Wiener-Hopf analysis of the parallel plate waveguide with soft boundaries having finite length impedance loading

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Jawad Ahmed

Department of Mathematics
Quaid-I-Azam University
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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTER OF PHILOSOPHY

We accept this dissertation as conforming to the required standard.

1. ____________________
Prof. Dr. Muhammad Ayub
(Chairman)

2. ____________________
Prof. Dr. Muhammad Ayub
(Supervisor)

3. ____________________
Prof. Dr. Aftab khan
(External Examiner)

Department of Mathematics
Quaid-I-Azam University
Islamabad, Pakistan

2011

Dedicated

To

my

Mother and Father

Whose affection is reason of every success in my life.
Who've always given me perpetual love,
care, and cheers. Whose prayers have
always been a source of great
inspiration for me and whose
sustained hope in me led me
to where I stand today.

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I would like to extend my heartfelt gratitude to my loving Mother for her boundless love and my Father who has been always on my side and give me strength in successful completion of my assignment.

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May "Almighty" Allah shower His choicest blessing and prosperity on all those who assisted me in any way during completion of my thesis.

Jawad Ahmed

Preface

Parallel plate wave guide is very important structure for studying the scattering of waves in electromagnetic theory. Engineers, physicist and mathematicians have shown a very keen interest in these types of problems. The grooved and impedance loaded wave guide structures have been frequently used for microwave filter and antenna applications. It has been verified experimentally that rectangular wave guide with finite length impedance loaded can be used as a band-stop filter. Hence, the solution of this problem is of importance for band-stop filter design using a rectangular wave guide with finite impedance loading.

The scattering coefficients for wall impedance change in parallel-plate wave guides have been investigated by several authors. Among them one can cite, for example, Johansen [1] who considered the case where the part $x<0$ of the parallel plates are perfectly conducting while the part $x>0$ has the same surface impedance. Heins and Feshbach [2] provided a Wiener-Hopf solution to the problem of coupling of two ducts. Karajala and Mittra [3] have considered the scattering at the junction of two semi-infinite parallel plate waveguides with impedance walls by mode matching method. Arora and Vijayaraghavan [4] used the Wiener-Hopf technique to compute the scattering of shielded surface wave in a parallel-plate waveguide consisting of inductively reactive guiding surfaces and characterized by an abrupt wall reactance discontinuity. Finally Tayyar et al. [5] have considered the parallel plate wave guide having finite length impedance loading.

In present dissertation some mathematical concepts and theorems [6, 7, 8] are given in the first chapter, which are necessary to understand the subsequent chapters. In the second chapter the paper by Tayyar et al. [5] is reproduced with more details. In third chapter we have extended the work of Tayyar et al. [14] by considering the soft boundaries having finite length impedance loading. The representation of the solution to the boundary-value problem in terms of Fourier integrals leads to two simultaneous modified Wiener-Hopf equations which are uncoupled by using the pole removal technique. The solution involves four infinite sets of unknown coefficients satisfying four infinite systems of linear algebraic equations. At the end reflection and transmission coefficients are calculated from these infinite systems of linear algebraic equations.

Contents

Chapter 1

Preliminaries

In this chapter some mathematical preliminaries, used in the subsequent chapters, are presented. These include Decomposition theorem, Factorization theorem, Extended Livoulle's theorem, Wiener-Hopf technique and analytic continuation principle [6, 7, 8].

1.1 Extended Livouville's Theorem[6]

If $f(\xi)$ is an integral function such that $\mid f(\xi) \mid \leq M \mid \xi \mid^p$ as $\mid \xi \mid \rightarrow \infty$ where M, p are constants, then $f(\xi)$ is a polynomial of degree less than or equal to $[p]$ where $[p]$ is the integral part of p.

1.2 Decomposition Theorem[6]

Let $f(\nu)$ be an analytic function of $\nu = \sigma + i\tau$ and regular in the strip $\tau_- < \tau < \tau_+$. Within this strip, $|f(\nu)| \rightarrow 0$ uniformly as $|\sigma| \rightarrow \infty$. Then $f(\nu)$ can be decomposed such that

$$f(\nu) = f_+(\nu) + f_-(\nu), \tag{1.1}$$

where

$$f_+(\nu) = \frac{1}{2\pi i} \int_{-\infty+ic}^{\infty+ic} \frac{f(\xi)}{\xi - \nu} d\xi, \qquad \tau_- < c < \tau < \tau_+, \tag{1.2}$$

is regular in the upper half ν-plane defined by $\tau_- < \tau$ and

$$f_-(\nu) = \frac{-1}{2\pi i} \int_{-\infty+id}^{\infty+id} \frac{f(\xi)}{\xi - \nu} d\xi, \qquad \tau_- < \tau < d < \tau_+, \tag{1.3}$$

is regular in the lower half ν-plane defined by $\tau < \tau_+$.

1.3 Factorization Theorem[7]

Let $f(\nu)$ be an analytic function of $\nu = \sigma + i\tau$ and be regular and be non zero in the strip $\tau_- < \tau < \tau_+$. Within this strip, $|f(\nu)| \to 1$ uniformly as $|\sigma| \to \infty$. Then $g(\nu)$ can be factorized such that

$$g(\nu) = g_+(\nu) g_-(\nu), \tag{1.4}$$

where

$$g_+(\nu) = \exp\left[\frac{1}{2\pi i} \int_{-\infty+ic}^{\infty+ic} \frac{\ln g(\xi)}{\xi - \nu} d\xi\right], \qquad \tau_- < c < \tau < \tau_+, \tag{1.5}$$

is regular and non zero in the upper half ν-plane defined by $\tau_- < \tau$ and

$$g_-(\nu) = \exp\left[\frac{-1}{2\pi i} \int_{-\infty+id}^{\infty+id} \frac{\ln g(\xi)}{\xi - \nu} d\xi\right], \qquad \tau_- < \tau < d < \tau_+, \tag{1.6}$$

is regular and non zero in the lower half ν-plane defined by $\tau < \tau_+$.

1.4 Analytic Continuation[8]

The intersection of two domain H_1 and H_2 is the domain $H_1 \cap H_2$ consisting of all points that lies in both H_1 and H_2. If we have two domains H_1 and H_2 with points in common and a function f_1 that is analytic in H_1, there may exist a function f_2 which is analytic in H_2, such that $f_2(z) = f_1(z)$ for each z in the intersection $H_1 \cap H_2$. If so, we call f_2 an analytic continuation of f_1 into the second domain H_2.

1.5 General Scheme of Wiener-Hopf Technique[6]

Wiener-Hopf technique is used for solving certain integral equations and various boundary value problems of mathematical physics by means of integral transformation. In this technique, we require to determine the unknown function $\psi_+(\nu)$ and $\psi_-(\nu)$ of a complex variable ν from the equation given below. The function $\psi_+(\nu)$ and $\psi_-(\nu)$ are analytic, respectively, in the half planes Im $\nu > \tau_-$ and Im $\nu < \tau_+$ tend to zero as $|\nu| \longrightarrow \infty$ in both domains of analyticity and satisfy in the strip $\tau_- < \tau < \tau_+$,

$$D_1(\nu)\,\psi_+(\nu) + D_2(\nu)\,\psi_-(\nu) + D_3(\nu) = 0, \tag{1.7}$$

where the functions $D_1(\nu)$, $D_2(\nu)$ and $D_3(\nu)$ are known functions regular in the strip $\tau_- < \tau < \tau_+$ and $D_1(\nu)$ and $D_2(\nu)$ are non zero in the strip. The basic idea in this technique for solution of this equation is to substitute

$$\frac{D_1(\nu)}{D_2(\nu)} = \frac{T_+(\nu)}{T_-(\nu)}, \tag{1.8}$$

where the function $T_+(\nu)$ and $T_-(\nu)$ are analytic and different from zero, respectively, in $\tau > \tau_-$ and $\tau < \tau_+$. By using Eq. (1.7) and (1.8), we obtain

$$T_+(\nu)\,\psi_+(\nu) + T_-(\nu)\,\psi_-(\nu) + T_-(\nu)\,\frac{D_3(\nu)}{D_2(\nu)} = 0. \tag{1.9}$$

The last term of the above equation can be written as

$$T_-(\nu)\frac{D_3(\nu)}{D_2(\nu)} = D_+(\nu) + D_-(\nu), \tag{1.10}$$

where the functions $D_+(\nu)$ and $D_-(\nu)$ are analytic in the half planes $\tau > \tau_-$ and $\tau < \tau_+$. Thus Eq. (1.9) takes the form

$$T_+(\nu)\psi_+(\nu) + D_+(\nu) = -T_-(\nu)\psi_-(\nu) - D_-(\nu). \tag{1.11}$$

We observe that $L.H.S.$ of Eq. (1.11) is regular in $\tau > \tau_-$ while the $R.H.S$ is regular in $\tau < \tau_+$. Thus, both sides are equal to a certain integral function (polynomial) $J(\nu)$ in the strip $\tau_- < \tau < \tau_+$

$$J(\nu) = T_+(\nu)\psi_+(\nu) + D_+(\nu) = -T_-(\nu)\psi_-(\nu) - D_-(\nu). \tag{1.12}$$

By using analytic continuation, we can determine $J(\nu)$ which is regular in the whole complex $\nu - plane$. Let

$$| T_+(\nu)\psi_+(\nu) + D_+(\nu) | < | \nu |^p \; as \;\; \nu \longrightarrow \infty, \; \tau > \tau_-, \tag{1.13}$$

$$| -T_-(\nu)\psi_-(\nu) - D_-(\nu) | < | \nu |^p \; as \;\; \nu \longrightarrow \infty, \; \tau < \tau_+. \tag{1.14}$$

Chapter 2

A Wiener-Hopf analysis of the parallel plate waveguide with finite length impedance loading

In this chapter the review of recent paper by *Tayyar et al.* [5] is presented in detail. A Wiener-Hopf technique is used to study the band-stop filter characteristic of the parallel plate waveguide with finite length impedance loading. The boundary value problem gives two simultaneous modified Wiener-Hopf equations which are uncoupled by using pole removal technique. The solution involves four infinite sets of unknown coefficients satisfying four infinite system of linear algebraic equations. At the end reflection and transmission coefficients are determined from these system of linear algebraic equations.

2.1 Introduction

In the present analysis *Tayyar et al.* [5] have considered the parallel plates waveguide with a finite length impedance loading as depicted in figure below. The part $x < 0$ and $x > l$, are perfectly conducting at $y = 0$ and $y = b$, and the part $0 < x < l$, have constant surface impedances at $y = 0$ and $y = b$. The surface impedances of the lower and upper plates are different from each other and denoted by $Z_1 = \eta_1 Z_0$ and $Z_2 = \eta_2 Z_0$, respectively, with Z_0 being the characteristic impedance of the free space.

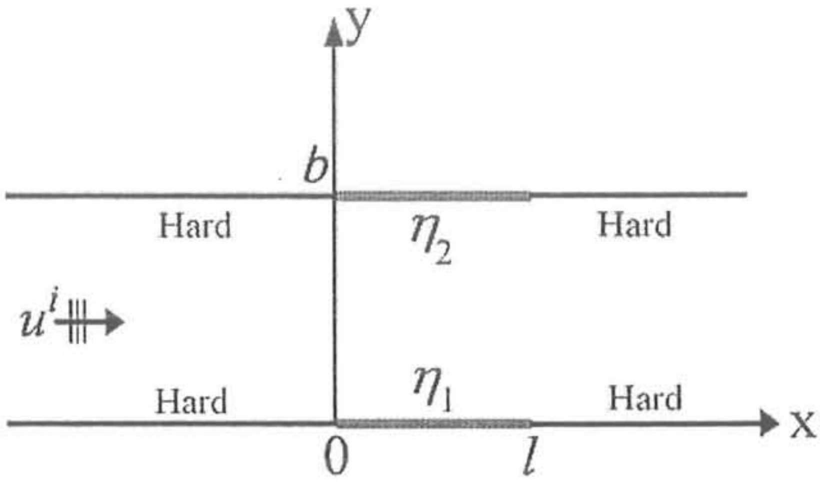

Figure 2.1: Geometry of the problem.

2.2 Mathematical Formulation

Let the incident TEM mode propagating in the positive x direction be given by

$$H_z^i = u^i = e^{ikx}. \tag{2.1}$$

The total field $u^T(x, y)$ can be written as

$$u^T(x, y) = u^i(x, y) + u(x, y), \qquad y\epsilon(0, b) \text{ and } x\epsilon(-\infty, \infty), \tag{2.2}$$

In Eq. (2.2), $u(x, y)$ is the unknown function which satisfied the Helmholtz equation

$$\Delta^2 u(x, y) + k^2 u(x, y) = 0. \tag{2.3}$$

The corresponding boundary conditions are

$$\frac{\partial u(x,0)}{\partial y} = 0, \qquad -\infty < x < 0 \text{ and } l < x < \infty, \tag{2.4}$$

$$\frac{\partial u(x,b)}{\partial y} = 0, \qquad -\infty < x < 0 \text{ and } l < x < \infty, \tag{2.5}$$

$$(1 + \frac{1}{ik\eta_1}\frac{\partial}{\partial y})u(x,0) = -e^{ikx}, \qquad 0 < x < l, \tag{2.6}$$

$$(1 - \frac{1}{ik\eta_2}\frac{\partial}{\partial y})u(x,b) = -e^{ikx}, \qquad 0 < x < l. \tag{2.7}$$

To obtain the unique solution to the mixed boundary value problem the edge and radiation conditions are

$$u^T(x,0) = \begin{cases} O(|\, x \,|^{1/2}), |\, x \,| \to 0 \\ O(|\, x - l \,|^{1/2}), |\, x \,| \to l \end{cases}, \tag{2.8}$$

$$u(x,y) = O(e^{ik|x|}), |\, x \,| \to \infty. \tag{2.9}$$

Taking Fourier transform of Eq. (2.3), we obtain

$$\left[\frac{d^2}{dy^2} + K^2(\alpha)\right]\bar{u}(\alpha,y) = 0, \tag{2.10}$$

where

$$K(\alpha) = \sqrt{k^2 - \alpha^2}, \tag{2.11}$$

and

$$\bar{u}(\alpha,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,y)\; e^{i\alpha x}dx. \tag{2.12}$$

Taking inverse Fourier transform, we get

$$u(x,y) = \int_{-\infty}^{\infty} \bar{u}(\alpha,y)e^{i\alpha x}dx, \tag{2.13}$$

Taking Fourier transform of Eqs. (2.4) and (2.5), we obtain

$$\frac{\partial \bar{u}(\alpha,0)}{\partial y}=0, \qquad -\infty<x<0 \ \text{ and } \ l<x<\infty, \tag{2.14}$$

$$\frac{\partial \bar{u}(\alpha,b)}{\partial y}=0, \qquad -\infty<x<0 \ \text{ and } \ l<x<\infty. \tag{2.15}$$

Taking Fourier transform of Eq. (2.6), we obtain

$$(1+\frac{1}{ik\eta_1}\frac{\partial}{\partial y})\bar{u}(\alpha,0)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}(1+\frac{1}{ik\eta_1}\frac{\partial}{\partial y})u(x,0)e^{i\alpha x}dx,$$

$$\begin{aligned}(1+\frac{1}{ik\eta_1}\frac{\partial}{\partial y})\bar{u}(\alpha,0) \ &= \ \frac{1}{2\pi}\int\limits_{-\infty}^{0}(1+\frac{1}{ik\eta_1}\frac{\partial}{\partial y})u(x,0)e^{i\alpha x}dx+\frac{1}{2\pi}\int\limits_{0}^{l}(1+\frac{1}{ik\eta_1}\frac{\partial}{\partial y})u(x,0)e^{i\alpha x}dx \\ &\quad +\frac{1}{2\pi}\int\limits_{l}^{\infty}(1+\frac{1}{ik\eta_1}\frac{\partial}{\partial y})u(x,0)e^{i\alpha x}dx,\end{aligned}$$

$$(1+\frac{1}{ik\eta_1}\frac{\partial}{\partial y})\bar{u}(\alpha,0)=\Phi_1^-(\alpha)-\frac{1}{2\pi i}\frac{e^{i(\alpha+k)l}-1}{\alpha+k}+e^{i\alpha l}\Phi_1^+(\alpha), \tag{2.16}$$

where

$$\Phi_1^-(\alpha)=\frac{1}{2\pi}\int\limits_{-\infty}^{0}(1+\frac{1}{ik\eta_1}\frac{\partial}{\partial y})u(x,0)e^{i\alpha x}dx, \tag{2.17}$$

$$\Phi_1^+(\alpha)=\frac{1}{2\pi}\int\limits_{l}^{\infty}(1+\frac{1}{ik\eta_1}\frac{\partial}{\partial y})u(x,0)e^{i\alpha(x-l)}dx. \tag{2.18}$$

Similarly taking Fourier transform of Eq. (2.7), we obtain

$$(1-\frac{1}{ik\eta_2}\frac{\partial}{\partial y})\bar{u}(\alpha,b)=\Phi_2^-(\alpha)-\frac{1}{2\pi i}\frac{e^{i(\alpha+k)l}-1}{\alpha+k}+e^{i\alpha l}\Phi_2^+(\alpha), \tag{2.19}$$

where

$$\Phi_2^-(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{0}(1 - \frac{1}{ik\eta_2}\frac{\partial}{\partial y})u(x,b)e^{i\alpha x}dx, \tag{2.20}$$

$$\Phi_2^+(\alpha) = \frac{1}{2\pi}\int_{l}^{\infty}(1 - \frac{1}{ik\eta_2}\frac{\partial}{\partial y})u(x,b)e^{i\alpha(x-l)}dx. \tag{2.21}$$

The complementary solution of Eq. (2.10) is of the form

$$\bar{u}(\alpha, y) = A(\alpha)\cos\left[K(\alpha)y\right] + B(\alpha)\sin\left[K(\alpha)y\right], \tag{2.22}$$

where A and B are the function of α. Putting Eq. (2.22) in Eq. (2.13), we obtain

$$u(x,y) = \int_{-\infty}^{\infty}\left\{A(\alpha)\cos\left[K(\alpha)y\right] + B(\alpha)\sin\left[K(\alpha)y\right]\right\}e^{-i\alpha x}d\alpha. \tag{2.23}$$

Using Eq. (2.14) in Eq. (2.22), we obtain

$$B(\alpha) = \frac{F_1(\alpha)}{K(\alpha)}, \tag{2.24}$$

where

$$F_1(\alpha) = \frac{1}{2\pi}\int_{0}^{l}\frac{\partial u(x,0)}{\partial y}e^{i\alpha x}dx. \tag{2.25}$$

Using Eq. (2.15) in Eq. (2.22), we obtain

$$-A(\alpha)\sin[K(\alpha)b] + B(\alpha)\cos[K(\alpha)b] = \frac{F_2(\alpha)}{K(\alpha)}, \tag{2.26}$$

where

$$F_2(\alpha) = \frac{1}{2\pi}\int_{0}^{l}\frac{\partial u(x,b)}{\partial y}e^{i\alpha x}dx. \tag{2.27}$$

Using Eq. (2.16) in Eq. (2.22), we obtain

$$A(\alpha) + \frac{B(\alpha)K(\alpha)}{ik\eta_1} = \Phi_1^-(\alpha) - \frac{1}{2\pi i}\frac{e^{i(\alpha+k)l}-1}{\alpha+k} + e^{i\alpha l}\Phi_1^+(\alpha). \tag{2.28}$$

Using Eq. (2.19) in Eq. (2.22), we obtain

$$\begin{aligned}&\frac{A(\alpha)}{\eta_2}\left[\eta_2\cos[K(\alpha)b] + \frac{K(\alpha)}{ik}\sin[K(\alpha)b]\right] + \frac{B(\alpha)}{\eta_2}\left[\eta_2\sin[K(\alpha)b] - \frac{K(\alpha)}{ik}\cos[K(\alpha)b]\right]\\ =\ &\Phi_2^-(\alpha) - \frac{1}{2\pi i}\frac{e^{i(\alpha+k)l}-1}{(\alpha+k)} + e^{i\alpha l}\Phi_2^+(\alpha). \end{aligned}\tag{2.29}$$

Incorporating value of $B(\alpha)$ from Eq. (2.24) in Eq. (2.26), we obtain

$$A(\alpha) = \frac{F_1(\alpha)\cos[K(\alpha)b] - F_2(\alpha)}{K(\alpha)\sin[K(\alpha)b]}. \tag{2.30}$$

Using Eqs. (2.24) and (2.30) in Eq. (2.28), we arrive at

$$\frac{F_1(\alpha)\cos[K(\alpha)b] - F_2(\alpha)}{K(\alpha)\sin[K(\alpha)b]} + \frac{F_1(\alpha)}{ik\eta_1} = \Phi_1^-(\alpha) - \frac{1}{2\pi i}\frac{e^{i(\alpha+k)l}-1}{(\alpha+k)} + e^{i\alpha l}\Phi_1^+(\alpha),$$

$$\begin{aligned}&\frac{F_1(\alpha)}{\eta_1 K(\alpha)\sin[K(\alpha)b]}\left[\eta_1\cos[K(\alpha)b] + \frac{K(\alpha)}{ik}\sin[K(\alpha)]b\right] - \frac{F_2(\alpha)}{K(\alpha)\sin[K(\alpha)b]}\\ &-\Phi_1^-(\alpha) - \frac{1}{2\pi i}\frac{1}{(\alpha+k)} = e^{i\alpha l}\left[\Phi_1^+(\alpha) - \frac{1}{2\pi i}\frac{e^{ilk}}{(\alpha+k)}\right],\end{aligned}\tag{2.31}$$

$$\frac{F_1(\alpha)M_1(\alpha)}{\eta_1 N(\alpha)K^2(\alpha)} - \frac{F_2(\alpha)}{N(\alpha)K^2(\alpha)} + P_*^-(\alpha) = e^{i\alpha l}R^+(\alpha), \tag{2.32}$$

where $M_1(\alpha)$, $N(\alpha)$, $P_*^-(\alpha)$ and $R^+(\alpha)$ are defined as respectively,

$$M_1(\alpha) = \eta_1\cos[K(\alpha)b] + \frac{K(\alpha)}{ik}\sin[K(\alpha)b], \tag{2.33}$$

$$N(\alpha) = \frac{\sin[K(\alpha)b]}{K(\alpha)}, \tag{2.34}$$

$$P_*^-(\alpha) = -\Phi_1^-(\alpha) - \frac{1}{2\pi i}\frac{1}{\alpha+k}, \tag{2.35}$$

and

$$R^+(\alpha) = \Phi_1^+(\alpha) - \frac{1}{2\pi i}\frac{e^{ilk}}{\alpha + k}. \tag{2.36}$$

Using Eqs. (2.24) and (2.30) in Eq. (2.29), we obtain

$$\frac{F_1(\alpha)\cos[K(\alpha)b] - F_2(\alpha)}{\eta_2 K(\alpha)\sin[K(\alpha)b]}\left[\eta_2\cos[K(\alpha)b] + \frac{K(\alpha)}{ik}\sin[K(\alpha)b\right]$$

$$+\frac{F_1(\alpha)}{\eta_2 K(\alpha)}\left[\eta_2\sin[K(\alpha)b] - \frac{K(\alpha)}{ik}\cos[K(\alpha)b]\right]$$

$$= \Phi_2^-(\alpha) + e^{i\alpha l}\Phi_2^+(\alpha) - \frac{1}{2\pi i}\frac{e^{i(\alpha+k)l} - 1}{\alpha + k},$$

$$\begin{aligned}
&\frac{F_1(\alpha)}{\eta_2 K(\alpha)\sin[K(\alpha)b]}\left\{\begin{array}{l} \eta_2\cos^2[K(\alpha)b] + \frac{K(\alpha)}{ik}\sin[K(\alpha)b]\cos[K(\alpha)b] \\ +\eta_2\sin^2[K(\alpha)b] - \frac{K(\alpha)}{ik}\sin[K(\alpha)b]\cos[K(\alpha)b] \end{array}\right\} \\
&-\frac{F_2(\alpha)M_2(\alpha)}{\eta_2 K(\alpha)\sin[K(\alpha)b]} - \Phi_2^-(\alpha) - \frac{1}{2\pi i}\frac{1}{\alpha + k} \\
= \; & e^{i\alpha l}\left[\Phi_2^+(\alpha) - \frac{1}{2\pi i}\frac{e^{ilk}}{\alpha + k}\right],
\end{aligned}$$

$$-\frac{F_2(\alpha)M_2(\alpha)}{\eta_2 K^2(\alpha)N(\alpha)} + \frac{F_1(\alpha)}{K^2(\alpha)N(\alpha)} + Q_*^-(\alpha) = e^{i\alpha l}S^+(\alpha), \tag{2.37}$$

where $M_2(\alpha), S^+(\alpha)$ and $Q_*^-(\alpha)$ are defined as respectively,

$$M_2(\alpha) = \eta_2\cos[K(\alpha)b] + \frac{K(\alpha)}{ik}\sin[K(\alpha)b],$$

$$Q_*^-(\alpha) = -\Phi_2^-(\alpha) - \frac{1}{2\pi i}\frac{1}{\alpha + k}, \tag{2.38}$$

$$S^+(\alpha) = \Phi_2^+(\alpha) - \frac{1}{2\pi i}\frac{e^{ilk}}{\alpha + k}. \tag{2.39}$$

Notice that $P_*^-(\alpha)$ and $Q_*^-(\alpha)$ are regular in the lower half plane except at the pole singularity occurring at $\alpha = -k$.

