Diffraction of sound in a moving fluid

Amjad Naeem

Department of Mathematics
Quaid-i-Azam University, Islamabad
PAKISTAN
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A Dissertation
Submitted in Partial Fulfillment of the
Requirements for the Degree of

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IN

MATHEMATICS

We accept this thesis as conforming to the required standard.

1. ______________________
Prof. Dr. Muhammad Ayub
(Chairman)

2. ______________________
Prof. Dr. Muhammad Ayub
(Supervisor)

3. ______________________
Prof. Dr. Muhammad Ozair Ahmad
(External Examiner)

4. ______________________
Dr. Muhammad Salahuddin
(External Examiner)

Department of Mathematics
Quaid-i-Azam University, Islamabad
Pakistan

2012

Dedicated to….

My Wife

And

My Daughters

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Nomenclature

$f(x)$	function of x
$\widetilde{F}(\alpha)$	Fourier transform of $f(x)$
α	Fourier transform parameter
σ, τ	real and imaginary parts of α
ω	oscillating frequency
t	time
c	speed of sound
$\widehat{p}$	acoustic pressure
M	A constant
ξ	complex variable
$K(\xi)$	kernel function
a	constant
$A(\alpha)$, $B(\alpha)$, $C(\alpha)$	functions of complex var. α
λ	± 1
π	constant
ψ_+	regular in upper half or in the domain q to ∞
ψ_1	regular in the domain p to q
ψ_-	regular in lower half or in the domain $-\infty$ to p
$L_\pm(\alpha)$, $D_\pm(\alpha)$	splitt functions
$J(\alpha)$, $P(\alpha)$	polynomials in α
τ'_-, τ'_+, A	limits of integration

$A_i(\alpha)$ $(i = 1-4)$	constants
K_m	substitution
$F_1(\alpha)$	
$F_r(b\eta^{1/2})$	Fresnel function
$A_i(\alpha, w)$ $(i = 1-4)$	functions of complex var. α, w
B	absorbing parameter
$\widetilde{a}$	constant
x, y, z	Cartesian coordinates
ϕ_t, ψ_t	total velocity potential
ϕ_i, ψ_i	incident wave
ϕ_r, ψ_r	reflected wave
ϕ, ψ	diffracted wave
D_1, D_2	constants
k	wave number
θ_0	angle of incidence
θ	angle of observer
$\gamma(\alpha)$	branch cut
$K(\alpha)$	branch cut
$H_\pm(\alpha)$	splitt functions
$C_i\,(i = 1-6)$	constants
$\chi_0 e^{i\vartheta}$	polar form of χ $(= \chi_0 e^{i\vartheta})$
β	complex specific admittance
u, v	real and imaginary parts of $h(z)$

$I,\ I_i\ (i = 1-6)$	integrals
$\widehat{J}$	max. value of $j(z)$
U	max. value of u on the path A to B
s	large parameter
Γ	gamma function
$\widehat{\mu}$	a scalar
$z_0 = x_0 + iy_o$	saddle point
$r_0,\ \theta_0$	polar coordinates
$l,\ m,\ n$	scalars
$g,\ f$	polynomials
$\mathcal{F},\ \widetilde{\mathcal{F}}$	Fresnel functions in line and point source solutions
$F(\alpha)$	a function in strip/slit problems
$a_1,\ a_2$	scalars
δ	Dirac delta function
ε	a small positive quantity
W	Wronskian
$(x_o,\ y_0)$	position of line source
n	unit normal in the positive direction
sgn	Signum function
$T_\pm(\alpha),\ S_\pm(\alpha)$	splitt functions
k_0	free space wave number
η	specific impedance of the surface
$\mathcal{L}$	linear operator
$H.$	Heaviside function
$H_0^{(1)}$	Hankel function of order zero, kind one

p and q	edges of strip/slit
$G(\alpha)$, $\bar{G}(\alpha)$	known functions of α in strip/slit geometries
$W_{m,n}$	Whittaker function
$R_i\,(i = 1-6)$	Functions of functions (functionals) in strip/slit geometries
$P_i\,(i = 1-6)$	functionals in strip/slit geometries
$G_i\,(i = 1-6)$	functionals in strip/slit geometries
C_1, C_2	known constants in strip/slit geometry
$k\rho$	observer distance from the point of observation (usually origin)
Z_α	acoustic impedance
κ	keppa

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Abstract

The purpose of this thesis is to discuss some problems of diffraction of sound waves by half plane, slit and strip, satisfying absorbing boundary conditions. The mathematical route to these problems consists of Wiener- Hopf technique, integral transforms, modified method of steepest descent etc. The diffraction of a cylindrical and spherical acoustic wave from a slit in a moving fluid using Myres' conditions [133] is investigated in chapters three (3) and four (4), respectively. The solutions of these problems provided the corrective terms which were not present in the previous work on this topic. The mathematical results obtained were well supported by the graphical discussion showing how the absorbing parameter and Mach number affect the amplitude of the velocity potential.

Point source consideration is important because it is regarded as a fundamental radiating device and the solution of the point source problem is called the fundamental solution of the given differential equation. The introduction of point source changes the incident field and the method of solution requires a careful analysis in calculating the diffracted field. The point sources are regarded as better substitutes for real sources than line sources or plane waves. The mathematical significance of the problem of point source is that it will introduce another variable. The difficulty, that arises in the solution of the integral occurs in the inverse transform. These integrals are normally difficult to handle because of the presence of the branch points and are only amenable to solution using asymptotic approximations.

Transient nature of the field is an important area in the theory of acoustic diffraction and provides a more complete picture of the wave phenomenon. In chapter five, the problem of diffraction due to an impulse line source by an absorbing half plane, satisfying Myers' impedance condition in the presence of a subsonic flow has been discussed. The problem of acoustic diffraction by an absorbing half plane in a moving fluid using Myers' condition was discussed by Ahmad [46]. He considered the diffraction of sound waves by a semi-infinite absorbing half plane, when the whole system was in a moving fluid. In [46], the time dependence was considered to be harmonic in nature and was suppressed throughout the analysis. While, in this chapter, the time dependence has been taken into account throughout. The temporal Fourier transform has been applied to obtain the transform function in the transformed plane using the Wiener-Hopf technique [13] and the method of modified stationary phase [12]. The time-dependence of field is introduced by a delta function with temporal and spatial Fourier transform. In line with the solution for diffracted field, asymptotic solutions are sought for spatial integrals in far-field approximation. It has been shown that how the frequency of incident wave is effected by the amplitude of the diffracted field in different limiting positions. Also, the effects of different parameters on the field can be seen through the graphs.

In chapter six, the diffraction of waves due to an impulsive line source by an absorbing half plane in a moving fluid using Myers' impedance condition in the presence of a subsonic fluid flow is studied and the effect of the Kutta-Joukowski condition has been examined by introducing the wake (trailing edge)

attached to the half plane. The time dependence of the field requires a temporal Fourier transform in addition to the spatial Fourier transform. Expressions for the total far field for the trailing edge (wake present) situation are given.

To the best of auther's knowledge, no attempt has been made to calculate the diffracted field at an intermediate range, from an absorbing half plane with a wake attached to it using Myres' condition. In chapter seven, the diffraction of waves, in the intermediate zone, due to a line source by an absorbing half plane with a wake attached to it, in a moving fluid using Myres' condition is analyzed and the effects of the Kutta-Joukowski condition has been examined by introducing the wake (trailing edge) attached to the half plane. The solution for the leading edge situation can be obtained if the wake, and consequently a Kutta-Joukowski edge condition, is ignored.

In chapter eight, the diffraction of an acoustic wave from a finite absorbing barrier at an intermediate range by using Myers' impedance boundary conditions have been investigated. In the calculations of the integrals [46] the terms of $1/(KR_0)^{(1/2)}$are retained. If we consider intermediate range approximation in terms of source position we need to retain the next terms of $1/(KR_0)^{(3/2)}$ in the expansion of the Hankel function. It is observed that the solution so obtained are much better than the solution obtained earlier.

Contents

Chapter 1

INTRODUCTION

Over the years, scientific research and development has genuinely been focusing on the study of 'acoustic waves', discovering their numerous applications. Today, these waves are commonly used to define fluids and solids in terms of mechanical parameters like compactness/stiffness and softness. Magnitude of such functions also expands over a variety of disciplines of which seismic study procedures/borehole survey, quantitative non-destructive assessment of mechanical structures and investigative/therapeutic usage in medical science may be cited as common examples.

The experimental observation enhanced the developments in acoustics. In this field, the problems are generally not agreeable to the direct theoretical treatment and their solutions are very complex. Moreover, the acoustical phenomena are not less than three dimensions and they are transient in nature too. The medium is neither at rest nor isotropic where the scattering of acoustic waves is taking place. A complex analytic treatment is required to solve the problem as the boundary circumstances to which wave equation is subjected are not necessarily on a regular shaped geometry. "The Theory of Sound' written by Rayleigh, was based on the mathematical investigation of the subject of acoustics in which he has considered the assumption that the scatters are small as compared to the wavelength. Morse treatises of 'Vibrations and Sound' are considered as another classical work in the same field in which he gave the solution of scattering by rigid, circular cylinders and spheres, not necessarily small compared to the wavelength [1].

Scattering is a physical process in which some forms of radiation / moving particles are made to diverge from a straight path by localized non-uniformities in the medium through which they pass. It also includes deviation of reflected radiation from the angle given by the law of reflection. Mathematical analysis of scattering of light by faceted objects was the center of attention for so many medieval scientists and the study was pioneered in 10th century AD by Ibn-al-Haitam who calculated the asymptotic field for diffraction of a wedge and came up with the theory of wave propagation referred as Poincare [2]. Then came Sommerfeld [3], who deliberated on exact solution of diffraction from plate using physical method of images and mathematical theory of Riemann surfaces [4]. The technique of constructing the mandatory valued solution of the wave equation was simplified by Sommerfeld in a succeeding paper [5]. Lamb [6] used parabolic coordinates and the results achieved by him were the same as incurred by Sommerfeld. Similarly, Macdonald [7] presented exact solution to diffraction by soft/ hard wedge in acoustic plane wave incidence of dual-dimension by summing the Fourier series representation of the Green's function. Next, Carslaw [8] introduced periodic Green's function to resolve diffraction problem. Schwinger [9, 10] used application of integral equation method for diffraction problems. Later, he also conducted research on the problems involving diffraction of plane wave by semi-infinite parallel cylinders or plates which led to Wiener-Hopf type of integral equations capable of exact solutions. Magnus [11] showed that the problem of diffraction of sound waves of small amplitude can be reduced to the solution of a singular integral equation. Copson [12] analyzed the diffraction case from plane screen by integral equation, solved it using Wiener-Hopf technique [13] and proved that his solution was in line with half plane problem of Sommerfeld [3].

Around 1931, W-H technique [13] was introduced to solve certain integral equations, of the type

$$\int_0^\infty f(\xi)\, K(x-\xi)\, d\xi = g(x) \qquad (0 < x < \infty). \tag{1.1}$$

where K and g are given and f is to be found. This technique is footed on the application of integral transforms and the theory of analytic continuation of complex valued functions. It was Copson [12] who applied this technique for the first time to solve the problem of diffraction of sound waves by a perfectly reflecting half plane, using the integral equation formulation. The procedure was practiced by N. Wiener and H. Hopf (1931) for solution of integral equations with a difference kernel for semi-infinite interval. It is also called general method of solving functional equations or factorization method and has been quite helpful for diffraction solutions in elasticity theory of boundary value problems for heat conduction equation etc. The procedure of W-H technique requires that the associated mathematical boundary value problem is to be transformed to the W-H functional equation which has two unknown functions of a complex variable. These unknown functions are normally Fourier transforms of the solution of a partial differential equation or an integral equation. These are analytic in two overlapping half planes and are considered as an entire function whose asymptotic behaviour helps to determine the Fourier transforms and finally the inverse transformation gives the solutions of the original equation.(1.1)

The solution of W.H. functional equation requires decomposition of kernel into product of functions - a known function of complex variable with numerous isolated singularities that characterize the underlying physical processes. The product consists of one analytic and of almost algebraic growth in the upper/right half of the complex plane, whereas the other is analytic and of almost algebraic growth in the lower/left half plane. In scalar case, this may be achieved using Cauchy's integral theorem. Relevant material along-with an extension of these techniques is found in Rawlins and Williams [14], Williams [15], Danielle [16] and Rawlins [17]. The factorization of scalar kernel is also suggested by

Carrier [18]. For numerical processing, Bates and Mittra [19] have suggested an integral representation for the factorization of a class of scalar functions and it has been successfully used for the radiation problems involving waveguide structures. The matched asymptotic expansions (MAE) method as proposed by Crighton [20] can also be used for the asymptotic factorization of W-H kernels.

The Wiener-Hopf method is useful for considering diffraction by half-plane e.g., Noble [13], Mittra & Lee [21]. This was easily extended to diffraction of two parallel half-planes like Abrahams [22] and Lee [23]. It makes a powerful tool to tackle 2 or 3 dimensional diffraction problems. This technique also comes up with many applications to extra-acoustics issues and electromagnetism as seen in Wickham [24]. Kuiken [25], Shaw [26], Davis [27] problems with axial symmetry using some interesting modifications to classical method, Koeg [28].

Acceptable procedure on Wiener-Hopf technique based problems, in wave propagation theory, can be found in selected literature like Copson [12], Noble [13], de Hoop [29], Jones [30–32], Mittra and Lee [21], Achenbach [33], Speck [34] and Abraham [22, 35].

The basic characteristics of the Wiener-Hopf technique are as follows:-

(a) The uniform asymptotic solution of the diffracted field doesn't impose any restriction on incident and observation angles as in the case of GTD, Kobayashi [37].

(b) The W-H technique does not have a restriction on the boundaries as compared to the arithmetical techniques which are valid only for boundaries of finite length.

Numerous authors have undertaken research into scattering of sound and electromagnetic waves by half plane. First, Sommerfeld [3] obtained valid solution of plane diffraction waves from half plane using image waves, though the answers get unbounded at incident or reflected shadow boundaries. Various investigations in the past were carried out for the study of classical problems in diffraction theory, e.g., electromagnetic waves diffraction due to a line source by a perfectly conducting half plane. Line source diffraction of acoustic waves by a hard half plane with a wake attached to it, was investigated by Jones [38] and acoustic waves diffraction by an absorbing half plane was studied by Rawlins [39]. Boersma and Lee [40] studied the electric line source diffraction by a half plane. Hongo and Nakajima [41] well thought out the diffraction problem of an anisotropic cylindrical wave by a cylindrical obstacle. Buyukaksoy and Uzgoren [42] studied the magnetic line source diffraction with different impedances. Rawlins et al [43] considered the line source diffraction by an electromagnetic dielectric semi-infinite plane. Asghar et al has discussed line source and point source diffraction by three half planes in a moving fluid [44]; line source diffraction by a rectangular cylinder on an infinite impedance plane has been examined by Tayyar and Buyukaksoy [45]; Acoustic diffraction due to a line source by an absorbing semi-infinite plane with Myres' conditions has been contributed by Ahmad [46]. In recent times, Ayub et al [47–49] investigated the line source diffraction phenomenon by a junction, reactive step and an impedance step and more recently Ahmed and Naqvi [50] studied the response of a coated circular cylinder subjected to electromagnetic radiation produced by a line source. Diffraction of line source and a point source field by an ideal half plane was obtained by MacDonald [51] and Carslaw [52] whereas solutions in terms of Fresnel functions bounded at shadow boundaries were presented by Clemmow [53] and Senior [54]. Using uniform geometrical diffraction, further advancement on half plane solution with ideal boundaries was rendered by Pathak and Kouyoumjian [55]. Williams [56] also referred to diffraction of waves by half planes with no ideal boundary conditions in reference to surfaces with identical point reacting impedance in infinite product. Then, Rawlins [57] offered closed form solution for diffraction of plane wave by rigid-soft half plane. Considering the importance of line source, the third chapter of the thesis comprises discussion on the "Line source diffraction by a slit in a moving fluid" and the contents of this chapter have been published in **Canadian Journal of Physics Vol. 87 (11), 1139-1149, 2009. Also**

Discovery of new physical applications enabled the study of scattering by half plane surfaces with more complicated boundary conditions. To cite an example, application as absorbent liners in aero engine exhausts may be attributed to impedance surfaces using Wiener-Hopf technique. Further to his earlier work on diffraction from an absorbing barrier Rawlins [39], Rawlins [58] calculated the diffraction from its edge with a model of an acoustically penetrable, but absorbing half plane barrier. He used boundary conditions with two parameters mixing the pressure and its normal derivative on sides. The boundary condition discontinuities are set by two parameters, chosen to give approximately the same reflection and transmission coefficients as those found for plane wave incident to a thin layer with scalar wave equation as governing formula.

Plane wave theory is taken as fairly accurate to predict wave scattering phenomena when sound source is far away from scatterer. However, a limitation is observed when a large source-to-scatterer separation remains wanting mainly due to obscurity regarding sufficiency of separation length. Wave front of incident wave gets spherical at large distances from sound source having finite aperture. With extension of scatterer over a large size, change in frequency and spatial characteristics may occur as sphericity of wave front causes significant bending stress. The source is more appropriately modeled as point source or array of point sources while presenting experimental situations. Significant difference in scattered field as compared to that of a plane incident would be observed if the point source is brought closer to the scatterer. Reasons of the difference in results may be attributed to the fact that geometrically, surface area of scatterer by spherical wave front is smaller than that of plane wave which reduces scattered field of specular reflection. Keeping in view the significance of point source, the same has been discussed in chapter four titled "Point source diffraction by a slit in moving fluid".

Due to their practical applications in science, engineering and communication systems, strips and slits are typical examples among a number of simple obstructions. The scattering and diffraction problems related to these geometries have been explained by many research scholars. The phenomenon of multiple diffraction often occurs from both the strip and slit geometries Kobayashi [37]. The problem of diffraction by a strip with parallel edges was first discussed by Fox [59] and he also studied the diffraction of pulses by a slit and a grating. Morse and Rubenstein [60] used the technique of separating the variables to study acoustic waves diffraction of ribbons and slits. Much of the work done on half planes, strips, slits and cylinders etc. has been summarized and reviewed by Bowman et al. [61]. John [62] studied the wave scattering by strip geometry using symmetry-like principles. The diffraction of electromagnetic waves by strips satisfying various types of boundary conditions has been studied by Senior [63], Tiberio and Kouyoumajian [64] and Tiberio et al. [65] which was based on the method called the geometrical theory of diffraction (GTD) and it was introduced by Keller [66].

The method of successive approximations was used by Bowman [67], Chakrabarti [68] and Wickham [24] to solve the problems related to strip configuration . Scientists, like Jones [32], Kobayashi [37], Noble [13], Faulkner [69], Chakrabarti [70], Asghar [71], Hayat and Asghar [72,73] and Ayub et al [74–79] have successfully implied the W-H technique [13] for scattering by strips. Serbest and Buyukaksoy has expounded a unique contribution regarding the studies of diffraction by strips using W-H technique in conjunction with ray optical method and spectral iteration technique (SIT) and this technique has also been applied by Serbest et al. [81], Buyukaksoy and Uzgoren [82], Erdogan et al [83], Buyukaksoy and Alkumru [84,85] and Cinar and Buyukaksoy [86]. Cinar and Buyukaksoy [87] have studied the electromagnetic plane wave diffraction by considering an important configuration comprising a slit in an impedance plane and a parallel complementary strip.

In both theoretical and engineering considerations, scattering from a slit is a well studied problem in diffraction theory. Several authors like Asestas and Kleinman [88] have treated diffraction from slit in finite thickness screen. Jones [32] and Nobel [13] discussed diffraction from a slit using Wiener-Hopf model. Still, the most extensive investigation is that of Lehman [89], who used analytical properties of finite Fourier transform. Numerical treatment of coupled integral equations was used by Hongo [90], Nero and Mur [91]. Similarly, Auckland and Harrington [92] worked on the problem in terms of a pair of coupled integral equations to secure transmission characteristics of slit in conducting screen of finite thickness placed between two different media.

A vital part of noise theory is related to interaction of acoustic sources with wave supporting structures. Spatial distributions of acoustic quadruples develop acoustic similarity which produces turbulance in the sound regions as seen in Light hill [93, 94] (Lighthill's famous M8 law) predicting that amplitude of the sound produced due to compact turbulent eddy is proportionate to eddy Mach number M's 8th power. Curle [95] extended this theory to consider the effect of solid boundaries. Here, Light hill's free space quadruples are supplemented by surface distribution of dipoles (leaky resonant system) with strength proportional to local fluid pressure in surface distribution of monopoles. Far more efficient radiators of sound in contrast with acoustic quadruples are marked by dipole and monopole sources. Therefore, it was previously thought that acoustic power radiated by a nearby turbulent flow would substantially increase due to presence of large homogeneous surfaces. It was proved by Powell [96] and Ffowcs-William [97, 98] that large plane homogeneous surfaces can release, at the most, quadruple, acoustic energy radiated from a region of turbulence. They also showed that genuine dipole sources may be enhanced in presence of ribs and struts in an otherwise homogeneous surface. Later, it was demonstrated by Ffowcs-William and Hall [99] that the radiated intensity is increased over its free field value by a large factor $[M]^{-3}$ if compact turbulent eddy is situated within acoustic wavelength of sharp edge of rigid half-plane.

An inquiry into surface inhomogeneities and semi-infinite geometries was also facilitated by Crighton [100], Crighton and Leppington [101, 102] and Leppington [103].

Studying compact turbulent eddy close to elastic half-plane surface, Crighton [104] indicated presence of a powerful edge-scattered wave with increase of $|M|^{-5/2}$ on a radiated intensity with free value. His analysis skips fluid -plate motions whereas the influence of fluid-loading is significant in underwater acoustics and energy from incident wave may be transferred to plate waves as well as the edge-scattered wave in an outgoing phenomenon. Scattering of such a surface-wave was analyzed by Cannell [105,106], who used thin elastic plate's edge for fluid-loading to examine diffraction of an incident wave. His observations for heavy fluid-loading were adopted by Abraham [107] to study plane wave diffraction by a large fixed plate. Crighton and Innes [108] further explained the motion of plane elastic structures in heavy fluid-loading and its applications in edge constraints and various geometries. Later, it was seen that this study of Ffowcs-Williams was based on assumption of a potential flow near edge with infinite velocity on the point. The finiteness of velocity can be described in two possible ways.

a. Application of linear formulae with source terms, as adopted by Navier-Stokes, and analysis of Alblas [109] model would reveals that in absence of main flow, small viscosity removes singularity in velocity at edge without affecting the far-field pressure.

b. As in aerofoil theory, we may use the equation of sound waves with brief amplitude on edge in Kutta-Joukowski condition.

Fluid adheres to bounding surface in viscous fluid. The effect of viscosity is confined to a small boundary layer having strong vorticity, at small distance from surface, where velocity reaches the free stream value. Vortex layer is thin in case of thin wings. Vortices wake behind the wing and Kutta-Joukowski condition may be applied to determine strength of wake. Jones [30] introduced wake to see the kutta-Joukowski' effect having low Mach number. His contribution also includes calculation of a field scattered from line source both in still air and moving fluid. He concluded that in moving medium beyond diffracting plane, Kutta-Joukowski condition does not exert appreciable influence on scattered field.

This condition yields a stronger field close to wake. In this way, intense sound can easily be transmitted away from source through wake. It was Balasubramanyam [110] who extended the case to the point source excitation whereas Rienstra achieved diffraction of cylindrical impulse. Considering the importance of wake condition, the sixth chapter of the thesis comprises discussion on the "Diffraction of an impulsive line source with wake" and the contents of this chapter have been published in **Physica Scripta 82 (4), 045402, 2010.**

Noise pollution is a serious threat to environmental protection and scientists like Butler [111], Kurze [112], Jones [113], Rawlins [134], Asghar [71] remained associated with its reduction. The efforts led to application of diffraction theory through barriers particularly to control highway noise in urban areas. Barriers with absorbing linings in satisfying absorbing boundary condition can be an ideal method of noise reduction. A mathematical model on these lines was first devised by Rawlins [134], who secured the mixed boundary condition through diffraction of acoustic waves from half-plane.

It is important to deliberate acoustic diffraction for moving fluids while handling the issue of noise produced by motor engines and other heavy machinery. Pioneering work of Rawlins [114, 134] may be applied here for acoustic diffraction by half-plane in moving fluids, while assuming flow as subsonic by definition. It concludes that sound intensity in shadow region of absorbing screen is attenuated in fluid flow. This study works out far field for trailing edge under wake [114] conditions while calculating the leading edge without it. Asghar [71] extended this research further to a finite plate and calculated the diffracted field due to both edges of plate. He came up with finite absorbing barrier which has great significance in aerodynamics and aero-acoustics disciplines. To solve the issues of underwater acoustics and of supersonic aircraft noise, a lot of work has been done on acoustic transients and transient structure-fluid interaction, for example, between "sonic boom" of supersonic aircraft and windowpane or between an underwater shock wave and a submersible.

Transient nature of the field, furnishing a better picture of wave processes, makes a significant trait of acoustic diffraction theory. Wave occurrence is time dependent and it varies harmoniously in time-period. In this context, time harmonic wave propagation is often applied, but inharmonic time variation also exists in some significant fields.

As a common rule, time harmonic wave propagation is of paramount importance; however, the basic wave propagation incidence from source to receiver through an ambient environment can better be comprehended through direct signal tracking in transient state. Here, time harmonic field comes as a special constant frequency in case of continuously emitted excitation. Therefore, transient wave phenomena stimulated by various applications requiring explicit treatment of time-dependent effects made a point of interest with Friedlander [59], Harris [115, 116] Reinstra [117], de Hoop [118], Jones [32] and Asghar [119, 120]. Considering the importance of transient nature of the field, the fifth chapter of the thesis comprises discussion on the "Sound due to an impulsive line source" whose contents have been published in **Computer and Mathematics with Application, VOL. 60 (12), 3123-3129, (2010).**.

