(301\\
&\quad -1146E + hM(-77 + 888E + 2hM(-4 + hM + (-95 + hM(-7 + 2hM))E))) + F(\\
&\quad -258 + 480E + hM(558 - 1214E + hM(-149 + 2hM(-8 + 2hM(1 + (-3 + hM)E\\
&\quad))))))),\\
k_{38} &= (-3 + M(h - F^2M(3 + M(hM(h - y) + 3y)) + hM(-y + h(-7 - 5My + hM(5 + 4M\\
&\quad y))) - F(2 + M(y + h(13 + M(8y + h(-8 + hM - 5My))))))),\\
k_{39} &= (-108 + M(-108 + F^2M^2(189 + h(156 + M(47 + 4h(66 + (-13 + 27h)M)))) + h\\
&\quad (-276 + M(-227 + h(-336 + M(49 + 4h(-57 + M(46 + h(-15 + 14M))))))) + F\\
&\quad (156 + M(-61 + h(312 + M(346 + M(231 + 4h(78 + M + 27hM))))))))),\\
k_{40} &= (1 - FM + FhM^2 + h^2M^2 + y + FM^2y + hM^2y),\\
k_{41} &= (15k_{14}(\cosh(3My) + (\sinh(3My)) - 240k_{15}(\cosh(3M(h + y)) + \sinh(3M(h + y)))\\
&\quad -120k_{16}(\cosh(5M(h + y)) + \sinh(5M(h + y))) - 5L_{97}(\cosh(M(2h + y)) + \sinh\\
&\quad (M(2h + y))) + 120k_{27}(\cosh(M(3h + y)) + \sinh(M(3h + y))) + 120L_{97}(\cosh(3M\\
&\quad (3h + y)) + \sinh(3M(3h + y))) - 5L_{101}(\cosh(M(4h + y)) + \sinh(M(4h + y))) +\\
&\quad 128k_{28}(\cosh(M(5h + y)) + \sinh(M(5h + y))) - 5L_{106}(\cosh(M(6h + y)) + \sinh(M\\
&\quad (6h + y))) + 120k_{29}(\cosh(M(7h + y)) + \sinh(M(7h + y)))),\\
k_{42} &= ((-1 + FM)(-6 + M(2h - y(4 + My))) + 2(1 + hM)(-8 + M(h + h^2M - y(4 + My)
\end{aligned}$$

$$))E),$$

$$k_{43} = (F^2hM^2(12+7h^2M^2+My(4+My))+2F(10-24E-My(4+My)(-1+3E)+h^4 M^4(2+3E)+h^2M^2(21+5E+My(4+My)(2+E)))+h(44-102E+M(y(4+M y)(5-12E)+h^2M(3+(70+8My(4+My))E)))),$$

$$k_{44} = ((-1+FM)(6+M(2h-h^2M+y(4+My)))-2(-1+hM)(8+M(h-h^2M+y(4 +My)))E),$$

$$k_{45} = (L_{99}(\cosh(M(2h+3y))+\sinh(M(2h+3y)))-2L_{102}(\cosh(M(4h+3y))+\sinh(M (4h+3y)))+240L_{103}(\cosh(M(5h+3y))+\sinh(M(5h+3y)))+240L_{104}(\cosh(M (7h+3y))+\sinh(M(7h+3y)))+L_{74}(\cosh(M(8h+3y))+\sinh(M(8h+3y)))+15 k_{37}(\cosh(M(10h+3y))+\sinh(M(10h+3y)))+960k_{35}(\cosh(M(3h+4y))+\sinh(M(3h+4y)))+960k_{32}M(\cosh(M(5h+4y))+\sinh(M(5h+4y)))+960L_{108}(\cosh (M(7h+4y))+\sinh(M(7h+4y)))),$$

$$k_{46} = (F^2M^2(-24+2hM-h^2M^2+7h^3M^3)+h^3M^3(-19+18E)+h^2M^2(-31+8E)+2F M(14+6E+h^3M^3(-10+3E)+h^2M^2(-8+3E)+h^4M^4(2+3E)+hM(-16+9E))),$$

$$k_{47} = (F^2M^2(-24-10hM-3h^2M^2+h^3M^3)+12(-1+E)-2h^4M^4E-3h^2M^2(-3+2E) -h^3M^3(1+2E)+2FM(2(-9+E)+3hM(-4+E)-3h^3M^3E+h^4M^4E-3h^2M^2 (2+E))),$$

$$k_{48} = (k_{41}+k_{45}-480k_{42}(-1+FM)(1+hM)(\cosh(M(h+2y))+\sinh(M(h+2y)))),$$

$$k_{49} = (-1+5FM+4hM+2h^2M^2-2Fh^2M^2-2M^2y^2+2FM^3y^2),$$

$$k_{50} = (1-FM+FhM^2+h^2M^2-FM^2y-hM^2y),$$

$$k_{51} = (-6+4My-M^2y^2+h^2M^2(1-4E)+16E+2h^3M^3E-8My+2M^2y^2-2hM(1+(7 -4My+M^2y^2)E)),$$

$$k_{52} = (k_{48}-960L_{109}(\cosh(M(3h+2y))+\sinh(M(3h+2y)))-960Mk_{43}(\cosh(M(5h+2y)) +\sinh(M(5h+2y)))-960k_{20}(\cosh(M(7h+2y))+\sinh(M(7h+2y)))-480k_{44}(\cosh$$

$(M(9h+2y))+\sinh(M(9h+2y)))-120k_{18}(\cosh(M(h+3y))+\sinh(M(h+3y)))+$
$k_{30}(\cosh(M(h+4y))+\sinh(M(h+4y)))+k_{51}(\cosh(M(9h+4y))+\sinh(M(9h+4y)))$
$-5L_{107}(\cosh(M(2h+5y))+\sinh(M(2h+5y)))-120k_{46}(\cosh(M(3h+5y))+\sinh(M$
$(3h+5y)))-5L_{100}(\cosh(M(4h+5y))+\sinh(M(4h+5y)))-5L_{90}(\cosh(M(6h+5y))$
$+\sinh(M(6h+5y)))-120k_{47}(\cosh(M(7h+5y))+\sinh(M(7h+5y)))-5L_{89}(\cosh($
$M(8h+5y))+\sinh(M(8h+5y)))),$

$$
\begin{aligned}
k_{53} = {} & (2L_{67}(\cosh(3M(2h+y))+\sinh(3M(2h+y)))+3k_{31}(\cosh(M(4h+y))+\sinh(M(4h\\
& +y)))+3k_{38}(\cosh(M(6h+y))+\sinh(M(6h+y)))-24k_{21}M(\cosh(M(3h+2y))+\\
& \sinh(M(3h+2y)))+24Mk_{22}(\cosh(M(7h+2y))+\sinh(M(7h+2y)))-12(F+h)M\\
& (1+FM)(\cosh(M(9h+2y))+\sinh(M(9h+2y)))+L_{95}(\cosh(M(2h+3y))+\sinh(M\\
& (2h+3y)))-2L_{82}(\cosh(M(4h+3y))+\sinh(M(4h+3y)))+L_{88}(\cosh(M(8h+3y))\\
& +\sinh(M(8h+3y)))-24ML_{94}(\cosh(M(3h+4y))+\sinh(M(3h+4y)))+24ML_{87}\\
& (\cosh(M(7h+4y))+\sinh(M(7h+4y)))+3k_{40}(-1+FM)(1+hM)(\cosh(M(2h+5\\
& y))+\sinh(M(2h+5y)))),
\end{aligned}
$$

$$
\begin{aligned}
k_{54} = {} & (k_{53}+3k_{49}(-1+FM)(1+hM)(\cosh(3My)+\sinh(3My))+3k_{50}(-1+FM)(-1+hM)\\
& (\cosh(M(h+2y))+\sinh(M(h+2y)))-3L_{93}(\cosh(M(4h+5y))+\sinh(M(4h+5y)))\\
& -3L_{92}(\cosh(M(6h+5y))+\sinh(M(6h+5y)))),
\end{aligned}
$$

$$
\begin{aligned}
k_{55} = {} & (-16(F+h)L_{41}(-3+M)M(1+hM)(\cosh(3Mh)+\sinh(3Mh))+32(F+h)(-3+M)M\\
& (1+FhM^2)(\cosh(5Mh)+\sinh(5Mh))+16(F+h)L_{42}(-3+M)(\cosh(7Mh)+\sinh(7Mh))\\
& -k_{39}(\cosh(3My)+\sinh(3My))+240L_{41}(h-y)(\cosh(3M(h+y))+\sinh(3M(h+y)))\\
& -4L_{85}(\cosh(3M(2h+y))+\sinh(3M(2h+y)))-9L_{83}(\cosh(M(6h+y))+\sinh(M(6h\\
& +y)))+144(F+h)L_{41}(1+hM)(\cosh(M(3h+2y))+\sinh(M(3h+2y)))),
\end{aligned}
$$

$$
\begin{aligned}
k_{56} = {} & (L_{97}(\cosh(M(2h+3y))+\sinh(M(2h+3y)))-4L_{80}(\cosh(M(4h+3y))+\sinh(M(4h\\
& +3y)))-2304M^2L_{42}(F+h)(-1+hM)(h-y)(\cosh(M(7h+3y))+\sinh(M(7h+3y)))\\
& +L_{77}(\cosh(M(8h+3y))+\sinh(M(8h+3y)))),
\end{aligned}
$$

$$
\begin{aligned}
k_{57} &= (144ML_{41}(F+h)(9+M)(1+hM)(\cosh(M(3h+4y))+\sinh(M(3h+4y)))-288M(F \\
&\quad +h)(9+M)(1+FhM^2)(\cosh(M(5h+4y))+\sinh(M(5h+4y)))-144ML_{42}(9+M) \\
&\quad (-1+hM)(\cosh(M(2h+5y))+\sinh(M(2h+5y)))), \\
k_{58} &= (144ML_{42}(F+h)(9+M)(-1+hM)(\cosh(M(7h+2y))+\sinh(M(7h+2y)))+L_{91} \\
&\quad (\cosh(M(10h+3y))+\sinh(M(10h+3y)))-9L_{44}(\cosh(M(4h+5y))+\sinh(M(4h \\
&\quad +5y)))), \\
k_{59} &= (9L_{78}(\cosh(M(6h+5y))+\sinh(M(6h+5y)))+9L_{45}(\cosh(M(8h+5y))+\sinh(M \\
&\quad (8h+5y)))), \\
k_{60} &= (32(F+h)M(3+M)(1+FhM^2)(\cosh(M(5h+6y))+\sinh(M(5h+6y)))+16ML_{42} \\
&\quad (3+M)(-1+hM)(\cosh(M(7h+6y))+\sinh(M(7h+6y)))).
\end{aligned}
$$

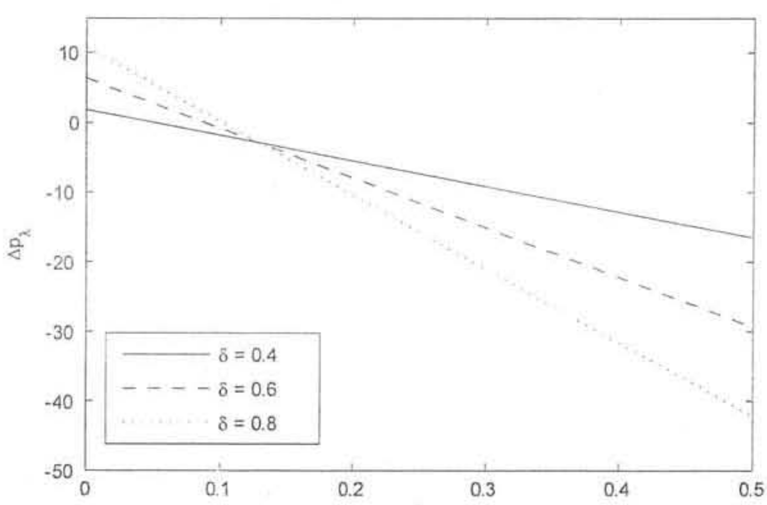

Figure 8.1: Plot showing Δp_λ versus flow rate θ when $\phi = 0.4$, $M = 0.5$, $E = 0.8$, $R_m = 1$, $R_e = 1$, $S_t = 1$, $\lambda_1 = 0.8$ and $\lambda_2 = 0.8$.

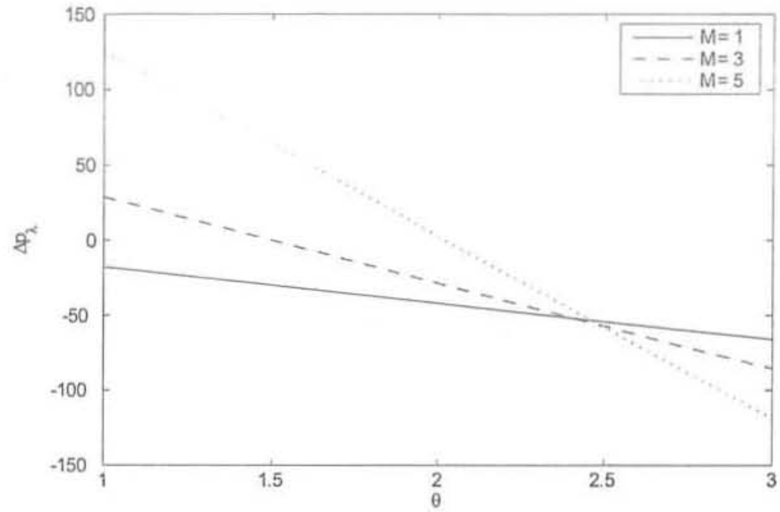

Figure 8.2: Plot showing Δp_λ versus flow rate θ when $\phi = 0.4$, $\delta = 0.6$, $E = 0.8$, $R_m = 1$, $R_e = 1$, $S_t = 1$, $\lambda_1 = 0.8$ and $\lambda_2 = 0.8$.

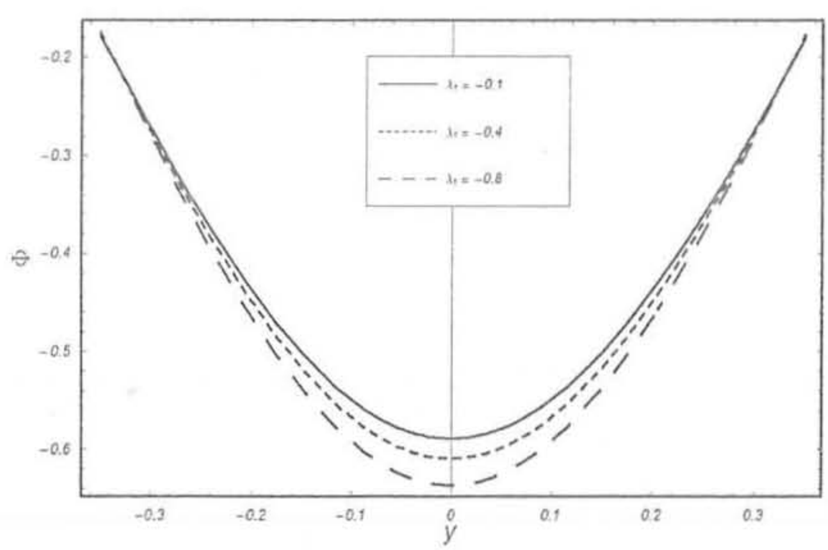

Figure 8.3a: Magnetic force function versus y when $\theta = 2$, $R_m = 1$, $R_e = 1$, $M = 1$, $E = 0.8$, $\phi = 0.6$, $\delta = 0.08$ and $x = \frac{\pi}{2}$.

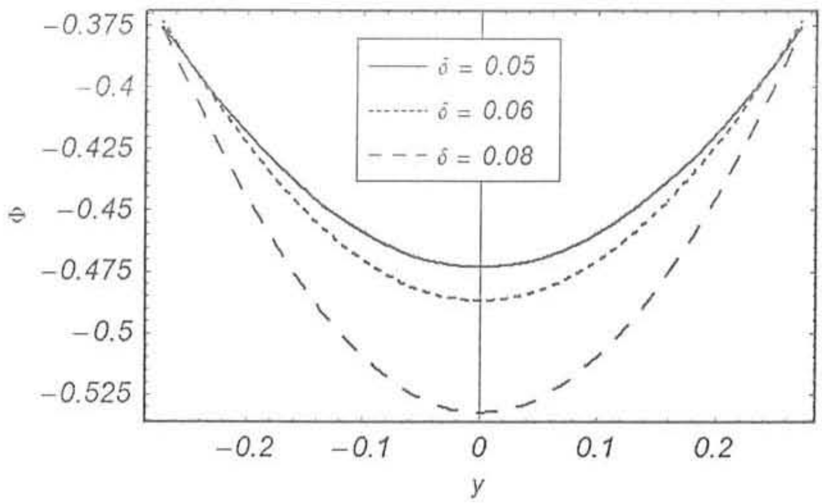

Figure 8.3b: Magnetic force function versus y when $\theta = 2$, $R_m = 1$, $R_e = 1$, $M = 1$, $E = 0.8$, $\phi = 0.6$, $\lambda_1 = -0.8$ and $x = \frac{\pi}{2}$.

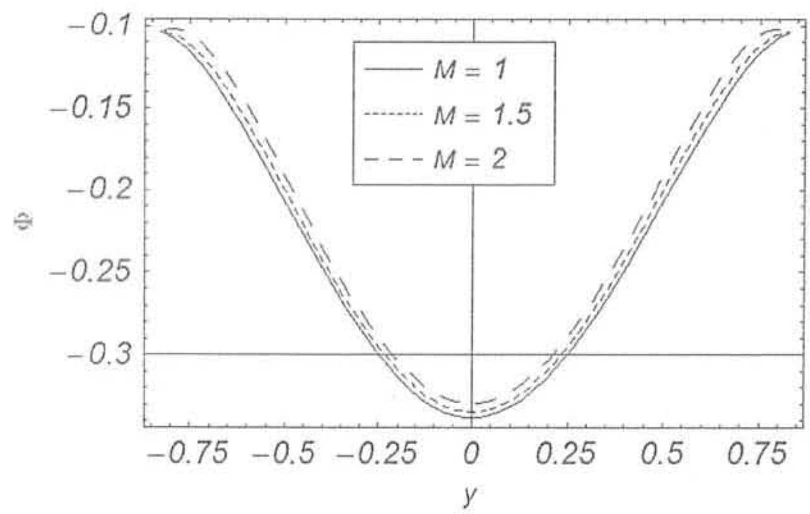

Figure 8.3c: Magnetic force function versus y when $\theta = 2$, $R_m = 1$, $R_e = 1$, $E = 0.8$, $\phi = 0.6$, $\delta = 0.08$, $\lambda_1 = -0.8$ and $x = \frac{\pi}{2}$.

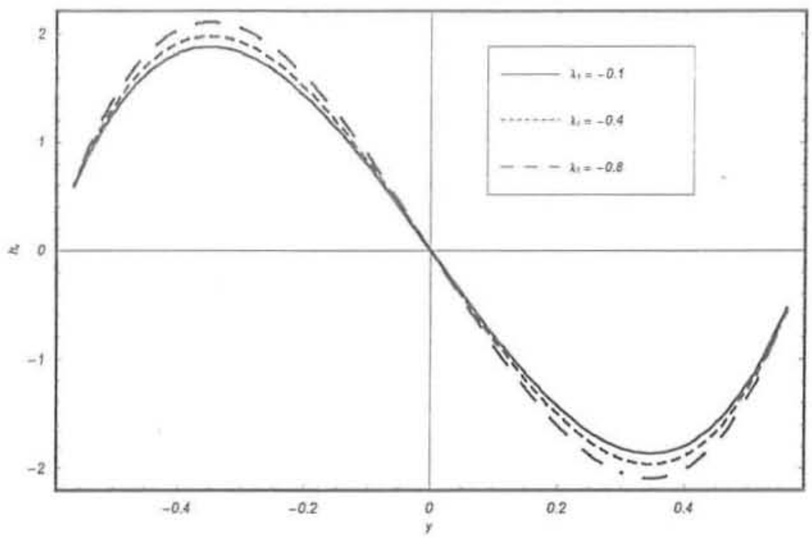

Figure 8.4a: Axial induced magnetic field h_x versus y when $\theta = 2$, $R_m = 1$, $R_e = 1$, $M = 1$, $E = 0.8$, $\phi = 0.6$, $\delta = 0.08$ and $x = \frac{\pi}{2}$.

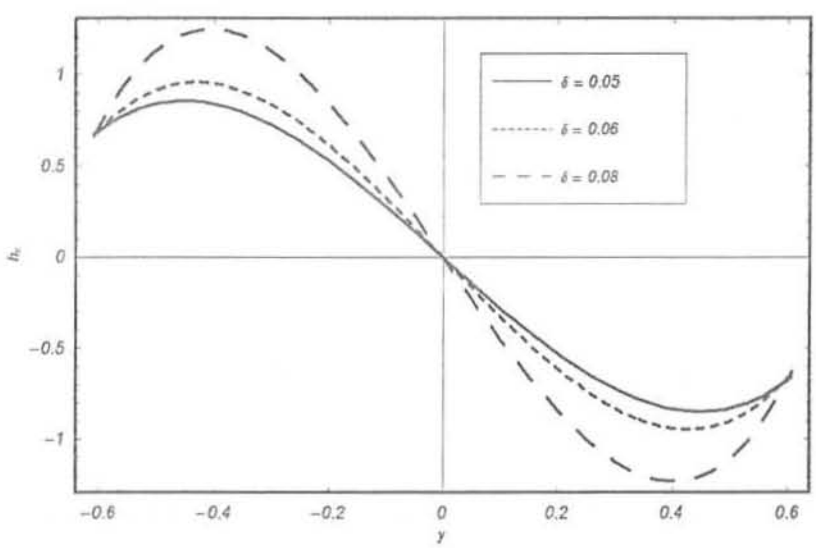

Figure 8.4b: Axial induced magnetic field h_x versus y when $\theta = 2$, $R_m = 1$, $R_e = 1$, $M = 1$, $E = 0.8$, $\phi = 0.6$, $\lambda_1 = -0.8$ and $x = \frac{\pi}{2}$.

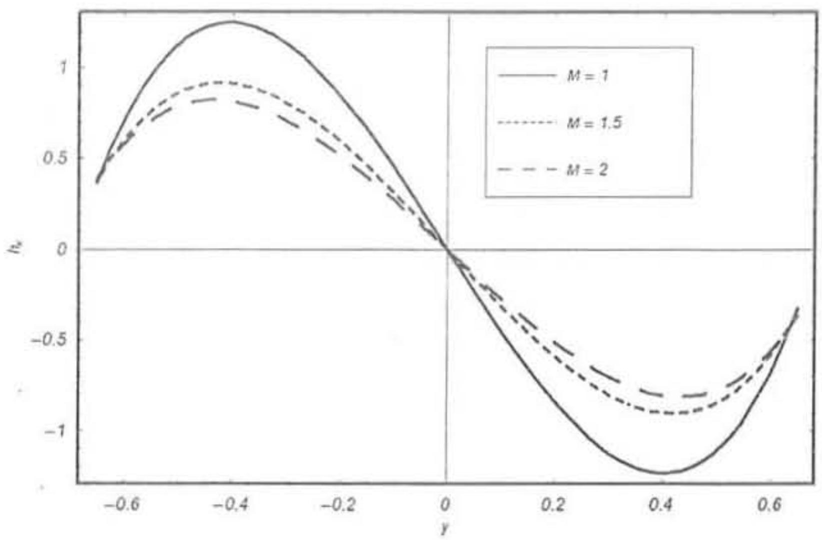

Figure 8.4c: Axial induced magnetic field h_x versus y when $\theta = 2$, $R_m = 1$, $R_e = 1$, $\delta = 0.08$, $E = 0.8$, $\phi = 0.6$, $\lambda_1 = -0.8$ and $x = \frac{\pi}{2}$.

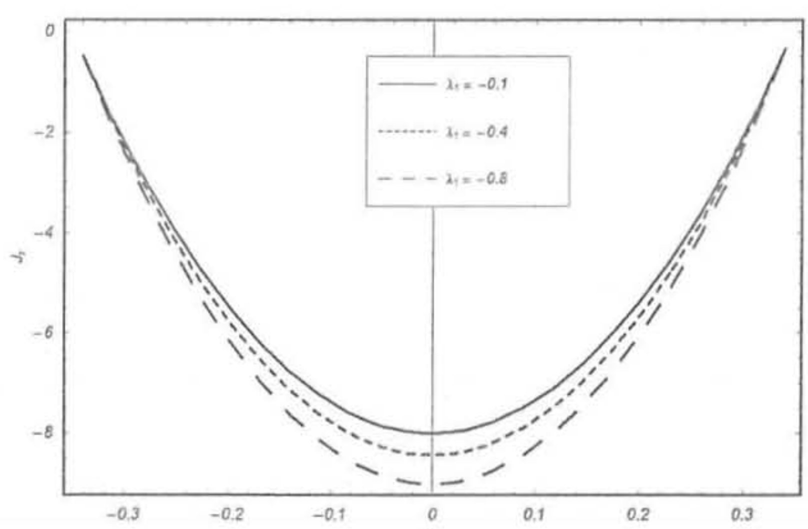

Figure 8.5a: Current density distribution versus y when $\theta = 2$, $R_m = 1$, $R_e = 1$, $M = 1$, $E = 0.8$, $\phi = 0.6$, $\delta = 0.08$ and $x = \frac{\pi}{2}$.

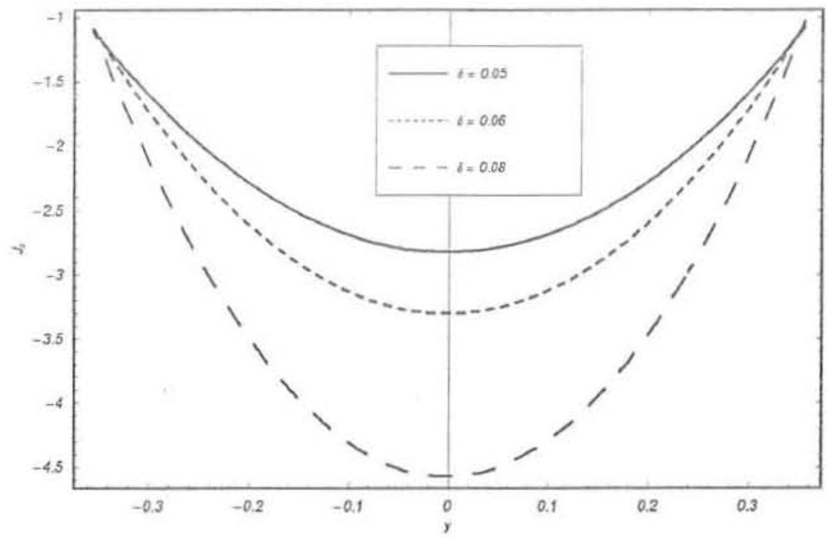

Figure 8.5b: Current density distribution versus y when $\theta = 2$, $R_m = 1$, $R_e = 1$, $M = 1$, $E = 0.8$, $\phi = 0.6$, $\lambda_1 = -0.8$ and $x = \frac{\pi}{2}$.

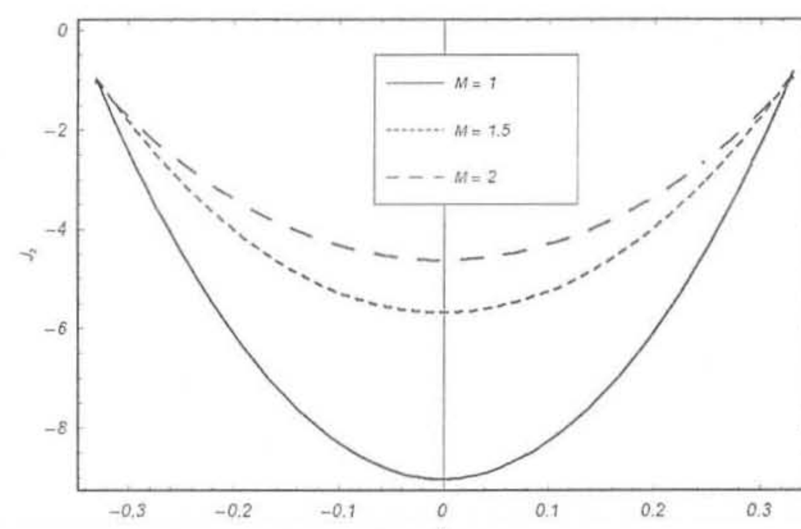

Figure 8.5c: Current density distribution versus y when $\theta = 2$, $R_m = 1$, $R_e = 1$, $\delta = 0.08$, $E = 0.8$, $\phi = 0.6$, $\lambda_1 = -0.8$ and $x = \frac{\pi}{2}$.