2.3 Solution of the Simultaneous Modified Wiener-Hopf Equations

The kernel factorization of $M_{1,2}(\alpha)$ and $N(\alpha)$ appearing in Eqs. (2.32) and (2.37), are as follow

$$M_{1,2}(\alpha) = M_{1,2}^{+}(\alpha)M_{1,2}^{-}(\alpha), \tag{2.40}$$

$$N(\alpha) = N^{+}(\alpha)N^{-}(\alpha). \tag{2.41}$$

The explicit expression of $M_{1,2}^{+}(\alpha)$ and $N^{+}(\alpha)$ can be written as procedure outlined by *Lee and Mittra* [7]

$$M_1^{+}(\alpha) = [\eta_1 \cos(kb) + \frac{1}{i}\sin(kb)]^{1/2}.\exp\{\frac{i\alpha b}{\pi}[1 - C - \ln(\frac{|\alpha| b}{\pi}) + i\frac{\pi}{2}]\}$$
$$\times \prod_{m=1}^{\infty}(1 + \frac{\alpha}{\beta_m})\exp(\frac{i\alpha b}{m\pi}), \tag{2.42}$$

$$M_2^{+}(\alpha) = [\eta_2 \cos(kb) + \frac{1}{i}\sin(kb)]^{1/2}.\exp\{\frac{i\alpha b}{\pi}[1 - C - \ln(\frac{|\alpha| b}{\pi}) + i\frac{\pi}{2}]\}$$
$$\times \prod_{m=1}^{\infty}(1 + \frac{\alpha}{\nu_m})\exp(\frac{i\alpha b}{m\pi}), \tag{2.43}$$

$$N^{+}(\alpha) = [\frac{\sin(kb)}{k}]^{1/2}.\exp\{\frac{i\alpha b}{\pi}[1 - C - \ln(\frac{|\alpha| b}{\pi}) + i\frac{\pi}{2}]\}$$
$$\times \prod_{m=1}^{\infty}(1 + \frac{\alpha}{\alpha_m})\exp(\frac{i\alpha b}{m\pi}). \tag{2.44}$$

Here $\beta_m's$, $\nu_m's$ and $\alpha_m's$ are the roots of the functions $M_{1,2}(\alpha)$ and $N(\alpha)$, respectively.

$$M_1(\pm\beta_m) = 0, \quad M_1(\pm\nu_m) = 0\ , \quad N(\pm\alpha_m) = 0 \qquad\qquad m = 1, 2, 3...\ , \tag{2.45}$$

with

$$M_{1,2}^{-}(\alpha) = M_{1,2}^{+}(-\alpha), \qquad N^{-}(\alpha) = N^{+}(-\alpha).$$

In Eqs. (2.42)- (2.44), C is the Euler's constant given by $C = 0.57721...$. It can be easily shown that one has

$$M^{\pm}_{1,2}(\alpha) = \mid \alpha \mid^{1/2}, \quad N^{\pm}(\alpha) = \mid \alpha \mid^{-1/2} .$$

Multiplying Eq. (2.32) with $\frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)}$, we obtain

$$\frac{F_1(\alpha)M_1^+(\alpha)}{\eta_1 N^+(\alpha)(k+\alpha)} - \frac{F_2(\alpha)M_1^+(\alpha)}{N^+(\alpha)(k+\alpha)M_1(\alpha)} + \frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)}P_*^-(\alpha) = e^{i\alpha l}\frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)}R^+(\alpha). \tag{2.46}$$

Now multiplying Eq. (2.32) with $e^{-i\alpha l}\frac{(k+\alpha)N^+(\alpha)}{M_1^+(\alpha)}$, we obtain

$$e^{-i\alpha l}\frac{F_1(\alpha)M_1^-(\alpha)}{\eta_1 N^-(\alpha)(k-\alpha)} - e^{-i\alpha l}\frac{F_2(\alpha)M_1^-(\alpha)}{N^-(\alpha)(k-\alpha)M_1(\alpha)} + e^{-i\alpha l}\frac{(k+\alpha)N^+(\alpha)}{M_1^+(\alpha)}P_*^-(\alpha)$$

$$= \frac{(k+\alpha)N^+(\alpha)}{M_1^+(\alpha)}R^+(\alpha). \tag{2.47}$$

Multiplying Eq. (2.37) with $\frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)}$, we obtain

$$-\frac{F_2(\alpha)M_2^+(\alpha)}{\eta_2 N^+(\alpha)(k+\alpha)} + \frac{F_1(\alpha)M_2^+(\alpha)}{N^+(\alpha)(k+\alpha)M_2(\alpha)} + \frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)}Q_*^-(\alpha) = e^{i\alpha l}\frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)}S^+(\alpha). \tag{2.48}$$

Now multiplying Eq. (2.37) with $e^{-i\alpha l}\frac{(k+\alpha)N^+(\alpha)}{M_2^+(\alpha)}$, we obtain

$$-e^{-i\alpha l}\frac{F_2(\alpha)M_2^-(\alpha)}{\eta_2 N^-(\alpha)(k-\alpha)} + e^{-i\alpha l}\frac{F_1(\alpha)M_2^-(\alpha)}{N^-(\alpha)(k-\alpha)M_2(\alpha)} + e^{-i\alpha l}\frac{(k+\alpha)N^+(\alpha)}{M_2^+(\alpha)}Q_*^-(\alpha)$$

$$= \frac{(k+\alpha)N^+(\alpha)}{M_2^+(\alpha)}S^+(\alpha). \tag{2.49}$$

The first term appearing in the left hand side of Eq. (2.46) is evidently regular in the upper half plane. The third term and the R.H.S. of same equation have singularities in both half-planes. Hence one has to apply the Wiener-Hopf decomposition procedure on these terms. Consider the third term of Eq. (2.46) (we want to make it regular in the lower half plane)

$$\frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)}P_*^-(\alpha) = -\frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)}\Phi_1^-(\alpha) - \frac{1}{2\pi i}\frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)}\frac{1}{(\alpha+k)}. \quad (2.50)$$

Consider the 2nd term of Eq. (2.50). Let

$$f(\alpha) = \frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)}\frac{1}{(\alpha+k)}, \quad (2.51)$$

be decomposed by decomposition theorem

$$f(\alpha) = f_+(\alpha) + f_-(\alpha), \quad (2.52)$$

where

$$f_+(\alpha) = \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{f(\xi)}{\xi-\alpha}d\xi,$$

$$f_+(\alpha) = \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{(k-\xi)N^-(\xi)}{M_1^-(\xi)(\xi-\alpha)(\xi+k)}d\xi.$$

Completing the contour by semi circle in the upper half plane then $\xi=\alpha$ and $\xi=-k$ are the simple poles which gives

$$\operatorname*{Res}_{\xi=\alpha}[f_+(\alpha)] = \frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)}\frac{1}{(\alpha+k)},$$

$$\operatorname*{Res}_{\xi=-k}[f_+(\alpha)] = \frac{-(2k)N^+(k)}{M_1^+(k)(\alpha+k)},$$

so that

$$f_+(\alpha) = \frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)}\frac{1}{(\alpha+k)} - \frac{(2k)N^+(k)}{M_1^+(k)(\alpha+k)}.$$

Define

$$f_-(\alpha) = -\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id}\frac{f(\xi)}{\xi-\alpha}d\xi,$$

$$f_-(\alpha) = -\frac{1}{2\pi i} \int\limits_{-\infty+id}^{\infty+id} \frac{(k-\xi)N^-(\xi)}{M_1^-(\xi)(\xi-\alpha)(\xi+k)} d\xi.$$

Completing the contour by semi circle in the upper half plane then $\xi = -k$ is the simple pole which gives

$$f_-(\alpha) = \frac{(2k)N^+(k)}{M_1^+(k)(\alpha+k)}.$$

Thus, Eq. (2.52) becomes

$$\frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)} \frac{1}{(\alpha+k)} = \frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)} \frac{1}{(\alpha+k)} - \frac{(2k)\,N^+(k)}{M_1^+(k)(\alpha+k)} + \frac{(2k)N^+(k)}{M_1^+(k)(\alpha+k)}. \tag{2.53}$$

Using Eq. (2.53) in Eq. (2.50), we obtain

$$\frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)} P_*^-(\alpha) = \frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)} P_*^-(\alpha) + \frac{k}{\pi i} \frac{N^+(k)}{M_1^+(k)(\alpha+k)} - \frac{k}{\pi i} \frac{N^+(k)}{M_1^+(k)(\alpha+k)}. \tag{2.54}$$

Now consider the R.H.S. of Eq. (2.46) (we want to make it regular in the upper half plane.)

$$e^{i\alpha l} \frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)} R^+(\alpha) = e^{i\alpha l} \frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)} \Phi_1^+(\alpha) - \frac{1}{2\pi i} e^{i\alpha l} \frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)} \frac{e^{ilk}}{(\alpha+k)}. \tag{2.55}$$

Consider the first term of Eq. (2.55) and let

$$p(\alpha) = e^{i\alpha l} \frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)} \Phi_1^+(\alpha), \tag{2.56}$$

be decomposed by decomposition theorem

$$p(\alpha) = p_+(\alpha) + p_-(\alpha), \tag{2.57}$$

where

$$p_+(\alpha) = \frac{1}{2\pi i} \int\limits_{-\infty+ic}^{\infty+ic} \frac{p(\xi)}{\xi-\alpha} d\xi,$$

$$p_+(\alpha) = \frac{1}{2\pi i} \int_{-\infty+ic}^{\infty+ic} e^{i\xi l} \frac{(k-\xi)N^-(\xi)}{M_1^-(\xi)(\xi-\alpha)} \Phi_1^+(\xi) d\xi.$$

Completing the contour in upper half plane by semicircle then $\xi = \alpha$ and zeros of $M_1(\xi)$ are the singularities which gives

$$\operatorname*{Res}_{\xi=\alpha}[p_+(\alpha)] = e^{i\alpha l} \frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)} \Phi_1^+(\alpha),$$

$$\operatorname*{Res}_{\xi=\beta_m}[p_+(\alpha)] = e^{i\xi l} \frac{(k-\xi)N(\xi)M_1^+(\xi)}{\frac{d}{d\xi}[M_1(\xi)N^+(\xi)(\xi-\alpha)]} \Phi_1^+(\xi) \Big|_{\xi=\beta_m},$$

$$= -\sum_{m=1}^{\infty} \frac{e^{i\beta_m l}(k-\beta_m)N(\beta_m)M_1^+(\beta_m)}{M_1^{/}(\beta_m)N^+(\beta_m)(\alpha-\beta_m)} \Phi_1^+(\beta_m),$$

so that

$$p_+(\alpha) = e^{i\alpha l} \frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)} \Phi_1^+(\alpha) - \sum_{m=1}^{\infty} \frac{e^{i\beta_m l}(k-\beta_m)N(\beta_m)M_1^+(\beta_m)}{M_1^{/}(\beta_m)N^+(\beta_m)(\alpha-\beta_m)} \Phi_1^+(\beta_m).$$

Define

$$p_-(\alpha) = -\frac{1}{2\pi i} \int_{-\infty+id}^{\infty+id} \frac{p(\xi)}{\xi-\alpha} d\xi,$$

$$p_-(\alpha) = -\frac{1}{2\pi i} \int_{-\infty+id}^{\infty+id} e^{i\xi l} \frac{(k-\xi)N^-(\xi)}{M_1^-(\xi)(\xi-\alpha)} \Phi_1^+(\xi) d\xi.$$

If we close the contour in upper half plane by semicircle then zeros of $M_1(\xi)$ are the singularities so that

$$p_-(\alpha) = \sum_{m=1}^{\infty} \frac{e^{i\beta_m l}(k-\beta_m)N(\beta_m)M_1^+(\beta_m)}{M_1^{/}(\beta_m)N^+(\beta_m)(\alpha-\beta_m)} \Phi_1^+(\beta_m).$$

Thus, Eq. (2.57) gives

$$e^{i\alpha l} \frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)} \Phi_1^+(\alpha) = e^{i\alpha l} \frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)} \Phi_1^+(\alpha) - \sum_{m=1}^{\infty} \frac{e^{i\beta_m l}(k-\beta_m)N(\beta_m)M_1^+(\beta_m)}{M_1^{/}(\beta_m)N^+(\beta_m)(\alpha-\beta_m)} \Phi_1^+(\beta_m)$$

$$+\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}(k-\beta_m)N(\beta_m)M_1^+(\beta_m)}{M_1^{/}(\beta_m)N^+(\beta_m)(\alpha-\beta_m)}\Phi_1^+(\beta_m). \tag{2.58}$$

Consider the term of Eq. (2.55) and let

$$l(\alpha)=e^{i\alpha l}\frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)}\frac{e^{ilk}}{(\alpha+k)}, \tag{2.59}$$

be decomposed by decomposition theorem

$$l(\alpha)=l_+(\alpha)+l_-(\alpha), \tag{2.60}$$

$$l_+(\alpha)=\frac{1}{2\pi i}\int\limits_{-\infty+ic}^{\infty+ic}\frac{l(\xi)}{\xi-\alpha}d\xi,$$

$$l_+(\alpha)=\frac{1}{2\pi i}\int\limits_{-\infty+ic}^{\infty+ic}e^{i\xi l}\frac{(k-\xi)N^-(\xi)}{M_1^-(\xi)(\xi-\alpha)}\frac{e^{ilk}}{(\xi+k)}d\xi.$$

If we close the contour in upper half plane by semicircle then $\xi=\alpha$ and zeros of $M_1(\xi)$ are the singularities which gives

$$\operatorname*{Res}_{\xi=\alpha}[l_+(\alpha)]=e^{i\alpha l}\frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)}\frac{e^{ilk}}{(\alpha+k)},$$

$$\operatorname*{Res}_{\xi=\beta_m}[l_+(\alpha)]=-\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}M_1^+(\beta_m)(k-\beta_m)N(\beta_m)e^{ilk}}{M_1^{/}(\beta_m)(\alpha-\beta_m)N^+(\beta_m)(\beta_m+k)},$$

so that

$$l_+(\alpha)=e^{i\alpha l}\frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)}\frac{e^{ilk}}{(\alpha+k)}-\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}M_1^+(\beta_m)(k-\beta_m)N(\beta_m)e^{ilk}}{M_1^{/}(\beta_m)(\alpha-\beta_m)N^+(\beta_m)(\beta_m+k)}.$$

Define

$$l_-(\alpha)=-\frac{1}{2\pi i}\int\limits_{-\infty+id}^{\infty+id}\frac{l(\xi)}{\xi-\alpha}d\xi$$

$$l_-(\alpha) = -\frac{1}{2\pi i}\int\limits_{-\infty+id}^{\infty+id} e^{i\xi l}\frac{(k-\xi)N^-(\xi)}{M_1^-(\xi)(\xi-\alpha)}\frac{e^{ilk}}{(\xi+k)}d\xi.$$

If we close the contour in upper half plane by semicircle then zeros of $M_1(\xi)$ are the singularities which gives

$$l_-(\alpha) = \sum_{m=1}^{\infty}\frac{e^{i\beta_m l}M_1^+(\beta_m)(k-\beta_m)N(\beta_m)e^{ilk}}{M_1^/(\beta_m)(\alpha-\beta_m)N^+(\beta_m)(\beta_m+k)}.$$

Thus, Eq. (2.60) becomes

$$e^{i\alpha l}\frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)}\frac{e^{ilk}}{(\alpha+k)} = e^{i\alpha l}\frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)}\frac{e^{ilk}}{(\alpha+k)} - \sum_{m=1}^{\infty}\frac{e^{i\beta_m l}M_1^+(\beta_m)(k-\beta_m)N(\beta_m)e^{ilk}}{M_1^/(\beta_m)(\alpha-\beta_m)N^+(\beta_m)(\beta_m+k)}$$

$$+\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}M_1^+(\beta_m)(k-\beta_m)N(\beta_m)e^{ilk}}{M_1^/(\beta_m)(\alpha-\beta_m)N^+(\beta_m)(\beta_m+k)}. \quad (2.61)$$

Putting Eqs.(2.58) and (2.61) in Eq. (2.55), we obtain

$$e^{i\alpha l}\frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)}R^+(\alpha) = e^{i\alpha l}\frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)}\Phi_1^+(\alpha) - \frac{1}{2\pi i}\frac{e^{i\alpha l}(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)}\frac{e^{ilk}}{(\alpha+k)}$$

$$-\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}(k-\beta_m)N(\beta_m)M_1^+(\beta_m)}{M_1^/(\beta_m)N^+(\beta_m)(\alpha-\beta_m)}\left[\Phi_1^+(\beta_m) - \frac{1}{2\pi i}\frac{e^{ilk}}{(k+\beta_m)}\right]$$

$$+\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}(k-\beta_m)N(\beta_m)M_1^+(\beta_m)}{M_1^/(\beta_m)N^+(\beta_m)(\alpha-\beta_m)}\left[\Phi_1^+(\beta_m) - \frac{1}{2\pi i}\frac{e^{ilk}}{(k+\beta_m)}\right]. \quad (2.62)$$

Using Eqs. (2.55) and (2.36) in Eq. (2.62), we obtain

$$e^{i\alpha l}\frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)}R^+(\alpha) = e^{i\alpha l}\frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)}R^+(\alpha) - \sum_{m=1}^{\infty}\frac{e^{i\beta_m l}(k-\beta_m)N(\beta_m)M_1^+(\beta_m)R^+(\beta_m)}{M_1^/(\beta_m)N^+(\beta_m)(\alpha-\beta_m)}$$

$$+\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}(k-\beta_m)N(\beta_m)M_1^+(\beta_m)R^+(\beta_m)}{M_1^/(\beta_m)N^+(\beta_m)(\alpha-\beta_m)}. \quad (2.63)$$

Putting Eqs. (2.54) and (2.63) in Eq. (2.46), we obtain

$$\frac{F_1(\alpha)M_1^+(\alpha)}{\eta_1 N^+(\alpha)(k+\alpha)} - \frac{F_2(\alpha)M_1^+(\alpha)}{N^+(\alpha)(k+\alpha)M_1(\alpha)} - \frac{k}{\pi i}\frac{N^+(k)}{M_1^+(k)(\alpha+k)} - e^{i\alpha l}\frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)}R^+(\alpha)$$

$$+\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}(k-\beta_m)N(\beta_m)M_1^+(\beta_m)R^+(\beta_m)}{M_1^{/}(\beta_m)N^+(\beta_m)(\alpha-\beta_m)}$$

$$= -\frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)}P_*^-(\alpha) - \frac{k}{\pi i}\frac{N^+(k)}{M_1^+(k)(\alpha+k)} + \sum_{m=1}^{\infty}\frac{e^{i\beta_m l}(k-\beta_m)N(\beta_m)M_1^+(\beta_m)R^+(\beta_m)}{M_1^{/}(\beta_m)N^+(\beta_m)(\alpha-\beta_m)}. \tag{2.64}$$

The regularity of the L.H.S. of Eq. (2.64) in upper half plane may be violated by the simple pole occurring at zeros of $M_1(\alpha)$ lying in the upper half plane namely $\alpha = \beta_m$, $m = 1, 2, 3, \ldots$. Consider the the second term of Eq. (2.64). Let

$$D(\alpha) = \frac{F_2(\alpha)M_1^+(\alpha)}{N^+(\alpha)(k+\alpha)M_1(\alpha)}. \tag{2.65}$$

By decomposition theorem we can write

$$D(\alpha) = D_+(\alpha) + D_-(\alpha), \tag{2.66}$$

where

$$D_+(\alpha) = \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{D(\xi)}{\xi-\alpha}d\xi,$$

$$D_+(\alpha) = \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{F_2(\xi)M_1^+(\xi)}{N^+(\xi)(k+\xi)M_1(\xi)(\xi-\alpha)}d\xi.$$

If we close the contour in upper half plane by semicircle then $\xi = \alpha$ and zeros of $M_1(\xi)$ are the singularities which gives

$$\operatorname*{Res}_{\xi=\alpha}[D_+(\alpha)] = \frac{F_2(\alpha)M_1^+(\alpha)}{N^+(\alpha)(k+\alpha)M_1(\alpha)},$$

$$\operatorname*{Res}_{\xi=\beta_m}[D_+(\alpha)] = -\sum_{m=1}^{\infty}\frac{F_2(\beta_m)M_1^+(\beta_m)}{N^+(\beta_m)(k+\beta_m)M_1^{/}(\beta_m)(\alpha-\beta_m)},$$

so that

$$D_+(\alpha) = \frac{F_2(\alpha)M_1^+(\alpha)}{N^+(\alpha)(k+\alpha)M_1(\alpha)} - \sum_{m=1}^{\infty}\frac{F_2(\beta_m)M_1^+(\beta_m)}{N^+(\beta_m)(k+\beta_m)M_1^{/}(\beta_m)(\alpha-\beta_m)}.$$

Define

$$D_-(\alpha) = -\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id}\frac{D(\xi)}{\xi-\alpha}d\xi,$$

$$D_-(\alpha) = -\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id}\frac{F_2(\xi)M_1^+(\xi)}{N^+(\xi)(k+\xi)M_1(\xi)(\xi-\alpha)}d\xi.$$

closing the contour in upper half plane by semicircle then zeros of $M_1(\xi)$ are the singularities which gives

$$D_-(\alpha) = \sum_{m=1}^{\infty}\frac{F_2(\beta_m)M_1^+(\beta_m)}{N^+(\beta_m)(k+\beta_m)M_1^{/}(\beta_m)(\alpha-\beta_m)}.$$

Thus, Eq. (2.66) becomes

$$\frac{F_2(\alpha)M_1^+(\alpha)}{N^+(\alpha)(k+\alpha)M_1(\alpha)} = \frac{F_2(\alpha)M_1^+(\alpha)}{N^+(\alpha)(k+\alpha)M_1(\alpha)} - \sum_{m=1}^{\infty}\frac{F_2(\beta_m)M_1^+(\beta_m)}{N^+(\beta_m)(k+\beta_m)M_1^{/}(\beta_m)(\alpha-\beta_m)}$$

$$+\sum_{m=1}^{\infty}\frac{F_2(\beta_m)M_1^+(\beta_m)}{N^+(\beta_m)(k+\beta_m)M_1^{/}(\beta_m)(\alpha-\beta_m)}. \tag{2.67}$$

Putting Eq. (2.67) in Eq. (2.64), we get

$$\frac{F_1(\alpha)M_1^+(\alpha)}{\eta_1 N^+(\alpha)(k+\alpha)} - \frac{F_2(\alpha)M_1^+(\alpha)}{N^+(\alpha)(k+\alpha)M_1(\alpha)} - \sum_{m=1}^{\infty}\frac{a_m}{\alpha-\beta_m} - \frac{k}{\pi i}\frac{N^+(k)}{M_1^+(k)(\alpha+k)}$$

$$-e^{i\alpha l}\frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)}R^+(\alpha) + \sum_{m=1}^{\infty}\frac{e^{i\beta_m l}(k-\beta_m)N(\beta_m)M_1^+(\beta_m)R^+(\beta_m)}{M_1^{/}(\beta_m)N^+(\beta_m)(\alpha-\beta_m)}$$

$$= -\frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)}P_*^-(\alpha) - \frac{k}{\pi i}\frac{N^+(k)}{M_1^+(k)(\alpha+k)} + \sum_{m=1}^{\infty}\frac{e^{i\beta_m l}(k-\beta_m)N(\beta_m)M_1^+(\beta_m)R^+(\beta_m)}{M_1^{/}(\beta_m)N^+(\beta_m)(\alpha-\beta_m)}$$

$$-\sum_{m=1}^{\infty}\frac{a_m}{\alpha-\beta_m}, \quad (2.68)$$

where

$$a_m = \frac{-F_2(\beta_m)M_1^+(\beta_m)}{N^+(\beta_m)(k+\beta_m)M_1^{/}(\beta_m)}. \quad (2.69)$$

