Axisymmetric flow in a third-grade fluid

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A Dissertation Submitted in the Partial Fulfillment of the Requirements for the

Degree of

MASTER OF PHILOSOPHY

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CERTIFICATE

A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTER OF PHILOSOPHY

We accept this dissertation as conforming to the required standard

1. Prof. Dr. Muhammad Ayub
(Chairman)

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Department of Mathematics
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2011

DEDICATION

To the four pillars of my life: ALLAH, my Parents and my Siblings. Without you, my life would fall apart.

I humbly thank ALLAH Almighty, the Merciful and the Beneficent, who gave me health, thoughts and co-operative people to enable me achieve this goal. I might not know where the life's road will take me, but walking with You, Allah, through this journey has given me strength.

Ammi, you have given me so much, thanks for your faith in me, and for teaching me that I should never surrender.

Abbu, you always told me to "reach for the stars." I think I got my first one. Thanks for inspiring my love for transportation. We made it…..

I also thank my brothers and sisters for their never ending moral support and prayers which always acted as a catalyst in my academic life.

Acknowledgement

Praise is due to Allah whose worth cannot be described by speakers, whose rewards cannot be counted by calculators and whose claim (to obedience) cannot be satisfied by those who attempt to do so, whom the height of intellectual courage cannot appreciate, and the dividing of understanding cannot reach. He whose description no limits has been laid down, no tribute and praise exists, no time is ordained and no duration is fixed. All my respect goes to the Holy prophet Hazrat Muhammad (peace be upon Him), who emphasized the significance of knowledge from cradle to grave.

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Where would I be without my family? My parents deserve special mention for their inseparable support and prayers. My Father, Muhammad Shafiq, in the first place is the person who put the fundament my learning character, showing me the joy of intellectual pursuit ever since I was a child. Although he is no longer with us, he is forever remembered. I am sure he shares our joy and happiness in the heaven. My Mother, Razia, is the one who sincerely raised me with her caring and gently love. Thanks for all my brother and sister for being supportive and caring siblings. I feel proud of my sister, Shahnaz, for her talents. She had been role model for me to follow unconsciously when I was a teenager and always been one of my best counselors. I am greatful to all my brothers and sisters because due to their prayers my research near to an end.

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Lastly, I offer my regards and blessing to all of those who supported me in any respect during the completion of the dissertation.

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Anum Shafiq

Preface

Non-Newtonian fluids have been a subject of great interest to recent researchers because of their varied applications in industry and engineering. Unlike the viscous fluids, the non-Newtonian fluids cannot be described by the single constitutive relationship between stress and strain rate. This is due to diverse characteristics of such fluids in nature. In general, the mathematical problems in non-Newtonian fluids are more complicated, non-linear and higher order than the viscous fluids. Several mathematicians, engineers, modelers and numerical simulists are still engaged even after knowing the various interesting involved complexities in non-Newtonian fluid dynamics. Some recent contributions in this direction can be mentioned in the studies [1-15].

Flows over a disk or between disks are popular amongest the researchers because of their applications in engineering. Disk-shaped bodies are quite popular in rotating heat exchangers, rotating disk reactor for bio-fluids production, gas or marine turbine and chemical and automobile industries. Because of such interests several investigators have analyzed such flows under various aspects since the pioneering work by Von Karman [16]. He considered hydrodynamic flow over an infinite disk. Cochran [17] presented asymptotic solutions for the steady hydrodynamic flow over a rotating infinite disk. Benton [18] extended Cochran's work to initial value flow problem governed by impulsively started disk. Takhar et al [19] investigated effect of magnetic field on unsteady mixed convection flow from a rotating vertical cone. Maleque et al. [20] studied fully developed laminar flow of a viscous fluid with variable properties. Stuart [21] examined the effect of uniform suction on the steady flow due to rotating disk. Sparrow et al. [22] considered flow over a porous disk. Miklavcic and Wang [23] studied the flow by a rough rotating disk. Takhar et al [24] computed numerical solution for nonlinear coupled system of boundary value problems describing heat transfer in the flow of a micropolar fluid between porous disks. Recently Hayat and Nawaz [25] discussed the unsteady stagnation point flow over a rotating disk.

Further, the heat transfer applications is encountered in many industrial and engineering processes including geo thermal processes, in the design of turbines and turbo-machines in estimating the flight path of rotating wheels and stabilized missiles. The topic is quite important and many investigators [26-34] have considered the heat transfer characteristics in the flow of non-Newtonian fluids.

Existing literature on the topic indicates that an axisymmtric flow of third grade fluid between two porous disks with heat transfer is not investigated so far. The main interest in this dissertation is to examine the effect of heat transfer in third grade fluid when Joule heating and viscous dissipation are also present. This dissertation consists of three chapters. Chapter one includes basics of fluid mechanics. Chapter two discusses MHD axisymmetric flow of a third grade fluid between two porous disks. Chapter three contains the analysis of heat transfer for flow of a third grad fluid between two porous disks. The viscous dissipation and Joule heating are present. Series solution is constructed by using homotopy analysis method. Convergence of the developed solutions is checked. The plots are presented for the analysis of various embedded flow parameters.

Contents

Chapter 1

Basic definations and equations

This chapter consists of relevant definations for heat transfer and fundamental and Maxwell's laws for the convenience of readers. Note that the definations and equations are standared and have been included here for the convenience of the readers.

1.1 Heat Transfer

It is a branch of thermal engineering that deals with the exchange of thermal energy from one physical system to another. The transfer of energy as heat is always from the higher temperature medium to the lower temperature one, and heat transfer stops when the two mediums have the same temperature.

1.1.1 Heat transfer mechanisms

Such mechanisms can be achieved through the three categories given below.

(i) Conduction

(ii) Convection

(iii) Radiation.

(i) Conduction

Condition is the transfer of energy from the more energetic particles of a substance to the adjacent less energetic ones as a results of interactions between the particles. Conduction is possible in solids, liquids, or gases. In gases and liquids, conduction is due to the collisions and diffusion of the molecules during their random motion. In solids, it is due to the combination of vibrationns of the molecules in a lattice and the energy transport by free electrons.

The rate of heat conduction depends through a medium depends on the geometry of the medium, its thickness and the material of the medium, as well as the temperature difference across the medium. It can be easily expressed as follows :

$$\text{Rate of heat conduction} \quad \propto \quad \frac{\text{(Area) (Temperature difference)}}{\text{Thickness}}$$

$$\dot{Q}_{cond} \quad = \quad kA\frac{T_1 - T_2}{\Delta x} = -kA\frac{\Delta T}{\Delta x} \tag{1.1}$$

In above expression the constant of proportionality k is the thermal conductivity of the material, that measure the ability of a material to conduct heat. when $\Delta x \to 0$, then Eq (1.1) yields

$$\dot{Q}_{cond} = -kA\frac{dT}{dx} \tag{1.2}$$

The above expression is known as **Fourier's law of heat conduction.**

(ii) Convection

Convection is the mode of energy transfer between a solid surface and the adjacent liquid or gas that is in motion, and it involves the combined effects of conduction and fluid motion. It is quite obvious that the faster the fluid motion, the greater the convection heat transfer. Further, convection is called **forced convection** if the fluid is forced to flow over the surface by external means such as a fan, pump, or the wind. In comparison, convection is called **natural** (or **free**) **convection** if the fluid motion is caused by buoyancy forces that are induced by density differences due to the varitation of temperature in the fluid.

(iii) Radiation

Radiation is the energy emitted by matter in the form of electromagnetic waves (or photons) as a result of the changes in the electronic configrations of the atoms or molecules. Distinct from conduction and convection, energy transfer by radiation does not require the presence of an interving medium.

1.1.2 Thermal conductivity

It is defined as the rate of heat transfer through a unit thickness of the material per unit area per unit temperature difference. i. e.

$$k = \frac{Qb}{tA\,(T_u - T_l)}. \tag{1.3}$$

In above expression, k is thermal conductivity, Q is the quantity of heat, t is time, A is surface area and $T_u - T_l$ are the temperature diffreence along the distance b.

1.1.3 Specific heat

It is the amount of heat requried to raise the temperature of a unit mass of the fluid by one degree. That is

$$c = \frac{\partial Q}{\partial T}, \tag{1.4}$$

where ∂Q is the amount of heat added to raise the temperature by ∂T.

The specific heats of the above processes are denoted and defined as

$$\text{Specific heat at constant volume} \quad = \quad c_v = \left(\frac{\partial Q}{\partial T}\right)_v. \tag{1.5}$$

$$\text{Specific heat at constant pressure} \quad = \quad c_p = \left(\frac{\partial Q}{\partial T}\right)_p. \tag{1.6}$$

The ratio of specific heats is denoted by γ and thus one can write

$$\gamma = \frac{C_p}{C_v}. \tag{1.7}$$

1.2 Newtonian fluids

A fluid obeying the Newton's law of viscosity is called the **Newtonian fluid.** For such fluids shear stress is then directly and linearly proportional to the deformation rate. In mathematical form we have

$$\tau_{xy} \propto \frac{du}{dy}, \tag{1.8}$$

or

$$\tau_{xy} = \mu \frac{du}{dy}. \tag{1.9}$$

where τ_{xy} is the shear stress applied by the fluid particle, μ is the fluid viscocity or the constant of proportionality and du/dy is the velocity gradient perpendicular to the direction of shear, or equivalently the strain rate. The examples of Newtonian fluids are water, milk, suger solvents, mineral and ethyl alcahol.

1.2.1 Non-Newtonian fluids

A non-Newtonian fluid is a fluid whose flow properties are different from the Newtonian fluids. Such fluids do not obey the Newton's law of viscosity (i.e. shear stress is not linearly proportional to the deformation rate). There is much diversity of fluids in nature. As a consequence, various models of non-Newtonian fluids have been proposed. In this dissertation we consider the third grade fluid for which Cauchy stress tensor can be expressed as

$$\tau = -pI + \mu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2 + \beta_1 \mathbf{A}_3 + \beta_2 \left(\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1\right) + \beta_3 \left(tr \mathbf{A}_1^2\right) \mathbf{A}_1, \tag{1.10}$$

with the Rivlin-Ericksen tensors defined as

$$\mathbf{A}_1 = (\operatorname{grad}\mathbf{V}) + (\operatorname{grad}\mathbf{V})^T, \tag{1.11}$$

$$\mathbf{A}_n = \frac{d\mathbf{A}_{n-1}}{dt} + \mathbf{A}_{n-1}(\operatorname{grad}\mathbf{V}) + (\operatorname{grad}\mathbf{V})^T\mathbf{A}_{n-1}, \qquad n = 2, 3. \tag{1.12}$$

Here $\mathbf{V}$ is the velocity, t is time, μ is dynamic viscosity and α_1, α_2, β_1, β_2 and β_3 are material constants. These material constants obey the following conditions

$$\mu \geq 0, \qquad \alpha_1 \geq 0, \qquad |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3},$$

$$\beta_1 = 0, \qquad \beta_2 = 0, \qquad \beta_3 \geq 0. \tag{1.13}$$

1.3 Fundamental equations

1.3.1 Equation of continuity

In fluid dynamics, the continuity equation (conservation of mass equation) is a mathematical statement that, in any steady state process, the rate at which mass enters a system is equal to the rate at which mass leaves the system. The differential form of the continuity equation is expressed as

$$\frac{\partial\rho}{\partial t} + \nabla.(\rho\mathbf{V}) = 0, \tag{1.14}$$

where ρ is fluid density, t is time and $\mathbf{V}$ is the flow velocity vector. If density (ρ) is a constant, as in the case of **incompressible** flow, the continuity equation simplifies to the following equation.

$$\nabla.\mathbf{V} = 0. \tag{1.15}$$

which means that the divergence of velocity field is zero everywhere.

1.3.2 Equation of motion

In mathematical form the **Navier-Stokes** equations for incompressible viscous flow are

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}.\nabla\mathbf{V}\right) = -\nabla p + \mu\nabla^2.\mathbf{V}+\rho\mathbf{b}, \tag{1.16}$$

The equation of motion is

$$\rho\frac{d\mathbf{V}}{dt} = \nabla.\boldsymbol{\tau}+\rho\mathbf{b}. \tag{1.17}$$

with Cauchy stress tensor $\boldsymbol{\tau}$ as given below

$$\tau = -p\mathbf{I} + \mu\mathbf{A}_1, \tag{1.18}$$

$$\mathbf{A}_1 = \operatorname{grad}\mathbf{V} + (\operatorname{grad}\mathbf{V})^t, \tag{1.19}$$

where ρ is the fluid density, $\mathbf{V}$ is the velocity field, $\mathbf{b}$ is the body force, p is the pressure and μ is the dynamic viscosity. The Cauchy stress tensor can be written in matrix form as

$$\boldsymbol{\tau} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}, \tag{1.20}$$

where σ_{xx}, σ_{yy} and σ_{zz} are the normal stresses and τ_{xy}, τ_{xz}, τ_{yx}, τ_{yz}, τ_{zx} and τ_{zy} are the shear stresses.

The components form of equation of motion is

$$\rho\frac{du}{dt} = \frac{\partial(\sigma_{xx})}{\partial x}+\frac{\partial(\tau_{xy})}{\partial y}+\frac{\partial(\tau_{xz})}{\partial z}+\rho b_x, \tag{1.21}$$

$$\rho\frac{dv}{dt} = \frac{\partial(\tau_{yx})}{\partial x}+\frac{\partial(\sigma_{yy})}{\partial y}+\frac{\partial(\tau_{yz})}{\partial z}+\rho b_y, \tag{1.22}$$

$$\rho\frac{dw}{dt} = \frac{\partial(\tau_{zx})}{\partial x}+\frac{\partial(\tau_{zy})}{\partial y}+\frac{\partial(\sigma_{zz})}{\partial z}+\rho b_z, \tag{1.23}$$

where b_x, b_y and b_z are the body forces in the x, y and z directions respectively.

1.3.3 The law of conservation of energy

The law of conservation of energy states that energy may neither be created nor destroyed. it can only be transformed from one state to another. Therefore the sum of all the energies in the system is a constant.

Mathematically we have

$$\rho c_p\frac{dT}{dt} = K\nabla^2T+tr(\tau\mathbf{L})+\frac{1}{\sigma}\mathbf{J}.\mathbf{J} \tag{1.24}$$

$$\mathbf{J} = \sigma(\mathbf{V}\times\mathbf{B}), \tag{1.25}$$

in which c_p is the specific heat, T is the temperature, K is the thermal conductivity, τ is the Cauchy stress tensor, d/dt is the material derivative and L is the velocity gradient.