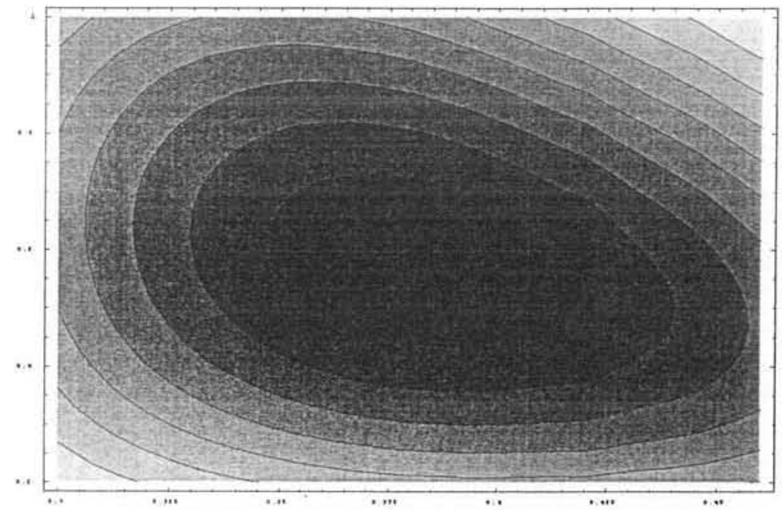

Figure 8.6a: Streamlines for $M = 0.2$ when $\phi = 0.6$, $R_e = 1$, $R_m = 0.5$, $\delta = 0.09$, $\theta = 1$, $E = 0.8$ and $\lambda_1 = 0.9$.

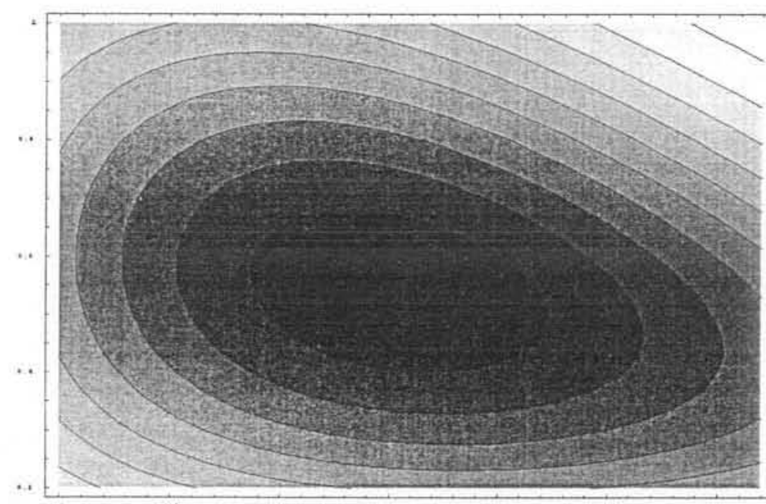

Figure 8.6b: Streamlines for $M = 0.8$ when $\phi = 0.6$, $R_e = 1$, $R_m = 0.5$, $\delta = 0.09$, $\theta = 1$, $E = 0.8$ and $\lambda_1 = 0.9$.

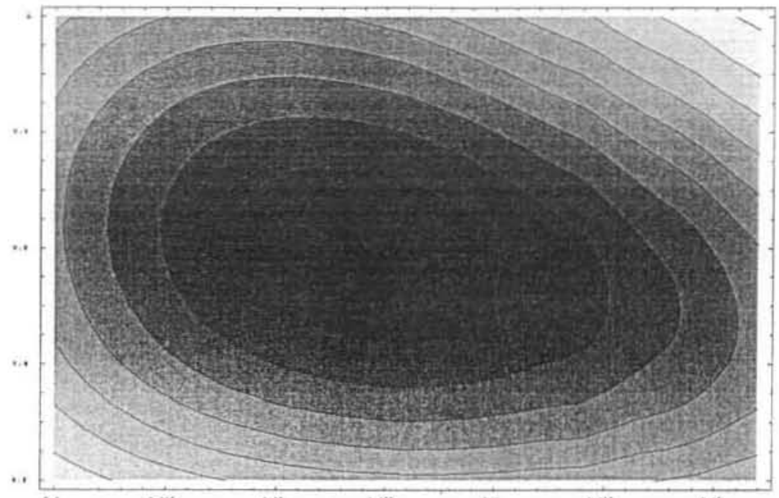

Figure 8.7a: Streamlines for $\theta = 0.8$ when $\phi = 0.6$, $R_e = 1$, $R_m = 0.5$, $\delta = 0.09$, $M = 0.2$, $E = 0.8$ and $\lambda_1 = 0.9$.

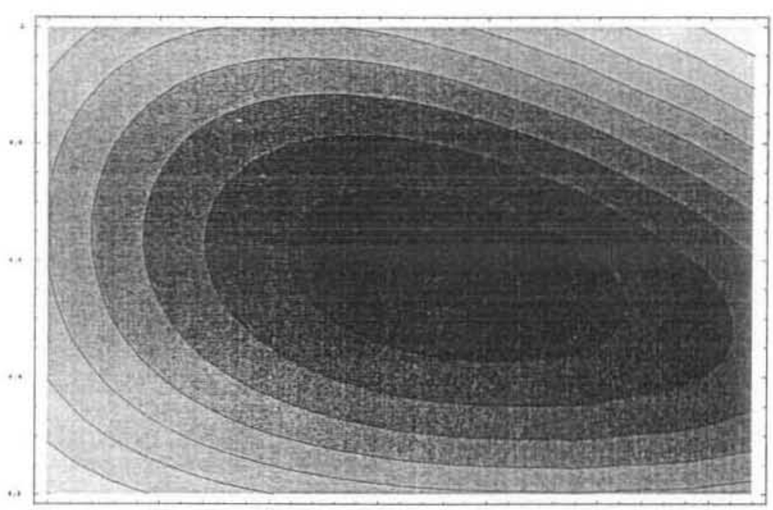

Figure 8.7b: Streamlines for $\theta = 1.2$ when $\phi = 0.6$, $R_e = 1$, $R_m = 0.5$, $\delta = 0.09$, $M = 0.2$, $E = 0.8$ and $\lambda_1 = 0.9$.

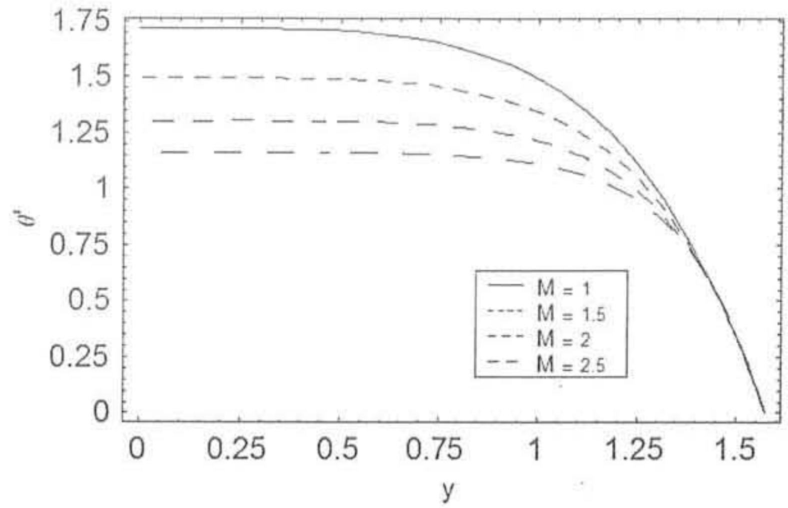

Figure 8.8a: Variation of M on θ' when $Br = 1$, $Pr = 1$, $\phi = 0.6$, $R_e = 1$, $R_m = 1$, $\delta = 0.01$, $\theta = 1$, $E = 0.4$ and $\lambda_1 = 0.09$.

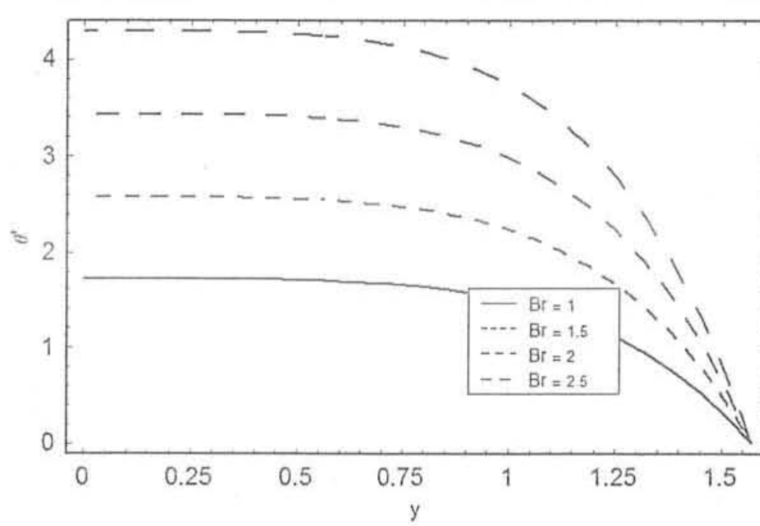

Figure 8.8b: Variation of Br on θ' when $M = 1$, $Pr = 1$, $\phi = 0.6$, $R_e = 1$, $R_m = 1$, $\delta = 0.01$, $\theta = 1$, $E = 0.4$ and $\lambda_1 = 0.09$.

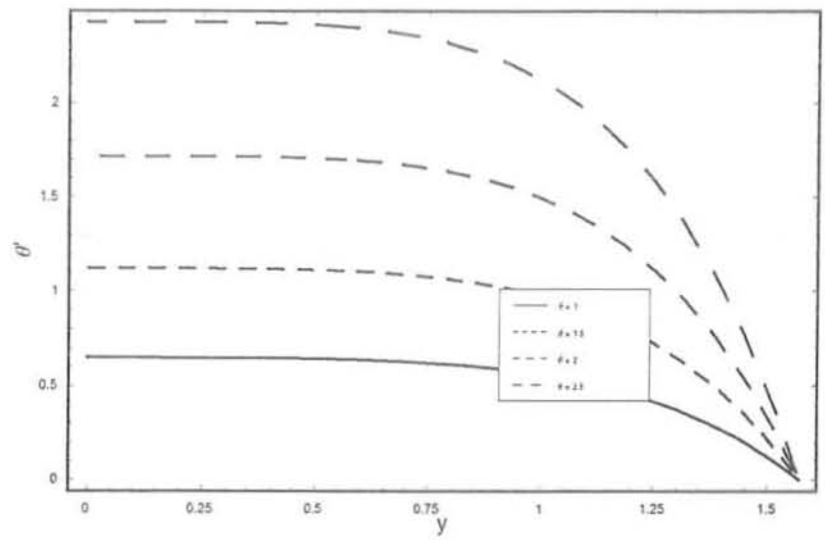

Figure 8.8c: Variation of θ on θ' when $Br = 1$, $Pr = 1$, $\phi = 0.6$, $R_e = 1$, $R_m = 1$, $\delta = 0.01$, $\theta = 1$, $E = 0.4$ and $\lambda_1 = 0.09$.

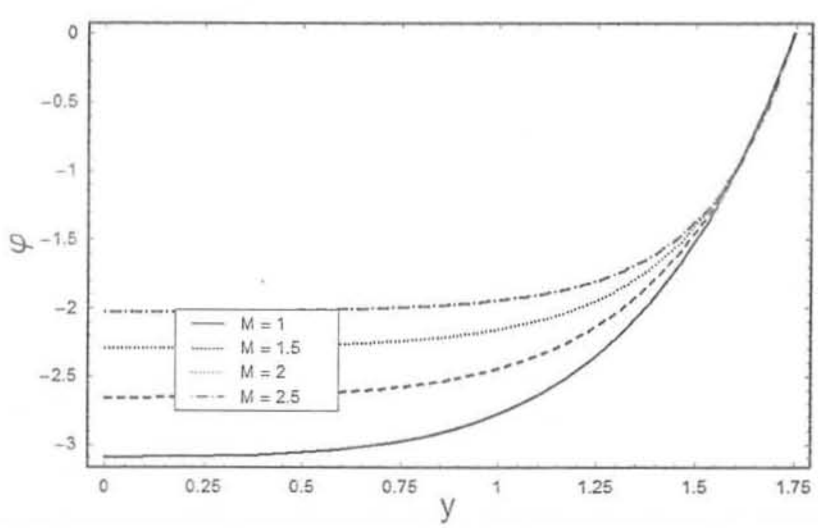

Figure 8.9a: Variation of M on φ when $Br = 1$, $Pr = 1$, $Sc = 1$, $Sr = 1$, $\phi = 0.6$, $R_e = 1$, $R_m = 1$, $\delta = 0.01$, $\theta = 3$, $E = 0.4$ and $\lambda_1 = 0.09$.

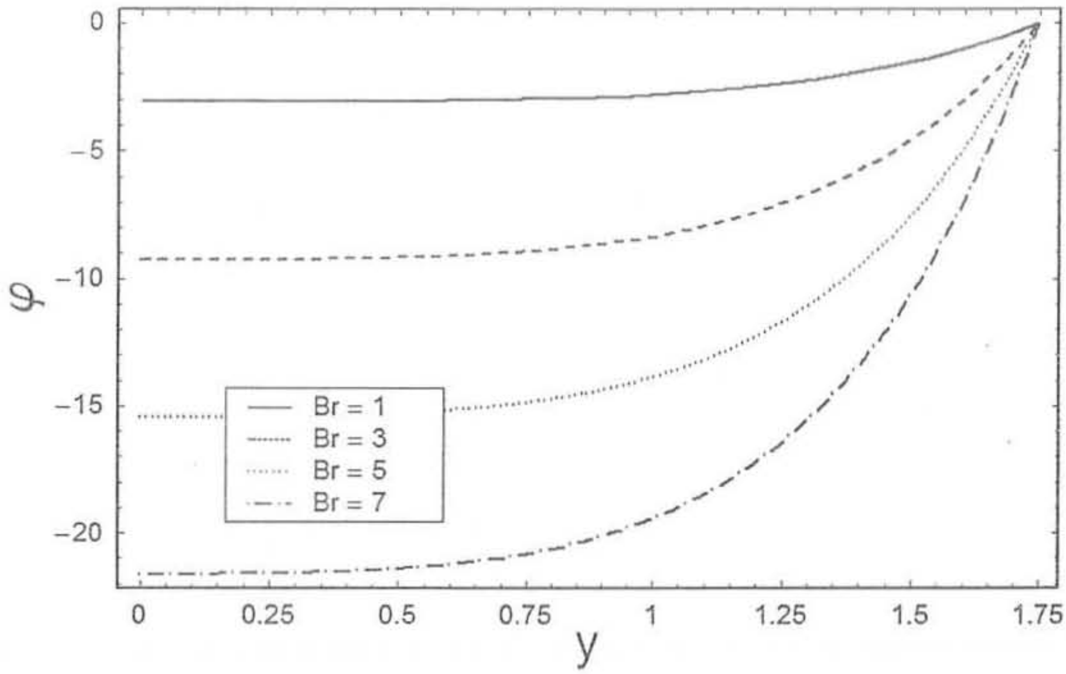

Figure 8.9b: Variation of Br on φ when $M = 1$, $Pr = 1$, $Sc = 1$, $Sr = 1$, $\phi = 0.6$, $R_e = 1$, $R_m = 1$, $\delta = 0.01$, $\theta = 3$, $E = 0.4$ and $\lambda_1 = 0.09$.

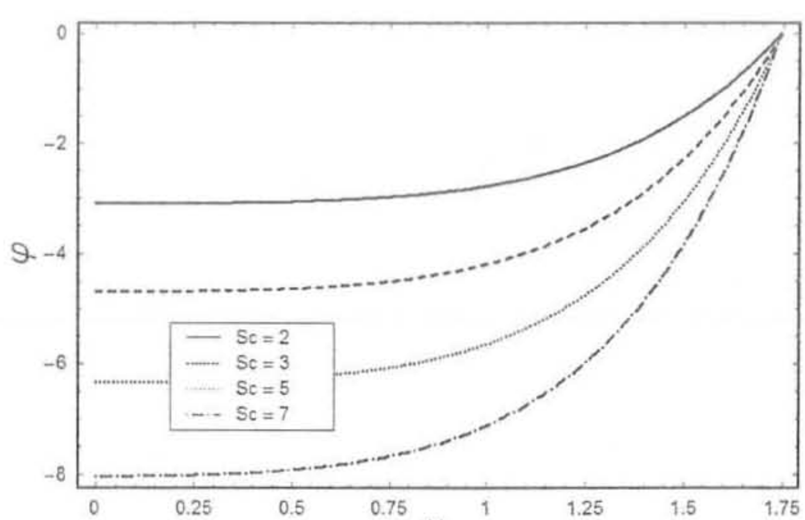

Figure 8.9c: Variation of Sc on φ when $M = 1$, $Pr = 1$, $Br = 1$, $Sr = 1$, $\phi = 0.6$, $R_e = 1$, $R_m = 1$, $\delta = 0.01$, $\theta = 3$, $E = 0.4$ and $\lambda_1 = 0.09$.

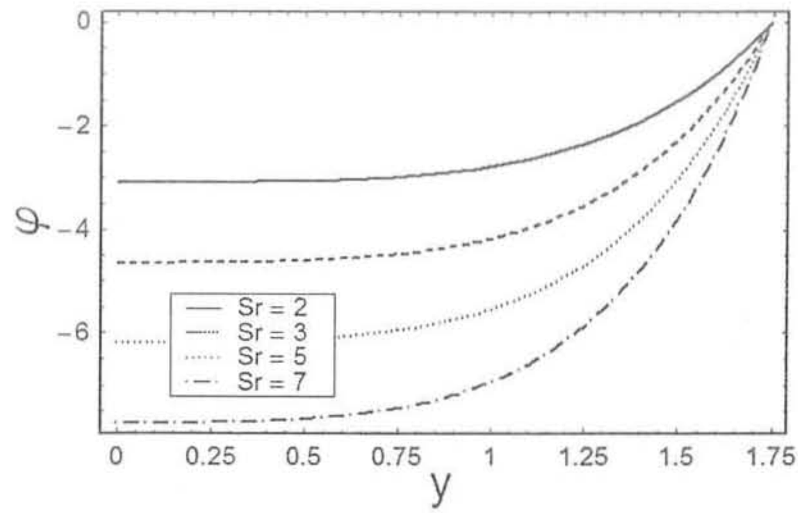

Figure 8.9d: Variation of Sr on φ when $M = 1$, $Pr = 1$, $Br = 1$, $Sc = 1$, $\phi = 0.6$, $R_e = 1$, $R_m = 1$, $\delta = 0.01$, $\theta = 3$, $E = 0.4$ and $\lambda_1 = 0.09$.

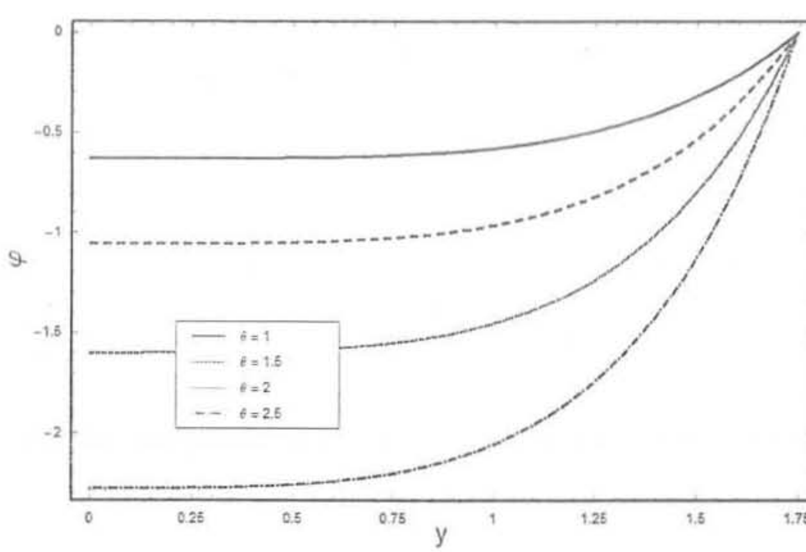

Figure 8.9e: Variation of Br on φ when $M = 1$, $Pr = 1$, $Sc = 1$, $Sr = 1$, $\phi = 0.6$, $R_e = 1$, $R_m = 1$, $\delta = 0.01$, $\theta = 3$, $E = 0.4$ and $\lambda_1 = 0.09$.

8.3 Results and discussion

In order to analyze the variations of the pertinent parameters of interest on the various flow quantities such as the pressure rise per wavelength (Δp_λ), the stream function (Ψ), the magnetic force function (Φ), the axial induced magnetic field (h_x), the current density distribution (J_z), the temperature distribution (θ') and the concentration distribution (φ), we have prepared Figures 8.1-8.9.

The expression for the pressure rise per wavelength is computed numerically and presented graphically. Figures 8.1 and 8.2 illustrate the variations of pressure rise per wavelength versus y for different values of wave number (δ) and Hartman number (M). In Figure 8.1, we have noticed that Δp_λ increases with an increase in δ in the peristaltic pumping and free pumping regions. However in the copumping region, an opposite behaviour is observed. Here Δp_λ decreases when δ is increased. Moreover the volume flow rate is maximum for small values of δ. In Figure 8.2, it is found that Δp_λ increases for large values of M in the pumping and free pumping regions. For the appropriate values of volume flow rate (θ), Δp_λ decreases with an increase of M in the copumping region. The volume flow rate (θ) is maximum for small values of M.

Figures 8.3-8.6 discuss the variations of magnetic force function Φ, axial induced magnetic field h_x and current density distribution J_z versus y for different values of λ_1, δ and M. Figures 8.3(a-c) reveal that the magnetic force function Φ increases as λ_1 and M increase but decreases when large values of δ are taken into account. In Figures 8.4(a-c), it is shown that the axial induced magnetic field h_x is symmetric about the channel and decreases with an increase in λ_1 and M. However it decreases when δ is decreased. Moreover, in the half region $(y > 0)$, h_x has one direction and in the other half region $(y < 0)$, it is in the opposite direction. It is equal to zero for $y = 0$. The Figures 8.5(a-c) are plotted to describe that the current density distribution J_z increases with an increase of λ_1 and M. Moreover it is noticed that J_z increases with a decrease of δ.

In peristalsis, trapping is one of the more important and interesting phenomenon. Figures 8.6 and 8.7 explain this phenomenon for different values of M and θ. Here we have seen that the size of the trapped bolus is going to be squeezed when M is increased. The volume flow rate (θ) has an increasing effect on trapping. Here the trapped bolus increases in size with an increase of θ.

Figures 8.8$(a-c)$ depict the variations of the temperature distribution (θ') for different values of M, Br and θ. Figure 8a indicates that the temperature is decreasing function of M. Figures 8.8 $(b$ & $c)$ present an opposite picture when compared to Figure 8.8a. Here Br and θ show increasing effect on the temperature profile. The temperature increases by increasing Br and θ. In these figures, we have observed that the temperature profile parabolic in nature.

Figures 8.9$(a-e)$ describe the variations of the concentration distribution (φ) for different values of M, Br, Sc, Sr and θ. Figure 8.9a shows that the concentration distribution increases with an increase in M. However in Figures 9 $(b-e)$, the situation is quite opposite. In these figures, the concentration distribution increases for the small values of Br, Sc, Sr and θ.

Chapter 9

A mathematical model for the study of slip effects on the peristaltic motion of micropolar fluid with heat and mass transfer

In this chapter, we present a mathematical model with an interest to examine the peristaltic motion in an asymmetric channel by taking into account the partial slip and heat and mass transfer effects. Constitutive relationships for a micropolar fluid are used. The solution procedure for nonlinear analysis is given under long wavelength and low Reynolds number approximations. The effects of sundry parameters entering into the expressions of axial velocity, temperature and concentration profiles are explored. Pumping and trapping phenomena are discussed.

9.1 Development of the mathematical problem

We examine the peristaltic flow of an incompressible micropolar fluid in an asymmetric channel having width $d_1 + d_2$. The sinusoidal waves traveling down its walls are given by the following expressions:

$$h_1^{'}(X,Y,t) = d_1 + a_1 \cos\left[\frac{2\pi}{\lambda}(X - c\acute{t})\right]; \quad \text{upper wall}, \tag{9.1}$$

$$h_2^{'}(X,Y,t) = -d_2 - b_1 \cos\left[\frac{2\pi}{\lambda}(X - c\acute{t}) + \phi\right]; \quad \text{lower wall}. \tag{9.2}$$

Here a_1 and b_1 are the wave amplitudes, λ is the wavelength, $\acute{t}$ is the time, $\phi\,(0 \leq \phi \leq \pi)$ is the phase difference. It is noticed that the case $\phi = 0$ corresponds to symmetric channel with waves out of phase and for $\phi = \pi$, it correponds to the situation when waves are in phase. Also a_1, b_1, d_1 and d_2 satisfy the following condition

$$a_1^2 + b_1^2 + 2a_1 b_1 \cos\phi \leq (d_1 + d_2)^2. \tag{9.3}$$

We also assume that the walls have only transverse motion. The flow in laboratory (X', Y') and wave (x', y') frames are treated unsteady and steady, respectively. The transformations between the two frames are given below:

$$x' = X' - ct', \quad y' = Y',$$
$$u'(x', y') = U' - c, \quad v'(x', y') = V', \tag{9.4}$$

in which (U', V') and (u', v') are the respective velocity components in the laboratory and wave frames.

Neglecting body force and body couple, the equations governing the present flow are

$$\nabla \cdot \overline{\mathbf{V'}} = 0, \tag{9.5}$$

$$\rho\left(\overline{\mathbf{V'}} \cdot \nabla\right)\overline{\mathbf{V'}} = -\nabla p' + k\nabla \times \overline{\mathbf{w'}} + (\mu + k)\nabla^2 \overline{\mathbf{V'}}, \tag{9.6}$$

$$\rho j'\left(\overline{\mathbf{V'}} \cdot \nabla\right)\overline{\mathbf{w'}} = -2k\overline{\mathbf{w'}} + k\nabla \times \mathbf{V}' - \gamma\left(\nabla \times \nabla \times \overline{\mathbf{w'}}\right) + (\alpha + \beta + \gamma)\nabla\left(\nabla \cdot \overline{\mathbf{w'}}\right), \tag{9.7}$$

$$\rho\varsigma\frac{dT}{dt} = \kappa\nabla^2 T + \tau.\mathrm{L}, \tag{9.8}$$

$$\frac{dC}{dt} = D\nabla^2 C + \frac{DK_T}{T_m}\nabla^2 T, \tag{9.9}$$

where $\overline{\mathrm{V}'}$ is velocity vector, $\overline{\mathrm{w}'}$ is the microrotation vector, p' is the pressure, ρ is the density, j' is the microgyration parameter, ς is the specific heat, ν is the kinematic viscosity, κ is thermal conductivity, τ is the Cauchy stress tensor, D is the coefficient of mass diffusivity, T_m is the mean temperature, K_T is the thermal-diffusion ratio, T is the temperature and C is the concentration of fluid. Further, the material constants satisfy the following inequalities

$$2\mu + k \geq 0, \quad k \geq 0, \quad 3\alpha + \beta + \gamma \geq 0, \quad \gamma \geq |\beta|. \tag{9.10}$$

The velocity field and microrotation vector are defined as

$$\overline{\mathrm{V}'} = (u', v', 0), \quad \overline{\mathrm{w}'} = (0, 0, w'). \tag{9.11}$$

We introduce the non-dimensional variables given below

$$\begin{aligned} &x = \frac{x'}{\lambda}, \; y = \frac{y'}{d_1}, \; u = \frac{u'}{c}, \; v = \frac{\lambda v'}{d_1 c}, \; t = \frac{t' c}{\lambda}, \; p = \frac{d_1^2 p'}{c\lambda\mu}, \; \mathrm{Re} = \frac{\rho c a}{\mu}, \; \tau = \frac{d_1 \tau'}{\mu c}, \\ &h_1 = \frac{h_1'}{d_1}, \; h_2 = \frac{h_2'}{d_1}, \; \Psi = \frac{\Psi'}{c d_1}, \; \delta = \frac{d_1}{\lambda}, \; d = \frac{d_2}{d_1}, \; a = \frac{a_1}{d_1}, \; b = \frac{a_2}{d_1}, \; \theta' = \frac{T - T_0}{T_1 - T_0}, \\ &\varphi = \frac{C - C_0}{C_0}, \quad \mathrm{Pr} = \frac{\rho\nu\varsigma}{\kappa}, \quad E = \frac{c^2}{\varsigma(T_1 - T_0)}, \quad Sc = \frac{\mu}{\rho D}, \quad Sr = \frac{\rho D K_T T_0}{\mu T_m C_0}, \end{aligned} \tag{9.12}$$

$$u = \frac{\partial\Psi}{\partial y}, \quad v = -\frac{\partial\Psi}{\partial x}, \quad \nabla^2 = \delta^2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{9.13}$$

in which δ is the wave number, Re the Reynolds number, Pr the Prandtl number, E the Eckert number, Br the Brinkman number, Sc the Schmidt number and Sr the Soret number.