A unique effort has been made in chapter seven and eight by considering the diffraction for the intermediate range. In chapter seven the diffraction by a half plane with wake for an intermediate range has been examined while in chapter eight sound waves diffraction by a strip for an intermediate zone has been discussed in detail. As far as available record, no such attempt has been made to solve the above mentioned problem, particularly for an intermediate range.

Noise reduction by means of barriers is achieved through diffraction theory. Much attention is being paid to design and performance of noise barriers, particularly for reduction of traffic noise e.g., Butler [111] and Kurze [112]. Equally important is application of noise protecting by barriers Jones [38]. Noise level can also be effectively reduced in shadow region of barrier by lining the barrier with absorbent substance. The absorbing linning of the surface is depicted by the relation between acoustic pressure of the wave and normal velocity changes (Morse and Ingard [121]). The 'Conclusions' also include that dependence of field on absorption parameter tantamount that intensity of sound can be quite reduced if absorption parameter is carefully selected.

Chapter 2

PRELIMINARIES

The term acoustics is associated with the production, transmission, reception etc. of sound. A vibrational disturbance is called sound if its frequency lies in the range of 20 to 20,000 Hertz. However, the ultrasonic frequencies above 20,000 Hertz and infrasonic frequencies below 20 Hertz are also included in acoustics. Actually, sound is a disturbance of mechanical energy which propagates through matter as a longitudinal wave. The frequency, wavelength, amplitude, period and speed are the characteristics of sound.

The phenomenon of winding of waves about small hurdles or the dispersion of waves past small openings is called diffraction. A sound wave is scattered not only by a solid object but also by a region in which the acoustic properties differ, in their values, from the rest of the medium. When an object scatters sound, some of the energy carried by the incident wave is dispersed. This energy loss is either absorbed by the scatterer or deflected from its original course and results in the reduction of intensity of plane wave.

Some of the definitions and the mathematical preliminaries are presented here.

2.1 Fourier Transform [122]

Fourier transforms are very useful in solving partial differential equations. The Fourier transform technique can be used in solving the integral equations or boundary value problems when the domain of the problem is infinite or semi-infinite.

The Fourier transformation of $f(x)$ is given by

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x} dx, \tag{2.1}$$

provided that the above integral exists and $f(x)$ is bounded in the given domain for all x . Here, $\alpha = \sigma + i\tau$ is a Fourier transform variable and is complex in general. The Fourier transform of $f(x)$ is represented by $F(\alpha)$. The properties of the function $F(\alpha)$. are discussed by writing the Fourier transform integral as a sum of the two integrals and each one of them is defined in the semi-infinite range i.e.,

$$f(x) = f_+(x) + f_-(x), \tag{2.2}$$

where

$$f_+(x) = \begin{cases} 0 & x < 0 \\ f(x) & x > 0 \end{cases}, \tag{2.3}$$

or

$$F_+(\alpha) = \frac{1}{2\pi} \int_{0}^{\infty} f_+(x)\, e^{i\alpha x} dx. \tag{2.4}$$

If the exponential order of the function $f_+(x)$ is as

$$|f_+(x)| < M\ e^{\tau - x} \quad as \quad x \longrightarrow \infty, \tag{2.5}$$

then the function $F_+(\alpha)$ is an analytic function of the complex variable $\alpha = \sigma + i\tau$, and is regular in the upper half $\alpha-$plane, i.e., $\tau > \tau_-$. In this domain $F_+(\alpha) \longrightarrow 0$ as $|\alpha| \longrightarrow \infty$. Now by taking the inverse Fourier transform

$$f_+(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_+(\alpha) e^{-i\alpha x} d\alpha, \tag{2.6}$$

and the integration will be over any straight line $\tau > \tau_-$ which is parallel to the real axis in the complex $\alpha - plane$. The strips of analyticity will be found by taking the domain of analyticity of $F_+(\alpha)$ as the real axis when $f_+(x)$ decreases at infinity, i.e., $\tau_- < 0$, and the Eq. (2.6) would be integrated along the real axis and when $f_+(x)$ increases at infinity, i.e., $\tau_- > 0$, but not faster than the exponential function with the linear exponent, the domain of analyticity of $F_+(\alpha)$ lies above the real axis of the complex $\alpha - plane$ and the Eq. (2.6) would be integrated above the real axis.

Similarly for the function

$$f_-(x) = \begin{cases} f(x) & x < 0 \\ 0 & x > 0 \end{cases}, \tag{2.7}$$

or

$$F_-(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} f_-(x) e^{i\alpha x} dx. \tag{2.8}$$

If the exponential order of the function $f_-(x)$ is as

$$|f_-(x)| < M\ e^{\tau_+ x} \quad \text{as} \quad x \longrightarrow \infty, \tag{2.9}$$

then the function $F_-(\alpha)$ is an analytic function of the complex variable $\alpha = \sigma + i\tau$ and is regular in the lower half $\alpha-$plane, i.e., $\tau < \tau_+$. In this domain $F_-(\alpha) \longrightarrow 0$ as $|\alpha| \longrightarrow \infty$.

We take the inverse Fourier transform

$$f_-(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_-(\alpha) e^{-i\alpha x} d\alpha, \tag{2.10}$$

and the domain of analyticity of $F_-(\alpha)$ contains the real axis for $\tau_+ > 0$, and it does not contain the real axis for $\tau_+ < 0$. So, the Eq. (2.10) is analytic in the domain $\tau_- < \tau < \tau_+$. The inverse Fourier transform relates the functions $f(x)$ and $F(\alpha)$ as under

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\tau}^{\infty+i\tau} F(\alpha) e^{-i\alpha x} d\alpha, \tag{2.11}$$

where the integration is performed on the complex $\alpha-plane$ lying in the strip $\tau_- < \tau < \tau_+$. The function $F(\alpha)$ is analytic for $\tau_+ > 0$ and $\tau_- < 0$ in the strip containing the real axis of the complex $\alpha - plane$.

2.2 Decomposition Theorem [13]

Let $f(\alpha)$ be an analytic function of $\alpha = \sigma + i\tau$ regular in the strip $\tau_- < \tau < \tau_+$. If $|f(\alpha)| \to 0$ uniformly, within this strip as $|\sigma| \to \infty$, then $f(\alpha)$ can be decomposed as

$$f(\alpha) = f_+(\alpha) + f_-(\alpha), \tag{2.12}$$

where

$$f_+(\alpha) = \frac{1}{2\pi i} \int_{-\infty+ic}^{\infty+ic} \frac{f(\upsilon)}{\upsilon - \alpha} d\upsilon \qquad \tau_- < c < \tau < \tau_+, \tag{2.13}$$

is non zero and analytic in the α-plane defined by $\tau_- < \tau$ and

$$f_-(\alpha) = \frac{-1}{2\pi i} \int_{-\infty+id}^{\infty+id} \frac{f(\upsilon)}{\upsilon - \alpha} d\alpha \qquad \tau_- < \tau < d < \tau_+, \tag{2.14}$$

is non zero and analytic in the α-plane $\tau < \tau_+$.

2.3 Factorization Theorem [13]

Let $f(\alpha)$ be an analytic function of $\alpha = \sigma + i\tau$, regular and non zero in the strip $\tau_- < \tau < \tau_+$. If $|f(\alpha)| \to 1$ uniformly, within this strip, as $|\sigma| \to \infty$, then $f(\alpha)$ can be factorized as

$$f(\alpha) = f_+(\alpha) f_-(\alpha), \tag{2.15}$$

where

$$f_+(\alpha) = \exp\left[\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} \frac{\ln f(\upsilon)}{\upsilon - \alpha} d\upsilon\right] \qquad \tau_- < c < \tau < \tau_+, \tag{2.16}$$

is regular and non zero in the upper half α-plane defined by $\tau_- < \tau$ and

$$f_-(\alpha) = \exp\left[\frac{-1}{2\pi i}\int_{-\infty+id}^{\infty+id} \frac{\ln f(\upsilon)}{\upsilon - \alpha} d\upsilon\right] \qquad \tau_- < \tau < d < \tau_+, \tag{2.17}$$

is regular and non zero in the lower half α-plane defined by $\tau < \tau_+$.

2.4 Kutta-Joukowski Condition

The flow is assumed to be trailing off the edge at the fluid- fluid interface and to cater this situation a condition is imposed which is called the Kutta-Joukowski condition. Imposition of this condition requires the discontinuity of the velocity potential and continuity of its normal derivative. The Kutta- Joukowski condition allows an aerodynamicist to incorporate a significant effect of viscosity while neglecting viscous effects in the conservation of momentum equation. It is vital in practical calculation for the lift of a wing. The equation of conservation of mass and conservation of momentum applied to inviscid fluid flow, such as potential flow around a solid body, results in an infinite number of valid solutions. One of the ways to choose the correct solution is to apply the viscous equation, in the form of Navier -Stokes equation. However, the results may not provide the correct solutions in the closed form. The boundary condition for the flow around a body, in case of a non-viscous flow, is that the surface normal component vanishes. But for viscous fluid, the actual boundary condition is that the fluid adheres to the bounding surface and this requires that both velocity components (normal and tangential) subject to the body should disappear. For a small space off the edge, the velocity becomes equal to free stream value and the viscosity is confined to a very small boundary layer along the surface. But, as the vortex layer is thin for thin wings, so it make a wake behind the wing whose strength is approximated by what is known as the Kutta-Joukowsky condition.

2.5 The Wiener-Hopf Technique [13, 32]

The Wiener-Hopf technique is a powerful tool for solving the problems involving diffraction by semi-infinite / finite planes. The technique is footed on the implication of integral transforms and the analytic continuation of complex valued function. Carleman invented this technique in 1922. Later, N. Wiener and E. Hopf (1931) applied this technique to solve the differential equation governing a problem of radiation equilibrium of stars [123]. They also found a tool to solve such type of problems and published a paper [124] in that regard. Originally, W-H technique [13] was used to solve the integral equation of the type

$$\int_{0}^{\infty} f(\xi) K(x-\xi) d\xi = g(x) \qquad (0 < x < \infty), \tag{2.18}$$

where K and g are given and f is to be found. D. S. Jones made a modification in 1952 by applying this technique to boundary value problems [125]. The Jones' method was based on the direct application of integral transformations to the partial differential equation and the associated boundary conditions. This method overlooked the initial derivation of the integral equation and resulted into formation of a complex W-H functional equation directly. Since then, it is used for solving certain integral equations and various boundary value problems in Mathematical Physics, Elasticity and Heat conduction by means of integral, Fourier, Laplace and other transforms.

Partial differential equations are very useful to find the solution of certain geometrical shapes such as circle and infinite strips, with the help of method of separation of variables. However, difficulty comes across in finding solutions for shapes not covered by the method of separation of variables. The Wiener Hopf technique provides a significant extension of the range of problems that can be solved by Fourier, Laplace and Mellin transforms.

General Scheme Of Wiener Hopf Technique

As an example, we apply Fourier transforms to partial differential equations which results in two unknown functions $\psi_+(\alpha)$ and $\psi_-(\alpha)$ of a complex variable α,. These unknown functions are analytic in the regions $\operatorname{Im}\alpha > \tau_-$ and $\operatorname{Im}\alpha < \tau_+$ $(\tau_- < \tau_+)$ respectively. The unknown functions approaches zero as $|\alpha| \to \infty$ in their respective domains of analyticity. They satisfy the functional equation

$$A(\alpha)\psi_+(\alpha) + B(\alpha)\psi_-(\alpha) + C(\alpha) = 0, \tag{2.19}$$

in the strip $\tau_- < \operatorname{Im}\alpha < \tau_+$, $A(\alpha)$, $B(\alpha)$ and $C(\alpha)$ are given complex valued functions, analytic in the strip, $\tau_- < \operatorname{Im}\alpha < \tau_+$ and, moreover, $A(\alpha)$ and $B(\alpha)$ are non zero in the strip [13] .

The core step in the Wiener-Hopf method is to find $L_+(\alpha)$ and $L_-(\alpha)$ in above equations, i.e.,

$$A(\alpha)/B(\alpha) = L_+(\alpha)/L_-(\alpha), \tag{2.20}$$

where the function $L_+(\alpha)$ and $L_-(\alpha)$ are analytic and different from zero, for half planes

$\operatorname{Im}\alpha > \tau'_-$ and $\operatorname{Im}\alpha < \tau'_1$ and strips $\tau_- < \operatorname{Im}\alpha < \tau_1$ and $\tau'_- < \operatorname{Im}\alpha < \tau'_+$ respectively, and have a common portion. We use Eq. (2.20) in Eq. (2.19) to write

$$L_+(\alpha)\psi_+(\alpha) + L_-(\alpha)\psi_-(\alpha) + L_-(\alpha)\frac{C(\alpha)}{B(\alpha)} = 0. \tag{2.21}$$

We write the last term in Eq. (2.21) as

$$L_-(\alpha)\frac{C(\alpha)}{B(\alpha)} = D_+(\alpha) + D_-(\alpha), \tag{2.22}$$

where the functions $D_+(\alpha)$ and $D_-(\alpha)$ are analytic in the half planes $\operatorname{Im}\alpha > \tau''_-$ and $\operatorname{Im}\alpha < \tau''_+$ respectively, and all the three strips, $\tau_- < \operatorname{Im}\alpha < \tau_+$, $\tau'_- < \operatorname{Im}\alpha < \tau'_+$ and $\tau''_- < \operatorname{Im}\alpha < \tau''_+$ have a common portion in the strip $\tau''_- < \operatorname{Im}\alpha < \tau''_+$. The following equation holds true in the given strip

$$L_+(\alpha)\psi_+(\alpha) + D_+(\alpha) = -L_-(\alpha)\psi_-(\alpha) - D_-(\alpha). \tag{2.23}$$

The left hand side of Eq. (2.23) is a function analytic in $\tau^0_- < \operatorname{Im}\alpha$, and the right hand side is a function which is analytic in the domain $\operatorname{Im}\alpha < \tau^0_+$. The equality of these functions in the strip $\tau^0_- < \operatorname{Im}\alpha < \tau^0_+$, with analytic continuation shows that there exists a unique entire function $P(\alpha)$, coinciding with left and right side of Eq. (2.23) respectively, in the domains of their analyticity. We suppose that

$$\left|L_+(\alpha)\psi_+(\alpha) + D_+(\alpha)\right| < |\alpha|^p \qquad \text{as } \alpha \longrightarrow \infty, \quad \operatorname{Im}\alpha > \tau_-, \tag{2.24}$$

$$\left|L_-(\alpha)\psi_-(\alpha) + D_-(\alpha)\right| < |\alpha|^q \qquad \text{as } \alpha \longrightarrow \infty, \quad \operatorname{Im}\alpha < \tau_+. \tag{2.25}$$

Then by the extended Liouville's theorem [13] , If $P(\alpha)$ is an integral function such as $|P(\alpha)| < M\,|\alpha|^p$ as $\alpha \longrightarrow \infty$ where M and p are constants, then $P(\alpha)$ is a polynomial of degree less than or equal to $\lfloor p \rfloor$ where $\lfloor p \rfloor$ is the integral part of p. So the equations

$$\psi_+(\alpha) = \frac{P_n(\alpha) - D_+(\alpha)}{L_+(\alpha)}, \tag{2.26}$$

$$\psi_-(\alpha) = \frac{-P_n(\alpha) - D_-(\alpha)}{L_-(\alpha)}, \tag{2.27}$$

define the desired functions with constants and the supplementary conditions of the problem help us to determined these unknowns. So, we can say that the representations of the Eqs.(2.26) and (2.27) form the basis to use the Wiener-Hopf method.

2.6 Jones' Method [13, 126]

Jones' method, as established by D.S. Jones in 1952, is more appropriate than the integral equation formulation method for solving the boundary value problems using W-H technique. The integral equation formulation method provides to apply the boundary conditions before the Fourier transform is taken. But , in Jones' method, the Fourier transform is applied directly to the wave equation and all the boundary conditions are applied in the transformed domain. As an example, let us take a half plane problem in which steady state waves exist in two dimensional (x, y) space and the harmonic time dependence is $e^{-i\omega t}$. Suppose the boundary along the negative real axis is rigid and the plane waves

$$\phi_i(x, y) = e^{-ikx\cos\theta - iky\sin\theta} \qquad (0 < \theta < \pi), \tag{2.28}$$

are incident on the barrier. Let the total velocity potential be $\phi_t(x, y)$ as

$$\phi_t(x, y) = \phi_i(x, y) + \phi(x, y), \tag{2.29}$$

where $\phi(x, y)$ is the diffracted potential and satisfies the Helmholtz's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + k^2\phi = 0, \tag{2.30}$$

and we assume k to have positive imaginary part [13]. The boundary conditions are

$$\frac{\partial\phi_t}{\partial y} = 0 \quad \text{on } y = 0, \qquad -\infty < x \leq 0, \tag{2.31}$$

or

$$\frac{\partial\phi}{\partial y} = ik\sin\theta e^{-ikx\cos\theta}, \ y = 0, \qquad -\infty < x \leq 0, \tag{2.32}$$

also

$$\frac{\partial\phi_t}{\partial y} \text{ and } \frac{\partial\phi}{\partial y} \text{ are continuous on } y = 0, \qquad -\infty < x < \infty, \tag{2.33}$$

and

$$\phi_t \text{ and } \phi \text{ are continuous on } y = 0, \qquad 0 < x < \infty. \tag{2.34}$$

We define the Fourier transform of a certain function $\phi(x, y)$ as:

$$\begin{aligned} \Psi(\alpha, y) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(x, y)\, e^{i\alpha x}dx, \\ \text{and also } \Psi(\alpha, y) &= \Psi_+(\alpha, y) + \Psi_-(\alpha, y) \end{aligned} \tag{2.35}$$

where

$$\Psi_+(\alpha, y) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\phi(x, y)\, e^{i\alpha x}dx, \tag{2.36}$$

and

$$\Psi_-(\alpha, y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\phi(x, y)\, e^{i\alpha x}dx. \tag{2.37}$$

Further

$$|\Psi| < D_1 e^{-k_2 x} \qquad \text{as} \qquad x \longrightarrow \infty, \tag{2.38}$$

$$|\Psi| < D_1 e^{k_2 x \cos\theta} \qquad \text{as } x \longrightarrow -\infty, \tag{2.39}$$

so that Ψ_+ is analytic for $\operatorname{Im}\alpha > -k_2\cos\theta$ and Ψ_- is analytic for $\operatorname{Im}\alpha < k_2\cos\theta$. Applying Fourier transform on Eq. (2.30),we obtain

$$\frac{d^2\Psi(\alpha, y)}{dy^2} - \gamma^2\Psi(\alpha, y) = 0, \tag{2.40}$$

where

$$\gamma = \left(\alpha^2 - k^2\right)^{1/2}. \tag{2.41}$$

The solution of Eq (2.40) is given as

$$\Psi(\alpha, y) = \begin{cases} A_1(\alpha)e^{-\gamma y} + B_1(\alpha)e^{\gamma y}, & y \geq 0, \\ A_2(\alpha)e^{-\gamma y} + B_2(\alpha)e^{\gamma y}, & y \leq 0. \end{cases} \tag{2.42}$$

The real part of γ is always positive in the strip $-k_2 < \operatorname{Im}\alpha < k_2\cos\theta$ and so $A_2 = B_1 = 0$ in Eq.(2.42) and using Eq. (2.33) we write

$$\frac{d\Psi(\alpha, 0^+)}{dy} = -\gamma A_1(\alpha), \tag{2.43}$$

$$\frac{d\Psi(\alpha, 0^-)}{dy} = \gamma B_2(\alpha). \tag{2.44}$$

and also, we write $-A_1 = B_2 = A$, in Eq. (2.42) and get

$$\Psi(\alpha, y) = \begin{cases} A(\alpha)e^{-\gamma y}, & y \geq 0, \\ -A(\alpha)e^{\gamma y}, & y \leq 0. \end{cases} \tag{2.45}$$

When a transform is discontinuous across $y = 0$, we use the following notation

$$\Psi_-(\alpha, 0^\pm) = \Psi_-(0^\pm) = \lim_{y \longrightarrow \pm 0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \phi e^{i\alpha x} dx, \tag{2.46}$$

and the use of Eq. (2.34) results in

$$\Psi_+(0^+) = \Psi_+(0^-) = \Psi_+(0), \tag{2.47}$$

and using Eq. (2.33), we get

$$\Psi_+^{'}(0^+) = \Psi_+^{'}(0^-) = \Psi_+^{'}(0). \tag{2.48}$$

Similarly

$$\Psi_-^{'}(0^+) = \Psi_-^{'}(0^-) = \Psi_-^{'}(0). \tag{2.49}$$

Using Eqs. (2.47) to (2.49) in Eq. (2.45), we obtain

$$\Psi_+(\alpha, 0) + \Psi_-\left(\alpha, 0^+\right) = A(\alpha), \tag{2.50}$$

$$\Psi_+(\alpha, 0) + \Psi_-\left(\alpha, 0^-\right) = -A(\alpha), \tag{2.51}$$

$$\Psi_+^{\prime}(\alpha, 0) + \Psi_-^{\prime}(\alpha, 0) = -\gamma A(\alpha). \tag{2.52}$$

Actually, we are concerned with the equations which contain the functions whose regions of regularity are known.

By adding Eqs. (2.50) and (2.51), we obtain

$$2\Psi_+(\alpha, 0) = -\Psi_-\left(\alpha, 0^+\right) - \Psi_-\left(\alpha, 0^-\right). \tag{2.53}$$

By subtracting Eqs. (2.50) and (2.51), we get

$$A(\alpha) = \frac{1}{2}\left[\Psi_{-}\left(\alpha, 0^{+}\right) - \Psi_{-}\left(\alpha, 0^{-}\right)\right], \tag{2.54}$$

$$\Psi'_{+}(\alpha, 0) + \Psi'_{-}(\alpha, 0) = -\frac{1}{2}\gamma\left[\Psi_{-}\left(\alpha, 0^{+}\right) - \Psi_{-}\left(\alpha, 0^{-}\right)\right]. \tag{2.55}$$

In above equation, $\Psi'_{-}(\alpha, 0)$ is known and is given by

$$\Psi'_{-}(\alpha, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{i\alpha x}\left(ik \sin\theta e^{-ikx\cos\theta}\right) dx,$$

or

$$\Psi'_{-}(\alpha, 0) = \frac{k\sin\theta}{\sqrt{2\pi}\left(\alpha - k\cos\theta\right)}. \tag{2.56}$$

Let us write for convenience

$$\Psi_{-}\left(\alpha, 0^{+}\right) - \Psi_{-}\left(\alpha, 0^{-}\right) = 2\Psi^{*}(\alpha), \tag{2.57}$$

and

$$\Psi_{-}\left(\alpha, 0^{+}\right) + \Psi_{-}\left(\alpha, 0^{-}\right) = 2\Phi^{*}(\alpha). \tag{2.58}$$

From Eqs. (2.53) and (2.58), we obtain

$$\Psi_{+}(\alpha, 0) = -\Phi^{*}(\alpha). \tag{2.59}$$

Using Eqs. (2.56) and (2.57) in Eq. (2.58), we get

$$\Psi'_{+}(\alpha, 0) + \frac{k\sin\theta}{\sqrt{2\pi}\left(\alpha - k\cos\theta\right)} = -\gamma\Psi^{*}(\alpha). \tag{2.60}$$

The unknown functions in Eqs. (2.59) and (2.60) are $\Psi_+(0)$, $\Psi'_+(0)$, $\Psi^*(\alpha)$ and $\Phi^*(\alpha)$. Both the equations hold in the strip of analyticity $-k_2 < \operatorname{Im}\alpha < k_2\cos\theta$ and are in standard W-H form. We substitute $\gamma = \sqrt{\alpha+k}\sqrt{\alpha-k}$ in Eq. (2.60) and arrange it as

$$\frac{\Psi'_+(0)}{\sqrt{\alpha+k}} + \frac{k\sin\theta}{\sqrt{2\pi}\sqrt{\alpha+k}\,(\alpha-k\cos\theta)} = -\sqrt{\alpha-k}\Phi^*(\alpha), \tag{2.61}$$

where $\sqrt{\alpha+k}$ is non-zero and analytic in $\operatorname{Im}\alpha > -k_2$ and $\sqrt{\alpha-k}$ is non-zero and analytic in $\operatorname{Im}\alpha < k_2\cos\theta$. In above equation,we observe that the first term on the left hand side is regular in $\operatorname{Im}\alpha > -k_2$ and the second term is regular in the strip $-k_2 < \operatorname{Im}\alpha < k_2\cos\theta$, whereas its right side is regular in $\operatorname{Im}\alpha < k_2\cos\theta$. The middle term is splitted as

$$\begin{aligned}\frac{k\sin\theta}{\sqrt{2\pi}\sqrt{\alpha+k}\,(\alpha-k\cos\theta)} &= \frac{k\sin\theta}{\sqrt{2\pi}\,(\alpha-k\cos\theta)}\left[\frac{1}{\sqrt{\alpha+k}} - \frac{1}{\sqrt{k+k\cos\theta}}\right] \\ &+\frac{k\sin\theta}{\sqrt{2\pi}\,(\alpha-k\cos\theta)\sqrt{k+k\cos\theta}},\end{aligned} \tag{2.62}$$

or

$$\frac{k\sin\theta}{\sqrt{2\pi}\sqrt{\alpha+k}\,(\alpha-k\cos\theta)} = P_+(\alpha) + P_-(\alpha), \tag{2.63}$$

where

$$P_+(\alpha) = \frac{k\sin\theta}{\sqrt{2\pi}\,(\alpha-k\cos\theta)}\left\{\frac{1}{\sqrt{\alpha+k}} - \frac{1}{\sqrt{k+k\cos\theta}}\right\}, \tag{2.64}$$

is regular in $\tau > -k_2$, and

$$P_-(\alpha) = \frac{k\sin\theta}{\sqrt{2\pi}\,(\alpha-k\cos\theta)\sqrt{k+k\cos\theta}}. \tag{2.65}$$

is regular in $\tau < k_2\cos\theta$.