1.4 Magnetohydrodynamics

The word magnetohydrodynamics (MHD) is derived from magneto- meaning magnetic field, and hydro- meaning liquid, and -dynamics meaning movement. Magnetohydrodynamics, or MHD, is a branch of the science of the dynamics of matter moving in an electromagnetic field.

1.4.1 Maxwell's equations

The differential form of Maxwell's equations are given by

$$\operatorname{div} \mathbf{E}_{app} = \nabla . \mathbf{E} = \frac{\rho_e}{\epsilon_0}, \quad \text{Guass' law} \tag{1.26}$$

$$\operatorname{div} \mathbf{B} = \nabla . \mathbf{B} = 0, \quad \text{Solenoidal nature of } \mathbf{B} \tag{1.27}$$

$$\operatorname{curl} \mathbf{E}_{ind} = \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \text{Faraday's law} \tag{1.28}$$

$$\operatorname{curl} \mathbf{B} = \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}_{app}}{\partial t}, \quad \text{Ampere Maxwell equation} \tag{1.29}$$

In above expressions ρ_e is the charge density, $\mathbf{J}$ is the current density, μ_0 is the magnetic permeability, ϵ_0 is the permittivity of the free space, $\mathbf{E}_{app}$, $\mathbf{E}_{ind}$ is the applied and induced electric field respectively, $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$ ($\mathbf{B}_0$ and $\mathbf{b}$ are the applied and induced magnetic fields respectively), is total magnetic field.

1.4.2 Ohm's law

Ohms' and Lorentz force laws are characterized by the following equations

$$\mathbf{J} = \sigma \mathbf{E}, \tag{1.30}$$

$$\mathbf{F} = \mathbf{J} \times \mathbf{B}, \tag{1.31}$$

in which σ is the electrical conductivity of the fluid and $\mathbf{E} = \mathbf{E}_{app} + \mathbf{V} \times \mathbf{B}$, is the electric field.

In absence of applied electric field $\mathbf{E}_{app}$, Eq. (1.30) reduce to

$$\mathbf{J} = \sigma \left(\mathbf{V} \times \mathbf{B} \right). \tag{1.32}$$

1.5 Some useful dimensionless numbers

1.5.1 Reynolds number (Re)

It gives a measure of the ratio of inertial forces to viscous forces. Reynolds number can be defined for a number of different situations where a fluid is in relative motion to a surface. Reynolds number is also used to define different flow regimes, such as laminar or turbulent flow. Laminar flow occurs when Reynolds numbers are very small, where viscous forces are dominant, and is describe by smooth, constant fluid motion, while turbulent flow occurs at high Reynolds numbers and is dominated by inertial force. It can be defined as

$$\text{Reynolds number} = \frac{\text{Inertial force}}{\text{Viscous force}}, \tag{1.33}$$

$$\text{Re} = \frac{\rho \mathbf{V} L}{\mu}, \tag{1.34}$$

$$\text{Re} = \frac{\mathbf{V} L}{\nu}. \tag{1.35}$$

in which $\mathbf{V}$ is the mean velocity of the object relative to the fluid, L is a characteristic length, μ is the dynamic viscosity of the fluid, ν is the kinematic viscosity ($\nu = \mu/\rho$) and ρ is the density of the fluid.

1.5.2 Prandtl number (Pr)

It is the ratio of momentum diffusivity (kinematic viscosity) to thermal diffusivity. It is named after the German physicist Ludwig Prandtl. It can be written as

$$\text{Pr} = \frac{\text{viscous diffusion rate}}{\text{thermal diffusion rate}}, \tag{1.36}$$

or

$$\text{Pr} = \frac{\nu}{\alpha}, \tag{1.37}$$

or

$$\text{Pr} = \frac{\left(\frac{\mu}{\rho}\right)}{\left(\frac{k}{\rho c_p}\right)}, \tag{1.38}$$

or

$$\Pr = \frac{c_p \mu}{k}, \tag{1.39}$$

where α is the thermal diffusivity given by $\alpha = k/(\rho c_p)$, K is the thermal conductivity and c_p is the specific heat and ρ is the density.

1.5.3 Eckert number (Ec)

It expresses the relationship between a flow's kinetic energy and enthalpy, and is used to characterize dissipation. It is named after Ernst R. G. Eckert. It is written as

$$E_c = \frac{\text{Kinetic energy}}{\text{Enthalpy}} = \frac{V^2}{c_p \Delta T}. \tag{1.40}$$

Here V is a characteristic velocity of the fluid, c_p is the specific heat at constant pressure and ΔT is temperature difference.

1.5.4 Nusselt number (Nu)

The Nusselt number is the ratio of convective to conductive heat transfer across (normal to) the boundary. Named after Wilhelm Nusselt, it is a dimensionless number. The conductive component is measured under the same conditions as the heat convection but with a (hypothetically) stagnant (or motionless) fluid. The convection and conduction heat flows are parallel to each other and to the surface normal of the boundary surface, and are all perpendicular to the mean fluid flow in the simple case. Mathematically it can be written as

$$N_u = \frac{\text{Convective heat transfer coefficient}}{\text{Conductive heat transfer coefficient}}, \tag{1.41}$$

or

$$N_u = \frac{hL}{K_f}, \tag{1.42}$$

where L is characteristic length, K_f is thermal conductivity of the fluid and h is convective heat transfer coefficient.

1.5.5 Hartman number (M)

It is the ratio of magnetic body force to the viscous force. it can be written as

$$M = \frac{\text{Magnetic forces}}{\text{Viscous forces}}, \tag{1.43}$$

or

$$M = \sqrt{\frac{B_0^2 d^2}{\mu_0 \rho \nu \lambda}}, \tag{1.44}$$

where μ_0 is the magnetic permeability, ρ is the density of the fluid, ν is the kinematic viscosity and λ is the magnetic diffusivity. Further B_0 and d are a characteristic magnetic field and a length scale of the system respectively.

Chapter 2

Magnetohydrodynamic axisymmetric flow of a third-grade fluid between porous disks

2.1 Introduction

Magnetohydrodynamic (MHD) axisymmetric flow of an incompressible third-grade fluid between two porous disks is investigated in this chapter. Solution expression to the nonlinear mathematical problem is developed by a homotopy analysis method (HAM). The effect of dimensionless parameters on radial and axial velocities are observed through graphs. The skin friction coefficients at the upper and lower disks are tabulated for various values of dimensionless physical parameters.

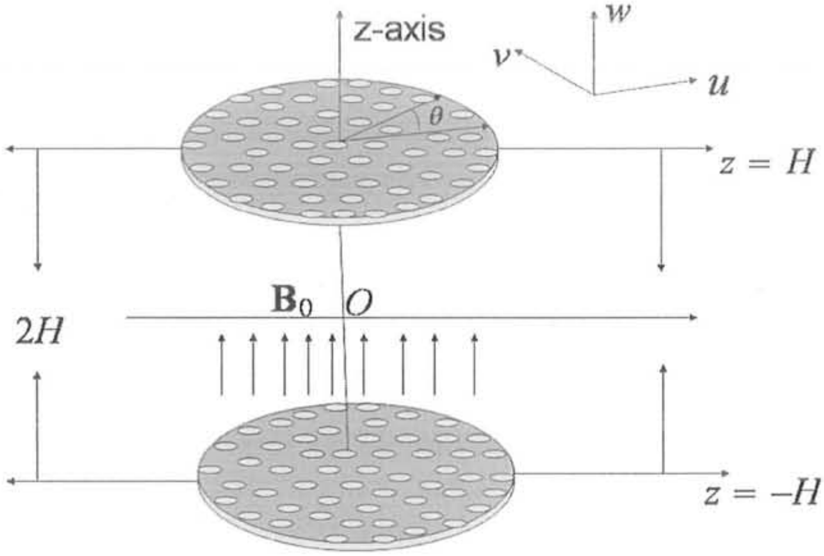

Fig.2.1 Geometry of the problem.

2.2 Mathematical formulation

Consider the steady and axisymmetric flow of an electrically conducting fluid between two porous disks at $z = \pm H$. A flow is induced by suction/injection. A constant magnetic field $\mathbf{B}_0$ is applied perpendicular to plane of disks *i.e* along $z-$axis. There is no external electric field and induced magnetic field is neglected under the assumption of small magnetic Reynolds number. Physical model of the problem is shown in Fig. 2.1. The equations which can govern the MHD flow are

$$\nabla.\mathbf{V}=0, \tag{2.1}$$

$$\rho\frac{d\mathbf{V}}{dt}=\operatorname{div}\boldsymbol{\tau}+\mathbf{J}\times\mathbf{B}_0, \tag{2.2}$$

$$\mathbf{J}=\sigma\left(\mathbf{V}\times\mathbf{B}_0\right), \tag{2.3}$$

where $\mathbf{V}$ is the velocity field, ρ is the fluid density, p is pressure, J is the current density, σ is the electrical conductivity of the fluid, d/dt is the material derivative. Cauchy stress tensor $\boldsymbol{\tau}$ in third-grade fluid is given by

$$\boldsymbol{\tau}=-p\mathbf{I}+\mu\mathbf{A}_1+\alpha_1\mathbf{A}_2+\alpha_2\mathbf{A}_1^2+\beta_3\left(tr\mathbf{A}_1^2\right)\mathbf{A}_1, \tag{2.4}$$

in which $\mathbf{I}$ is the identity tensor, μ is the fluid viscosity and α_1, α_2, β_1, β_2, β_3 are material constants. These material constants satisfy the following constraints

$$\mu\geq0,\alpha_1\geq0,\beta_1=\beta_2=0,\beta_3\geq0,\alpha_1+\alpha_2\leq\sqrt{24\mu\beta_3}, \tag{2.5}$$

$\mathbf{A}_1$ and $\mathbf{A}_2$ are Rivlin-Ericksen tensors which are defined by

$$\mathbf{A}_1=\nabla\mathbf{V}+\left(\nabla\mathbf{V}\right)^*, \tag{2.6}$$

$$\mathbf{A}_2=\frac{d\mathbf{A}_1}{dt}+\mathbf{A}_1\left(\nabla\mathbf{V}\right)+\left(\nabla\mathbf{V}\right)^*\mathbf{A}_1. \tag{2.7}$$

The velocity field for the flow under consideration is

$$\mathbf{V}=\left[u\left(r,z\right),\ 0,\ w\left(r,z\right)\right]. \tag{2.8}$$

Using Eqs.(2.3) – (2.8) in Eq. (2.2) we get

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0, \tag{2.9}$$

$$\begin{aligned}
\rho\left(u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}\right) &= -\frac{\partial p}{\partial r}+\mu\left(\frac{\partial^2 u}{\partial r^2}+\frac{\partial^2 u}{\partial z^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right)\\
&+\alpha_1\left[-\frac{2u^2}{r^3}-2\frac{w}{r^2}\frac{\partial u}{\partial z}+\frac{4}{r}\left(\frac{\partial u}{\partial r}\right)^2+\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2}+w\frac{\partial^3 u}{\partial z^3}-2\frac{u}{r^2}\frac{\partial u}{\partial r}\right.\\
&+3\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial z^2}+3\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial z^2}+\frac{2}{r}\left(\frac{\partial w}{\partial r}\right)^2+\frac{2w}{r}\frac{\partial^2 u}{\partial r\partial z}+5\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z}\\
&+4\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+\frac{\partial u}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+u\frac{\partial^3 u}{\partial r\partial z^2}+w\frac{\partial^3 w}{\partial r\partial z^2}+\frac{2u}{r}\frac{\partial^2 u}{\partial r^2}\\
&+10\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r^2}+3\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r^2}+4\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}+2w\frac{\partial^3 u}{\partial r^2\partial z}+u\frac{\partial^3 w}{\partial r^2\partial z}+2u\frac{\partial^3 u}{\partial r^3}\\
&\left.+\frac{2}{r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}+4\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}\right]+\alpha_2\left[-\frac{4u^2}{r^3}+\frac{4}{r}\left(\frac{\partial u}{\partial r}\right)^2+2\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2}\right.\\
&+2\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial z^2}+2\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial z^2}+4\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+\frac{1}{r}\left(\frac{\partial w}{\partial r}\right)^2+\frac{1}{r}\left(\frac{\partial u}{\partial z}\right)^2+2\frac{\partial u}{\partial r}\frac{\partial^2 w}{\partial r\partial z}\\
&+2\frac{\partial w}{\partial z}+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}+8\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r^2}+2\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r^2}+2\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}\\
&\left.+\frac{2}{r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}+4\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}\right]+\beta_3\left[-\frac{8u^3}{r^4}+\frac{8}{r}\left(\frac{\partial u}{\partial r}\right)^3-\frac{8u^2}{r^3}\frac{\partial u}{\partial r}\right.\\
&+\frac{8}{r}\frac{\partial u}{\partial r}\left(\frac{\partial w}{\partial z}\right)^2+\frac{4}{r}\frac{\partial u}{\partial r}\left(\frac{\partial u}{\partial z}\right)^2+\frac{4}{r}\frac{\partial u}{\partial r}\left(\frac{\partial w}{\partial r}\right)^2+\frac{8}{r}\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\\
&+24\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 u}{\partial r^2}+\frac{8u^2}{r^2}\frac{\partial^2 u}{\partial r^2}+\frac{16u}{r^2}\frac{\partial u}{\partial r}+8\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 u}{\partial r^2}+16\frac{\partial u}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}\\
&+4\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r^2}+4\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r^2}+16\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}+8\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r^2}\\
&+16\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+8\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}+4\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 u}{\partial z^2}+4\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 w}{\partial r\partial z}+\frac{4u^2}{r^2}\frac{\partial^2 u}{\partial z^2}\\
&+\frac{4u^2}{r^2}\frac{\partial^2 w}{\partial r\partial z}+\frac{4u}{r^2}\left(\frac{\partial u}{\partial z}\right)^2+8\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}+8\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}+4\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 u}{\partial z^2}
\end{aligned}$$