The application of analytic continuation principle together with Liouville's theorem to the Eq. (2.68) yields

$$\frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)}P_*^-(\alpha) = \sum_{m=1}^{\infty}\frac{e^{i\beta_m l}(k-\beta_m)N(\beta_m)M_1^+(\beta_m)R^+(\beta_m)}{M_1^{/}(\beta_m)N^+(\beta_m)(\alpha-\beta_m)} - \frac{k}{\pi i}\frac{N^+(k)}{M_1^+(k)(\alpha+k)}$$

$$-\sum_{m=1}^{\infty}\frac{a_m}{\alpha-\beta_m}. \quad (2.70)$$

Now we will apply similar treatment to Eq. (2.47). In Eq. (2.47), first term is regular in the lower half plane, while right hand side is regular in the upper half plane. Second and third term on L.H.S. have singularities in the lower half plane. Consider the third term of Eq. (2.47) on L.H.S. (we want to make it regular in the lower half)

$$\frac{e^{-i\alpha l}(k+\alpha)N^+(\alpha)}{M_1^+(\alpha)}P_*^-(\alpha) = -\frac{e^{-i\alpha l}(k+\alpha)N^+(\alpha)\Phi_1^-(\alpha)}{M_1^+(\alpha)} - \frac{1}{2\pi i}\frac{e^{-i\alpha l}(k+\alpha)N^+(\alpha)}{M_1^+(\alpha)(\alpha+k)}. \quad (2.71)$$

Consider the first term on R.H.S. of Eq. (2.71). Let

$$E(\alpha) = \frac{e^{-i\alpha l}(k+\alpha)N^+(\alpha)\Phi_1^-(\alpha)}{M_1^+(\alpha)}. \quad (2.72)$$

By decomposition theorem we can write

$$E(\alpha) = E_+(\alpha) + E_-(\alpha), \quad (2.73)$$

where

$$E_{+}(\alpha)=\frac{e^{-i\alpha l}(k+\alpha)N^{+}(\alpha)\Phi_{1}^{-}(\alpha)}{M_{1}^{+}(\alpha)}-\sum_{m=1}^{\infty}\frac{e^{i\beta_{m}l}(k-\beta_{m})N(-\beta_{m})M_{1}^{+}(\beta_{m})\Phi_{1}^{-}(-\beta_{m})}{M_{1}^{\prime}(-\beta_{m})N^{+}(\beta_{m})(\alpha+\beta_{m})},$$

and

$$E_{-}(\alpha)=\sum_{m=1}^{\infty}\frac{e^{i\beta_{m}l}(k-\beta_{m})N(-\beta_{m})M_{1}^{+}(\beta_{m})\Phi_{1}^{-}(-\beta_{m})}{M_{1}^{\prime}(-\beta_{m})N^{+}(\beta_{m})(\alpha+\beta_{m})}.$$

Therefore, Eq. (2.73) becomes

$$\frac{e^{-i\alpha l}(k+\alpha)N^{+}(\alpha)\Phi_{1}^{-}(\alpha)}{M_{1}^{+}(\alpha)}=\frac{e^{-i\alpha l}(k+\alpha)N^{+}(\alpha)\Phi_{1}^{-}(\alpha)}{M_{1}^{+}(\alpha)}-\sum_{m=1}^{\infty}\frac{e^{i\beta_{m}l}(k-\beta_{m})N(-\beta_{m})M_{1}^{+}(\beta_{m})\Phi_{1}^{-}(-\beta_{m})}{M_{1}^{\prime}(-\beta_{m})N^{+}(\beta_{m})(\alpha+\beta_{m})}$$

$$+\sum_{m=1}^{\infty}\frac{e^{i\beta_{m}l}(k-\beta_{m})N(-\beta_{m})M_{1}^{+}(\beta_{m})\Phi_{1}^{-}(-\beta_{m})}{M_{1}^{\prime}(-\beta_{m})N^{+}(\beta_{m})(\alpha+\beta_{m})}. \tag{2.74}$$

Now consider the second term of Eq. (2.71). Let

$$H(\alpha)=\frac{e^{-i\alpha l}(k+\alpha)N^{+}(\alpha)}{M_{1}^{+}(\alpha)(\alpha+k)}. \tag{2.75}$$

By decomposition theorem we can write

$$H(\alpha)=H_{+}(\alpha)+H_{-}(\alpha), \tag{2.76}$$

where

$$H_{+}(\alpha)=\frac{e^{-i\alpha l}(k+\alpha)N^{+}(\alpha)}{M_{1}^{+}(\alpha)(\alpha+k)}-\sum_{m=1}^{\infty}\frac{e^{i\beta_{m}l}(k-\beta_{m})N(-\beta_{m})M_{1}^{+}(\beta_{m})}{M_{1}^{\prime}(-\beta_{m})(\alpha+\beta_{m})N^{+}(\beta_{m})(k-\beta_{m})},$$

and

$$H_{-}(\alpha)=\sum_{m=1}^{\infty}\frac{e^{i\beta_{m}l}(k-\beta_{m})N(-\beta_{m})M_{1}^{+}(\beta_{m})}{M_{1}^{\prime}(-\beta_{m})(\alpha+\beta_{m})N^{+}(\beta_{m})(k-\beta_{m})}.$$

Thus, Eq. (2.76) becomes

$$\frac{e^{-i\alpha l}(k+\alpha)N^+(\alpha)}{M_1^+(\alpha)(\alpha+k)} = \frac{e^{-i\alpha l}(k+\alpha)N^+(\alpha)}{M_1^+(\alpha)(\alpha+k)} - \sum_{m=1}^{\infty} \frac{e^{i\beta_m l}(k-\beta_m)N(-\beta_m)M_1^+(\beta_m)}{M_1^{/}(-\beta_m)(\alpha+\beta_m)N^+(\beta_m)(k-\beta_m)}$$

$$- \sum_{m=1}^{\infty} \frac{e^{i\beta_m l}(k-\beta_m)N(-\beta_m)M_1^+(\beta_m)}{M_1^{/}(-\beta_m)(\alpha+\beta_m)N^+(\beta_m)(k-\beta_m)}. \tag{2.77}$$

Putting Eqs. (2.74) and (2.77) in Eq. (2.71), we get

$$\frac{e^{-i\alpha l}(k+\alpha)N^+(\alpha)}{M_1^+(\alpha)} P_*^-(\alpha) = \frac{e^{-i\alpha l}(k+\alpha)N^+(\alpha)}{M_1^+(\alpha)} P_*^-(\alpha) - \sum_{m=1}^{\infty} \frac{e^{i\beta_m l}(k-\beta_m)N(-\beta_m)M_1^+(\beta_m)P_*^-(-\beta_m)}{M_1^{/}(-\beta_m)N^+(\beta_m)(\alpha+\beta_m)}$$

$$+ \sum_{m=1}^{\infty} \frac{e^{i\beta_m l}(k-\beta_m)N(-\beta_m)M_1^+(\beta_m)P_*^-(-\beta_m)}{M_1^{/}(-\beta_m)N^+(\beta_m)(\alpha+\beta_m)}. \tag{2.78}$$

Now we consider the second term of Eq. (2.47). Let (we want to make it regular in the lower half plane)

$$Z(\alpha) = e^{-i\alpha l} \frac{F_2(\alpha)M_1^-(\alpha)}{N^-(\alpha)(k-\alpha)M_1(\alpha)}. \tag{2.79}$$

By decomposition theorem we can write

$$Z(\alpha) = Z_+(\alpha) + Z_-(\alpha), \tag{2.80}$$

where

$$Z_+(\alpha) = \frac{e^{-i\alpha l}F_2(\alpha)M_1^-(\alpha)}{N^-(\alpha)(k-\alpha)M_1(\alpha)} - \sum_{m=1}^{\infty} \frac{e^{i\beta_m l}F_2(-\beta_m)M_1^+(\beta_m)}{M_1^{/}(-\beta_m)(k+\beta_m)N^+(\beta_m)(\alpha+\beta_m)},$$

and

$$Z_-(\alpha) = \sum_{m=1}^{\infty} \frac{e^{i\beta_m l}F_2(-\beta_m)M_1^+(\beta_m)}{M_1^{/}(-\beta_m)(k+\beta_m)N^+(\beta_m)(\alpha+\beta_m)}.$$

Thus, Eq. (2.80) becomes

$$e^{-i\alpha l}\frac{F_2(\alpha)M_1^-(\alpha)}{N^-(\alpha)(k-\alpha)M_1(\alpha)} = \frac{e^{-i\alpha l}F_2(\alpha)M_1^-(\alpha)}{N^-(\alpha)(k-\alpha)M_1(\alpha)} - \sum_{m=1}^{\infty}\frac{e^{i\beta_m l}F_2(-\beta_m)M_1^+(\beta_m)}{M_1^{\prime}(-\beta_m)(k+\beta_m)N^+(\beta_m)(\alpha+\beta_m)}$$

$$+\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}F_2(-\beta_m)M_1^+(\beta_m)}{M_1^{\prime}(-\beta_m)(k+\beta_m)N^+(\beta_m)(\alpha+\beta_m)}. \tag{2.81}$$

Putting Eqs. (2.78) and (2.81) in Eq. (2.47), we get

$$e^{-i\alpha l}\frac{F_1(\alpha)M_1^-(\alpha)}{\eta_1 N^-(\alpha)(k-\alpha)} - e^{-i\alpha l}\frac{F_2(\alpha)M_1^-(\alpha)}{N^-(\alpha)(k-\alpha)M_1(\alpha)} - \sum_{m=1}^{\infty}\frac{b_m}{\alpha+\beta_m}$$

$$+e^{-i\alpha l}\frac{(k+\alpha)N^+(\alpha)}{M_1^+(\alpha)}P_*^-(\alpha) - \sum_{m=1}^{\infty}\frac{e^{i\beta_m l}(k-\beta_m)N(-\beta_m)M_1^+(\beta_m)P_*^-(-\beta_m)}{M_1^{\prime}(-\beta_m)N^+(\beta_m)(\alpha+\beta_m)}$$

$$= \frac{(k+\alpha)N^+(\alpha)}{M_1^+(\alpha)}R^+(\alpha) - \sum_{m=1}^{\infty}\frac{b_m}{\alpha+\beta_m} - \sum_{m=1}^{\infty}\frac{e^{i\beta_m l}(k-\beta_m)N(-\beta_m)M_1^+(\beta_m)P_*^-(-\beta_m)}{M_1^{\prime}(-\beta_m)N^+(\beta_m)(\alpha+\beta_m)}, \tag{2.82}$$

where

$$b_m = -\frac{e^{i\beta_m l}F_2(-\beta_m)M_1^+(\beta_m)}{M_1^{\prime}(-\beta_m)(k+\beta_m)N^+(\beta_m)}. \tag{2.83}$$

The application of analytic continuation principle together with Liouville's theorem to Eq. (2.82), yields

$$\frac{(k+\alpha)N^+(\alpha)}{M_1^+(\alpha)}R^+(\alpha) = \sum_{m=1}^{\infty}\frac{e^{i\beta_m l}(k-\beta_m)N(-\beta_m)M_1^+(\beta_m)P_*^-(-\beta_m)}{M_1^{\prime}(-\beta_m)N^+(\beta_m)(\alpha+\beta_m)} + \sum_{m=1}^{\infty}\frac{b_m}{\alpha+\beta_m}. \tag{2.84}$$

Now we apply similar treatment to Eq. (2.48). In Eq. (2.48) first term is regular in the upper half plane, the third term and the R.H.S. have singularties in both half planes. Consider the third term on L.H.S. (we want to make it regular in the lower half plane). Let

$$\frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)}Q_*^-(\alpha) = -\frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)}\Phi_2^-(\alpha) - \frac{1}{2\pi i}\frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)}\frac{1}{(\alpha+k)}. \tag{2.85}$$

Consider the second term

$$T(\alpha) = \frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)} \frac{1}{(\alpha+k)}. \tag{2.86}$$

By decomposition theorem we can write

$$T(\alpha) = T_+(\alpha) + T_-(\alpha), \tag{2.87}$$

where

$$T_+(\alpha) = \frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)(\alpha+k)} - \frac{2kN^+(k)}{M_2^+(k)(\alpha+k)},$$

and

$$T_-(\alpha) = \frac{2kN^+(k)}{M_2^+(k)(\alpha+k)}.$$

Thus, Eq. (2.87) becomes

$$\frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)} \frac{1}{(\alpha+k)} = \frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)(\alpha+k)} - \frac{2kN^+(k)}{M_2^+(k)(\alpha+k)} + \frac{2kN^+(k)}{M_2^+(k)(\alpha+k)}. \tag{2.88}$$

Putting Eq. (2.88) in Eq. (2.85), we have

$$\frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)} Q_*^-(\alpha) = \frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)} Q_*^-(\alpha) - \frac{k}{\pi i} \frac{N^+(k)}{M_2^+(k)(\alpha+k)} + \frac{k}{\pi i} \frac{N^+(k)}{M_2^+(k)(\alpha+k)}. \tag{2.89}$$

Now consider the R.H.S. of Eq. (2.48) (we want to make it regular in the upper half plane)

$$e^{i\alpha l} \frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)} S^+(\alpha) = e^{i\alpha l} \frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)} \Phi_2^+(\alpha) - \frac{1}{2\pi i} e^{i\alpha l} \frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)} \frac{e^{ilk}}{(\alpha+k)}. \tag{2.90}$$

Consider

$$U(\alpha) = e^{i\alpha l} \frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)} \Phi_2^+(\alpha). \tag{2.91}$$

Applying decomposition on $U(\alpha)$, we get

$$U_+(\alpha) = e^{i\alpha l}\frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)}\Phi_2^+(\alpha) - \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}(k-\nu_m)N(\nu_m)M_2^+(\nu_m)\Phi_2^+(\nu_m)}{M_2^{/}(\nu_m)N^+(\nu_m)(\alpha-\nu_m)},$$

and

$$U_-(\alpha) = \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}(k-\nu_m)N(\nu_m)M_2^+(\nu_m)\Phi_2^+(\nu_m)}{M_2^{/}(\nu_m)N^+(\nu_m)(\alpha-\nu_m)}.$$

Thus, Eq. (2.91) becomes

$$\begin{aligned} e^{i\alpha l}\frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)}\Phi_2^+(\alpha) &= e^{i\alpha l}\frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)}\Phi_2^+(\alpha) - \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}(k-\nu_m)N(\nu_m)M_2^+(\nu_m)\Phi_2^+(\nu_m)}{M_2^{/}(\nu_m)N^+(\nu_m)(\alpha-\nu_m)} \\ &+ \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}(k-\nu_m)N(\nu_m)M_2^+(\nu_m)\Phi_2^+(\nu_m)}{M_2^{/}(\nu_m)N^+(\nu_m)(\alpha-\nu_m)}. \end{aligned} \tag{2.92}$$

Consider the term of Eq. (2.90),

$$V(\alpha) = e^{i\alpha l}\frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)}\frac{e^{ilk}}{(\alpha+k)}. \tag{2.93}$$

Applying decomposition on $V(\alpha)$, we get

$$V_+(\alpha) = e^{i\alpha l}\frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)}\frac{e^{ilk}}{(\alpha+k)} - \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}(k-\nu_m)N(\nu_m)M_2^+(\nu_m)e^{ilk}}{M_2^{/}(\nu_m)N^+(\nu_m)(k+\nu_m)(\alpha-\nu_m)},$$

and

$$V_-(\alpha) = \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}(k-\nu_m)N(\nu_m)M_2^+(\nu_m)e^{ilk}}{M_2^{/}(\nu_m)N^+(\nu_m)(k+\nu_m)(\alpha-\nu_m)}.$$

Thus, Eq. (2.93) becomes

$$\begin{aligned} e^{i\alpha l}\frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)}\frac{e^{ilk}}{(\alpha+k)} &= e^{i\alpha l}\frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)}\frac{e^{ilk}}{(\alpha+k)} - \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}(k-\nu_m)N(\nu_m)M_2^+(\nu_m)e^{ilk}}{M_2^{/}(\nu_m)N^+(\nu_m)(k+\nu_m)(\alpha-\nu_m)} \\ &+ \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}(k-\nu_m)N(\nu_m)M_2^+(\nu_m)e^{ilk}}{M_2^{/}(\nu_m)N^+(\nu_m)(k+\nu_m)(\alpha-\nu_m)}. \end{aligned} \tag{2.94}$$

Putting Eqs. (2.92) and (2.94) in Eq. (2.90), we reach at

$$e^{i\alpha l}\frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)}S^+(\alpha) = e^{i\alpha l}\frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)}S^+(\alpha) - \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}(k-\nu_m)N(\nu_m)M_2^+(\nu_m)S^+(\nu_m)}{M_2^/(\nu_m)N^+(\nu_m)(\alpha-\nu_m)} + \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}(k-\nu_m)N(\nu_m)M_2^+(\nu_m)S^+(\nu_m)}{M_2^/(\nu_m)N^+(\nu_m)(\alpha-\nu_m)}. \quad (2.95)$$

Now consider the second term of Eq. (2.48). (we want to make it regular in upper half plane)

$$I(\alpha) = \frac{F_1(\alpha)M_2^+(\alpha)}{N^+(\alpha)(k+\alpha)M_2(\alpha)}. \quad (2.96)$$

Applying decomposition theorem on $I(\alpha)$, we get

$$I_+(\alpha) = \frac{F_1(\alpha)M_2^+(\alpha)}{N^+(\alpha)(k+\alpha)M_2(\alpha)} - \sum_{m=1}^{\infty}\frac{F_1(\nu_m)M_2^+(\nu_m)}{N^+(\nu_m)(k+\nu_m)M_2^/(\nu_m)(\alpha-\nu_m)},$$

and

$$I_-(\alpha) = \sum_{m=1}^{\infty}\frac{F_1(\nu_m)M_2^+(\nu_m)}{N^+(\nu_m)(k+\nu_m)M_2^/(\nu_m)(\alpha-\nu_m)}.$$

Thus, Eq. (2.96) becomes

$$\frac{F_1(\alpha)M_2^+(\alpha)}{N^+(\alpha)(k+\alpha)M_2(\alpha)} = \frac{F_1(\alpha)M_2^+(\alpha)}{N^+(\alpha)(k+\alpha)M_2(\alpha)} - \sum_{m=1}^{\infty}\frac{F_1(\nu_m)M_2^+(\nu_m)}{N^+(\nu_m)(k+\nu_m)M_2^/(\nu_m)(\alpha-\nu_m)} + \sum_{m=1}^{\infty}\frac{F_1(\nu_m)M_2^+(\nu_m)}{N^+(\nu_m)(k+\nu_m)M_2^/(\nu_m)(\alpha-\nu_m)},$$

$$\frac{F_1(\alpha)M_2^+(\alpha)}{N^+(\alpha)(k+\alpha)M_2(\alpha)} = \frac{F_1(\alpha)M_2^+(\alpha)}{N^+(\alpha)(k+\alpha)M_2(\alpha)} - \sum_{m=1}^{\infty}\frac{c_m}{\alpha-\nu_m} + \sum_{m=1}^{\infty}\frac{c_m}{\alpha-\nu_m}, \quad (2.97)$$

where

$$c_m = \frac{F_1(\nu_m)M_2^+(\nu_m)}{N^+(\nu_m)(k+\nu_m)M_2^/(\nu_m)}. \quad (2.98)$$

Putting Eqs. (2.89), (2.95) and (2.97) in Eq. (2.48), we obtain

$$-\frac{F_2(\alpha)M_2^+(\alpha)}{\eta_2 N^+(\alpha)(k+\alpha)}+\frac{F_1(\alpha)M_2^+(\alpha)}{N^+(\alpha)(k+\alpha)M_2(\alpha)}-\sum_{m=1}^{\infty}\frac{c_m}{\alpha-\nu_m}+\sum_{m=1}^{\infty}\frac{e^{i\nu_m l}(k-\nu_m)N(\nu_m)M_2^+(\nu_m)S^+(\nu_m)}{M_2^{/}(\nu_m)N^+(\nu_m)(\alpha-\nu_m)}$$

$$-\frac{k}{\pi i}\frac{N^+(k)}{M_2^+(k)(\alpha+k)}-e^{i\alpha l}\frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)}S^+(\alpha)=$$

$$-\frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)}Q_*^-(\alpha)-\sum_{m=1}^{\infty}\frac{e^{i\nu_m l}(k-\nu_m)N(\nu_m)M_2^+(\nu_m)S^+(\nu_m)}{M_2^{/}(\nu_m)N^+(\nu_m)(\alpha-\nu_m)}-\frac{k}{\pi i}\frac{N^+(k)}{M_2^+(k)(\alpha+k)}-\sum_{m=1}^{\infty}\frac{c_m}{\alpha-\nu_m}. \tag{2.99}$$

The application of analytic continuation principle together with Liouville's theorem to Eq. (2.99) yields

$$\frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)}Q_*^-(\alpha)=\sum_{m=1}^{\infty}\frac{e^{i\nu_m l}(k-\nu_m)N(\nu_m)M_2^+(\nu_m)S^+(\nu_m)}{M_2^{/}(\nu_m)N^+(\nu_m)(\alpha-\nu_m)}-\frac{k}{\pi i}\frac{N^+(k)}{M_2^+(k)(\alpha+k)}$$

$$-\sum_{m=1}^{\infty}\frac{c_m}{\alpha-\nu_m}. \tag{2.100}$$

Now we will apply similar treatment to Eq. (2.49). In Eq. (2.49) first term is regular in the lower half plane, while right hand side is regular in the upper half plane. Second and third term on L.H.S have singularities in the lower half plane. Consider the third term on L.H.S. of Eq. (2.49). (we want to make it regular in lower half plane)

$$\frac{e^{-i\alpha l}(k+\alpha)N^+(\alpha)}{M_2^+(\alpha)}Q_*^-(\alpha)=-\frac{e^{-i\alpha l}(k+\alpha)N^+(\alpha)\Phi_2^-(\alpha)}{M_2^+(\alpha)}-\frac{1}{2\pi i}\frac{e^{-i\alpha l}(k+\alpha)N^+(\alpha)}{M_2^+(\alpha)(\alpha+k)}. \tag{2.101}$$

Consider the first term on R.H.S. of Eq. (2.101)

$$E_1(\alpha)=\frac{e^{-i\alpha l}(k+\alpha)N^+(\alpha)\Phi_2^-(\alpha)}{M_2^+(\alpha)}. \tag{2.102}$$

By decomposition theorem, we can write

$$E_1(\alpha)=E_{1+}(\alpha)+E_{1-}(\alpha), \tag{2.103}$$

where

$$E_{1+}(\alpha) = \frac{e^{-i\alpha l}(k+\alpha)N^+(\alpha)\Phi_2^-(\alpha)}{M_2^+(\alpha)} - \sum_{m=1}^{\infty} \frac{e^{i\nu_m l}(k-\nu_m)N(-\nu_m)M_2^+(\nu_m)\Phi_2^-(-\nu_m)}{M_2^/(-\nu_m)N^+(\nu_m)(\alpha+\nu_m)}, \tag{2.104}$$

and

$$E_{1-}(\alpha) = \sum_{m=1}^{\infty} \frac{e^{i\nu_m l}(k-\nu_m)N(-\nu_m)M_2^+(\nu_m)\Phi_2^-(-\nu_m)}{M_2^/(-\nu_m)N^+(\nu_m)(\alpha+\nu_m)}. \tag{2.105}$$

Thus, Eq. (2.103) becomes

$$\begin{aligned} \frac{e^{-i\alpha l}(k+\alpha)N^+(\alpha)\Phi_2^-(\alpha)}{M_2^+(\alpha)} &= \frac{e^{-i\alpha l}(k+\alpha)N^+(\alpha)\Phi_2^-(\alpha)}{M_2^+(\alpha)} \\ &- \sum_{m=1}^{\infty} \frac{e^{i\nu_m l}(k-\nu_m)N(-\nu_m)M_2^+(\nu_m)\Phi_2^-(-\nu_m)}{M_2^/(-\nu_m)N^+(\nu_m)(\alpha+\nu_m)} \\ &+ \sum_{m=1}^{\infty} \frac{e^{i\nu_m l}(k-\nu_m)N(-\nu_m)M_2^+(\nu_m)\Phi_2^-(-\nu_m)}{M_2^/(-\nu_m)N^+(\nu_m)(\alpha+\nu_m)}. \end{aligned} \tag{2.106}$$

Now consider the second term of Eq. (2.101)

$$H_1(\alpha) = \frac{e^{-i\alpha l}(k+\alpha)N^+(\alpha)}{M_2^+(\alpha)(\alpha+k)}. \tag{2.107}$$