Consider a function $J(\alpha)$ which is regular in the whole $\alpha - plane$ as

$$J(\alpha) = \begin{cases} \Psi'_{+}(0)\,(\alpha + k)^{-1/2} + P_{+}(\alpha) \\ -\sqrt{\alpha - k}\,\Psi^{*}(\alpha) - P_{-}(\alpha) \end{cases} \tag{2.66}$$

$J(\alpha)$ is regular in $\operatorname{Im}\alpha > -k_2$ and in $\operatorname{Im}\alpha < k_2\cos\theta$, and hence regular in the strip $-k_2 < \operatorname{Im}\alpha < k_2\cos\theta$, provided that $J(\alpha)$ has algebraic behaviour as $\alpha \longrightarrow \infty$. The exact form of $J(\alpha)$ can be found with the help of extended form of the Liouville's theorem. Now, using the edge conditions [13, 126], the behaviour of the functions appearing in Eq. (2.66) as $\alpha \longrightarrow \infty$ can be examined as follows

$$\begin{aligned} \left|\Psi_{-}(0^{+})\right| &< C_1\left|\alpha\right|^{-1} \text{ as } \alpha \longrightarrow \infty \text{ in } \tau < k_2\cos\theta, \\ \left|\Psi'_{+}(0)\right| &< C_2\left|\alpha\right|^{-\frac{1}{2}} \text{ as } \alpha \longrightarrow \infty \text{ in } \tau > -k_2, \\ P_{-}(\alpha) &< C_3\left|\alpha\right|^{-1}, \text{ as } \alpha \longrightarrow \infty \text{ in } \tau < k_2\cos\theta, \\ P_{+}(\alpha) &< C_4\left|\alpha\right|^{-1}, \text{ as } \alpha \longrightarrow \infty \text{ in } \tau > -k_2. \end{aligned} \tag{2.67}$$

Using the above asymptotic approximations in Eq. (2.66) and we observe that

$$\begin{aligned} &J(\alpha) < C_5\left|\alpha\right|^{-\frac{1}{2}} \text{ as } \alpha \longrightarrow \infty \text{ in } \tau < k_2\cos\theta, \\ &J(\alpha) < C_6\left|\alpha\right|^{-1} \text{ as } \alpha \longrightarrow \infty \text{ in } \tau > -k_2, \end{aligned} \tag{2.68}$$

which shows that $J(\alpha)$ is regular in the whole $\alpha - plane$ and tends to zero as $\alpha \longrightarrow \infty$. Thus by the extended Liouville's theorem, $J(\alpha)$ must be identically equal to a constant . Therefore, we have

$$\Psi'_{+}(0) = -(\alpha + k)^{\frac{1}{2}}\,P_{+}(\alpha), \tag{2.69}$$

$$\Phi^{*}(0) = -(\alpha - k)^{\frac{1}{2}}\,P_{-}(\alpha). \tag{2.70}$$

We substitute $\Psi'_+(0)$ and $\Psi'_-(0)$ in Eq. (2.52), and find $A(\alpha)$ as

$$A(\alpha) = -\frac{k\sin\theta}{\sqrt{2\pi}\sqrt{\alpha-k}\,(\alpha-k\cos\theta)\,\sqrt{k+k\cos\theta}}. \tag{2.71}$$

Now, substituting the value of $A(\alpha)$ in Eq. (2.45) and then applying inverse Fourier transform, we get

$$\phi(x,y) = \mp\frac{1}{2\pi}\sqrt{k-k\cos\theta}\int_{-\infty+ia}^{\infty+ia}\frac{e^{-i\alpha x\mp\gamma y}}{\sqrt{\alpha-k}\,(\alpha-k\cos\theta)}d\alpha. \tag{2.72}$$

which can be solved by using asymptotic methods.

2.7 Green's Function Method

The Green's function is the response at x at time t due to a source located at time t_0 and only depends on time after the occurrence of the concentrated source. The Green's function method may be utilized to solve partial differential equation and we use a unit source (impulse Dirac Delta) as the driving function. An integral operator is found which produces a solution satisfying all the given boundary conditions for our non-homogeneous differential equation. The integral operator comprises a kernel which is known as the Green function and we denote it by $G(x, x_0)$. This kernel is multiplied by the non-homogeneous term and the equation is integrated by one of the variables. The Green's function is also known as the impulse response of the system or the '*transfer function*' [127].

Let $\mathcal{L}$ be a given differential operator and $g(x)$ be a given continuous function. We need to find the unknown function $y(x)$ which satisfies

$$\mathcal{L}\,[y(x)] = g(x), \tag{2.73}$$

and specified boundary conditions. If the operator $\mathcal{L}$ is one-one, then $\mathcal{L}^{-1}$ also exists and we can write

$$y(x) = \mathcal{L}^{-1}[g(x)], \tag{2.74}$$

or

$$y(x) = \int \mathcal{L}^{-1}\left[g(x_0)\delta(x - x_0)\, dx_0\right]. \tag{2.75}$$

The solution of Eq. (2.73) when

$$g(x) = \delta(x - x_0), \tag{2.76}$$

is called the Green's function $G(x, x_0)$ and it satisfies

$$\mathcal{L}\left[G(x, x_0)\right] = \delta(x - x_0), \tag{2.77}$$

for the same boundary conditions as on the function $y(x)$.

Construction of the Green's function

The Green's function is different for different equations. We construct the Green's function for the S-L boundary value problem, as every second order non-homogenous differential equation can be converted into Sturm-Liouville (S-L) form, subject to the homogeneous boundary conditions [128].

$$\left[\frac{d}{dx}\left\{p(x)\frac{dy}{dx}\right\} - q(x)y\right] + \lambda r(x)y = g(x). \tag{2.78}$$

We can write Eq. (2.78) as

$$[\mathcal{L} + \lambda r(x)] = g(x), \tag{2.79}$$

where λ is an eigen value of the corresponding S-L system. The Green's function will not exist for all those values of λ for which the above equation has trivial solution. Now, suppose the Green's function exists, then the solution of Eq. (2.78) can be written as

$$y(x) = \int_a^b g(x_0) G(x, x_0)\, dx_0. \tag{2.80}$$

For a unit impulse deriving function, Eq.(2.78) will take the form

$$\left[\frac{d}{dx}\left\{p(x)\frac{dG}{dx}\right\} - q(x)G\right] + \lambda r(x)G = \delta(x - x_0), \tag{2.81}$$

At $x \neq x_0$ Eq. (2.81) can be written as

$$\left[\frac{d}{dx}\left\{p(x)\frac{dG}{dx}\right\} - q(x)G\right] + \lambda r(x)G = 0. \tag{2.82}$$

Suppose Eq. (2.82) has the solution as

$$G(x, x_0) = \begin{cases} A_1 y_1(x) & a \leq x \leq x_0, \\ A_2 y_2(x) & x_0 \leq x \leq b. \end{cases} \tag{2.83}$$

where the constants A_1 and A_2 can be found by using the various properties of the Green's function.

The continuity of $G(x, x_0)$ at $x = x_0$ implies that

$$A_1 y_1(x_0) - A_2 y_2(x_0) = 0. \tag{2.84}$$

To establish the discontinuity of the derivative of $G(x, x_0)$ at $x = x_0$, we integrate Eq. (2.81) from $x = x_0 - \varepsilon$ to $x_0 + \varepsilon$, i.e.,

$$\lim_{\varepsilon \longrightarrow 0} \int_{x_0-\varepsilon}^{x_0+\varepsilon} \left[\frac{d}{dx}\left(p(x)\frac{dG}{dx}\right) + \{-q(x) + \varrho r(x)\}\, G\right] dx = \lim_{\varepsilon \longrightarrow 0} \int_{x_0-\varepsilon}^{x_0+\varepsilon} \delta(x - x_0)\, dx. \tag{2.85}$$

Since $q(x)$, $r(x)$ and $G(x, x_0)$ are continuous at $x = x_0$, we write Eq. (2.85) as

$$\lim_{\varepsilon \longrightarrow 0} \left[p(x) \left\{ \frac{dG(x_0 + \varepsilon, x_0)}{dx} - \frac{dG(x_0 - \varepsilon, x_0)}{dx} \right\} \right] = 1, \tag{2.86}$$

which leads to

$$\left\{ \frac{dG(x_{0+}, x_0)}{dx} - \frac{dG(x_{0-}, x_0)}{dx} \right\} = \frac{1}{p(x_0)} \tag{2.87}$$

or

$$A_2 y_2'(x_0) - A_1 y_1'(x_0) = \frac{1}{p(x_0)}. \tag{2.88}$$

We Solve Eqs. (2.84) and (2.88) to obtain

$$A_1 = \frac{y_2'(x_0)}{W(x_0)p(x_0)}, \qquad A_2 = \frac{y_1'(x_0)}{W(x_0)p(x_0)}. \tag{2.89}$$

where $W(x_0)$ is the Wronskian.

We get the Green's function by substituting the values of A_1 and A_2 into Eq. (2.86)

$$G(x, x_0) = \begin{cases} \frac{y_2'(x_0)}{W(x_0)p(x_0)} y_1(x) & a \le x \le x_0, \\ \frac{y_1'(x_0)}{W(x_0)p(x_0)} y_2(x) & x_0 \le x \le b. \end{cases} \tag{2.90}$$

The Green's function thus obtained is in the closed form and not in the form of an infinite series of orthogonal functions.

The properties of the Green's function are summarized as follows:

- $G(x, x_0)$ satisfies the homogeneous differential equation except at $x = x_0$.
- $G(x, x_0)$ is symmetric with respect to x and x_0.
- $G(x, x_0)$ satisfies the homogeneous boundary conditions.
- $G(x, x_0)$ is continuous at $x = x_0$.

- The derivative of $G(x, x_0)$ is discontinuous at $x = x_0$.

- $G(x, x_0)$ satisfies $\mathcal{L}G(x, x_0) = 0$, $x \neq x_0$.

2.8 The Method of Steepest Descent [129 – 132]

The method of Steepest Descent is used to approximate several integrals which comes across in different boundary value problems and represent the diffraction and radiation fields. The integrand in these integrals usually contains a large parameter and the integrals are approximated in terms of that parameter because they have a large value contribution in the integrals. It is very difficult to handle these integrals and sometimes it is impossible to evaluate them in closed form due to the diverging nature of the parameter. A path of integration with a special geometrical property is chosen to solve these types of integrals. Riemann was the one who originated this method and Debey later developed it further [131].

For example, let us find the asymptotic expansion of a function defined by an integral of the form

$$\int e^{\lambda w_1(z)}\phi(z)dz, \tag{2.91}$$

where λ is very large and positive parameter, and the path A to B of integration is an arc or a closed curve in the $z-$ plane. The functions $w_1(z)$ and $\phi(z)$ are independent of λ , analytic in functions of z and regular in a domain which contains the path of integration. We take λ to be real, otherwise $\lambda = \lambda_0 e^{i\beta}$ and $e^{i\beta}$ can be absorbed into $w_1(z)$. Writing $w_1(z) = u(x, y) + iv(x, y)$, where x, y, u and v are real, then

$$\left|e^{\lambda w_1(z)}\right| = e^{\lambda u}. \tag{2.92}$$

The magnitude of the integral mainly depends upon the real part of $w_1(z)$. The core idea of this method is to deform the path of integration so that the path passes through the zero of $w_1'(z)$ whereas the imaginary part of $w_1(z)$ is constant on the path.

The bound for the integral I is

$$|I| \leq L\widehat{J}e^{\lambda u}, \tag{2.93}$$

where L is the length of the contour, $\widehat{J}$ is the maximum value of $|\phi(z)|$ on the path and u is the maximum value of u on the path from A to B. The Eq. (2.96) might be an over estimate since the deformation of the path might produce a much less value of u. So, the best bound is selected by taking the path of integration in such a manner so as to take u as small as possible.

When λ is large, a small displacement in v will produce a rapid oscillation of the sinusoidal terms in $exp(i\theta)$. In general, the contribution from any other part of the path of integration will be almost the same as from this part. However, if the path of integration is chosen in such a way on which v is constant, then the rapid oscillation will disappear and the most quickly varying part of the integration will be $exp(-\lambda u\ )$. In this case, the contribution will then come from the neighbourhood of the points where u is the smallest. So, the core of the method consists of deforming the contour, as far as possible, into a curve $v = constt.$ passing through the point where u is the greatest.

The points at which $w_1'(z) = 0$ are called *saddle points. Let* $z = z_0 = x_0 + iy_0$ be a saddle point. As $w_1'(z) = 0$ so

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} = 0. \tag{2.94}$$

u and so v cannot have a maxima or minima at a saddle point (x_0, y_0) because

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0. \tag{2.95}$$

Further

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{2.96}$$

which gives

$$u_{xx} = -u_{yy} \tag{2.97}$$

i.e., both the quantities u_{xx} and u_{yy} have opposite signs. Also, we have

$$u_{xx}u_{yy} - u_{xy}^2 = -\left(u_{xx}^2 + u_{xy}^2\right) \leq 0 \tag{2.98}$$

which is a condition for a saddle point. The above relation shows that $z_0 = (x_0, y_0)$ is neither a maxima nor a minima.

Now, we show that the path along $v = constt$ is the path of the steepest descent. At a position defined by $z = z_1$, we consider a local coordinate S in a direction defined by the angle θ with the position z-axis. Then

$$\frac{\partial u}{\partial s} = u_x \cos\theta + u_y \sin\theta \tag{2.99}$$

where

$$u_x = \left|\frac{\partial u}{\partial x}\right|_{z=z_1}, \; u_y = \left|\frac{\partial u}{\partial y}\right|_{z=z_1} \tag{2.100}$$

If $\frac{\partial u}{\partial s}$ is to be the maximum of variable θ , then we must have

$$-u_x \sin\theta + u_y \cos\theta = 0 \tag{2.101}$$

Using Cauchy-Rehman equations, the Eq. (2.101) takes the form

$$-v_x \sin\theta + v_y \cos\theta = -\frac{\partial v}{\partial s} = 0 \tag{2.102}$$

Since z_1 is arbitrary, v must be constant along the path of steepest descent.

The saddle point links the *valleys* and *ridges on the surface* $u(x, y)$. The curve $v =$ *constant* will go either up a *ridge* or down a *valley* as these are the derivatives of the greatest change. These are the points of thesteepest descent for which the neighbourhood of the saddle point produces the most significant contribution. We have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \tag{2.103}$$

at a saddle point. i.e., where $w_1^{'}(z_0) = 0$. We can expand $w_1(z)$ in a Taylor's series along the path of steepest descent by assuming that $w_1^{''}(z_0) \neq 0$ i.e.,

$$w_1(z) = w_1(z_0) + \frac{1}{2}(z - z_0)^2 w_1^{''}(z_0) + ..., \tag{2.104}$$

or

$$w_1(z) - w_1(z_0) = \frac{1}{2}(z - z_0)^2 w_1^{''}(z_0). \tag{2.105}$$

In above, right hand side is purely real as $v = constant$ and we write

$$w_1(z) - w_1(z_0) = -t^2, \tag{2.106}$$

and it changes the complex variable z to a real variable t and describes the path from valley to valley along a route that descends most rapidly on the either side of the point

z_0. Now using Eq. (2.106) in Eq. (2.91), we have

$$I(\lambda) = \int_A^B \phi(z) e^{\lambda w_1(z) - t^2} dz. \tag{2.107}$$

or

$$I(\lambda) \approx e^{\lambda w_1(z)} \int_A^B \phi(z) e^{-\lambda t^2} \frac{dz}{dt} dt. \tag{2.108}$$

The exponential in the above integrand decays more rapidly as compared to the function $\phi(z)$. We write

$$I(\lambda) \approx e^{\lambda w_1(z)} \phi(z_0) \int_{-t_1}^{t_1} e^{-\lambda t^2} \frac{dz}{dt} dt, \tag{2.109}$$

with the condition that $\phi(z_0)$ is not singular in the vicinity of $z = z_0$. We write Eq. (2.106) as

$$-t^2 = \frac{1}{2}(z - z_0)^2 w_1^{"}(z_0). \tag{2.110}$$

and using the polar coordinates

$$z - z_0 = r_0 e^{i\theta_0}, \tag{2.111}$$

in Eq. (2.110) we get

$$\arg\left[\frac{1}{2} r_0^2 e^{2i\theta_0} w_1^{"}(z_0)\right] = 0. \tag{2.112}$$

This leads to

$$r_0^2 = \frac{-t^2}{\left|\frac{1}{2} w_1^{"}(z_0)\right|}, \tag{2.113}$$

or

$$r_0 = e^{\frac{i\pi}{2}} t \left|\frac{1}{2} w_1^{"}(z_0)\right|^{-\frac{1}{2}}. \tag{2.114}$$

By using Eq. (2.114) in Eq. (2.111) , we find

$$\frac{dz}{dt} = \left|\frac{1}{2} w_1^{"}(z_0)\right|^{-\frac{1}{2}} e^{\frac{i\pi}{2} + i\theta_0}. \tag{2.115}$$

Substituting Eq. (2.115) in Eq. (2.109) results in

$$I(\lambda) \approx e^{\lambda w_1(z_0)} \phi(z_0) \frac{1}{\left|\frac{1}{2} w_1^{"}(z_0)\right|^{\frac{1}{2}}} \sqrt{\frac{\pi}{\lambda}} e^{\frac{i\pi}{2} + i\theta_0}, \tag{2.116}$$

or

$$I(\lambda) \approx e^{\lambda w_1(z_0)} \phi(z_0) \sqrt{\frac{2\pi}{\lambda \left|w_1^{"}(z_0)\right|}} e^{i\left(\frac{\pi}{2} + \theta_0\right)}. \tag{2.117}$$

Chapter 3

LINE SOURCE DIFFRACTION BY A SLIT IN A MOVING FLUID

In this chapter the diffraction of a cylindrical acoustic wave from a slit in a moving fluid using Myers' condition [133] is investigated to present an improved form of the analytical solution for the diffracted field. The mathematical results are well supported by graphical discussion showing how the absorbing parameter and Mach number affect the amplitude of the velocity potential. We have drawn the numerical results and we get a corrective term in the solution of our problem which was altogether missing in [134]. The graphs give a clear picture of the variation of the velocity potential for various parameters. The graphs also verify our mathematical calculations. The asymptotic analysis of the resulting integrals is carried far enough to permit the calculations of diffracted fields far from the slit. To the best of author's knowledge, this problem of diffraction has not been discussed earlier and so it seems to be the first attempt. The method of solution consists of Fourier transform, Wiener-Hopf technique [13, 32] and the scattered field is found by taking the inverse Fourier transform and using the method of steepest descent [129–132]. Here, unequivocal expressions are obtained for the singly diffracted field, i.e., separated field, and doubly diffracted field, i.e., interaction of one edge upon the other.

A classical topic in electromagnetic and acoustic wave theory is diffraction by slits and strips. The diffraction patterns from them involves a large number of analytical, numerical or a combination of both methods, like separation of variables [60], geometrical theory of diffraction (GTD) [66, 136], Kobayashi's potential method [137, 138], spectral iteration technique (SIT) [80], method of successive approximations [24, 68], and the W-H technique [71, 139]. Bessel's potential spaces [140] and Maliuzhinetz-Sommerfeld integral representation [141] are some of the recent developments. The microwave, optical instrumentation and guiding structures containing thick slits or slots has made diffraction of plane acoustic/electromagnetic waves by a slit as an important topic [142]. Clemmow [143] studied diffraction of H-polarized plane wave by a wide slit and a normally incident E-polarized plane wave by a narrow slit by using the method of plane wave spectrum representation. Similarly, Achenbach [33] analysed diffraction of a plane horizontally polarized shear wave and a plane longitudinal wave by a semi-infinite slit using integral transforms with the W-H technique and the Cagniard de-Hoop method.

3.1 Formulation of the Problem

We consider the diffraction of an acoustic wave due to a line source from a slit occupying a space $\{p \le x \le q,\ y = 0,\ z \in (-\infty, \infty)\}$. The positions of the planes located on both sides of the slit are given by $\{-\infty < x \le p,\ y = 0,\ z \in (-\infty, \infty)\}$ and $\{q \le x < \infty,\ y = 0,\ z \in (-\infty, \infty)\}$, respectively and these are assumed to have vanishing thicknesses. The line source is located at (x_0, y_0) and the system is placed in a fluid moving with subsonic velocity U parallel to the x-axis. The time dependence is considered to be of harmonic type $e^{-i\omega t}$ (ω is the angular frequency) and is suppressed throughout the manuscript. The plane is assumed to be satisfying the Myers' condition [133]

$$u_n = \frac{-\widetilde{p}}{Z_a} + \frac{U}{i\omega Z_a}\frac{\partial \widetilde{p}}{\partial x}, \tag{3.1}$$

where u_n is the normal derivative of the perturbation velocity, $\widetilde{p}$ is the surface pressure, Z_a is the acoustic impedance of the surface and $-\mathbf{n}$ a normal pointing from the fluid into the surface. The perturbation velocity $\mathbf{u}$ of the irrotational sound wave can be written in terms of the velocity potential Φ as $\mathbf{u} = \nabla\Phi$. The resulting pressure $\widetilde{p}$ of the sound field can be written as

$$\widetilde{p} = -\rho_0 \left(-iw + U\frac{\partial}{\partial x} \right) \Phi(x, y), \tag{3.2}$$

where ρ_0 is the density of the undisturbed stream. The geometry of the problem is shown in figure 1.

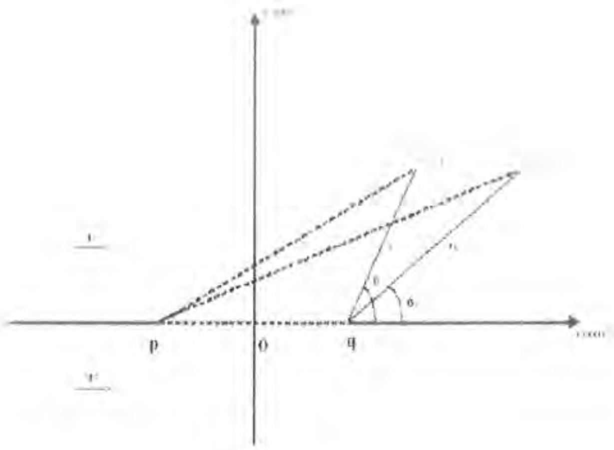


Fig. 3.1: Geometry of the diffraction problem

The wave equation satisfied by the total velocity potential Φ in the presence of the line source is given by

$$\left[(1 - M^2)\frac{\partial^2}{\partial x^2} + 2ikM\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} + k^2 \right] \Phi(x, y) = \delta(x - x_0)\delta(y - y_0), \tag{3.3}$$

and at $y = 0$, the Eqs. (3.1) and (3.2) lead to the following boundary conditions

$$\left[\frac{\partial}{\partial y} \mp 2\beta M \frac{\partial}{\partial x} \pm ik\beta \mp \frac{i\beta M^2}{k}\frac{\partial^2}{\partial x^2}\right]\Phi(x, 0^{\pm}) = 0 \quad -\infty < x < p, \; q < x < \infty. \quad (3.4)$$

In Eqs. (3.4) , the quantity 0^+ refers to the situation that $y \to 0$ through positive $y - axis$ and the quantity 0^- refers to the situation $y \to 0$ through negative $y - axis$. For analytic ease, we shall assume that the wave number $k = k_1 + ik_2$ has a small positive imaginary part to ensure the regularity of the Fourier transform integrals and that k_2 is the loss factor of the medium. The specific complex admittance is $\beta = \frac{\rho_0 c}{Z_a}$ and $M = \frac{U}{c}$ (c being velocity of sound) is the Mach number. It is assumed that the flow is subsonic, i.e., $|M| < 1$ and $\operatorname{Re}\beta > 0$, which is a necessary condition for an absorbing surface [39]. We remark that $\beta = 0$ corresponds to the rigid barrier and $\beta = \infty$ corresponds to the pressure release barrier.

Also, the potential Φ and its derivative $\frac{\partial \Phi}{\partial y}$ are continuous on the slit, i.e.,

$$\frac{\partial \Phi}{\partial y}(x, 0^+) = \frac{\partial \Phi}{\partial y}(x, 0^-) \quad p \leq x \leq q,$$

$$\Phi(x, 0^+) = \Phi(x, 0^-) \quad p \leq x \leq q. \quad (3.5)$$

Further, as $r \to \infty$, if Φ represents an out going wave at infinity; and if the time factor is considered as $e^{-i\omega t}$ then

$$(r)^{\frac{1}{2}}\left(\frac{\partial \Phi}{\partial r} - ik\Phi\right) \to 0, \quad (3.6)$$

and if the time factor is taken as $e^{i\omega t}$, then

$$(r)^{\frac{1}{2}}\left(\frac{\partial \Phi}{\partial r} + ik\Phi\right) \to 0, \quad (3.7)$$

is the radiation condition [13].