$$
\begin{aligned}
&+4\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 w}{\partial r\partial z}+6\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 u}{\partial z^2}+6\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 u}{\partial z^2}+12\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial z}\\
&+12\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial z^2}-\frac{8u}{r^2}\left(\frac{\partial u}{\partial r}\right)^2-\frac{8u}{r^2}\left(\frac{\partial w}{\partial z}\right)^2-\frac{4u}{r^2}\left(\frac{\partial w}{\partial r}\right)^2\\
&+6\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 w}{\partial r\partial z}+6\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 w}{\partial r\partial z}\Bigg]-\sigma B_0^2 u, \qquad (2.10)
\end{aligned}
$$

$$
\begin{aligned}
\rho\left(u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right) = {} & -\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial r^2}+\frac{\partial^2 w}{\partial z^2}\right]\\
&+\alpha_1\left[\frac{u}{r}\frac{\partial^2 w}{\partial r^2}+\frac{w}{r}\frac{\partial^2 u}{\partial z^2}+\frac{u}{r}\frac{\partial^2 u}{\partial r\partial z}+\frac{w}{r}\frac{\partial^2 w}{\partial r\partial z}+\frac{3}{r}\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\right.\\
&+\frac{3}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}+\frac{1}{r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}+\frac{1}{r}\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}+2\frac{\partial u}{\partial r}\frac{\partial^2 w}{\partial r^2}+u\frac{\partial^3 w}{\partial r^3}\\
&+3\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial z^2}+w\frac{\partial^3 u}{\partial r\partial z^2}+4\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+u\frac{\partial^3 u}{\partial r^2\partial z}+4\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial z}\\
&+w\frac{\partial^3 w}{\partial r^2\partial z}+3\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r^2}+3\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r^2}+5\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial r\partial z}\\
&+\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r^2}+2u\frac{\partial^3 w}{\partial r\partial z^2}+2\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}+2w\frac{\partial^3 w}{\partial z^3}+8\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}\\
&\left.+4\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}\right]+\alpha_2\left[\frac{2}{r}\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}+\frac{2}{r}\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}+\frac{2}{r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\right.\\
&+\frac{2}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r^2}+2\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+2\frac{\partial^2 u}{\partial r^2}\frac{\partial w}{\partial r}+2\frac{\partial u}{\partial r}\frac{\partial^2 w}{\partial r^2}\\
&+4\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial r\partial z}+2\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+2\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r^2}+8\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}\\
&\left.+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}+2\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial z^2}+2\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial z}\right]+\beta_3\left[\frac{4}{r}\frac{\partial u}{\partial z}\left(\frac{\partial u}{\partial r}\right)^2\right.\\
&+\frac{4}{r}\frac{\partial w}{\partial r}\left(\frac{\partial u}{\partial r}\right)^2+\frac{4u^2}{r^3}\frac{\partial u}{\partial z}+\frac{4u^2}{r^3}\frac{\partial w}{\partial r}+\frac{4}{r}\frac{\partial u}{\partial z}\left(\frac{\partial w}{\partial z}\right)^2\\
&+\frac{4}{r}\frac{\partial w}{\partial r}\left(\frac{\partial w}{\partial z}\right)^2+\frac{2}{r}\left(\frac{\partial u}{\partial z}\right)^3+\frac{2}{r}\left(\frac{\partial w}{\partial r}\right)^3+\frac{6}{r}\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial w}{\partial r}\\
&+\frac{6}{r}\frac{\partial u}{\partial z}\left(\frac{\partial w}{\partial r}\right)^2+4\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 u}{\partial r\partial z}+4\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 w}{\partial r^2}+8\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r^2}\\
&+8\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r^2}+\frac{8u}{r^2}\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}+\frac{8u}{r^2}\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}+\frac{4u^2}{r^2}\frac{\partial^2 u}{\partial r\partial z}+\frac{4u^2}{r^2}\frac{\partial^2 w}{\partial r^2}\\
&+4\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 u}{\partial r\partial z}+4\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 w}{\partial r^2}+8\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+8\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}
\end{aligned}
$$

$$
\begin{aligned}
&+6\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 u}{\partial r\partial z}+6\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 w}{\partial r^2}+6\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 u}{\partial r\partial z}+6\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 w}{\partial r^2}\\
&+12\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+12\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}+8\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 w}{\partial z^2}+16\frac{\partial u}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial r\partial z}\\
&+\frac{8u^2}{r^2}\frac{\partial^2 w}{\partial z^2}+\frac{16u}{r^2}\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}+24\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 w}{\partial z^2}+4\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 w}{\partial z^2}\\
&+4\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 w}{\partial z^2}+8\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial z^2}+8\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}+8\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}\\
&+8\frac{\partial w}{\partial r}+8\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}\Bigg]. \qquad (2.11)
\end{aligned}
$$

The relevant boundary conditions are

$$
u(r,H)=0,\ u(r,-H)=0,\ w(r,H)=-V_0,\ w(r,-H)=V_0, \qquad (2.12)
$$

where V_0 is the constant velocity. Here $V_0 > 0$ corresponds to the case when fluid is being sucked but $V_0 < 0$ leads to situation when there is blowing.

Let us define the dimensionless quantities

$$
u(r,z)=\frac{V_0 r}{2H}f'(\eta),\ w(r,z)=-V_0 f(\eta),\ \eta=\frac{z}{H} \qquad (2.13)
$$

By using the above quantities, the continuity Eq. (2.9) is identically satisfied and Eqs (2.10) – (2.12) yields

$$
\begin{aligned}
&f^{(iv)}+\mathrm{Re}\, f f'''-\alpha\left[2f''f'''+f'f^{(iv)}+ff^{(v)}\right]-\gamma\left[2f''f'''+f'f^{(iv)}\right]\\
&+\beta\left[7f''^3+24f'f''f'''+3f'^2f^{(iv)}+\frac{3}{2}\delta^2f''f'''^2+\frac{3}{4}\delta^2f''^2f^{(iv)}\right]-\mathrm{Re}\,M^2f''=0\ .(2.14)
\end{aligned}
$$

The subjected boundary conditions are

$$
f'(1)=0,\quad f'(-1)=0\quad f(1)=1,\quad f(-1)=-1, \qquad (2.15)
$$

in which the primes denote the derivative with respect to η and

$$\begin{aligned} \mathrm{Re} &= \frac{V_0 H}{\nu}, \quad M^2 = \frac{\sigma B_0^2 H}{\rho V_0}, \quad \nu = \frac{\mu}{\rho}, \quad \alpha = \frac{\alpha_1 V_0}{\mu H}, \\ \gamma &= \frac{\alpha_2 V_0}{\mu H}, \quad \beta = \frac{2\beta_3 V_0^2}{\mu H^2}, \quad \delta = \frac{r}{H}. \end{aligned} \tag{2.16}$$

Here Re, M, α, β, γ and δ are the Reynolds number (Re), the Hartman number (M), third-grade parameters (α, β, γ) and the dimensionless radial distance (δ). It is important to note that $\mathrm{Re} > 0$ corresponds to case when there is suction at upper disk and blowing at lower disk. However $\mathrm{Re} < 0$ leads to the situtation when there is blowing at upper disk and suction at lower disk.

The respective skin friction coefficients C_{1f} and C_{2f} at the upper and lower disks are

$$C_{1f} = \frac{\tau_w}{\frac{1}{2}\rho (V_0)^2} = \frac{\tau_{rz}|_{z=H}}{\frac{1}{2}\rho (V_0)^2} = \mathrm{Re}_r^{-1/2} \left[f''(1) - \alpha f'''(1) + \frac{\beta \delta^2}{4} f''^3(1) \right], \tag{2.17}$$

$$C_{2f} = \frac{\tau_w}{\frac{1}{2}\rho (V_0)^2} = \frac{\tau_{rz}|_{z=-H}}{\frac{1}{2}\rho (V_0)^2} = \mathrm{Re}_r^{-1/2} \left[f''(-1) + \alpha f'''(-1) + \frac{\beta \delta^2}{4} f''^3(-1) \right], \tag{2.18}$$

in which $\mathrm{Re}_r = V_0 H^2 / \nu r$ is the local Reynolds number.

2.3 Solution by homotopy analysis method (HAM)

In this section we will find out the homotopy solution (HAM). For this purpose we choose the base functions

$$\left\{\eta^{2n+1}; n \geqslant 0\right\} \tag{2.19}$$

and write

$$f(\eta) = \sum_{n=0}^{\infty} a_n \eta^{2n+1}, \tag{2.20}$$

where a_n are the constant coefficients. We take initial guess as

$$f_0(\eta) = \frac{3}{2}\eta - \frac{1}{2}\eta^3, \tag{2.21}$$

with the following auxiliary linear operator

$$\mathcal{L}_1[f(\eta)] = \frac{d^4 f}{d\eta^4}. \tag{2.22}$$

Above linear operator preserves the following property

$$\mathcal{L}_1\left[\frac{C_1}{6}\eta^3 + \frac{C_2}{2}\eta^2 + C_3\eta + C_4\right] = 0, \tag{2.23}$$

where $C_i\,(i = 1 - 4)$ are the constants.

2.3.1 Zeroth order deformation equation

The corresponding problems at the zeroth order deformation are

$$(1-q)\,\mathcal{L}_1\left[\hat{f}(\eta, q) - f_0(\eta)\right] = q\hbar_f \mathcal{N}_1\left[\hat{f}(\eta, q)\right], \tag{2.24}$$

$$\hat{f'}(1, q) = 0, \quad \hat{f}'(-1, q) = 0, \quad \hat{f}(1, q) = 1. \quad \hat{f}(-1, q) = -1, \tag{2.25}$$

where $\hbar_f \neq 0$ and $q \in [0,1]$ are respectively the auxiliary and embedding parameters. The non-linear operator is given by

$$\begin{aligned}\mathcal{N}_1\left[f\left(\eta,q\right)\right] &= \frac{\partial^4\hat{f}}{\partial\eta^4}+\operatorname{Re}\hat{f}\frac{\partial^3\hat{f}}{\partial\eta^3}-\alpha\left[2\frac{\partial^2\hat{f}}{\partial\eta^2}\frac{\partial^3\hat{f}}{\partial\eta^3}+\frac{\partial\hat{f}}{\partial\eta}\frac{\partial^4\hat{f}}{\partial\eta^4}+f\frac{\partial^5\hat{f}}{\partial\eta^5}\right]-\gamma\left[2\frac{\partial^2\hat{f}}{\partial\eta^2}\frac{\partial^3\hat{f}}{\partial\eta^3}+\frac{\partial\hat{f}}{\partial\eta}\frac{\partial^4\hat{f}}{\partial\eta^4}\right]\\ &+\beta\left[7\left(\frac{\partial^2\hat{f}}{\partial\eta^2}\right)^3+24\frac{\partial\hat{f}}{\partial\eta}\frac{\partial^2\hat{f}}{\partial\eta^2}\frac{\partial^3\hat{f}}{\partial\eta^3}+3\left(\frac{\partial\hat{f}}{\partial\eta}\right)^2\frac{\partial^4\hat{f}}{\partial\eta^4}\right.\\ &\left.+\frac{3}{2}\delta^2\frac{\partial^2\hat{f}}{\partial\eta^2}\left(\frac{\partial^3\hat{f}}{\partial\eta^3}\right)^2+\frac{3}{4}\delta^2\left(\frac{\partial^2\hat{f}}{\partial\eta^2}\right)^2\frac{\partial^4\hat{f}}{\partial\eta^4}\right]-\operatorname{Re}M^2\frac{\partial^2\hat{f}}{\partial\eta^2}. \qquad (2.26)\end{aligned}$$

When $q=0$ and $q=1$ then

$$\widehat{f}\left(\eta;0\right)=f_0\left(\eta\right) \text{ and } \widehat{f}\left(\eta;1\right)=f\left(\eta\right). \qquad (2.27)$$

It is noticed that when q increases from 0 to 1 then $f(\eta;p)$ vary from the initial guess $f_0\left(\eta\right)$ to the final solutions $f\left(\eta\right)$. Using Taylor series we may write

$$\hat{f}\left(\eta,q\right)=f_0\left(\eta\right)+\sum_{m=1}^{\infty}f_m\left(\eta\right)q^m, \qquad (2.28)$$

in which

$$\hat{f}_m\left(\eta\right)=\frac{1}{m!}\frac{\partial^m f\left(\eta,q\right)}{\partial q^m}\bigg|_{q=0}. \qquad (2.29)$$

Obviously Eq. (2.24) has a non-zero auxiliary parameter h_f. The convergence of the series (2.28) is dependent upon h_f. The value of h_f is chosen properly in such a way so that the Eq. (2.28) are convergent at $q=1$. In view of Eq. (2.28) we have

$$\hat{f}\left(\eta\right)=f_0\left(\eta\right)+\sum_{m=1}^{\infty}f_m\left(\eta\right). \qquad (2.30)$$

2.3.2 mth order deformation equations

The m^{th} order deformation problems are obtained by differentiating zeroth-order Eqs. (2.24) and (2.25) m times with respect to q and then dividing them by $m!$. Finally setting $q=0$ we get the following higher-order deformation equation

$$\mathcal{L}_1\left[\hat{f}_m(\eta)-\chi_m\hat{f}_{m-1}(\eta)\right]=\hbar_f\mathcal{R}_m^f(\eta), \tag{2.31}$$

$$\left.\frac{\partial\hat{f}_m(\eta,q)}{\partial\eta}\right|_{\eta=1}=0,\qquad \left.\frac{\partial\hat{f}_m(\eta,q)}{\partial\eta}\right|_{\eta=-1}=0,\qquad \hat{f}_m(1,q)=0.\quad \hat{f}_m(-1,q)=0, \tag{2.32}$$

$$\chi_m=\begin{cases}0, & m\leqslant 1,\\ 1, & m>1,\end{cases} \tag{2.33}$$

$$\begin{aligned}\mathcal{R}_m^f(\eta) \;=\;& f_{m-1}^{(iv)}(\eta)+\sum_{k=0}^{m-1}\Big[\operatorname{Re} f_{m-1-k}f_k'''-\alpha\left(2f_{m-1-k}f_k'''+f_{m-1-k}'f_k^{(iv)}+f_{m-1-k}f_k^{(v)}\right)\\ &-\gamma\left(2f_{m-1-k}''f_k'''+f_{m-1-k}f_k^{(iv)}\right)+\beta\sum_{l=0}^{k}\{7f_{m-1-k}''f_{k-l}''f_l''+24f_{m-1-k}'f_{k-l}''f_l'''\\ &+3f_{m-1-k}'f_{k-l}f_l'^{(iv)}+\frac{3}{2}\delta^2f_{m-1-k}''f_{k-l}'''f_l'''+\frac{3}{4}\delta^2f_{m-1-k}''f_{k-l}''f_l^{(iv)}\}\Big]\\ &-\operatorname{Re}M^2f_{m-1}''.\end{aligned} \tag{2.34}$$

The general solution of the Eq. (2.31) is of the form

$$f(\eta)=f^*+\frac{1}{6}C_1^m\eta^3+\frac{1}{2}C_2^m\eta^2+C_3^m\eta+C_4^m, \tag{2.35}$$

where f^* denotes the special solution and C_i^m $(i=1-4)$ are the arbitrary constants which can be determined by the boundary conditions given in Eq. (2.32).