By decomposition theorem, we can write

$$H_1(\alpha) = H_{1+}(\alpha) + H_{1-}(\alpha), \tag{2.108}$$

where

$$H_+(\alpha) = \frac{e^{-i\alpha l}(k+\alpha)N^+(\alpha)}{M_2^+(\alpha)(\alpha+k)} - \sum_{m=1}^{\infty} \frac{e^{i\nu_m l}(k-\nu_m)N(-\nu_m)M_2^+(\nu_m)}{M_2^/(-\nu_m)(\alpha+\nu_m)N^+(\nu_m)(k-\nu_m)}, \tag{2.109}$$

and

$$H_{1-}(\alpha) = \sum_{m=1}^{\infty} \frac{e^{i\nu_m l}(k-\nu_m)N(-\nu_m)M_2^+(\nu_m)}{M_2^/(-\nu_m)(\alpha+\nu_m)N^+(\nu_m)(k-\nu_m)}. \tag{2.110}$$

Thus, Eq. (2.108) becomes

$$\frac{e^{-i\alpha l}(k+\alpha)N^+(\alpha)}{M_2^+(\alpha)(\alpha+k)} = \frac{e^{-i\alpha l}(k+\alpha)N^+(\alpha)}{M_2^+(\alpha)(\alpha+k)} - \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}(k-\nu_m)N(-\nu_m)M_2^+(\nu_m)}{M_2^{/}(-\nu_m)(\alpha+\nu_m)N^+(\nu_m)(k-\nu_m)} + \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}(k-\nu_m)N(-\nu_m)M_2^+(\nu_m)}{M_2^{/}(-\nu_m)(\alpha+\nu_m)N^+(\nu_m)(k-\nu_m)} \tag{2.111}$$

Putting Eqs. (2.106) and (2.111) in Eq. (2.101), we obtain

$$\frac{e^{-i\alpha l}(k+\alpha)N^+(\alpha)}{M_2^+(\alpha)}Q_*^-(\alpha) = \frac{e^{-i\alpha l}(k+\alpha)N^+(\alpha)}{M_2^+(\alpha)}Q_*^-(\alpha) - \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}(k-\nu_m)N(-\nu_m)M_2^+(\nu_m)Q_*^-(-\nu_m)}{M_2^{/}(-\nu_m)N^+(\nu_m)(\alpha+\nu_m)} + \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}(k-\nu_m)N(-\nu_m)M_2^+(\nu_m)Q_*^-(-\nu_m)}{M_2^{/}(-\nu_m)N^+(\nu_m)(\alpha+\nu_m)}. \tag{2.112}$$

Now we consider the second term of Eq. (2.49) (we want to make it regular in the lower half plane)

$$Z_1(\alpha) = e^{-i\alpha l}\frac{F_1(\alpha)M_2^-(\alpha)}{N^-(\alpha)(k-\alpha)M_2(\alpha)}. \tag{2.113}$$

By decomposition theorem, we can write

$$Z_1(\alpha) = Z_{1+}(\alpha) + Z_{1-}(\alpha), \tag{2.114}$$

where

$$Z_{1+}(\alpha) = \frac{e^{-i\alpha l}F_1(\alpha)M_2^-(\alpha)}{N^-(\alpha)(k-\alpha)M_2(\alpha)} - \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}F_1(-\nu_m)M_2^+(\nu_m)}{M_2^{/}(-\nu_m)(k+\nu_m)N^+(\nu_m)(\alpha+\nu_m)}, \tag{2.115}$$

and

$$Z_{1-}(\alpha) = \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}F_1(-\nu_m)M_2^+(\nu_m)}{M_2^{/}(-\nu_m)(k+\nu_m)N^+(\nu_m)(\alpha+\nu_m)}. \tag{2.116}$$

Thus, Eq. (2.114) becomes

$$\begin{aligned} e^{-i\alpha l}\frac{F_1(\alpha)M_2^-(\alpha)}{N^-(\alpha)(k-\alpha)M_2(\alpha)} &= \frac{e^{-i\alpha l}F_1(\alpha)M_2^-(\alpha)}{N^-(\alpha)(k-\alpha)M_2(\alpha)} \\ &-\sum_{m=1}^{\infty}\frac{e^{i\nu_m l}F_1(-\nu_m)M_2^+(\nu_m)}{M_2^{/}(-\nu_m)(k+\nu_m)N^+(\nu_m)(\alpha+\nu_m)} \\ &+\sum_{m=1}^{\infty}\frac{e^{i\nu_m l}F_1(-\nu_m)M_2^+(\nu_m)}{M_2^{/}(-\nu_m)(k+\nu_m)N^+(\nu_m)(\alpha+\nu_m)}. \end{aligned} \quad (2.117)$$

Putting Eqs. (2.112) and (2.117) in Eq. (2.49), we obtain

$$\begin{aligned} &-e^{-i\alpha l}\frac{F_2(\alpha)M_2^-(\alpha)}{\eta_2 N^-(\alpha)(k-\alpha)} + e^{-i\alpha l}\frac{F_1(\alpha)M_2^-(\alpha)}{N^-(\alpha)(k-\alpha)M_2(\alpha)} - \sum_{m=1}^{\infty}\frac{d_m}{\alpha+\nu_m} \\ &+e^{-i\alpha l}\frac{(k+\alpha)N^+(\alpha)}{M_2^+(\alpha)}Q_*^-(\alpha) - \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}(k-\nu_m)N(-\nu_m)M_2^+(\nu_m)Q_*^-(-\nu_m)}{M_2^{/}(-\nu_m)N^+(\nu_m)(\alpha+\nu_m)} \\ = &\ \frac{(k+\alpha)N^+(\alpha)}{M_2^+(\alpha)}S^+(\alpha) - \sum_{m=1}^{\infty}\frac{d_m}{\alpha+\nu_m} \\ &-\sum_{m=1}^{\infty}\frac{e^{i\nu_m l}(k-\nu_m)N(-\nu_m)M_2^+(\nu_m)Q_*^-(-\nu_m)}{M_2^{/}(-\nu_m)N^+(\nu_m)(\alpha+\nu_m)}, \end{aligned} \quad (2.118)$$

where

$$d_m = \frac{e^{i\nu_m l}F_1(-\nu_m)M_2^+(\nu_m)}{M_2^{/}(-\nu_m)(k+\nu_m)N^+(\nu_m)}. \quad (2.119)$$

The application of analytic continuation principle together with Liouville's theorem to Eq. (2.118) yields

$$\frac{(k+\alpha)N^+(\alpha)}{M_2^+(\alpha)}S^+(\alpha) = \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}(k-\nu_m)N(-\nu_m)M_2^+(\nu_m)Q_*^-(-\nu_m)}{M_2^{/}(-\nu_m)N^+(\nu_m)(\alpha+\nu_m)} + \sum_{m=1}^{\infty}\frac{d_m}{\alpha+\nu_m}. \quad (2.120)$$

Putting Eqs. (2.70) and (2.84) in Eq. (2.46), we obtain

$$\frac{F_1(\alpha)M_1^+(\alpha)}{\eta_1 N^+(\alpha)(k+\alpha)} - \frac{F_2(\alpha)M_1^+(\alpha)}{N^+(\alpha)(k+\alpha)M_1(\alpha)}$$

$$= -\sum_{m=1}^{\infty} \frac{e^{i\beta_m l}(k-\beta_m)N(\beta_m)M_1^+(\beta_m)R^+(\beta_m)}{M_1^/(\beta_m)N^+(\beta_m)(\alpha-\beta_m)} + \frac{k}{\pi i}\frac{N^+(k)}{M_1^+(k)(\alpha+k)} + \sum_{m=1}^{\infty}\frac{a_m}{\alpha-\beta_m}$$

$$+e^{i\alpha l}\frac{(k-\alpha)N^-(\alpha)M_1^+(\alpha)}{(k+\alpha)N^+(\alpha)M_1^-(\alpha)}$$

$$\left[\sum_{m=1}^{\infty} \frac{e^{i\beta_m l}(k-\beta_m)N(-\beta_m)M_1^+(\beta_m)P_*^-(-\beta_m)}{M_1^/(-\beta_m)N^+(\beta_m)(\alpha+\beta_m)} + \sum_{m=1}^{\infty}\frac{b_m}{\alpha+\beta_m}\right]. \qquad (2.121)$$

Putting Eq. (2.70) and Eq. (2.84) in Eq. (2.47), we obtain

$$e^{-i\alpha l}\frac{F_1(\alpha)M_1^-(\alpha)}{\eta_1 N^-(\alpha)(k-\alpha)} - e^{-i\alpha l}\frac{F_2(\alpha)M_1^-(\alpha)}{N^-(\alpha)(k-\alpha)M_1(\alpha)} =$$

$$\sum_{m=1}^{\infty} \frac{e^{i\beta_m l}(k-\beta_m)N(-\beta_m)M_1^+(\beta_m)P_*^-(-\beta_m)}{M_1^/(-\beta_m)N^+(\beta_m)(\alpha+\beta_m)} + \sum_{m=1}^{\infty}\frac{b_m}{\alpha+\beta_m} - e^{-i\alpha l}\frac{(k+\alpha)N^+(\alpha)M_1^-(\alpha)}{(k-\alpha)N^-(\alpha)M_1^+(\alpha)}$$

$$\left[\sum_{m=1}^{\infty} \frac{e^{i\beta_m l}(k-\beta_m)N(\beta_m)M_1^+(\beta_m)R^+(\beta_m)}{M_1^/(\beta_m)N^+(\beta_m)(\alpha-\beta_m)} - \frac{k}{\pi i}\frac{N^+(k)}{M_1^+(k)(\alpha+k)} - \sum_{m=1}^{\infty}\frac{a_m}{\alpha-\beta_m}\right]. \qquad (2.122)$$

Putting Eqs. (2.100) and (2.120) in Eq. (2.48), we obtain

$$-\frac{F_2(\alpha)M_2^+(\alpha)}{\eta_2 N^+(\alpha)(k+\alpha)} + \frac{F_1(\alpha)M_2^+(\alpha)}{N^+(\alpha)(k+\alpha)M_2(\alpha)} = -\sum_{m=1}^{\infty} \frac{e^{i\nu_m l}(k-\nu_m)N(\nu_m)M_2^+(\nu_m)S^+(\nu_m)}{M_2^/(\nu_m)N^+(\nu_m)(\alpha-\nu_m)}$$

$$+\frac{k}{\pi i}\frac{N^+(k)}{M_2^+(k)(\alpha+k)} + \sum_{m=1}^{\infty}\frac{c_m}{\alpha-\nu_m} + e^{i\alpha l}\frac{(k-\alpha)N^-(\alpha)M_2^+(\alpha)}{(k+\alpha)N^+(\alpha)M_2^-(\alpha)}$$

$$\left[\sum_{m=1}^{\infty} \frac{e^{i\nu_m l}(k-\nu_m)N(-\nu_m)M_2^+(\nu_m)Q_*^-(-\nu_m)}{M_2^/(-\nu_m)N^+(\nu_m)(\alpha+\nu_m)} + \sum_{m=1}^{\infty}\frac{d_m}{\alpha+\nu_m}\right]. \qquad (2.123)$$

Putting Eqs. (2.100) and (2.120) in Eq. (2.49), we obtain

$$e^{-i\alpha l}\frac{F_2(\alpha)M_2^-(\alpha)}{\eta_2 N^-(\alpha)(k-\alpha)} - e^{-i\alpha l}\frac{F_1(\alpha)M_2^-(\alpha)}{N^-(\alpha)(k-\alpha)M_2(\alpha)} = \sum_{m=1}^{\infty} \frac{e^{i\nu_m l}(k-\nu_m)N(-\nu_m)M_2^+(\nu_m)Q_*^-(-\nu_m)}{M_2^/(-\nu_m)N^+(\nu_m)(\alpha+\nu_m)}$$

$$+\sum_{m=1}^{\infty}\frac{b_m}{\alpha+\nu_m} - e^{-i\alpha l}\frac{(k+\alpha)N^+(\alpha)M_2^-(\alpha)}{(k-\alpha)N^-(\alpha)M_2^+(\alpha)}$$

$$\left[\sum_{m=1}^{\infty} \frac{e^{i\nu_m l}(k-\nu_m)N(\nu_m)M_2^+(\nu_m)S^+(\nu_m)}{M_2^{/}(\nu_m)N^+(\nu_m)(\alpha-\nu_m)]} - \frac{k}{\pi i}\frac{N^+(k)}{M_2^+(k)(\alpha+k)} - \sum_{m=1}^{\infty}\frac{c_m}{\alpha-\nu_m}\right]. \tag{2.124}$$

Put $\alpha = \nu_m$ in Eq. (2.121), we obtain

$$\frac{F_1(\nu_m)M_1^+(\nu_m)}{\eta_1 N^+(\nu_m)(k+\nu_m)} - \frac{F_2(\nu_m)M_1^+(\nu_m)}{N^+(\nu_m)(k+\nu_m)M_1(\nu_m)} - e^{i\nu_m l}\frac{(k-\nu_m)N^-(\nu_m)M_1^+(\nu_m)}{(k+\nu_m)N^+(\nu_m)M_1^-(\nu_m)}$$
$$\left[\sum_{n=1}^{\infty}\frac{e^{i\beta nl}(k-\beta_n)N(-\beta_n)M_1^+(\beta_n)P_*^-(-\beta_n)}{M_1^{/}(-\beta_n)N^+(\beta_n)(\nu_m+\beta_n)} + \sum_{n=1}^{\infty}\frac{b_n}{\nu_m+\beta_n}\right]$$
$$+\sum_{n=1}^{\infty}\frac{e^{i\beta_n l}(k-\beta_n)N(\beta_n)M_1^+(\beta_n)R^+(\beta_n)}{M_1^{/}(\beta_n)N^+(\beta_n)(\nu_m-\beta_n)} - \sum_{n=1}^{\infty}\frac{a_n}{\nu_m-\beta_n} = \frac{k}{\pi i}\frac{N^+(k)}{M_1^+(k)(\nu_m+k)}. \tag{2.125}$$

Put $\alpha = -\nu_m$ in Eq. (2.122), we obtain

$$e^{i\nu_m l}\frac{F_1(-\nu_m)M_1^+(\nu_m)}{\eta_1 N^+(\nu_m)(k+\nu_m)} - e^{i\nu_m l}\frac{F_2(-\nu_m)M_1^+(\nu_m)}{N^+(\nu_m)(k+\nu_m)M_1(-\nu_m)} + e^{i\nu_m l}\frac{(k-\nu_m)N^-(\nu_m)M_1^+(\nu_m)}{(k+\nu_m)N^+(\nu_m)M_1^+(-\nu_m)}$$
$$\left[-\sum_{n=1}^{\infty}\frac{e^{i\beta_n l}(k-\beta_n)N(\beta_n)M_1^+(\beta_n)R^+(\beta_n)}{M_1^{/}(\beta_n)N^+(\beta_n)(\nu_m+\beta_n)} + \sum_{n=1}^{\infty}\frac{a_n}{\nu_m+\beta_n}\right]$$
$$+\sum_{n=1}^{\infty}\frac{e^{i\beta_n l}(k-\beta_n)N(-\beta_n)M_1^+(\beta_n)P_*^-(-\beta_n)}{M_1^{/}(-\beta_n)N^+(\beta_n)(\nu_m-\beta_n)} + \sum_{n=1}^{\infty}\frac{b_n}{\nu_m-\beta_n}$$
$$= \frac{k}{\pi i}\frac{e^{i\nu_m l}N^+(k)N^-(\nu_m)M_1^+(\nu_m)}{M_1^+(k)(k+\nu_m)N^+(\nu_m)M_1^+(-\nu_m)}. \tag{2.126}$$

Put $\alpha = \beta_m$ in Eq. (2.123), we obtain

$$-\frac{F_2(\beta_m)M_2^+(\beta_m)}{\eta_2 N^+(\beta_m)(k+\beta_m)} + \frac{F_1(\beta_m)M_2^+(\beta_m)}{N^+(\beta_m)(k+\beta_m)M_2(\beta_m)} - e^{i\beta_m l}\frac{(k-\beta_m)N^-(\beta_m)M_2^+(\beta_m)}{(k+\beta_m)N^+(\beta_m)M_2^-(\beta_m)}$$
$$\left[\sum_{n=1}^{\infty}\frac{e^{i\nu_n l}(k-\nu_n)N(-\nu_n)M_2^+(\nu_n)Q_*^-(-\nu_n)}{M_2^{/}(-\nu_n)N^+(\nu_n)(\beta_m+\nu_n)} + \sum_{n=1}^{\infty}\frac{d_n}{\beta_m+\nu_n}\right]$$
$$+\sum_{n=1}^{\infty}\frac{e^{i\nu_n l}(k-\nu_n)N(\nu_n)M_2^+(\nu_n)S^+(\nu_n)}{M_2^{/}(\nu_n)N^+(\nu_n)(\beta_m-\nu_n)} - \sum_{n=1}^{\infty}\frac{c_n}{\beta_m-\nu_n} = \frac{k}{\pi i}\frac{N^+(k)}{M_2^+(k)(\beta_m+k)}. \tag{2.127}$$

Put $\alpha = -\beta_m$ in Eq. (2.124), we obtain

$$-\frac{e^{i\beta_m l}F_2(-\beta_m)M_2^+(\beta_m)}{\eta_2 N^+(\beta_m)(k+\beta_m)}+\frac{e^{i\beta_m l}F_1(-\beta_m)M_2^+(\beta_m)}{N^+(\beta_m)(k+\beta_m)M_2(-\beta_m)}+e^{i\beta_m l}\frac{(k-\beta_m)N^-(\beta_m)M_2^+(\beta_m)}{(k+\beta_m)N^+(\beta_m)M_2^+(-\beta_m)}$$

$$\left[-\sum_{n=1}^{\infty}\frac{e^{i\nu_n l}(k-\nu_n)N(\nu_n)M_2^+(\nu_n)S^+(\nu_n)}{M_2^/(\nu_n)N^+(\nu_n)(\beta_m+\nu_n)}+\sum_{n=1}^{\infty}\frac{c_n}{\beta_m+\nu_n}\right]$$

$$+\sum_{n=1}^{\infty}\frac{e^{i\nu_n l}(k-\nu_n)N(-\nu_n)M_2^+(\nu_n)Q_*^-(-\nu_n)}{M_2^/(-\nu_n)N^+(\nu_n)(\beta_m-\nu_n)}+\sum_{n=1}^{\infty}\frac{d_n}{\beta_m-\nu_n}$$

$$=\frac{k}{\pi i}\frac{e^{i\beta_m l}N^+(k)N^-(\beta_m)M_2^+(\beta_m)}{M_2^+(k)(k+\beta_m)N^+(\beta_m)M_2^+(-\beta_m)}. \tag{2.128}$$

In Eqs. (2.125)-(2.128), $P_*^-(-\beta_m)$, $R^+(\beta_m)$, $Q_*^-(-\nu_m)$ and $S^+(\nu_m)$ stands for

Put $\alpha = -\beta_m$ in Eq. (2.54)

$$\frac{(k+\beta_m)N^+(\beta_m)}{M_1^+(\beta_m)}P_*^-(-\beta_m)=-\sum_{n=1}^{\infty}\frac{e^{i\beta_n l}(k-\beta_n)N(\beta_n)M_1^+(\beta_n)R^+(\beta_n)}{M_1^/(\beta_n)N^+(\beta_n)(\beta_m+\beta_n)}-\frac{k}{\pi i}\frac{N^+(k)}{M_1^+(k)(k-\beta_m)}$$

$$+\sum_{n=1}^{\infty}\frac{a_n}{\beta_m+\beta_n}. \tag{2.129}$$

Put $\alpha = \beta_m$ in Eq. (2.69)

$$\frac{(k+\beta_m)N^+(\beta_m)}{M_1^+(\beta_m)}R^+(\beta_m)=\sum_{n=1}^{\infty}\frac{e^{i\beta_n l}(k-\beta_n)N(-\beta_n)M_1^+(\beta_n)P_*^-(-\beta_n)}{M_1^/(-\beta_n)N^+(\beta_n)(\beta_m+\beta_n)}+\sum_{n=1}^{\infty}\frac{b_n}{\beta_m+\beta_n}. \tag{2.130}$$

Put $\alpha = -\nu_m$ in Eq. (2.100)

$$\frac{(k+\nu_m)N^+(\nu_m)}{M_2^+(\nu_m)}Q_*^-(-\nu_m)=-\sum_{n=1}^{\infty}\frac{e^{i\nu_n l}(k-\nu_n)N(\nu_n)M_2^+(\nu_n)S^+(\nu_n)}{M_2^/(\nu_n)N^+(\nu_n)(\nu_m+\nu_n)}-\frac{k}{\pi i}\frac{N^+(k)}{M_2^+(k)(k-\nu_m)}$$

$$+\sum_{n=1}^{\infty}\frac{c_n}{\nu_m+\nu_n}. \tag{2.131}$$

Put $\alpha = \nu_m$ in Eq. (2.120)

$$\frac{(k+\nu_m)N^+(\nu_m)}{M_2^+(\nu_m)}S^+(\nu_m) = \sum_{n=1}^{\infty} \frac{e^{i\nu_n l}(k-\nu_n)N(-\nu_n)M_2^+(\nu_n)Q_*^-(-\nu_n)}{M_2^{/}(-\nu_n)N^+(\nu_n)(\nu_m+\nu_n)} + \sum_{n=1}^{\infty} \frac{d_n}{\nu_m+\nu_n}. \tag{2.132}$$

2.4 Scattered Field

2.4.1 Reflected Field

Putting value of $A(\alpha)$ and $B(\alpha)$ from Eqs. (2.24) and (2.30) in Eq. (2.23), we get

$$u(x,y) = \int_L \left[\frac{F_1(\alpha)\cos[K(\alpha)b] - F_2(\alpha)}{K(\alpha)\sin[K(\alpha)b]}\cos K(\alpha)y + \frac{F_1(\alpha)}{K(\alpha)}\sin K(\alpha)y\right] e^{-i\alpha x}d\alpha, \tag{2.133}$$

$$\Longrightarrow u(x,y) = \int_L \left[\frac{\{F_1(\alpha)\cos[K(\alpha)b] - F_2(\alpha)\}\cos K(\alpha)y + F_1(\alpha)\sin[K(\alpha)b]\sin K(\alpha)y}{K(\alpha)\sin[K(\alpha)b]}\right] e^{-i\alpha x}d\alpha, \tag{2.134}$$

where L is a straight line parallel to the real $\alpha - axis$, lying in the strip $Im(k) < Im(a) < Im(k)$. The above integral is calculated by closing the contour in the upper half plane and evaluating the residue contributions from the simple poles occurring at the zeros of $K(\alpha)sin[K(\alpha)b]$ lying in the upper half-plane. The reflection coefficient R of the fundamental mode is defined as the complex coefficient multiplying the travelling wave term $exp(ikx)$ and is computed from the contribution of the first pole at $\alpha = k$. Put

$$K(\alpha)\sin[K(\alpha)b] = 0,$$

$$K^2(\alpha)b\left[1 - \frac{K(\alpha)b^2}{3!} + \frac{K(\alpha)b^4}{5!}.....\right] = 0,$$

$$\alpha = \pm k,$$

$$u(x,y)=\pi i\left[\frac{\begin{array}{c}\{F_1(\alpha)\cos[\sqrt{k^2-\alpha^2}b]-F_2(\alpha)\}\cos\sqrt{k^2-\alpha^2}y\\+F_1(\alpha)\sin[\sqrt{k^2-\alpha^2}b]\sin\sqrt{k^2-\alpha^2}y\end{array}}{\frac{d}{d\alpha}\{\sqrt{k^2-\alpha^2}\sin[\sqrt{k^2-\alpha^2}b]\}}\Bigg|_{\alpha=k}\right],$$

$$R=\frac{\pi i}{kb}[F_2(k)-F_1(k)]. \tag{2.135}$$

Now we will find $F_2(k)-F_1(k)$. Put $\alpha=k$ in Eqs. (2.121) and (2.123), solving simultaneously for $F_1(k)$ and $F_2(k)$ we finally reach at

$$F_2(k)=\frac{\eta_1\eta_2M_1(k)N^+(k)(2k)}{M_1^+(k)[M_1(k)M_2(k)-\eta_1\eta_2]}$$

$$\left[-\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}N(\beta_m)M_1^+(\beta_m)R^+(\beta_m)}{M_1^{/}(\beta_m)N^+(\beta_m)}+\frac{k}{\pi i}\frac{N^+(k)}{M_1^+(k)(2k)}+\sum_{m=1}^{\infty}\frac{a_m}{k-\beta_m}\right]$$

$$-\frac{M_1(k)M_2(k)\eta_2N^+(k)(2k)}{M_2^+(k)[M_1(k)M_2(k)-\eta_1\eta_2]}$$

$$\left[-\sum_{m=1}^{\infty}\frac{e^{i\nu_m l}N(\nu_m)M_2^+(\nu_m)S^+(\nu_m)}{M_2^{/}(\nu_m)N^+(\nu_m)}+\frac{k}{\pi i}\frac{N^+(k)}{M_2^+(k)(2k)}+\sum_{m=1}^{\infty}\frac{c_m}{k-\nu_m}\right], \tag{2.136}$$

and

$$F_1(k)=\frac{\eta_1\eta_1\eta_2N^+(k)(2k)}{M_1^+(k)[M_1(k)M_2(k)-\eta_1\eta_2]}$$

$$\left[-\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}N(\beta_m)M_1^+(\beta_m)R^+(\beta_m)}{M_1^{/}(\beta_m)N^+(\beta_m)}+\frac{k}{\pi i}\frac{N^+(k)}{M_1^+(k)(2k)}+\sum_{m=1}^{\infty}\frac{a_m}{k-\beta_m}\right]$$

$$-\frac{\eta_1M_2(k)\eta_2N^+(k)(2k)}{M_2^+(k)[M_1(k)M_2(k)-\eta_1\eta_2]}$$

$$\left[-\sum_{m=1}^{\infty}\frac{e^{i\nu_m l}N(\nu_m)M_2^+(\nu_m)S^+(\nu_m)}{M_2^{/}(\nu_m)N^+(\nu_m)}+\frac{k}{\pi i}\frac{N^+(k)}{M_2^+(k)(2k)}+\sum_{m=1}^{\infty}\frac{c_m}{k-\nu_m}\right]+\frac{\eta_1N^+(k)(2k)}{M_1^+(k)}$$

$$\left[-\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}N(\beta_m)M_1^+(\beta_m)R^+(\beta_m)}{M_1^{/}(\beta_m)N^+(\beta_m)}+\frac{k}{\pi i}\frac{N^+(k)}{M_1^+(k)(2k)}+\sum_{m=1}^{\infty}\frac{a_m}{k-\beta_m}\right], \tag{2.137}$$