The edge conditions require that Φ_t and its normal derivative must be bounded for a unique solution and should satisfy [57, 144].

$$\Phi_t(x,0) = \begin{cases} -1 + O(x-p)^{\frac{1}{4}} & \text{as } x \to p^-, \\ -1 + O(x-q)^{\frac{1}{4}} & \text{as } x \to q^+. \end{cases} \tag{3.8}$$

$$\frac{\partial \Phi_t(x,0)}{\partial y} = \begin{cases} O(x-p)^{-\frac{3}{4}} & \text{as } x \to p^-, \\ O(x-q)^{-\frac{3}{4}} & \text{as } x \to q^+. \end{cases} \tag{3.9}$$

In the above equations, a negative sign indicates a limit taken from the left of the point p and a positive sign indicates that a limit taken from the right of the point q on the $x - axis$ [145, 146].

For the subsonic flow, we can make the following real substitutions

$$x = \sqrt{1-M^2}X,\ x_0 = \sqrt{1-M^2}X_0,\ y = Y,\ y_0 = Y_0,\ \beta = \sqrt{1-M^2}B,\ k = \sqrt{1-M^2}K, \tag{3.10}$$

and

$$\Phi(x,y) = \psi_t(X,Y)e^{-iKMX}. \tag{3.11}$$

Now, using the relations (3.10) and (3.11), the Eqs. (3.3) to (3.5) can be written as

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + k^2\right)\psi_t(X,Y) = \frac{\delta(X-X_0)\delta(Y-Y_0)}{\sqrt{1-M^2}}e^{-iKMX_0}, \tag{3.12}$$

$$\left[\frac{\partial}{\partial Y} \mp 2BM\frac{\partial}{\partial X} \pm iKB(1+M^2) \mp \frac{iBM^2}{(1-M^2)K}\frac{\partial^2}{\partial x^2}\right]\psi_t(X,0^{\pm}) = 0,\ -\infty < X < p,\ q < X < \infty$$

and

$$\begin{aligned} \frac{\partial \psi_t}{\partial Y}(X,0^+) &= \frac{\partial \psi_t}{\partial Y}(X,0^-) \qquad p \leq X \leq q, \\ \psi_t(X,0^+) &= \psi_t(X,0^-) \qquad p \leq X \leq q. \end{aligned} \tag{3.13}$$

For the analysis purpose, it is convenient to express the total field ψ_t as

$$\psi_t(X,Y) = \begin{cases} \Psi_i(X,Y) + \Psi_r(X,Y) + \Psi(X,Y), & Y \geq 0 \\ \Psi(X,Y), & Y \leq 0 \end{cases}, \tag{3.14}$$

where $\Psi_i(X,Y)$ is the incident field (corresponding to the inhomogeneous equation), $\Psi(X,Y)$ is the diffracted field (corresponding to the homogenous equation) and $\Psi_r(X,Y)$ is the reflected field from the soft surface.

Thus, $\Psi_i(X,Y)$ satisfies the equation

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + K^2\right)\Psi_i(X,Y) = \frac{\delta(X-X_0)\delta(Y-Y_0)}{\sqrt{1-M^2}}e^{-iKMX_0}, \tag{3.15}$$

and the scattered field $\Psi(X,Y)$ satisfies the Helmholtz equation

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + K^2\right)\Psi(X,Y) = 0, \tag{3.16}$$

The solution of inhomogeneous Eq. (3.15) can be obtained by using the Green's function method [127] as follows:

The spatial Fourier transform over the variable X is defined as

$$\overline{\Psi}(\alpha,Y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi(X,Y)e^{i\alpha X}dX. \tag{3.17}$$

Now, as

$$\begin{aligned} \delta(Y-Y_0) &= 0 \quad \text{when} \quad Y \neq Y_0 \\ &= \infty \quad \text{when} \quad Y = Y_0 \end{aligned} \tag{3.18}$$

such that

$$\int_{-\infty}^{\infty} \delta(Y - Y_0)\, dY = 1.$$

Thus, applying Eq. (3.17) to Eq. (3.15) will result in

$$\left(\frac{d^2}{dX^2} + K^2\right) G(Y; Y_0) = e^{i\alpha X_0} \delta(Y - Y_0), \tag{3.19}$$

with

$$G(-\infty; Y_0) = G(\infty; Y_0) \tag{3.20}$$

where $K^2 = (k^2 - \alpha^2)$ and $G(Y; Y_0)$ is the Green's function corresponding to the concentrated source located at (X_0, Y_0).

The homogenous solution of Eq. (3.19) can be written as

$$G(Y; Y_0) = A_1(\alpha) e^{-\gamma(\alpha)Y} + A_2(\alpha) e^{\gamma(\alpha)Y}, \tag{3.21}$$

where

$$\gamma = -iK(\alpha) \qquad or \qquad K(\alpha) = i\gamma \tag{3.22}$$

or

$$\begin{aligned} G(Y; Y_0) &= A_2(\alpha) e^{\gamma(\alpha)Y} \qquad Y \geq Y_0 \\ &= A_1(\alpha) e^{-\gamma(\alpha)Y} \qquad Y \leq Y_0 \end{aligned} \tag{3.23}$$

Since Green's function is continuous at $Y = Y_0$ Thus, we can write

$$A_1(\alpha) e^{-\gamma(\alpha)Y_0} = A_2(\alpha) e^{\gamma(\alpha)Y_0} = \widehat{\rho} \tag{3.24}$$

where $\widehat{\rho}$ is an unknown constant and is taken to be the same for both cases by using the property that the Green's function is continuous across the boundary $Y = Y_0$.

A suitable form of the radiated field can be expressed as

$$G(Y;Y_0) = \begin{cases} \widehat{\rho}e^{\gamma(Y-Y_0)} & Y \leq Y_0 \\ \widehat{\rho}e^{-\gamma(Y-Y_0)} & Y \geq Y_0 \end{cases}, \tag{3.25}$$

The constant $\widehat{\rho}$ can be determined by using the property of the Green's function that the derivative of Green's function is discontinuous at $Y = Y_0$ and it comes to be

$$\widehat{\rho} = \frac{e^{-iX_0(\alpha-KM)}}{iK(\alpha)\sqrt{\pi}\,(1-M^2)}, \tag{3.26}$$

also

$$\gamma(\alpha) = -iK(\alpha) \tag{3.27}$$

By using Eqs. (3.26) and (3.25), the Green's function i.e., the influence function due to a line source located at (X_0, Y_0) is determined to be

$$\overline{\Psi_i}(\alpha, Y) = \frac{1}{2iK(\alpha)}e^{i\alpha X_0 + iK(\alpha)|Y-Y_0|}, \tag{3.28}$$

where $K(\alpha) = \sqrt{k^2-\alpha^2}$ is the square root function and can be considered as that branch which reduces to $+k$ when $\alpha = 0$ and the complex $\alpha-$plane is cut either from $\alpha = k$ to $\alpha = k\infty$ or from $\alpha = -k$ to $\alpha = -k\infty$ where α is the Fourier transform variable. Now, to represent Eq. (3.15) in the Hankel's function, we put

$$\alpha = K\cos(\theta_0 + it) \tag{3.29}$$

$$X - X_0 = R\cos\theta_0$$

$$|Y - Y_0| = R\sin\theta_0 \tag{3.30}$$

and after simplification we get

$$\Psi_i(X,Y) = \frac{a}{4i} H_0^1(KR) \tag{3.31}$$

So the solution of Eq. (3.15), obtained by Green's function method, is

$$\Psi_i(X,Y) = \frac{a}{4i} H_0^1(KR) = \frac{a}{4\pi i} \int_{-\infty}^{\infty} \frac{1}{\kappa} e^{-i[\alpha(X-X_0)-\kappa|Y-Y_0|]} d\alpha, \tag{3.32}$$

where $a = \frac{e^{iKMX_0}}{\sqrt{1-M^2}}$, and $R = \sqrt{(X-X_0)^2 + (Y-Y_0)^2}$.

Now, by making the following substitutions

$$X_0 = R_0 \cos\theta_0, \quad Y_0 = R_0 \sin\theta_0, \quad 0 < \theta_0 < \pi \tag{3.33}$$

in Eq. (3.32) and letting $KR_0 \to \infty$, we can write

$$\Psi_i(X,Y) = be^{-iKX\cos\theta_0 - i\kappa Y\sin\theta_0}, \tag{3.34}$$

and

$$\Psi_r(X,Y) = be^{-iKX\cos\theta_0 + i\kappa Y\sin\theta_0}, \tag{3.35}$$

where

$$b = \frac{a}{4i}\left(\frac{2}{\pi K R_0}\right)^{\frac{1}{2}} e^{i(KR_0 - \frac{\pi}{4})}. \tag{3.36}$$

3.2 Wiener-Hopf Equations

The spatial Fourier transform over the variable X is defined as

$$\overline{\Psi}(\alpha, Y) = \int_{-\infty}^{\infty} \Psi(X,Y) e^{i\alpha X} dX, \tag{3.37}$$

In order to accommodate three part boundary conditions on $Y = 0$, we split $\overline{\Psi}(\alpha, Y)$ as

$$\overline{\Psi}(\alpha, Y) = \overline{\Psi}_+(\alpha, Y)e^{i\alpha q} + \overline{\Psi}_1(\alpha, Y) + \overline{\Psi}_-(\alpha, Y)e^{i\alpha p}, \tag{3.38}$$

where

$$\overline{\Psi}_-(\alpha, Y) = \int_{-\infty}^{p} \Psi(X, Y)e^{i\alpha(X-p)}dX, \tag{3.39}$$

$$\overline{\Psi}_1(\alpha, Y) = \int_{p}^{q} \Psi(X, Y)e^{i\alpha X}dX, \tag{3.40}$$

$$\overline{\Psi}_+(\alpha, Y) = \int_{q}^{\infty} \Psi(X, Y)e^{i\alpha(X-q)}dX. \tag{3.41}$$

Here $\overline{\Psi}_-(\alpha, Y)$ is regular for $\operatorname{Im}\alpha < \operatorname{Im}K$, and $\overline{\Psi}_+(\alpha, Y)$ is regular for $\operatorname{Im}\alpha > -\operatorname{Im}K$ while $\overline{\Psi}_1(\alpha, Y)$ is an integral function and is analytic in the common region $-\operatorname{Im}K < \alpha < \operatorname{Im}K$.

Now, taking the Fourier transform of Eq. (3.12), we obtain

$$\left(\frac{d^2}{dY^2} + \kappa^2\right)\overline{\Psi}(\alpha, Y) = 0, \tag{3.42}$$

and α plane is cut such that $\operatorname{Im}\kappa > 0$ (for bounded solution) [13].

The solution of Eq. (3.42), representing the outgoing waves at infinity, can formally be written as

$$\overline{\Psi}(\alpha, Y) = \begin{cases} A_3(\alpha)e^{i\kappa Y} & if\ Y \geq 0, \\ A_4(\alpha)e^{-i\kappa Y} & if\ Y < 0. \end{cases} \tag{3.43}$$

where $A_3(\alpha)$ and $A_4(\alpha)$ are the unknown coefficients which are to be determined. Taking the derivative with respect to "Y" of Eq.(3.43) ,we get

$$\overline{\Psi}'(\alpha,Y)=\begin{cases} i\kappa A_3(\alpha)e^{i\kappa Y} & if\ Y\geq 0, \\ -i\kappa A_4(\alpha)e^{-i\kappa Y} & if\ Y<0. \end{cases} \tag{3.44}$$

and using Eq. (3.43), we get

$$\overline{\Psi}'(\alpha,Y)=\begin{cases} i\kappa\overline{\Psi}(\alpha,Y) & if\ Y\geq 0, \\ -i\kappa\overline{\Psi}(\alpha,Y) & if\ Y<0. \end{cases} \tag{3.45a}$$

The Fourier transform of Eq. (**??**), when $X<p$, gives

$$\overline{\Psi}'_-(\alpha,0^+)=-iB\left(K\left(1+M^2\right)+2\alpha M+\frac{\alpha^2M^2}{K\left(1-M^2\right)}\right)\overline{\Psi}_-(\alpha,0^+)$$

$$-iB\left(K\left(1+M^2\right)+2\alpha M+\frac{\alpha^2M^2}{K\left(1-M^2\right)}\right)\left[\overline{\Psi}_{i-}(\alpha,0)+\overline{\Psi}_{r-}(\alpha)\right], \tag{3.46}$$

and

$$\overline{\Psi}'_-(\alpha,0^-)=iB\left(K\left(1+M^2\right)+2\alpha M+\frac{\alpha^2M^2}{K\left(1-M^2\right)}\right)\overline{\Psi}_-(\alpha,0^-), \tag{3.47}$$

and the Fourier transform of Eq. (**??**), when $X>q$, gives

$$\overline{\Psi}'_+(\alpha,0^+)=-iB\left(K\left(1+M^2\right)+2\alpha M+\frac{\alpha^2M^2}{K\left(1-M^2\right)}\right)\overline{\Psi}_+(\alpha,0^+)$$

$$-iB\left(K\left(1+M^2\right)+2\alpha M+\frac{\alpha^2M^2}{K\left(1-M^2\right)}\right)\left[\overline{\Psi}_{i+}(\alpha,0)+\overline{\Psi}_{r+}(\alpha)\right], \tag{3.48}$$

$$\overline{\Psi}'_+(\alpha,0^-)=iB\left(K\left(1+M^2\right)+2\alpha M+\frac{\alpha^2M^2}{K\left(1-M^2\right)}\right)\overline{\Psi}_+(\alpha,0^-), \tag{3.49}$$

where

$$\overline{\Psi}_{i-}(\alpha,0)=b\int_{-\infty}^{p}\Psi_i(X,0)e^{i\alpha(X-p)}dX, \tag{3.50}$$

$$\overline{\Psi}_{i+}(\alpha,0)=b\int_{q}^{\infty}\Psi_i(X,0)e^{i\alpha(X-q)}dX, \tag{3.51a}$$

where " $'$ " denotes differentiation w.r.t Y. The Fourier transform of Eq. (3.13) gives

$$\overline{\Psi}_1'(\alpha,0^+)=\overline{\Psi}_1'(\alpha,0^-)=\overline{\Psi}_1'(\alpha), \tag{3.52}$$

$$\overline{\Psi}_1(\alpha,0^+)-\overline{\Psi}_1(\alpha,0^-)=-\left[\overline{\Psi}_{i1}(\alpha,0)+\overline{\Psi}_{r1}(\alpha)\right]. \tag{3.53}$$

Now, with the help of Eqs. (3.16) and (3.45a), we can write

$$\overline{\Psi}_+'(\alpha,0^+)e^{i\alpha q}+\overline{\Psi}_1'(\alpha,0^+)+\overline{\Psi}_-'(\alpha,0^+)e^{i\alpha p}$$

$$=i\kappa\left[\overline{\Psi}_+(\alpha,0^+)e^{i\alpha q}+\overline{\Psi}_1(\alpha,0^+)+\overline{\Psi}_-(\alpha,0^+)e^{i\alpha p}\right], \tag{3.54}$$

and

$$\overline{\Psi}_+'(\alpha,0^-)e^{i\alpha q}+\overline{\Psi}_1'(\alpha,0^-)+\overline{\Psi}_-'(\alpha,0^-)e^{i\alpha p}$$

$$=-i\kappa\left[\overline{\Psi}_+(\alpha.0^-)e^{i\alpha q}+\overline{\Psi}_1(\alpha,0^-)+\overline{\Psi}_-(\alpha,0^-)e^{i\alpha p}\right]. \tag{3.55}$$

Making use of Eqs. (3.46), (3.47), (3.48), (3.49), (3.52) and (3.53) in Eqs. (3.54) and (3.55) and then adding the resulting equations, we get

$$-i\left[B\left(K\left(1+M^2\right)+2\alpha M+\frac{\alpha^2M^2}{K\left(1-M^2\right)}\right)+\kappa\right]\rho_+(\alpha)e^{i\alpha q}$$

$$-i\left[B\left(K\left(1+M^2\right)+2\alpha M+\frac{\alpha^2M^2}{K\left(1-M^2\right)}\right)+\kappa\right]\rho_-(\alpha)e^{i\alpha p}+\overline{\Psi}_1'(\alpha)$$

$$=iB\left(K\left(1+M^2\right)+2\alpha M+\frac{\alpha^2M^2}{K\left(1-M^2\right)}\right)\left[\overline{\Psi}_{i+}(\alpha)e^{i\alpha q}+\overline{\Psi}_{i-}(\alpha)e^{i\alpha p}\right]$$

$$+\frac{i\kappa}{2}\left[\overline{\Psi}_1(\alpha,0^+)-\overline{\Psi}_1(\alpha,0^-)\right], \tag{3.56}$$

where

$$\overline{\Psi}_{\pm}(\alpha,0^{+})-\overline{\Psi}_{\pm}(\alpha,0^{-})=2\rho_{\pm}(\alpha), \tag{3.57}$$

and

$$\overline{\Psi}_{i\pm}(\alpha)=\frac{\pm ibe^{-iK\cos\theta_{0}q}}{(\alpha-K\cos\theta_{0})}. \tag{3.58}$$

From Eq. (3.53) , we have

$$\overline{\Psi}_{1}(\alpha,0^{+})-\overline{\Psi}_{1}(\alpha,0^{-})=-\left[\overline{\Psi}_{i1}(\alpha)+\overline{\Psi}_{r1}(\alpha)\right]=-2\overline{\Psi}_{i1}(\alpha)=2ibG(\alpha), \tag{3.59}$$

where

$$G(\alpha)=\frac{1}{(\alpha-K\cos\theta_{0})}\left[e^{i(\alpha-K\cos\theta_{0})q}-e^{i(\alpha-K\cos\theta_{0})p}\right], \tag{3.60}$$

Now making use of Eqs.(3.58) and (3.59) in Eq.(3.56)., we get

$$\rho_{+}(\alpha)e^{i\alpha q}+\rho_{-}(\alpha)e^{i\alpha p}+\frac{i\overline{\Psi}_{1}'(\alpha)}{\kappa L(\alpha)}=-ibG(\alpha), \tag{3.61}$$

where

$$L_{\pm}(\alpha)=1+\frac{B}{\kappa_{\pm}(\alpha)}\left[K\left(1+M^{2}\right)+2\alpha M+\frac{\alpha^{2}M^{2}}{(1-M^{2})\,K}\right]. \tag{3.62}$$

Eq. (3.61) is analogous to the Eq. (5.60) available in [13]. In Eq. (3.61), $\kappa L(\alpha)$ is the kernel and in order to solve it, we have to factorize $\kappa L(\alpha)$ as the product of two non-singular factors such that one factor being regular in the lower half plane and the other factor being regular in the upper half plane with the additional requirements that both the factors as well as their inverses contains elements of algebraic growth at infinity and both of these factors should commute with each other. Note that Eq. (3.61) is a standard Wiener- Hopf equation. The solution to this equation is found in the subsequent section.

3.3 Solution of the Wiener-Hopf Equation

The W-H technique is not only independent of the incidence and reflection angles [37] , but it also gives an insight into the physical structure of the diffracted field [13]. For the solution of Eq. (3.61), we make the following factorization,

$$\kappa L(\alpha) = S(\alpha) = S_+(\alpha)S_-(\alpha), \tag{3.63}$$

and

$$S_\pm(\alpha) = \kappa_\pm(\alpha)L_\pm(\alpha), \tag{3.64}$$

$L_+(\alpha)$ and $\kappa_+(\alpha)$ are regular for $\operatorname{Im}\alpha > -\operatorname{Im} \mathrm{K}$, i.e., upper half plane and $L_-(\alpha)$ and $\kappa_-(\alpha)$ are regular for $\operatorname{Im}\ \alpha < \operatorname{Im} K$, i.e., lower half plane. So, $S_+(\alpha)$ is regular in the upper half plane and $S_-(\alpha)$ is regular in the lower half plane.

We write Eq. (3.61) as

$$\rho_+(\alpha)e^{i\alpha q} + \frac{i\overline{\Psi}_1'(\alpha)}{S_+(\alpha)S_-(\alpha)} + \rho_-(\alpha)e^{i\alpha p} = AG(\alpha), \tag{3.65}$$

where

$$G(\alpha) = \frac{1}{(\alpha - K_m)}\left[e^{i(\alpha-K_m)q} - e^{i(\alpha-K_m)p}\right]. \tag{3.66}$$

and for simplification, we write

$$A = -ib \quad \text{and} \quad K_m = K\cos\theta_0. \tag{3.67}$$

We multiply Eq. (3.65) by $S_+(\alpha)e^{-i\alpha q}$, put the value of $G(\alpha)$, add and subtract pole contribution to get

$$S_+(\alpha)\rho_+(\alpha)+\frac{i\overline{\Psi}_1'(\alpha)}{S_-(\alpha)}e^{-i\alpha q}+S_+(\alpha)\rho_-(\alpha)e^{i\alpha(p-q)}$$

$$=\frac{AS_+(\alpha)e^{-i\alpha q}}{(\alpha-K_m)}\left[\left(e^{i(\alpha-K_m)q}-e^{i(\alpha-K_m)p}\right)\right]. \tag{3.68}$$

The first term on the left hand side of above equation is regular in the upper half plane and the terms whose gender is not known, can be decomposed according to the procedure defined in [13], as under

$$S_+(\alpha)\rho_-(\alpha)e^{i\alpha(p-q)}=U_+(\alpha)+U_-(\alpha), \tag{3.69}$$

and

$$\frac{AS_+(\alpha)e^{i\alpha(p-q)-iK_mp}}{(\alpha-K_m)}=V_+(\alpha)+V_-(\alpha). \tag{3.70}$$

Invoking Eqs. (3.69) and (3.70) in Eq. (3.68) , we obtain

$$S_+(\alpha)\rho_+(\alpha)-\frac{Ae^{-iK_mq}}{(\alpha-K_m)}\left[S_+(\alpha)-S_+(K_m)\right]+U_+(\alpha)+V_+(\alpha)$$

$$=\frac{AS_+(K_m)e^{-iK_mq}}{(\alpha-K_m)}-\frac{i\overline{\Psi}_1'(\alpha)}{S_-(\alpha)}e^{-i\alpha q}-U_-(\alpha)-V_-(\alpha). \tag{3.71}$$

Now, multiplying Eq. (3.65) by $S_-(\alpha)e^{-i\alpha p}$ on both sides and putting the value of $G(\alpha)$, we get

$$S_-(\alpha)\rho_-(\alpha)+\frac{AS_-(\alpha)e^{-iK_mp}}{(\alpha-K_m)}+R_-(\alpha)-P_-(\alpha)$$

$$=-\frac{i\overline{\Psi}_1'(\alpha)e^{-i\alpha p}}{S_+(\alpha)}-R_+(\alpha)+P_+(\alpha). \tag{3.72}$$

where

$$S_-(\alpha)\rho_+(\alpha)e^{i\alpha(q-p)}=R_+(\alpha)+R_-(\alpha), \tag{3.73}$$

and

$$\frac{AS_-(\alpha)e^{i\alpha(q-p)-iK_m q}}{\alpha - K_m} = P_+(\alpha) + P_-(\alpha). \tag{3.74}$$

Let $J(\alpha)$ be a function equal to both sides of Eq. (3.68). We observe that left hand side of Eq. (3.68) is regular in $\operatorname{Im}\alpha > -\operatorname{Im}K$ and the right hand side is regular for $\operatorname{Im}\alpha < K\cos\theta_0$ respectively. We use analytical continuation so that the definition of $J(\alpha)$ can be extended throughout the complex α plane. We examine the asymptotic behaviour of Eq. (3.68) to ascertain the form of $J(\alpha)$ as $|\alpha| \to \infty$. We note $|L_\pm(\alpha)| \sim O(1)$, [46] as $|\alpha| \to \infty$ and find with the help of edge condition that $\rho_+(\alpha)$ and $\rho_-(\alpha)$ should be at least of $O(|\alpha|^{\frac{-1}{2}})$ as $|\alpha| \to \infty$. So, using the extended form of Liouville's theorem [13], we see $J(\alpha) \sim O(|\alpha|^{\frac{-1}{2}})$ and so a polynomial that represent $J(\alpha)$ can only be a constant equal to zero. Similarly, the same conclusion can be made for Eq. (3.69), and therefore each side of Eq. (3.68) and (3.69) is equal to zero, i.e.,

$$S_+(\alpha)\rho_+(\alpha) - \frac{Ae^{-iK_m q}}{(\alpha - K_m)}\left(S_+(\alpha) - S_+(K_m)\right) + U_+(\alpha) + V_+(\alpha) = 0, \tag{3.75}$$

and

$$S_-(\alpha)\rho_-(\alpha) + \frac{AS_-(\alpha)e^{-iK_m p}}{(\alpha - K_m)} + R_-(\alpha) - P_-(\alpha) = 0, \tag{3.76}$$

where

$$U_+(\alpha) = \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} \frac{S_+(\eta)\rho_-(\eta)e^{i\eta(p-q)}}{(\eta - \alpha)}d\eta, \tag{3.77}$$

$$V_+(\alpha) = \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} \frac{AS_+(\eta)e^{i\eta(p-q)-iK_m p}}{(\eta - K_m)(\eta - \alpha)}d\eta, \tag{3.78}$$

and

$$R_-(\alpha) = -\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id} \frac{S_-(\eta)\rho_+(\eta)e^{i\eta(q-p)}}{(\eta - \alpha)}d\eta, \tag{3.79}$$

$$P_-(\alpha) = -\frac{1}{2\pi i}\int\limits_{-\infty+id}^{\infty+id}\frac{AS_-(\eta)e^{i\eta(q-p)-iK_mq}}{(\eta-K_m)(\eta-\alpha)}d\eta, \tag{3.80}$$

here $-K_2 < c < \tau < d < K_2\cos\theta_0$, where $\tau = \operatorname{Im}\alpha$ and $K_2 = \operatorname{Im}K$