2.4 Convergence of the homotopy solutions

The series solution (2.31) contains the auxiliary parameter $\hbar_f$. Obviously the convergence of series solution (2.35) strongly depends upon the suitable range of this auxiliary parameter $\hbar_f$. To obtain the range for suitable values of $\hbar_f$, we ploted $\hbar$-curve for 28th order of approximation in Fig. (2.2). It is noted from Fig. (2.2) that admissible values of h_f is $-0.85 \leq h_f \leq -0.35$. Table 2.1 is made just to decide that how much order of approximations are necessary for a convergent solution. It is noticed that $10th$ order of approximations up to 6 decimal places are enough.

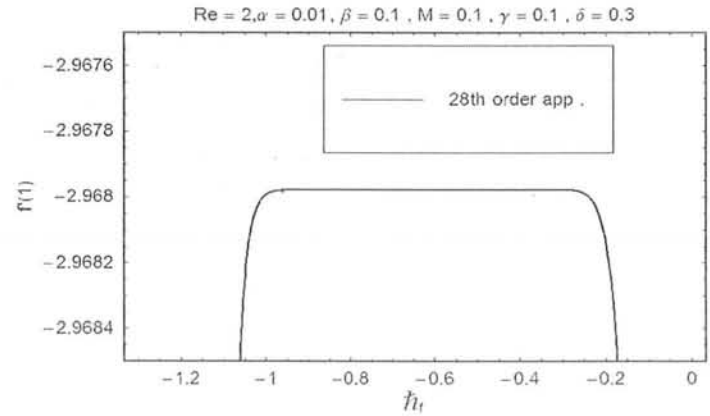

Fig. 2.2. $\hbar$–curve of $f'(1)$ for $28th$ order of approximations.

Table 2.1. Convergence of homotopy solutions when $\alpha = 0.01, \beta = 0.1, \gamma = 0.1, \delta = 0.3$, $M = 0.1, \mathrm{Re} = 2$ and $\hbar_f = -0.6$.

Order of approximations	$f''(1)$
1	−3.03822
2	−2.99133
5	−2.96884
8	−2.96795
10	−2.96798
15	−2.96798
20	−2.96798
30	−2.96798
40	−2.96798
50	−2.96798
60	−2.96798

2.5 Results and discussion

In this section we have examined the influence of physical parameters on dimensionless radial and axial velocities. Figs. 2.2 and 2.3 depict the variation of radial velocity $f'(\eta)$ for various values of Hartman number M. Fig. 2.2 shows the behavior of Hartman number M on radial velocity $f'(\eta)$ when upper disk is subjected to suction and there is blowing at lower disk. However Fig. 2.3 is displayed in order to see the influence of Hartman number M on radial velocity $f'(\eta)$ when upper disk is subjected to blowing and fluid is being sucked from lower disk. It is noted from Fig 2.2 that in the vicnity of disks the radial velocity increases with an increase in Hartman number M whereas it is decreasing function of M in the central region between disks. This behavior of radial velocity $f'(\eta)$ is due to mass conservation constraint. An increase in $f'(\eta)$ near the disks is compensated by decrease in $f'(\eta)$ in the central region. Fig. 2.3 reveals that behaviour of M on $f'(\eta)$ in case of blowing at upper disk and suction at lower disk is opposite that on $f'(\eta)$ for the case of suction at upper disk and blowing at lower disk. Figs. 2.4 and 2.5 are sketched to examine the influence of suction/injection on radial velocity $f'(\eta)$. Here $\mathrm{Re} > 0$ corresponds to the case when fluid is being sucked from upper disk and lower disk is subjected to blowing and vice versa for $\mathrm{Re} < 0$. From these Figs. it is noted that the effect of $\mathrm{Re} > 0$ on $f'(\eta)$ is opposite to that of $\mathrm{Re} < 0$ on $f'(\eta)$. Figs. 2.6 and 2.7 demonstrate the behaviour of third-grade parameter β on radial velocity $f'(\eta)$ for both cases $\mathrm{Re} > 0$ and $\mathrm{Re} < 0$. Figs. $2.8 - 2.11$ are displayed in order to analyze the effect of parameters α and γ on radial velocity $f'(\eta)$ when $\mathrm{Re} > 0$ and $\mathrm{Re} < 0$. It is observed from these Figs. that radial velocity $f'(\eta)$ decreases in the vicinity of disks whereas it increases in the central region between disks for both $\mathrm{Re} > 0$ and $\mathrm{Re} < 0$. Figs. $2.12 - 2.21$ reflect the influence of emerging parameters Re, M, α, β and γ respectively. Figs. 2.12 and 2.13 show that the magnitude of axial velocity $f(\eta)$ is a decreasing function of Re. The effect of Hartman number M on $f(\eta)$ for $\mathrm{Re} > 0$ is opposite to that of M on $f(\eta)$ when $\mathrm{Re} < 0$ as shown in Figs. 2.14 and 2.15. Magnitude of axial velocity f is an increasing function of α, β and γ for both cases $\mathrm{Re} > 0$ and $\mathrm{Re} < 0$.

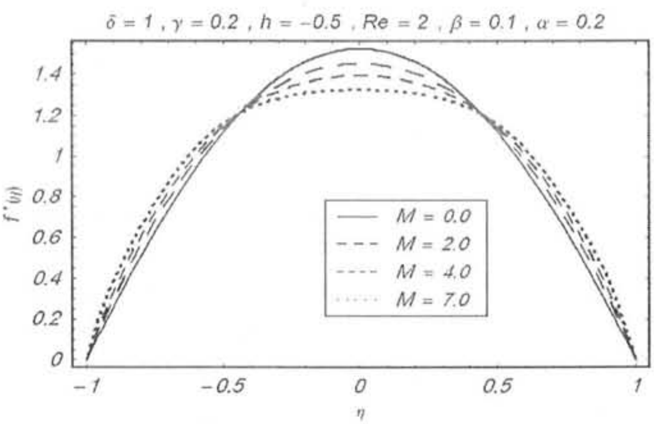

Fig.2.3. Influence of Hartman number M on dimensionless radial velocity $f'(\eta)$ when $\mathrm{Re} > 0$.

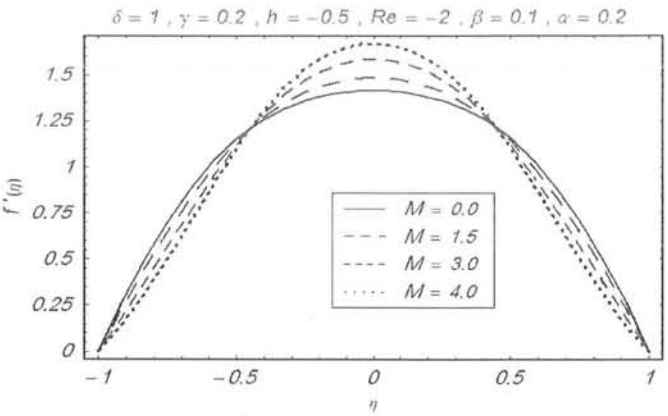

Fig. 2.4. Influence of Hartman number M on dimensionless radial velocity $f'(\eta)$ when $\mathrm{Re} < 0$.

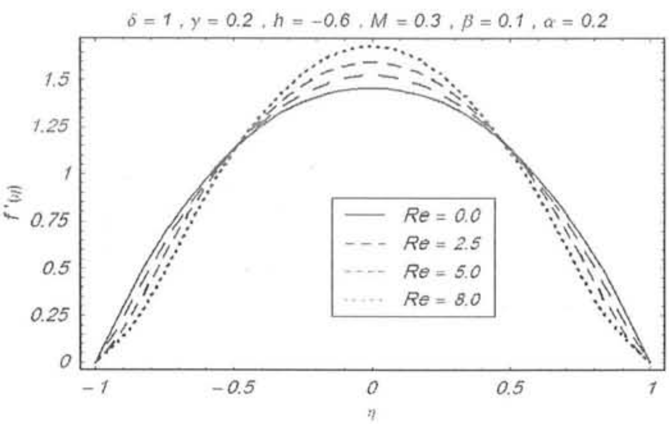

Fig. 2.5. Influence of Reynolds number $\mathrm{Re} > 0$ on dimensionless radial velocity $f'(\eta)$.

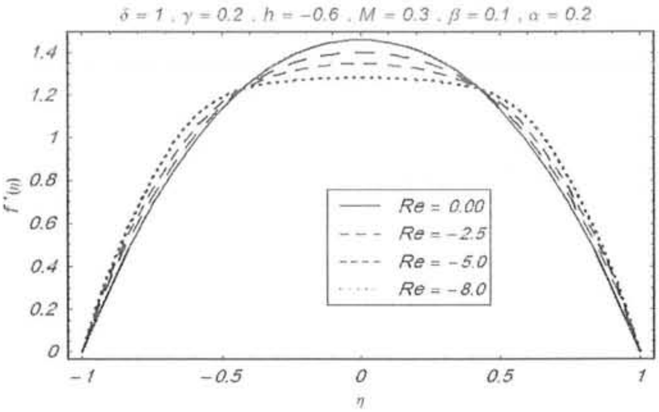

Fig. 2.6. Influence of Reynolds number $\mathrm{Re} < 0$ on dimensionless radial velocity $f'(\eta)$.

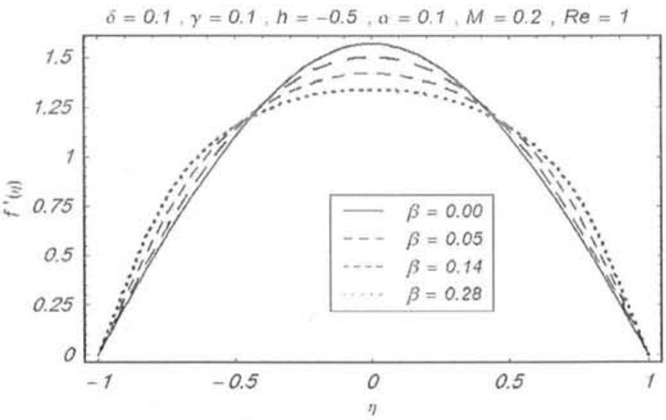

Fig. 2.7. Influence of third-grade parameter β on dimensionless radial velocity $f'(\eta)$ when $\mathrm{Re} > 0$.

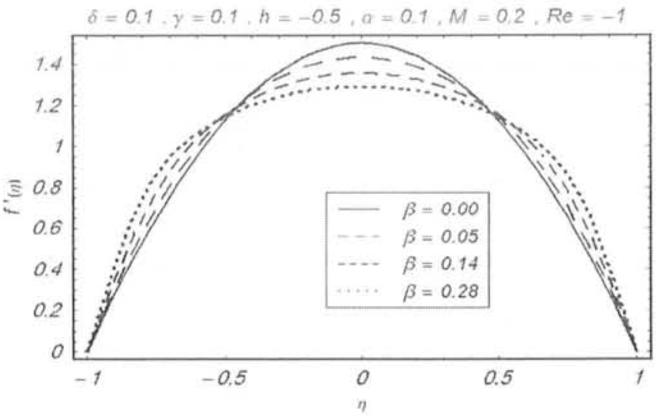

Fig. 2.8. Influence of third-grade parameter β on dimensionless radial velocity $f'(\eta)$ when $\mathrm{Re} < 0$.

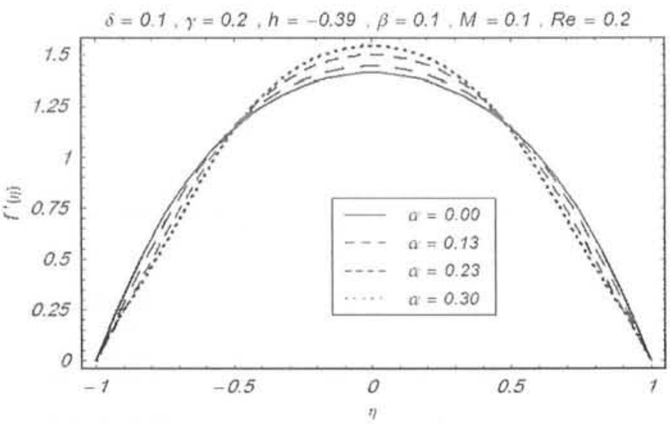

Fig. 2.9. Influence of second-grade parameter α on dimensionless radial velocity $f'(\eta)$ when $\mathrm{Re} > 0$.

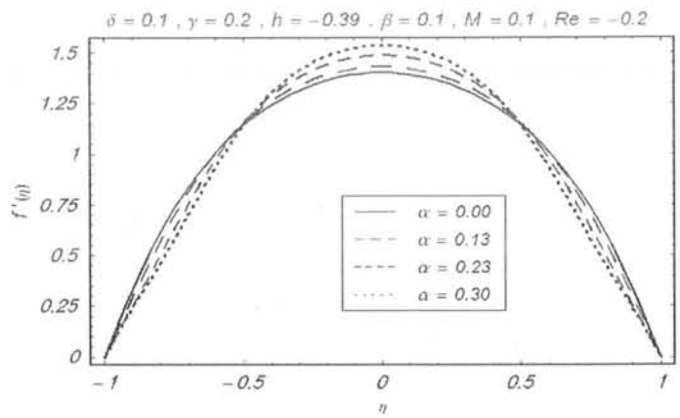

Fig. 2.10. Influence of second-grade parameter α on dimensionless radial velocity $f'(\eta)$ when $\mathrm{Re} < 0$.