Subtracting Eq. (2.136) from Eq. (2.137), we get

$$\begin{aligned} F_2(k) - F_1(k) &= \frac{-\eta_1 2kN^+(k)}{M_1^+(k)} \left[-\sum_{m=1}^{\infty} \frac{e^{i\beta_m l} N(\beta_m) M_1^+(\beta_m) R^+(\beta_m)}{M_1^{/}(\beta_m) N^+(\beta_m)} \right. \\ &\left. + \frac{1}{2\pi i} \frac{N^+(k)}{M_1^+(k)} + \sum_{m=1}^{\infty} \frac{a_m}{k - \beta_m} \right] \end{aligned}$$

$$\begin{aligned} &+ \frac{-\eta_2 2kN^+(k)}{M_2^+(k)} \left[-\sum_{m=1}^{\infty} \frac{e^{i\nu_m l} N(\nu_m) M_2^+(\nu_m) S^+(\nu_m)}{M_2^{/}(\nu_m) N^+(\nu_m)} \right. \\ &\left. + \frac{1}{2\pi i} \frac{N^+(k)}{M_2^+(k)} + \sum_{m=1}^{\infty} \frac{c_m}{k - \nu_m} \right]. \end{aligned}$$

2.4.2 Transmitted Field

The transmission coefficient T of the fundamental mode which is defined as to be complex coefficient of $exp(ikx)$ and is computed from the contribution of the pole at $\alpha = -k$ in Eq. (2.134) the result is

$$T = -\frac{\pi i}{kb}[F_2(-k) - F_1(-k)]. \qquad (2.138)$$

Now we will find $F_2(-k) - F_1(-k)$. Put $\alpha = -k$ in Eqs. (2.122) and (2.124) solving simultaneously for $F_1(-k)$ and $F_2(-k)$ we finally have

$$\begin{aligned} F_2(-k) - F_1(-k). &= \frac{-2k\eta_1 N^+(k) e^{-ikl}}{M_1^+(k)} \left[-\sum_{m=1}^{\infty} \frac{e^{i\beta_m l} N(-\beta_m) M_1^+(\beta_m) P_*^-(-\beta_m)}{M_1^{/}(-\beta_m) N^+(\beta_m)} \right. \\ &\left. + \frac{1}{2\pi i} \frac{e^{ikl} N^-(k)}{M_1^+(-k)} - \sum_{m=1}^{\infty} \frac{b_m}{k - \beta_m} \right] \\ &+ \frac{-2k\eta_2 N^+(k) e^{-ikl}}{M_2^+(k)} \left[-\sum_{m=1}^{\infty} \frac{e^{i\nu_m l} N(-\nu_m) M_2^+(\nu_m) Q_*^-(-\nu_m)}{M_2^{/}(-\nu_m) N^+(\nu_m)} \right. \\ &\left. + \frac{1}{2\pi i} \frac{e^{ikl} N^-(k)}{M_2^+(-k)} - \sum_{m=1}^{\infty} \frac{d_m}{k - \nu_m} \right]. \end{aligned}$$

Chapter 3

Wiener-Hopf analysis of the parallel plate waveguide with soft boundaries having finite length impedance loading

In this chapter we have extended the work of *Tayyar et al.* [5] by considering the soft boundaries of parallel plate waveguide having finite length impedance loading. The boundary value problem gives two simultaneous modified Wiener-Hopf equations which are uncoupled by using pole removal technique. The solution involves four infinite sets of unknown coefficients satisfying four infinite system of linear algebraic equations. At the end reflection and transmission coefficients are determined from these system of linear algebraic equations.

3.1 Introduction

Numerous past investigation have been made to study the scattering coefficients for the wall impedance change in parallel plate waveguides. For example *Johansen* [1] have considered the part $x < 0$ of the parallel plates are perfectly conducting while the part $x > 0$ has the same surface impedance. *Heins and Feshbach* [2] have considered the problem of coupling of two ducts. *Karajala and Mittra* [3] provided the mode matching method to the scattering at the junction of two semi-infinite parallel plate waveguides with impedance wall. *Arora and Vijayaraghavan* [4] have considered Scattering of shielded surface wave in a parallel plate waveguide consisting of inductively reactive guiding surfaces and characterized by an abrupt wall reactance discontinuity. Finally *Tayyar et al.* [5] have considered the parallel plate waveguide by taking different impedance of upper and lower parallel plates.

We consider the infinite parallel plate waveguide having soft boundaries for $x < 0$ and $x > l$ at at $y = 0$ and $y = b$, and the part $0 < x < l$ have constant surface impedances at $y = 0$ and $y = b$. The surface impedances of the lower and upper plates are different from each other and denoted by $Z_1 = \eta_1 Z_0$ and $Z_2 = \eta_2 Z_0$, respectively with Z_0 being the characteristic impedance of the free space.

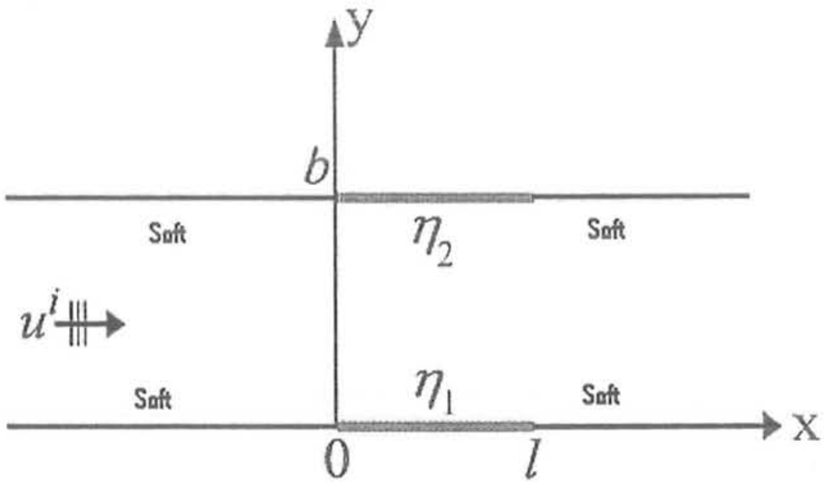

Figure 3.1: Geometry of the problem.

3.2 Mathematical Formulation

Let the incident TEM mode propagating in the positive x direction be given by

$$H_z^i = u^i = e^{ikx}. \tag{3.1}$$

The total field $u^T(x,y)$ can be written as

$$u^T(x,y) = u^i(x,y) + u(x,y) \qquad y\epsilon(0,b) \text{ and } x\epsilon(-\infty,\infty). \tag{3.2}$$

In Eq. (3.2) $u(x,y)$ is the unknown function which satisfied the Helmholtz equation

$$\Delta^2 u(x,y) + k^2 u(x,y) = 0. \tag{3.3}$$

The corresponding boundary conditions are

$$u(x,0) = 0, \qquad -\infty < x < 0 \text{ and } l < x < \infty, \tag{3.4}$$

$$u(x,b) = 0, \qquad -\infty < x < 0 \text{ and } l < x < \infty, \tag{3.5}$$

$$(1 + \frac{1}{ik\eta_1}\frac{\partial}{\partial y})u(x,0) = -e^{ikx}, \qquad 0 < x < l, \tag{3.6}$$

$$(1 - \frac{1}{ik\eta_2}\frac{\partial}{\partial y})u(x,b) = -e^{ikx}, \qquad 0 < x < l. \tag{3.7}$$

To obtain the unique solution to the mixed boundary value problem the edge and radiation conditions are

$$u^T(x,0) = \begin{cases} O(\mid x \mid^{1/2}), \mid x \mid \rightarrow 0 \\ O(\mid x - l \mid^{1/2}), \mid x \mid \rightarrow l \end{cases}, \tag{3.8}$$

$$u(x,y) = O(e^{ik|x|}), \mid x \mid \rightarrow \infty. \tag{3.9}$$

Taking Fourier transform of Eq. (3.3), we obtain

$$\left[\frac{d^2}{dy^2} + K^2(\alpha)\right] \bar{u}(\alpha, y) = 0, \tag{3.10}$$

where

$$\bar{u}(\alpha, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x, y)\ e^{i\alpha x} dx, \tag{3.11}$$

and

$$K(\alpha) = \sqrt{k^2 - \alpha^2}. \tag{3.12}$$

Taking inverse Fourier transform of Eq. (3.11), we get

$$u(x, y) = \int_{-\infty}^{\infty} \bar{u}(\alpha, y) e^{-i\alpha x} d\alpha. \tag{3.13}$$

Taking Fourier transform of Eqs. (3.4) and (3.5) respectively, we obtain

$$\bar{u}(\alpha, 0) = 0, \qquad -\infty < x < 0 \ \text{ and } \ l < x < \infty, \tag{3.14}$$

$$\bar{u}(\alpha, b) = 0, \qquad -\infty < x < 0 \ \text{ and } \ l < x < \infty. \tag{3.15}$$

Taking Fourier transform of Eq. (3.6), we obtain

$$(1 + \frac{1}{ik\eta_1}\frac{\partial}{\partial y})\bar{u}(\alpha, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 + \frac{1}{ik\eta_1}\frac{\partial}{\partial y}) u(x, 0) e^{i\alpha x} dx,$$

$$\begin{aligned}(1 + \frac{1}{ik\eta_1}\frac{\partial}{\partial y})\bar{u}(\alpha, 0) &= \frac{1}{2\pi} \int_{-\infty}^{0} (1 + \frac{1}{ik\eta_1}\frac{\partial}{\partial y}) u(x, 0) e^{i\alpha x} dx + \frac{1}{2\pi} \int_{0}^{l} (1 + \frac{1}{ik\eta_1}\frac{\partial}{\partial y}) u(x, 0) e^{i\alpha x} dx \\ &\quad + \frac{1}{2\pi} \int_{l}^{\infty} (1 + \frac{1}{ik\eta_1}\frac{\partial}{\partial y}) u(x, 0) e^{i\alpha x} dx,\end{aligned}$$

$$(1+\frac{1}{ik\eta_1}\frac{\partial}{\partial y})\bar{u}(\alpha,0)=\Phi_1^-(\alpha)-\frac{1}{2\pi i}\frac{e^{i(\alpha+k)l}-1}{(\alpha+k)}+e^{i\alpha l}\Phi_1^+(\alpha), \quad (3.16)$$

where

$$\Phi_1^-(\alpha)=\frac{1}{2\pi}\int_{-\infty}^{0}(1+\frac{1}{ik\eta_1}\frac{\partial}{\partial y})u(x,0)e^{i\alpha x}dx, \quad (3.17)$$

$$\Phi_1^+(\alpha)=\frac{1}{2\pi}\int_{l}^{\infty}(1+\frac{1}{ik\eta_1}\frac{\partial}{\partial y})u(x,0)e^{i\alpha(x-l)}dx. \quad (3.18)$$

Similarly taking Fourier transform of Eq. (3.7), we obtain

$$(1-\frac{1}{ik\eta_2}\frac{\partial}{\partial y})\bar{u}(\alpha,b)=\Phi_2^-(\alpha)-\frac{1}{2\pi i}\frac{e^{i(\alpha+k)l}-1}{(\alpha+k)}+e^{i\alpha l}\Phi_2^+(\alpha), \quad (3.19)$$

where

$$\Phi_2^-(\alpha)=\frac{1}{2\pi}\int_{-\infty}^{0}(1-\frac{1}{ik\eta_2}\frac{\partial}{\partial y})u(x,b)e^{i\alpha x}dx, \quad (3.20)$$

$$\Phi_2^+(\alpha)=\frac{1}{2\pi}\int_{l}^{\infty}(1-\frac{1}{ik\eta_2}\frac{\partial}{\partial y})u(x,b)e^{i\alpha(x-l)}dx. \quad (3.21)$$

The complementary solution of Eq. (3.10) is of the form

$$\bar{u}(\alpha,y)=A(\alpha)\cos\left[K(\alpha)y\right]+B(\alpha)\sin\left[K(\alpha)y\right], \quad (3.22)$$

where A and B are the function of α. Putting Eq. (3.22) in Eq. (3.13), we obtain

$$u(x,y)=\int_{-\infty}^{\infty}\left\{A(\alpha)\cos\left[K(\alpha)y\right]+B(\alpha)\sin\left[K(\alpha)y\right]\right\}e^{-i\alpha x}d\alpha. \quad (3.23)$$

Using Eq. (3.14) in Eq. (3.22), we obtain

$$A(\alpha)=F_1(\alpha), \quad (3.24)$$

where

$$F_1(\alpha) = \frac{1}{2\pi}\int_0^l u(x,0)e^{i\alpha x}dx. \tag{3.25}$$

Using Eq. (3.15) in Eq. (3.22), we obtain

$$A(\alpha)\cos[K(\alpha)b] + B(\alpha)\sin[K(\alpha)b] = F_2(\alpha), \tag{3.26}$$

where

$$F_2(\alpha) = \frac{1}{2\pi}\int_0^l u(x,b)e^{i\alpha x}dx. \tag{3.27}$$

Using Eq. (3.16) in Eq. (3.22), we obtain

$$A(\alpha) + \frac{B(\alpha)K(\alpha)}{ik\eta_1} = \Phi_1^-(\alpha) - \frac{1}{2\pi i}\frac{e^{i(\alpha+k)l}-1}{(\alpha+k)} + e^{i\alpha l}\Phi_1^+(\alpha). \tag{3.28}$$

Using Eq. (3.19) in Eq. (3.22), we obtain

$$\begin{aligned}
&\frac{A(\alpha)}{\eta_2}\left[\eta_2\cos[K(\alpha)b] + \frac{K(\alpha)}{ik}\sin[K(\alpha)b]\right] + \frac{B(\alpha)}{\eta_2}\left[\eta_2\sin[K(\alpha)b] - \frac{K(\alpha)}{ik}\cos[K(\alpha)b]\right] \\
= \;& \Phi_2^-(\alpha) - \frac{1}{2\pi i}\frac{e^{i(\alpha+k)l}-1}{(\alpha+k)} + e^{i\alpha l}\Phi_2^+(\alpha). \qquad (3.29)
\end{aligned}$$

Incorporating value of $A(\alpha)$ from Eq. (3.24) in Eq. (3.26), we obtain

$$B(\alpha) = \frac{F_2(\alpha) - F_1(\alpha)\cos[K(\alpha)b]}{\sin[K(\alpha)b]}. \tag{3.30}$$

Putting Eqs. (3.24) and (3.30) in Eq. (3.28), we arrive at

$$F_1(\alpha) + \frac{K(\alpha)}{ik\eta_1}\left[\frac{F_2(\alpha) - F_1(\alpha)\cos[K(\alpha)b]}{\sin[K(\alpha)b]}\right] = \Phi_1^-(\alpha) - \frac{1}{2\pi i}\frac{e^{i(\alpha+k)l}-1}{(\alpha+k)} + e^{i\alpha l}\Phi_1^+(\alpha),$$

$$-\frac{F_1(\alpha)}{\eta_1\sin[K(\alpha)b]}\left[-\eta_1\sin[K(\alpha)b] + \frac{K(\alpha)}{ik}\cos[K(\alpha)]b\right] + \frac{K(\alpha)F_2(\alpha)}{ik\eta_1\sin[K(\alpha)b]}$$

$$-\Phi_1^-(\alpha) - \frac{1}{2\pi i}\frac{1}{(\alpha+k)} = e^{i\alpha l}\left[\Phi_1^+(\alpha) - \frac{1}{2\pi i}\frac{e^{ilk}}{(\alpha+k)}\right],$$

$$-\frac{F_1(\alpha)M_1(\alpha)}{\eta_1 N(\alpha)K(\alpha)} + \frac{F_2(\alpha)}{ik\eta_1 N(\alpha)} + P_*^-(\alpha) = e^{i\alpha l}R^+(\alpha), \tag{3.31}$$

where $M_1(\alpha)$, $N(\alpha)$, $P_*^-(\alpha)$ and $R^+(\alpha)$ are defined as respectively,

$$M_1(\alpha) = -\eta_1 \sin[K(\alpha)b] + \frac{K(\alpha)}{ik}\cos[K(\alpha)]b, \tag{3.32}$$

$$N(\alpha) = \frac{\sin[K(\alpha)b]}{K(\alpha)}, \tag{3.33}$$

$$P_*^-(\alpha) = -\Phi_1^-(\alpha) - \frac{1}{2\pi i}\frac{1}{(\alpha+k)}, \tag{3.34}$$

and

$$R^+(\alpha) = \Phi_1^+(\alpha) - \frac{1}{2\pi i}\frac{e^{ilk}}{(\alpha+k)}. \tag{3.35}$$

Putting Eqs. (3.24) and (3.30) in Eq. (3.29), we obtain

$$\begin{aligned}
&\frac{F_1(\alpha)}{\eta_2}\left[\eta_2\cos[K(\alpha)b] + \frac{K(\alpha)}{ik}\sin[K(\alpha)b]\right] \\
&+\frac{F_2(\alpha) - F_1(\alpha)\cos[K(\alpha)b]}{\eta_2\sin[K(\alpha)b]}\left[\eta_2\sin[K(\alpha)b] - \frac{K(\alpha)}{ik}\cos[K(\alpha)b]\right] \\
&= \Phi_2^-(\alpha) + e^{i\alpha l}\Phi_2^+(\alpha) - \frac{1}{2\pi i}\frac{e^{i(\alpha+k)l}-1}{(\alpha+k)},
\end{aligned}$$

$$\begin{aligned}
&\frac{F_1(\alpha)K(\alpha)}{\eta_2 ik\sin[K(\alpha)b]}\left\{\cos^2[K(\alpha)b] + \sin^2[K(\alpha)b]\right\} \\
&-\frac{F_2(\alpha)}{\eta_2\sin[K(\alpha)b]}\left[-\eta_2\sin[K(\alpha)b] + \frac{K(\alpha)}{ik}\cos[K(\alpha)b]\right] - \Phi_2^-(\alpha) - \frac{1}{2\pi i}\frac{1}{(\alpha+k)} \\
= \; & e^{i\alpha l}\left[\Phi_2^+(\alpha) - \frac{1}{2\pi i}\frac{e^{ilk}}{(\alpha+k)}\right],
\end{aligned}$$

$$\frac{F_1(\alpha)}{ik\eta_2 N(\alpha)} - \frac{F_2(\alpha)M_2(\alpha)}{\eta_2 K(\alpha)N(\alpha)} + Q_*^-(\alpha) = e^{i\alpha l}S^+(\alpha), \tag{3.36}$$

where $M_2(\alpha)$, $Q_*^-(\alpha)$ and $S^+(\alpha)$ are defined as respectively,

$$M_2(\alpha) = -\eta_2 \sin[K(\alpha)b] + \frac{K(\alpha)}{ik}\cos[K(\alpha)b], \tag{3.37}$$

$$Q_*^-(\alpha) = -\Phi_2^-(\alpha) - \frac{1}{2\pi i}\frac{1}{(\alpha+k)}, \tag{3.38}$$

$$S^+(\alpha) = \Phi_2^+(\alpha) - \frac{1}{2\pi i}\frac{e^{ilk}}{(\alpha+k)}. \tag{3.39}$$

Notice that $P_*^-(\alpha)$ and $Q_*^-(\alpha)$ are regular in the lower half plane except at the pole singularity occurring at $\alpha = -k$.

3.3 Solution of Simultaneous Modified Wiener-Hopf Equations

The kernel factorization of $M_{1,2}(\alpha)$ and $N(\alpha)$ appearing in Eqs. (3.32), (3.37) and (3.33) are as follow

$$M_{1,2}(\alpha) = M_{1,2}^+(\alpha)M_{1,2}^-(\alpha), \tag{3.40}$$

$$N(\alpha) = N^+(\alpha)N^-(\alpha). \tag{3.41}$$

The explicit expression of $M_{1,2}^+(\alpha)$ and $N^+(\alpha)$ can be written as procedure outlined by *Lee and Mittra* [7]

$$M_1^+(\alpha) = [-\eta_1 \sin(kb) + \frac{1}{i}\cos(kb)]^{1/2}.\exp\{\frac{i\alpha b}{\pi}[1 - C - \ln(\frac{|\alpha| b}{\pi}) + i\frac{\pi}{2}]\}$$
$$\times \prod_{m=1}^{\infty}(1+\frac{\alpha}{\beta_m})\exp(\frac{i\alpha b}{m\pi}), \tag{3.42}$$

$$M_2^+(\alpha) = [-\eta_2 \sin(kb) + \frac{1}{i}\cos(kb)]^{1/2}.\exp\{\frac{i\alpha b}{\pi}[1 - C - \ln(\frac{|\alpha| b}{\pi}) + i\frac{\pi}{2}]\}$$
$$\times \prod_{m=1}^{\infty}(1+\frac{\alpha}{\nu_m})\exp(\frac{i\alpha b}{m\pi}), \tag{3.43}$$

$$N^+(\alpha) = [\frac{\sin(kb)}{k}]^{1/2}.\exp\{\frac{i\alpha b}{\pi}[1 - C - \ln(\frac{|\alpha| b}{\pi}) + i\frac{\pi}{2}]\}$$
$$\times \prod_{m=1}^{\infty}(1 + \frac{\alpha}{\alpha_m})\exp(\frac{i\alpha b}{m\pi}). \quad (3.44)$$

Here $\beta_m's$, $\nu_m's$ and $\alpha_m's$ are the roots of the functions $M_{1,2}(\alpha)$ and $N(\alpha)$, respectively.

$$M_1(\pm\beta_m) = 0, \quad M_1(\pm\nu_m) = 0\ , \quad N(\pm\alpha_m) = 0 \qquad m = 1, 2, 3...\ , \quad (3.45)$$

with

$$M_{1,2}^-(\alpha) = M_{1,2}^+(-\alpha), \qquad N^-(\alpha) = N^+(-\alpha).$$

In Eqs. (3.42)-(3.44), C is the Euler's constant given by $C = 0.57721...$. It can be easily shown that one has

$$M_{1,2}^{\pm}(\alpha) = |\alpha|^{1/2} \qquad N^{\pm}(\alpha) = |\alpha|^{-1/2}.$$

Multiplying Eq. (3.31) with $\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_1^-(\alpha)}$, we obtain

$$-\frac{F_1(\alpha)M_1^+(\alpha)}{\eta_1 N^+(\alpha)\sqrt{k+\alpha}} + \frac{F_2(\alpha)\sqrt{k-\alpha}M_1^+(\alpha)}{ik\eta_1 N^+(\alpha)M_1(\alpha)} + \frac{\sqrt{k-\alpha}N^-(\alpha)}{M_1^-(\alpha)}P_*^-(\alpha) = e^{i\alpha l}\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_1^-(\alpha)}R^+(\alpha). \quad (3.46)$$