By using Eqs. (9.12) and (9.13) into Eqs. (9.5)-(9.9), we get

$$\mathrm{Re}\,\delta\left\{\left(\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial\Psi}{\partial y}\right\} = -\frac{\partial p}{\partial x} + \frac{1}{1-N}\left\{N\frac{\partial w}{\partial y} + \delta^2\frac{\partial^3\Psi}{\partial x^2\partial y} + \frac{\partial^3\Psi}{\partial y^3}\right\} \tag{9.14}$$

$$-\mathrm{Re}\,\delta^3\left\{\left(\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial\Psi}{\partial x}\right\} = -\frac{\partial p}{\partial y} + \frac{\delta^2}{1-N}\left\{N\frac{\partial w}{\partial x} - \delta^2\frac{\partial^3\Psi}{\partial x^3} + \frac{\partial^3\Psi}{\partial y^3}\right\} \tag{9.15}$$

$$\mathrm{Re}\,\delta\,\mathrm{Pr}\left(\frac{\partial\Psi}{\partial y}\frac{\partial\theta'}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial\theta'}{\partial y}\right)=\left(\delta^2\frac{\partial^2\theta'}{\partial x^2}+\frac{\partial^2\theta'}{\partial y^2}\right)+Br\left\{\begin{array}{c}\delta\left(\tau_{xx}-\tau_{yy}\right)\frac{\partial^2\Psi}{\partial x\partial y}+\\ \tau_{xy}\left(\frac{\partial^2\Psi}{\partial y^2}-\delta^2\frac{\partial^2\Psi}{\partial x^2}\right)\end{array}\right\} \tag{9.16}$$

$$\mathrm{Re}\,\delta\left(\frac{\partial\Psi}{\partial y}\frac{\partial\varphi'}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial\varphi'}{\partial y}\right)=\frac{1}{Sc}\left(\delta^2\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\right)+Sr\left(\delta^2\frac{\partial^2\theta'}{\partial x^2}+\frac{\partial^2\theta'}{\partial y^2}\right). \tag{9.17}$$

Adopting long wavelength and low Reynolds number procedure one obtains

$$N\frac{\partial w}{\partial y}+\frac{\partial^3\Psi}{\partial y^3}=(1-N)\frac{\partial p}{\partial x}, \tag{9.18}$$

$$\frac{\partial p}{\partial y}=0, \tag{9.19}$$

$$-2w-\frac{\partial^2\Psi}{\partial y^2}+(\frac{2-N}{M^2})\frac{\partial^2 w}{\partial y^2}=0, \tag{9.20}$$

$$\frac{\partial^2\theta'}{\partial y^2}+Br\left\{\left(\frac{1}{1-N}\frac{\partial^2\Psi}{\partial y^2}-\frac{N}{1-N}w\right)\frac{\partial^2\Psi}{\partial y^2}\right\}=0, \tag{9.21}$$

$$\frac{1}{Sc}\frac{\partial^2\varphi}{\partial y^2}+Sr\frac{\partial^2\theta'}{\partial y^2}=0, \tag{9.22}$$

where Eq. (9.19) shows that $p\neq p(y)$ and hence $p=p(x)$.

From Eqs. (9.18) and (9.19), one can write

$$N\frac{\partial^2 w}{\partial y^2}+\frac{\partial^4\Psi}{\partial y^4}=0, \tag{9.23}$$

where $N=k/(\mu+k)$ is the coupling number $(0\leq N\leq 1)$, $M^2=d_1^2k(2\mu+k)/\left(\gamma(\mu+k)\right)$ is the micropolar parameter and α, β do not arise in the governing equation as the microrotation vector is solenoidal. When $k\to 0$, these equations reduce to the Navier-Stokes equations.

The dimensionless boundary conditions are

$$\Psi=\frac{F}{2},\quad \frac{\partial\Psi}{\partial y}=-\beta\left(\frac{1}{1-N}\frac{\partial^2\Psi}{\partial y^2}+\frac{N}{1-N}w\right)-1,\quad \theta'=0,\quad \varphi=0\quad \text{at } y=h_1, \tag{9.24}$$

$$\Psi=-\frac{F}{2},\quad \frac{\partial\Psi}{\partial y}=\beta\left(\frac{1}{1-N}\frac{\partial^2\Psi}{\partial y^2}+\frac{N}{1-N}w\right)-1,\quad \theta'=1,\quad \varphi=0\quad \text{at } y=h_2, \tag{9.25}$$

$$
\begin{aligned}
h_1 &= 1 + a\cos(2\pi x), \\
h_2 &= -d - b\cos(2\pi x + \phi),
\end{aligned} \tag{9.26}
$$

where $\beta\ (= \frac{L}{d_1})$ is the dimensionless slip parameter. L is dimensional slip parameter and h_1 and h_2 are the dimensional form of the peristaltic walls.

The dimensionless pressure rise per wavelength is

$$
\Delta p_\lambda = \int_0^1 \left(\frac{dp}{dx}\right) dx. \tag{9.27}
$$

The dimensionless mean flows in the laboratory (θ) and wave (F) frames are related by the following expressions

$$
\theta = F + d + 1, \tag{9.28}
$$

$$
F = \int_{h_2(x)}^{h_1(x)} \frac{\partial \Psi}{\partial y} dy = \Psi(h_1(x)) - \Psi(h_2(x)). \tag{9.29}
$$

9.2 Exact solution

With the help of Eqs. (9.18) and (9.20)-(9.25), we have the following expressions

$$
\begin{aligned}
u &= -\frac{1}{3M^2(N-2)}\{(-3(C_1+2C_6)N\cosh\left(\frac{M(N-2)y}{\sqrt{(N-2)^2}}\right) + \frac{1}{\sqrt{(N-2)^2}}(3(\sqrt{(N-2)^2} \\
&\quad (M^2(2C_4+2C_5y+C_1y^2)+N(C_1+2C_6+M^2(-C_4+2C_2y+C_6y^2))) - (2C_2+C_5) \\
&\quad M(-2+N)N\sinh\left(\frac{M(N-2)y}{\sqrt{(N-2)^2}}\right)))\},
\end{aligned} \tag{9.30}
$$

$$
\begin{aligned}
w &= -\frac{1}{(N-2)}\{-NC_2+C_5-y(C_1+NC_6)+(2C_2-C_5)\cosh\left(\frac{M(N-2)y}{\sqrt{(N-2)^2}}\right) + \frac{1}{M\sqrt{(N-2)^2}} \\
&\quad ((N-2)(C_1+2C_6)\sinh\left(\frac{M(N-2)y}{\sqrt{(N-2)^2}}\right))\},
\end{aligned} \tag{9.31}
$$

$$
\left.\frac{dp}{dx}\right|_{y=0} = \frac{C_1+NC_6}{1-N}, \tag{9.32}
$$

$$\Psi = \frac{1}{3M^2(N-2)}\{3N(2C_2+M^2y^2C_2-M^2C_3+y(C_1-M^2C_4)+C_5)+M^2(y^3C_1+6C_3 +6yC_4+3y^2C_5)+Ny(6+M^2y^2)C_6-3N(2C_2+C_5)\cosh\left(\frac{M(N-2)y}{\sqrt{(N-2)^2}}\right)- \frac{1}{M\sqrt{(N-2)^2}}(3(N-2)N(C_1+2C_2)\sinh\left(\frac{M(N-2)y}{\sqrt{(N-2)^2}}\right))\}, \tag{9.33}$$

$$\theta' = \frac{1}{L_{59}}((4-N^2)^{3/2}(-\frac{1}{(4-N^2)^{3/2}}L_{60}+Br(\frac{1}{(4-N^2)^{3/2}}L_{61}+48M(N-6)N(C_6N-C_1) +L_{62}L_{58}))+h_1(-48BrM(N-6)N(-C_1+6C_6)\sinh(\frac{L_{57}}{2}(y-h_2))+((-C_1+2C_6) (4-N^2)^{1/2}\sinh(\frac{L_{57}}{2}(y-h_2)))-\frac{1}{(4-N^2)^{3/2}}L_{63}+3BrNL_{57}L_{64})), \tag{9.34}$$

$$\varphi = \frac{1}{2L_{85}}(Br(4-N^2)^{3/2}ScSr(\cosh(2(y+h_1+h_2)L_{57})-\sinh(2(y+h_1+h_2)L_{57}))(L_{67}+h_1 (24M(6-N)N(C_6N-C_1)(\cosh(yL_{57})-\cosh(h_2L_{57})+\sinh(yL_{57})-\sinh(h_2L_{57})((2C_6 +C_1)(4-N^2)^{1/2}(-1+\cosh((y+h_2)\,L_{57})+\sinh((y+h_2)\,L_{57})+2C_2M(2+N)(1+\cosh ((y+h_2)\,L_{57})+\sinh((y+h_2)\,L_{57}))+C_5M(2+N)(1+\cosh((y+h_2)\,L_{57})+\sinh((y+h_2) L_{57}))))(\cosh((y+2h_1+h_2)\,L_{57})+\sinh((y+2h_1+h_2)\,L_{57})-\frac{1}{(N-2)(4-N^2)^{1/2}}(4M^3(2 +N)^2(\cosh((y+2h_1+h_2)\,L_{57})+\sinh((y+2h_1+h_2)\,L_{57})(y^2(-3C_5^2M^2(8+N^3)-2C_5M^2 (2+N)(3C_2N(3N-4)+2C_6(N-2)Ny+C_1(3N^2+4y-2Ny))+C_1^2(-4M^2y^2-N^3(3+ 2M^2y)+N^2(6+M^2(y-4)y))+N^2(-24C_2^2M^2(2+N)+2C_2C_6M^2(-8-2N+N^2)y+C_6^2 (N-2)(-12+M^2(2+N)y^2))+2C_1N(C_2M^2(2+N)+C_6(12N+4M^2y^2)))+3(C_5^2M^2(8+N^3)+2C_5M^2N+N^2(C_1^2(N-2)+C_1(4C_6(N-2)-2C_2M^2N(2+N))))+3NL_{57}L_{82})+3Ny(-8M(N-6)(NC_6-C_1)(\cosh(h_1L_{57})-\cosh(h_2L_{57})+\sinh(h_1L_{57})-\sinh(h_2L_{57}))((2C_6+C_1)(4-N^2)^{1/2}(-1+\cosh((h_1+h_2)\,L_{57})+\sinh((h_1+h_2)\,L_{57})))+M(2+N)(2C_2+C_5)(1+\cosh((h_1+h_2)\,L_{57})+\sinh((h_1+h_2)\,L_{57})))(\cosh((2y+h_1+h_2)\,L_{57})+\sinh ((2y+h_1+h_2)\,L_{57})+L_{57}L_{83})+h_2(-24(C_6N-C_1)(\cosh(yL_{57})-\cosh(h_1L_{57})+\sinh(yL_{57}) -\sinh(h_1L_{57}))+(\cosh(2\,(y+h_1+h_2)\,L_{57})+\sinh(2\,(y+h_1+h_2)\,L_{57}))L_{66})))), \tag{9.35}$$

Now we write the definitions of $C_1 - C_6$ and several values therein.

$$
\begin{aligned}
C_1 &= \left(\frac{(C_7(C_8+C_9+C_{10}))}{(C_{11}+C_{12}+C_{13}+C_{14})}\right),\\
C_2 &= \left(\frac{1}{C_{19}}(C_{15}(\frac{C_{16}+C_{17}+C_{18}}{C_{20}}))\right),\\
C_3 &= \left(\frac{(C_{21}+C_{22}+C_{23})}{C_{24}}\right),\\
C_4 &= \left(\frac{1}{C_{29}}\left(\frac{L_5C_{25}(C_{26}+C_{27}+C_{28})}{(h_1-h_2)}\right)\right),\\
C_5 &= \left(\frac{C_{30}(C_{31}+C_{32})}{(C_{33}+C_{34}+C_{35}+C_{36}+C_{37})}\right),\\
C_6 &= \left(\frac{-(C_7(C_{38}+C_{39}))}{C_{40}}\right),\\
C_7 &= (3M^2(N-1)(F+h_1-h_2)),\\
C_8 &= (2(N-2)^2(N-1)NL_1^2(2+N-4L_3)+2(N-2)^2(N-1)NL_2^2(2+N-4L_4)),\\
C_9 &= -(N-2)L_1(4N(2-3N+N^2)L_2(2+N-2L_3-2L_4)+M(h_2(N(-8+6N+N^2\\
&\quad +N^3)-4(-2+N+N^3)L_3)-(N-1)h_1(-4(2+N)L_3+N(8+2N+N^2-4N\\
&\quad L_4))-4\beta(-2NL_3^2+L_3(4-4N-2NL_4)+N(-4+NL_4)))L_5),\\
C_{10} &= -M(N-2)L_2((N-1)((N-1)h_2(N(8+2N+N^2)-4N^2L_3-4(N-2)L_4+h_1\\
&\quad (N(-8+6N+N^2+N^3)-4(-2+N+N^3)L_4)+4\beta(-4N+NL_3(N-2L_4)+h_1\\
&\quad (N(-8+6N+N^2+N^3)-4(-2+N+N^3)L_4)+4\beta(-4N+NL_3(N-2L_4)+4(\\
&\quad N+1)L_4-2NL_4^2))L_5+N(2(2-3N+N^2)(L_3-L_4)^2-M^2(N-1)h_2^2(-4N+(\\
&\quad 6+N)L_3+(N-2)L_4)-M^2(N-1)h_1^2(-4N+(N-2)L_3+(6+N)L_4)-M^2\beta\\
&\quad h_2(-8N-4NL_3^2+(-4+4N+N^2)L_4+L_3(12+4N+N^2-4NL_4))+M^2h_1(2\\
&\quad (N-1)h_2(-4N+(N+2)L_3+(N+2)L_4)+\beta(-8N+(12+4N+N^2)L_4-4N\\
&\quad L_4^2+L_3(-4+4N+N^2-4NL_4))))L_5))),\\
C_{11} &= (M^3(-2+N)(-1+N)^2h_1^4(L_1(-2N+(2+N)L_3)-L_2(-2N+(2+N)L_4))L_5+\\
&\quad M^2(-1+N)h_1^3((-2+N)^2(-1+N)NL_1^2(-10+N+8L_3)+(-2+N)^2(-1+N)
\end{aligned}
$$

$NL_2^2(2+N-4L_3)+2(-2+N)L_1(2N(2-3N+N^2)L_2(-1+N+L_3-2L_4)+$
$M(-2(-1+N)h_2(-2N+(2+N)L_3)+\beta(-2NL_3^2+N(8-3N+NL_4)+L_3(-8$
$-2N+4NL_4)))L_5^{1/2}))),$

$$
\begin{aligned}
C_{12} = {} & -2M(N-2)L_2(-2(N-1)h_2(-2N(N+2)L_4)+\beta((8-3N)N+NL_3(N-2L_4) \\
& -2(4+N)L_4+4NL_4^2))L_5+N((2-3N+N^2)L_3^2+2(N-1)L_3(-3N+2(N+ \\
& 1)L_4+(-1+N)L_4(-6N+(10+N)L_4))L_5)+h_2(N(2-3N+N^2)^2(3N-M^2 \\
& h_2^2)L_1^2(2+N-4L_3)+(N-2)^2(N-1)NL_2^2(3(N-1)N(2+N-4L_3)-24M^2 \\
& \beta h_2(L_4-1)-M^2(N-1)h_2(-10+N+8L_4))-(N-2)L_1(2N(2-3N+N^2 \\
&)L_2(3(N-1)N(2+N-4L_3)-3M\beta^2h_2(N^2-4(N-1)L_3)+2M^2(N-1)h_2^2 \\
& (N-1-2L_3+L_4)+M(M^2(N-1)^2h_2^3(-2N+(2+N)L_3)-6(N-1)N\beta(- \\
& 4(N-1)L_3^2+L_3(4+2N+N^2-4L_4)+2N(L_4-2))+2M^2(N-1)\beta h_2^2(4N \\
& L_3^2+N(8-3N+NL_4)-2L_3(4+N+NL_4))+3h_2(-4NL_6-2N(-2(N-2 \\
& N^2+N^3+2M^2\beta^2-M^2N\beta^2)+(N-1)^2NL_4)+L_3(N^5-8M^2\beta^2+N^3(M^2 \\
& \beta^2-7)-4N(M^2\beta^2+1)+2N^2(5+M^2\beta^2)+4(N-1)NL_4)))L_5), \\
C_{13} = {} & (ML_2(-M^2(N-1)^2h_2^3(-2N+(2+N)L_4)+2M^2(N-1)\beta h_2^2((8-3N)-2(4+N)L_4-2 \\
& NL_4^2+NL_3(N+4L_4))-6(N-1)N\beta(-4N+L_4(4+2N+N^2)-4L_4^2+L_3(2N-4(N- \\
& 1)L_4))+3h_2(4N(N-2N^2+N^3+2M^2\beta^2-M^2N\beta^2)+(N^5-8M^2\beta^2+N^3(M^2\beta^2-7) \\
& -4N(1+M^2\beta^2)+2N^2(5+M^2\beta^2))L_4+4(N-1)^2NL_4^2-2NL_3((N-1)^2N+2(1-N \\
& -N^2+N^3-2M^2\beta^2+M^2N\beta^2)L_4)))L_5), \\
C_{14} = {} & -(((N-1)N(3(N-1)N(2+N-4L_3)(L_3-L_4)^2+M^2h_2^2((10+9N+N^2)L_3^2+(N-1) \\
& L_4(-6N+(N-2)L_4)+(N-1)L_3(-3N+2(1+N)L_4))+6M^2\beta h_2(-2NL_4+NL_3(-2 \\
& +(4+N)L_4)))L_5)+h_1(-3(N-2)^2(L_{15}-2M(N-2)L_2(-2M^2(N-2)^2h_2^3(-2N+(N \\
& +2)L_4)-3(N-1)N\beta+L_{30}+3h_2(4N(N-2N^2+N^3+2M^2\beta^2-M^2\beta^2N)+(N^5-8 \\
& M^2\beta^2+N^3(M^2\beta^2-7)-4N(1+M^2\beta^2)+2N^2(5+M^2\beta^2))L_4-2NL_4^2+2N^2(5+M^2 \\
& \beta^2)L_4)))L_5)+L_{16}),
\end{aligned}
$$

$$C_{15} = 3M^2(F + h_1 - h_2),$$

$$C_{16} = (2-3N+N^2)(h_1(2(N-2)^2(N-1)NL_1^2 - (N-2)L_1L_2(2N(2-3N+N^2)+M(2(N-1)Nh_2-(N-2)\beta(N-2L_3))L_5)+(M(N-2)L_2(-2(N-1)Nh_2+(N-2)\beta(N-2L_4))+(N-1)N(N-2L_4)(L_4-L_3)L_5)L_5+M(2-3N+N^2)h_1^2(L_1(N+2L_3)+L_2(N-2L_4))L_5+h_2(2(N-2)^2(N-1)NL_2^2-(N-2)L_1(2N(2-3N+N^2)L_2+M((-2+N)\beta-(N-1)h_2)(N-2L_3)L_5)-M(-2+N)((-2+N)\beta-(N-1)h_2)L_2(N+2L_4)L_5+(N-1)N(N-2L_3)(L_3-L_4)L_5))L_8),$$

$$C_{17} = (-((-2+N)^2(N-1)(L_1h_1-L_2h_2)((-2+N)^2(N-1)NL_1^2(2+N-4L_3)+(N-2)^2(N-1)NL_2^2(2+N-4L_4)-(N-2)L_1(2N(2-3N+N^2)L_2(2+N-2L_3-2L_4)+M(-1+N)h_2(N(6+N)-2(2+3N)L_3)(N-1)h_1(-2(N+2)L_3+N(6+N-4L_4))-2\beta(-2NL_3^2+L_3(4+4N-2NL_4)+N(-4+NL_4)))L_5)-(N(6+N)-2(2+3N)L_4)+2\beta(-4N+NL_3(N-2L_4)+4(1+N)L_4-2NL_4^2))L_5),$$

$$C_{18} = ((N(2-3N+N^2)(L_3-L_4)^2-M^2(N-1)h_2^2(-4N+(6+N)L_3+(N-2)L_4)-M^2(N-1)h_1^2(-4N+(-2+N)L_3+(6+N)L_4)-M^2\beta h_2(-8N-4NL_3^2+(-4+4N+N^2)L_4+L_3(12+4N+N^2-4NL_4))+M^2h_1(2(-1+N)h_2(-4N+(2+N)L_3+(2+N)L_4)+\beta(-8N+(12+4N+N^2)L_4-4NL_4^2+L_3(-4+4N+N^2-4NL_4))))L_5),$$

$$C_{19} = L_7L_{30},$$

$$C_{20} = (M^3L_5L_9 - 3Mh_1^2L_{22} + L_{20} + L_{21}),$$

$$C_{21} = (M(-1+N)h_1^3(L_{31} - L_{32} + L_{33})L_5),$$

$$C_{22} = (h_2(FN(2-3N+N^2)^2(3N-M^2h_2^2)L_1^2(2+N-4L_3)+(-2+N)^2(N-1)NL_2^2L_{34}+L_{35}+L_{36}+(L_{37}-3M(N-1)h_2^2L_{38}+2M(N-2)L_2L_{39}+(N-1)NL_{40}+h_1^2L_{41}-2M^3(N-2)(N-1)^2h_2^3L_2L_{41}-2M^3(N-2)(N-1)^2h_2^3L_2L_{42}-6M^2N(-(2+N)^3L_2^2+2M(N-2)N\beta L_2(-2+L_3+L_4)L_5-(N-1)(L_3-L_4)(-4N+(2+N)L_3+(2+N)L_4)L_5)-(N(-2-7N^2+4FM^2\beta+N(9-3FM^2\beta))+(3N^3-4FM^2\beta+2N(-3+FM^2\beta)+N^2(3+FM^2\beta))L_4-2FM^2\beta L_4^2+NL_3(N-4(N-1)L_4))L_5$$

$$+N^2(3+FM^2\beta))L_4+L_3N(N(-2+2N+FM^2\beta)-4(N-1)L_4))L_5),$$

$$C_{23} = +FMN(2(N-1)L_3(-3N+(4+N)L_4)-(N-1)L_4(-2N+(6+N)L_4))+M^3$$
$$(F-2h_2)L_5L_{25},$$

$$C_{24} = (2(h_2L_{43}+M(N-2)L_2L_5L_{44})+h_1(L_{45}+2M^2(N-2)(N-1)^2h_2^2L_2(-2N+($$
$$2+N)L_4)L_5((2-N)L_2(4N(N-2N^2+N^3+2M^2\beta^2+NM^2\beta^2)-2(N-2L_4)$$
$$+((2-N)L_2(4N(N-2N^2+N^3+2M^2\beta^2+NM^2\beta^2)L_4^2)+L_{46}L_4)L_5M(N-1)$$
$$h_2((N-2)^3(N-1)NL_2^2-2M(N-2)\beta L_2((8-3N)N+N^2L_3-2(4+N)L_4)+$$
$$N((-2+N+N^2)+2L_3(N-1)(-3N+2(N+1)L_4)+(N-1)L_4(-6N+(6+N$$
$$)L_4))L_5))-M^3L_4L_{25}-M^2(N-1)h_1^3((N-1)NL_1^2(-10+8L_3+N)+2(N-2)$$
$$L_1(2N(2-3N+N^2)L_2(-1+N+L_3-2L_4)+M(-2(N-1)h_2(-2N+(2+N)L_3$$
$$)+\beta(-2NL_3^2+N(8-3N+NL_4)+L_3(-8-2N+4NL_4)))L_5)-2M(N-2)L_2($$
$$-2(N-1)h_2(-2N+(N+2)L_4)+\beta((8-3N)N+NL_3(N-2L_4)-2(4+N)L_4$$
$$+4NL_4^2))L_5+L_{28}))),$$

$$C_{25} = (N-1)(F+h_1-h_2),$$

$$C_{26} = 3N(N-2)(L_1-L_2)+M(3Nh_1+M^2h_1^3-3Nh_2-M^2h_2^3)L_5)(2(N-2)^2(N-1$$
$$)NL_1^2(2+N-4L_3)+2(N-2)^2(N-1)NL_2^2(2+N-4L_4)-(N-2)L_1(4N(2-$$
$$3N+N^2)L_2(2+N-2L_3-2L_4)+M(h_2(N(-8+6N+N^2+N^3)-4(-2+N+$$
$$N^3)L_3-(N-1)h_1(-4(N+2)L_3+N(8+2N+N^2-4NL_4))-4\beta(-2NL_3^2+L_3$$
$$(4+4N-2NL_4)+N(N-4L_4)))L_5),$$

$$C_{27} = (-M(N-2)L_2(-(N-1)h_2(N(8+2N+N^2)-4N^2L_3-4(2+N)L_4)+h_1(N(-8$$
$$+6N+N^2+N^3)-4(2+N+N^3)L_4)+4\beta(-4N+NL_3(N-2L_4)+4(1+N)L_4$$
$$-2NL_4^2))L_5),$$

$$C_{28} = N(2(2-3N+N^2)(L_3-L_4)^2-M^2(N-1)^2h_2^2(-4N+(6+N)L_3+(N-2)L_4)-$$
$$M^2(N-1)h_1^2(-4N+(N-2)L_3(6+N)L_4)-M^2\beta h_2(-8N-4NL_3^2+(-4+4N+$$
$$N^2)L_4+L_3(12+4N+N^2-4NL_4))+M^2h_1(2(N-1)h_2(-4N+(2+N)L_3+($$

$$2+N)L_4)+\beta(-8N+12+4N+N^2)L_4-4NL_4^2+L_3(-4+4N+N^2-4NL_4))),$$

$$C_{29} = (L_{47}+(L_{50}L_{51}-M^2(N-2)L_2(-M^2h_2^3(-2N+L_4(2+N))+2M^2(N-1)\beta h_2^2((8-3N)N-2(4+N)L_4-2NL_4^2+NL_3(N+4L_4))-6(N-1)N\beta(-4N+L_3(2N-4(N-1)L_4))-4L_4^2+L_3(2N-4(N-1)L_4))+3h_2(4N(N-2N^2+N^3+2M^2\beta^2-NM^2\beta^2)+(N^5-8M^2\beta^2+N^3(-7+M^2\beta^2)-4N(1+M^2\beta^2)+2N^2(5+M^2\beta^2))L_4-2NL_3((N-1)^2N+2(1-N-N^2+N^3-2M^2\beta^2+NM^2\beta^2)L_4)))L_5+L_{52}(N-2)((N-2)L_1(N-2L_3)-(N-2)L_2(N-2L_4)+M(-h_2(N-2L_3)+h_1(N-2L_4))L_5)),$$

$$C_{30} = 3M^2(N-2)(N-1)(F+h_1-h_2),$$

$$C_{31} = (2(N-2)^2(N-1)N(L_1-L_2)(h_1L_1-h_2L_2)+M(N-2)((N-2)\beta h_1-(N-1)h_2)(h_1(L_1(N+2L_3)+L_2(N-2L_4))-h_2(L_1(N-2L_3)+L_2(N+2L_4)))L_5,$$