Now making use of Eqs. (3.77) and (3.78) in Eq. (3.75) and using Eqs. (3.79) and (3.80) in Eq. (3.76), we get

$$S_+(\alpha)\rho_+^*(\eta) + \frac{AS_+(\alpha)e^{-iK_mq}}{(\alpha-K_m)}) + \frac{1}{2\pi i}\int\limits_{-\infty+ic}^{\infty+ic}\frac{S_+(\eta)\rho_-^*(\eta)e^{i\eta(p-q)}}{(\eta-\alpha)}d\eta = 0, \tag{3.81}$$

and

$$S_-(\alpha)\rho_-^*(\eta) - \frac{1}{2\pi i}\int\limits_{-\infty+id}^{\infty+id}\frac{S_-(\eta)\rho_+^*(\eta)e^{i\eta(q-p)}}{(\eta-\alpha)}d\eta = 0. \tag{3.82}$$

where

$$\rho_+(\alpha) - \frac{Ae^{-iK_mq}}{\alpha-K_m} = \rho_+^*(\eta), \tag{3.83}$$

and

$$\rho_-(\alpha) + \frac{Ae^{-iK_mp}}{\alpha-K_m} = \rho_-^*(\eta). \tag{3.84}$$

The assumption $0 < \theta_0 < \pi$, allows us to choose a so that $-K_2\cos\theta_0 < a < K_2\cos\theta_0$. Also take $c = d = a$, replace η by $-\eta$, α by $-\alpha$ in Eqs. (3.39 & 3.40), respectively, and use

$$S_+(-\alpha) = S_-(\alpha) \tag{3.85}$$

to get

$$S_+(\alpha)\left[F_+(\alpha) - \frac{Ae^{-iK_mq}}{\alpha-K_m} + \lambda\frac{Ae^{-iK_mp}}{\alpha+K_m}\right]$$

$$+\frac{\lambda}{2\pi i}\int\limits_{-\infty+ia}^{\infty+ia}\frac{S_-(\eta)e^{i\eta(q-p)}}{(\eta+\alpha)}\left[F_+(\eta) - \frac{Ae^{-iK_mq}}{(\eta-K_m)} - \lambda\frac{Ae^{-iK_mp}}{(\eta+K_m)}\right]d\eta$$

$$+\frac{AS_+(K_m)e^{-iK_mq}}{(\eta-K_m)} = 0. \tag{3.86}$$

where

$$\rho_+^*(\alpha) \pm \rho_-^*(\alpha) = F_+^*(\alpha) = F_+(\alpha) - \frac{Ae^{-iK_m q}}{\alpha - K_m} - \lambda \frac{Ae^{-iK_m p}}{\alpha + K_m}, \qquad for \ \ \lambda = \mp 1 \quad (3.87)$$

and $F_+^*(\alpha)$ can be calculated by using Eqs. (3.83), (3.84), (3.87), and combining the results, i.e.,

$$F_+^*(\alpha) = F_+(\alpha) - \frac{Ae^{-iK_m q}}{\alpha - K_m} - \lambda \frac{Ae^{-iK_m p}}{\alpha + K_m}, \tag{3.88}$$

where

$$F_+(\alpha) = \rho_+(\alpha) - \lambda \rho_-(-\alpha). \tag{3.89}$$

Now consider the integral appearing in Eq.(3.41) i.e.,

$$I = \int_{-\infty+ia}^{\infty+ia} \frac{S_-(\eta)e^{i\eta(q-p)}}{(\eta+\alpha)} \left[F_+(\eta) - \frac{Ae^{-iK_m q}}{(\eta - K_m)} - \lambda \frac{Ae^{-iK_m p}}{(\eta + K_m)} \right] d\eta,$$

or

$$I = I_1 - Ae^{-iK_m q} I_2 - \lambda Ae^{-iK_m p} I_3, \tag{3.90}$$

where

$$I_1 = \int_{-\infty+ia}^{\infty+ia} \frac{S_-(\eta)F_+(\eta)e^{i\eta l}}{(\eta+\alpha)} d\eta, \tag{3.91}$$

$$I_2 = \int_{-\infty+ia}^{\infty+ia} \frac{S_-(\eta)e^{i\eta l}}{(\eta+\alpha)(\eta - K_m)} d\eta, \tag{3.92}$$

$$I_3 = \int_{-\infty+ia}^{\infty+ia} \frac{S_-(\eta)e^{i\eta l}}{(\eta+\alpha)(\eta + K_m)} d\eta, \tag{3.93}$$

and

$$l = (q-p).$$

We observe that $F_+(\alpha)$ is regular in $\tau > -K_2$, so we can expect that $F_+(\alpha)$ will have a branch point at $\alpha = -K$. But for large l, it is sufficiently far from the point $\alpha = K$, which enables us to evaluate the above integrals in the asymptotic expansion [13].

Now, using the procedure as in [13], we get

$$I_1 = \int_{-\infty+ia}^{\infty+ia} \frac{S_-(\eta)F_+(\eta)e^{i\eta l}}{(\eta+\alpha)} d\eta \approx E_0 W_0\left[-i(K+\alpha)l\right] F_+(K)L_-(K), \tag{3.94}$$

or

$$I_1 \approx 2\pi i T(\alpha) F_+(K) L_-(K), \tag{3.95}$$

where

$$T(\alpha) = \frac{E_0 W_0\left[-i(K+\alpha)l\right]}{2\pi i}, \; E_0 = 2e^{\frac{i\pi}{4}} e^{iKl} l^{-1/2} e^{\frac{i\pi}{4}},$$

$$h_0 = .$$

and

$$W_0(z) = \int_0^{\infty} \frac{u^{-1/2}e^{-u}}{u+z} du = \frac{\sqrt{\pi}}{2} e^{z/2} z^{-1/4} W_{-3/4,1/4}(z), \; z = -i(K+\alpha)l,$$

where

and $W_{i,j}(z)$ is a Whittaker function [147, 148]. Since the slit width has been considered to be larger as compared to the incident wavelength, the integrals have been evaluated asymptotically in terms of Whittaker functions. But in that case the width of the slit is taken smaller as compared to the incident wavelength, the Whittaker functions can be replaced by the Fresnel Integrals [142]. Now, the other integrals are calculated as follows [13]

$$I_2 = \int_{-\infty+ia}^{\infty+ia} \frac{S_-(\eta)e^{i\eta l}}{(\eta+\alpha)(\eta-K_m)}d\eta \approx 2\pi i\left[\frac{S_-(K_m)e^{iK_m l}}{(\alpha+K_m)} + R_2(\alpha)\right]. \tag{3.96}$$

and

$$I_3 = \int_{-\infty+ia}^{\infty+ia} \frac{S_-(\eta)e^{i\eta l}}{(\eta+\alpha)(\eta+K_m)}d\eta \approx 2\pi i R_1(\alpha), \tag{3.97}$$

where

$$R_{1,2}(\alpha) = \frac{E_0\left[W_0\left\{-i(K\pm K_m)l\right\} - W_0\left\{-i(K+\alpha)l\right\}\right]}{2\pi i(\alpha\mp K_m)}, \tag{3.98}$$

Thus, Eq. (3.90) takes the form

$$I = 2\pi i\left[T(\alpha)F_+(K)L_-(K) - AR_2(\alpha)e^{-iK_m q} - \frac{AS_-(K_m)e^{-iK_m p}}{(\alpha+K_m)} - \lambda Ae^{-iK_m p}R_1(\alpha)\right]. \tag{3.99}$$

Using the above expression in Eq. (3.86), we get

$$S_+(\alpha)F_+(\alpha) = A\left[G_1(\alpha) - \lambda G_2(\alpha)\right] - \lambda T(\alpha)F_+(K)L_-(K) \tag{3.100}$$

or

$$F_+(\alpha) = \frac{A}{S_+(\alpha)}\left[G_1(\alpha) - \lambda G_2(\alpha)\right] - \frac{A\lambda T(\alpha)}{S_+(\alpha)}\left[\frac{G_1(K) - \lambda G_2(K)}{S_+(K) + \lambda T(K)L_-(K)}\right], \tag{3.101}$$

where

$$G_1(\alpha) = P_1(\alpha)e^{-iK_m q} - R_1(\alpha)e^{-iK_m p}, \tag{3.102}$$

$$G_2(\alpha) = P_2(\alpha)e^{-iK_m q} - R_2(\alpha)e^{-iK_m p}. \tag{3.103}$$

and

$$P_1(\alpha) = \frac{S_+(\alpha) - S_+(K_m)}{\alpha - K_m}, \tag{3.104}$$

$$P_2(\alpha) = \frac{S_+(\alpha) - S_+(K_m)}{\alpha + K_m}, \tag{3.105}$$

and where $F_+(K)$ can be calculated from Eq. (3.101) by putting $\alpha = K$.

Now using $\lambda = \mp 1$, in Eqs. (3.89) and (3.101) and adding the resultant equations, we obtain

$$\rho_+(\alpha) = A\frac{G_1(\alpha)}{S_+(\alpha)} + \frac{AT(\alpha)}{S_+(\alpha)S_+(K)\left(1 - \frac{T^2(K)L_-^2(K)}{S_+^2(K)}\right)}\left[G_2(K) + \frac{G_1(K)T(K)L_-(K)}{S_+(K)}\right]. \tag{3.106}$$

and by subtracting the same resultant equations with $\alpha = -\alpha$, we get $\rho_-(\alpha)$ or replacing G_1 by G_2 and G_2 by G_1, changing the sign of α in Eq. (3.106) and using $S_+(-\alpha) = S_-(\alpha)$, we get

$$\rho_-(\alpha) = \frac{AG_2(-\alpha)}{S_-(\alpha)} + \frac{AT(-\alpha)}{S_-(\alpha)S_+(K)\left[\left(1 - \frac{T^2(K)L_-^2(K)}{S_+^2(K)}\right)\right]}\left[G_1(K) + \frac{G_2(K)T(K)L_-(K)}{S_+(K)}\right]. \tag{3.107}$$

Using Eq. (3.57) in Eqs. (3.106) and (3.107), respectively, we get

$$\overline{\Psi}_+(\alpha, 0^+) - \overline{\Psi}_+(\alpha, 0^-) = \frac{2A}{S_+(\alpha)}\left[G_1(\alpha) + C_1(K)T(\alpha)\right] \tag{3.108}$$

and

$$\overline{\Psi}_-(\alpha, 0^+) - \overline{\Psi}_-(\alpha, 0^-) = \frac{2A}{S_-(\alpha)}\left[G_2(-\alpha) + C_2(K)T(-\alpha)\right], \tag{3.109}$$

where

$$C_1(K) = \frac{1}{S_+(K)\left(1 - \frac{T^2(K)L_-^2(K)}{S_+^2(K)}\right)}\left[G_2(K) + \frac{G_1(K)T(K)L_-(K)}{S_+(K)}\right], \tag{3.110}$$

$$C_2(K) = \frac{1}{S_+(K)\left[\left(1 - \frac{T^2(K)L_-^2(K)}{S_+^2(K)}\right)\right]}\left[G_1(K) + \frac{G_2(K)T(K)L_-(K)}{S_+(K)}\right]. \tag{3.111}$$

Using Eqs. (3.38), (3.43), (3.45a), (3.59), (3.108), (3.109), we get

$$\begin{aligned} A_3(\alpha) &= -A_4(\alpha) = \frac{A}{S_+(\alpha)}\left[G_1(\alpha) + C_1(K)T(\alpha)\right]e^{i\alpha q} - AG(\alpha) \\ &+ \frac{A}{S_-(\alpha)}\left[G_2(-\alpha) + C_2(K)T(-\alpha)\right]e^{i\alpha p}. \end{aligned} \tag{3.112}$$

where $A_3(\alpha)$ corresponds to $Y \geqslant 0$ and $A_4(\alpha)$ corresponds to $Y < 0$.

We substitute these values of $A_3(\alpha)$ and $A_4(\alpha)$ into Eq. (3.22) then obtain $\Psi(X, Y)$ by taking the inverse Fourier transform as

$$\Psi(X,Y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\begin{array}{c} \frac{A}{S_+(\alpha)}\left[G_1(\alpha) + C_1(K)T(\alpha)\right]e^{i\alpha q} - AG(\alpha) \\ +\frac{A}{S_-(\alpha)}\left[G_2(-\alpha) + C_2(K)T(-\alpha)\right]e^{i\alpha p} \end{array}\right] e^{-i\alpha X + i\kappa|Y|}d\alpha. \tag{3.113}$$

Invoking Eqs. (3.102), (3.103), (3.104) and (3.105) in Eq. (3.113), we get

$$\Psi(X,Y) = \frac{A}{2\pi}\int_{-\infty}^{\infty}\left[\left\{\begin{array}{c} \frac{e^{i(\alpha-K_m)q}S_+(K_m)}{S_+(\alpha)(\alpha-K_m)} + \frac{e^{-iK_m p}R_1(\alpha)e^{i\alpha q}}{S_+(\alpha)} - \frac{C_1(K)T(\alpha)e^{i\alpha q}}{S_+(\alpha)} \\ -\frac{e^{i(\alpha-K_m)p}S_-(K_m)e^{-i\alpha X - i\kappa|Y|}}{S_-(\alpha)(\alpha-K_m)} \\ +\frac{e^{-iK_m q}R_2(-\alpha)e^{i\alpha p}}{S_-(\alpha)} - \frac{C_2(K)T(-\alpha)e^{i\alpha p}}{S_-(\alpha)} \end{array}\right\}\right] e^{-i\alpha X + i\kappa|Y|}d\alpha. \tag{3.114}$$

We can break up the field $\Psi(X, Y)$ into two parts

$$\Psi(X,Y) = \Psi^{sep}(X,Y) + \Psi^{int}(X,Y), \tag{3.115}$$

where

$$\begin{aligned} \Psi^{sep}(X,Y) &= \frac{ib}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i(\alpha-K_m)q}S_+(K_m)e^{-i\alpha X + i\kappa|Y|}}{S_+(\alpha)(\alpha-K_m)}d\alpha \\ &- \frac{ib}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i(\alpha-K_m)p}S_-(K_m)e^{-i\alpha X + i\kappa|Y|}}{S_-(\alpha)(\alpha-K_m)}d\alpha, \end{aligned} \tag{3.116}$$

and

$$\Psi^{int}(X,Y) = \frac{ib}{2\pi}\int\limits_{-\infty}^{\infty}\left[\begin{array}{c} \frac{e^{-iKmp}R_1(\alpha)e^{i\alpha q}}{S_+(\alpha)} - \frac{C_1(K)T(\alpha)e^{i\alpha q}}{S_+(\alpha)} \\ +\frac{e^{-iKmq}R_2(-\alpha)e^{i\alpha p}}{S_-(\alpha)} - \frac{C_2(K)T(-\alpha)e^{i\alpha p}}{S_-(\alpha)} \end{array}\right] e^{-i\alpha X + i\kappa|Y|}d\alpha. \qquad (3.117)$$

3.4 Far Field Approximation

We calculate the far field by evaluating the integrals appearing in Eqs. (3.57) and (3.58) with the help of the following substitutions.

$$X = r\cos\theta, \quad Y = r\sin\theta \quad \alpha = -K\cos(\theta + it) \qquad (3.118)$$

and deform the contour by the transformation $\alpha = -K\cos(\theta + it)$ which changes the contour of integration over α into a hyperbole through the point $\alpha = -K\cos\theta$, where $(0 < \theta < \pi, \ -\infty < t < \infty)$.

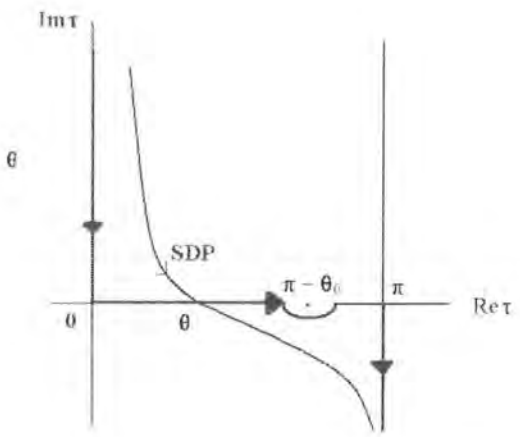


Fig. 3.2: Steepest descent path in the complex

$\tau-$plane.

So, we have

$$\Psi^{sep}(X,Y)=\frac{bK}{2\pi}\int_{-\infty}^{\infty}f_1\left[K\cos\left(\theta+it\right)\right]\left[\exp(-iKr\cosh t)\right]\sin(\theta+it)dt, \tag{3.119}$$

where

$$f_1\left[K\cos\left(\theta+it\right)\right]=\frac{K}{\left[K\cos\left(\theta+it\right)-K_m\right]}\left[\begin{array}{c}\frac{e^{i(K\cos(\theta+it)-K_m)d}S_+(K_m)}{S_+[K\cos(\theta+it)]}\\ -\frac{e^{i(K\cos(\theta+it)-K_m)d}S_-(K_m)}{S_-[K\cos(\theta+it)]}\end{array}\right]. \tag{3.120}$$

Similarly, we have

$$\Psi^{int}(X,Y)=\frac{bK}{2\pi}\int_{-\infty}^{\infty}f_2\left[K\cos\left(\theta+it\right)\right]\exp(-iKr\cosh t)\sin(\theta+it)dt, \tag{3.121}$$

where

$$f_2\left[K\cos\left(\theta+it\right)\right]=\left[\begin{array}{c}\frac{R_1[K\cos(\theta+it)]e^{i(K\cos(\theta+it)-K_m)d}}{S_+[K\cos(\theta+it)]}\\ -\frac{C_1(K)T(K\cos(\theta+it))e^{iK\cos(\theta+it)d}}{S_+[K\cos(\theta+it)]}\end{array}\right.$$

$$\left.+\frac{R_2\left[-K\cos\left(\theta+it\right)\right]e^{i[K\cos(\theta+it)-K_m]d}}{S_-\left[K\cos\left(\theta+it\right)\right]}-\frac{C_2(K)T\left[-K\cos\left(\theta+it\right)\right]e^{iK\cos(\theta+it)d}}{S_-\left[K\cos\left(\theta+it\right)\right]}\right], \tag{3.122}$$

we see that, in deforming the contour into a hyperbola, the pole $\alpha=-K\cos\theta$ may be crossed and we make the transformation $\alpha=-K\cos\left(\theta+it\right)$ so that the contour over α also goes into a hyperbola. The two hyperbolae will not cross each other if $\theta<\theta_0$ but if the inequality is reversed, there will be a contribution from the pole which actually cancels the incident wave in the shadow region. So, a modification of the method of stationary phase is required because the pole may come close to the stationary point [39]. Hence, for large Kr, by using modified method of stationary phase, Eqs. (3.60) and (3.62), become

$$\Psi^{sep}(X,Y)=\frac{-ibK\sin\theta}{\sqrt{2\pi Kr}}f_1(K\cos\theta)\exp i(Kr-\frac{\pi}{4}), \tag{3.123}$$

and

$$\Psi^{int}(X,Y)=\frac{-ibK\sin\theta}{\sqrt{2\pi Kr}}f_2(K\cos\theta)\exp i(Kr-\frac{\pi}{4}), \tag{3.124}$$

where $f_1(K\cos\theta)$ and $f_2(K\cos\theta)$ are given, by setting $t=0$, in Eqs. (3.120) and (3.122). Now, use of Eqs. (3.123) and (3.124) in Eq. (3.11) implies that

$$\Phi^{sep}(x,y)=\frac{-ibK\sin\theta e^{-iKMX}}{\sqrt{2\pi Kr}}f_1(K\cos\theta)\exp i(Kr-\frac{\pi}{4}), \tag{3.125}$$

and

$$\Phi^{int}(x,y)=\frac{-ibK\sin\theta e^{-iKMX}}{\sqrt{2\pi Kr}}f_2(K\cos\theta)\exp i(Kr-\frac{\pi}{4}). \tag{3.126}$$

Here, $\Psi^{sep}(X,Y)$ consists of two parts, each representing the diffracted field produced by the edges at $x=p$ and $x=q$ respectively, as though the other edge was absent while $\Psi^{int}(X,Y)$ gives the interaction of one edge upon the other.

Thus, using Eqs. (3.11), (3.14), (3.32), (3.125) and (3.126), the total far field is given by

$$\Phi(x,y)\sim\frac{a}{4i}H_0^1(KR)e^{-iKMX}-\frac{ibK\sin\theta e^{-iKMX}}{\sqrt{2\pi Kr}}\left\{\exp i(Kr-\frac{\pi}{4})\right\}f_{1,2}(K\cos\theta). \tag{3.127}$$

3.5 Graphical Results

A computer program MATHEMATICA has been used for the numerical evaluation and Graphical plotting of the separated field given by the expression (3.123). Some graphs are presented which show the effects of parameters kr i.e., distance of the observer from the point of observation, kr_0 i.e., distance of the source from the point of observation and θ_0 i.e.,the angle of incidence, on the diffracted field. The absorbing parameter B is to be taken such that $\operatorname{Re}B>0$ for an absorbing surface and $|M|<1$ for a subsonic flow. Positive Mach number indicates that the stream flow is from left to right and negative

Mach number indicates that the stream flow is from right to left. The following situations are considered:

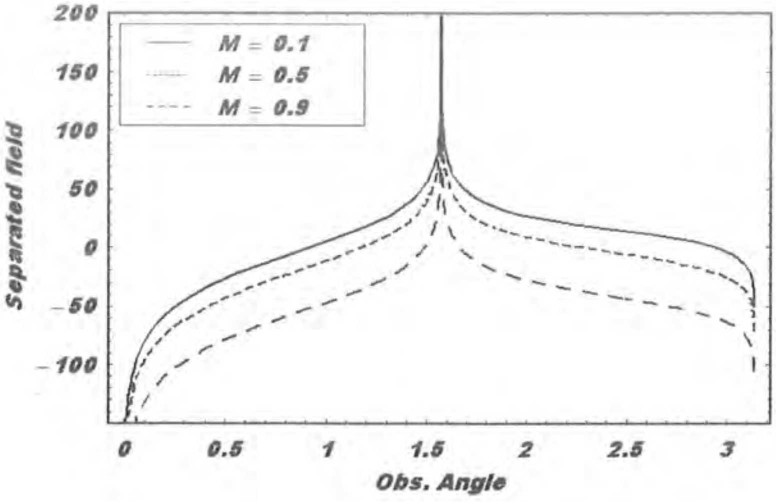
(i) When the source is fixed in one position (for all Mach numbers), relative to the slit, ($\theta_0 = 90^o, 60^o$, M and θ are allowed to vary).

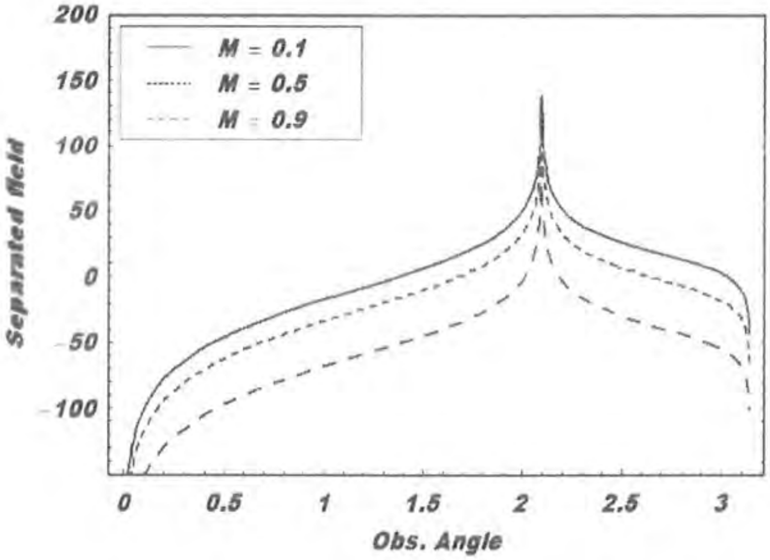
(ii) When the source is fixed in one position, ($\theta_0 = 90^o, 60^o$, B and θ are allowed to vary).

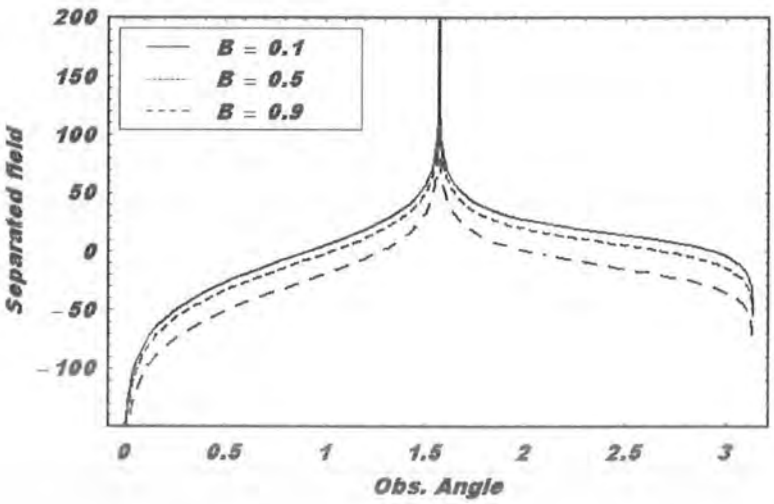
One can see from the figures (3), (4), (5)and (6) that the field in the region $0 < \theta < \pi$, is most affected by the changes in M, B and K. The main features of the graphical results are as follows:

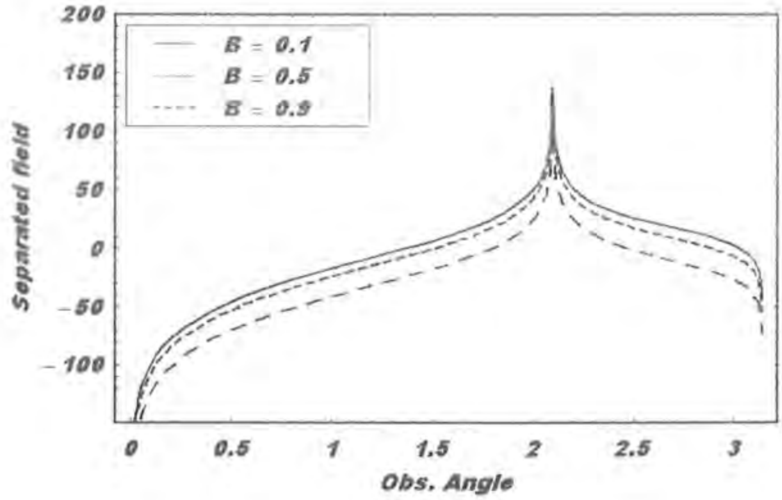
(a) Since we are considering the subsonic flow i.e., $U < c$, and by increasing the Mach number M ,fixing all other parameters and by taking $K = 1, 2$ respectively, the velocity of the fluid is coming close to the velocity of sound. We observe from Figs.(3) & (4) that the amplitude of the velocity potential decreases i.e., the sound intensity reduces. Thus a clear variation of the velocity potential with respect to Mach number can be observed from the graphs.