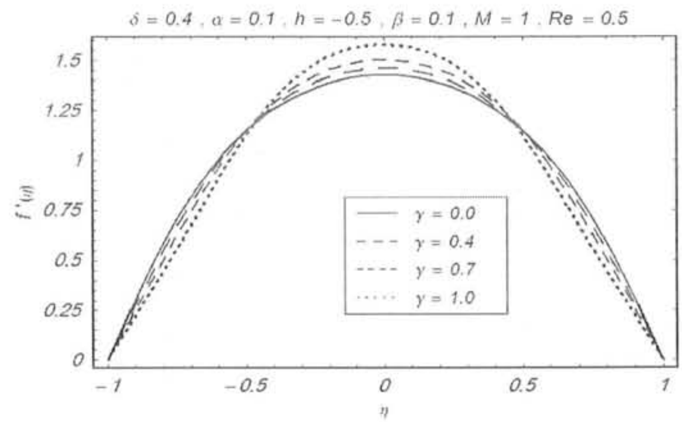

Fig. 2.11. Influence of second-grade parameter γ on dimensionless radial velocity $f'(\eta)$ when $\mathrm{Re} > 0$.

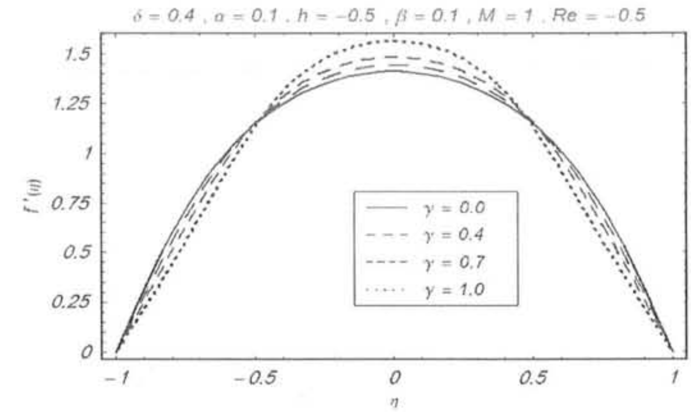

Fig. 2.12. Influence of second-grade parameter γ on dimensionless radial velocity $f'(\eta)$ when $\mathrm{Re} < 0$.

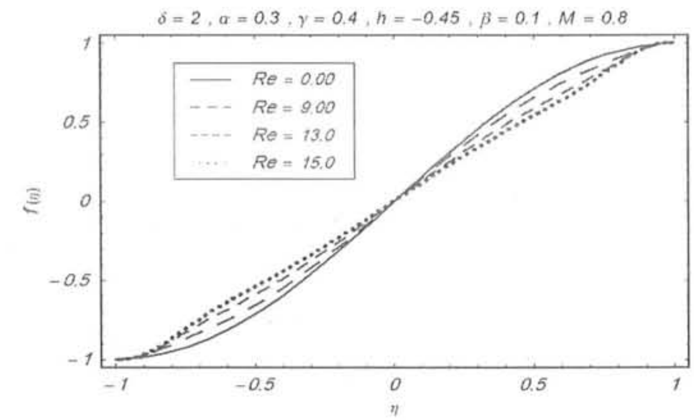

Fig. 2.13. Influence of $\mathrm{Re} > 0$ on the dimensionless axial velocity $f(\eta)$.

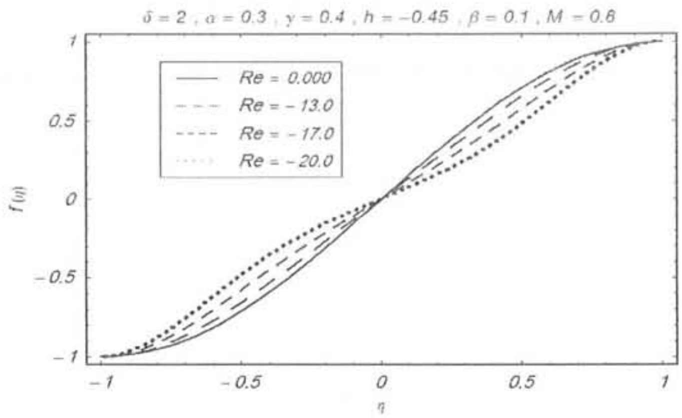

Fig. 2.14. Influence of Re < 0 on dimensionless axial velocity $f(\eta)$.

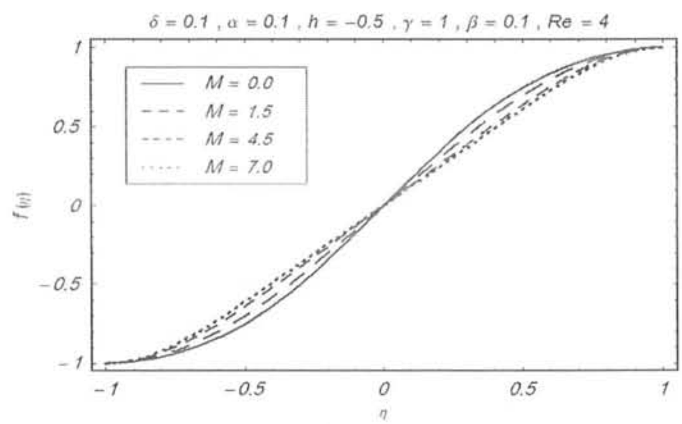

Fig. 2.15 Influence of Hartman number M on dimensionless axial velocity $f(\eta)$ when Re > 0.

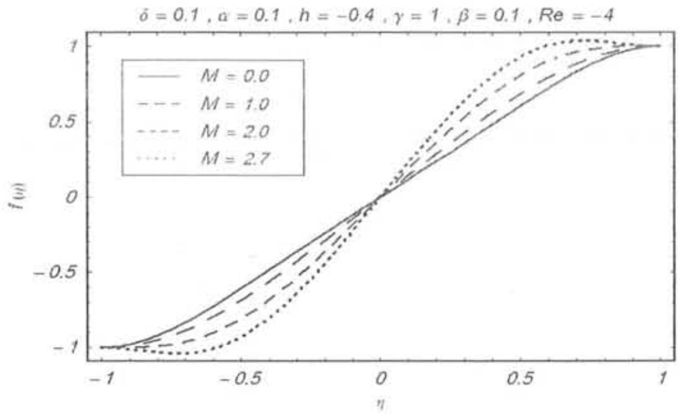

Fig. 2.16 Influence of Hartman number M on dimensionless axial velocity $f(\eta)$ when $\mathrm{Re} < 0$.

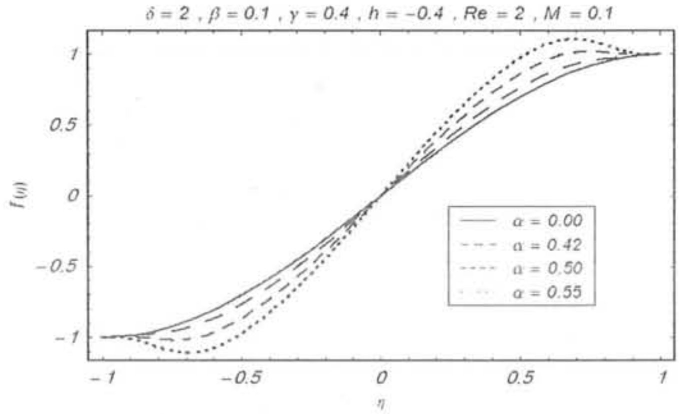

Fig. 2.17 Influence of second-grade parameter α on dimensionless axial velocity $f(\eta)$ when $\mathrm{Re} > 0$.

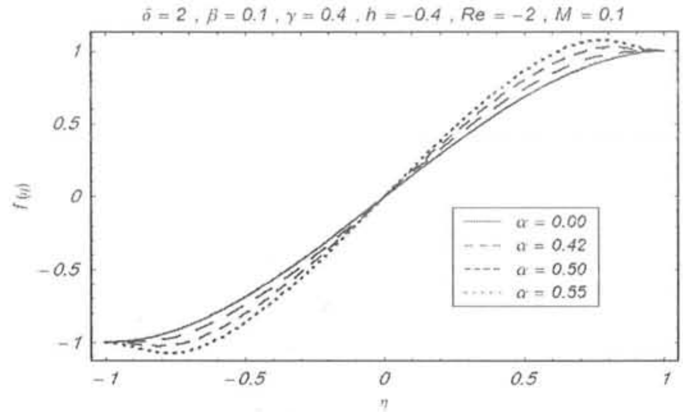

Fig. 2.18 Influence of second-grade parameter α on dimensionless axial velocity $f(\eta)$ when $\mathrm{Re} < 0$.

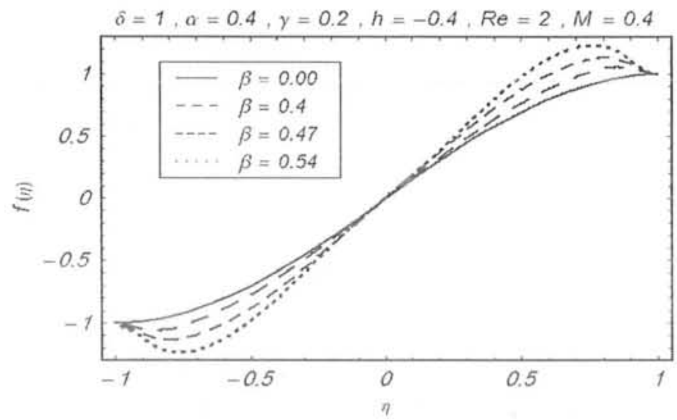

Fig. 2.19 Influence of third-grade parameter β on dimensionless axial velocity $f(\eta)$ when $\mathrm{Re} > 0$.

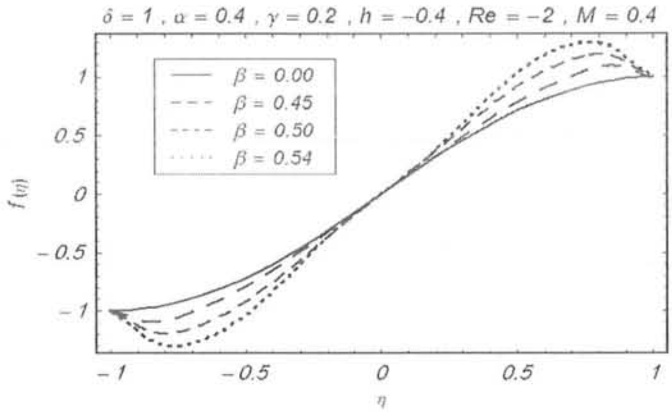

Fig. 2.20 Influence of third-grade parameter β on dimensionless axial velocity $f(\eta)$ when $\mathrm{Re} < 0$.

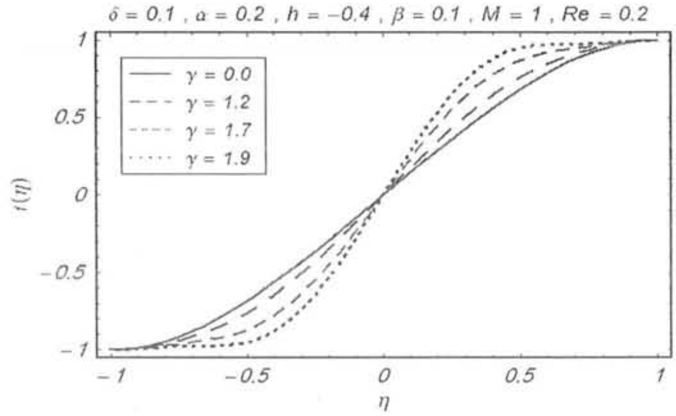

Fig. 2.21 Influence of second-grade parameter γ on dimensionless axial velocity $f(\eta)$ when $\mathrm{Re} > 0$.

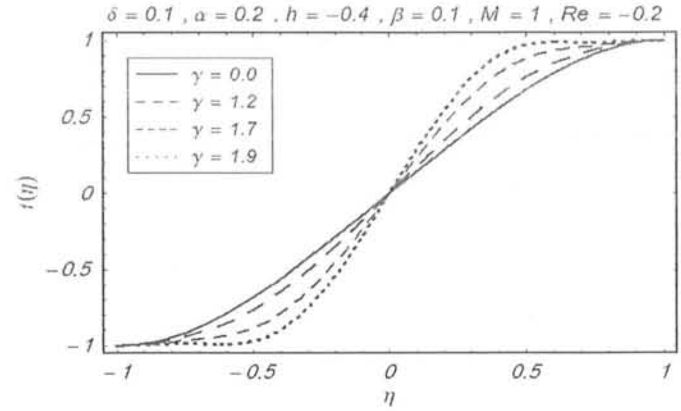

Fig. 2.22 Influence of second-grade γ on dimensionless axial velocity $f(\eta)$ when $\mathrm{Re} < 0$.

Table. 2.2: Numerical values of skin friction coefficients $Re_r C_{1f}$ at upper disk and $Re_r C_{2f}$ at lower disk when $\mathrm{Re} > 0$.

α	γ	β	R	M	$-\mathrm{Re}\, C_{1f}$	$\mathrm{Re}\, C_{2f}$
0.00	0.01	0.1	2	0.1	3.06726	3.06726
0.01					3.05566	3.05566
0.02					3.02173	3.02173
0.03					2.97896	2.97896
0.01	0.0	0.1	2	0.1	3.05325	3.05325
	0.1				2.95380	2.95380
	0.2				2.85605	2.85605
	0.3				2.75956	2.75956
0.01	0.1	0.0	2	0.1	2.50211	2.50211
		0.1			2.97327	2.97327
		0.2			3.42907	3.42907
		0.3			3.97299	3.97299
0.01	0.01	0.1	0.0	0.3	3.51253	3.51253
			0.1		3.48955	3.48955
			0.2		3.46019	3.46019
			0.3		3.43428	3.43428
0.01	0.1	0.1	2	0.0	2.96128	2.96128
				0.1	2.96470	2.96470
				0.2	2.97496	2.97496
				0.3	2.99211	2.99211

Table. 2.3: Numerical values of skin friction coefficients $Re_r C_{1f}$ at upper disk and $Re_r C_{2f}$ at lower disk when $\mathrm{Re} < 0$

α	γ	β	R	M	$-\mathrm{Re}\, C_{1f}$	$\mathrm{Re}\, C_{2f}$
0.00	0.01	0.1	−0.2	0.1	3.66567	3.66567
0.01					3.81843	3.81843
0.02					3.80093	3.80093
0.03					3.67106	3.67106
0.01	0.0	0.1	−0.2	0.1	3.75011	3.75011
	0.1				3.63775	3.63775
	0.2				3.52178	3.52178
	0.3				3.40317	3.40317
0.01	0.1	0.0	−0.2	0.1	2.97450	2.97450
		0.1			3.74536	3.74536
		0.2			4.32452	4.32452
		0.21			4.37740	4.37740
0.01	0.01	0.1	0.0	0.3	3.51253	3.51253
			−0.1		3.53571	3.53571
			−0.2		3.56554	3.56554
			−0.3		3.59227	3.59227
0.01	0.1	0.1	−0.2	0.0	3.70103	3.70103
				0.1	3.70057	3.70057
				0.2	3.69916	3.69916
				0.3	3.69681	3.69681

Table. 2.1 ensures the convergence of the obtained series solution. It is observed that 10^{th}-order of approximations are enough for a series solution. Tables. 2.2 and 2.3 are constructed to see the effects of third-grade parameter β, second-grade parameters α and γ and the Reynolds number Re and the Hartman number M on the variation of skin friction coefficientC_f.