$$C_{32} = (-(N-1)N(-h_2(N-2L_3)+h_1(N-2L_4))(L_3-L_4)L_5),$$

$$C_{33} = ((M^3(N-2)(N-1)^2h_1^4(L_1(-2N+(2+N)L_3)+L_2(2N-(2+N)L_4))L_5+M^2(N-1)h_1^3(NL_1^2(N-2)^2(N-1)(-10+8L_3+N)+(N-2)^2NL_2(2+N-4L_4)+2(N-2)L_1(2(N-2)NL_2(N-1)+M(-2(N-1)h_2(-2N+(2+N)L_3)+\beta((8-3N)N-2L_3(4+N+NL_3)+N(N+4L_3)L_4))L_5)-2ML_2(-2(N-1)h_2(-2N+(2+N)L_4)+\beta(N(8-3N+NL_3)-2(4+N+NL_4)L_4+4NL_4^2))L_5+(N-1)N((N-2)L_3^2+2L_3(-3N+2(N+1)L_4)+L_4(-6N+(10+N)L_4))L_5),$$

$$C_{34} = (h_2(N(2+3N+N^2)+(N-2)^2NL_2^2+(3(N-1)N-M^2(N-1)h_2^2(-10+N+8L_4))-(N-2)L_1(2(N-2)NL_2(3(N-1)N(2+N-4L_3)+2M^2(N-1)h_2^2(-1+N-2L_3+L_4))+M(M^2(N-1)^2h_2^3(2N-(2+N)L_3)-6(N-1)(-4(N-1)L_3^2+L_3(4+N(N-2)-4L_4)+2N(-2+L_4))+3h_2(4N((N-1)^2N-M^2\beta^2(N-2))+L_3((N-1)^2N(-4+N(N-2))+M^2\beta^2(N-2))L_3)-2N(N-1)L_4)+2M^2(N-1)\beta^2h_2^2(4NL_3^2-2L_3(N-1)))L_5)),$$

$$C_{35} = M(N-2)L_2(M^2(N-1)^2+6(N-1)\beta(-2N(L_3-2)+4L_3^2)+3h_2(4N((N-1)-M^2$$

$$(N-2)\beta^2) + L_4(N(N-1) + 4(N-1)NL_4) + 2NL_3(-(N-1)N - 2((N-2) + M^2\beta^2$$
$$(N-1))L_4))),$$

$$C_{36} = ((-(N-1)N(3N(N-1) + 6M^2\beta^2(-2(N-2L_3) + N(-2 + (4+N)L_3)L_4) + M^2$$
$$(N-1)h_2((10+N)L_3^2 + 2L_3(-3N + 2(1+N)L_4)))L_5) - 3Mh_1^2(M(N-2) + (N$$
$$-2)L_1(2M(N-1)NL_2 + L_3((N-1)^2N(-4 + N(2+N)) + 4(N-1)NL_3) - 2$$
$$N((N-1)^2N + 2((N-1)^2(N+1) + M^2\beta^2(N-2)L_3) + 2M^2\beta h_2((8-3N)N +$$
$$N(N-2L_3)L_4))L_5))),$$

$$C_{37} = -(N-2)L_2(4N((N-1)^2 - M^2(N-2)\beta^2) - 4N((N-1)^2(N-2)\beta^2)L_4) + MN$$
$$(N-1)((-2+N+N^2)h_2L_3^2 + L_4(N(N-1) + h_2(-6N + (6+N)L_4)) + 2L_3(h_2$$
$$(-3(N-1) + 2NL_4) + N\beta(-2 + (4+N-4L_4)L_4)))L_5) + h_1(-3(N-2)NL_1^2(-$$
$$M^2(N-2)h_2^2 + N(N-1)) + 2(N-2)L_1(3(N-2)NL_2(2M^2(N-1-L_3) + 2M^2$$
$$\beta^2(-N^2 + 2(N-1)L_3 + 2(N-1)L_4)) + M(-2M^2h_2^3 + 3h_2(4N((N-1)^2 - M^2\beta^2$$
$$(N-2)) + L_3(N(N-1)^2 - 2N((N-1)N + M^2\beta^2(N-2))L_3) - 2N((N-1)^2 +$$
$$((N-1)^2N + M^2(N-2)\beta^2)L_3)L_4) - 3N\beta(-4L_3^2 + L_3(4 - 4(N-1)L_4)) + 3M^2$$
$$h_2^2(-2(4+N)L_3 + 2NL_3^2))L_5) + L_4) + 3h_2(4N((N-1)^2 - M^2\beta^2) + (N-2)M^2$$
$$\beta^2)L_4 - 2NL_3((N-1)^2N + L_4)))L_5 + 3N(N-1)(N(N-1) + M^2(N-1)h_2^2((6$$
$$+N)L_3^2 + L_4(-6N + (2+N)L_4)) + 4M^2\beta h_2(2L_4(N-L_4) + 2L_3^2(1-NL_4) + NL_3(-2+$$
$$(4+N-2L_4)L_4)))L_5),$$

$$C_{38} = ((N-2)^2(N-1)N(L_1 - L_2)(L_1(2+N-4L_3) - L_2(2+N-4L_4))),$$

$$C_{39} = (M(N-2)(L_2(-2N\beta(-4+NL_3) + 4\beta(-2-2N+NL_3)L_4 + 4N\beta L_4^2 - (N-1)h_1$$
$$(N(6+N) - 2(2+3N)L_4) + h_2(((N-1)N(6+N-4L_3) - 2(-2+N+N^2)L_4)) + L_1$$
$$(8\beta(L_3 - N) - (N-1)h_2(N(6+N) - 2(2+3N)L_3) + (N-1)h_1(-2(N+2)L_3 + N(6$$
$$+N-4L_4)) + 2N\beta(NL_4 - 2L_3(-2+L_3+L_4))))L_5 + ((N-2)(N-1)N(L_3-L_4)^2$$
$$-M^2(N-1)h_2^2(-4N + (6+N)L_3 + (N-2)L_4) - M^2(N-1)h_1^2(-4N + (N-2)L_3 +$$
$$(6+N)L_4) + M^2\beta h_2(8N + 4NL_3^2 - (4+N(N+4))L_4 - L_3(12 + N(N+4) - 4NL_4))$$

$$+M^2h_1(-4\beta(2N+L_3)+2(N-1)h_2(-4N+(2+N)L_3+(2+N)L_4)+\beta(12L_4+N$$
$$(4+N-4L_4)(L_3-L_4))))L_5),$$

$$
\begin{aligned}
C_{40} &= (M^3(N-2)(N-1)^2h_1^4(L_1(-2N+(N+2)L_3)+L_2(2N-(2+N)L_4))L_5+L_{55})+M \\
&\quad (N-2)L_2(M^2(N-1)^2h_2^3(2N-(2+N)L_4)+(N-1)N\beta(-2N(L_3-2)-(4-(N-1 \\
&\quad)L_3)L_4)+3h_2(4N((N-1)^2N-M^2\beta^2)+L_4((N-1)^2N+4(N-1)NL_4)+2NL_3(-(N \\
&\quad -1)N-2((N-1)(N+1)+4NL_4)+2NL_3((N-1)^2N-2((N-1)^2(N+1)))L_4))+ \\
&\quad L_{56}))),\\
L_1 &= \sinh\left(\frac{M(-2+N)h_2}{\sqrt{(-2+N)^2}}\right),\\
L_2 &= \sinh\left(\frac{M(-2+N)h_1}{\sqrt{(-2+N)^2}}\right),\\
L_3 &= \cosh\left(\frac{M(-2+N)h_1}{\sqrt{(-2+N)^2}}\right),\\
L_4 &= \cosh\left(\frac{M(-2+N)h_2}{\sqrt{(-2+N)^2}}\right),\\
L_5 &= (4-4N+N^2),\\
L_6 &= ((1-N^2+N^3-2M^2\beta^2+N(-1+M^2\beta^2))L_3^2),\\
L_7 &= ((N-2)L_1(N-2L_3)-(N-2)L_2(N-2L_4)+M(-h_2(N-2L_3)+h_1(N-2L_4))L_5),\\
L_8 &= ((N-2)L_1(L_3-1)-(N-2)L_2(L_4-1)+M(-h_2(L_3-1)+h_1(L_4-1))L_5),\\
L_9 &= (1-N^2+N^3-2M^2\beta^2+N(-1+M^2\beta^2)),\\
L_{10} &= (N-2)(N-1)^2h_1^4(L_1(-2N+(2+N)L_3)-L_2(-2N+(2+N)L_4)),\\
L_{11} &= ((2-3N+N^2)L_3^2+2(N-1)L_3(-3N+2(N+1)L_4+(N-1)L_4(-6N+(10+N)L_4)),\\
L_{12} &= (M^2(2-3N+N^2)h_2^2-8M^2\beta h_2(L_3-1)+(N-1)N(2+N-4L_4)),\\
L_{13} &= ((N-1)N(2+N-4L_4)-8M^2\beta h_2(L_4-1)-M^2(N-1)h_2^2(-6+N+4L_4)),\\
L_{14} &= (3N(2-3N+N^2)L_2(2M^2(N-1)h_2^2(N-1-L_3)+(N-1)N(2+N-4L_4)-2M^2\beta h_2(\\
&\quad N^2-2(N-1)L_3-2(N-1)L_4))+M(-2M^2(N-1)^2h_2^3(-2N+(2+N)L_3)-3(N-1) \\
&\quad N\beta(4L_3^2+2N(L_4-2)+L_3(4+2N+N^2-4(N-1)L_4))+3M^2(N-1)\beta h_2^2(-2(4+N) \\
&\quad L_3+2NL_3^2+N(8-3N+NL_4))+3h_2(-2N(N-2N^2+N^3-2M^2\beta^2+M^2\beta^2N)L_3^2-2
\end{aligned}
$$

$$N(-2(N-2N^2+N^3+2M^2\beta^2-M^2\beta^2N)+(N-1)^2NL_4)+L_3(N^5-8M^2\beta^2+N^3(M^2\beta^2-7)-4N(1+M^2\beta^2)+2N^2(5+M^2\beta^2)-2N(N-2N^2+N^3-2M^2\beta^2+M^2\beta^2N)L_4)))L_5),$$

$$L_{15} = ((-3(N-2)^2(N-1)NL_1^2(M^2(2-3N+N^2)h_2^2-8M^2\beta h2(L_3-1)+(N-1)N(2+N-4L_4))-3(-2+N)^2(N-1)NL_2^2((N-1)N(2+N-4L_4)-8M^2\beta h_2(L_4-1)-M^2(N-1)h_2^2(-6+N+4L_4))+(2(N-2)L_1(3N(2-3N+N^2)L_2(2M^2(N-1)h_2^2(N-1-L_3)+(N-1)N(2+N-4L_4)-2M^2\beta h_2(N^2-2(N-1)L_3-2(N-1)L_4))+M(-2M^2(2+N)-3(N-1)N\beta+L_3(N^5-8M^2\beta^2-4N(1+M^2\beta^2)-2N(N-2N^2+N^3-2M^2\beta^2+NM^2\beta^2)L_4)))L_5))+L_4),$$

$$L_{16} = (-4N+2L_3+(4+2N+N^2)L_4(N-1)L_4^2)+\beta h_2((8N-3N)N-2(4+N)L_4+NL_3(N+2L_4)),$$

$$L_{17} = (N-2)^2(N-1)NL_1^2(8\beta(L_3-1)+(N-1)h_2(-6+N+4L_3))),$$

$$L_{18} = 2M^2(N-2)(N-1)^2h_2^2L_2(-2N+(2+N)L_4)L_5+M(N-1)h_2((N-2)^3(N-1)h_2^2L_2(-2N+(2+N)L_4)L_5+M(N-1)h_2((N-2)^3(N-1)NL_2^2-2M(N-2)\beta L_2((8-3N)N+N^2L_3-2(4+N)L_4+2NL_4^2)L_5+N((-2+N+N^2)L_3^2+2(N-1)L_3(-3N+(N+1)L_4(-6N+(6+N)L_4))L_5),$$

$$L_{19} = L_5(2M(N-1)N\beta(-2(N-2L_4)L_4+NL_3(-2+(4+N)L_4-4L_4^2))L_5+(2-N)L_2(4N(N-2L_4)L_4+NL_3(-2+(4+N)L_4-4L_4^2))L_5+(N-2)L_2(4N(N-2N^2+N^3+2M^2\beta^2-M^2\beta^2N)-2(N-1)^2NL_3(N-2L_4)+(N^5-8M^2\beta^2+N^3(-7+M^2\beta^2)-4N(1+M^2\beta^2)+2N^2(5+M^2\beta^2))L_4-4NL_4^2L_9))+(N-2)L_1(-2MN(2-3N+N^2)L_2(-2(N-1)h_2(-1+N-L_4)+\beta(N^2-4(N-1)L_4))+L_5NL_3^2(4(N-1)^2-2M^2(N-1)^2h_2^2(-2N+(2+N)L_3)-2N(N-1)\beta h_2(N(8-3N+NL_4)+2(-4-N+NL_4))L_3+L_3(N^5-8M^2\beta^2+N^3(M^2\beta^2-7)-4N(1+M^2\beta^2)+2N^2(5+M^2\beta^2)-4NL_4L_9))),$$

$$L_{20} = M^2(N-1)h_1^3((N-2)^2(N-1)NL_1^2(-10+N+8L_3)+(N-2)^2(N-1)h_2(-2N+(2$$

$$+N)L_3 + \beta(-2NL_3^2 + N(8 - 3N + NL_4) + L_3(-8 - 2N + 4NL_4)))L_5) - 2M(N-2)$$
$$L_2L_5L_{10} + NL_5L_{11}),$$

$$L_{21} = (h_1(-2M(N-2)L_2(-2M^2(N-1)^2h_2^3(-2N + (N+2)L_4) - 3(N-1)N\beta(-4N + 2L_3$$
$$(N - 2L_4) + (4 + 2N + N^2)L_4 - 4(N-1)L_4^2) + 3M^2(N-1)\beta h_2^2((8-3N)N - 2(4$$
$$+N)L_4 + NL_3(N + 2L_4)) + 3h_2(4N(N - 2N^2 + N^3 + 2M^2\beta^2 - M^2\beta^2 N) + (N^5 -$$
$$8M^2\beta^2 + N^3(M^2\beta^2 - 7) - 4N(1 + M^2\beta^2) + 2N^2(5 + M^2\beta^2))L_4 - 2N(N - 2N^2 +$$
$$N^3 - 2M^2\beta^2 + M^2\beta^2 N)L_4^2 - 2NL_3((N-1)^2N + (N - 2N^2 + N^3 - 2M^2\beta^2 + M^2$$
$$\beta^2 N)L_4^2 - 2NL_3((N-1)^2N + (N - 2N^2 + N^3 + 2M^2\beta^2 - NM^2\beta^2) + N^5 - 8M^2\beta^2$$
$$+N^3(M^2\beta^2 - 7) - 4N(1 + M^2\beta^2) + 2N^2(5 + M^2\beta^2))L_4 - 2N(N - 2N^2 + N^3 - 2$$
$$M^2\beta^2 + NM^2\beta^2)L_4^2 - 2NL_3((N-1)^2N + (N - 2N^2 + N^3 - 2M^2\beta^2 + NM^2\beta^2)L_4$$
$$)))L_5 + 3(N-1)N((N-1)N(2 + N - 4L_4)(L_3 - L_4)^2 + M^2h_2^2((5N - 6 + N^2)L_3^2$$
$$+2(N-1)L_3(-3N + 2(N+1)L_4)) + 6M^2\beta h_2(-2NL_4 + 4L_3^2(1 - NL_4) + NL_3$$
$$(-2 + (4+N)L_4)))L_5 - (N-2)L_1(2N(2 - 3N + N^2)L_2(3(N-1)N(2 + N - 4$$
$$L_3) - 3M^2\beta h_2(N^2 - 4(N-1)L_3) + 2M^2(N-1)h_2^2(N - 1 - 2L_3 + L_4)) + ML_5^{1/2}$$
$$(-M^2(N-1)^2h_2^3(-2N + (N+2)L_3) - 6(N-1)N\beta(-4(N-1)L_3^2 + L_3(4 + 2N +$$
$$N^2 + 4L_4)) + 2N(L_4 - 2)) + 2M^2(N-1)\beta h_2^2(4NL_3^2 + N(8 - 3N + NL_4) - 2L_3$$
$$(4 + N + NL_4)) + 2M^2(N-1)\beta h_2^2(4NL_3^2 + N(8 - 3N + NL_4) - 2L_3(4 + N + N$$
$$L_4)) + 3h_2(-2N(-2(N - 2N^2 + N^3 + 2M^2\beta^2 - M^2\beta^2 N) + (N-1)^2NL_4) - 4N$$
$$L_3^2L_9))) + NL_{15} + M(N-2)L_2L_5(-M^2(N-1)h_2^3 + 2M^2(-1+N)\beta((8-3N)N$$
$$-2(4+N)L_4 - 2NL_4^2 + NL_3(N + 4L_4)) + 4N(1 + M^2\beta^2) + 2N^2(M^2\beta^2 + 5))L_4$$
$$+4(N-1)^2NL_4^2 - 2NL_3((N-1)^2N + 2(1 - N - N^2 + N^3 - 2M^2\beta^2 + NM^2\beta^2)$$
$$L_4)) - 6L_{16}) + L_{16})),$$

$$L_{22} = 3(N-2)^2(N-1)NL_1^2(-M^2(2 - 3N + N^2)h_2^2 + (N-1)N(2 + N - 4L_4)) + 3(N -$$
$$2)^2(N-1)NL_2^2((N-1)N - 8M^2\beta h_2(L_4 - 1) - M^2(N-1)(6 + N + 4L_4)) - (N$$
$$-2)L_1(3NL_2(2M^2(N-1)h_2^2(N - 1 - L_3) + (N-1)N(2 + N - 2L_4)(N^2 - 2(N$$

$-1)L_3 - 2(N-1)L_4)) + M(-2M^2(N-1)^2h_2^3(-2N+L_3) - 3(N-1)N\beta(-4L_3^2$
$+L_3(4+2N+N^2-4(N-1)L_4)) + 3M^2(N-1)\beta h_2^2(-2(4+N)L_3 + N(8-3N$
$+NL_4)) + 3h_2(-2N(N-2N^2+N^3-2M^2\beta^2+NM^2\beta^2)L_3^2 - 2N(-2(N-2N+$
$N^3+2M^2\beta^2-NM^2\beta^2)+NL_4) + L_3(N^5-8M^2\beta^2+N^3(-7+M^2\beta^2)-4N(1+$
$M^2\beta^2)+2N^2(5+M^2\beta^2)-2N(N-1)L_4)))L_5),$

$$L_{23} = 3(N-1)N\beta(-4N+2L_3(N-2L_4)+(4+2N+N^2)L_4-4(N-1)L_4^2),$$

$$L_{24} = (N^5-8M^2\beta^2+N^3(M^2\beta^2-7)-4N(1+M^2\beta^2)+2N^2(5+M^2\beta^2)),$$

$$L_{25} = (N-2)(N-1)^2h_1^4(L_1(-2N+(2+N)L_3)-L_2(-2N+(2+N)L_4)),$$

$$L_{26} = (M^3(N-2)(N-1)^2h_1^4(F-2h_2)(L_1(-2N+(2+N)L_3)-L_2(-2N+(2+N)L_4))L_5),$$

$$L_{27} = (M^2(N-1)h_1^3((N-2)^2(N-1)NL_1^2(-10+N+8L_3)+(N-2)^2(N-1)NL_2^2(2+ N-4L_4)+2L_1(2N+(-2(N-1)h_2)+\beta(-2NL_3^2++L_3(-8-2N+4NL_4)))L_5)),$$

$$L_{28} = (N((2-3N+N^2)L_3^2+2(N-1)L_3(-3N+2(1+N)L_4)+(N-1)L_4(-6N+(10 +N)L_4))L_5),$$

$$L_{29} = 3Mh_1^2(M(N-2)^2(N-1)NL_1^2(8\beta(L_3-1)+(N-1)h_2(-6+N+4L_3))+(N- 2)L_1(-2MN(2-3N+N^2+L_2)(-2(N-1)h_2(L_4-1+N)+\beta(N^2-4(N-1)L_4))+(4(N-1)^2NL_3^2-2M^2(N-1)^2h_2^2(-2N+(2+N)L_3)(-2(N-2N^2-N^3)+ (N-1)^2NL_4)+L_3(N^5-8M^2\beta^2+N^3(M^2\beta^2-7)-4N(M^2\beta^2+1)-4N(1-N^2 +N^3-2M^2\beta^2+N(1+M^2\beta^2))L_4)+2M^2N(N-1)\beta h_2(N(8-3N+NL_4)+2L_3 (-4-N+NL_4))L_5)+2M^2(N-1)(N-2)h_2^2L_2(-2N+(2+N)L_4)L_5)+(-(N- 2)L_2(4N(1-N^2-N^3+2M^2\beta^2+NM^2\beta^2)-2(N-1)^2L_3N(N-2L_4)(N^5-8M^2 \beta^2+N^3(M^2\beta^2-7)-4N(1+M^2\beta^2)+2N^2(5+M^2\beta^2))L_4-4N(1-N^2+N^3- 2M^2\beta^2+N(-1+M^2\beta^2))L_4^2+2M(N-1)N\beta(-2(N-2L_4)L_4+NL_3(-2(4+N) L_4-4L_4^2))+L_5)+M(-1+N)h_2((-2+N)^3(N-1)NL_2^2-2M(N-2)\beta L_2((8- 3N)N+N^2L_3-2(4+N)L_4+2NL_4^2)L_5+N((-2+N+N^2)L_5))),$$

$$
\begin{aligned}
L_{30} &= (-M^3L_5L_9+3Mh_1^2(ML_{17}+L_{18}+L_{19})-L_{20}+h_1(-3(N-1)N((N-1)N(2+N\\
&\quad -4L_4)(L_3-L_4)^2+M^2h_2^2((-6+5N+N^2)L_3^2+2(N-1)L_3(-3N+2(N+1)L_4)\\
&\quad +(N-1)L_4(-6N+(N+2)L_4))+4M^2\beta h_2(2L_4(-N+L_4)+L_3^2(2-2NL_4)+N\\
&\quad L_3(-2+(4+N)L_4-2L_4^2)))L_5+3(N-2)^2(N-1)NL_1^2L_2+3(N-2)^2(N-1)N\\
&\quad L_2^2L_3-2(N-2)L_1L_4+2M(N-2)L_2L_5(-2M^2(N-1)^2h_2^3(-2N+(2+N)L_4)-\\
&\quad 3(N-1)N\beta(-4N+2L_3(N-2L_4)+(4+2N+N^2)L_4-4(N-1)L_4^2)+3M^2(N\\
&\quad -1)\beta h_2^2((8-3N)N-2(N+4)L_4+NL_3(N+2L_4))+3M^2(N-1)\beta h_2^2((8-3\\
&\quad N)N-2(4+N)L_4+NL_3(N+2L_4))+3h_2(4N(N-2N^2+N^3+2M^2\beta^2-NM^2\\
&\quad \beta^2)-2N(N-2N^2+N^3-2M^2\beta^2+NM^2\beta^2)L_4^2-2NL_3((-1+N)^2N+(N-2\\
&\quad N^2+N^3-2M^2\beta^2+NM^2\beta^2)L_4)+L_4L_2)))+h_2((N-1)N(3(N-1)N(2+N\\
&\quad -4L_3)(L_3-L_4)^2+M^2h_2^2((-10+9N+N^2)L_3^2+(N-1)L_4(-6N+(N-2)L_4)\\
&\quad +2(N-1)L_3(-3N+2(N+1)L_4))+6M^2\beta h_2(-2NL_4+4L_3^2(1-NL_4)+NL_3\\
&\quad (-2+(4+N))L_4)))L_5-6M^2\beta h_2(NL_3(-2+(4+N)L_4)))L_5-NL_{15}-L_{16}-\\
&\quad M(N-2)L_2(-M^2(N-1)^2h_2^3+(2+N)L_4+2(4+N)L_4-2NL_4^2+NL_3(N+4\\
&\quad L_4))-6L_{16}+3h_2(4N(N-2N^2+N^3+2M^2\beta^2-NM^2\beta^2)+4(N-1)NL_4^2-\\
&\quad 2NL_3(N+2(1-N^2+N^3-2M^2\beta^2+NM^2\beta^2)L_4)+L_4))+L_1(2NL_2(3(N-\\
&\quad 1)(2+N-4L_3)-3M^2\beta h_2(N^2-4(N-1)L_3)+2M^2h_2^2(4NL_3^2+N(8-3N+\\
&\quad NL_4)-2L_3(4+N+NL_4))+3h_2(-2N(-2(N-2N^2+N^3+2M^2\beta^2-NM^2\beta^2\\
&\quad)+(N-1)^2NL_4)-4NL_3^2L_9+L_3(4(N-1)^2NL_4+L_{24}))))))),\\
L_{31} &= (M(N-2)^2(N-1)NL_1^2(4h_2(-4+N+2L_3)+F(-10+N+8L_3))+M(-2+\\
&\quad N^2)(-1+N)N(F-2h_2)L_2^2(2+N-4L_4)-2(N-2)L_1(-MN(2-3N+N^2)L_2\\
&\quad (2F(-1+N+L_3-2L_4)-h_2(-10+N+4L_3+4L_4))+(2FM^2N\beta L_3^2-M^2(\\
&\quad N-1)h_2^2(-2N+(2+N)L_3)+L_3(-21N^2+3N^3+8FM^2\beta+2N(9+FM^2\beta)+\\
&\quad 4N(-3+3N-FM^2\beta)L_4)+N(-18+33N-15N^2-8FM^2\beta+3FM^2N\beta+N(
\end{aligned}
$$

$$-6+6N-FM^2\beta)L_4)+M^2h_2(-4N\beta L_3^2+L_3(2F(-2+N+N^2)+(-4+8N-3N^2)\beta-4N\beta L_4)+2N(-2F(N-1)+(2+3N)\beta-2N\beta L_4)))L_5)),$$

$$L_{32} = (2(N-2)L_2(6N-3N^2-3N^3+8FM^2N\beta-3FM^2N\beta L_3-18NL_4+15N^2L_4+3N^3L_4-8FM^2L_4\beta-2FM^2N\beta L_4-2FM^2N\beta L_3L_4+12NL_4^2-12N^2L_4^2+4FM^2N\beta L_4^2+M^2(N-1)h_2^2(-2N(2+N0L_4)+M^2h_2(2N(2F(N-1)+(-2+3N)\beta)-2N\beta L_3(N-2L_4)-(2F(-2+N+N^2)+(-4+8N+N^2)\beta)L_4+4N\beta L_4^2))L_5),$$

$$L_{33} = (M(N-1)N((-N-2)(F-2h_2)L_3^2+L_4(4h_2(-3N+(4+N)L_4)+F(-6N+(10+N)L_4))+2L_3(F(-3N+2(N+1)L_4)-h_2(-6N+(10+N)L_4))),$$

$$L_{34} = (3F(N-1)N(2+N+-4L_3)+FM^2(N-1)h_2^2(-10+N+8L_4)-6h_2(-2N+N^2+N^3+4FM^2\beta-4(N-1)NL_3-4FM^2L_4\beta))+(N-2)L_1(3h_2(2N(2-3N+N^2)L_2(N(-2+N+N^2-FM^2N\beta)-4(N-1)(N-FM^2\beta)L_3)+M(-4N(-2(2-3N+N^2)\beta+F(-1-3N^2+N^3-2M^2\beta^2+N(3+M^2\beta^2)))L_3^2)),$$

$$L_{35} = (M(N-2)L_2(FM^2(N-1)^2h_2^3(-2N+(2+N)L_4+6F(N-1)N\beta(12-4N+N^2-2L_3(N-2L_4)+2(-8+N)L_4+4L_4^2))-2(N-1)h_2^2(N(-18+33N-15N^2-8FM^2\beta+3FM^2N\beta)+(-21N^2+3N^3+8FM^2\beta+2N(9+FM^2\beta))L_4+2FM^2\beta NL_4+NL_3(N(-6+6N-FM^2\beta)+4(-3+3N-FM^2\beta)L_4))+3h_2(-2(N-2)N((6-7N+N^2)\beta+F(3-6N+3N^2+2M^2\beta^2))+(-2N(12-16N+3N^2+N^3)\beta+F(3-6N+3N^2+2M^2\beta^2))+(-2N(12-16N+3N^2+N^3)\beta+F(2N^4+N^5-8M^2\beta^2+N^3-4N(4+M^2\beta^2)+2N^2(18+M^2\beta^2)))L_4+4F(N-1)^2NL_4^2+2NL_3(F(N-1)^2N-2(-2(2-3N+N^2)\beta+F(-1+3N-3N^2+N^3-2M^2\beta^2+NM^2\beta^2)L_4)))),$$