(b) In Figs. (5) and (6), the effects of absorbing parameter B on the velocity potential has been shown. By increasing the absorbing parameter B, the intensity of the diffracted wave will be decreased and consequently the amplitude of the velocity potential will decrease which can be seen from the graphs i.e., the sound intensity reduces, by fixing all other parameters and taking $K = 1, 2$ respectively.









3.6 Conclusion

In this chapter, the line source diffraction of acoustic waves by the slit in a moving fluid using Myers' conditions has been pondered upon. While using Fourier transform technique, the boundary value problem is reduced to the functional Wiener-Hopf equation whose solution is obtained by considering the factorization of the kernel. The Myers' conditions give rise to a corrective term $\left(\frac{BM\cos^2\theta}{\sin\theta(1-M^2)}\right)$ in the (third term of the kernel) solution. If this corrective term is ignored, the results of [135] can be achieved which shows that our results vary by a multiplicative factor. Further, a numerical solution of the problem is obtained showing the effect of sundry parameters (for various values of Mach number M and the absorbing parameter B against the velocity potential). The graphs show a clear variation of the velocity potential against these parameters. It is found that if a line source is removed to a far-off distance, i.e., infinity, the graphical results of plane wave situation [135]can be recovered. The Myers' conditions give better attenuation results for the separated and inter-active diffracted fields as compared to Ingards' conditions. This route of solution of diffraction problem is more rigorous and involves tedious mathematical calculations.

- The mathematical importance of the above solved problem rests upon the fact that we have to find four unknown functions $\left(\rho_{\pm}(\alpha), \overline{\Psi}_{\pm}(\alpha)\right)$ against the case of strip geometry where we have to find the two unknown functions.

- The above solved boundary value problem (a three part boundary value problem) is a very substantial one in the diffraction theory as it involves tedious mathematical calculations.

- Two diffracted fields i.e., one from each edge and the other from the interaction of one edge upon the other edge were obtained.

- The above solved problem of a slit in a moving fluid will help understand acoustic diffraction and will go a step further to complete the discussion for the slit geometry.

- The far-field situation for the diffracted field is presented and some graphs showing the effects of various parameters on the separated field are also plotted and discussed.

- To the best of author's knowledge, the above mentioned problem has not been solved so far by using the Noble's approach [13] which uses Jones' method [125] based on the W-H technique and avoids the integral equations approach.

- If we put $M = \beta = 0$, we get the results of [13] for a rigid barrier in still fluid.

Chapter 4

POINT SOURCE DIFFRACTION BY A SLIT IN A MOVING FLUID

In this chapter, we shall discuss the diffraction of a spherical acoustic wave from a slit in a moving fluid using Myers' impedance condition [133]. It is a new diffraction problem, under the influence of Myers' condition, which is not available in the existing literature; hence, it seems to be the first attempt of the kind. The method of solution consists of Fourier transform and Wiener-Hopf technique [13,32] where the scattered field is found by taking the inverse Fourier transform and using the method of steepest descent [129–132]. Unequivocal expressions are obtained for the singly diffracted field i.e., separated field, and doubly diffracted field i.e., interaction of one edge upon the other. The numerical results are drawn and the graphs verify our mathematical calculations and give a clear picture of the variation of the velocity potential for various parameters. The mathematical significance of the problem of point source is that it will introduce another variable. The difficulty, that arises in the solution of the integral, occurs in the inverse transform. These integrals are normally difficult to handle because of the presence of the branch points and are only amenable to solution using asymptotic approximations.

Scattering from a slit or strip is a well known problem in diffraction theory. Asestas and Kleinman [161] have done a lot of contribution by summarizing and reviewing the work done in this regard. Jones [125] and Noble [13] have studied the diffraction from a slit / strip using the Wiener- Hopf method. The point source situation arises when there is a finite opening in an infinite barrier which intercepts the line of sight from the source to the receiver. The introduction of point source changes the incident field and the method of solution requires a careful analysis in calculating the diffracted field. Point source consideration is important because it is regarded as a fundamental radiating device [32] and the solution of the point source problem is called the fundamental solution of the given differential equation. The point sources are regarded as better substitutes for real sources than line sources or plane waves. Some contributions regarding the point source scattering situations can be found in the works of Vlaar [162], Ghosh [163], Wenzel [164], Chattopadhyay et al [165], Balasubramanyam [110].

4.1 Formulation of the Problem

We consider the diffraction of an acoustic wave incident on the slit occupying a space $y = 0,\ \ p \leq x \leq q,\ -\infty < z < \infty$. The point source is located at (x_0, y_0, z_0) and the system is placed in a fluid moving with subsonic velocity U parallel to the x-axis. The time dependence is taken to be of harmonic type $e^{-i\omega t}$(ω is the angular frequency) and the plane is assumed to be satisfying the Myers' impedance condition [133].

$$u_n = \frac{-\widetilde{p}}{Z_a} + \frac{U}{i\omega Z_a}\frac{\partial \widetilde{p}}{\partial x}, \tag{4.1}$$

where u_n is the normal derivative of the perturbation velocity, $\widetilde{p}$ is the surface pressure, Z_a is the acoustic impedance of the surface and $-\mathbf{n}$ a normal pointing from the fluid into the surface. The perturbation velocity $\mathbf{u}$ of the irrotational sound wave can be written in terms of the velocity potential Φ as $\mathbf{u} = \nabla\Phi$. Then the resulting pressure $\tilde{p}$ of the sound field is

$$\tilde{p} = -\rho_0 \left(-i\omega + U\frac{\partial}{\partial x} \right) \Phi(x, y, z), \tag{4.2}$$

where ρ_0 is the density of the undisturbed stream.

The wave equation satisfied by the total velocity potential Φ in the presence of point source is given by

$$\left[(1 - M^2)\frac{\partial^2}{\partial x^2} + 2ikM\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2 \right] \Phi_t(x, y, z) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0), \tag{4.3}$$

subject to the following boundary conditions on the absorbing barriers

$$\left[\frac{\partial}{\partial y} \mp 2\beta M\frac{\partial}{\partial x} \pm ik\beta \mp \frac{i\beta M^2}{k}\frac{\partial^2}{\partial x^2} \right] \Phi_t(x, 0^{\pm}, z) = 0 \qquad \left. \begin{array}{c} -\infty < x < p, \\ q < x < \infty, \\ -\infty < z < \infty \end{array} \right] , \tag{4.4}$$

$$\left. \begin{array}{r} \Phi_t(x, 0^+, z) = \Phi_t(x, 0^-, z) \\ \frac{\partial \Phi_t}{\partial y}(x, 0^+, z) = \frac{\partial \Phi_t}{\partial y}(x, 0^-, z) \end{array} \right] \qquad p \leq x \leq q, \tag{4.5}$$

In Eqs. (4.4), the quantity 0^+ refers to the situation that $y \to 0$ through positive $y - axis$ and the quantity 0^- refers to the situation that $y \to 0$ through negative $y - axis$. For analytic ease, we shall assume that the wave number $k = k_1 + ik_2$ has a small positive imaginary part to ensure the regularity of the Fourier transform integrals and that k_2 is the loss factor of the medium. The specific complex admittance is $\beta = \frac{\rho_0 c}{Z_a}$ and the Mach number is $M = \frac{U}{c}$.

It is assumed that the flow is subsonic, i.e., $|M| < 1$, and $\operatorname{Re}\beta > 0$, which is a necessary condition for an absorbing surface [134]. We remark that $\beta = 0$ corresponds to the rigid barrier and $\beta = \infty$ corresponds to the pressure release barrier. Eq. (4.5) shows that the velocity potential Φ and its derivative $\frac{\partial \Phi}{\partial y}$ are continuous on the slit.

The edge conditions requires that Φ_t and its normal derivative must be bounded for a unique solution and should satisfy [57, 144].

$$\Phi_t(x,0) = \begin{cases} -1 + O(x-p)^{\frac{1}{4}} & \text{as } x \to p^-, \\ -1 + O(x-q)^{\frac{1}{4}} & \text{as } x \to q^+, \end{cases} \tag{4.6}$$

$$\frac{\partial \Phi_t(x,0)}{\partial y} = \begin{cases} O(x-p)^{-\frac{3}{4}} & \text{as } x \to p^-, \\ O(x-q)^{-\frac{3}{4}} & \text{as } x \to q^+. \end{cases} \tag{4.7}$$

In the above equations, the negative sign indicates a limit taken from the left of the point p and the positive sign indicates that a limit is taken from the right of the point q on the $x-axis$ [145, 146].

We define

$$\Phi_t(x,y,z) = \begin{cases} \Phi_i(x,y,z) + \Phi_r(x,y,z) + \Phi(x,y,z), & y \geqslant 0 \\ \Phi(x,y,z), & y < 0 \end{cases}. \tag{4.8}$$

and the Fourier transform pair over the variable $'z'$ as

$$\psi_t(x,y,w) = \int_{-\infty}^{\infty} \Phi_t(x,y,z) e^{-ikwz} dz, \tag{4.9}$$

$$\Phi_t(x,y,z) = \frac{k}{2\pi} \int_{-\infty}^{\infty} \psi_t(x,y,w)) e^{ikwz} dw. \tag{4.10}$$

Applying Eq. (4.9) on Eqs. (4.1), (4.2) and (4.4), we get

$$\left[(1-M^2)\frac{\partial^2}{\partial x^2} + 2ikM\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} + k^2(1-w^2)\right] \psi_t(x,y,w) = \delta(x-x_0)\delta(y-y_0)e^{-ikwz_o}, \tag{4.11}$$

subject to the boundary conditions

$$\left[\frac{\partial}{\partial y} \mp 2\beta M\frac{\partial}{\partial x} \pm ik\beta \mp \frac{i\beta M^2}{k}\frac{\partial^2}{\partial x^2}\right]\psi_t(x,0^{\pm},w) = 0 \qquad -\infty < x < p,\ q < x < \infty, \tag{4.12}$$

$$\left.\begin{array}{l} \frac{\partial \psi_t}{\partial y}(x,0^+,w) = \frac{\partial \psi_t}{\partial y}(x,0^-,w) \\ \psi_t(x,0^+,w) = \psi_t(x,0^-,w) \end{array}\right] \qquad p \leq x \leq q. \tag{4.13}$$

From Eq. (4.8) we can write

$$\psi_t(x,y,w) = \begin{cases} \psi_i(x,y,w) + \psi_r(x,y,w) + \psi(x,y,w), & y \geqslant 0 \\ \psi(x,y,w), & y < 0 \end{cases}. \tag{4.14}$$

For the subsonic flow, we can make the following real substitutions

$$x = \sqrt{1-M^2}X, \quad x_0 = \sqrt{1-M^2}X_0, \qquad \beta = \sqrt{1-M^2}B, \quad k = \sqrt{1-M^2}K,$$

$$y = Y, \quad y_0 = Y_0, \quad z_0 = Z_0, \tag{4.15}$$

and

$$\psi_t(x,y,w) = \Psi_t(X,Y,w)e^{-iKMX}. \tag{4.16}$$

Now using the Eqs. (4.15), (4.16) and the substitution

$$\gamma^2 = 1 - w^2\left(1-M^2\right), \tag{4.17}$$

Eqs. (4.11) and (4.14) can be written as

$$\left[\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + k^2\gamma^2\right]\Psi_t(X,Y,w) = \frac{\delta(X-X_0)\delta(Y-Y_0)e^{-iKMX-i\sqrt{1-M^2}KwZ_0}}{\sqrt{1-M^2}}. \tag{4.18}$$

Also

$$\Psi_t(X,Y,w) = \begin{cases} \Psi_i(X,Y,w) + \Psi_r(X,Y,w) + \Psi(X,Y,w), & y \geqslant 0 \\ \Psi(X,Y,w) & y < 0 \end{cases} . \tag{4.19}$$

where $\Psi_i(X,Y,w)$ is the incident wave corresponding to the non-homogeneous wave equation, $\Psi(X,Y,w)$ is the solution of the homogeneous wave equation, i.e. $\Psi(X,Y,w)$ is the diffracted wave and $\Psi_r(X,Y,w)$ is the reflected field. Now from Eqs. (4.18) and (4.19) , we can say that the incident field satisfies the equation

$$\left[\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + K^2\gamma^2\right] \Psi_i(X,Y,w) = \widetilde{a}\delta(X - X_0)\delta(Y - Y_0), \tag{4.20}$$

and the diffracted field satisfies the equation

$$\left[\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + K^2\gamma^2\right] \Psi(X,Y,w) = 0, \tag{4.21}$$

where

$$\widetilde{a} = \frac{e^{iKMX_0 - i\sqrt{1-M^2}KwZ_0}}{\sqrt{1-M^2}}. \tag{4.22}$$

Now, we define the Fourier transform over the variable $'X'$ as

$$\overline{\Psi}(\alpha,Y,w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(X,Y,w) e^{i\alpha X} dX,$$

$$\Psi(X,Y,w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{\Psi}(\alpha,Y,w) e^{-i\alpha X} d\alpha. \tag{4.23}$$

By applying the Fourier transform on Eq. (4.20), we get

$$\left(\frac{d^2}{dY^2} + \kappa^2\right) \overline{\Psi}_i(\alpha,Y,w) = \frac{\widetilde{a}\delta(Y - Y_0)e^{i\alpha X_0}}{\sqrt{2\pi}}, \tag{4.24}$$

where

$$\kappa^2 = \left(K^2\gamma^2 - \alpha^2\right), \tag{4.25}$$

is the square root function and can be considered as that branch which reduces to $+k$ when $\alpha = 0$ and the complex $\alpha-$plane is cut either from $\alpha = k$ to $\alpha = k\infty$ or from $\alpha = -k$ to $\alpha = -k\infty$ and α is the Fourier transform variable.

The solution of Eq. (4.24) is given in [13], as

$$\overline{\Psi}_i(\alpha, Y, w) = \frac{\widetilde{a}}{2\sqrt{2\pi}\kappa} e^{-i\alpha X_0 + i\kappa|Y-Y_0|}, \tag{4.26}$$

and taking inverse Fourier transform w.r.t. $'X'$, we get

$$\Psi_i(X, Y, w) = \frac{\widetilde{a}}{4\pi i} \int_{-\infty}^{\infty} \frac{e^{-i\alpha(X-X_0)+i\left(K^2\gamma^2-\alpha^2\right)^{1/2}|Y-Y_0|}}{\left(K^2\gamma^2 - \alpha^2\right)^{1/2}} d\alpha. \tag{4.27}$$

Now , we put

$$X - X_0 = R\cos\theta, \qquad Y - Y_0 = R\sin\theta, \qquad \alpha = -K\gamma\cos(\theta + it), \tag{4.28}$$

in Eq. (4.27), to represent it in Hankel's function and get

$$\Psi_i(X, Y, w) = \frac{-\widetilde{a}}{4i} H_0^1(K\gamma R), \tag{4.29}$$

where

$$R = \sqrt{(X - X_0)^2 + (Y - Y_0)^2}. \tag{4.30}$$

Now, using Eqs. (4.22) and (4.29), taking inverse Fourier transform in $'z'$ and putting

$$w = \frac{\cos(\theta_1 + i\eta)}{\sqrt{1 - M^2}}, \qquad \text{where } -\infty < \eta < \infty,\ 0 < \theta_1 < \pi, \tag{4.31}$$

$$\gamma = \sqrt{1 - w^2 (1 - M^2)} = \sin(\theta_1 + i\eta), \tag{4.32}$$

$$R = R_1 \sin\theta_1, \qquad z - z_0 = R_1 \cos\theta_1, \tag{4.33}$$

we get

$$\Phi_i(X, Y, Z) = \frac{ie^{-\frac{i\pi}{4}} e^{-iKM(X-X_0)} K}{8\pi i\sqrt{1-M^2}} \sqrt{\frac{2}{\pi K R_1 \sin\theta_1}} \int_{-\infty}^{\infty} e^{iKR_1 \cosh\eta} \sqrt{\sin(\theta_1 + i\eta)} d\eta, \tag{4.34}$$

and using the method of steepest descent, we obtain

$$\Phi_i(X, Y, Z) = \frac{e^{-iKM(X-X_0)}}{4\pi\sqrt{1-M^2}} \frac{e^{iKR_1}}{R_1}, \tag{4.35}$$

where

$$R_1 = \sqrt{(X - X_0)^2 + (Y - Y_0)^2 + (Z - Z_0)^2}. \tag{4.36}$$

Similarly, we obtain

$$\Phi_r(X, Y, Z) = \frac{e^{-iKM(X-X_0)}}{4\pi\sqrt{1-M^2}} \frac{e^{iK\hat{R}_1}}{\hat{R}_1}, \tag{4.37}$$

where

$$\hat{R}_1 = \sqrt{(X - X_0)^2 + (Y + Y_0)^2 + (Z - Z_0)^2}. \tag{4.38}$$

Now, we find the diffracted field, i.e., the solution of Eq. (4.21)

$$\left[\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + K^2\gamma^2\right] \Psi(X, Y, w) = 0, \tag{4.39}$$

subject to boundary conditions

$$\left[\frac{\partial}{\partial Y} - 2BM\frac{\partial}{\partial X} + iKB(1 + M^2) - \frac{iBM^2}{K(1 - M^2)} \frac{\partial^2}{\partial X^2}\right] \left[\begin{array}{c} \Psi_i(X, 0, w) + \Psi_r(X, 0, w) \\ + \Psi(X, 0^+, w) \end{array}\right] = 0, \tag{4.40}$$

$$\left[\frac{\partial}{\partial Y}+2BM\frac{\partial}{\partial X}-iKB(1+M^2)+\frac{iBM^2}{K(1-M^2)}\frac{\partial^2}{\partial X^2}\right]\Psi(X,0^-,w)=0,\quad \left.\begin{array}{l}-\infty<x<p,\\ q<x<\infty\end{array}\right],\tag{4.41}$$

where

$$\Psi(X,0^+,w)+\Psi_i(X,0,w)+\Psi_r(X,0,w)=\Psi(X,0^-,w),$$

$$\Psi(X,0^+,w)-\Psi(X,0^-,w)=-\left[\Psi_i(X,0,w)+\Psi_r(X,0,w)\right],\tag{4.42}$$

and

$$\left.\begin{array}{l}\frac{\partial\Psi}{\partial Y}(X,0^+,w)=\frac{\partial\Psi}{\partial Y}(X,0^-,w)\\ \Psi(X,0^+,w)=\Psi(X,0^-,w)\end{array}\right]\quad p\leq x\leq q.\tag{4.43}$$

Now, we introduce the spatial Fourier transform over the variable 'X' by

$$\overline{\Psi}(\alpha,Y,w)=\int_{-\infty}^{\infty}\Psi(X,Y,w)e^{i\alpha X}dX.\tag{4.44}$$

In order to accommodate three part boundary conditions on $Y=0$, we split $\overline{\Psi}(\alpha,Y,w)$ as

$$\overline{\Psi}(\alpha,Y,w)=\overline{\Psi}_+(\alpha,Y,w)e^{i\alpha q}+\overline{\Psi}_1(\alpha,Y,w)+\overline{\Psi}_-(\alpha,Y,w)e^{i\alpha p},\tag{4.45}$$

where

$$\overline{\Psi}_-(\alpha,Y,w)=\int_{-\infty}^{p}\Psi(X,Y,w)e^{i\alpha(X-p)}dX,$$

$$\overline{\Psi}_1(\alpha,Y,w)=\int_{p}^{q}\Psi(X,Y,w)e^{i\alpha X}dX,$$

$$\overline{\Psi}_-(\alpha,Y,w)=\int_{q}^{\infty}\Psi(X,Y,w)e^{i\alpha(X-q)}dX,\tag{4.46}$$

where $\overline{\Psi}_-(\alpha,Y,w)$ is regular for $\operatorname{Im}\alpha<\operatorname{Im}K$, $\overline{\Psi}_+(\alpha,Y,w)$ is regular for $\operatorname{Im}\alpha>-\operatorname{Im}K$ while $\overline{\Psi}_1(\alpha,Y,w)$ is an integral function which is analytic in the common region

$-\operatorname{Im} K < \alpha < \operatorname{Im} K$.

The Fourier transform of Eq. (4.21) w.r.t. $'X'$ results in

$$\left(\frac{d^2}{dY^2}+\kappa^2\right)\overline{\Psi}(\alpha,Y,w)=0, \tag{4.47}$$

where $\kappa=\sqrt{(K^2\gamma^2-\alpha^2)}$ and α- plane is cut such that $\operatorname{Im}\alpha>0$. The solution of Eq. (4.22) satisfying radiation condition is given by

$$\overline{\Psi}(\alpha,Y,w)=\left[\begin{array}{lll} A_1(\alpha,w)e^{i\kappa Y} & if & Y\geq 0 \\ A_2(\alpha,w)e^{-i\kappa Y} & if & Y<0 \end{array}\right., \tag{4.48}$$

where $A_1(\alpha,w)$and $A_2(\alpha,w)$ are unknown which are to be determined. We take the derivative of Eq. (4.48) and then using the Eq.(4.48) , we get

$$\Psi'(\alpha,Y,w)=\left[\begin{array}{lll} i\kappa\overline{\Psi}(\alpha,Y,w) & if & Y\geq 0 \\ -i\kappa\Psi(\alpha,Y,w & if & Y<0, \end{array}\right. \tag{4.49}$$

where prime " $'$ " denotes differentiation w.r.t. $'Y'$.