Multiplying Eq. (3.31) with $e^{-i\alpha l}\frac{\sqrt{k+\alpha}N^+(\alpha)}{M_1^+(\alpha)}$, we get

$$-\frac{e^{-i\alpha l}F_1(\alpha)M_1^-(\alpha)}{\eta_1 N^-(\alpha)\sqrt{k-\alpha}} + \frac{e^{-i\alpha l}F_2(\alpha)\sqrt{k+\alpha}M_1^-(\alpha)}{ik\eta_1 N^-(\alpha)M_1(\alpha)} + e^{-i\alpha l}\frac{\sqrt{k+\alpha}N^+(\alpha)}{M_1^+(\alpha)}P_*^-(\alpha)$$
$$= \frac{\sqrt{k+\alpha}N^+(\alpha)}{M_1^+(\alpha)}R^+(\alpha). \quad (3.47)$$

Now multiplying Eq. (3.36) with $\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_2^-(\alpha)}$, we get

$$-\frac{F_2(\alpha)M_2^+(\alpha)}{\eta_2 N^+(\alpha)\sqrt{k+\alpha}} + \frac{F_1(\alpha)\sqrt{k-\alpha}M_2^+(\alpha)}{ik\eta_2 N^+(\alpha)M_2(\alpha)} + \frac{\sqrt{k-\alpha}N^-(\alpha)}{M_2^-(\alpha)}Q_*^-(\alpha) = e^{i\alpha l}\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_2^-(\alpha)}S^+(\alpha). \quad (3.48)$$

Multiplying Eq. (3.36) with $e^{-i\alpha l}\frac{\sqrt{k+\alpha}N^+(\alpha)}{M_2^+(\alpha)}$, we obtain

$$-e^{-i\alpha l}\frac{F_2(\alpha)M_2^-(\alpha)}{\eta_2 N^-(\alpha)\sqrt{k-\alpha}}+e^{-i\alpha l}\frac{F_1(\alpha)\sqrt{k+\alpha}M_2^-(\alpha)}{ik\eta_2 N^-(\alpha)M_2(\alpha)}+e^{-i\alpha l}\frac{\sqrt{k+\alpha}N^+(\alpha)}{M_2^+(\alpha)}Q_*^-(\alpha)$$

$$=\frac{\sqrt{k+\alpha}N^+(\alpha)}{M_2^+(\alpha)}S^+(\alpha). \tag{3.49}$$

The first term appearing in the left hand side of Eq. (3.46) is evidently regular in the upper half plane. The third term and the R.H.S. of same equation have singularities in both half planes. Hence one has to apply the Wiener-Hopf decomposition procedure on these terms. Consider the third term of Eq. (3.46) (we want to make it regular in the lower half plane)

$$\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_1^-(\alpha)}P_*^-(\alpha)=-\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_1^-(\alpha)}\Phi_1^-(\alpha)-\frac{1}{2\pi i}\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_1^-(\alpha)}\frac{1}{\alpha+k}. \tag{3.50}$$

Consider the 2nd term of Eq. (3.50). Let

$$f(\alpha)=\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_1^-(\alpha)}\frac{1}{\alpha+k}, \tag{3.51}$$

be decomposed by decomposition theorem

$$f(\alpha)=f_+(\alpha)+f_-(\alpha), \tag{3.52}$$

where

$$f_+(\alpha)=\frac{1}{2\pi i}\int\limits_{-\infty+ic}^{\infty+ic}\frac{f(\xi)}{\xi-\alpha}d\xi$$

$$f_+(\alpha)=\frac{1}{2\pi i}\int\limits_{-\infty+ic}^{\infty+ic}\frac{\sqrt{k-\xi}N^-(\xi)}{M_1^-(\xi)(\xi-\alpha)(\xi+k)}d\xi.$$

Completing the contour by semi circle in the upper half plane then $\xi = \alpha$ and $\xi = -k$ are the simple poles which gives

$$\operatorname*{Res}_{\xi=\alpha}[f_+(\alpha)] = \frac{\sqrt{k-\alpha}N^-(\alpha)}{M_1^-(\alpha)}\frac{1}{\alpha+k},$$

$$\operatorname*{Res}_{\xi=-k}[f_+(\alpha)] = \frac{-\sqrt{2k}N^+(k)}{M_1^+(k)(\alpha+k)},$$

so that

$$f_+(\alpha) = \frac{\sqrt{k-\alpha}N^-(\alpha)}{M_1^-(\alpha)}\frac{1}{(\alpha+k)} - \frac{\sqrt{2k}N^+(k)}{M_1^+(k)(\alpha+k)}.$$

Define

$$f_-(\alpha) = -\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id}\frac{f(\xi)}{\xi-\alpha}d\xi,$$

$$f_-(\alpha) = -\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id}\frac{\sqrt{k-\xi}N^-(\xi)}{M_1^-(\xi)(\xi-\alpha)(\xi+k)}d\xi.$$

Completing the contour by semi circle in the upper half plane then $\xi = -k$ is the simple pole which gives

$$f_-(\alpha) = \frac{\sqrt{2k}N^+(k)}{M_1^+(k)(\alpha+k)}$$

Thus, Eq. (3.52) becomes

$$\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_1^-(\alpha)}\frac{1}{\alpha+k} = \frac{\sqrt{k-\alpha}N^-(\alpha)}{M_1^-(\alpha)}\frac{1}{\alpha+k} - \frac{\sqrt{2k}N^+(k)}{M_1^+(k)(\alpha+k)} + \frac{\sqrt{2k}N^+(k)}{M_1^+(k)(\alpha+k)}. \quad (3.53)$$

Putting Eq. (3.53) in Eq. (3.50), we obtain

$$\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_1^-(\alpha)}P_*^-(\alpha) = \frac{(k-\alpha)N^-(\alpha)}{M_1^-(\alpha)}P_*^-(\alpha) + \sqrt{\frac{k}{2}}\frac{1}{\pi i}\frac{N^+(k)}{M_1^+(k)(\alpha+k)} - \sqrt{\frac{k}{2}}\frac{1}{\pi i}\frac{N^+(k)}{M_1^+(k)(\alpha+k)}. \quad (3.54)$$

Now consider the R.H.S of Eq. (3.46). (we want to make it regular in the upper half plane)

$$e^{i\alpha l}\frac{\sqrt{k-\alpha}N^{-}(\alpha)}{M_1^{-}(\alpha)}R^{+}(\alpha)=e^{i\alpha l}\frac{\sqrt{k-\alpha}N^{-}(\alpha)}{M_1^{-}(\alpha)}\Phi_1^{+}(\alpha)-\frac{1}{2\pi i}e^{i\alpha l}\frac{\sqrt{k-\alpha}N^{-}(\alpha)}{M_1^{-}(\alpha)}\frac{e^{ilk}}{(\alpha+k)}. \tag{3.55}$$

Consider the first term of Eq. (3.55) and let

$$p(\alpha)=e^{i\alpha l}\frac{\sqrt{k-\alpha}N^{-}(\alpha)}{M_1^{-}(\alpha)}\Phi_1^{+}(\alpha), \tag{3.56}$$

be decomposed by

$$p(\alpha)=p_{+}(\alpha)+p_{-}(\alpha), \tag{3.57}$$

where

$$p_{+}(\alpha)=\frac{1}{2\pi i}\int\limits_{-\infty+ic}^{\infty+ic}\frac{p(\xi)}{\xi-\alpha}d\xi,$$

$$p_{+}(\alpha)=\frac{1}{2\pi i}\int\limits_{-\infty+ic}^{\infty+ic}e^{i\xi l}\frac{\sqrt{k-\xi}N^{-}(\xi)}{M_1^{-}(\xi)(\xi-\alpha)}\Phi_1^{+}(\xi)d\xi.$$

If we close the contour in upper half plane by semicircle then $\xi=\alpha$ and zeros of $M_1(\xi)$ are the singularities which gives

$$\operatorname*{Res}_{\xi=\alpha}[p_{+}(\alpha)]=e^{i\alpha l}\frac{\sqrt{k-\alpha}N^{-}(\alpha)}{M_1^{-}(\alpha)}\Phi_1^{+}(\alpha),$$

$$\begin{aligned}\operatorname*{Re\,s}_{\xi=\beta_m}[p_{+}(\alpha)]&=e^{i\xi l}\frac{\sqrt{k-\xi}N(\xi)M_1^{+}(\xi)}{\frac{d}{d\xi}[M_1(\xi)N^{+}(\xi)(\xi-\alpha)]}\Phi_1^{+}(\xi)\Big|_{\xi=\beta_m},\\ &=-\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}\sqrt{k-\beta_m}N(\beta_m)M_1^{+}(\beta_m)}{M_1^{\prime}(\beta_m)N^{+}(\beta_m)(\alpha-\beta_m)}\Phi_1^{+}(\beta_m),\end{aligned}$$

so that

$$p_{+}(\alpha)=e^{i\alpha l}\frac{\sqrt{k-\alpha}N^{-}(\alpha)}{M_1^{-}(\alpha)}\Phi_1^{+}(\alpha)-\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}\sqrt{k-\beta_m}N(\beta_m)M_1^{+}(\beta_m)}{M_1^{\prime}(\beta_m)N^{+}(\beta_m)(\alpha-\beta_m)}\Phi_1^{+}(\beta_m).$$

Define

$$p_-(\alpha) = -\frac{1}{2\pi i}\int\limits_{-\infty+id}^{\infty+id} \frac{p(\xi)}{\xi-\alpha}d\xi,$$

$$p_-(\alpha) = -\frac{1}{2\pi i}\int\limits_{-\infty+ic}^{\infty+ic} e^{i\xi l}\frac{\sqrt{k-\xi}N^-(\xi)}{M_1^-(\xi)(\xi-\alpha)}\Phi_1^+(\xi)d\xi.$$

If we close the contour in upper half plane by semicircle then zeros of $M_1(\xi)$ are the singularities which gives

$$p_-(\alpha) = \sum_{m=1}^{\infty}\frac{e^{i\beta_m l}\sqrt{k-\beta_m}N(\beta_m)M_1^+(\beta_m)}{M_1^{/}(\beta_m)N^+(\beta_m)(\alpha-\beta_m)}\Phi_1^+(\beta_m)$$

Thus, Eq. (3.57) becomes

$$e^{i\alpha l}\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_1^-(\alpha)}\Phi_1^+(\alpha) = e^{i\alpha l}\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_1^-(\alpha)}\Phi_1^+(\alpha) - \sum_{m=1}^{\infty}\frac{e^{i\beta_m l}\sqrt{k-\beta_m}N(\beta_m)M_1^+(\beta_m)}{M_1^{/}(\beta_m)N^+(\beta_m)(\alpha-\beta_m)}\Phi_1^+(\beta_m)$$

$$+\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}\sqrt{k-\beta_m}N(\beta_m)M_1^+(\beta_m)}{M_1^{/}(\beta_m)N^+(\beta_m)(\alpha-\beta_m)}\Phi_1^+(\beta_m). \tag{3.58}$$

Consider the term of Eq. (3.55) and let

$$l(\alpha) = e^{i\alpha l}\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_1^-(\alpha)}\frac{e^{ilk}}{\alpha+k}. \tag{3.59}$$

Applying decomposition theorem, we can write

$$l(\alpha) = l_+(\alpha) + l_-(\alpha), \tag{3.60}$$

where

$$l_+(\alpha) = \frac{1}{2\pi i}\int\limits_{-\infty+ic}^{\infty+ic} \frac{l(\xi)}{\xi-\alpha}d\xi,$$

$$l_+(\alpha) = \frac{1}{2\pi i}\int\limits_{-\infty+ic}^{\infty+ic} e^{i\xi l}\frac{\sqrt{k-\xi}N^-(\xi)}{M_1^-(\xi)(\xi-\alpha)}\frac{e^{ilk}}{\xi+k}d\xi.$$

If we close the contour in upper half plane by semicircle then $\xi = \alpha$ and zeros of $M_1(\xi)$ are the singularities which gives

$$\operatorname*{Res}_{\xi=\alpha}[l_+(\alpha)] = e^{i\alpha l}\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_1^-(\alpha)}\frac{e^{ilk}}{\alpha+k},$$

$$\operatorname*{Res}_{\xi=\beta_m}[l_+(\alpha)] = -\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}M_1^+(\beta_m)\sqrt{k-\beta_m}N(\beta_m)e^{ilk}}{M_1^{/}(\beta_m)(\alpha-\beta_m)N^+(\beta_m)(\beta_m+k)},$$

so that

$$l_+(\alpha) = e^{i\alpha l}\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_1^-(\alpha)}\frac{e^{ilk}}{\alpha+k} - \sum_{m=1}^{\infty}\frac{e^{i\beta_m l}M_1^+(\beta_m)\sqrt{k-\beta_m}N(\beta_m)e^{ilk}}{M_1^{/}(\beta_m)(\alpha-\beta_m)N^+(\beta_m)(\beta_m+k)}.$$

Define

$$l_-(\alpha) = -\frac{1}{2\pi i}\int\limits_{-\infty+id}^{\infty+id}\frac{l(\xi)}{\xi-\alpha}d\xi,$$

$$= -\frac{1}{2\pi i}\int\limits_{-\infty+id}^{\infty+id} e^{i\xi l}\frac{\sqrt{k-\xi}N^-(\xi)}{M_1^-(\xi)(\xi-\alpha)}\frac{e^{ilk}}{\xi+k}d\xi.$$

If we close the contour in upper half plane by semicircle then zeros of $M_1(\xi)$ are the singularities which gives

$$l_-(\alpha) = \sum_{m=1}^{\infty}\frac{e^{i\beta_m l}M_1^+(\beta_m)\sqrt{k-\beta_m}N(\beta_m)e^{ilk}}{M_1^{/}(\beta_m)(\alpha-\beta_m)N^+(\beta_m)(\beta_m+k)}.$$

Thus, Eq. (3.60) becomes

$$e^{i\alpha l}\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_1^-(\alpha)}\frac{e^{ilk}}{\alpha+k} = e^{i\alpha l}\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_1^-(\alpha)}\frac{e^{ilk}}{\alpha+k} - \sum_{m=1}^{\infty}\frac{e^{i\beta_m l}M_1^+(\beta_m)\sqrt{k-\beta_m}N(\beta_m)e^{ilk}}{M_1^{/}(\beta_m)(\alpha-\beta_m)N^+(\beta_m)(\beta_m+k)}$$

$$+\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}M_1^+(\beta_m)\sqrt{k-\beta_m}N(\beta_m)e^{ilk}}{M_1^{/}(\beta_m)(\alpha-\beta_m)N^+(\beta_m)(\beta_m+k)}. \tag{3.61}$$

Putting Eqs. (3.58) and (3.61) in Eq. (3.55), we obtain

$$e^{i\alpha l}\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_1^-(\alpha)}R^+(\alpha)=e^{i\alpha l}\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_1^-(\alpha)}R^+(\alpha)-\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}\sqrt{k-\beta_m}N(\beta_m)M_1^+(\beta_m)R^+(\beta_m)}{M_1^{/}(\beta_m)N^+(\beta_m)(\alpha-\beta_m)}$$

$$+\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}\sqrt{k-\beta_m}N(\beta_m)M_1^+(\beta_m)R^+(\beta_m)}{M_1^{/}(\beta_m)N^+(\beta_m)(\alpha-\beta_m)}. \tag{3.62}$$

Putting Eqs. (3.54) and (3.62) in Eq. (3.46), we get

$$-\frac{F_1(\alpha)M_1^+(\alpha)}{\eta_1 N^+(\alpha)\sqrt{k+\alpha}}+\frac{F_2(\alpha)\sqrt{k-\alpha}M_1^+(\alpha)}{ik\eta_1 N^+(\alpha)M_1(\alpha)}-\sqrt{\frac{k}{2}}\frac{1}{\pi i}\frac{N^+(k)}{M_1^+(k)(\alpha+k)}-e^{i\alpha l}\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_1^-(\alpha)}R^+(\alpha)$$

$$+\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}\sqrt{k-\beta_m}N(\beta_m)M_1^+(\beta_m)R^+(\beta_m)}{M_1^{/}(\beta_m)N^+(\beta_m)(\alpha-\beta_m)}$$

$$=-\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_1^-(\alpha)}P_*^-(\alpha)-\sqrt{\frac{k}{2}}\frac{1}{\pi i}\frac{N^+(k)}{M_1^+(k)(\alpha+k)}+\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}\sqrt{k-\beta_m}N(\beta_m)M_1^+(\beta_m)R^+(\beta_m)}{M_1^{/}(\beta_m)N^+(\beta_m)(\alpha-\beta_m)}. \tag{3.63}$$

The regularity of the L.H.S. of Eq. (3.63) in upper half plane may be violated by the simple pole occurring at zeros of $M_1(\alpha)$ lying in the upper half plane namely $\alpha=\beta_m$, $m=1,\ 2,\ 3\ldots$. Consider the the second term of Eq. (3.63). Let

$$D(\alpha)=\frac{F_2(\alpha)\sqrt{k-\alpha}M_1^+(\alpha)}{ik\eta_1 N^+(\alpha)M_1(\alpha)}. \tag{3.64}$$

Applying decomposition theorem, we can write

$$D(\alpha)=D_+(\alpha)+D_-(\alpha), \tag{3.65}$$

where

$$D_+(\alpha)=\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{D(\xi)}{\xi-\alpha}d\xi,$$

$$D_+(\alpha) = \frac{1}{2\pi i} \int\limits_{-\infty+ic}^{\infty+ic} \frac{F_2(\xi)\sqrt{k-\xi}M_1^+(\xi)}{ik\eta_1 N^+(\xi)M_1(\xi)(\xi-\alpha)} d\xi.$$

If we close the contour in upper half plane by semicircle then $\xi = \alpha$ and zeros of $M_1(\xi)$ are the singularities which gives

$$\operatorname*{Res}_{\xi=\alpha}[D_+(\alpha)] = \frac{F_2(\alpha)\sqrt{k-\alpha}M_1^+(\alpha)}{ik\eta_1 N^+(\alpha)M_1(\alpha)},$$

$$\operatorname*{Res}_{\xi=\beta_m}[D_+(\alpha)] = -\sum_{m=1}^{\infty} \frac{F_2(\beta_m)\sqrt{k-\beta_m}M_1^+(\beta_m)}{ik\eta_1 N^+(\beta_m)M_1^{\prime}(\beta_m)(\alpha-\beta_m)},$$

so that

$$D_+(\alpha) = \frac{F_2(\alpha)\sqrt{k-\alpha}M_1^+(\alpha)}{ik\eta_1 N^+(\alpha)M_1(\alpha)} - \sum_{m=1}^{\infty} \frac{F_2(\beta_m)\sqrt{k-\beta_m}M_1^+(\beta_m)}{ik\eta_1 N^+(\beta_m)M_1^{\prime}(\beta_m)(\alpha-\beta_m)}.$$

Define

$$D_-(\alpha) = -\frac{1}{2\pi i} \int\limits_{-\infty+id}^{\infty+id} \frac{D(\xi)}{\xi-\alpha} d\xi,$$

$$D_-(\alpha) = -\frac{1}{2\pi i} \int\limits_{-\infty+id}^{\infty+id} \frac{F_2(\xi)\sqrt{k-\xi}M_1^+(\xi)}{ik\eta_1 N^+(\xi)M_1(\xi)(\xi-\alpha)} d\xi.$$

Closing the contour in upper half plane by semicircle then zeros of $M_1(\xi)$ are the singularities which gives

$$D_-(\alpha) = \sum_{m=1}^{\infty} \frac{F_2(\beta_m)\sqrt{k-\beta_m}M_1^+(\beta_m)}{ik\eta_1 N^+(\beta_m)M_1^{\prime}(\beta_m)(\alpha-\beta_m)}.$$

Thus, Eq. (3.65) gives

$$\frac{F_2(\alpha)\sqrt{k-\alpha}M_1^+(\alpha)}{ik\eta_1 N^+(\alpha)M_1(\alpha)} = \frac{F_2(\alpha)\sqrt{k-\alpha}M_1^+(\alpha)}{ik\eta_1 N^+(\alpha)M_1(\alpha)} - \sum_{m=1}^{\infty} \frac{F_2(\beta_m)\sqrt{k-\beta_m}M_1^+(\beta_m)}{ik\eta_1 N^+(\beta_m)M_1^{\prime}(\beta_m)(\alpha-\beta_m)}$$

$$+\sum_{m=1}^{\infty}\frac{F_2(\beta_m)\sqrt{k-\beta_m}M_1^+(\beta_m)}{ik\eta_1N^+(\beta_m)M_1^{/}(\beta_m)(\alpha-\beta_m)}. \tag{3.66}$$

Putting Eq. (3.66) in Eq. (3.63), we get

$$-\frac{F_1(\alpha)M_1^+(\alpha)}{\eta_1N^+(\alpha)\sqrt{k+\alpha}}+\frac{F_2(\alpha)\sqrt{k-\alpha}M_1^+(\alpha)}{ik\eta_1N^+(\alpha)M_1(\alpha)}-\sum_{m=1}^{\infty}\frac{a_m}{\alpha-\beta_m}-\sqrt{\frac{k}{2}}\frac{1}{\pi i}\frac{N^+(k)}{M_1^+(k)(\alpha+k)}$$

$$-e^{i\alpha l}\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_1^-(\alpha)}R^+(\alpha)+\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}\sqrt{k-\beta_m}N(\beta_m)M_1^+(\beta_m)R^+(\beta_m)}{M_1^{/}(\beta_m)N^+(\beta_m)(\alpha-\beta_m)}$$

$$=-\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_1^-(\alpha)}P_*^-(\alpha)-\sqrt{\frac{k}{2}}\frac{1}{\pi i}\frac{N^+(k)}{M_1^+(k)(\alpha+k)}+\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}\sqrt{k-\beta_m}N(\beta_m)M_1^+(\beta_m)R^+(\beta_m)}{M_1^{/}(\beta_m)N^+(\beta_m)(\alpha-\beta_m)}$$

$$-\sum_{m=1}^{\infty}\frac{a_m}{\alpha-\beta_m}, \tag{3.67}$$

where

$$a_m=\frac{F_2(\beta_m)\sqrt{k-\beta_m}M_1^+(\beta_m)}{ik\eta_1N^+(\beta_m)M_1^{/}(\beta_m)}. \tag{3.68}$$

The application of analytic continuation principle together with Liouville's theorem to the Eq. (3.67) yields

$$\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_1^-(\alpha)}P_*^-(\alpha)=\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}\sqrt{k-\beta_m}N(\beta_m)M_1^+(\beta_m)R^+(\beta_m)}{M_1^{/}(\beta_m)N^+(\beta_m)(\alpha-\beta_m)}-\sqrt{\frac{k}{2}}\frac{1}{\pi i}\frac{N^+(k)}{M_1^+(k)(\alpha+k)}$$

$$-\sum_{m=1}^{\infty}\frac{a_m}{\alpha-\beta_m}. \tag{3.69}$$

Now we will apply similar treatment to Eq. (3.47). In Eq. (3.47) first term is regular in the lower half plane, while right hand side is regular in the upper half plane. Second and third term on L.H.S. have singularities in the lower half plane. Consider the third term of Eq. (3.47) on L.H.S. (we want to make it regular in the lower half)

$$\frac{e^{-i\alpha l}\sqrt{k+\alpha}N^+(\alpha)}{M_1^+(\alpha)}P_*^-(\alpha)=-\frac{e^{-i\alpha l}\sqrt{k+\alpha}N^+(\alpha)\Phi_1^-(\alpha)}{M_1^+(\alpha)}-\frac{1}{2\pi i}\frac{e^{-i\alpha l}N^+(\alpha)}{M_1^+(\alpha)\sqrt{\alpha+k}}. \tag{3.70}$$

Consider the first term on R.H.S. of Eq. (3.70). Let

$$E(\alpha) = \frac{e^{-i\alpha l}\sqrt{k+\alpha}N^{+}(\alpha)\Phi_1^{-}(\alpha)}{M_1^{+}(\alpha)}. \tag{3.71}$$

Applying the decomposition theorem we can write

$$E(\alpha) = E_{+}(\alpha) + E_{-}(\alpha), \tag{3.72}$$

where

$$E_{+}(\alpha) = \frac{e^{-i\alpha l}\sqrt{k+\alpha}N^{+}(\alpha)\Phi_1^{-}(\alpha)}{M_1^{+}(\alpha)} - \sum_{m=1}^{\infty}\frac{e^{i\beta_m l}\sqrt{k-\beta_m}N(-\beta_m)M_1^{+}(\beta_m)\Phi_1^{-}(-\beta_m)}{M_1^{/}(-\beta_m)N^{+}(\beta_m)(\alpha+\beta_m)},$$

and

$$E_{-}(\alpha) = \sum_{m=1}^{\infty}\frac{e^{i\beta_m l}\sqrt{k-\beta_m}N(-\beta_m)M_1^{+}(\beta_m)\Phi_1^{-}(-\beta_m)}{M_1^{/}(-\beta_m)N^{+}(\beta_m)(\alpha+\beta_m)}.$$