$$L_{36} = -(N-1)N(-12(N-1)N(F-2h_2)L_3^3-L_4(6N(-2+N+N^2-2FM^2\beta)h_2+3FN(-2+N+N^2)(-2+L_4)+FM^2(N-1)h_2^2(-6N+(N-2)L_4))+L_3^2(3FN(-10+9N+N^2)-FM^2(-10+9N+N^2)h_2^2-6h_2(-6N+5N^2+N^3+4FM^2\beta+4N$$

$$
\begin{aligned}
& (-1+N-FM^2\beta)L_4)) - 2L_3(FM^2(N-1)h_2^2(-3N+2(N+1)L_4) - 3Nh_2(-2+ \\
& N+N^2+2FM^2\beta+(-6+5N+N^2-4FM^2N\beta)L_4)+3FN(-2+N+N^2+4(N \\
& -1)L_4-2(N-1)L_4^2)))L_5), \\
L_{37} &= h_1((-2+N)^2(N-1)NL_1^2(-3FM^2(2-3N+N^2)h_2^2+2M^2(N-1)h_2^3(2+N \\
& -4L_3)-6h_2(-2N+N^2+N^3-4FM^2\beta+4(N-N^2+FM^2\beta)L_3)+3F(N \\
& -1)N(2+N-4L_3)-6h_2(-2N+N^2+N^3-4FM^2\beta+4(N-N^2+FM^2\beta \\
&)L_3)+3F(N-1)N(2+N-4L_4))-(N-2)^2(N-1)NL_2^2(3F(N-1)N(2 \\
& +N-4L_4)+4M^2(N-1)h_2^3(-4+N+2L_4)-3FM^2(N-1)h_2^2(-6+N+ \\
& 4L_4)-6h_2(-2N+N^2+N^3-4FM^2\beta+4(N-N^2+FM^2\beta)L_4))+2(N- \\
& 2)(M^2h_2^3(NL_2(-10+N+4L_3+4L_4)+M(-4N\beta L_4)+2N(-2F(N-1)+ \\
& (2-3N)\beta+N\beta L_4))L_5), \\
L_{38} &= (FMN(2-3N+N^2)L_2(-4+N+2L_3)+(-2FM^2N\beta L_3^2+L_3(3N^3-4FM^2 \\
& \beta+2N(-3+FM^2\beta)+N^2(3+FM^2\beta)-4(N-1)NL_4)+N(-2-7N^2+4 \\
& FM^2\beta+N(9-3FM^2\beta)+N(-2+2N+2FM^2\beta)L_4))L_5)-6Nh_2(2(N-2) \\
& (N-1)^2(N-FM^2\beta)L_2(L_3-L_4)+M((-2(2-3N+N^2)\beta+F(N-2N^2+ \\
& N^3-2M^2\beta^2+M^2\beta^2N))L_3^2-(N-1)(F(-2+5N-3N^2)-(-2+N)^2\beta+ \\
& F(N-1)NL_4)-L_3(F(N-1)^2N+(-2(2-3N+N^2)\beta+F(N-2N^2+N^3 \\
& -2M^2\beta^2+NM^2\beta^2))L_4))L_5)-M^3(-1+N)^2h_2^4(-2N+(2+N)L_3)L_5)+ \\
& 3FM(N-1)N\beta(12-4N+N^2+4L_3^2-2NL_4+2L_3(-8+N+2L_4))L_5^2), \\
L_{39} &= ((M^2(N-1)^2h_2^4(-2N+(2+N)L_4+3F(N-1)N\beta(-4-4N+N^2+2L_3(N \\
& -2L_4)-2(-8+NL_4)-4L_4^2)-(M^2(N-1)h_2^3(2N(-2F(N-1)+(2+3N) \\
& \beta)+(2F+(-4+8N-3N^2)\beta)L_4-4N\beta L_3(N+L_4))-3N^2(-9+FM^2\beta)) \\
& +NL_3(N-(-4+4N+FM^2\beta)+2FM^2\beta L_4))+6Nh_2((N-1)(F(2-5N+ \\
& 3N^2)+(N-2)\beta)-(-2(2-3N+N^2)\beta+F(N+2N^2+N^3+2M^2\beta^2+NM^2 \\
& \beta^2)+2FM^2\beta L_4))+L_3(FN+(-2(2-3N+N^2)\beta+F(N+2N^2+N^3+2M^2
\end{aligned}
$$

$$\beta^2 + NM^2\beta^2))L_4)))L_5),$$

$$\begin{aligned}
L_{40} &= (L_4(-3F(N-1)N(2+N-4L_4)(L_4-2)+2M^2(N-1)h_2^3(-6N+(N-2)L_4) \\
&\quad -3FM^2(N-1)h_2^2(-6+(N+2)L_4)+6h_2(-2N(-2+N+N^2)+(-6N+5N^2 \\
&\quad +N^3-4FM^2\beta)L_4))+L_3^2(3FM^2(5N-6+N^2)h_2^2-4M^2(-4+3N+N^2)h_2^3 \\
&\quad +3F(N-1)N(2+N-4L_4)-6h_2(-6N+5N^2+N^3-4FM^2\beta+4N(2-2N \\
&\quad +FM^2\beta)L_4))-2L_3(3F(N-1)N(2+N-4L_4)+3FM^2(N-1)h_2^2(N+(4+ \\
&\quad N)L_4)-M^2(N-1)h_2^3(6N+(10+N)L_4)-6Nh_2(-2+N+N^2+(4-4N+2 \\
&\quad FM^2\beta)L_4^2N)))L_5),
\end{aligned}$$

$$\begin{aligned}
L_{41} &= (3(N-2)^2(N-1)NL_1^2(-2M^2(2-3N+N^2)h_2^2+FM^2(N-1)h_2(-6+N+4 \\
&\quad L_3)+h_1^2(3(N-2)^2NL_1^2(-2M^2(2-3N+N^2)h_2^2+FM^2(N-1)h_2(-6+N+ \\
&\quad 4L_3)+2(-2N+N^2+N^3+4FM^2\beta-4FM^2\beta L_3-4(N-1)NL_4))-(N-2 \\
&\quad)L_1(6NL_2(N(-2+N+N^2-FM^2\beta N)-4(N-1)L_4FM^2h_2(N-1))+M(- \\
&\quad 2M^2(N-1)h_2^3(-2N+(N+2)L_3)+12M^2(N-1)N^2\beta h_2^2-6(N-1)h_2(4(N \\
&\quad -1)NL_3^2+L_3(3N^3+4FM^2\beta+N(6-2FM^2\beta)+N^2(-9+FM^2\beta))+N(-1 \\
&\quad 0+21N-11N^2-4FM^2\beta-FM^2\beta N+N(-4+4N+FM^2\beta)L_4))+3(4F(N \\
&\quad -1)^2NL_3^2+2N((N-2)((6-7N+N^2)\beta+F(3-6N+3N^2+2M^2\beta^2))+F \\
&\quad L_4N)+L_3(-2N(12-16N+3N^2+N^3)\beta+F(2N^4+N^5-8M^2\beta^2+N^3-4N \\
&\quad (4+M^2\beta^2)+2N^2)-4N(-2(2-3N+N^2)\beta+F(-1+3N-3N^2+N^3-2M^2 \\
&\quad \beta^2+M^2\beta^2N))L_4))L_5),
\end{aligned}$$

$$\begin{aligned}
L_{42} &= ((-2N+(N+2)L_4)L_5-3(-M(N-2)L_2(2N((4-4N-N^2+N^3)\beta+F(2-N \\
&\quad -4N^2+3M^2\beta^2-2M^2\beta^2N))-2F(N-1)^2NL_3(N-2L_4)+(-2N(12-16N \\
&\quad +3N^2+N^3)\beta+F(-2N^4+N^5-8M^2\beta^2+2N^2(16+M^2\beta^2)))L_4-4N(-2\beta+ \\
&\quad F(-1+3N-3N^2+N^3-2M^2\beta^2+NM^2\beta^2))L_4^2)-2(N-1)N(L_4(N(L_4(N(-2 \\
&\quad +N+N^2+2FM^2\beta)-L_4+4(N-1)L_4^2)+NL_3(2-N-N^2+2FM^2\beta+(-6 \\
&\quad +5N-4M^2\beta-FM^2\beta N)L_4+(4-4N+4FM^2\beta)L_4^2))L_5)L_5)),
\end{aligned}$$

$$L_{43} = (-N(2-3N+N^2)^2(3N-M^2h_2^2)L_1^2(2+N-4L_3)-(N-2)^2(N-1)NL_2^2(3(N-1)N(2+N-4L_3)-24M^2\beta h_2-M^2(N-1)h_2^2(-10+N+8L_4))+L_1(2N(2-3N+N^2)L_2(3(N-1)N-3M^2\beta(N^2-4(N-1)L_3)+2M^2(N-1)h_2^2(N-1-2L_3+L_4))+M(-M^2(N-1)(-2N+(2+N)L_3)-6N\beta(N-1)(-4(N-1)L_3+2N(L_4-2))+2M^2(N-1)\beta h_2^2(4NL_3^2-2L_3(4+N+NL_4))+3h_2(-4N(N-2N^2+N^3+2M^2\beta^2-NM^2\beta^2))-2N(-2(N-2N^2+N^3+2M^2\beta^2-NM^2\beta^2)+(N-1)NL_4)+L_3(N^5-8M^2\beta^2+N^3(7-M^2\beta^2)-4N(1+M^2\beta^2)+2N^2(5+M^2\beta^2)+4(N-1)^2NL_4)))L_5),$$

$$L_{44} = (-M^2(N-1)^2h_2^3(-2N+(2+N)L_4)+2M^2(N-1)\beta h_2^2((8-3N)N-2(4+N)L_4-2NL_4^2)-6N\beta(N-1)(-4N-4L_4^2+L_3(2N-4(N-1)L_4))+3h_2(4N(N-2N^2+N^3+2M^2\beta^2-NM^2\beta^2)+(N^5-8M^2\beta^2-4N(1+M^2\beta^2)+2N^2(5+M^2\beta^2))L_4+4(N-1)^2NL_4-2N((N-1)^2N+2(1-N-N^2+N^3-2M^2\beta^2+NM^2\beta^2)L_4)))L_5+N(3(N-1)N(2+N-4L_3)(L_3-L_4)^2+M^2h_2^2((10-N^2)+L_4(-6N+(N-2)L_4)+(N-1)L_3(-3N+2(N+1)L_4))+6M^2\beta h_2(-2NL_4+L_3^2(4-4NL_4)+NL_3(-2+(4+N)L_4))),$$

$$L_{45} = (2M(N-2)L_{22}L_2(-2M^2(N-1)^2h_2^3(-2N+(2+N)L_4)-3(N-1)N\beta(-4N+2L_3(N-2L_4)-4(N-1)L_4^2)+3M^2(N-1)\beta h_2^2((8-3N)N-2(4+N)L_4+NL_3(N+2L_4))+3h_2(4N(N-2N^2+N^3+2M^2\beta^2-NM^2\beta^2)+(N^5-8M^2\beta^2+N^3(-7+M^2\beta^2)-4N(1+M^2\beta^2)+2N^2(5+M^2\beta^2))L_4-2NL_4((N-1)^2N+(N-2N^2+N^3-2M^2\beta^2+NM^2\beta^2)L_4)))L_5-3(N-1)N((N-1)N(L_3-L_4)^2+M^2((-6+5N+N^2)L_3^2+2(N-1)(-3N+2(N-1)L_4)+L_4(N-1)(-6N+(2+N)L_4))+4M^2\beta h_2(2L_4(L_4-N)+2L_3^2(1-NL_4)+NL_3(-2+(4+N)L_4-2L_4^2)))L_5)+3Mh_1^2(M(N-2)^2(N-1)NL_1^2(8\beta(L_3-1)+(N-1)h_2(-6+N+4L_3))+L_1(N-2)(-2MNL_2(-2(N-1)h_2+\beta(N^2-4(N-1)L_4))+(4(N-1)^2NL_3^2-2M^2(N-1)^2h_2^2(-2N+(2+N))-2N(-2(N$$

$$-2N^2 + N^3 - 2M^2\beta^2 + NM^2\beta^2) + (N-1)L_4) + L_3(N^5 - 8M^2\beta^2 + N^3(-7$$
$$+M^2\beta^2) - 4N(1 + M^2\beta^2) + 2N^2(5 + M^2\beta^2) - 4NL_4) + 2M^2(N-1)\beta h_2(N($$
$$8 - 3N + NL_4) + 2L_3(-4 - N - NL_4)))L_5),$$

$$L_{46} = (2M(N-1)N\beta(-2(N-2L_4)L_4 + NL_3(-2 + (4+N)L_4 - 4L_4^2))),$$

$$L_{47} = (M(N-2)L_5L_{48} + M(N-2)L_2(-M^2(N-1)h_2^3(-2N + (2+N)L_4) + 2M^2(N$$
$$-1)\beta h_2^2 L_{49} - 2M(N-2)L_2(-2M^2(N+1)^2h_2^3(-2N + (N+2)L_4) - 3(N-1$$
$$)N\beta(-4N + 2L_3(N-2L_4) - 4(N-1)L_4^2) + 3M^2(N-1)\beta h_2^2((8-3N)N - 2($$
$$N+4)L_4) + 3h_2(4N(N - 2N^2 + N^3 - 2M^2\beta^2 + NM^2\beta^2) + (N^5 - 8M^2\beta^2 +$$
$$N^3(-7 + M^2\beta^2) - 4N(1 + M^2\beta^2) + 2N^2(5 + M^2\beta^2))L_4 - 2NL_3((N-1)N +$$
$$(N - 2N^2 + N^3 - 2M^2\beta^2 + NM^2\beta^2)L_4)))L_5),$$

$$L_{48} = (M^3(N-2)(N-1)^2h_1^4(L_1(-2N + (N+2)L_3) - (-2N + (2+N)L_4))L_5 +$$
$$M^2(N-1)h_1^3((N-1)(N-2)^2NL_1^2(-10 + N + 8L_3) + (N-2)^2(N-1)NL_2^2$$
$$(2 + N - 4L_4) + 2(N-2)L_1(2NL_2(2 - 3N + N^2)(-1 + N + L_3 - 2L_4) + M$$
$$(-2(N-1)h_2(-2N + (2+N)L_3 + \beta(-2NL_3^2 + N(8 - 3N + NL_4(N-1)))L_5$$
$$) - 2ML_2((N-1)h_2(-2N + (2+N)L_4) + \beta((8-3N)N + 4NL_4))L_5 + N(($$
$$2 - 3N + N^2)L_3^2 + 2(N-1)L_3(-3N + 2(N+1)L_4)(N-1)L_4(-6N + (10 +$$
$$N)L_4))L_5) + h_2(N(2 - 3N + N^2)^2(3N - M^2h_2^2)L_1^2(2 + N - 4L_3) + (N-2)^2$$
$$(3(N-1)N(2 + N - 4L_3) - 24M^2\beta h_2(L_4 - 1) - M^2(N-1)h_2^2) - (N-2)L_1$$
$$(3(N-1)(2 + N - 4L_3) + 2M^2(N-1)h_2^2(N - 1 - 2L_3 + L_4)))) + M(-M^2h_2^3$$
$$-6(N-1)\beta N(-4(N-1)L_3^2 + L_3(4 + 2N + N^2 - 4L_4) + 2N(L_4 - 2)) + 2M^2$$
$$(4NL_3^2 + N(8 - 3N + NL_4) - 2L_3(4 + N + NL_4)) + 3h_2(-4N(1 - N^2 + N^3 -$$
$$2M^2\beta^2 + N(1 + M^2\beta^2) + L_3(N^5 - 8M^2\beta^2 + N^3(-7 + M^2\beta^2) - 4N(1 + M^2\beta^2$$
$$) + 2N^2(5 + M^2\beta^2) + 4(N-1)^2NL_4)))L_5),$$

$$L_{49} = (6(-1+N)N\beta(-4N + (4 + 2N + N^2)L_4 + L_3(2N - 4(N-1)L_4)) + 3h_2(4N$$
$$(N - 2N^2 + N^3 + 2M^2\beta^2 - NM^2\beta^2) + (N^5 - 8M^2\beta^2 + N^3(-7 + M^2\beta^2) - 4N$$

$$
\begin{aligned}
& (1+M^2\beta^2)+2N^2(5+M^2\beta^2))L_4+4(N-1)NL_4^2-2NL_3((N-1)^2N+2(N \\
& -1)+(1-N-N^3-2M^2\beta^2+NM^2\beta^2)L_4)))L_5-N(3N(N-1)(L_3-L_4)^2 \\
& +M^2h_2^2(((-10+9N+N^2)L_3^2+L_4(-6N+(N-2)L_4)+2(N-1)L_3(-3N+ \\
& 2L_4(N-1)))+6M^2\beta h_2(-2NL_4+NL_3(-2+(4+N)L_4)))L_5)+h_1(-3(N \\
& -4L_4)-M^2(N-1)h_2^2(-6+N+4L_4))+2L_1(3N(2-3N+N^2)L_2(2M^2(\\
& -1+N-L_3)-2M^2\beta h_2(N^2-2(N-1)L_4))+M(-2M^2(N-1)^2h_2^3(-2N \\
& +(N+2)L_3)-3(N-1)(2N(N-1)+L_3(4+2N+N^2-4L_4))+3h_2(-2 \\
& N(N-2N^2+N^3-2M^2\beta^2+NM^2\beta^2)L_3^2-2N(-2(N-2N^2+N^3+2M^2\beta^2 \\
& -NM^2\beta^2)+NL_4)+L_3(N^5+N^3(-7+M^2\beta^2)-4N(1+M^2\beta^2)+2N^2(5 \\
& +M^2\beta^2)-2N(N-2N^2+N^3-2M^2\beta^2+NM^2\beta^2)L_4))L_5), \\
L_{50} \;=\; & (3(N-1)N((N-1)N(L_3-L_4)^2+M^2h_2^2((-6+5N+N^2)L_3^2+2(N-1)L_3 \\
& ((-3N+2(N+1)L_4)+L_4(N-1)(-6N+(2+N)L_4))+4M^2\beta h_2(2L_4(L_4 \\
& -N)+2L_3^2(1-NL_4)+NL_3(-2(4+N)L_4-2L_4^2)))L_5)-3Mh_1^2(M(N-2 \\
&)^2(N-1)NL_1^2(8(\beta(L_3-1)+(N-1)h_2)+(2-N)L_1(-2MN(2-3N+N^2 \\
&)L_2(-2(N-1)h_2+\beta(N^2-4(N-1)L_4))+(4(N-1)^2-2N(-2(N-2N^2 \\
& +N^3+2M^2\beta^2-NM^2\beta^2)+(N-1)^2NL_4)+L_3(N^5-8M^2\beta^2+N^3(-7+ \\
& M^2\beta^2)-4N(1+M^2\beta^2)+2N^2(5+M^2\beta^2)-4N(1-N^2+N^3-2M^2\beta^2+ \\
& N(-1+M^2\beta^2))L_4)+2M^2(N-1)\beta h_2(N-3N+NL_4)+2L_3(-4-N+N \\
& L_4)))L_5)+((-(N-2)L_2(4N(N-2N^2+N^3+2M^2\beta^2-NM^2\beta^2)+(N^5- \\
& 8M^2\beta^2+N^3(-7+M^2\beta^2)-4N(1+M^2\beta^2)+2N^2(5+M^2\beta^2))L_4-4N(1 \\
& -N^2+N^3-2M^2\beta^2+N(-1+M^2\beta^2))L_4^2)+2MN\beta(-2(N-2)L_4+NL_3(\\
& -2+(4+N)L_4))L_5)+M(N-1)h_2((N-2)^3(N-1)NL_2^2-2M(N-2)\beta \\
& L_2((8-3N)N+N^2L_3-2(4+N)L_4+2NL_4^2)L_5+N((-2+N+N^2)L_3^2+(\\
& N-1)L_4(-6N+(6+N)L_4))L_5))))-((N-1)N(F+h_1-h_2)(6(N-2)L_1 \\
& -6(N-2)L_2+ML_5)((N-2)^2(N-1)NL_1^2-(N-2)L_1(2N(2-3N+N^2
\end{aligned}
$$

$$)L_2 + M((N-1)h_2 - (N-1)h_1(-2(2+N)L_3 + N(6+N-4L_4)) - 2\beta(-$$
$$2NL_3^2 + L_3(4+4N-2NL_4) + N(-4+NL_4)))L_5) - M(N-2)L_2(-Nh_2(N($$
$$6+N) - 2(2+N)L_4) + 2\beta(-4N + 4(1+N)L_4 - 2NL_4^2))L_5 - (N(L_3-L_4)^2$$
$$-M^2(N-1)h_2^2(-4N + (6+N)L_3 + (N-2)L_4) - M^2\beta h_2(-8N - 4NL_3^2 + L_3$$
$$(12+4N+N^2-4NL_4)) + M^2h_1(2(N-1)h_2(-4N+(2+N)L_4) + \beta(-8N$$
$$+L_4(12+4N+N^2) + L_3(-4+4N+N^2-4NL_4))))L_5)),$$

$$\begin{aligned}
L_{51} &= (-M^3(N-2)(N-1)h_1^4(L_1(2N+(2+N)L_3) - L_2(-2N+(2+N)L_4))L_5 \\
&\quad -M^2(N-1)h_1^3((N-2)^2(N-1)NL_1^2(-10+N+8L_3) + (-2+N)^2NL_2^2(\\
&\quad 2+N-4L_4) + 2(N-2)L_1(2N(2-3N+N^2)L_2 + M(-2(N-1)h_2 + \beta(- \\
&\quad 2NL_3^2 + N(8-3N+NL_4) + L_3(-8-2N+4NL_4)))L_5) - 2ML_2)(-2(N- \\
&\quad 1)h_2(-2N+(2+N)L_4) + \beta((8-3N)N + NL_3(N-2L_4) - 2(4+N)L_4 + \\
&\quad 4NL_4^2))L_5 + N(2-3N+N^2)L_3^2 + (N-1)L_4(-6N+(10+N)L_4))L_5) + h_2 \\
&\quad (-N(2-3N+N^2)^2(3(N-1)N(2+N-4L_3) - 24M^2\beta h_2(L_4-1)) + M(\\
&\quad -M^2(N-1)^2h_2^3 - 6(N-1)N\beta + (-4(N-1)L_3^2 + 2NL_3(-2+L_4)) + 3 \\
&\quad h_2(-4N(1-N^2+N^3-2M^2\beta^2+N(-1+M^2\beta^2))L_3^2 - 2N(-2(N-2N^2+ \\
&\quad N^3 - 2M^2\beta^2 - NM^2\beta^2) + (N-1)NL_4) + L_3(N^5 - 8M^2\beta^2 + N^3(-7+M^2 \\
&\quad \beta^2) - 4N(1+M^2\beta^2) + 2N^2(5+M^2\beta^2) - 4(N-1)^2NL_4))), \\
L_{52} &= 2M(N-2)L_2(-2M^2(N-1)^2h_2^3(-2N+(2+N)L_4) - 3(N-1)N\beta(-4N+ \\
&\quad 2L_3(N-2L_4) + (4+2N+N^2)L_4 - 4(N-1)L_4^2) + 3M^2(N-1)\beta h_2^2((8- \\
&\quad 3N)N - 2(4+N)L_4 + NL_3(N-2L_4)) + 3h_2(4N(N-2N^2+N^3+2M^2\beta^2 \\
&\quad -NM^2\beta^2) + (N^5 - 8M^2\beta^2 + N^3(-7+M^2\beta^2) - 4N(1+M^2\beta^2) + 2N^2(5 \\
&\quad +M^2\beta^2))L_4 + 3Mh_2(M(N-2)^2(N-1)NL_1^2(8\beta(L_3-1) + (N-1)h_2(4 \\
&\quad L_3+N-6)) + L_1(-2MNL_2(-2(N-1)h_2(N-1-L_4) + \beta(N^2-4(N- \\
&\quad 1)L_4)) + (4(N-1)NL_3^2 - 2M^2(N-1)^2h_2^2(-2N+(2+N)L_3) - 2N - 2N \\
&\quad +(N-1)^2NL_4) + L_3(N^5 - 8M^2\beta^2 + N^3(-7+M^2\beta^2) - 4N(1+M^2\beta^2)
\end{aligned}$$

$$+2N^2(5+M^2\beta^2)-4N(1-N^2+N^3-2M^2\beta^2+N(-1+M^2\beta^2))L_4)+2$$
$$M^2h_2(N(8-3N+NL_4)+2L_3(-4-N+NL_4)))L_5)+(-(N-2)L_2(4N$$
$$(N-2N^2+N^3+2M^2\beta^2-NM^2\beta^2)+(N^5-8M^2\beta^2+N^3(-7+M^2\beta^2)$$
$$-4N(1+M^2\beta^2)+2N^2(5+M^2\beta^2))L_4-4N(1-N^2+N^3-2M^2\beta^2+N$$
$$(-1+M^2\beta^2))L_4)+M(N-1)h_2((N-2)^3(N-1)-2Mh_2((8-3N)N+$$
$$N^2L_3+2NL_4^2)L_5+N((-2+N+N^2)L_3^2+2(N-1)L_3(-3N+2(1+N)L_4)$$
$$+(N-1)L_4(-6N+(6+N)L_4))L_5),$$

$$L_{53} = (L_{54}+M(N-2)L_2(M^2(N-1)^2+6(N-1)\beta(-2N(L_3-2)+4L_3^2)+3$$
$$h_2(4N((N-1)-M^2(N-2)\beta^2)+L_4(N(N-1)+4(N-1)NL_4)+2N$$
$$L_3(-(N-1)N-2((N-2)+M^2\beta^2(N-1))L_4)))-(N-1)N(3N(N-$$
$$1)+6M^2\beta^2(-2(N-2L_3)+N(-2+(4+N)L_3)L_4)+M^2(N-1)h_2(($$
$$10+N)L_3^2+2L_3(-3N+2(1+N)L_4)))L_5)-3Mh_1^2(M(N-2)+(N-$$
$$2)L_1(2M(N-1)NL_2+L_3((N-1)^2N(-4+N(2+N))+4(N-1)NL_3$$
$$)-2N((N-1)^2N+2((N-1)^2(N+1)+M^2\beta^2(N-2)L_3)+2M^2\beta h_2$$
$$((8-3N)N+N(N-2L_3)L_4))L_5)-(N-2)L_2(4N((N-1)^2-M^2(N-$$
$$2)\beta^2)-4N((N-1)^2(N-2)\beta^2)L_4)+MN(N-1)((-2+N+N^2)h_2L_3^2$$
$$+L_4(N(N-1)+h_2(-6N+(6+N)L_4))+2L_3(h_2(-3(N-1)+2NL_4$$
$$)+N\beta(-2+(4+N-4L_4)L_4)))L_5)+h_1(-3(N-2)NL_1^2(-M^2(N-2)$$
$$h_2^2+N(N-1))+2(N-2)L_1(3(N-2)NL_2(2M^2(N-1-L_3)+2M^2$$
$$\beta^2(-N^2+2(N-1)L_3+2(N-1)L_4))+M(-2M^2h_2^3+3h_2(4N((N-$$
$$1)^2-M^2\beta^2(N-2))+L_3(N(N-1)^2-2N((N-1)N+M^2\beta^2(N-2$$
$$))L_3)-2N((N-1)^2+((N-1)^2N+M^2(N-2)\beta^2)L_3)L_4)-3N\beta(-4$$
$$L_3^2+L_3(4-4(N-1)L_4))+3M^2h_2^2(-2(4+N)L_3+2NL_3^2))L_5)+L_4)$$
$$+3h_2(4N((N-1)^2-M^2\beta^2)+(N-2)M^2\beta^2)L_4-2NL_3((N-1)^2N$$
$$+L_4)))L_5+3N(N-1)(N(N-1)+M^2(N-1)h_2^2((6+N)L_3^2+L_4(-6$$

$$N + (2+N)L_4)) + 4M^2\beta h_2(2L_4(N-L_4) + 2L_3^2(1-NL_4) + NL_3(-2$$
$$+(4+N-2L_4)L_4)))L_5),$$