Table 2.2 represents the variation of skin friction coefficients $\mathrm{Re}_r C_{1f}$ and $\mathrm{Re}_r C_{2f}$ at the upper and lower disks when $\mathrm{Re} > 0$ whereas Table 2.3 depicts the behavior of $\mathrm{Re}_r C_{1f}$ and $\mathrm{Re}_r C_{2f}$ when $\mathrm{Re} < 0$. It can be observed from Table 2.2 that $\mathrm{Re}_r C_{1f}$ and $\mathrm{Re}_r C_{2f}$ are increasing function of third-grade parameter β and Hartman number M whereas $\mathrm{Re}_r C_{1f}$ and $\mathrm{Re}_r C_{2f}$ are decreased when α, γ and Re are increased. Hence it can be concluded that tangential stresses at both upper and lower disks are increasing functions of β and M. However tangential stresses on the surface of disks can be reduced by increasing α, γ and Re. Since Re corresponds to suction/blowing phenomena. therefore stresses on the surface of disks can be reduced or adjusted by suction or injection mechanism. Table 2.2 also demonstrates that in second-grade fluid $(\beta = 0)$ the stresses on the surface of disks are smaller than those in third-grade fluid $(\beta \neq 0)$. Furthermore, an increase in strength of external magnetic field results in an increase of stresses at the surface of disks.

2.6 Concluding remarks

In this chapter the magnetohydrodynamic (MHD) axisymmetric flow of a third-grade fluid between two permeable disks is investigated. Expression of velocity field f' is determined. The main points can be summarized from the presented analysis as follow:

- For $\mathrm{Re} > 0$, the behavior of M on radial velocity $f'(\eta)$ and axial velocity is opposite to that of $f'(\eta)$ when $\mathrm{Re} < 0$.
- The influence of third-grade parameter β on radial velocity $f'(\eta)$ is similar for both cases of $\mathrm{Re} > 0$ and $\mathrm{Re} < 0$.
- The effects of second grade parameters α and γ on radial velocity $f'(\eta)$ are similar in a qualitative sense.

- Qualitatively α, β and γ have similar effect on axial velocity $f(\eta)$.
- $\mathrm{Re}_r\, C_{1f}$ and $\mathrm{Re}_r\, C_{2f}$ are increasing functions of third-grade parameter (β) and Hartman number (Re) where as $\mathrm{Re}_r\, C_{1f}$ and $\mathrm{Re}_r\, C_{2f}$ are decreased when α, γ and Re are increased.

Chapter 3

Magnetohydrodynamic axisymmetric flow of third-grade fluid between porous disks with heat transfer

3.1 Introduction

This chapter describes the magnetohydrodynamic (MHD) axisymmetric flow of third-grade fluid between two permeable disks with heat transfer. The resulting nonlinear problem is computed for velocity and temperature fields. Expressions for skin friction coefficients and Nusselt number are computed. The governing non-linear problems have been solved by homotopy analysis method (HAM). Convergence of the obtained series solutions is explicitly discussed. The dimensionless velocity and temperature fields are presented for various parameters of interest. Skin friction coefficient and Nusselt number are tabulated and discussed for dimensionless emerging parameters.

3.2 Heat transfer analysis

Let us examine the heat transfer characteristics in the flow of an electrically conducting third-grade fluid between porous disks at $z = \pm H$. The disks are non-conducting. Flow is caused by suction/injection and constant magnetic field $\mathbf{B}_0$ is applied perpendicular to planes of disk (along $z-$direction). There is no external electric field. Induced magnetic field is neglected under the assumption of small magnetic Reynolds number. Both the disks are maintained at constant temperature T_w. Joul heating and viscous dissipation are taken into account. All material properties are assumed constant. The equations which can govern the flow are

$$\rho c_p \frac{dT}{dt} = K\nabla^2 T + tr(\tau \mathbf{L}) + \frac{1}{\sigma}\mathbf{J}.\mathbf{J}, \tag{3.1}$$

$$\mathbf{J} = \sigma\left(\mathbf{V} \times \mathbf{B}\right), \tag{3.2}$$

where ρ the fluid density, $\mathbf{J}$ the current density, σ the electrical conductivity of the fluid, K the thermal conductivity, T the temperature, c_p the specific heat, d/dt the material derivative and τ the Cauchy stress tensor in third-grade fluid is

$$\begin{aligned} \tau &= -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2\left(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1\right) \\ &\quad + \beta_3\left(tr\mathbf{A}_1^2\right)\mathbf{A}_1, \end{aligned} \tag{3.3}$$

in which $\mathbf{I}$ is the identity tensor, μ the fluid viscosity and α_1, α_2, β_1, β_2, β_3 the material constants. Note that Eq. (3.3) is compatible with thermodynamic model when the material constants satisfy

$$\mu \geq 0, \alpha_1 \geq 0, \beta_1 = \beta_2 = 0, \beta_3 \geq 0, \alpha_1 + \alpha_2 \leq \sqrt{24\mu\beta_3}. \tag{3.4}$$

The Rivlin-Ericksen tensors $\mathbf{A}_1$ and $\mathbf{A}_2$ are

$$\mathbf{A}_1 = \nabla\mathbf{V} + (\nabla\mathbf{V})^T, \tag{3.5}$$

$$\mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1(\nabla\mathbf{V}) + (\nabla\mathbf{V})^T\mathbf{A}_1, \tag{3.6}$$

The velocity and temperature fields are defined as

$$\mathbf{V} = [u(r,z),\ 0,\ w(r,z)],\quad T = T(r,z). \tag{3.7}$$

The governing equation is

$$\begin{aligned}
\rho c_p\left(u\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial z}\right) &= K\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \sigma B_0^2 u^2 \\
&+\mu\left[2\frac{u^2}{r^2} + 2\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 + 2\frac{\partial u}{\partial z}\frac{\partial w}{\partial r} + \left(\frac{\partial w}{\partial r}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2\right] \\
&+\alpha_1\left[2\frac{u^3}{r^3} + 2\frac{u^2}{r^2}\frac{\partial u}{\partial r} + 4\left(\frac{\partial u}{\partial r}\right)^3 + 2u\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r^2} + 2w\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r\partial z} + 2\frac{uw}{r^2}\frac{\partial u}{\partial z}\right. \\
&+u\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z} + 3\frac{\partial u}{\partial r}\left(\frac{\partial u}{\partial z}\right)^2 + w\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2} + u\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z} + 6\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r} \\
&+w\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial z^2} + 3\frac{\partial u}{\partial r}\left(\frac{\partial w}{\partial r}\right)^2 + u\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r^2} + u\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2} + w\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r\partial z} \\
&+w\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial z} + 3\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial w}{\partial z} + 4\left(\frac{\partial w}{\partial z}\right)^3 + 6\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial w}{\partial z} + 3\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial w}{\partial z} \\
&\left.+2u\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z} + 2w\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}\right] + \alpha_2\left[4\frac{u^3}{r^3} + 4\left(\frac{\partial u}{\partial r}\right)^3 + 3\frac{\partial u}{\partial r}\left(\frac{\partial u}{\partial z}\right)^2\right. \\
&+6\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r} + 4\left(\frac{\partial w}{\partial z}\right)^3 + 3\frac{\partial u}{\partial r}\left(\frac{\partial w}{\partial r}\right)^2 + 3\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial w}{\partial z} + 6\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}
\end{aligned}$$

$$+3\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial w}{\partial z}\Bigg]+\beta_3\Bigg[8\frac{u^4}{r^4}+8\left(\frac{\partial u}{\partial r}\right)^4+16\frac{u^2}{r^2}\left(\frac{\partial u}{\partial r}\right)^2+8\frac{u^2}{r^2}\left(\frac{\partial u}{\partial z}\right)^2$$
$$+8\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{\partial u}{\partial z}\right)^2+2\left(\frac{\partial u}{\partial z}\right)^4+16\frac{u^2}{r^2}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+8\left(\frac{\partial w}{\partial z}\right)^4$$
$$+16\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+8\left(\frac{\partial u}{\partial z}\right)^3\frac{\partial w}{\partial r}+8\frac{u^2}{r^2}\left(\frac{\partial w}{\partial r}\right)^2+8\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{\partial w}{\partial r}\right)^2$$
$$+12\left(\frac{\partial u}{\partial z}\right)^2\left(\frac{\partial w}{\partial r}\right)^2+8\frac{\partial u}{\partial z}\left(\frac{\partial w}{\partial r}\right)^3+2\left(\frac{\partial w}{\partial r}\right)^4+16\frac{u^2}{r^2}\left(\frac{\partial w}{\partial z}\right)^2$$
$$+16\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{\partial w}{\partial z}\right)^2+8\left(\frac{\partial u}{\partial z}\right)^2\left(\frac{\partial w}{\partial z}\right)^2+16\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\left(\frac{\partial w}{\partial z}\right)^2$$
$$+8\left(\frac{\partial w}{\partial r}\right)^2\left(\frac{\partial w}{\partial z}\right)^2\Bigg]. \tag{3.8}$$

The relevant boundary conditions are

$$T(r,H)=T_w, \qquad T(r,-H)=T_w, \tag{3.9}$$

where T_w is the constant temperature at both the disks.

We define

$$u(r,z)=\frac{V_0 r}{2H}f'(\eta),\ w(r,z)=-V_0 f(\eta),\quad \theta(\eta)=\frac{T}{T_w},\quad \eta=\frac{z}{H}, \tag{3.10}$$

Using Eqs. (2.13) and (3.10), The Eqs. (3.8) and (3.9) reduces to

$$\theta''+\Pr\operatorname{Re} f\theta'+\frac{1}{4}Ec\Pr\left[12f'^2+\delta^2f''^2-\alpha\left\{12f'^3+12ff'f''+\delta^2f'f''^2+\delta^2ff''f'''\right\}\right.$$
$$\left.-\frac{3}{2}\gamma\left\{8f'^3+\delta^2f'f''^2\right\}+\frac{\beta}{4}\left\{144f'^4+24\delta^2f'^2f''^2+\delta^4f''^4\right\}+\frac{\operatorname{Re}M^2}{2}f'^2\right]=0, \tag{3.11}$$

$$\theta(1)=1,\ \theta(-1)=1. \tag{3.12}$$

The dimensionless quantities

$$\begin{aligned}\mathrm{Re} &= \frac{V_0 H}{\upsilon},\ M^2 = \frac{\sigma B_0^2 H}{\rho V_0},\ \upsilon = \frac{\mu}{\rho},\ \alpha = \frac{\alpha_1 V_0}{\mu H},\ \delta = \frac{r}{H}, \\ \gamma &= \frac{\alpha_2 V_0}{\mu H},\ \beta = \frac{2\beta_3 V_0^2}{\mu H^2},\ \mathrm{Pr} = \frac{\mu c_p}{K},\ Ec = \frac{V_0^2}{c_p T_w},\end{aligned} \tag{3.13}$$

respectively indicate the Reynolds number (Re), the Prandtl number (Pr), the Eckert number (Ec), the Hartman number (M), third-grade parameters (α, β, γ) and the dimensionless radial distance (δ). It is important to note that $\mathrm{Re} > 0$ corresponds to the case when there is suction at upper disk and blowing at lower disk. However $\mathrm{Re} < 0$ leads to the situtation when there is blowing at upper disk and suction at lower disk.

The Nusselt number at upper and the lower disks are defined as

$$Nu_1 = \frac{Hq_w}{KT_w} = -\frac{HK\ \frac{\partial T}{\partial z}\big|_{z=H}}{KT_w} = -\theta'(1), \tag{3.14}$$

$$Nu_2 = \frac{Hq_w}{KT_w} = -\frac{HK\ \frac{\partial T}{\partial z}\big|_{z=-H}}{KT_w} = -\theta'(-1). \tag{3.15}$$

3.3 Solution procedure

We choose the base function

$$\left\{\eta^{2n}; n \geqslant 0\right\}, \tag{3.16}$$

and write

$$\theta(\eta) = \sum_{n=0}^{\infty} b_n \eta^{2n}, \tag{3.17}$$

where b_n are the coefficients to be dertermined.

For the series solution of $\theta(\eta)$, the initial guess be taken

$$\theta_0(\eta) = 1, \tag{3.18}$$

and an auxiliary linear operator $\mathcal{L}_\theta$ as

$$\mathcal{L}_\theta[\theta(\eta)] = \frac{d^2 f}{d\eta^2}. \tag{3.19}$$

The above linear operator have the following properties

$$\mathcal{L}_\theta[D_1 + D_2\eta] = 0,. \tag{3.20}$$

where $D_i\,(i = 1, 2)$ are the arbitrary constants.