Therefore, Eq. (3.72) becomes

$$\frac{e^{-i\alpha l}\sqrt{k+\alpha}N^{+}(\alpha)\Phi_1^{-}(\alpha)}{M_1^{+}(\alpha)} = \frac{e^{-i\alpha l}\sqrt{k+\alpha}N^{+}(\alpha)\Phi_1^{-}(\alpha)}{M_1^{+}(\alpha)} - \sum_{m=1}^{\infty}\frac{e^{i\beta_m l}\sqrt{k-\beta_m}N(-\beta_m)M_1^{+}(\beta_m)\Phi_1^{-}(-\beta_m}{M_1^{/}(-\beta_m)N^{+}(\beta_m)(\alpha+\beta_m)}$$

$$+\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}\sqrt{k-\beta_m}N(-\beta_m)M_1^{+}(\beta_m)\Phi_1^{-}(-\beta_m)}{M_1^{/}(-\beta_m)N^{+}(\beta_m)(\alpha+\beta_m)}. \tag{3.73}$$

Now consider the second term of Eq. (3.70) and let

$$H(\alpha) = \frac{e^{-i\alpha l}N^{+}(\alpha)}{M_1^{+}(\alpha)\sqrt{\alpha+k}}, \tag{3.74}$$

be decomposed by decomposition theorem

$$H(\alpha) = H_{+}(\alpha) + H_{-}(\alpha), \tag{3.75}$$

where

$$H_+(\alpha) = \frac{e^{-i\alpha l}N^+(\alpha)}{M_1^+(\alpha)\sqrt{\alpha+k}} - \sum_{m=1}^{\infty} \frac{e^{i\beta_m l}N(-\beta_m)M_1^+(\beta_m)}{M_1^{/}(-\beta_m)(\alpha+\beta_m)N^+(\beta_m)\sqrt{k-\beta_m}},$$

and

$$H_-(\alpha) = \sum_{m=1}^{\infty} \frac{e^{i\beta_m l}N(-\beta_m)M_1^+(\beta_m)}{M_1^{/}(-\beta_m)(\alpha+\beta_m)N^+(\beta_m)\sqrt{k-\beta_m}}.$$

Thus, Eq. (3.75) becomes

$$\frac{e^{-i\alpha l}N^+(\alpha)}{M_1^+(\alpha)\sqrt{\alpha+k}} = \frac{e^{-i\alpha l}N^+(\alpha)}{M_1^+(\alpha)\sqrt{\alpha+k}} - \sum_{m=1}^{\infty} \frac{e^{i\beta_m l}N(-\beta_m)M_1^+(\beta_m)}{M_1^{/}(-\beta_m)(\alpha+\beta_m)N^+(\beta_m)\sqrt{k-\beta_m}}$$

$$-\sum_{m=1}^{\infty} \frac{e^{i\beta_m l}(k-\beta_m)N(-\beta_m)M_1^+(\beta_m)}{M_1^{/}(-\beta_m)(\alpha+\beta_m)N^+(\beta_m)\sqrt{k-\beta_m}}. \tag{3.76}$$

Putting Eqs. (3.73) and (3.76) in Eq. (3.70), we obtain

$$\frac{e^{-i\alpha l}\sqrt{k+\alpha}N^+(\alpha)}{M_1^+(\alpha)}P_*^-(\alpha) = \frac{e^{-i\alpha l}\sqrt{k+\alpha}N^+(\alpha)}{M_1^+(\alpha)}P_*^-(\alpha) - \sum_{m=1}^{\infty} \frac{e^{i\beta_m l}\sqrt{k-\beta_m}N(-\beta_m)M_1^+(\beta_m)P_*^-(-\beta_m)}{M_1^{/}(-\beta_m)N^+(\beta_m)(\alpha+\beta_m)}$$

$$+\sum_{m=1}^{\infty} \frac{e^{i\beta_m l}\sqrt{k-\beta_m}N(-\beta_m)M_1^+(\beta_m)P_*^-(-\beta_m)}{M_1^{/}(-\beta_m)N^+(\beta_m)(\alpha+\beta_m)}. \tag{3.77}$$

Now we consider the second term of Eq. (3.47). (we want to make it regular in the lower half plane)

$$Z(\alpha) = \frac{e^{-i\alpha l}F_2(\alpha)\sqrt{k+\alpha}M_1^-(\alpha)}{ik\eta_1 N^-(\alpha)M_1(\alpha)}. \tag{3.78}$$

Applying the decomposition theorem we can write

$$Z(\alpha) = Z_+(\alpha) + Z_-(\alpha), \tag{3.79}$$

where

$$Z_+(\alpha) = \frac{e^{-i\alpha l} F_2(\alpha)\sqrt{k+\alpha} M_1^-(\alpha)}{ik\eta_1 N^-(\alpha) M_1(\alpha)} - \sum_{m=1}^{\infty} \frac{e^{i\beta_m l} F_2(-\beta_m)\sqrt{k-\beta_m} M_1^+(\beta_m)}{ik\eta_1 M_1^{/}(-\beta_m) N^+(\beta_m)(\alpha+\beta_m)},$$

and

$$Z_-(\alpha) = \sum_{m=1}^{\infty} \frac{e^{i\beta_m l} F_2(-\beta_m)\sqrt{k-\beta_m} M_1^+(\beta_m)}{ik\eta_1 M_1^{/}(-\beta_m) N^+(\beta_m)(\alpha+\beta_m)}.$$

Thus, Eq. (3.79) becomes

$$\frac{e^{-i\alpha l} F_2(\alpha)\sqrt{k+\alpha} M_1^-(\alpha)}{ik\eta_1 N^-(\alpha) M_1(\alpha)} = \frac{e^{-i\alpha l} F_2(\alpha)\sqrt{k+\alpha} M_1^-(\alpha)}{ik\eta_1 N^-(\alpha) M_1(\alpha)} - \sum_{m=1}^{\infty} \frac{e^{i\beta_m l} F_2(-\beta_m)\sqrt{k-\beta_m} M_1^+(\beta_m)}{ik\eta_1 M_1^{/}(-\beta_m) N^+(\beta_m)(\alpha+\beta_m)}$$

$$+ \sum_{m=1}^{\infty} \frac{e^{i\beta_m l} F_2(-\beta_m)\sqrt{k-\beta_m} M_1^+(\beta_m)}{ik\eta_1 M_1^{/}(-\beta_m) N^+(\beta_m)(\alpha+\beta_m)}. \tag{3.80}$$

Putting Eqs. (3.77) and (3.80) in Eq. (3.47), we get

$$-\frac{e^{-i\alpha l} F_1(\alpha) M_1^-(\alpha)}{\eta_1 N^-(\alpha)\sqrt{k-\alpha}} + \frac{e^{-i\alpha l} F_2(\alpha)\sqrt{k+\alpha} M_1^-(\alpha)}{ik\eta_1 N^-(\alpha) M_1(\alpha)} - \sum_{m=1}^{\infty} \frac{b_m}{\alpha+\beta_m}$$

$$+e^{-i\alpha l} \frac{\sqrt{k+\alpha} N^+(\alpha)}{M_1^+(\alpha)} P_*^-(\alpha) - \sum_{m=1}^{\infty} \frac{e^{i\beta_m l}\sqrt{k-\beta_m} N(-\beta_m) M_1^+(\beta_m) P_*^-(-\beta_m)}{M_1^{/}(-\beta_m) N^+(\beta_m)(\alpha+\beta_m)}$$

$$= \frac{\sqrt{k+\alpha} N^+(\alpha)}{M_1^+(\alpha)} R^+(\alpha) - \sum_{m=1}^{\infty} \frac{b_m}{\alpha+\beta_m} - \sum_{m=1}^{\infty} \frac{e^{i\beta_m l}\sqrt{k-\beta_m} N(-\beta_m) M_1^+(\beta_m) P_*^-(-\beta_m)}{M_1^{/}(-\beta_m) N^+(\beta_m)(\alpha+\beta_m)}, \tag{3.81}$$

where

$$b_m = \frac{e^{i\beta_m l} F_2(-\beta_m)\sqrt{k-\beta_m} M_1^+(\beta_m)}{ik\eta_1 M_1^{/}(-\beta_m) N^+(\beta_m)}, \tag{3.82}$$

The application of analytic continuation principle together with Liouville's theorem to Eq. (3.81) yields

$$\frac{\sqrt{k+\alpha} N^+(\alpha)}{M_1^+(\alpha)} R^+(\alpha) = \sum_{m=1}^{\infty} \frac{e^{i\beta_m l}\sqrt{k-\beta_m} N(-\beta_m) M_1^+(\beta_m) P_*^-(-\beta_m)}{M_1^{/}(-\beta_m) N^+(\beta_m)(\alpha+\beta_m)} + \sum_{m=1}^{\infty} \frac{b_m}{\alpha+\beta_m}. \tag{3.83}$$

Now we apply similar treatment to Eq. (3.48). In Eq. (3.48) first term is regular in the upper half plane, the third term and the R.H.S. have singularities in both half planes. Consider the third term on L.H.S. (we want to make it regular in the lower half plane). Let

$$\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_2^-(\alpha)}Q_*^-(\alpha) = -\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_2^-(\alpha)}\Phi_2^-(\alpha) - \frac{1}{2\pi i}\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_2^-(\alpha)}\frac{1}{\alpha+k}. \tag{3.84}$$

Consider the second term of Eq. ([?]) and let

$$T(\alpha) = \frac{\sqrt{k-\alpha}N^-(\alpha)}{M_2^-(\alpha)}\frac{1}{\alpha+k}. \tag{3.85}$$

be decomposed by decomposition theorem

$$T(\alpha) = T_+(\alpha) + T_-(\alpha), \tag{3.86}$$

where

$$T_+(\alpha) = \frac{\sqrt{k-\alpha}N^-(\alpha)}{M_2^-(\alpha)(\alpha+k)} - \frac{\sqrt{2k}N^+(k)}{M_2^+(k)(\alpha+k)},$$

and

$$T_-(\alpha) = \frac{\sqrt{2k}N^+(k)}{M_2^+(k)(\alpha+k)}.$$

Therefore, Eq. (3.86) becomes

$$\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_2^-(\alpha)}\frac{1}{\alpha+k} = \frac{\sqrt{k-\alpha}N^-(\alpha)}{M_2^-(\alpha)}\frac{1}{\alpha+k} - \frac{\sqrt{2k}N^+(k)}{M_2^+(k)(\alpha+k)} + \frac{\sqrt{2k}N^+(k)}{M_2^+(k)(\alpha+k)}. \tag{3.87}$$

Putting Eq. (3.87) in Eq. (3.84), we arrive at

$$\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_2^-(\alpha)}Q_*^-(\alpha) = \frac{\sqrt{k-\alpha}N^-(\alpha)}{M_2^-(\alpha)}Q_*^-(\alpha) - \sqrt{\frac{k}{2}}\frac{1}{\pi i}\frac{N^+(k)}{M_2^+(k)(\alpha+k)} + \sqrt{\frac{k}{2}}\frac{1}{\pi i}\frac{N^+(k)}{M_2^+(k)(\alpha+k)}. \tag{3.88}$$

Now consider the R.H.S. of Eq. (3.48) (we want to make it regular in the upper half plane)

$$e^{i\alpha l}\frac{\sqrt{k-\alpha}N^{-}(\alpha)}{M_2^{-}(\alpha)}S^{+}(\alpha)=e^{i\alpha l}\frac{\sqrt{k-\alpha}N^{-}(\alpha)}{M_2^{-}(\alpha)}\Phi_2^{+}(\alpha)-\frac{1}{2\pi i}e^{i\alpha l}\frac{\sqrt{k-\alpha}N^{-}(\alpha)}{M_2^{-}(\alpha)}\frac{e^{ilk}}{\alpha+k}, \tag{3.89}$$

consider

$$U(\alpha)=e^{i\alpha l}\frac{\sqrt{k-\alpha}N^{-}(\alpha)}{M_2^{-}(\alpha)}\Phi_2^{+}(\alpha), \tag{3.90}$$

Applying the decomposition theorem we can write

$$U(\alpha)=U_{+}(\alpha)+U_{-}(\alpha), \tag{3.91}$$

where

$$U_{+}(\alpha)=e^{i\alpha l}\frac{\sqrt{k-\alpha}N^{-}(\alpha)}{M_2^{-}(\alpha)}\Phi_2^{+}(\alpha)-\sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}N(\nu_m)M_2^{+}(\nu_m)\Phi_2^{+}(\nu_m)}{M_2^{/}(\nu_m)N^{+}(\nu_m)(\alpha-\nu_m)},$$

and

$$U_{-}(\alpha)=\sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}N(\nu_m)M_2^{+}(\nu_m)\Phi_2^{+}(\nu_m)}{M_2^{/}(\nu_m)N^{+}(\nu_m)(\alpha-\nu_m)}.$$

Thus, Eq. (3.91) becomes

$$\begin{aligned}e^{i\alpha l}\frac{\sqrt{k-\alpha}N^{-}(\alpha)}{M_2^{-}(\alpha)}\Phi_2^{+}(\alpha) &= e^{i\alpha l}\frac{\sqrt{k-\alpha}N^{-}(\alpha)}{M_2^{-}(\alpha)}\Phi_2^{+}(\alpha)-\sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}N(\nu_m)M_2^{+}(\nu_m)\Phi_2^{+}(\nu_m)}{M_2^{/}(\nu_m)N^{+}(\nu_m)(\alpha-\nu_m)}\\ &\quad+\sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}N(\nu_m)M_2^{+}(\nu_m)\Phi_2^{+}(\nu_m)}{M_2^{/}(\nu_m)N^{+}(\nu_m)(\alpha-\nu_m)}.\end{aligned} \tag{3.92}$$

Consider the term of Eq. (3.89),

$$V(\alpha)=e^{i\alpha l}\frac{\sqrt{k-\alpha}N^{-}(\alpha)}{M_2^{-}(\alpha)}\frac{e^{ilk}}{(\alpha+k)}. \tag{3.93}$$

Applying the decomposition theorem we can write

$$V(\alpha) = V_+(\alpha) + V_-(\alpha), \tag{3.94}$$

where

$$V_+(\alpha) = e^{i\alpha l}\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_2^-(\alpha)}\frac{e^{ilk}}{(\alpha+k)} - \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}N(\nu_m)M_2^+(\nu_m)e^{ilk}}{M_2^{/}(\nu_m)N^+(\nu_m)(k+\nu_m)(\alpha-\nu_m)},$$

and

$$V_-(\alpha) = \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{(k-\nu_m)}N(\nu_m)M_2^+(\nu_m)e^{ilk}}{M_2^{/}(\nu_m)N^+(\nu_m)(k+\nu_m)(\alpha-\nu_m)}.$$

Thus, Eq. (3.94) becomes

$$\begin{aligned} e^{i\alpha l}\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_2^-(\alpha)}\frac{e^{ilk}}{(\alpha+k)} &= \frac{e^{i\alpha l}\sqrt{k-\alpha}N^-(\alpha)e^{ilk}}{M_2^-(\alpha)(\alpha+k)} - \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}N(\nu_m)M_2^+(\nu_m)e^{ilk}}{M_2^{/}(\nu_m)N^+(\nu_m)(k+\nu_m)(\alpha-\nu_m)} \\ &+ \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}N(\nu_m)M_2^+(\nu_m)e^{ilk}}{M_2^{/}(\nu_m)N^+(\nu_m)(k+\nu_m)(\alpha-\nu_m)}. \end{aligned} \tag{3.95}$$

Putting Eqs. (3.92) and (3.95) in Eq. (3.89), we obtain

$$\begin{aligned} e^{i\alpha l}\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_2^-(\alpha)}S^+(\alpha) &= e^{i\alpha l}\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_2^-(\alpha)}S^+(\alpha) - \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}N(\nu_m)M_2^+(\nu_m)S^+(\nu_m)}{M_2^{/}(\nu_m)N^+(\nu_m)(\alpha-\nu_m)} \\ &+ \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}N(\nu_m)M_2^+(\nu_m)S^+(\nu_m)}{M_2^{/}(\nu_m)N^+(\nu_m)(\alpha-\nu_m)}. \end{aligned} \tag{3.96}$$

Now consider the second term of Eq. (3.48), (we want to make it regular in upper half plane)

$$I(\alpha) = \frac{F_1(\alpha)\sqrt{k-\alpha}M_2^+(\alpha)}{ik\eta_2 N^+(\alpha)M_2(\alpha)}. \tag{3.97}$$

Applying the decomposition theorem we can write

$$I(\alpha) = I_+(\alpha) + I_-(\alpha), \tag{3.98}$$

where

$$I_+(\alpha) = \frac{F_1(\alpha)\sqrt{k-\alpha}M_2^+(\alpha)}{ik\eta_2 N^+(\alpha)M_2(\alpha)} - \sum_{m=1}^{\infty} \frac{F_1(\nu_m)\sqrt{k-\nu_m}M_2^+(\nu_m)}{ik\eta_2 N^+(\nu_m)M_2^{/}(\nu_m)(\alpha-\nu_m)},$$

and

$$I_-(\alpha) = \sum_{m=1}^{\infty} \frac{F_1(\nu_m)\sqrt{k-\nu_m}M_2^+(\nu_m)}{ik\eta_2 N^+(\nu_m)M_2^{/}(\nu_m)(\alpha-\nu_m)}.$$

Thus, Eq. (3.98) becomes

$$\begin{aligned}\frac{F_1(\alpha)\sqrt{k-\alpha}M_2^+(\alpha)}{ik\eta_2 N^+(\alpha)M_2(\alpha)} &= \frac{F_1(\alpha)\sqrt{k-\alpha}M_2^+(\alpha)}{ik\eta_2 N^+(\alpha)M_2(\alpha)} - \sum_{m=1}^{\infty} \frac{F_1(\nu_m)\sqrt{k-\nu_m}M_2^+(\nu_m)}{ik\eta_2 N^+(\nu_m)M_2^{/}(\nu_m)(\alpha-\nu_m)} \\ &+ \sum_{m=1}^{\infty} \frac{F_1(\nu_m)\sqrt{k-\nu_m}M_2^+(\nu_m)}{ik\eta_2 N^+(\nu_m)M_2^{/}(\nu_m)(\alpha-\nu_m)},\end{aligned}$$

$$\frac{F_1(\alpha)M_2^+(\alpha)}{N^+(\alpha)(k+\alpha)M_2(\alpha)} = \frac{F_1(\alpha)M_2^+(\alpha)}{N^+(\alpha)(k+\alpha)M_2(\alpha)} - \sum_{m=1}^{\infty}\frac{c_m}{\alpha-\nu_m} + \sum_{m=1}^{\infty}\frac{c_m}{\alpha-\nu_m}, \tag{3.99}$$

where

$$c_m = \frac{F_1(\nu_m)\sqrt{k-\nu_m}M_2^+(\nu_m)}{ik\eta_2 N^+(\nu_m)M_2^{/}(\nu_m)}. \tag{3.100}$$

Using Eqs. (3.88), (3.96) and (3.99) in Eq. (3.48), we arrive at

$$\begin{aligned} &-\frac{F_2(\alpha)M_2^+(\alpha)}{\eta_2 N^+(\alpha)\sqrt{k+\alpha}} + \frac{F_1(\alpha)\sqrt{k-\alpha}M_2^+(\alpha)}{ik\eta_2 N^+(\alpha)M_2(\alpha)} - \sum_{m=1}^{\infty}\frac{c_m}{\alpha-\nu_m} - \sqrt{\frac{k}{2}}\frac{1}{\pi i}\frac{N^+(k)}{M_2^+(k)(\alpha+k)} \\ &-e^{i\alpha l}\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_2^-(\alpha)}S^+(\alpha) + \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}N(\nu_m)M_2^+(\nu_m)S^+(\nu_m)}{M_2^{/}(\nu_m)N^+(\nu_m)(\alpha-\nu_m)} \\ =\ &-\frac{(k-\alpha)N^-(\alpha)}{M_2^-(\alpha)}Q_*^-(\alpha) - \sqrt{\frac{k}{2}}\frac{1}{\pi i}\frac{N^+(k)}{M_2^+(k)(\alpha+k)} + \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}N(\nu_m)M_2^+(\nu_m)S^+(\nu_m)}{M_2^{/}(\nu_m)N^+(\nu_m)(\alpha-\nu_m)} \\ &-\sum_{m=1}^{\infty}\frac{c_m}{\alpha-\nu_m}. \end{aligned} \tag{3.101}$$

The application of analytic continuation principle together with Liouville's theorem to Eq. (3.101) yields

$$\frac{\sqrt{k-\alpha}N^-(\alpha)}{M_2^-(\alpha)}Q_*^-(\alpha) = \sum_{m=1}^{\infty} \frac{e^{i\nu_m l}\sqrt{k-\nu_m}N(\nu_m)M_2^+(\nu_m)S^+(\nu_m)}{M_2^{/}(\nu_m)N^+(\nu_m)(\alpha-\nu_m)} - \sqrt{\frac{k}{2}}\frac{1}{\pi i}\frac{N^+(k)}{M_2^+(k)(\alpha+k)}$$

$$-\sum_{m=1}^{\infty}\frac{c_m}{\alpha-\nu_m}. \tag{3.102}$$

Now we will apply similar treatment to Eq. (3.49). In Eq. (3.49) first term is regular in the lower half plane, while right hand side is regular in the upper half plane. Second and third term on L.H.S have singularities in the lower half plane. Let

$$\frac{e^{-i\alpha l}\sqrt{k+\alpha}N^+(\alpha)}{M_2^+(\alpha)}Q_*^-(\alpha) = -\frac{e^{-i\alpha l}\sqrt{k+\alpha}N^+(\alpha)\Phi_2^-(\alpha)}{M_2^+(\alpha)} - \frac{1}{2\pi i}\frac{e^{-i\alpha l}N^+(\alpha)}{M_2^+(\alpha)\sqrt{\alpha+k}}. \tag{3.103}$$

Consider the first term on R.H.S. of Eq. (3.103)

$$E_1(\alpha) = \frac{e^{-i\alpha l}\sqrt{k+\alpha}N^+(\alpha)\Phi_2^-(\alpha)}{M_2^+(\alpha)}. \tag{3.104}$$

By decomposition theorem, we can write

$$E_1(\alpha) = E_{1+}(\alpha) + E_{1-}(\alpha), \tag{3.105}$$

where

$$E_{1+}(\alpha) = \frac{e^{-i\alpha l}\sqrt{k+\alpha}N^+(\alpha)\Phi_2^-(\alpha)}{M_2^+(\alpha)} - \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}N(-\nu_m)M_2^+(\nu_m)\Phi_2^-(-\nu_m)}{M_2^{/}(-\nu_m)N^+(\nu_m)(\alpha+\nu_m)}, \tag{3.106}$$

and

$$E_{1-}(\alpha) = \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}N(-\nu_m)M_2^+(\nu_m)\Phi_2^-(-\nu_m)}{M_2^{/}(-\nu_m)N^+(\nu_m)(\alpha+\nu_m)}. \tag{3.107}$$

Therefore, Eq. (3.105) gives

$$\frac{e^{-i\alpha l}\sqrt{k+\alpha}N^{+}(\alpha)\Phi_2^{-}(\alpha)}{M_2^{+}(\alpha)} = \frac{e^{-i\alpha l}\sqrt{k+\alpha}N^{+}(\alpha)\Phi_2^{-}(\alpha)}{M_2^{+}(\alpha)}.$$

$$-\sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}N(-\nu_m)M_2^{+}(\nu_m)\Phi_2^{-}(-\nu_m)}{M_2^{/}(-\nu_m)N^{+}(\nu_m)(\alpha+\nu_m)}$$

$$+\sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}N(-\nu_m)M_2^{+}(\nu_m)\Phi_2^{-}(-\nu_m)}{M_2^{/}(-\nu_m)N^{+}(\nu_m)(\alpha+\nu_m)} \qquad (3.108)$$

Now consider the second term of Eq. (3.103). Let

$$H_1(\alpha) = \frac{e^{-i\alpha l}N^{+}(\alpha)}{M_2^{+}(\alpha)\sqrt{\alpha+k}}. \qquad (3.109)$$

Applying decomposition theorem, we can write

$$H_1(\alpha) = H_{1+}(\alpha) + H_{1-}(\alpha), \qquad (3.110)$$

where

$$H_{+}(\alpha) = \frac{e^{-i\alpha l}N^{+}(\alpha)}{M_2^{+}(\alpha)\sqrt{\alpha+k}} - \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}N(-\nu_m)M_2^{+}(\nu_m)}{M_2^{/}(-\nu_m)(\alpha+\nu_m)N^{+}(\nu_m)\sqrt{k-\nu_m}}, \qquad (3.111)$$

and

$$H_{1-}(\alpha) = \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}N(-\nu_m)M_2^{+}(\nu_m)}{M_2^{/}(-\nu_m)(\alpha+\nu_m)N^{+}(\nu_m)\sqrt{k-\nu_m}}. \qquad (3.112)$$