Using Eqs. (4.45) and (4.49), we get

$$\overline{\Psi}'_+(\alpha,Y,w)e^{i\alpha q}+\overline{\Psi}'_1(\alpha,Y,w)+\overline{\Psi}'_-(\alpha,Y,w)e^{i\alpha p}=$$

$$i\kappa\left[\overline{\Psi}_+(\alpha,Y,w)e^{i\alpha q}+\overline{\Psi}_1(\alpha,Y,w)+\overline{\Psi}_-(\alpha,Y,w)e^{i\alpha p}\right]\quad Y\geq 0, \tag{4.50}$$

and

$$\overline{\Psi}'_+(\alpha,Y,w)e^{i\alpha q}+\overline{\Psi}'_1(\alpha,Y,w)+\overline{\Psi}'_-(\alpha,Y,w)e^{i\alpha p}=$$

$$-i\kappa\left[\overline{\Psi}_+(\alpha,Y,w)e^{i\alpha q}+\overline{\Psi}_1(\alpha,Y,w)+\overline{\Psi}_-(\alpha,Y,w)e^{i\alpha p}\right]\qquad Y<0. \tag{4.51}$$

4.2 Transformation of Boundary Conditions

Taking the Fourier transform of Eq. (4.41) with respect to $'X'$ in the region, i.e., for $X < p$, we get

$$\overline{\Psi}'_{-}(\alpha, 0^{+}, w) = -iB\left[K\left(1+M^{2}\right)+2\alpha M+\frac{\alpha^{2}M^{2}}{K\left(1-M^{2}\right)}\right]\overline{\Psi}_{-}(\alpha, 0^{+}, w)$$

$$-iB\left[K\left(1+M^{2}\right)+2\alpha M+\frac{\alpha^{2}M^{2}}{K\left(1-M^{2}\right)}\right]\left[\overline{\Psi}_{i\_}(\alpha, 0, w)+\overline{\Psi}_{r\_}(\alpha, 0, w)\right], \quad (4.52)$$

and

$$\overline{\Psi}'_{-}(\alpha, 0^{-}, w) = iB\left[K\left(1+M^{2}\right)+2\alpha M+\frac{\alpha^{2}M^{2}}{K\left(1-M^{2}\right)}\right]\overline{\Psi}_{-}(\alpha, 0^{-}, w). \quad (4.53)$$

Now, the Fourier transform for $X > q$ using Eq. (4.41), gives

$$\overline{\Psi}'_{+}(\alpha, 0^{+}, w) = -iB\left[K\left(1+M^{2}\right)+2\alpha M+\frac{\alpha^{2}M^{2}}{K\left(1-M^{2}\right)}\right]\overline{\Psi}_{+}(\alpha, 0^{+}, w)$$

$$-iB\left[K\left(1+M^{2}\right)+2\alpha M+\frac{\alpha^{2}M^{2}}{K\left(1-M^{2}\right)}\right]\left[\overline{\Psi}_{i+}(\alpha, 0, w)+\overline{\Psi}_{r+}(\alpha, 0, w)\right], \quad (4.54)$$

and

$$\overline{\Psi}'_{+}(\alpha, 0^{-}, w) = iB\left[K\left(1+M^{2}\right)+2\alpha M+\frac{\alpha^{2}M^{2}}{K\left(1-M^{2}\right)}\right]\overline{\Psi}_{+}(\alpha, 0^{-}, w), \quad (4.55)$$

where

$$\overline{\Psi}_{i\_}(\alpha, 0, w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{p}\Psi_{i}(X, 0, w)e^{i\alpha(X-p)}dX, \quad (4.56)$$

$$\overline{\Psi}_{i+}(\alpha, 0, w) = \frac{1}{\sqrt{2\pi}}\int_{q}^{\infty}\Psi_{i}(X, 0, w)e^{i\alpha(X-q)}dX. \quad (4.57)$$

Now from Eq. (4.42), we get

$$\overline{\Psi}_1(\alpha,0^+,w)-\overline{\Psi}_1(\alpha,0^-,w)=-\left[\overline{\Psi}_{i_1}(\alpha,0,w)+\overline{\Psi}_{r_1}(\alpha,0,w)\right],$$

$$=-2\overline{\Psi}_{i_1}(\alpha,0,w)=-2\overline{\Psi}_{i_1}(\alpha,w) \tag{4.58}$$

and

$$\overline{\Psi}'_1(\alpha,0^+,w)=\overline{\Psi}'_1(\alpha,0^-,w)=\overline{\Psi}'_1(\alpha,w), \tag{4.59}$$

where

$$\overline{\Psi}_{i1}(\alpha,w)=\frac{1}{\sqrt{2\pi}}\int\limits_p^q \Psi_i(X,w)e^{i\alpha X}dX, \tag{4.60}$$

and note that, for the sake of simplification, we write

$$\overline{\Psi}_{i1}(\alpha,0,w)=\overline{\Psi}_{i1}(\alpha,w).$$

Making use of Eqs. (4.53) and (4.54), we get the following from Eqs. (4.50) and (4.51)

$$\left[-iB\left\{K\left(1+M^2\right)+2\alpha M+\frac{\alpha^2M^2}{K\left(1-M^2\right)}\right\}\overline{\Psi}_+(\alpha,0^+,w)\right.$$

$$\left.-iB\left\{K\left(1+M^2\right)+2\alpha M+\frac{\alpha^2M^2}{K\left(1-M^2\right)}\right\}\left\{\overline{\Psi}_{i+}(\alpha,w)+\overline{\Psi}_{r+}(\alpha,w)\right\}\right]e^{i\alpha q}$$

$$+\overline{\Psi}'_1(X,0^+,w)+\left[iB\left\{K\left(1+M^2\right)+2\alpha M+\frac{\alpha^2M^2}{K\left(1-M^2\right)}\right\}\overline{\Psi}_-(\alpha,0^+,w)\right.$$

$$\left.-iB\left\{K\left(1+M^2\right)+2\alpha M+\frac{\alpha^2M^2}{K\left(1-M^2\right)}\right\}\left\{\overline{\Psi}_{i\_}(\alpha,w)+\overline{\Psi}_{r\_}(\alpha,w)\right\}\right]e^{i\alpha p}$$

$$=i\kappa\left[\overline{\Psi}_+(\alpha,0^+,w)e^{i\alpha q}+\overline{\Psi}_1(\alpha,0^+,w)+\overline{\Psi}_-(\alpha,0^+,w)e^{i\alpha p}\right], \tag{4.61}$$

and

$$iB\left\{K\left(1+M^2\right)+2\alpha M+\frac{\alpha^2M^2}{K\left(1-M^2\right)}\right\}\overline{\Psi}_{-}(\alpha,0^-,w)e^{i\alpha q}+\overline{\Psi}_1'(\alpha,w)$$
$$+iB\left\{K\left(1+M^2\right)+2\alpha M+\frac{\alpha^2M^2}{K\left(1-M^2\right)}\right\}\overline{\Psi}_{-}(\alpha,0^-,w)e^{i\alpha p}$$

$$=-i\kappa\left[\overline{\Psi}_{+}(\alpha,0^-,w)e^{i\alpha q}+\overline{\Psi}_1(\alpha,0^-,w)+\overline{\Psi}_{-}(\alpha,0^-,w)e^{i\alpha p}\right]. \tag{4.62}$$

Adding Eqs. (4.61) and (4.62), we attain

$$\left[i\kappa+iB\left\{K\left(1+M^2\right)+2\alpha M+\frac{\alpha^2M^2}{K\left(1-M^2\right)}\right\}\right]\rho_{+}(\alpha,w)e^{i\alpha q}+\overline{\Psi}_1'(\alpha,w)$$

$$+\left[i\kappa+iB\left\{K\left(1+M^2\right)+2\alpha M+\frac{\alpha^2M^2}{K\left(1-M^2\right)}\right\}\right]\rho_{-}(\alpha,w)e^{i\alpha p}$$

$$= iB\left[K\left(1+M^2\right)+2\alpha M+\frac{\alpha^2M^2}{K\left(1-M^2\right)}\right]\left\{\overline{\Psi}_{i+}(\alpha,w)e^{i\alpha q}+\overline{\Psi}_{i\_}(\alpha,w)e^{i\alpha p}\right\}$$
$$-\frac{i\kappa}{2}\left[\overline{\Psi}_1(\alpha,0^+,w)-\overline{\Psi}_1(\alpha,0^-,w)\right] \tag{4.63}$$

where

$$\overline{\Psi}_{-}(\alpha,0^+,w)-\overline{\Psi}_{-}(\alpha,0^-,w)=2\rho_{\pm}(\alpha,w). \tag{4.64}$$

Using Eq. (4.29), asymptotic form of the Hankel function and the following substitutions

$$X_0=R_0\cos\theta_0,\qquad Y_0=R_0\sin\theta_0,\qquad R_0^2=X_0^2+Y_0^2, \tag{4.65}$$

we obtain

$$\Psi_i(X,Y,w)=\tilde{b}e^{-iK\gamma(X\cos\theta_0+Y\sin\theta_0)}, \tag{4.66}$$

$$\Psi_r(X,Y,w)=\tilde{b}e^{-iK\gamma(X\cos\theta_0-Y\sin\theta_0)}, \tag{4.67}$$

where

$$\tilde{b} = -\frac{\tilde{a}}{4i}\left(\frac{2}{\pi K\gamma R_0}\right)^{\frac{1}{2}} e^{i(K\gamma R_0 - \frac{\pi}{4})}. \tag{4.68}$$

Thus

$$\overline{\Psi}_{i+}(\alpha, w) = \frac{1}{\sqrt{2\pi}}\int_q^\infty \Psi_i(X, w)e^{i\alpha(X-q)}dX,$$

or

$$\overline{\Psi}_{i+}(\alpha, w) = \frac{ibe^{-iK\gamma\cos\theta_0 q}}{\sqrt{2\pi}(\alpha - K\gamma\cos\theta_0)}, \tag{4.69}$$

and

$$\overline{\Psi}_{i\_}(\alpha, w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^p \Psi_i(X, w)e^{i\alpha(X-p)}dX,$$

gives

$$\overline{\Psi}_{i\_}(\alpha, w) = \frac{-ibe^{-iK\gamma\cos\theta_0 p}}{\sqrt{2\pi}(\alpha - K\gamma\cos\theta_0)}. \tag{4.70}$$

Also

$$\overline{\Psi}_{i1}(\alpha, w) = \frac{1}{\sqrt{2\pi}}\int_p^q \Psi_i(X, w)e^{i\alpha X}dX,$$

gives

$$\overline{\Psi}_{i1}(\alpha, w) = -ibG(\alpha, w), \tag{4.71}$$

where

$$G(\alpha, w) = \frac{1}{\sqrt{2\pi}(\alpha - K\gamma\cos\theta_0)}\left[e^{i(\alpha - K\gamma\cos\theta_0)q} - e^{i(\alpha - K\gamma\cos\theta_0)p}\right]. \tag{4.72}$$

Using Eqs. (4.48) and (4.72), we get

$$\overline{\Psi}_1(\alpha, 0^+, w) - \overline{\Psi}_1(\alpha, 0^-, w) = 2ibG(\alpha, w). \tag{4.73}$$

and using Eqs. (4.48), (4.70) and (4.73) in Eq. (4.63), we get

$$\rho_+(\alpha, w)e^{i\alpha q} + \rho_-(\alpha, w)e^{i\alpha p} - \frac{i\overline{\Psi}_1'(\alpha, w)}{\kappa L(\alpha, w)} = ibG(\alpha, w), \tag{4.74}$$

where

$$L(\alpha, w) = \left[1 + \frac{B\left\{K\left(1+M^2\right) + 2\alpha M + \frac{\alpha^2 M^2}{K(1-M^2)}\right\}}{\kappa}\right] \tag{4.75}$$

We observe that Eq. (4.74) is a standard Wiener-Hopf equation. The solution of this equation is discussed in the next section.

4.3 Solution of the Wiener-Hopf equation

For the solution of Eq. (4.74), we factorize $\kappa L(\alpha, w)$

$$\kappa L(\alpha, w) = S(\alpha, w) = S_+(\alpha, w)S_-(\alpha, w), \tag{4.76}$$

where

$$L(\alpha, w) = L_+(\alpha, w)L_-(\alpha, w), \tag{4.77}$$

and

$$\kappa(\alpha, w) = \kappa_+(\alpha, w)\kappa_-(\alpha, w), \tag{4.78}$$

where $L_+(\alpha, w)$ and $\kappa_+(\alpha, w)$ are regular for $Im\ \alpha > -\ Im\ K$, i.e., upper half plane and $L_-(\alpha, w)$ and $\kappa_-(\alpha, w)$ are regular for $Im\ \alpha < Im\ K$, i.e., lower half plane.

With the help of Eqs. (4.76) (4.77) and (4.78), we can write

$$S_+(\alpha, w) = \kappa_+(\alpha, w)L_+(\alpha, w), \tag{4.79}$$

and

$$S_-(\alpha, w) = \kappa_-(\alpha, w)L_-(\alpha, w), \tag{4.80}$$

where $S_+(\alpha, w)$ is regular in the upper half plane and $S_-(\alpha, w)$ is regular in the lower half plane.

For the sake of simplification, let us write

$$A = ib \text{ and } K_m = K\gamma\cos\theta_0, \tag{4.81}$$

so that, we get

$$\rho_+(\alpha, w)e^{i\alpha q} + \rho_-(\alpha, w)e^{i\alpha p} - \frac{i\overline{\Psi}_1'(\alpha, w)}{S_+(\alpha, w)S_-(\alpha, w)} = AG(\alpha, w), \tag{4.82}$$

where

$$G(\alpha, w) = \frac{1}{\sqrt{2\pi}(\alpha - K_m)}\left[e^{i(\alpha-K_m)q} - e^{i(\alpha-K_m)p}\right]. \tag{4.83}$$

Multiplying Eq. (4.82) by $S_+(\alpha, w)e^{-i\alpha q}$, putting the value of $G(\alpha, w)$, adding and subtracting pole contribution and get

$$S_+(\alpha, w)\rho_+(\alpha, w) - \frac{i\Psi_1'(\alpha, w)}{S_-(\alpha, w)}e^{-i\alpha q} + \rho_-(\alpha, w)S_+(\alpha, w)e^{i\alpha(p-q)} =$$

$$\frac{Ae^{-iK_m q}}{\sqrt{2\pi}(\alpha - K_m)}\left[S_+(\alpha, w) - S_+(K_m, w)\right] + \begin{pmatrix} \frac{AS_+(K_m,w)e^{-iK_m q}}{\sqrt{2\pi}(\alpha-K_m)} \\ -\frac{AS_+(\alpha,w)e^{i\alpha(p-q)-iK_m p}}{\sqrt{2\pi}(\alpha-K_m)} \end{pmatrix}. \tag{4.84}$$

The first term on the left hand side is regular in the upper half plane and the terms whose gender is not known [13] can be written as

$$\rho_-(\alpha, w)S_+(\alpha, w)e^{i\alpha(p-q)} = U_+(\alpha, w) + U_-(\alpha, w), \tag{4.85}$$

and

$$\frac{AS_+(\alpha, w)e^{i\alpha(p-q)-iK_m p}}{\sqrt{2\pi}(\alpha - K_m)} = V_+(\alpha, w) + V_-(\alpha, w). \tag{4.86}$$

Invoking Eqs. (4.85) and (4.86) in Eq. (4.84), we obtain

$$S_+(\alpha, w)\rho_+(\alpha, w) - \frac{Ae^{-iK_m q}}{\sqrt{2\pi}\,(\alpha - K_m)}\left[S_+(\alpha, w) - S_+(K_m)\right] + U_+(\alpha, w) + V_+(\alpha, w)$$

$$= \frac{AS_+(K_m)e^{-iK_m q}}{\sqrt{2\pi}\,(\alpha - K_m)} + \frac{i\overline{\Psi}_1'(\alpha, w)e^{-i\alpha q}}{S_-(\alpha, w)} - U_-(\alpha, w) - V_-(\alpha, w). \tag{4.87}$$

Now multiplying Eq. (4.82) on both sides by $S_-(\alpha, w)e^{-i\alpha p}$ and putting the value of $G(\alpha, w)$, we get

$$S_-(\alpha, w)\rho_-(\alpha, w) + S_-(\alpha, w)\rho_+(\alpha, w)e^{i\alpha(q-p)} - \frac{i\overline{\Psi}_1'(\alpha, w)e^{-i\alpha p}}{S_+(\alpha, w)}$$

$$= \frac{AS_-(\alpha, w)e^{i\alpha(q-p)-iK_m q}}{\sqrt{2\pi}\,(\alpha - K_m)} - \frac{AS_-(\alpha, w)e^{-iK_m p}}{\sqrt{2\pi}\,(\alpha - K_m)}. \tag{4.88}$$

The first term on the left hand side of the above equation is regular in the lower half plane. We write, again by using [13],

$$S_-(\alpha, w)\rho_+(\alpha, w)e^{i\alpha(q-p)} = R_+(\alpha, w) + R_-(\alpha, w), \tag{4.89}$$

and

$$\frac{AS_-(\alpha, w)e^{i\alpha(q-p)-iK_m q}}{\sqrt{2\pi}\,(\alpha - K_m)} = S_+(\alpha, w) + S_-(\alpha, w), \tag{4.90}$$

and putting the values of Eqs.(4.89), (4.90) in Eq. (4.88), we get

$$\begin{aligned} & S_-(\alpha, w)\rho_-(\alpha, 0, w) + \frac{AS_-(\alpha, w)e^{-iK_m p}}{\sqrt{2\pi}\,(\alpha - K_m)} + R_-(\alpha) - S_-(\alpha) \\ = \; & \frac{i\Psi_1'(\alpha, 0, w)e^{-i\alpha p}}{S_+(\alpha, w)} - R_+(\alpha) + S_+(\alpha). \end{aligned} \tag{4.91}$$

We observe that the LHS of Eq. (4.87) and RHS of Eq. (4.91) are regular in $\operatorname{Im}\alpha > -\operatorname{Im}K$ and the RHS of Eq. (4.87) and the LHS of Eq. (4.91) are regular in $\operatorname{Im}\alpha < \operatorname{Im}K\cos\theta_0$. Hence, using the extended form of Liouville's theorem, each side of Eqs. (4.87) and (4.91) is equal to zero, i.e.,

$$S_+(\alpha,w)\rho_+(\alpha,w) - \frac{Ae^{-iK_m q}}{\sqrt{2\pi}\,(\alpha-K_m)}\left[S_+(\alpha,w)-S_+(K_m)\right] + U_+(\alpha,w) + V_+(\alpha,w) = 0, \tag{4.92}$$

and

$$S_-(\alpha,w)\rho_-(\alpha,w) + \frac{AS_-(\alpha,w)e^{-iK_m p}}{\sqrt{2\pi}\,(\alpha-K_m)} + R_-(\alpha,w) - S_-(\alpha,w) = 0, \tag{4.93}$$

where

$$U_+(\alpha,w) = \frac{1}{2\pi i}\int\limits_{-\infty+ic}^{\infty+ic} \frac{S_+(\nu,w)\rho_-(\nu,w)e^{i\nu(p-q)}}{(\nu-\alpha)}d\nu, \tag{4.94}$$

$$V_+(\alpha,w) = \frac{1}{2\pi i}\int\limits_{-\infty+ic}^{\infty+ic} \frac{AS_+(\nu,w)e^{i\nu(p-q)-iK_m p}}{\sqrt{2\pi}(\nu-K_m)(\nu-\alpha)}d\nu, \tag{4.95}$$

$$R_-(\alpha,w) = -\frac{1}{2\pi i}\int\limits_{-\infty+id}^{\infty+id} \frac{S_-(\nu,w)\rho_+(\nu,w)e^{i\nu(q-p)}}{(\nu-\alpha)}d\nu, \tag{4.96}$$

$$S_-(\alpha,w) = -\frac{1}{2\pi i}\int\limits_{-\infty+id}^{\infty+id} \frac{AS_-(\nu,w)e^{i\nu(q-p)-iK_m q}}{\sqrt{2\pi}(\nu-K_m)(\nu-\alpha)}d\nu, \tag{4.97}$$

such that $-K_2 < c < \tau < d < K_2\cos\theta_0$ and $\tau = \operatorname{Im}\alpha$ and $K_2 = \operatorname{Im}K$.

Now making use of Eqs. (4.94) and (4.95) in Eq. (4.92) and Eqs. (4.96) and (4.97) in Eq. (4.93), we get

$$S_+(\alpha,w)\rho_+^*(\alpha,w) + \frac{AS_+(K_m)e^{-iK_m q}}{\sqrt{2\pi}\,(\alpha-K_m)} + \frac{1}{2\pi i}\int\limits_{-\infty+ic}^{\infty+ic} \frac{S_+(\nu,w)\rho_-^*(\nu,w)e^{i\nu(p-q)}}{(\nu-\alpha)}d\nu = 0, \tag{4.98}$$

$$S_-(\alpha,w)\rho_-^*(\nu,w) - \frac{1}{2\pi i}\int\limits_{-\infty+id}^{\infty+id} \frac{S_-(\nu,w)\rho_+^*(\nu,w)e^{i\nu(q-p)}}{(\nu-\alpha)}d\nu = 0, \tag{4.99}$$

where

$$\rho_+(\alpha, w) - \frac{Ae^{-iK_m q}}{\sqrt{2\pi}\,(\alpha - K_m)} = \rho_+^*(\alpha, w), \tag{4.100}$$

$$\rho_-(\alpha, w) + \frac{Ae^{-iK_m p}}{\sqrt{2\pi}\,(\alpha - K_m)} = \rho_-^*(\alpha, w). \tag{4.101}$$

Now the assumption $0 < \theta_0 < \pi$ allows us to choose $'a'$ so that $-K_2 \cos\theta_0 < a < K_2 \cos\theta_0$. Also take $d = c = a$, replace ν by $-\nu$, α by $-\alpha$ in Eq. (4.98) and Eq. (4.99) respectively, and use

$$S_+(-\alpha, w) = S_-(\alpha, w), \tag{4.102}$$

to get

$$S_+(\alpha, w)F_+^*(\alpha, w) + \frac{\lambda}{2\pi i}\int_{-\infty+ia}^{\infty+ia} \frac{S_-(\nu, w)F_+^*(\nu, w)e^{i\nu(q-p)}}{(\nu+\alpha)}d\nu + \frac{AS_+(K_m)e^{-iK_m q}}{\sqrt{2\pi}\,(\alpha - K_m)} = 0, \tag{4.103}$$

where

$$F_+^*(\alpha, w) = \rho_+^*(\alpha, w) - \lambda\rho_-^*(-\alpha, w) \qquad \text{for } \lambda = \pm 1, \tag{4.104}$$

We can calculate $F_+^*(\alpha, w)$ from Eqs. (4.100), (4.101) and (4.104), as

$$F_+^*(\alpha, w) = F_+(\alpha, w) - \frac{Ae^{-iK_m q}}{\sqrt{2\pi}\,(\alpha - K_m)} + \lambda\frac{Ae^{-iK_m p}}{\sqrt{2\pi}\,(\alpha + K_m)}, \tag{4.105}$$

where

$$F_+(\alpha, w) = \rho_+(\alpha, w) - \lambda\rho_-(-\alpha, w). \tag{4.106}$$

Now using Eq. (4.105) in Eq. (4.103), we obtain

$$S_+(\alpha, w)\left[F_+(\alpha, w) - \frac{Ae^{-iK_m q}}{\sqrt{2\pi}\,(\alpha - K_m)} + \frac{A\lambda e^{-iK_m p}}{\sqrt{2\pi}\,(\alpha + K_m)}\right] + \frac{AS_+(K_m)e^{-iK_m q}}{\sqrt{2\pi}\,(\alpha - K_m)}$$

$$+\frac{\lambda}{2\pi i}\int\limits_{-\infty+ia}^{\infty+ia}\frac{S_-(\nu,w)e^{i\nu(q-p)}}{(\nu+\alpha)}\left[F_+(\nu,w)-\frac{Ae^{-iK_mq}}{\sqrt{2\pi}\,(\nu-K_m)}+\frac{A\lambda e^{-iK_mp}}{\sqrt{2\pi}\,(\nu+K_m)}\right]d\nu=0. \tag{4.107}$$

Now, consider the integral appearing in Eq. (4.107) i.e.,

$$I=\int\limits_{-\infty+ia}^{\infty+ia}\frac{S_-(\nu,w)e^{i\nu(q-p)}}{(\nu+\alpha)}\left[F_+(\nu,w)-\frac{Ae^{-iK_mq}}{\sqrt{2\pi}\,(\nu-K_m)}+\frac{A\lambda e^{-iK_mp}}{\sqrt{2\pi}\,(\nu+K_m)}\right]d\nu,$$

which can also be written as

$$I=I_1-\frac{Ae^{-iK_mq}}{\sqrt{2\pi}}I_2+\frac{A\lambda e^{-iK_mp}}{\sqrt{2\pi}}I_3, \tag{4.108}$$

where

$$I_1=\int\limits_{-\infty+ia}^{\infty+ia}\frac{S_-(\nu,w)F_+(\nu,w)e^{i\nu l}}{(\nu+\alpha)}d\nu, \tag{4.109}$$

$$I_2=\int\limits_{-\infty+ia}^{\infty+ia}\frac{S_-(\nu,w)e^{i\nu l}}{(\nu+\alpha)\,(\nu-K_m)}d\nu, \tag{4.110}$$

$$I_3=\int\limits_{-\infty+ia}^{\infty+ia}\frac{S_-(\nu,w)e^{i\nu l}}{(\nu+\alpha)\,(\nu+K_m)}d\nu, \tag{4.111}$$

and

$$l=(q-p).$$

We observe that $F_+(\alpha)$ is regular in $\tau>-K_2$, so we can expect that $F_+(\alpha)$ will have a branch point at $\alpha=-K$. But for large l, it is sufficiently far from the point $\alpha=+K$, which enables us to evaluate the above integrals in the asymptotic expansion [13]. Now

$$I_1=\int\limits_{-\infty+ia}^{\infty+ia}\frac{S_-(\nu,w)F_+(\nu,w)e^{i\nu l}}{(\nu+\alpha)}d\nu\approx 2\pi iT(\alpha,w)F_+(K\gamma,w)L_-(K,w),$$

where

$$T(\alpha, w) = \frac{E_0 W_0\left[-i(K\gamma + \alpha)l\right]}{2\pi i},$$

$$E_0 = 2\, e^{\frac{i\pi}{4}}\, e^{iK\gamma l}\, l^{-1/2} h_0,$$

$$h_0 = e^{\frac{i\pi}{4}},$$

and

$$W_0(z) = \int_0^\infty \frac{u^{-1/2}e^{-u}}{u+z}du = \frac{\sqrt{\pi}}{2}\; e^{z/2}\; z^{-1/4}\; W_{-3/4,1/4}(z),$$

which is obtained by using

$$W_{j-\frac{1}{2}}(z) = \int_0^\infty \frac{u^j e^{-u}}{u+z}du = \Gamma(j+1)e^{z/2}z^{\frac{1}{2}j-\frac{1}{2}}W_{-\frac{1}{2}(j+1),\frac{1}{2}j}(z),$$

where

$$z = -i(K+\alpha)l,$$

and $W_{i,j}(z)$ is a Whittaker function. The other integrals are calculated as

$$I_2 = \int_{-\infty+ia}^{\infty+ia} \frac{S_-(\nu,w)e^{i\nu l}}{(\nu+\alpha)\,(\nu-K_m)}d\nu \approx 2\pi i\left[\frac{S_-(K_m,w)e^{iK_m l}}{(\alpha+K_m)} + R_2(\alpha,w)\right],$$

and

$$I_3 = \int_{-\infty+ia}^{\infty+ia} \frac{S_-(\nu,w)e^{i\nu l}}{(\nu+\alpha)\,(\nu+K_m)}d\nu \approx 2\pi i R_1(\alpha,w),$$

where

$$R_{1,2}(\alpha,w) = \frac{E_0\left[W_0\left\{-i(K\gamma \pm K_m)l\right\} - W_0\left\{-i(K\gamma+\alpha)l\right\}\right]}{2\pi i(\alpha \mp K_m)},$$

which is obtained by using

$$R_{1,2}(\alpha) = \frac{E_r\left[W_r\left\{-i(K \pm K\cos\theta_0)l\right\} - W_r\left\{-i(K+\alpha)l\right\}\right]}{2\pi i(\alpha \mp K\cos\theta_0)},$$

Thus, Eq. (4.108) takes the form

$$I = 2\pi i \begin{bmatrix} T(\alpha, w)F_+(K\gamma, w)L_-(K, w) - \frac{Ae^{-iK_m q}}{\sqrt{2\pi}} R_2(\alpha, w) \\ -\frac{Ae^{-iK_m p}S_-(K_m)}{\sqrt{2\pi}(\alpha + K_m)} + \lambda \frac{Ae^{-iK_m p}}{\sqrt{2\pi}} R_1(\alpha, w) \end{bmatrix}. \tag{4.112}$$