$$L_{54} = M^3(N-2)(N-1)^2h_1^4(L_1(-2N+(2+N)L_3) + L_2(2N-(2+N)L_4))L_5$$
$$+M^2(N-1)h_1^3(NL_1^2(N-2)^2(N-1)(-10+8L_3+N) + (N-2)^2NL_2($$
$$2+N-4L_4) + 2(N-2)L_1(2(N-2)NL_2(N-1) + M(-2(N-1)h_2(-2$$
$$N+(2+N)L_3) + \beta((8-3N)N - 2L_3(4+N+NL_3) + N(N+4L_3)L_4))$$
$$L_5) - 2ML_2(-2(N-1)h_2(-2N+(2+N)L_4) + \beta(N(8-3N+NL_3) - 2$$
$$(4+N+NL_4)L_4 + 4NL_4^2))L_5 + (N-1)N((N-2)L_3^2 + 2L_3(-3N+2(N$$
$$+1)L_4) + L_4(-6N+(10+N)L_4))L_5) + h_2(N(2+3N+N^2) + (N-2)^2$$
$$NL_2^2 + (3(N-1)N - M^2(N-1)h_2^2(-10+N+8L_4)) - (N-2)L_1(2(N-$$
$$2)NL_2(3(N-1)N(2+N-4L_3) + 2M^2(N-1)h_2^2(-1+N-2L_3+L_4))$$
$$+M(M^2(N-1)^2h_2^3(2N-(2+N)L_3) - 6(N-1)(-4(N-1)L_3^2 + L_3(4+$$
$$N(N-2) - 4L_4) + 2N(-2+L_4)) + 3h_2(4N((N-1)^2N - M^2\beta^2(N-2))$$
$$+L_3((N-1)^2N(-4+N(N-2)) + M^2\beta^2(N-2))L_3) - 2N(N-1)L_4) +$$
$$2M^2(N-1)\beta^2h_2^2(4NL_3^2 - 2L_3(N-1)))L_5),$$

$$L_{55} = M^2(N-1)h_1^3((N-2)^2(N-1)NL_1^2 + (N-2)^2(N-1)NL_2^2 + 2(N-2)L_1$$
$$(2(N-2)(N-1)NL_2(-1+N+L_3-2L_4) + M(-2(N-1)h_2(-2N+(2$$
$$+N)L_3) + \beta((8-3N)N - 2L_3(4+N+NL_3) + N(N+4L_3)L_4))L_5) - 2$$
$$M(N-2)L_2(-2(N-1)h_2 + \beta(N(8-3N+NL_3) - 2(4+N+NL_3)L_4))$$
$$L_5 + (N-1)N((N-2)L_3^2 + 2L_3(-6N(10+N)L_4))L_5) + h_2(N(2-3N+$$
$$N^2) + L_2(N-2)(N-1) + 3M^2\beta h_2(N^2 - 4(N-1)L_3) + 2M^2(N-1)h_2^2$$
$$) + M(M^2(N-1)h_2^3(2N - L_3(N-2)) - 5N(N-1)(-4(N-1)L_3^2 + 2N$$
$$(L_4-2)) + 3h_2(4N((N-1)^2N - M^2(N-2)\beta^2) + L_3((N-1)N - 4N($$
$$(N-1)^2(N+1) + M^2(N-2)\beta^2)L_3) - 2N(N-1)L_4) + 2M^2\beta h_2^2(4NL_3^2$$
$$+N(8-3N) - 2L_3(4+N+NL_4))) + L_5),$$

$$
\begin{aligned}
L_{56} &= -(N-1)N(3(N-1)N(2+N-4L_3)(L_3-L_4)^2+6M^2\beta h_2(-2(N-2L_3)\\
&\quad L_3+NL_4+2M^2h_2^3(N-1)((10+N)L_3^2+2L_3(-3N+2(1+N)L_4)))L_5)\\
&\quad -3Mh_1^2(M(N-2)^3(N-1)^2Nh_2L_2^2+MNL_1^2(8\beta(L_3-1)+(N-1)h_2(\\
&\quad -6+N+4L_3))+(N-2)L_1(2M(N-2)(N-1)NL_2+(4N((N-1)^2\\
&\quad N-M^2(N-2)\beta^2)+L_3((N-1)^2N+4N(N-1)L_3)+2M^2(N-1)^2h_2^2\\
&\quad (2N-(N-2)L_3)-2\beta M^2(N-1)h_2((8-3N)N-2(4+N)L_3))L_5)-2\\
&\quad M^2(N-1)\beta h_2(N(8-3N+NL_3)-2(4+3)L_4+2NL_4^2))L_5+M(N-1\\
&\quad)N((-2+N+N^2)h_2L_3^2+L_4(-4\beta(N-2L_4)+(N-1)h_2(-6N+(6+N\\
&\quad)L_4))+2L_3(h_2+N\beta(-2+(4+N-4L_4)L_4)))L_5-2M(N-2)L_2(-2\\
&\quad M^2(N-1)h_2^3+3M^2h_2^2(N-1)((8-3N)N-2(4+N)L_4+NL_3)+3N\beta\\
&\quad (-4N+2L_3(N-2L_4))+3h_2(4N+((N-1)^2N(-4+N(2+N))+M^2\\
&\quad (2+N)\beta^2)L_4-2NL_3((N-1)^2N+((N-1)^2N+M^2\beta^2)L_4)))L_5+3N\\
&\quad (N-1)((N-1)N(L_3-L_4)^2+M^2h_2^2((6+N)L_3^2+2L_3+L_4(-6N+(2\\
&\quad +N)L_4))+4M^2\beta h_2(2L_4(-N+L_4)+2L_3^2(1-NL_4)+NL_3(-2+(4+\\
&\quad N-2L_4)L_4)))L_5),\\
L_{57} &= \frac{M(2+N)}{(4-N^2)^{1/2}},\\
L_{58} &= M(4-N^2)^{1/2},\\
L_{59} &= (12M^5(N-1)(N+2)^6(h_1-h_2)),\\
L_{60} &= (6BrM^3(2+N^3)(C_5^2M^2(8+N^3)+2C_5M^2N(2+N)(C_1N+C_2(3N-4))+\\
&\quad N^2(C_1^2(N-2)+4(C_6^2(N-2)+2C_2^2M^2(2+N))+C_1(4C_6(N-2)-2C_2M^2\\
&\quad N(2+N))))h_1^2(y-h_2))+C_1^2(2+N)-\frac{1}{(4-N^2)^{3/2}}(4MBr(2+N)^4(C_1-N\\
&\quad C_6)(-2C_5(N-2)+N(C_2(N-4)+C_1N))h_1^3(y-h_2))-\frac{1}{(4-N^2)^{1/2}}(2BrM^5\\
&\quad (2+N^3)(C_1-C_6N)h_1^4(y-h_2))+\frac{1}{(4-N^2)^{3/2}}(6MBr(C_5^2M^2(8+N^3)+2C_5\\
&\quad M^2N(C_1N+C_2(3N-4))+N^2(C_1^2(N-2)+4(C_6^2(N-2)+2C_2^2M^2)+C_1(\\
&\quad 4C_6(N-2)-2C_2)))yh_2^2)+\frac{1}{(4-N^2)^{3/2}}(2BrM^5(2+N^3)(C_1-C_6N)yh_2^4)-
\end{aligned}
$$

$$3y(\frac{1}{(4-N^2)^{3/2}}(4M^2(N-1)Nh_2) - 16BrM((2C_2+C_5)M(2+N)\cosh(\frac{L_{57}}{2}$$

$$(h_1-h_2)) + (C_1+2C_6)(4+N)\cosh(\frac{L_{57}}{2}(h_1-h_2))) + BrNL_{57}(2(\cosh($$

$$h_1L_{57}) - \cosh(h_2L_{57}))(2(-C_5(N-6)+(-2C_1+C_2(N-4))N) - N(C_1^2(N$$

$$-2) - M^2(N-2))\cosh(h_1L_{57}) - N(C_1^2(N-2) - (2C_2-C_5)^2M^2(N-2))$$

$$\cosh(h_1L_{57}) + 4(C_12C_6)(-2(C_5(N-6)+N)\cosh(\frac{L_{57}}{2}(h_1-h_2)) + (2C_2$$

$$+C_5)N(\cosh(\frac{L_{57}}{2}(3h_1-h_2)) + \cosh(\frac{L_{57}}{2}(h_1-3h_2))))\sinh(\frac{L_{57}}{2}(h_1-$$

$$h_2))L_{58})),$$

$$L_{61} = (2BrM^5(2+N)^5(C_1-C_6N)^2h_1^4(y-h_2)),$$

$$L_{62} = (2M^3y^2(3C_5^2M^2(8+N^3) + 2C_5M^2(N+2)(3C_2(3N-4) + 2C_6(N-2)Ny) +$$

$$C_1^2(4M^2y^2 + N^3(3+2M^2y) - N^2(6+M^2y)) + N^2(24C_2^2M^2 - 2C_6^2(N-2))$$

$$-2C_1N(C_2M^2(2+N)(3M^2+4y-N) + C_6(12N+M^2N^3y+4M^2y^2)))) +$$

$$2C_1^2N\cosh(2yL_{57}) + 8C_1C_6N\cosh(2yL_{57}) + 2C_6^2N\cosh(2yL_{57}) + 8C_2^2M^2$$

$$N\cosh(2yL_{57}) + 8C_2C_5M^2N\cosh(2yL_{57}) + 2C_5^2M^2N\cosh(2yL_{57}) - C_1^2N^2$$

$$\cosh(2yL_{57}) - 4C_1C_6M^2\cosh(2yL_{57}) - 4C_6^2N^2\cosh(2yL_{57}) + 4C_2C_5M^2$$

$$N^2\cosh(2yL_{57}) - 96C_2C_5M^2\cosh(h_1L_{57}) - 48C_5^2M^2\cosh(h_1L_{57}) + 32C_1$$

$$C_2M^2N\cosh(h_1L_{57}) + 64C_2^2M^2N\cosh(h_1L_{57}) + 16C_1C_5M^2N\cosh(h_1L_{57})$$

$$-16C_5^2M^2N\cosh(h_1L_{57}) + 16C_2^2M^2N^2\cosh(h_1L_{57}) + 8C_1C_5M^2N^2\cosh$$

$$(h_1L_{57}) + 16C_2C_5M^2N^2\cosh(h_1L_{57}) - 4C_5C_2M^2N^3\cosh(h_1L_{57}) - 2C_1^2N$$

$$\cosh(h_1L_{57}) - 8C_1C_6N\cosh(2h_1L_{57}) - 8C_6^2N\cosh(2h_1L_{57}) - 8C_5C_2M^2$$

$$N\cosh(2h_1L_{57}) - 4C_2^2M^2N^2\cosh(2h_1L_{57}) - C_5^2M^2N^2\cosh(2h_1L_{57}) - 96$$

$$C_1C_2M^2y\cosh(h_2L_{57}) - 48C_1C_5M^2y\cosh(h_2L_{57}) - 32C_1C_2M^2Ny\cosh$$

$$(h_2L_{57}) - 16C_1C_5M^2Ny\cosh(h_2L_{57}) + 32C_1C_2M^2N^2y\cosh(h_2L_{57}) + 4$$

$$C_1C_5M^2N^2y\cosh(h_2L_{57}) + 32C_6C_2M^2N^2y\cosh(h_2L_{57}) + 16C_5C_6M^2N^2$$

$$y\cosh(h_2L_{57}) - (-2(C_5(6-N) + C_1(-6y+N(2+y)))\sinh(yL_{57}) + (2C_2$$

$$+C_5)N\sinh(2h_1L_{57}) - 12C_5\sinh(h_1L_{57}) + 4C_1N\sinh(h_1L_{57}) - 2C_2N^2$$

$$\sinh(h_1L_{57}) - 2C_2N\sinh(2h_1L_{57}) - C_5N\sinh(2h_2L_{57}) - 12C_1y\sinh(h_2L_{57}) + 12C_6Ny\sinh(h_2L_{57}) - 2C_6N^2y\sinh(h_2L_{57})),$$

$$L_{63} = (2M^3(2+N)^3(-3Br(C_1+2C_6)y^2 + (N-2)N^2y^2 + M^2(2+N)(BrNy^2(6C_5 + 4C_5(C_1+2C_6)y + C_1(C_1+4C_6)y^2 + 8C_2(3C_5+yC_1)) - 2(-12+Bry^2(6C_5^2+4C_1C_5y)) + N^3(-6+Bry^2(C_6y(2C_2+C_6y)+2C_1)) - N^2(18+Bry^2(24C_2^2+6C_1C_5+2C_1^2y+4C_6C_5+2C_1C_6+2C_6^2y^2))) + 3Br(C_5^2M^2(8+N^3) + 2C_5M^2N(2+N)(C_1N+C_2(3N-4)) + N^2(C_1^2(N-2)+4(C_6^2(N-2) + 2C_2^2M^2) + C_1(4C_6(N-2)+C_1N)) - BrM^2(C_1+2C_6)^2(N^2-4)),$$

$$L_{64} = (96C_2C_5M^2\cosh(yL_{57}) + 48C_5^2\cosh(yL_{57}) - 32C_1C_2M^2N\cosh(yL_{57}) - 64C_2^2M^2N\cosh(yL_{57}) - 16C_5C_2M^2N\cosh(yL_{57}) + 16C_5^2M^2N\cosh(yL_{57}) - 16C_1C_2M^2N^2\cosh(yL_{57}) - 16C_2^2M^2N^2\cosh(yL_{57}) - 8C_1C_5M^2N^2\cosh(yL_{57}) - 16C_2C_5M^2N^2\cosh(yL_{57}) - 4C_5^2M^2N^2\cosh(yL_{57}) + 88C_2^2M^2N^3\cosh(yL_{57}) + 4C_2C_5M^2N^3\cosh(yL_{57}) + 96C_1C_2M^2y\cosh(yL_{57}) + 48C_1C_5M^2\cosh(yL_{57}) - 48C_5C_6M^2N\cosh(yL_{57}) - 88C_1C_2M^2N^2y\cosh(yL_{57}) - 4C_1C_5M^2N^2y\cosh(yL_{57}) - 32C_2C_6yM^2N^2\cosh(yL_{57}) - 16C_5C_6yM^2N^2\cosh(yL_{57}) + 8C_2C_6yM^2N^3\cosh(yL_{57}) + 4C_5C_6yM^2N^3\cosh(yL_{57}) + 8C_6^2yN\cosh(2yL_{57}) + 8C_2C_5yM^2N\cosh(2yL_{57}) - C_1^2N^2\cosh(2yL_{57}) - 4C_1C_6N^2\cosh(2yL_{57}) + 4C_2^2M^2N^2\cosh(2yL_{57}) + 4C_2C_5M^2N^2\cosh(2yL_{57}) + 4C_2C_5M^2N^2\cosh(2yL_{57}) + C_5^2M^2N^2\cosh(2yL_{57}) - 96C_1C_2M^2y\cosh(h_1L_{57}) - 48C_1C_5M^2y\cosh(h_1L_{57}) - 32C_1C_2M^2Ny\cosh(h_1L_{57}) - 16C_1C_2M^2Ny\cosh(h_1L_{57}) + 48C_5C_6M^2Ny\cosh(h_1L_{57}) + 32C_1C_2M^2N^2y\cosh(h_1L_{57}) + 16C_5C_6M^2N^2y\cosh(h_1L_{57}) - 4C_5C_6M^2N^3y\cosh(h_1L_{57}) + 64C_2^2M^2N\cosh(h_2L_{57}) + 16C_1C_5M^2N\cosh(h_2L_{57}) - 16C_5^2M^2N\cosh(h_2L_{57}) - 2C_1^2N\cosh(2h_2L_{57}) - 8C_1C_6N\cosh(2h_2L_{57}) - 8C_6^2N\cosh(2h_2L_{57}) - 8C_2^2M^2N\cosh(2h_2L_{57}) - 8C_2C_5M^2N\cosh(2h_2L_{57}) - 2C_5^2M^2N\cosh(2h_2L_{57}) + 4C_1C_6N^2\cosh(2h_2L_{57}) + 4C_6^2M^2N\cosh(2h_2L_{57}) - C_5^2M^2N^2\cosh(2h_2L_{57}) + 8M(N-6)(NC_6-C_1)\sinh(\frac{L_{57}}{2}(h_1-h_2))((2C_6+C_1)\cosh($$

$$\frac{L_{57}}{2}(h_1 - h_2)) + (2C_2 + C_5)M(N+2)\sinh(\frac{L_{57}}{2}(h_1 - h_2)))h_2 + (2C_6 + C_1)$$
$$\sinh(yL_{57}) + (2C_2 + C_5)N\sinh(2yL_{57}) - 12C_1 y\sinh(h_1 L_{57}) - 2C_1 Ny\sinh(h_1 L_{57}$$
$$) - 2C_6 N^2 y\sinh(h_1 L_{57}) + 4C_1 Ny\sinh(h_2 L_{57}) + 8C_2 Ny\sinh(h_2 L_{57}) - 2C_2 N^2 \sinh$$
$$(h_2 L_{57}) - 2C_2 N\sinh(2h_2 L_{57}) - C_5 N\sinh(2h_2 L_{57})L_{58}),$$

$$
\begin{aligned}
L_{65} &= ((N-2)(4-N^2)^{1/2}) + \frac{1}{(4-N^2)^{1/2}}(4M^5(2+N^3)(C_1 - C_6 N)^2 h_1^4(y-h_2)) - \\
&\frac{1}{(N-2)(4-N^2)^{1/2}}(12M^3(-C_5^2 M^2(8+N^3) - 2C_5 M^2 N(2+N) + N^2(C_1^2(N-2) \\
&-y(\cosh(2(y+h_1+h_2)L_{57}) + \sinh(2(y+h_1+h_2)L_{57})) - (8M^5(2+N)^3(C_1 \\
&-C_6 N)(-2C_5(N-2) + N(C_2 - C_1 N))y(\cosh(2(y+h_1+h_2)L_{57}) + \sinh(2(y+ \\
&h_1+h_2)L_{57}))h_2^3), \\
L_{66} &= (4M^3(2+N^2)y^2(-3C_5^2 M^2(8+N^3) - 2C_5 M^2(2+N)(3C_2 N(3N-4) + 2C_6 \\
&(N-2)Ny + C_1(3N^2 + 4y - 2Ny)) + C_1^2(-4M^2y^2 - N^3(3+2M^2y) + N^2(\\
&-24C_2^2 M^2(2+N) + 2C_2 C_6 M^2(-8-2N+N^2)y + C_6^2(N-2)(-12+M^2(2 \\
&+N)y^2))2C_1 N(C_2 M^2(2+N) + C_6(12N + M^2N^3y + 4M^2y^2 - N^2(6+M^2 \\
&(y-2)y))))), \\
L_{67} &= (\frac{1}{(N-2)(4-N^2)^{1/2}}(12M^3(2+N)^2(-C_5^2 M^2(8+N^3) - 2C_5 M^2 N(2+N) + \\
&N^2(-C_1^2(N-2) - 4(C_6^2(N-2) + 2C_2^2 M^2(2+N)) + C_1(-4C_6 + 2C_2 M^2 N(\\
&2+N))))(\cosh(2(y+h_1+h_2)L_{57}) + \sinh(2(y+h_1+h_2)L_{57}))h_1^2) + \\
&\frac{1}{(N-2)(4-N^2)^{1/2}}(8M^5(2+N)^3(NC_6 - C_1)(-2C_5(N-2) + N(C_2(N-2 \\
&)(\cosh(2(y+h_1+h_2)L_{57}) + \sinh(2(y+h_1+h_2)L_{57})))h_1^2)) + \frac{1}{(4-N^2)^{1/2}} \\
&(4M^5(2+N)^3(NC_6 - C_1)(\cosh(2(y+h_1+h_2)L_{57}) + \sinh(2(y+h_1+h_2)L_{57} \\
&))h_1^2)) - \frac{1}{(N-2)(4-N^2)^{1/2}}(12M^3(2+N)^2(C_5^2 M^2(8+N^3) - 2C_5 M^2 N - 4(\\
&C_6^2(N-2) + C_2(2+N) + 2C_2^2 M^2(2+N))))y(\cosh(2(y+h_1+h_2)L_{57}) + \sinh \\
&(2(y+h_1+h_2)L_{57})))h_2^2)) - \frac{1}{(4-N^2)^{1/2}}(4M^5(2+N)^3(NC_6 - C_1)^2 y(\cosh(2 \\
&(y+h_1+h_2)L_{57}) + \sinh(2(y+h_1+h_2)L_{57})))h_2^4)),
\end{aligned}
$$

$$
\begin{aligned}
L_{68} = {} & ((C_2C_5M^2N\cosh(2(y+h_1)L_{57}) + 2C_5^2M^2N\cosh(2(y+h_1)L_{57}) - C_1^2N^2\cosh \\
& (2(y+h_1)L_{57}) - 4C_1C_6N^2\cosh(2(y+h_1)L_{57}) + 4C_2^2M^2N^2\cosh(2(y+h_1)L_{57} \\
&) + 4C_2C_5M^2N^2\cosh(2(y+h_1)L_{57}) + C_5^2M^2N^2\cosh(2(y+h_1)L_{57}) - 8C_1C_6 \\
& N\cosh(2(y+h_1)L_{57}) - 2C_5^2M^2N\cosh(2(y+h_1)L_{57}) + C_1^2N^2\cosh(2(y+h_1) \\
& L_{57}) + 4C_1C_6N^2\cosh(2(h_1+h_2)L_{57}) - 4C_2^2M^2N^2\cosh(2(h_1+h_2)L_{57}) - 4C_2 \\
& C_5MN(4-N^2)^{1/2}\cosh(2(h_1+h_2)L_{57}) - 8C_6^2N\cosh(2(2y+h_1+h_2)L_{57}) - 8 \\
& C_2^2M^2N\cosh(2(2y+h_1+h_2)L_{57}) - 8C_2C_5M^2N\cosh(2(2y+h_1+h_2)L_{57}) - 2 \\
& C_5^2M^2N\cosh(2(2y+h_1+h_2)L_{57}) + C_1^2N^2\cosh(2(2y+h_1+h_2)L_{57}) + 4C_1C_6 \\
& N^2\cosh(2(2y+h_1+h_2)L_{57}) + 4C_6^2N^2\cosh(2(2y+h_1+h_2)L_{57}) - 4C_2^2M^2N^2 \\
& \cosh(2(2y+h_1+h_2)L_{57}) - 4C_2C_5M^2N^2\cosh(2(2y+h_1+h_2)L_{57}) - C_5^2M^2N^2 \\
& \cosh((2y+2h_1+h_2)L_{57}) + 96C_2C_5M^2\cosh((2y+2h_1+h_2)L_{57}) + 48C_5^2M^2 \\
& \cosh((2y+2h_1+h_2)L_{57}) + 32C_1C_2M^2N\cosh((2y+2h_1+h_2)L_{57}) - 64C_2^2M^2 \\
& N\cosh((2y+2h_1+h_2)L_{57}) - 16C_1C_5M^2N\cosh((2y+2h_1+h_2)L_{57}) + 16C_5^2 \\
& M^2N\cosh((2y+2h_1+h_2)L_{57}) + 16C_1C_2M^2N^2\cosh((2y+2h_1+h_2)L_{57}) - 1 \\
& 6C_2^2M^2N^2\cosh((2y+2h_1+h_2)L_{57}) + 8C_5^2M^2N^2\cosh((2y+2h_1+h_2)L_{57}) + \\
& 8C_2^2M^2N^3\cosh((2y+2h_1+h_2)L_{57}) + 4C_2C_5M^2N^3\cosh((2y+2h_1+h_2)L_{57}) \\
& +32C_2C_6MN(4-N^2)^{1/2}\cosh((2y+2h_1+h_2)L_{57}) - 4C_1C_2M^2N(4-N^2)^{1/2} \\
& \cosh((2y+2h_1+h_2)L_{57}) - 8C_2C_6M^2N^2\cosh((2y+2h_1+h_2) \\
& L_{57}) + 2C_5^2M^2N\cosh(2(y+h_1+2h_2)L_{57}) - C_1^2N^2\cosh(2(y+h_1+2h_2)L_{57}) + \\
& 4C_1C_2M^2N^2\cosh(2(y+h_1+2h_2)L_{57}) + 4C_1C_2MN(4-N^2)^{1/2}\cosh(2(y+h_1 \\
& +2h_2)L_{57}) + 8C_2C_6MN(4-N^2)^{1/2}\cosh(2(y+h_1+2h_2)L_{57}) - 96C_2C_5M^2 \\
& \cosh((y+2h_1+2h_2)L_{57}) - 48C_5^2M^2\cosh((y+2h_1+2h_2)L_{57}) + 32C_1C_2M^2 \\
& \cosh((y+2h_1+2h_2)L_{57}) + 16C_2C_5M^2N^2\cosh((y+2h_1+2h_2)L_{57}) + 4C_5^2 \\
& M^2N^2\cosh((y+2h_1+2h_2)L_{57}) + 8C_2^2M^2N^3\cosh((y+2h_1+2h_2)L_{57}) - 4 \\
& C_2C_5M^2N^3\cosh((y+2h_1+2h_2)L_{57}) - 16C_1C_2M(4-N^2)^{1/2}\cosh((y+2h_1
\end{aligned}
$$