Thus, Eq. (3.110) becomes

$$\frac{e^{-i\alpha l}N^{+}(\alpha)}{M_2^{+}(\alpha)\sqrt{\alpha+k}} = \frac{e^{-i\alpha l}N^{+}(\alpha)}{M_2^{+}(\alpha)\sqrt{\alpha+k}} - \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}N(-\nu_m)M_2^{+}(\nu_m)}{M_2^{/}(-\nu_m)(\alpha+\nu_m)N^{+}(\nu_m)\sqrt{k-\nu_m}}$$

$$+\sum_{m=1}^{\infty}\frac{e^{i\nu_m l}N(-\nu_m)M_2^{+}(\nu_m)}{M_2^{/}(-\nu_m)(\alpha+\nu_m)N^{+}(\nu_m)\sqrt{k-\nu_m}}. \qquad (3.113)$$

Putting Eqs. (3.108) and (3.113) in Eq. (3.103), we obtain

$$\frac{e^{-i\alpha l}\sqrt{k+\alpha}N^{+}(\alpha)}{M_2^{+}(\alpha)}Q_*^{-}(\alpha) = \frac{e^{-i\alpha l}\sqrt{k+\alpha}N^{+}(\alpha)}{M_2^{+}(\alpha)}Q_*^{-}(\alpha)$$

$$-\sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}N(-\nu_m)M_2^{+}(\nu_m)Q_*^{-}(-\nu_m)}{M_2^{/}(-\nu_m)N^{+}(\nu_m)(\alpha+\nu_m)}$$

$$+\sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}N(-\nu_m)M_2^{+}(\nu_m)Q_*^{-}(-\nu_m)}{M_2^{/}(-\nu_m)N^{+}(\nu_m)(\alpha+\nu_m)} \quad (3.114)$$

Now we consider the second term of Eq. (3.49) (we want to make it regular in the lower half plane)

$$Z_1(\alpha) = e^{-i\alpha l}\frac{F_1(\alpha)\sqrt{k+\alpha}M_2^{-}(\alpha)}{ik\eta_2 N^{-}(\alpha)M_2(\alpha)}. \quad (3.115)$$

By decomposition theorem, we can write

$$Z_1(\alpha) = Z_{1+}(\alpha) + Z_{1-}(\alpha), \quad (3.116)$$

where

$$Z_{1+}(\alpha) = e^{-i\alpha l}\frac{F1(\alpha)\sqrt{k+\alpha}M_2^{-}(\alpha)}{ik\eta_2 N^{-}(\alpha)M_2(\alpha)} - \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}F_1(-\nu_m)M_2^{+}(\nu_m)}{ik\eta_2 M_2^{/}(-\nu_m)N^{+}(\nu_m)(\alpha+\nu_m)}, \quad (3.117)$$

and

$$Z_{1-}(\alpha) = \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}F_1(-\nu_m)M_2^{+}(\nu_m)}{ik\eta_2 M_2^{/}(-\nu_m)N^{+}(\nu_m)(\alpha+\nu_m)}. \quad (3.118)$$

Thus, Eq. (3.116) becomes

$$e^{-i\alpha l}\frac{F1(\alpha)\sqrt{k+\alpha}M_2^{-}(\alpha)}{ik\eta_2 N^{-}(\alpha)M_2(\alpha)} = e^{-i\alpha l}\frac{F1(\alpha)\sqrt{k+\alpha}M_2^{-}(\alpha)}{ik\eta_2 N^{-}(\alpha)M_2(\alpha)} - \sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}F_1(-\nu_m)M_2^{+}(\nu_m)}{ik\eta_2 M_2^{/}(-\nu_m)N^{+}(\nu_m)(\alpha+\nu_m)}$$

$$+\sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}F_1(-\nu_m)M_2^{+}(\nu_m)}{ik\eta_2 M_2^{/}(-\nu_m)N^{+}(\nu_m)(\alpha+\nu_m)}. \quad (3.119)$$

Putting Eqs. (3.114) and (3.119) in Eq. (3.49), we obtain

$$-e^{-i\alpha l}\frac{F_2(\alpha)M_2^-(\alpha)}{\eta_2 N^-(\alpha)\sqrt{k-\alpha}}+e^{-i\alpha l}\frac{F1(\alpha)\sqrt{k+\alpha}M_2^-(\alpha)}{ik\eta_2 N^-(\alpha)M_2(\alpha)}-\sum_{m=1}^{\infty}\frac{d_m}{\alpha+\nu_m}$$

$$\frac{e^{-i\alpha l}\sqrt{k+\alpha}N^+(\alpha)}{M_2^+(\alpha)}Q_*^-(\alpha)-\sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}N(-\nu_m)M_2^+(\nu_m)Q_*^-(-\nu_m)}{M_2^{/}(-\nu_m)N^+(\nu_m)(\alpha+\nu_m)}$$

$$=\frac{\sqrt{k+\alpha}N^+(\alpha)}{M_2^+(\alpha)}S^+(\alpha)-\sum_{m=1}^{\infty}\frac{d_m}{\alpha+\nu_m}$$

$$-\sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}N(-\nu_m)M_2^+(\nu_m)Q_*^-(-\nu_m)}{M_2^{/}(-\nu_m)N^+(\nu_m)(\alpha+\nu_m)}, \tag{3.120}$$

where

$$d_m=\frac{e^{i\nu_m l}\sqrt{k-\nu_m}F_1(-\nu_m)M_2^+(\nu_m)}{ik\eta_2 M_2^{/}(-\nu_m)N^+(\nu_m)}. \tag{3.121}$$

The application of analytic continuation principle together with Liouville's theorem to Eq. (3.120) yields

$$\frac{\sqrt{k+\alpha}N^+(\alpha)}{M_2^+(\alpha)}S^+(\alpha)=\sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}N(-\nu_m)M_2^+(\nu_m)Q_*^-(-\nu_m)}{M_2^{/}(-\nu_m)N^+(\nu_m)(\alpha+\nu_m)}+\sum_{m=1}^{\infty}\frac{d_m}{\alpha+\nu_m}. \tag{3.122}$$

Putting Eqs. (3.69) and (3.83) in Eq. (3.46), we obtain

$$-\frac{F_1(\alpha)M_1^+(\alpha)}{\eta_1 N^+(\alpha)\sqrt{k+\alpha}}+\frac{F_2(\alpha)\sqrt{k-\alpha}M_1^+(\alpha)}{ik\eta_1 N^+(\alpha)M_1(\alpha)}$$

$$=-\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}\sqrt{k-\beta_m}N(\beta_m)M_1^+(\beta_m)R^+(\beta_m)}{M_1^{/}(\beta_m)N^+(\beta_m)(\alpha-\beta_m)}+\sqrt{\frac{k}{2}}\frac{1}{\pi i}\frac{N^+(k)}{M_1^+(k)(\alpha+k)}+\sum_{m=1}^{\infty}\frac{a_m}{\alpha-\beta_m}$$

$$+e^{i\alpha l}\frac{\sqrt{k-\alpha}N^-(\alpha)M_1^+(\alpha)}{\sqrt{k+\alpha}N^+(\alpha)M_1^-(\alpha)}$$

$$\left[\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}\sqrt{k-\beta_m}N(-\beta_m)M_1^+(\beta_m)P_*^-(-\beta_m)}{M_1^{/}(-\beta_m)N^+(\beta_m)(\alpha+\beta_m)}+\sum_{m=1}^{\infty}\frac{b_m}{\alpha+\beta_m}\right]. \tag{3.123}$$

Putting Eqs. (3.69) and (3.83) in Eq. (3.47), we obtain

$$-\frac{e^{-i\alpha l}F_1(\alpha)M_1^-(\alpha)}{\eta_1 N^-(\alpha)\sqrt{k-\alpha}}+\frac{e^{-i\alpha l}F_2(\alpha)\sqrt{k+\alpha}M_1^-(\alpha)}{ik\eta_1 N^-(\alpha)M_1(\alpha)}=$$

$$\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}\sqrt{k-\beta_m}N(-\beta_m)M_1^+(\beta_m)P_*^-(-\beta_m)}{M_1^{/}(-\beta_m)N^+(\beta_m)(\alpha+\beta_m)}+\sum_{m=1}^{\infty}\frac{b_m}{\alpha+\beta_m}-e^{-i\alpha l}\frac{\sqrt{k+\alpha}N^+(\alpha)M_1^-(\alpha)}{\sqrt{k-\alpha}N^-(\alpha)M_1^+(\alpha)}$$
$$\left[\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}\sqrt{k-\beta_m}N(\beta_m)M_1^+(\beta_m)R^+(\beta_m)}{M_1^{/}(\beta_m)N^+(\beta_m)(\alpha-\beta_m)}-\sqrt{\frac{k}{2}}\frac{1}{\pi i}\frac{N^+(k)}{M_1^+(k)(\alpha+k)}-\sum_{m=1}^{\infty}\frac{a_m}{\alpha-\beta_m}\right]. \tag{3.124}$$

Putting Eqs. (3.102) and (3.122) in Eq. (3.48), we obtain

$$-\frac{F_2(\alpha)M_2^+(\alpha)}{\eta_2 N^+(\alpha)\sqrt{k+\alpha}}+\frac{F_1(\alpha)\sqrt{k-\alpha}M_2^+(\alpha)}{ik\eta_2 N^+(\alpha)M_2(\alpha)}=-\sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}N(\nu_m)M_2^+(\nu_m)S^+(\nu_m)}{M_2^{/}(\nu_m)N^+(\nu_m)(\alpha-\nu_m)}$$
$$+\sqrt{\frac{k}{2}}\frac{1}{\pi i}\frac{N^+(k)}{M_2^+(k)(\alpha+k)}+\sum_{m=1}^{\infty}\frac{c_m}{\alpha-\nu_m}+e^{i\alpha l}\frac{\sqrt{k-\alpha}N^-(\alpha)M_2^+(\alpha)}{\sqrt{k+\alpha}N^+(\alpha)M_2^-(\alpha)}$$
$$\left[\sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}N(-\nu_m)M_2^+(\nu_m)Q_*^-(-\nu_m)}{M_2^{/}(-\nu_m)N^+(\nu_m)(\alpha+\nu_m)}+\sum_{m=1}^{\infty}\frac{d_m}{\alpha+\nu_m}\right]. \tag{3.125}$$

Putting Eqs. (3.102) and (3.122) in Eq. (3.49), we obtain

$$-e^{-i\alpha l}\frac{F_2(\alpha)M_2^-(\alpha)}{\eta_2 N^-(\alpha)\sqrt{k-\alpha}}+e^{-i\alpha l}\frac{F1(\alpha)\sqrt{k+\alpha}M_2^-(\alpha)}{ik\eta_2 N^-(\alpha)M_2(\alpha)}=\sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}N(-\nu_m)M_2^+(\nu_m)Q_*^-(-\nu_m)}{M_2^{/}(-\nu_m)N^+(\nu_m)(\alpha+\nu_m)}$$
$$+\sum_{m=1}^{\infty}\frac{d_m}{\alpha+\nu_m}-e^{-i\alpha l}\frac{\sqrt{(k+\alpha)}N^+(\alpha)M_2^-(\alpha)}{\sqrt{(k-\alpha)}N^-(\alpha)M_2^+(\alpha)}$$
$$\left[\sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}N(\nu_m)M_2^+(\nu_m)S^+(\nu_m)}{M_2^{/}(\nu_m)N^+(\nu_m)(\alpha-\nu_m)}-\sqrt{\frac{k}{2}}\frac{1}{\pi i}\frac{N^+(k)}{M_2^+(k)(\alpha+k)}-\sum_{m=1}^{\infty}\frac{c_m}{\alpha-\nu_m}\right]. \tag{3.126}$$

3.4 Scattered Field

3.4.1 Reflected Field

Putting value of $A(\alpha)$ and $B(\alpha)$ from Eqs. (3.24) and (3.30) in Eq. (3.23), we obtain

$$u(x,y) = \int_L \left\{ F_1(\alpha)\cos K(\alpha)y + \frac{F_2(\alpha) - F_1(\alpha)\cos[K(\alpha)b]}{\sin[K(\alpha)b]} \sin K(\alpha)y \right\} e^{-i\alpha x} d\alpha, \tag{3.127}$$

$$\Longrightarrow u(x,y) = \int_L \left\{ \frac{F_1(\alpha)\cos[K(\alpha)y]\sin[K(\alpha)b] + \{F_2(\alpha) - F_1(\alpha)\cos[K(\alpha)b]\}\sin K(\alpha)y}{\sin[K(\alpha)b]} \right\} e^{-i\alpha x} d\alpha. \tag{3.128}$$

Put

$$\sin[K(\alpha)b] = 0,$$

$$\begin{aligned} \alpha &= \pm\sqrt{k^2 - \frac{n^2\pi^2}{b^2}} \quad where\ n = 0,1,2,\ldots, \\ \alpha &= \pm q. \end{aligned} \tag{3.129}$$

The reflection coefficient R of the fundamental mode is defined as the complex coefficient multiplying the travelling wave term $exp(ikx)$ and is computed from the contribution of the first pole at $\alpha = q$

$$R = \frac{(-1)^{-n-1} n\pi^2 i}{b^2 q}[F_2(q) - (-1)^n F_1(q)] \sin\left[\frac{n\pi}{b}y\right], \tag{3.130}$$

where

$$F_1(q) = \frac{\eta_1 k^2 M_1(q) M_2(q) N^+(q)\sqrt{k+q}}{M_1^+(q)\{k^2 M_1(q)M_2(q) - q^2 + k^2\}} \left[\begin{array}{c} \sum_{m=1}^{\infty} \frac{e^{i\beta_m l}\sqrt{k-\beta_m} N(\beta_m) M_1^+(\beta_m) R^+(\beta_m)}{M_1'(\beta_m) N^+(\beta_m)(q-\beta_m)} \\ -\sqrt{\frac{k}{2}}\frac{1}{\pi i}\frac{N^+(k)}{M_1^+(k)(q+k)} - \sum_{m=1}^{\infty}\frac{a_m}{q-\beta_m} \end{array} \right]$$

$$
-\frac{\eta_1 k^2 M_1(q) M_2(q) N^-(q) e^{iql}\sqrt{k-q}}{M_1^-(q)\{k^2 M_1(q) M_2(q) - q^2 + k^2\}} \left[\sum_{m=1}^{\infty} \frac{e^{i\beta_m l}\sqrt{k-\beta_m} N(-\beta_m) M_1^+(\beta_m) P_*^-(-\beta_m)}{M_1'(-\beta_m) N^+(\beta_m)(q+\beta_m)} + \sum_{m=1}^{\infty} \frac{b_m}{q+\beta_m} \right]
$$

$$
+\frac{\eta_2 k M_2(q) N^+(q)(k+q)\sqrt{k-q}}{iM_2^+(q)\{k^2 M_1(q) M_2(q) - q^2 + k^2\}} \left[\sum_{m=1}^{\infty} \frac{e^{i\nu_m l}\sqrt{k-\nu_m} N(\nu_m) M_2^+(\nu_m) S^+(\nu_m)}{M_2'(\nu_m) N^+(\nu_m)(q-\nu_m)} - \sqrt{\frac{k}{2}}\frac{1}{\pi i}\frac{N^+(k)}{M_2^+(k)(q+k)} - \sum_{m=1}^{\infty} \frac{c_m}{q-\nu_m} \right]
$$

$$
-\frac{\eta_2 k M_2(q) N^-(q)(k-q) e^{iql}\sqrt{k+q}}{iM_2^-(q)\{k^2 M_1(q) M_2(q) - q^2 + k^2\}} \left[\sum_{m=1}^{\infty} \frac{e^{i\nu_m l}\sqrt{k-\nu_m} N(-\nu_m) M_2^+(\nu_m) Q_*^-(-\nu_m)}{M_2'(-\nu_m) N^+(\nu_m)(q+\nu_m)} + \sum_{m=1}^{\infty} \frac{d_m}{q+\nu_m} \right] \qquad (3.131)
$$

and

$$
F_2(q) = \frac{\eta_1 k M_1(q) N^+(q)(k+q)\sqrt{k-q}}{iM_1^+(q)\{k^2 M_1(q) M_2(q) - q^2 + k^2\}} \left[\sum_{m=1}^{\infty} \frac{e^{i\beta_m l}\sqrt{k-\beta_m} N(\beta_m) M_1^+(\beta_m) R^+(\beta_m)}{M_1'(\beta_m) N^+(\beta_m)(q-\beta_m)} - \sqrt{\frac{k}{2}}\frac{1}{\pi i}\frac{N^+(k)}{M_1^+(k)(q+k)} - \sum_{m=1}^{\infty} \frac{a_m}{q-\beta_m} \right]
$$

$$
-\frac{\eta_1 k M_1(q) N^-(q) e^{iql}(k-q)\sqrt{k+q}}{iM_1^-(q)\{k^2 M_1(q) M_2(q) - q^2 + k^2\}} \left[\sum_{m=1}^{\infty} \frac{e^{i\beta_m l}\sqrt{k-\beta_m} N(-\beta_m) M_1^+(\beta_m) P_*^-(-\beta_m)}{M_1'(-\beta_m) N^+(\beta_m)(q+\beta_m)} + \sum_{m=1}^{\infty} \frac{b_m}{q+\beta_m} \right]
$$

$$
+\frac{\eta_2 k^2 M_1(q) M_2(q) N^+(q)\sqrt{k+q}}{M_2^+(q)\{k^2 M_1(q) M_2(q) - q^2 + k^2\}} \left[\sum_{m=1}^{\infty} \frac{e^{i\nu_m l}\sqrt{k-\nu_m} N(\nu_m) M_2^+(\nu_m) S^+(\nu_m)}{M_2'(\nu_m) N^+(\nu_m)(q-\nu_m)} - \sqrt{\frac{k}{2}}\frac{1}{\pi i}\frac{N^+(k)}{M_2^+(k)(q+k)} - \sum_{m=1}^{\infty} \frac{c_m}{q-\nu_m} \right]
$$

$$
-\frac{\eta_2 k^2 M_1(q) M_2(q) N^-(q) e^{iql}\sqrt{k-q}}{M_2^-(q)\{k^2 M_1(q) M_2(q) - q^2 + k^2\}} \left[\sum_{m=1}^{\infty} \frac{e^{i\nu_m l}\sqrt{k-\nu_m} N(-\nu_m) M_2^+(\nu_m) Q_*^-(-\nu_m)}{M_2'(-\nu_m) N^+(\nu_m)(q+\nu_m)} + \sum_{m=1}^{\infty} \frac{d_m}{q+\nu_m} \right] \qquad (3.132)
$$

3.4.2 Transmitted Field

The transmission coefficient T of the fundamental mode which is defined as to be complex coefficient of $exp(ikx)$ and is computed from the contribution of the pole at $\alpha = -q$ in Eq. (3.128) the result is

$$R = \frac{(-1)^{-n-1} n\pi^2 i}{b^2 q}[F_2(-q) - (-1)^n F_1(-q)] \sin\left[\frac{n\pi}{b}y\right], \tag{3.133}$$

where

$$\begin{aligned}
F_1(-q) \quad = \quad & \frac{\eta_2 k M_2(-q) N^+(q) e^{-iql}(k+q)\sqrt{k-q}}{iM_2^+(q)\{k^2 M_1(-q) M_2(-q) - q^2 + k^2\}} \left[\begin{array}{c} -\sum_{m=1}^{\infty} \frac{e^{i\nu_m l}\sqrt{k-\nu_m} N(-\nu_m) M_2^+(\nu_m) Q_*^-(-\nu_m)}{M_2^{/}(-\nu_m) N^+(\nu_m)(-q+\nu_m)} \\ -\sum_{m=1}^{\infty} \frac{d_m}{(-q+\nu_m)} \end{array} \right] \\
& + \frac{\eta_2 k M_2(-q) N^-(q)(k-q)\sqrt{k+q}}{iM_2^-(q)\{k^2 M_1(-q) M_2(-q) - q^2 + k^2\}} \left[\begin{array}{c} \sum_{m=1}^{\infty} \frac{e^{i\nu_m l}\sqrt{k-\nu_m} N(\nu_m) M_2^+(\nu_m) S^+(\nu_m)}{M_2^{/}(\nu_m) N^+(\nu_m)(-q-\nu_m)} \\ -\sqrt{\frac{k}{2}}\frac{1}{\pi i}\frac{N^+(k)}{M_2^+(k)(-q+k)} - \sum_{m=1}^{\infty} \frac{c_m}{(-q-\nu_m)} \end{array} \right] \\
& \frac{\eta_1 k^2 M_1(-q) M_2(-q) N^-(q) e^{-iql}\sqrt{k+q}}{M_1^+(q)\{k^2 M_1(q) M_2(q) - q^2 + k^2\}} \left[\begin{array}{c} -\sum_{m=1}^{\infty} \frac{e^{i\beta_m l}\sqrt{k-\beta_m} N(-\beta_m) M_1^+(\beta_m) P_*^-(-\beta_m)}{M_1^{/}(-\beta_m) N^+(\beta_m)(-q+\beta_m)} \\ -\sum_{m=1}^{\infty} \frac{b_m}{(-q+\beta_m)} \end{array} \right] \\
& \frac{\eta_1 k^2 M_1(-q) M_2(-q) N^+(q)\sqrt{k-q}}{M_1^-(q)\{k^2 M_1(-q) M_2(-q) - q^2 + k^2\}} \left[\begin{array}{c} \sum_{m=1}^{\infty} \frac{e^{i\beta_m l}\sqrt{k-\beta_m} N(\beta_m) M_1^+(\beta_m) R^+(\beta_m)}{M_1^{/}(\beta_m) N^+(\beta_m)(-q-\beta_m)} \\ -\sqrt{\frac{k}{2}}\frac{1}{\pi i}\frac{N^+(k)}{M_1^+(k)(-q+k)} - \sum_{m=1}^{\infty} \frac{a_m}{-q-\beta_m} \end{array} \right]
\end{aligned} \tag{3.134}$$

and

$$F_2(-q) = \frac{\eta_2 k^2 M_1(-q) M_2(-q) N^+(q) e^{-iql}\sqrt{k+q}}{M_2^+(q)\{k^2 M_1(-q) M_2(-q) - q^2 + k^2\}} \left[\begin{array}{c} -\sum_{m=1}^{\infty} \frac{e^{i\nu_m l}\sqrt{k-\nu_m} N(-\nu_m) M_2^+(\nu_m) Q_*^-(-\nu_m)}{M_2^{/}(-\nu_m) N^+(\nu_m)(-q+\nu_m)} \\ -\sum_{m=1}^{\infty} \frac{d_m}{(-q+\nu_m)} \end{array} \right]$$

$$
+\frac{\eta_2 k^2 M_1(-q)M_2(-q)N^-(q)\sqrt{k-q}}{M_2^-(q)\{k^2 M_1(-q)M_2(-q)-q^2+k^2\}}\left[\begin{array}{c}\sum_{m=1}^{\infty}\frac{e^{i\nu_m l}\sqrt{k-\nu_m}N(\nu_m)M_2^+(\nu_m)S^+(\nu_m)}{M_2'(\nu_m)N^+(\nu_m)(-q-\nu_m)}\\ -\sqrt{\frac{k}{2}}\frac{1}{\pi i}\frac{N^+(k)}{M_2^+(k)(-q+k)}-\sum_{m=1}^{\infty}\frac{c_m}{(-q-\nu_m)}\end{array}\right]
$$

$$
+\frac{\eta_1 k M_1(-q)N^+(q)e^{-iql}(k+q)\sqrt{k-q}}{iM_1^+(q)\{k^2 M_1(-q)M_2(-q)-q^2+k^2\}}\left[\begin{array}{c}-\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}\sqrt{k-\beta_m}N(-\beta_m)M_1^+(\beta_m)P_*^-(-\beta_m)}{M_1'(-\beta_m)N^+(\beta_m)(-q+\beta_m)}\\ -\sum_{m=1}^{\infty}\frac{b_m}{(-q+\beta_m)}\end{array}\right]
$$

$$
+\frac{\eta_1 k M_1(-q)N^-(q)(k-q)\sqrt{k+q}}{iM_1^-(q)\{k^2 M_1(-q)M_2(-q)-q^2+k^2\}}\left[\begin{array}{c}\sum_{m=1}^{\infty}\frac{e^{i\beta_m l}\sqrt{k-\beta_m}N(\beta_m)M_1^+(\beta_m)R^+(\beta_m)}{M_1'(\beta_m)N^+(\beta_m)(-q-\beta_m)}\\ -\sqrt{\frac{k}{2}}\frac{1}{\pi i}\frac{N^+(k)}{M_1^+(k)(-q+k)}-\sum_{m=1}^{\infty}\frac{a_m}{(-q-\beta_m)}\end{array}\right] \quad (3.135)
$$

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