3.3.1 Zeroth order deformation equation

The zeroth-order deformation problem is given by

$$(1 - q)\,\mathcal{L}_\theta\left[\hat{\theta}(\eta, q) - \theta_0(\eta)\right] = q\hbar_\theta\mathcal{N}_\theta\left[\hat{\theta}(\eta, q)\right], \tag{3.21}$$

$$\hat{\theta}(1, q) = 1, \quad \hat{\theta}(-1, q) = 1, \tag{3.22}$$

where $\hbar_\theta \neq 0$ and $q \in [0, 1]$ are respectively the auxiliary and embedding parameters. When q varies from 0 to 1, then $\hat{\theta}(\eta, q)$ varies from initial guess $\theta_0(\eta)$ to final solution $\theta(\eta)$. The non-linear operator is

$$
\begin{aligned}
\mathcal{N}_2\left[\theta\left(\eta,q\right),f\left(\eta,q\right)\right] &= \frac{\partial^2\hat{\theta}}{\partial\eta^2}+\Pr\operatorname{Re}f\frac{\partial\hat{\theta}}{\partial\eta}+\frac{1}{4}\Pr Ec\left[12\left(\frac{\partial\hat{f}}{\partial\eta}\right)^2+\delta^2\left(\frac{\partial^2\hat{f}}{\partial\eta^2}\right)^2-\alpha\left\{12\left(\frac{\partial\hat{f}}{\partial\eta}\right)^3\right.\right. \\
&\left.+12f\frac{\partial\hat{f}}{\partial\eta}\frac{\partial^2\hat{f}}{\partial\eta^2}+\delta^2\frac{\partial\hat{f}}{\partial\eta}\left(\frac{\partial^2\hat{f}}{\partial\eta^2}\right)^2+\delta^2f\frac{\partial^2\hat{f}}{\partial\eta^2}\frac{\partial^3\hat{f}}{\partial\eta^3}\right\}-\frac{3}{2}\gamma\left\{8\left(\frac{\partial\hat{f}}{\partial\eta}\right)^3\right. \\
&\left.+\delta^2\frac{\partial\hat{f}}{\partial\eta}\left(\frac{\partial^2\hat{f}}{\partial\eta^2}\right)^2\right\}+\frac{\beta}{4}\left\{144\left(\frac{\partial\hat{f}}{\partial\eta}\right)^4+24\delta^2\left(\frac{\partial\hat{f}}{\partial\eta}\right)^2\left(\frac{\partial^2\hat{f}}{\partial\eta^2}\right)^2\right. \\
&\left.\left.+\delta^2\left(\frac{\partial^2\hat{f}}{\partial\eta^2}\right)^4\right\}+\frac{\operatorname{Re}M^2\delta^2}{2}\left(\frac{\partial\hat{f}}{\partial\eta}\right)^2\right]. \qquad (3.23)
\end{aligned}
$$

In view of Taylor series expansion we write

$$\hat{\theta}_m\left(\eta\right)=\frac{1}{m!}\frac{\partial^m\theta\left(\eta,q\right)}{\partial q^m}\bigg|_{q=0}, \qquad (3.24)$$

$$\hat{\theta}\left(\eta,q\right)=\theta_0\left(\eta\right)+\sum_{m=1}^{\infty}\theta_m\left(\eta\right)q^m. \qquad (3.25)$$

Obviously Eq. (3.21) has a non-zero auxiliary parameters h_θ. The convergence of the series (3.25) is depend upon h_θ. The values of h_θ is chosen properly so that Eq. (3.25) is convergent at $q=1$. In view of Eq. (3.25) we have

$$\hat{\theta}\left(\eta\right)=\theta_0\left(\eta\right)+\sum_{m=1}^{\infty}\theta_m\left(\eta\right). \qquad (3.26)$$

3.3.2 mth order deformation equations

The m^{th} order deformation problems are obtained by differentiating Eqs. (3.21)$-$ m times with respect to q and then dividing by $m!$ and setting $q=0$. These are given by

$$\mathcal{L}_\theta\left[\hat{\theta}_m\left(\eta\right)-\chi_m\hat{\theta}_{m-1}\left(\eta\right)\right]=\hbar_\theta\mathcal{R}_m^\theta\left(\eta\right), \qquad (3.27)$$

$$\hat{\theta}_m(1,q) = 0, \quad \hat{\theta}_m(-1,q) = 0, \tag{3.28}$$

$$\begin{aligned}
\mathcal{R}_m^{\theta}(\eta) &= \theta''_{m-1}(\eta) + \Pr\operatorname{Re}\sum_{k=0}^{m-1} f_{m-1-k}\theta'_k + \frac{1}{4}Ec\Pr\sum_{k=0}^{m-1}\Big[12f'_{m-1-k}f'_k + \delta^2 f''_{m-1-k}f''_k \\
&-\alpha\sum_{l=0}^{k}\{12f'_{m-1-k}f'_{k-l}f'_l + 12f_{m-1-k}f'_{k-l}f''_l + \delta^2 f'_{m-1-k}f''_{k-l}f''_l \\
&+\delta^2 f_{m-1-k}f''_{k-l}f'''_l\} - \frac{3}{2}\gamma\sum_{l=0}^{k}\{8f'_{m-1-k}f'_{k-l}f'_l + \delta^2 f'_{m-1-k}f''_{k-l}f''_l\} \\
&+\frac{\beta}{4}\sum_{l=0}^{k}\left\{144f'_{m-1-k}f'_{k-l}\sum_{j=0}^{l}f'_{l-j}f'_j + 24\delta^2 f'_{m-1-k}f'_{k-l}\sum_{j=0}^{l}f''_{l-j}f''_j\right. \\
&\left.+\delta^4 f''_{m-1-k}f''_{k-l}\sum_{j=0}^{l}f''_{l-j}f''_j\right\} + \frac{\operatorname{Re}M^2\delta^2}{2}f'_{m-1-k}f'_k\Big],
\end{aligned} \tag{3.29}$$

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases} \tag{3.30}$$

Denoting $\theta^*(\eta)$ as the special solution, we have the following general solution

$$\theta(\eta) = \theta^* + C_5^m + C_6^m\eta, \tag{3.31}$$

in which C_i^m $(i = 1-6)$ are arbitrary constants which can be determined by the boundary conditions in Eq. (3.28).

3.4 Convergence of the derived solution

The derived series solution contain auxiliary parameter $\hbar_\theta$. The convergence of series solution (3.31) strongly depend upon the suitable range of these auxiliary parameter $\hbar_\theta$. For this purpose the $\hbar$-curves is plotted in the Figs. 3.1. From these Figs. it is noted that suitable range for $\hbar_\theta$ is $-0.9 \leq \hbar_\theta < -0.49$. Furthermore, convergence of series solution is checked and shown in Table. 3.1. This table shows that series solutions converge at $20th$ order of approximation up to six decimal places.

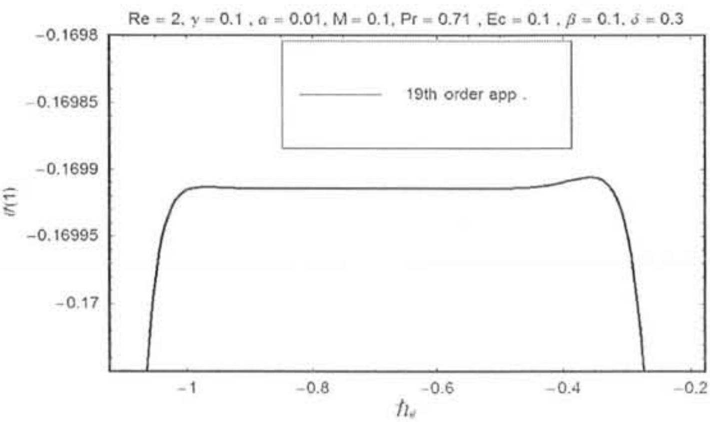

Fig. 3.1 $\hbar$−curve of $\theta'(1)$ at $19th$ order of approximation.

Table 3.1. Convergence of homotopy solutions when $\alpha = 0.01$, $\beta = 0.1$, $\gamma = 0.1$, $\delta = 0.3$, $M = 0.1$, $\mathrm{Re} = 2$, $\mathrm{Pr} = 0.71$, $Ec = 0.1$ and $\hbar_f = \hbar_\theta = -0.7$.

Order of approximations	$-f''(1)$	$-\theta'(1)$
1	3.03822	0.250999
2	2.99133	0.188300
5	2.96884	0.169592
8	2.96795	0.169908
10	2.96798	0.169914
15	2.96798	0.169914
20	2.96798	0.169914
27	2.96798	0.169914
30	2.96798	0.169914
35	2.96798	0.169914
40	2.96798	0.169914

3.5 Results and discussion

This section presents the nonlinear analysis describing heat transfer characteristics in the flow of third grade fluid between two porous disks. Here the effects of dimensionless parameters on dimensionless temperature profile $\theta(\eta)$ is analyzed. The variations of Nusselt number for various values of pertinent parameters are analyzed. Figs. 3.2 and 3.3 are sketched to analyze the influence of third-grade parameter β on dimensionless temperature $\theta(\eta)$. From these Figs. one can observe that $\theta(\eta)$ is an increasing function of third-grade parameter β for both the cases of $\mathrm{Re} > 0$ and $\mathrm{Re} < 0$ whereas $\theta(\eta)$ decreases when α is increased for both $\mathrm{Re} > 0$ and $\mathrm{Re} < 0$ (see Figs. 3.4 and 3.5). Figs 3.6 and 3.7 depict the behavior of Hartman number M on dimensionless temperature $\theta(\eta)$ when $\mathrm{Re} > 0$ and $\mathrm{Re} < 0$. Dimensionless temperature $\theta(\eta)$ increases when M is increased. In fact an increase in Hartman number corresponds to an increase in the applied magnetic field. This increase in external magnetic field results to increase in joule heating, Consequently the temperature increases.This fact is obvious from Figs. 3.8 and 3.9. Eckert number Ec is the ratio of kinetic energy to enthalpy. Thus an increase in Ec corresponds to an increase in kinetic energy of fluid particles. As expected the temperature of fluid increases with an increase in Ec. This fact is observed in the Figs. 3.10 and 3.11. The dimensionless temperature $\theta(\eta)$ is an increasing function of Prandtl number Pr for both cases ($\mathrm{Re} > 0$ and $\mathrm{Re} < 0$). In Figs. 3.12 and 3.13 the temperature profile decrease and increase for $\mathrm{Re} > 0$ and $\mathrm{Re} < 0$ respectively. The temperature profile of γ decreaces for both cases in Figs. 3.14 and 3.15. Table 3.2 is prepared to examine the effects of dimensionless parameters on Nusselt numbers Nu_1 and Nu_2 when $\mathrm{Re} > 0$. This table shows that Nu_1 and Nu_2 are increasing functions of M, Pr, β and Ec whereas Nu_1 and Nu_2 decrease when dimensionless material parameters α and γ are increased. These observations are noted when upper disk is subjected to suction where a lower disk is subjected to blowing. It means that for $\mathrm{Re} > 0$ the rate of heat transfer increases when M, Pr, β and Ec are increased. However rate of heat transfer decreases by increasing material constants α and γ. Similar observations are noted when $\mathrm{Re} < 0$ as shown in Table 3.3.

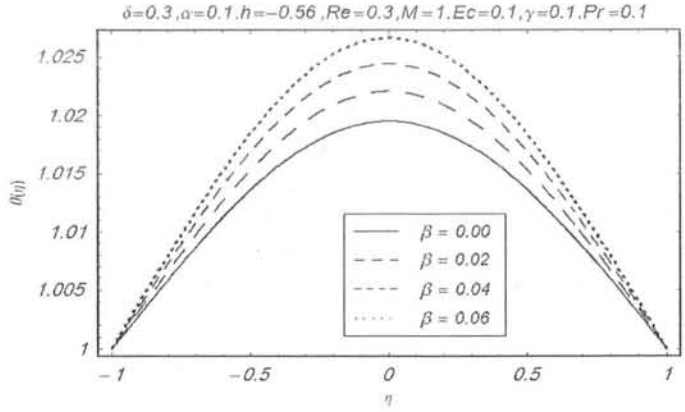

Fig. 3.2 Influence of β on dimensionless temperature $\theta(\eta)$ when $\mathrm{Re} > 0$.

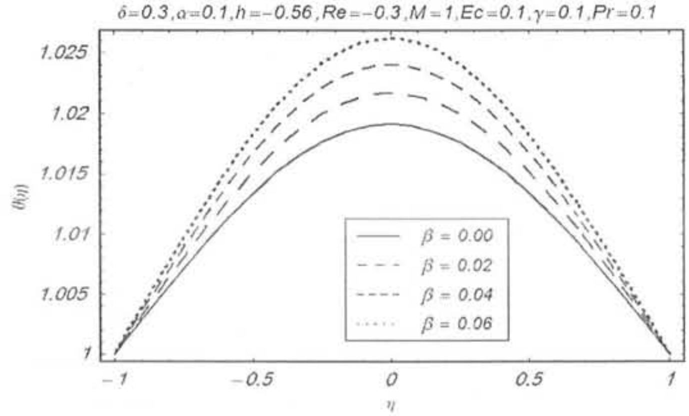

Fig. 3.3 Influence of β on dimensionless temperature $\theta(\eta)$ when $\mathrm{Re} < 0$.

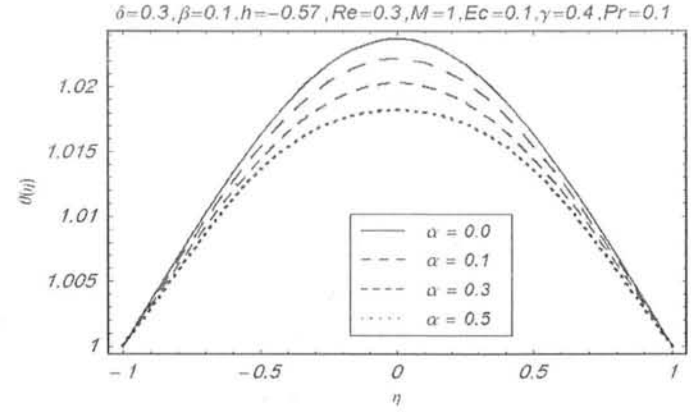

Fig. 3.4 Influence of α on dimensionless temperature $\theta(\eta)$ when $\mathrm{Re} > 0$.

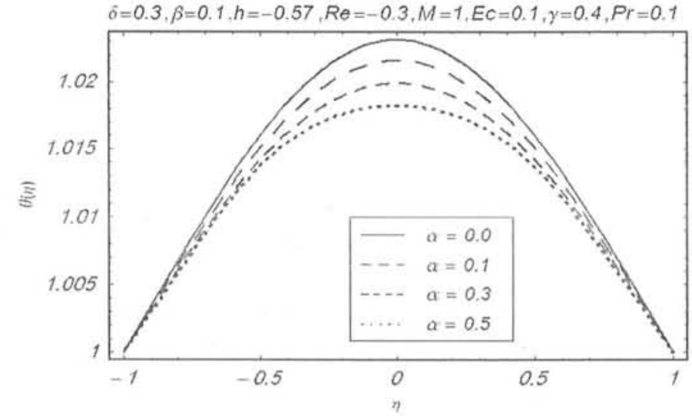

Fig. 3.5 Influence of α on dimensionless temperature $\theta(\eta)$ when $\mathrm{Re} < 0$.