$+2h_2)L_{57}) - 32C_2C_6MN(4-N^2)^{1/2}\cosh((y+2h_1+2h_2)L_{57}) + 4C_1C_2M$

$N^2(4-N^2)^{1/2}\cosh((y+2h_1+2h_2)L_{57})),$

$$
\begin{aligned}
L_{69} \;=\;& ((-2C_1^2N - 8C_1C_6N - 8C_6^2N - 2C_5^2M^2N + C_1^2N^2 + 4C_6^2N^2 + 4C_1C_6N^2 - C_5^2\\
& M^2N^2 + 2C_1^2M(4-N^2)^{1/2})\cosh(2(y+h_2)L_{57}) + (2C_1^2N + 8C_1C_6N + 8C_6^2N\\
& +2C_5^2M^2N - C_1^2N^2 - 4C_1C_6N^2 - 4C_6^2N^2 - 2C_1C_5MN(4-N^2)^{1/2} + 4C_5C_6\\
& MN(4-N^2)^{1/2})\cosh(2(h_1+h_2)L_{57}) + (2C_1^2N + 8C_1C_6N + 8C_6^2N + 2C_5^2M^2\\
& N - NC_1^2 + 4C_1C_6N^2 - 4C_6^2N + 5C_5^2M^2N^2 + 2C_1C_5MN(4-N^2)^{1/2})\cosh(2\\
& (2y+h_1+h_2)L_{57}) + (2C_5^2M^2N + 4C_1C_6N^2 + 4C_6^2N^2 - C_5^2M^2N^2 - 2C_1C_5M\\
& N(4-N^2)^{1/2})\cosh(2(y+2h_1+h_2)L_{57}) + (-48C_1C_5M^2y + 4C_1C_5M^2N^2y +\\
& 16C_5C_6M^2N^2y + 48C_5^2M^2 + 16C_1C_5M^2N - 4C_5^2M^2N^2 - 48C_5C_6M(4-N^2\\
&)^{1/2} + 4C_1C_5MN(4-N^2)^{1/2} + 8C_5C_6MN(4-N^2)^{1/2})\cosh((y+2h_1+2h_2)\\
& L_{57}) + (48C_5^2M^2 - 16C_1C_5M^2N - 4C_5^2M^2N + 24C_1C_5M - 48C_5C_6M - 4C_1\\
& C_5MN(4-N^2)^{1/2} - 48C_5C_6M^2y)\cosh((3y+2h_1+2h_2)L_{57}) + (16C_1C_5M^2\\
& Ny - 48C_5C_6NM^2y - 4C_1C_5M^2N^2y + 16C_5C_6M^2N^2y + 16C_1C_5M^2N - 16\\
& C_5^2M^2N + 8C_1C_2M^2N^2)\cosh((2y+2h_1+3h_2)L_{57})),
\end{aligned}
$$

$$
\begin{aligned}
L_{70} \;=\;& ((-C_2^2M^2(2+N)(\cosh(2yL_{57}) + \sinh(2yL_{57}))(\cosh(h_1L_{57}) + \cosh(h_2L_{57})\\
& - \cosh((2h_1+h_2)L_{57}) + \cosh((h_1+2h_2)L_{57}) + \cosh(h_1L_{57}) + \sinh(h_2L_{57}\\
&) - 8\cosh((2h_1+h_2)L_{57}) + 2N\cosh((h_1+2h_2)L_{57})(\cosh(h_1L_{57}) + \cosh(\\
& h_2L_{57}) - 8\cosh((h_1+h_2)L_{57}) + 2N\cosh((h_1+h_2)L_{57}) + \sinh(h_1L_{57}) +\\
& \sinh(h_2L_{57}) - 8\sinh((h_1+h_2)L_{57}) + 2N\sinh((h_1+h_2)L_{57}) + \sinh((2h_1+h_2\\
&)L_{57}) + \sinh((h_1+2h_2)L_{57}) - \cosh(2(y+2h_1+h_2)L_{57}) + \cosh(2(y+h_1+\\
& 2h_2)L_{57}) + \sinh(2(y+h_1)L_{57}) - \sinh(2(y+h_2)L_{57}) - \sinh(2(y+2h_1+h_2)L_{57})\\
& + \sinh(2(y+h_1+2h_2)L_{57})))),
\end{aligned}
$$

$$
\begin{aligned}
L_{71} \;=\;& (C_6(N-2)(\cosh(2(y+h_1)L_{57}) - \cosh(2(y+h_2)L_{57}) - \cosh(2(y+2h_1+h_2)\\
& L_{57}) + \cosh(2(y+h_1+2h_2)L_{57}) + \sinh(2(y+h_1)L_{57}) - \sinh(2(y+h_2)L_{57}) -
\end{aligned}
$$

$$
\begin{aligned}
&\sinh(2(y+2h_1+h_2)L_{57})+\sinh(2(y+h_1+2h_2)L_{57}))+MC_2((4-N^2)^{1/2}\cosh\\
&(2(y+h_1)L_{57})-(4-N^2)^{1/2}\cosh(2(y+h_2)L_{57})+(4-N^2)^{1/2}\cosh(2(y+2h_1+\\
&h_2)L_{57})-(4-N^2)^{1/2}\cosh(2(y+h_1+2h_2)L_{57})+(4-N^2)^{1/2}\sinh(2(y+h_1)L_{57}\\
&)-(4-N^2)^{1/2}\sinh(2(y+h_2)L_{57})+(4M(2+N)+(N-4)(4-N^2)^{1/2})(\cosh((2y\\
&+2h_1+h_2)L_{57})+\sinh((2y+2h_1+h_2)L_{57}))-(4-N^2)^{1/2}\sinh(2(y+h_1+2h_2)\\
&L_{57})+(4M(2+N)-(N-4)(4-N^2)^{1/2})(\cosh((2y+2h_1+3h_2)L_{57})+\sinh((2y\\
&+2h_1+3h_2)L_{57}))-(4M(2+N)-(N-4)(4-N^2)^{1/2})(\cosh((2y+3h_1+2h_2)\\
&L_{57})+\sinh((2y+3h_1+2h_2)L_{57})),+3h_2)L_{57}))-(4M(2+N)-(N-4)(4-N^2\\
&)^{1/2})(\cosh((2y+3h_1+2h_2)L_{57})+\sinh((2y+3h_1+2h_2)L_{57})),\\
L_{72} \;=\;& ((2C_1^2N+8C_1C_6+8C_6^2N)\cosh(2(y+h_1)L_{57})+(8C_2C_6M^2N^2(4-N^2)^{1/2}-96C_1\\
&C_2M^2y-48C_1C_5M^2y-16C_1C_5NM^2y+48C_5C_6NM^2y+4C_1C_5N^2M^2y-4C_5C_6N^3\\
&M^2y)\cosh((y+2h_1+2h_2)L_{57})+(96C_2C_5M^2-48C_5^2M^2+32C_1C_2M^2N+16C_1C_5\\
&M^2N-16C_5^2M^2N+16C_1C_2M^2N^2+16C_2^2M^2N^2+16C_1C_2MN(4-N^2)^{1/2}+32C_1C_2\\
&M^2N^3(4-N^2)^{1/2}+16C_1C_2MN(4-N^2)^{1/2}-8C_2C_6M^2N(4-N^2)^{1/2}-48C_1C_5M^2y\\
&+4C_1C_5N^2M^2y-4C_5C_6N^3M^2y)\cosh((3y+2h_1+2h_2)L_{57})+(96C_1C_5M^2+48C_5^2\\
&M^2+32C_1C_5NM^2+32C_1C_2NM^2+16C_5^2NM^2-16C_1C_2N^2M^2-8C_1C_5N^2M^2-4\\
&C_5^2N^2M^2+8C_2^2N^3M^2+16C_1C_2MN(4-N^2)^{1/2}-32C_2C_6MN(4-N^2)^{1/2}+4C_1C_2M\\
&N^2(4-N^2)^{1/2})\cosh((2y+2h_1+3h_2)L_{57})+(96C_1C_2M^2y+48C_1C_5M^2y+32C_1C_2\\
&M^2Ny+16C_1C_5M^2Ny-96C_2C_6M^2Ny-8C_1C_2M^2N^2y-4C_1C_5M^2N^2y-16C_5C_6N^2\\
&M^2y+4C_5C_6M^2y+48C_1C_5M^2y+16C_1C_5NM^2y-8C_1C_2M^2N^2y-4C_1C_5N^2M^2y-\\
&32C_2C_6M^2yN^2+4C_5C_6M^2N^3y)\cosh((2y+3h_1+2h_2)L_{57})(-8C_1C_6N-8C_6^2N-8C_2^2\\
&M^2N-8C_5C_2M^2N-2C_5^2M^2N+4C_1C_6N^2-4C_2^2M^2N^2+C_1C_2MN^2(4-N^2)^{1/2}+8\\
&C_6C_2MN^2(4-N^2)^{1/2}-8C_5C_6NM^2+C_1^2N+4C_6^2N+96C_5C_6M^2-32C_1C_2MN^2-1\\
&6C_1C_5MN^2-16C_1C_2M^2N^2-16C_2^2M^2N^2+4C_5C_2M^2N^3)\cosh((2y+2h_1+h_2)L_{57}\\
&)),
\end{aligned}
$$

$$L_{73} = ((-8C_2C_5M^2N(4-N^2)^{1/2} + 2C_1^2N + 8C_2C_6N + 8C_6^2N + 8C_2^2NM^2 + 8C_1C_5NM^2 - C_1^2 N^2 - 4C_6^2N^2 + 4C_2C_5N^2M^2 - 96C_2C_5M^2 + 8C_1C_5M^2N^2 + 4C_5^2M^2 - 4C_2C_5M^2N^3 - 48C_1C_5M^2Ny + 4C_1C_5M^2N^2y + 4C_2C_5M^2N^3 + 164C_1C_2MN(4-N^2)^{1/2} + 32C_2C_6N M(4-N^2)^{1/2} - 4C_1C_2MN^2 + 16C_1C_5M^2N)\cosh((3y+2h_1+2h_2)L_{57}));$$

$$L_{74} = ((24C_1C_5M(4-N^2)^{1/2} + 48C_6C_5M(4-N^2)^{1/2} - 4C_1C_5MN(4-N^2)^{1/2} - 8C_6C_5M(4 -N^2)^{1/2} - 48C_5^2M^2 + 16C_1C_5M^2N - C_5^2M^2N + 16C_1C_5M^2N + 16C_1C_5M^2N + C_5^2M^2 N^2 - 24C_1C_5M^2 + 8C_6C_5M^2N)\cosh((2y+3h_1+2h_2)L_{57}) + C_2^2M^2N(2+N)(\cosh(y L_{57}) + \cosh(h_1L_{57}) + \cosh((2y+h_1)L_{57}) - \cos((y+2h_1)L_{57}) - \sinh(yL_{57}) + \sinh(h_1 L_{57}) + \sinh((2y+h_1)L_{57}) + \sinh((y+2h_1)L_{57})) + (-2C_1^2N - 8C_1C_6N - 8C_6^2N - 2C_5^2 NM^2 + C_1^2N^2 + 4C_1C_6N^2 + 4C_6^2N^2 - C_5^2M^2N^2 + 2C_1C_5MN(4-N^2)^{1/2} + 4C_5C_6M(4- N^2)^{1/2})\sinh(2(y+h_2)L_{57}) + (2C_1^2N + 8C_1C_6N + 8C_6^2N + 2C_5^2M^2N - C_1^2N^2 - 4C_6^2N^2 +C_5^2M^2N^2 - 4C_5C_6MN)\sinh(2(h_1+h_2)L_{57}) + (C_1C_5MN(4-N^2)^{1/2} - 4C_5C_6MN(4 -N^2)^{1/2} - 48C_5^2M^2 - 16C_1C_5M^2N + 16C_5^2M^2N - 8C_1C_5M^2N^2 - 4C_5^2M^2N - 8C_1C_5M^2 N^2 - 24C_1C_5M(4-N^2)^{1/2})\sinh((y+2h_1+2h_2)L_{57}) + (4C_5C_6M^2N^3y + 48C_5^2M^2 + 16 C_1C_5M^2N^2 - 4C_5^2M^2N^2 + 24C_1C_5(4-N^2)^{1/2}M + 48C_5C_6M(4-N^2)^{1/2} - 4C_1C_5MN(4 -N^2)^{1/2})\sinh((3y+2h_1+2h_2)L_{57})),$$

$$L_{75} = (C_2M(2+N)(\cosh(2yL_{57}) + \sinh(2yL_{57}))(\cosh(h_1L_{57}) - \cosh(h_2L_{57}) - \cosh((2h_1+h_2)L_{57}) + \cosh((h_1+2h_2)L_{57}) + \sinh(h_1L_{57}) - \sinh(h_2L_{57}) - \sinh((2h_1+h_2)L_{57}) + \sinh ((h_1+2h_2)L_{57}))(N\cosh(h_1L_{57}) - N\cosh(h_2L_{57}) - 12\cosh((h_1+h_2)L_{57}) - 6N\cosh((h_1 +h_2)L_{57}) + N^2\cosh((h_1+h_2)L_{57}) + N\cosh((2h_1+h_2)L_{57}) + N\cosh((h_1+2h_2)L_{57}) + N\sinh(h_1L_{57}) + N\sinh(h_2L_{57}) + 12\sinh((h_1+h_2)L_{57}) - 6N\sinh((h_1+h_2)L_{57}) + N^2 \sinh((h_1+h_2)L_{57}) + N\sinh((2h_1+h_2)L_{57}) + 12\sinh((h_1+h_2)L_{57}) - 6N\sinh((h_1+h_2) L_{57}) + N^2\sinh((h_1+h_2)L_{57}) + N\sinh((2h_1+h_2)L_{57}) + N\sinh((2h_1+h_2)L_{57}) + N\sinh ((h_1+2h_2)L_{57}) + 2C_6(4-N^2)^{1/2}(\cosh(2yL_{57}) + \sinh(2yL_{57}))(\cosh(h_1L_{57}) - \cosh(h_2L_{57})$$

$$-\sinh(h_1 L_{57}) + \sinh(h_2 L_{57}))(N\cosh(h_1 L_{57}) + N\cosh(h_2 L_{57}) + 12\cosh((h_1 + h_2)$$
$$L_{57}) - 2N\cosh((h_1 + h_2)L_{57}) + 12\cosh(2(h_1 + h_2)L_{57}) - 2N\cosh(2(h_1 + h_2)L_{57})$$
$$+N\cosh((3h_1 + 2h_2)L_{57}) + N\cosh((2h_1 + 3h_2)L_{57}) + N\sinh(h_1 L_{57}) + N\sinh(h_2$$
$$L_{57}) + N\sinh((3h_1 + 2h_2)L_{57}) + N\sinh((2h_1 + 3h_2)L_{57})),$$

$$\begin{aligned}
L_{76} \;=\;& ((C_1 + 2C_6)(4 - N^2)^{1/2}(-1 + \cosh((h_1 + h_2)L_{57}) + \sinh((h_1 + h_2)L_{57})) + 2C_2 \\
& M(2 + N) + C_5 M(2 + N)(1 + \cosh((h_1 + h_2)L_{57}) + \sinh((h_1 + h_2)L_{57})))((\cosh \\
& ((2y + h_1 + h_2)L_{57}) + \sinh((2y + h_1 + h_2)L_{57}))) + L_{57}(C_5^2 M^2(2 + N)(\cosh(2y \\
& L_{57}) + \sinh(2yL_{57}))(-\cosh(h_1 L_{57}) + \cosh(h_2 L_{57}) + \cos((2h_1 + h_2)L_{57}) - \cosh \\
& ((h_1 + 2h_2)L_{57}) - \sinh(h_1 L_{57}) + \sinh(h_2 L_{57}) + \sinh((2h_1 + h_2)L_{57})) - \sinh((h_1 \\
& +2h_2)L_{57})))(N\cosh(h_1 L_{57}) + N\cosh(h_2 L_{57}) + 24\cosh((h_1 + h_2)L_{57}) - 4N \\
& \cosh((h_1 + 2h_2)L_{57}) + N\cosh((h_1 + 2h_2)L_{57}) + N\sinh(h_1 L_{57}) + N\sinh(h_2 L_{57}) \\
& +24\sinh((h_1 + h_2)L_{57}) + 4N\sinh((h_1 + h_2)L_{57}) + N\sinh((2h_1 + h_2)L_{57}) + N \\
& \sinh((h_1 + 2h_2)L_{57})) + N(C_1^2(N - 2)(\cosh((h_1 + y)L_{57}) - \cosh(2(y + h_2)L_{57}) + \\
& \sinh(2(y + h_1)L_{57}) - \sinh(2(y + h_2)L_{57})) - C_1(C_1 + 2C_6)(\cosh((2y + 2h_1 + h_2) \\
& L_{57}) - \cosh((2y + 2h_1 + 3h_2)L_{57}) - \cosh((2y + h_1 + 2h_2)L_{57}) + \cosh((2y + 3h_1 \\
& +2h_2)L_{57}) + \sinh((2y + 2h_1 + h_2)L_{57}) - \sinh((2y + 2h_1 + 3h_2)L_{57}) - \sinh((2y \\
& +h_1 + 2h_2)L_{57}) + \sinh((2y + 3h_1 + 2h_2)L_{57})))L_{58}))),
\end{aligned}$$

$$\begin{aligned}
L_{77} \;=\;& ((-8C_1C_5MN(4 - N^2)^{1/2} + 48C_5C_6M^2 + C_1C_5M^2N^2 - 16C_5C_6M^2N^2y + 4C_5C_6 \\
& M^2N^3y + 48C_1C_5M^2y - 16C_1C_5M^2Ny + 4C_1C_5M^2N^2y - 4C_5C_6M^2N^3y)\sinh((3 \\
& y + 2h_1 + 2h_2)L_{57}) + (48C_5^2M^2 + 16C_1C_5M^2N - 16C_5^2M^2N + 8C_1C_5M^2N^2 + 4C_5^2 \\
& M^2N^2 + 48C_5C_6M(4 - N^2)^{1/2} - 4C_1C_5MN(4 - N^2)^{1/2} - 8C_5C_6MN(4 - N^2)^{1/2}) \\
& \sinh((2y + h_1 + 2h_2)L_{57}) + (48C_5^2M^2 + 16C_1C_5M^2N + 8C_1C_5M^2N^2 - 24C_1C_5M \\
& (4 - N^2)^{1/2} - 48C_5C_6M(4 - N^2)^{1/2} + 4C_1C_5NM(4 - N^2)^{1/2} + 8C_5C_6MN(4 - N^2)^{1/2} \\
&)\sinh((2y + 3h_1 + 2h_2)L_{57})),
\end{aligned}$$

$$
\begin{aligned}
L_{78} &= ((-C_5M(2+N)(\cosh(yL_{57})-\cosh(h_1L_{57})-\cosh((2y+h_2)L_{57})+\cosh((y+2\\
&\quad h_2)L_{57})+\sinh(yL_{57})-\sinh(h_1L_{57})-2\sinh((2y+h_1)L_{57})+\sinh((y+2h_1)L_{57})\\
&\quad)(N\cosh(yL_{57})-N\cosh(h_1L_{57})-12\cosh((y+h_1)L_{57})-6N\cosh((y+h_1)L_{57})\\
&\quad +N^2\cosh((2y+h_1)L_{57})+N\cosh((y+2h_1)L_{57})+N\sinh(yL_{57})+N\sinh(h_1L_{57}\\
&\quad)+12\sinh((y+h_1)L_{57})-6N\sinh((y+h_1)L_{57})+N\sinh((y+2h_1)L_{57}))(\cosh(2\\
&\quad h_2L_{57})+\sinh(2h_2L_{57}))),\\
L_{79} &= ((N(4-N^2)^{1/2}\cosh(2(y+h_2)L_{57})-(N(4-N^2)^{1/2}\cosh(2(h_1+h_2)L_{57})+N\\
&\quad (4-N^2)^{1/2}\cosh(2(2y+h_1+h_2)L_{57})-N(4-N^2)^{1/2}\cosh(2(y+2h_1+h_2)L_{57}\\
&\quad)+N(4-N^2)^{1/2}\sinh(2(y+h_2)L_{57})-N(4-N^2)^{1/2}\sinh(2(h_1+h_2)L_{57})+N(4\\
&\quad -N^2)^{1/2}\sinh(2(2y+h_1+h_2)L_{57})-N(4-N^2)^{1/2}\sinh(2(y+2h_1+h_2)L_{57})-\\
&\quad ((N-4)N(4-N^2)^{1/2}+2M(2+N))+2M(-12-4N+N^2)(\cosh((2y+2h_1+\\
&\quad 3h_2)L_{57})+\sinh((2y+2h_1+3h_2)L_{57}))+N(4M(2+N)+(N-4)(4-N^2)^{1/2}\\
&\quad (\cosh((2y+h_1+2h_2)L_{57})+\sinh((2y+h_1+2h_2)L_{57}))-N(-4M(2+N)+(\\
&\quad N-4)(4-N^2)^{1/2})(\cosh((2y+h_1+2h_2)L_{57})+\sinh((2y+h_1+2h_2)L_{57})))),\\
L_{80} &= (48C_5C_6M^2Ny+8+C_1C_5M^2N^2y+32C_6C_2M^2N^2y+16C_5C_6M^2N^2y-8C_6C_2\\
&\quad M^2N^3y)\sinh((3y+2h_1+2h_2)L_{57})+(96C_2C_5M^2N-32C_1C_2M^2N-64C_2^2M^2N\\
&\quad -16C_1C_5M^2N+16C_5^2M^2N-16C_1C_2M^2N^2-8C_1C_5M^2N^2-16C_2C_5M^2N^2+\\
&\quad 4C_5C_2M^2N^2-16C_1C_2MN(4-N^2)^{1/2}-32C_6C_2MN(4-N^2)^{1/2}+4C_1C_2MN^2(\\
&\quad 4-N^2)^{1/2}+8C_2C_6MN^2(4-N^2)^{1/2})\sinh((2y+2h_1+3h_2)L_{57})+(16C_5C_6M^2N^3\\
&\quad y+96C_1C_2M^2y+48C_1C_5M^2y+32C_1C_2M^2Ny+16C_1C_5M^2Ny-96C_6C_2M^2N\\
&\quad y-8C_1C_2M^2N^2y-4C_1C_5M^2N^2y-32C_6C_2M^2N^2y-16C_5C_6M^2N^2y+8C_6C_2\\
&\quad M^2N^3y)\sinh((2y+3h_1+2h_2)L_{57})-4M(N-6)(-2C_2M(2+N)(\cosh((2y+2h_1\\
&\quad +h_2)L_{57})+\cosh((2y+2h_1+3h_2)L_{57})-\cosh((2y+3h_1+2h_2)L_{57})-\sinh((2\\
&\quad y+3h_1+2h_2)L_{57})+\sinh((2y+h_1+2h_2)L_{57})+\sinh((2y+2h_1+3h_2)L_{57})))),
\end{aligned}
$$

$$
\begin{aligned}
L_{81} &= (C_1+2C_6)(C_5(\cosh(h_1L_{57})+\sinh(h_1L_{57}))(-\cosh(yL_{57})+\cosh(h_2L_{57})-\sinh \\
&\quad (yL_{57})+\sinh(h_2L_{57}))(N\cosh(yL_{57})+\cosh(h_2L_{57})+12\cosh((y+h_2)L_{57})+2N \\
&\quad \cosh(2(y+h_2)L_{57})+N\cosh((2y+3h_2)L_{57})+N\sinh(yL_{57})+\sinh(h_2L_{57})+12 \\
&\quad \sinh((y+h_2)L_{57})+12\sinh(2(y+h_2)L_{57})N+N^2\sinh((2y+3h_2)L_{57}))-2(-C_6 \\
&\quad (N-6)Ny+C_1(-2N\cosh((2y+2h_1+h_2)L_{57})+2N\cosh((2y+2h_1+3h_2)L_{57}) \\
&\quad +(-6y+N(2+y))(\cosh((y+2h_1+2h_2)L_{57})+\sinh((y+2h_1+2h_2)L_{57}))+2 \\
&\quad N\sinh((2y+2h_1+3h_2)L_{57})-(N-6)y(\cosh((2y+h_1+2h_2)L_{57})+\sinh((2y+ \\
&\quad h_1+2h_2)L_{57}))+(N-6)y(\cosh((2y+3h_1+2h_2)L_{57})+\sinh((2y+3h_1+2h_2) \\
&\quad L_{57}))))), \\
L_{82} &= (48C_5C_6M^2Ny+8C_1C_2M^2N^2y+4C_1C_5M^2N^2y+32C_6C_2M^2N^2y-8C_6C_2M^2N^3 \\
&\quad y-4C_5C_6M^2N^2y)\sinh((3y+2h_1+2h_2)L_{57})+(96C_2C_5M^2+48C_5^2M^2-32C_1 \\
&\quad C_2M^2N-16C_1C_5M^2N-16C_1C_2M^2N^2-8C_1C_5M^2N^2-16C_2C_5M^2N^2-4C_5^2 \\
&\quad M^2N^2+8C_2^2M^2N^2+4C_2C_5M^2N^3)\sinh((3y+2h_1+2h_2)L_{57})+(96C_1C_2M^2y+ \\
&\quad 48C_1C_5M^2y+32C_1C_2M^2Ny+16C_1C_5M^2Ny-96C_2C_6NM^2y-48C_5C_6M^2Ny \\
&\quad -8C_1C_2M^2N^2y-32C_6C_2M^2N^2y-16C_5C_6M^2N^2y+8C_2C_6M^2N^3y+4C_5C_6 \\
&\quad M^2N^3y+96C_1C_2M^2y+32C_1C_2M^2Ny-96C_2C_6M^2Ny-48C_5C_6M^2Ny-32C_2 \\
&\quad C_6M^2Ny+8C_2C_6M^2N^3y+4C_5C_6M^2N^3y-4M(N-6)(C_6N-C_1)(-2N(2+N) \\
&\quad (\cosh((2y+2h_1+h_2)L_{57})+\sinh((2y+2h_1+h_2)L_{57})+\sinh((2y+h_1+2h_2)L_{57} \\
&\quad)-\sinh((2y+3h_1+h_2)L_{57}))-C_5M(2+N)(\cosh((2y+2h_1+h_2)L_{57})+\cosh((\\
&\quad 2y+2h_1+3h_2)L_{57})-\cosh((2y+h_1+2h_2)L_{57})+\sinh((2y+2h_1+h_2)L_{57})-\sinh \\
&\quad ((2y+h_1+2h_2)L_{57})-\sinh((2y+3h_1+2h_2)L_{57})))h_2+L_{68}+L_{72}+L_{73}+2L_{58} \\
&\quad L_{81}), \\
L_{83} &= C_5^2M^2(2+N)(\cosh(2yL_{57})+\sinh(2yL_{57}))(-\cosh(h_1L_{57})+\cosh(h_2L_{57})+\cosh \\
&\quad ((2h_1+h_2)L_{57})-\cosh((h_1+2h_2)L_{57})-\sinh(h_1L_{57})+\sinh(h_2L_{57})+\sinh((2
\end{aligned}
$$

$$
\begin{aligned}
& h_1 + h_2)L_{57})) - (N \cosh(h_1 L_{57}) + \cosh(h_2 L_{57}) + 24 \cosh((h_1 + h_2)L_{57}) - 4N \cosh \\
& ((h_1 + h_2)L_{57}) + N \cosh((2h_1 + h_2)L_{57}) + N \cosh((h_1 + 2h_2)L_{57}) + N \sinh(h_1 L_{57}) \\
& + N \sinh(h_2 L_{57}) + 24 \sinh((h_1 + h_2)L_{57}) + 4N \sinh((2h_1 + h_2)L_{57}) + N \sinh((h_1 \\
& + 2h_2)L_{57})) + N(C_1^2(N-2)(\cosh((y + h_1)L_{57}) - \cosh((y + h_2)L_{57}) - \cosh(2(y \\
& + 2h_1 + h_2)L_{57}) + \cosh(2(y + h_1 + 2h_2)L_{57}) + \sinh(2(y + h_1)L_{57}) - \sinh(2(y + \\
& h_2)L_{57}) - \sinh(2(y + 2h_1 + h_2)L_{57}) + \sinh(2(y + h_1 + 2h_2)L_{57})) + 4L_{70} + 4C_1 \\
& L_{71}) - 2C_5 M L_{75})),
\end{aligned}
$$

$$
\begin{aligned}
L_{84} \;=\; & (4(C_1 + 2C_6)(C_6(N-6)Ny(\cosh((2y + 2h_1 + h_2)L_{57}) - \cosh((y + 2h_1 + 2h_2)L_{57}) + \\
& \sinh((y + 2h + 2h_2)L_{57}) + \sinh((3y + 2h_1 + 2h_2)L_{57}) - \sinh((2y + 2h_1 + 3h_2)L_{57})) \\
& + C_1(-2N \cosh((2y + h_1 + 2h_2)L_{57}) + 2N \cosh((2y + 3h_1 + 2h_2)L_{57}) - (N-6)y(\\
& \cosh((2y + 2h_1 + h_2)L_{57}) + \sinh((2y + 2 + h_1 + h_2)L_{57})(-6y + N(2 + y))(\cosh((y + \\
& 2h_1 + 2h_2)L_{57}) + \sinh((y + 2h_1 + 2h_2)L_{57})) - (-6y + N(2 + y))(\cosh((3y + 2h_1 + 2 \\
& h_2)L_{57}) + \sinh((3y + 2h_1 + 2h_2)L_{57})) + (N-6)y(\cosh((2y + 2h_1 + 3h_2)L_{57}) + \sinh \\
& ((2y + 2h_1 + 3h_2)L_{57})) - 2N \sinh((2y + h_1 + 2h_2)L_{57}) + 2N \sinh((y + 3h_1 + 2h_2)L_{57} \\
&)))),
\end{aligned}
$$

$$
L_{85} \;=\; (6NL_{59}(L_{69} + L_{74} + L_{77} + 4C_2 M(L_{78} + C_1 L_{79}) + L_{58} + L_{84})).
$$

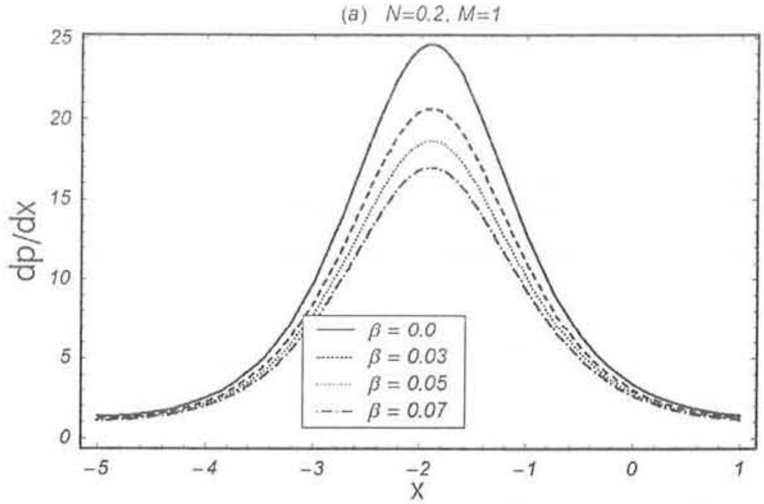

Figure 9.1: Plot showing dp/dx versus x. Here $N = 0.2$, $M = 1$, $a = 0.6$, $b = 0.2$, $d = 8$, $\theta = -2$ and $\phi = \frac{\pi}{2}$.