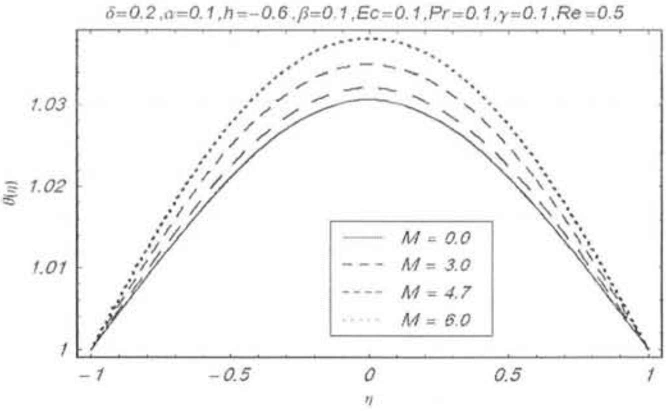

Fig. 3.6 Influence of M on dimensionless temperature $\theta(\eta)$ when $\mathrm{Re} > 0$.

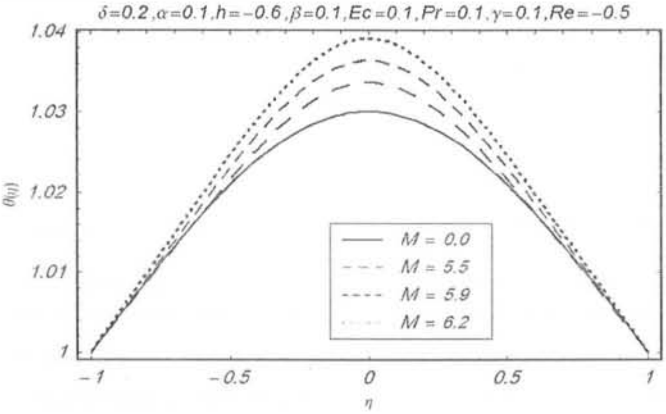

Fig. 3.7 Influence of M on dimensionless temperature $\theta(\eta)$ when $\mathrm{Re} < 0$.

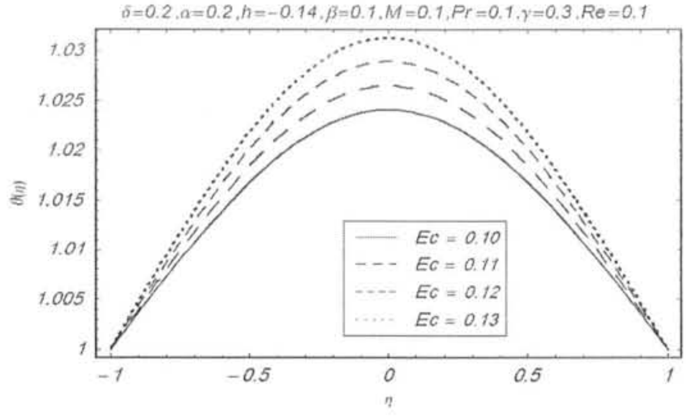

Fig. 3.8 Influence of Ec on dimensionless temperature $\theta\left(\eta\right)$ when $\mathrm{Re} > 0$.

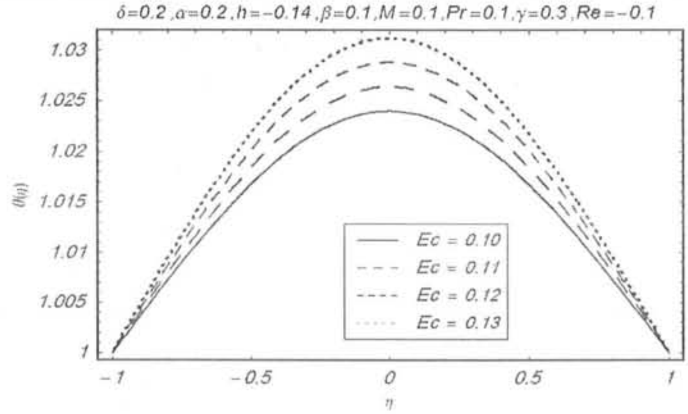

Fig. 3.9 Influence of Ec on dimensionless temperature $\theta\left(\eta\right)$ when $\mathrm{Re} < 0$.

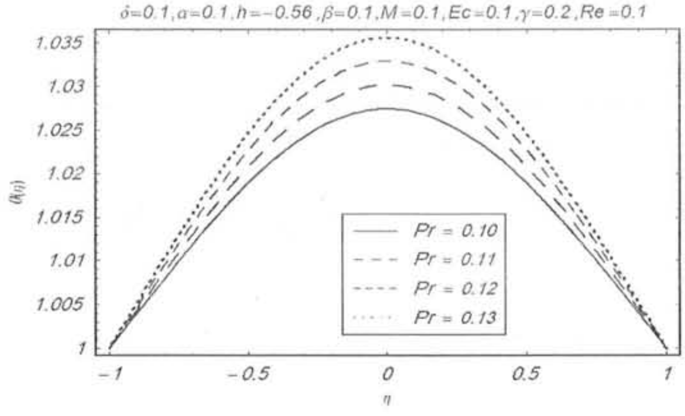

Fig. 3.10 Influence of Pr on dimensionless temperature $\theta(\eta)$ when $\mathrm{Re} > 0$.

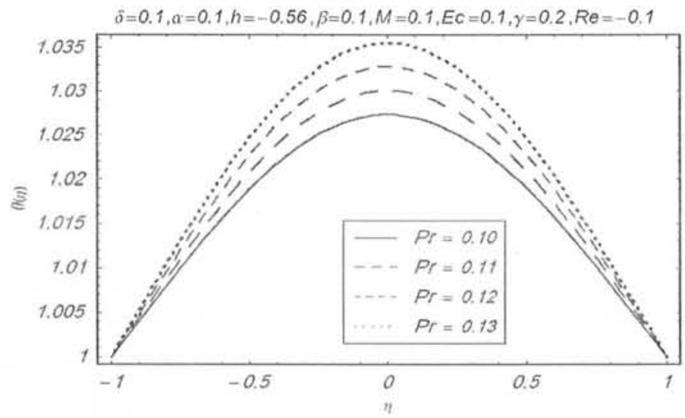

fig. 3.11 Influence of Pr on dimensionless temperature $\theta(\eta)$ when $\mathrm{Re} < 0$.

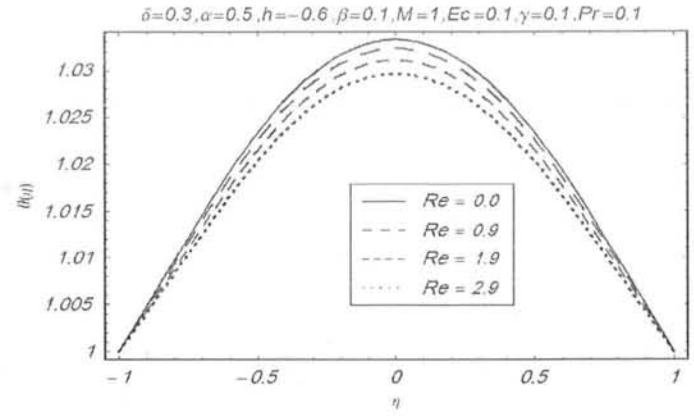

Fig. 3.12 Influence of Re > 0 on dimensionless temperature $\theta(\eta)$.

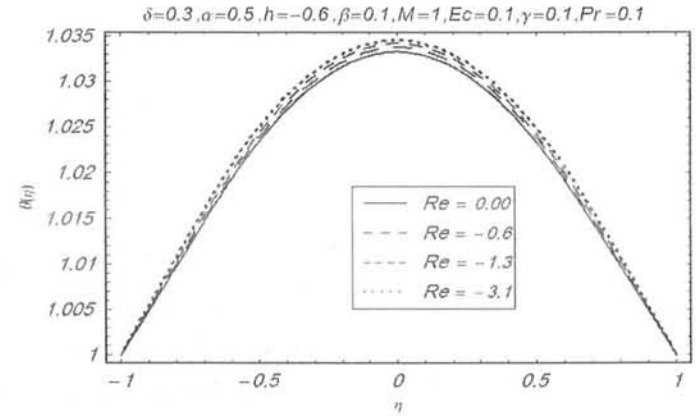

Fig. 3.13 Influence of Re < 0 on dimensionless temperature $\theta(\eta)$.

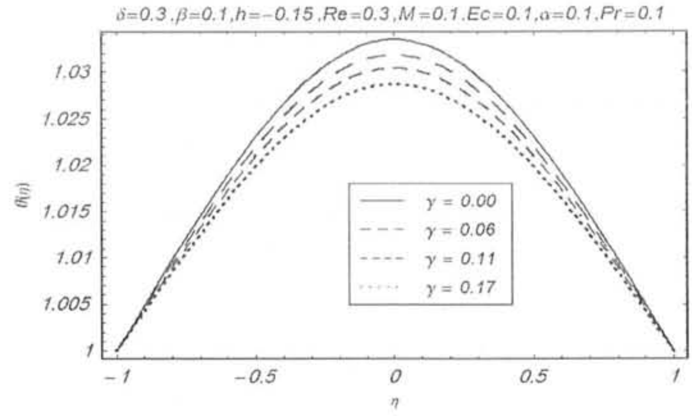

Fig. 3.14 Influence of γ on dimensionless temperature $\theta(\eta)$ when $\mathrm{Re} > 0$.

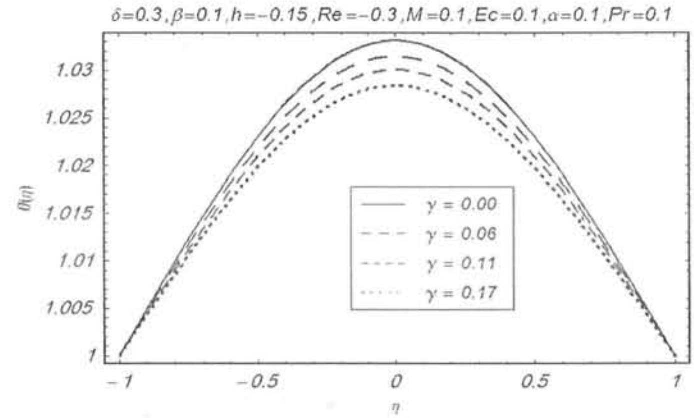

Fig. 3.15 Influence of γ on dimensionless temperature $\theta(\eta)$ when $\mathrm{Re} < 0$.

Table 3.2: Numerical values of Nusselt number Nu_1 at upper disk and Nu_2 at lower disk when $\text{Re} > 0$.

α	γ	β	Re	M	Pr	Ec	Nu_1	$-Nu_2$
0.0	0.1	0.1	1	0.1	0.71	0.1	0.265683	0.265683
0.01							0.264562	0.264562
0.02							0.263478	0.263478
0.03							0.262428	0.262428
0.01	0.0	0.1	2	0.1	0.71	0.1	0.207198	0.207198
	0.1						0.190906	0.190906
	0.2						0.174529	0.174529
	0.3						0.158050	0.158050
0.01	0.01	0.0	2	0.1	0.71	0.1	0.139697	0.139697
		0.1					0.204784	0.204784
		0.11					0.210884	0.210884
		0.12					0.216943	0.216943
0.01	0.01	0.1	0.0	0.1	0.71	0.1	0.397815	0.397815
			0.1				0.384897	0.384897
			0.2				0.372395	0.372395
			0.3				0.360298	0.360298
0.01	0.01	0.1	2	0.0	0.71	0.1	0.204695	0.204695
				0.1			0.204784	0.204784
				0.2			0.205049	0.205049
				0.3			0.205491	0.205491
0.01	0.01	0.1	2	0.1	0.1	0.1	0.0522148	0.0522148
					0.2		0.0945392	0.0945392
					0.3		0.128483	0.128483
0.01	0.71	0.1	0.1	0.1	0.71	0.1	0.164350	0.164350
						0.2	0.328700	0.328700
						0.3	0.493050	0.493050

Table 3.3: Numerical values of Nusselt number Nu_1 at upper disk and Nu_2 at lower disk when $\mathrm{Re} < 0$.

α	γ	β	Re	M	Pr	Ec	Nu_1	$-Nu_2$
0.0	0.1	0.1	−1	0.1	0.71	0.1	0.514328	0.514328
0.01							0.511209	0.511209
0.02							0.508202	0.508202
0.03							0.505320	0.505320
0.01	0.0	0.1	−2	0.1	0.71	0.1	0.774095	0.774095
	0.1						0.710417	0.710417
	0.2						0.646485	0.646485
	0.3						0.582276	0.582276
0.01	0.01	0.0	−2	0.1	0.71	0.1	0.539884	0.539884
		0.1					0.765731	0.765731
		0.11					0.787620	0.787620
		0.12					0.809448	0.809448
0.01	0.01	0.1	0.0	0.1	0.71	0.1	0.397815	0.397815
			−0.1				0.411164	0.411164
			−0.2				0.424956	0.424956
			−0.3				0.439207	0.439207
0.01	0.01	0.1	−2	0.0	0.71	0.1	0.765957	0.765957
				0.1			0.765731	0.765731
				0.2			0.765054	0.765054
				0.3			0.763930	0.763930
0.01	0.01	0.1	−2	0.1	0.1	0.1	0.0595431	0.0595431
					0.2		0.131010	0.131010
					0.3		0.216368	0.216368
0.01	0.71	0.1	−0.1	0.1	0.71	0.1	0.175163	0.175163
						0.2	0.350327	0.350327
						0.3	0.525490	0.525490

3.6 Concluding remarks

This investigation deals with the effect of heat transfer on the flow of a third-grade fluid between porous disks. The behavior of dimensionless parameters on Nusselt number is analyzed. Main findings can be summarized as follows:

- The skin friction coefficient are reduced by suction.
- Behavior of β is opposite to that of γ on $\theta(\eta)$ in a qualitative sense.
- Stresses on the surface of disks in second grade fluid $(\beta = 0)$ is smaller than those in third grade fluid $(\beta \neq 0)$.
- Rate of heat transfer from disks into fluid increases when M, Pr, β and Ec are increased by increasing α and γ.
- An increase in external magnetic field results in an increase of temperature.
- Dimensionless temperature $\theta(\eta)$ increases when β, M, Ec and Pr are increased whereas $\theta(\eta)$ is decreasing function of α and γ.

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