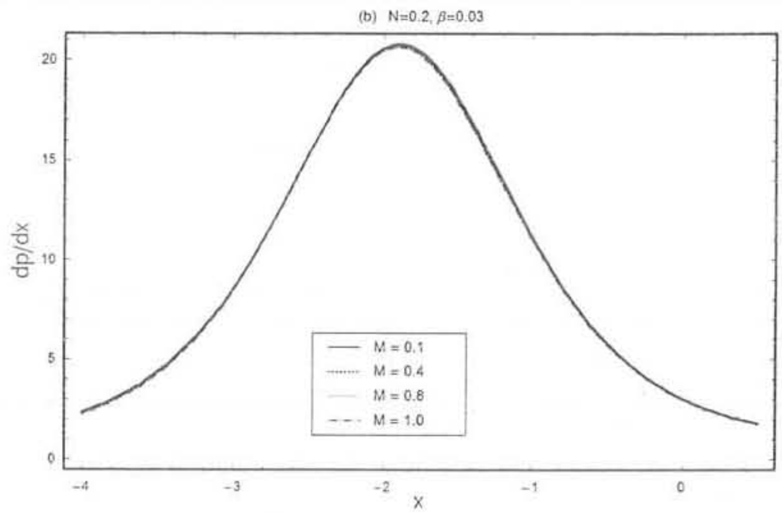

Figure 9.2: Plot showing dp/dx versus x. Here $N = 0.2$, $\beta = 0.03$, $a = 0.6$, $b = 0.2$, $d = 8$, $\theta = -2$ and $\phi = \frac{\pi}{2}$.

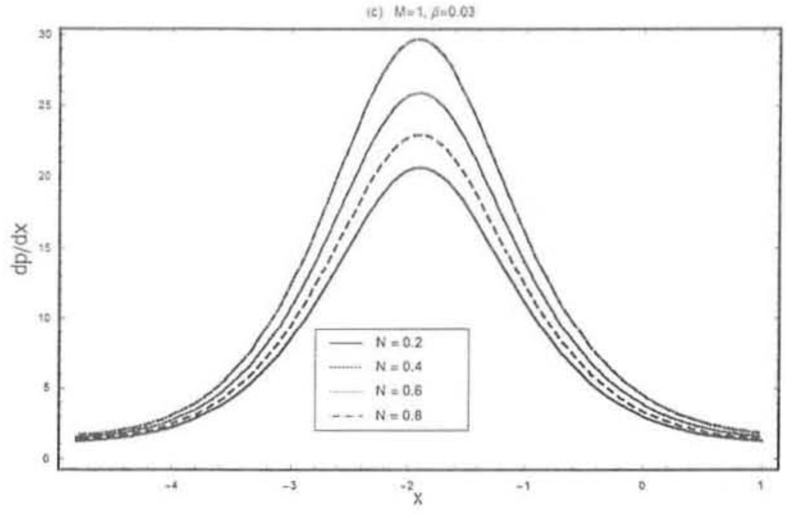

Figure 9.3: Plot showing dp/dx versus x. Here $\beta = 0.03$, $M = 1$, $a = 0.6$, $b = 0.2$, $d = 8$, $\theta = -2$ and $\phi = \frac{\pi}{2}$.

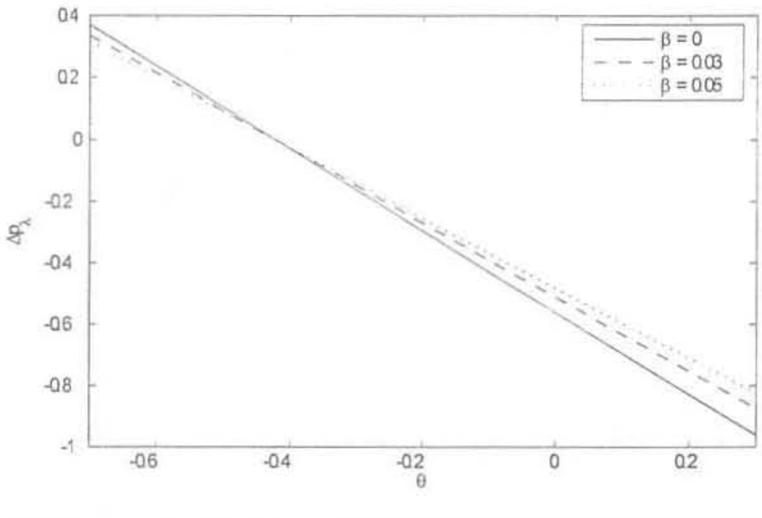

Figure 9.4: Plot showing Δp_λ versus y. Here $N = 0.2$, $M = 1$, $a = b = 0.4$, $d = 1.1$ and $\phi = \frac{\pi}{6}$.

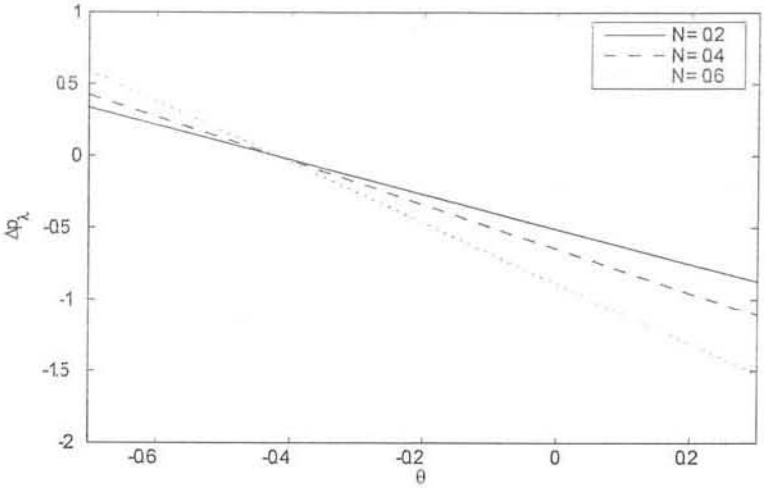

Figure 9.5: Plot showing Δp_λ versus y. Here $\beta = 0.03$, $M = 1$, $a = b = 0.4$, $d = 1.1$ and $\phi = \frac{\pi}{6}$.

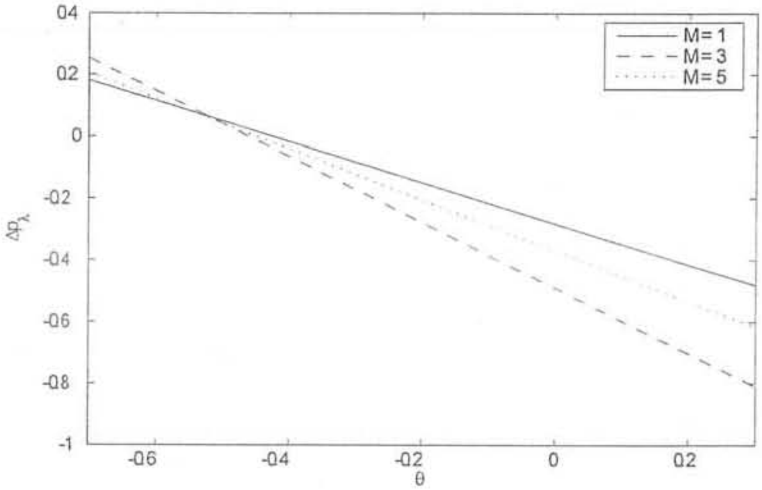

Figure 9.6: Plot showing Δp_λ versus y. Here $\beta = 0.03$, $N = 0.2$, $a = b = 0.4$, $d = 1.1$ and $\phi = \frac{\pi}{6}$.

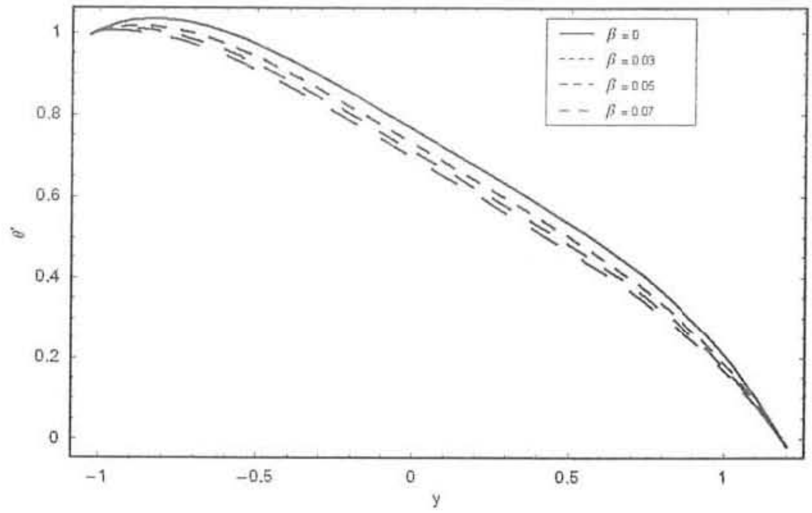

Figure 9.7: Temperature distribution versus y for different values of β. Here $N = 0.2$, $M = 1$, $Br = 0.3$, $a = 0.6$, $b = 0.3$, $d = 1.1$, $\phi = 0.6$, $\theta = 2$ and $x = 0.1$.

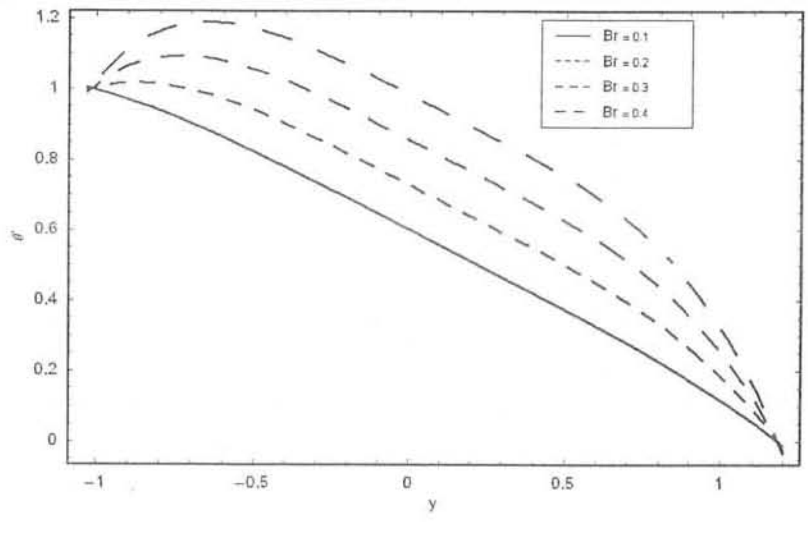

Figure 9.8: Temperature distribution versus y for different values of Br. Here $N = 0.2$, $M = 1$, $\beta = 0.03$, $a = 0.6$, $b = 0.3$, $d = 1.1$, $\phi = 0.6$, $\theta = 2$ and $x = 0.1$.

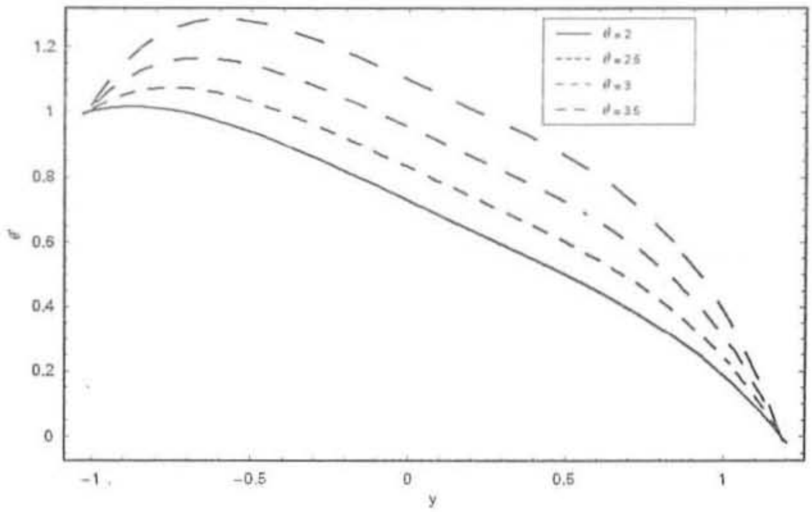

Figure 9.9: Temperature distribution versus y for different values of θ. Here $N = 0.2$, $M = 1$, $\beta = 0.03$, $a = 0.6$, $b = 0.3$, $d = 1.1$, $\phi = 0.6$, $Br = 0.3$ and $x = 0.1$.

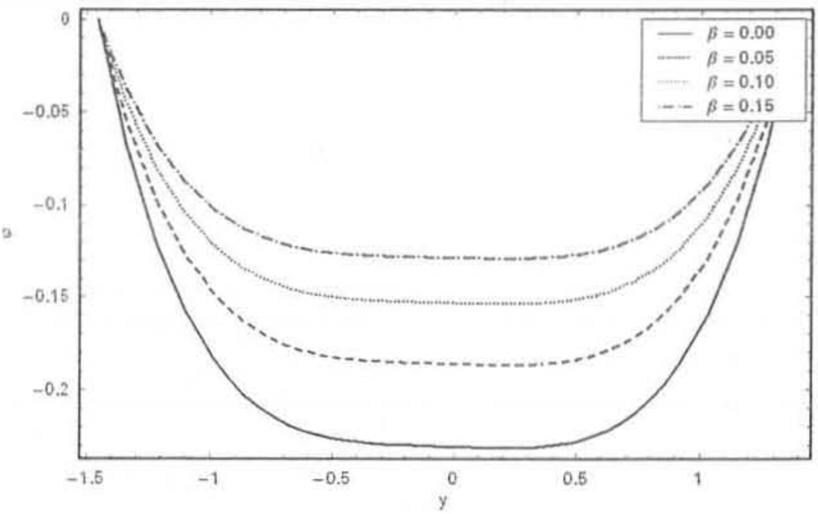

Figure 9.10: Concentration distribution versus y for different values of β. Here $N = 0.2$, $M = 1$, $Br = 4$, $ScSr = 1$, $a = b = 0.4$, $d = 1.1$, $\phi = 0.6$, $\theta = 2$ and $x = 0.1$.

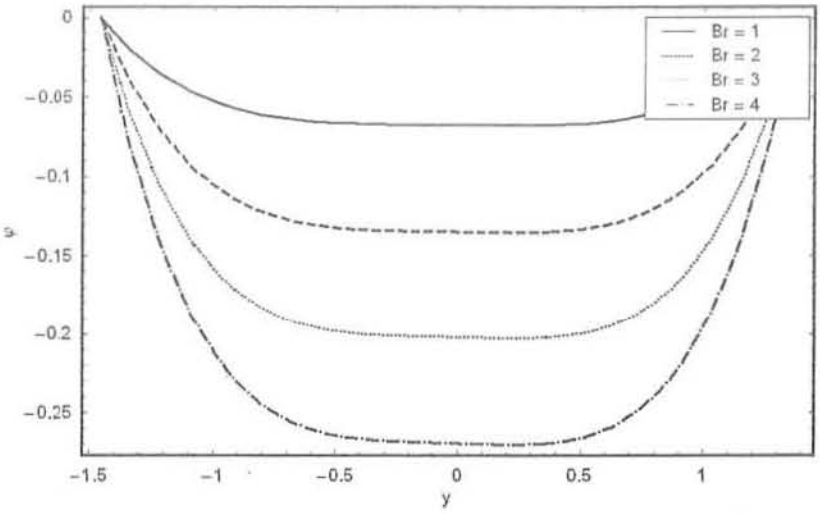

Figure 9.11: Concentration distribution versus y for different values of Br. Here $N = 0.2$, $M = 1$, $\beta = 0.03$, $ScSr = 1$, $a = b = 0.4$, $d = 1.1$, $\phi = 0.6$, $\theta = 2$ and $x = 0.1$.

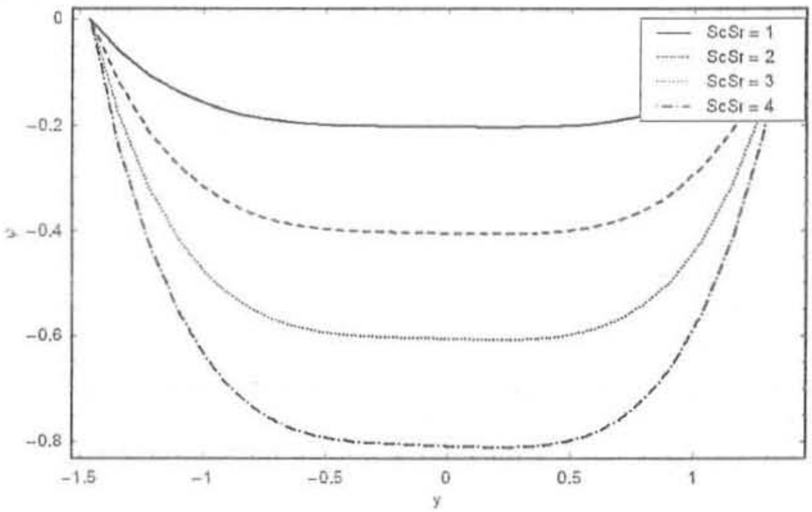

Figure 9.12: Concentration distribution versus y for different values of $ScSr$. Here $N = 0.2$, $M = 1$, $\beta = 0.03$, $Br = 4$, $a = b = 0.4$, $d = 1.1$, $\phi = 0.6$, $\theta = 2$ and $x = 0.1$.

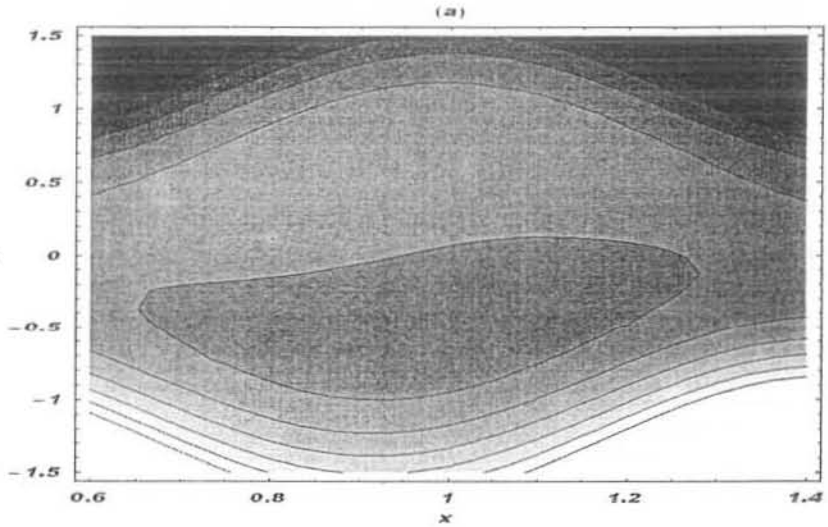

Figure 9.13a: Streamlines for $\beta = 0$. The other parameters are $N = 0.2$, $M = 1$, $a = b = 0.4$, $d = 0.9$, $\phi = \frac{\pi}{6}$ and $\theta = 1.5$.

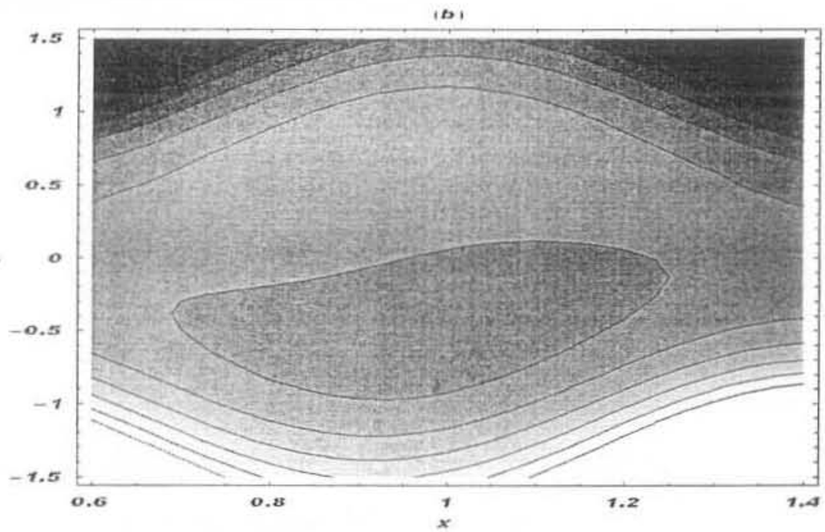

Figure 9.13b: Streamlines for $\beta = 0.03$. The other parameters are $N = 0.2$, $M = 1$, $a = b = 0.4$, $d = 0.9$, $\phi = \frac{\pi}{6}$ and $\theta = 1.5$.

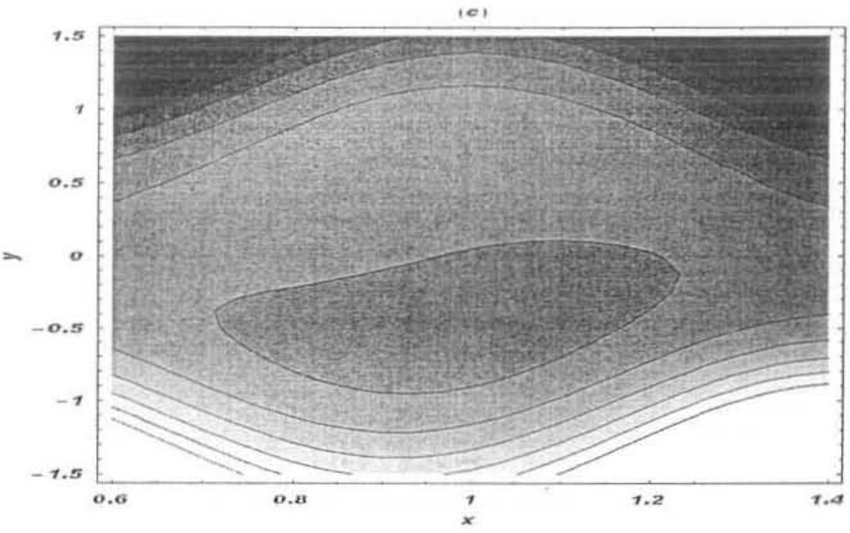

Figure 9.13c: Streamlines for $\beta = 0.05$. The other parameters are $N = 0.2$, $M = 1$, $a = b = 0.4$, $d = 0.9$, $\phi = \frac{\pi}{6}$ and $\theta = 1.5$.

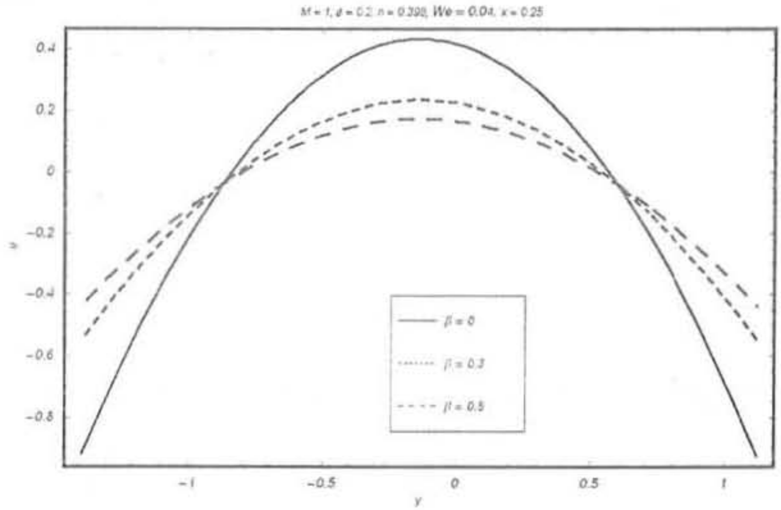

Figure 9.14: Plot showing velocity u versus y for different values of β.

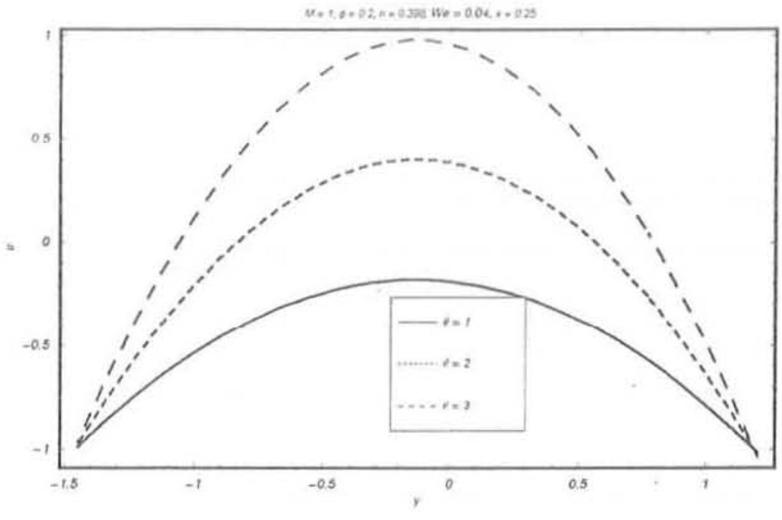

Figure 9.15: Plot showing velocity u versus y for different values of θ.

9.3 Discussion

Figures 9.1 − 9.15 present the effects of various parameters of interest on the flow quantities such as the pressure gradient (dp/dx), the pressure rise per wavelength (Δp_λ), the stream function (Ψ), the temperature (θ'), the concentration (φ) distribution and velocity (u) of the fluid.

Figures 9.1 − 9.3 show the variations of dp/dx versus x for different values of slip parameter (β), the microrotation parameter (M) and the coupling number (N). In Figures 9.1 and 9.2, we can see that dp/dx increases for small values of β and M. However the effect of M on dp/dx is insignificant. Figure 9.3 indicates that the behaviour of N on dp/dx is completely opposite when compared to β and M. Here dp/dx increases with an increase in N. In these figures, we can also observe that in the wider part of the channel, dp/dx is small. In such region, flow can occur without applying a large dp/dx. However in the narrow part of the channel, a much greater pressure gradient is needed in order to retain the same flux to pass it.

The effects of slip parameter (β), the microrotation parameter (M) and the coupling number (N) on Δp_λ have been illustrated in the Figures 9.4 − 9.6. From Figures 9.4 and 9.5, it is concluded that Δp_λ increases and decreases with the increase of β and M in the retrograde pumping region. For an appropriate value of θ, there is an increase in Δp_λ with the increase of β and M in the augmented pumping region. Figure 9.6 shows that the pressure rise per wavelength (Δp_λ) increases and decreases with the increase of N in the retrograde pumping region. However for an appropriate value of θ, Δp_λ increases for small values of N in the augmented pumping region.

The influence of various parameters on the temperature distribution is illustrated in the Figures $9.7-9.9$. Figure 9.7 depicts the temperature profile for different values of β. The slip effects have the ability to decrease the fluid temperature. In this Figure, we have noticed that θ' decreases when the slip parameter increases. Figures 9.8 and 9.9 represent the variation of θ' for different values of Brinkman number (Br) and dimensionless flow rate (θ). Here the temperature is an increasing function of Br and θ.

Figures $9.10-9.12$ are plotted to see the effects of slip parameter (β), the Brinkman number (Br) and the product ($ScSr$). In Figure 9.10, we have observed that the concentration distribution (φ) increases by increasing β. Figures 9.11 and 9.12 depict that the behaviour of $ScSr$ and Br are quite opposite to that given in Figure 9.10. In these figures it is noticed that φ decreases with the increase of Br and $ScSr$.

Another interesting phenomenon of peristalsis is trapping. The formation of internally closed circulating bolus of the fluid by closed streamlines is called trapping and this trapped bolus is pushed ahead along with the peristaltic wave. We have plotted Figures $9.13(a-c)$ to show the effect of slip parameter on trapping. These figures reveal that the size of trapped bolus decreases with an increase of β.

In Figures 9.14 and 9.15, we have plotted the axial velocity u versus y for different values of slip parameter (β) and dimensionless mean flow rate (θ). Figure 9.14 reflects that the axial velocity (u) increases for small values of β near the centre of the channel. However a reverse situation occurs near the channel walls. Here u increases with an increase in β. In Figure 9.15, it is noticed that u increases by increasing dimensionless mean flow rate θ as it supports the flow.

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