Using Eq. (4.112) in Eq. (4.107), we get

$$S_+(\alpha, w)F_+(\alpha, w) = \frac{A}{\sqrt{2\pi}}\left[G_1(\alpha, w) - \lambda G_2(\alpha, w)\right] - \lambda T(\alpha, w)F_+(K\gamma, w)L_-(K, w), \tag{4.113}$$

where

$$G_1(\alpha, w) = P_1(\alpha, w)e^{-iK_m q} - R_1(\alpha, w)e^{-iK_m p}, \tag{4.114}$$

$$G_2(\alpha, w) = P_2(\alpha, w)e^{-iK_m p} - R_2(\alpha, w)e^{-iK_m q}, \tag{4.115}$$

and

$$P_1(\alpha, w) = \frac{S_+(\alpha, w) - S_+(K_m)}{(\alpha - K_m)}, \tag{4.116}$$

$$P_2(\alpha, w) = \frac{S_+(\alpha, w) - S_+(K_m)}{(\alpha + K_m)}. \tag{4.117}$$

Now $F_+(K\gamma, w)$ can be calculated from Eq. (4.113) by putting $\alpha = K\gamma$

$$F_+(K\gamma, w) = \frac{A}{\sqrt{2\pi}}\left[\frac{G_1(K\gamma, w) - \lambda G_2(K\gamma, w)}{S_+(K\gamma, w) + \lambda T(K\gamma, w)L_-(K, w)}\right]. \tag{4.118}$$

Putting this value in Eq. (4.113), we get

$$F_+(\alpha, w) = \frac{A}{\sqrt{2\pi}S_+(\alpha, w)}\left[G_1(\alpha, w) - \lambda G_2(\alpha, w)\right]$$

$$-\frac{A\lambda T(\alpha, w)L_-(K, w)}{\sqrt{2\pi}S_+(\alpha, w)}\left[\frac{G_1(K\gamma, w) - \lambda G_2(K\gamma, w)}{S_+(K\gamma, w) + \lambda T(K\gamma, w)L_-(K, w)}\right]. \tag{4.119}$$

For $\lambda = -1$, from Eqs.(4.106) and (4.119), we get

$$\rho_+(\alpha, 0) + \rho_-(-\alpha, w) = \frac{A}{\sqrt{2\pi}S_+(\alpha, w)}\left[G_1(\alpha, w) + G_2(\alpha, w)\right]$$

$$+\frac{AT(\alpha, w)L_-(K, w)}{\sqrt{2\pi}S_+(\alpha, w)}\left[\frac{G_1(K\gamma, w) + G_2(K\gamma, w)}{S_+(K\gamma, w) - T(K\gamma, w)L_-(K, w)}\right], \tag{4.120}$$

and for $\lambda = 1$ from Eqs. (4.106) and (4.119), we get

$$\rho_+(\alpha, w) - \rho_-(-\alpha, w) = \frac{A}{\sqrt{2\pi}S_+(\alpha, w)}\left[G_1(\alpha, w) - G_2(\alpha, w)\right]$$

$$-\frac{AT(\alpha, w)L_-(K, w)}{\sqrt{2\pi}S_+(\alpha, w)}\left[\frac{G_1(K\gamma, w) - G_2(K\gamma, w)}{S_+(K\gamma, w) + T(K\gamma, w)L_-(K, w)}\right]. \tag{4.121}$$

Addition of Eqs. (4.120) and (4.121) results in

$$2\rho_+(\alpha, w) = \frac{2AG_1(\alpha, w)}{\sqrt{2\pi}S_+(\alpha, w)} + \frac{2AT(\alpha, w)L_-(K, w)}{\sqrt{2\pi}S_+(\alpha, w)S_+(K\gamma, w)\left(1 - \frac{T^2(K\gamma, w)L_-^2(K, w)}{S_+^2(K\gamma, w)}\right)}$$

$$\times\left[G_2(K\gamma, w) + \frac{G_1(K\gamma, w)T(K\gamma, w)L_-(K, w)}{S_+(K\gamma, w)}\right]. \tag{4.122}$$

Now, if we subtract Eq. (4.121) from Eq. (4.120) with $\alpha = -\alpha$, we get $\rho_-(\alpha, w)$ or if we replace G_1 by G_2 and G_2 by G_1 and also change sign of α in Eq. (4.121) and use $S_+(-\alpha) = S_-(\alpha, w)$, we get $\rho_-(\alpha, w)$,i.e.,

$$2\rho_-(\alpha, w) = \frac{2AG_2(-\alpha, w)}{\sqrt{2\pi}S_-(\alpha, w)} + \frac{2AT(-\alpha, w)L_-(K, w)}{\sqrt{2\pi}S_-(\alpha, w)S_+(K\gamma, w)\left(1 - \frac{T^2(K\gamma, w)L_-^2(K, w)}{S_+^2(K\gamma, w)}\right)}$$

$$\times\left[G_1(K\gamma, w) + \frac{G_2(K\gamma, w)T(K\gamma, w)L_-(K, w)}{S_+(K\gamma, w)}\right]. \tag{4.123}$$

Using Eq. (4.64) in Eqs. (4.122) and (4.123), we get

$$\overline{\Psi}_{+}(\alpha,0^{+},w)-\overline{\Psi}_{+}(\alpha,0^{-},w)=\frac{2A}{\sqrt{2\pi}S_{+}(\alpha,w)}\left[G_{1}(\alpha,w)+C_{1}T(\alpha,w)\right], \quad (4.124)$$

$$\overline{\Psi}_{-}(\alpha,0^{+},w)-\overline{\Psi}_{-}(\alpha,0^{-},w)=\frac{2A}{\sqrt{2\pi}S_{-}(\alpha,w)}\left[G_{2}(-\alpha,w)+C_{2}T(-\alpha,w)\right], \quad (4.125)$$

where

$$C_{1}(K\gamma,w)=\frac{L_{-}(K,w)}{S_{+}(K\gamma,w)\left(1-\frac{T^{2}(K\gamma,w)L_{-}^{2}(K,w)}{S_{+}^{2}(K\gamma,w)}\right)}\left[G_{2}(K\gamma,w)+\frac{G_{1}(K\gamma,w)T(K\gamma,w)L_{-}(K,w)}{S_{+}(K\gamma,w)}\right], \quad (4.126)$$

$$C_{2}(K\gamma,w)=\frac{L_{-}(K,w)}{S_{+}(K\gamma,w)\left(1-\frac{T^{2}(K\gamma,w)L_{-}^{2}(K,w)}{S_{+}^{2}(K\gamma,w)}\right)}\left[G_{1}(K\gamma,w)+\frac{G_{2}(K\gamma,w)T(K\gamma,w)L_{-}(K,w)}{S_{+}(K\gamma,w)}\right] \quad (4.127)$$

Now from Eq. (4.45) and (4.48), we obtain

$$A_{1}(\alpha,w)=\overline{\Psi}_{+}(\alpha,0^{+},w)e^{i\alpha q}+\overline{\Psi}_{1}(\alpha,0^{+},w)+\overline{\Psi}_{-}(\alpha,0^{+},w)e^{i\alpha p}, \quad (4.128)$$

$$A_{2}(\alpha,w)=\overline{\Psi}_{+}(\alpha,0^{-},w)e^{i\alpha q}+\overline{\Psi}_{1}(\alpha,0^{-},w)+\overline{\Psi}_{-}(\alpha,0^{-},w)e^{i\alpha p}, \quad (4.129)$$

where $A_1(\alpha,w)$ corresponds to $Y\geqslant 0$ and $A_2(\alpha,w)$ corresponds to $Y<0$.

Using Eqs. (4.48), (4.49), (4.73), (4.124) and (4.125), we get

$$A_{1}(\alpha,w)=-A_{2}(\alpha,w)=\frac{A}{\sqrt{2\pi}S_{+}(\alpha,w)}\left[G_{1}(\alpha,w)+C_{1}(K\gamma,w)T(\alpha,w)\right]e^{i\alpha q}+AG(\alpha,w)+\frac{A}{\sqrt{2\pi}S_{-}(\alpha,w)}\left[G_{2}(-\alpha,w)+C_{2}(K\gamma,w)T(-\alpha,w)\right]e^{i\alpha p}, \quad (4.130)$$

Using Eqs. (4.27) and (4.130), and taking the inverse Fourier transform w.r.t. $'X'$, we get

$$\Psi(X,Y,w) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left[\frac{A}{S_+(\alpha,w)}\left\{G_1(\alpha,w) + C_1(K\gamma,w)T(\alpha,w)\right\}e^{i\alpha q}\right]e^{-i\alpha X+i\kappa|Y|}d\alpha$$

$$+\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left[\frac{A}{S_-(\alpha,w)}\left[G_2(-\alpha,w) + C_2(K\gamma,w)T(-\alpha,w)\right]e^{i\alpha p}\right]e^{-i\alpha X+i\kappa|Y|}d\alpha. \tag{4.131}$$

From Eqs. (4.83), (4.114), (4.115), (4.116) and (4.117), we get

$$\Psi(X,Y,w) = \frac{A}{2\pi}\int\limits_{-\infty}^{\infty}\left[\left\{\begin{array}{c} -\frac{e^{i(\alpha-K_m)q}S_+(K_m)}{S_+(\alpha,w)(\alpha-K_m)} - \frac{e^{-iK_m p}R_1(\alpha,w)e^{i\alpha q}}{S_+(\alpha,w)} \\ +\frac{C_1(K\gamma,w)T(\alpha,w)e^{i\alpha q}}{S_+(\alpha,w)} - \frac{e^{i(\alpha-K_m)p}S_-(K_{m,w})}{S_-(\alpha.w)(\alpha-K_m)} \\ -\frac{e^{-iK_m q}R_2(-\alpha,w)e^{i\alpha p}}{S_-(\alpha,w)} + \frac{C_2(K\gamma,w)T(-\alpha,w)e^{i\alpha p}}{S_-(\alpha,w)} \end{array}\right\}\right]e^{-i\alpha X}e^{+i\kappa|Y|}d\alpha. \tag{4.132}$$

We can break up the field $\Psi(X,Y,w)$ into two parts

$$\Psi(X,Y,w) = \Psi^{sep}(X,Y,w) + \Psi^{int}(X,Y,w). \tag{4.133}$$

where

$$\Psi^{sep}(X,Y,w) = \frac{-ib}{2\pi}\int\limits_{-\infty}^{\infty}\frac{e^{i(\alpha-K_m)d}S_+(K_{m,}w)e^{-i\alpha X+i\kappa|Y|}}{S_+(\alpha,w)(\alpha-K_m)}d\alpha$$

$$-\frac{ib}{2\pi}\int\limits_{-\infty}^{\infty}\frac{e^{-i(\alpha-K_m)d}S_-(K_m)e^{-i\alpha X+i\kappa|Y|}}{S_-(\alpha,w)(\alpha-K_m)}d\alpha, \tag{4.134}$$

and

$$\Psi^{int}(X,Y,w) = \frac{ib}{2\pi}\int\limits_{-\infty}^{\infty}\left[\begin{array}{c} -\frac{e^{-iK_m d}R_1(\alpha,w)e^{i\alpha d}}{S_+(\alpha,w)} + \frac{C_1(K\gamma,w)T(\alpha,w)e^{i\alpha d}}{S_+(\alpha,w)} \\ +\frac{e^{-iK_m d}R_2(-\alpha,w)e^{-i\alpha d}}{S_-(\alpha,w)} - \frac{C_2(K\gamma,w)T(-\alpha,w)e^{-i\alpha d}}{S_-(\alpha,w)} \end{array}\right]e^{-i\alpha X+i\kappa|Y|}d\alpha. \tag{4.135}$$

Here $\Psi^{sep}(X,Y,w)$ consists of two parts each representing the diffracted field produced by the edges at $x=p$ and $x=q$, respectively, as though the other edge was absent while

$\Psi^{int}(X, Y, w)$ gives the interaction of one edge upon the other.

4.4 Far Field Approximation

The far field may now be calculated by evaluating the integrals in Eqs. (4.134) and (4.135) asymptotically. For that put $X = R_2 \cos\theta_2$, $Y = R_2 \sin\theta_2$ and deform the contour by the transformation $\alpha = -K\gamma\cos(\theta_2 + it)$, where $(0 < \theta_2 < \pi,\ -\infty < t < \infty)$.

$$\Psi^{sep}(X, Y, w) = \frac{-b}{2\pi}\int_{-\infty}^{\infty} f_1\left[-K\gamma\cos(\theta_2 + it)\right]\left[\exp(iK\gamma R_2\cosh t)\right]\sin(\theta_2 + it)dt, \quad (4.136)$$

where

$$f_1\left[-K\gamma\cos(\theta_2 + it)\right] = \frac{K\gamma}{\left[K\gamma\cos(\theta_2 + it) + K_m\right]}\left[\frac{e^{-i[K\gamma\cos(\theta_2+it)+K_m]d}S_+(K_m, w)}{S_+\left[-K\gamma\cos(\theta_2 + it), w\right]}\right.$$

$$\left.+\frac{e^{i[K\gamma\cos(\theta_2+it)+K_m]d}S_-(K_m, w)}{S_-\left[-K\gamma\cos(\theta_2 + it), w\right]}\right]. \quad (4.137)$$

Similarly, we get

$$\Psi^{int}(X, Y, w) = \frac{-b}{2\pi}\int_{-\infty}^{\infty} f_2\left[-K\gamma\cos(\theta_2 + it)\right]\left[\exp(iK\gamma R_2\cosh t)\right]\sin(\theta_2 + it)dt, \quad (4.138)$$

where

$$f_2\left[-K\gamma\cos(\theta_2 + it)\right] = K\gamma\left[-\frac{R_1\left[-K\gamma\cos(\theta_2 + it)\right]e^{-i[K\gamma\cos(\theta_2+it)+K_m]d}}{S_+\left[-K\gamma\cos(\theta_2 + it), w\right]}\right.$$

$$+\frac{C_1(K\gamma, w)T\left[-K\gamma\cos(\theta_2 + it)\right]e^{-iK\gamma\cos(\theta_2+it)d}}{S_+\left[-K\gamma\cos(\theta_2 + it), w\right]} + \frac{R_2\left[K\gamma\cos(\theta_2 + it)\right]e^{i[K\gamma\cos(\theta_2+it)-K_m]d}}{S_-\left[-K\gamma\cos(\theta_2 + it), w\right]}$$

$$\left.-\frac{C_2(K\gamma, w)T\left[K\gamma\cos(\theta_2 + it)\right]e^{iK\gamma\cos(\theta_2+it)d}}{S_-\left[-K\gamma\cos(\theta_2 + it), w\right]}\right]. \quad (4.139)$$

Hence, for large R_2, Eqs. (4.136) and (4.138) become

$$\Psi^{sep}(X,Y,w) = \frac{-ib\sin\theta_2}{\sqrt{2\pi K\gamma R_2}} f_1(-K\gamma\cos\theta_2)\exp i(K\gamma R_2 - \frac{\pi}{4}), \tag{4.140}$$

and

$$\Psi^{int}(X,Y,w) = \frac{-ib\sin\theta_2}{\sqrt{2\pi K\gamma R_2}} f_2(-K\gamma\cos\theta_2)\exp i(K\gamma R_2 - \frac{\pi}{4}), \tag{4.141}$$

where $f_1(-K\gamma\cos\theta_2)$ and $f_2(-K\gamma\cos\theta_2)$ are given by setting $t = 0$ in Eqs. (4.137) and (4.139) respectively.

Now using Eqs. (4.140) and (4.141) in Eq. (4.119), we obtain

$$\psi^{sep}(x,y,w) = \frac{-ib\sin\theta_2 e^{-iKMX}}{\sqrt{2\pi K\gamma r}} f_1(-K\gamma\cos\theta_2)\exp\left[i(K\gamma R_2 - \frac{\pi}{4})\right], \tag{4.142}$$

$$\psi^{int}(x,y,w) = \frac{-ib\sin\theta_2 e^{-iKMX}}{\sqrt{2\pi K\gamma R_2}} f_2(-K\gamma\cos\theta_2)\exp\left[i(K\gamma R_2 - \frac{\pi}{4})\right]. \tag{4.143}$$

Using the values of Eqs. (4.125) and (4.68) we get

$$\psi^{sep}(x,y,w) = \frac{-i\sin\theta_2 e^{-iKM(X-X_0)-iK\sqrt{1-M^2}wZ_0}}{4\pi K\gamma\sqrt{1-M^2}\sqrt{R_0R_2}} f_1(-K\gamma\cos\theta_2)\exp\left[iK\gamma(R_0+R_2)\right], \tag{4.144}$$

and

$$\psi^{int}(x,y,w) = \frac{-i\sin\theta_2 e^{-iKM(X-X_0)-iK\sqrt{1-M^2}wZ_0}}{4\pi K\gamma\sqrt{1-M^2}\sqrt{R_0R_2}} f_2(-K\gamma\cos\theta_2)\exp\left[iK\gamma(R_0+R_2)\right]. \tag{4.145}$$

Taking inverse Fourier transform w.r.t. z and using $k = K\sqrt{1-M^2}$, we obtain

$$\Phi^{sep}(x,y,z) = \frac{-i\sin\theta_2 e^{-iKM(X-X_0)}}{8\pi^2\sqrt{R_0R_2}} \int_{-\infty}^{\infty} \begin{bmatrix} \frac{f_1(-K\gamma\cos\theta_2)e^{iKw\sqrt{1-M^2}(Z-Z_0)}}{\gamma} \\ \times\exp\left[iK\gamma(R_0+R_2)\right] \end{bmatrix} dz. \tag{4.146}$$

Similarly

$$\Phi^{int}(x,y,z) = \frac{-i\sin\theta_2 e^{-iKM(X-X_0)}}{8\pi^2\sqrt{R_0R_2}} \int_{-\infty}^{\infty} \left[\begin{array}{c} \frac{f_2(-K\gamma\cos\theta_2)e^{iKw\sqrt{1-M^2}(Z-Z_0)}}{\gamma} \\ \times \exp\left[iK\gamma(R_0+R_2)\right] \end{array} \right] dz. \quad (4.147)$$

Now putting

$$w = \frac{\cos(\beta + i\eta)}{\sqrt{1-M^2}},$$

$$\gamma = \sqrt{1 - w^2(1-M^2)} = \sin(\beta + i\eta),$$

$$-\infty < \eta < \infty \qquad\qquad 0 < \beta < \pi,$$

$$Z - Z_0 = R_{12}\cos\beta \qquad R_0 + R_2 = R_{12}\sin\beta \qquad R_{12} = \sqrt{(Z-Z_0)^2 + (R_0+R_2)^2},$$

and using the steepest descent method, we get

$$\Phi^{sep}(x,y,z) = \frac{-i\sin\theta_2 e^{-iKM(X-X_0)} f_1(-K\sin\beta\cos\theta_2)e^{i(KR_{12}-\pi/4)}}{8\pi\sqrt{1-M^2}\sqrt{R_0R_2}} \sqrt{\frac{2}{\pi K R_{12}}}, \quad (4.148)$$

and

$$\Phi^{int}(x,y,z) = \frac{-i\sin\theta_2 e^{-iKM(X-X_0)} f_2(-K\sin\beta\cos\theta_2)e^{i(KR_{12}-\pi/4)}}{8\pi\sqrt{1-M^2}\sqrt{R_0R_2}} \sqrt{\frac{2}{\pi K R_{12}}}. \quad (4.149)$$

4.5 Graphical Results

In this section, we shall present illustrative numerical examples of the scattered far field for various physical parameters and investigate the scattering characteristics of the diffraction phenomenonin in detail. The absorbing parameter B is to be taken such that $\operatorname{Re} B > 0$ for an absorbing surface and $|M| < 1$ for a subsonic flow. Positive Mach number indicates that the stream flow is from left to right and negative Mach number indicates that the stream flow is from right to left. The following situations are considered:

(i) When the source is fixed in one position, $\theta_0 = 45^o$, while K and θ are allowed to vary.

(ii) When the source is fixed in one position, $\theta_0 = 45^o$, while M and θ are allowed to vary.

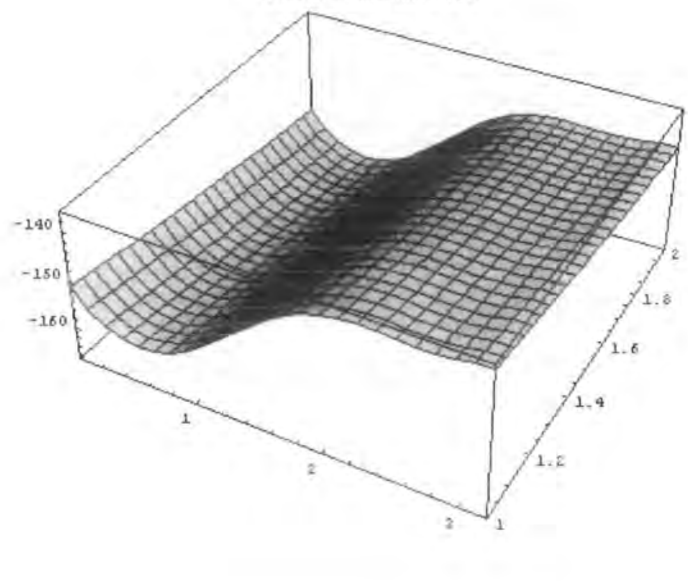
(iii) When the source is fixed in one position, $\theta_0 = 45^o$, while B and θ are allowed to vary.

For all the situations, $\theta_0 = 45^o$, see figures (1), (2), (3), (4), (5) and (6) which show that the field in the region $0 < \theta < \pi$, is most affected by the changes in K, M and B. The main features of the graphical results are as follows:

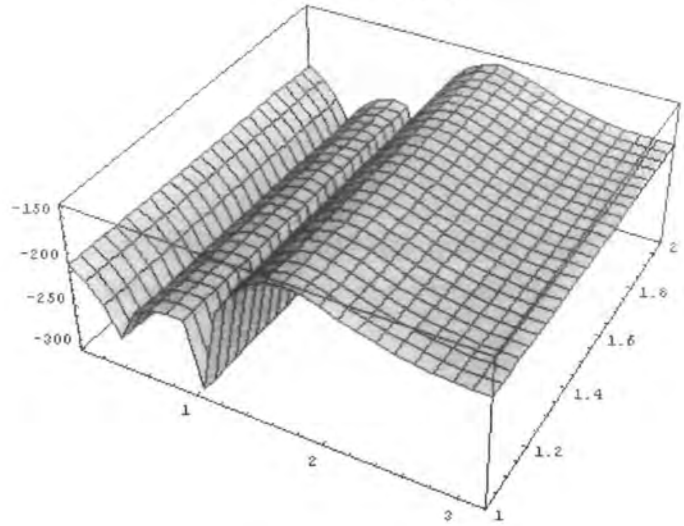
(a) In Figs.(1) and (2), by increasing the values of K, the number of oscillations increases, which is quite natural.

(b) In Figs.(3) and (4), as the value of M increases by fixing all other parameters, the amplitude of the separated field decreases, i.e., the sound intensity reduces.

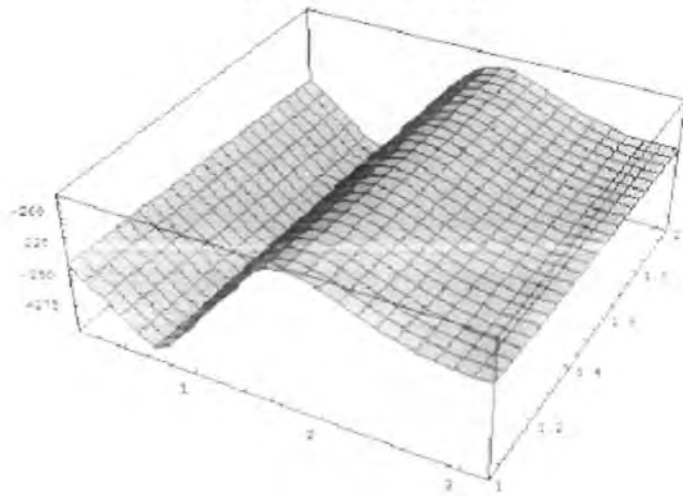
(c) In Figs.(5) and (6), by increasing the absorbing parameter B, again the amplitude of the separated field decreases, i.e., the sound intensity reduces, by fixing all other parameters.



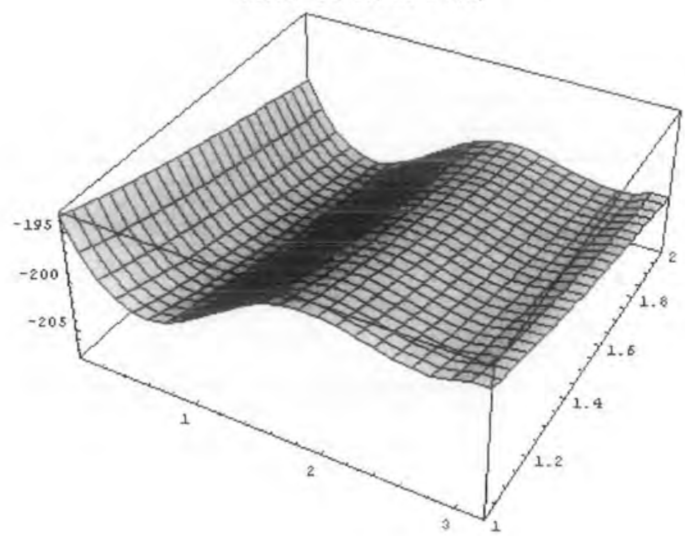
k=? d = 10, θ0 = π/4

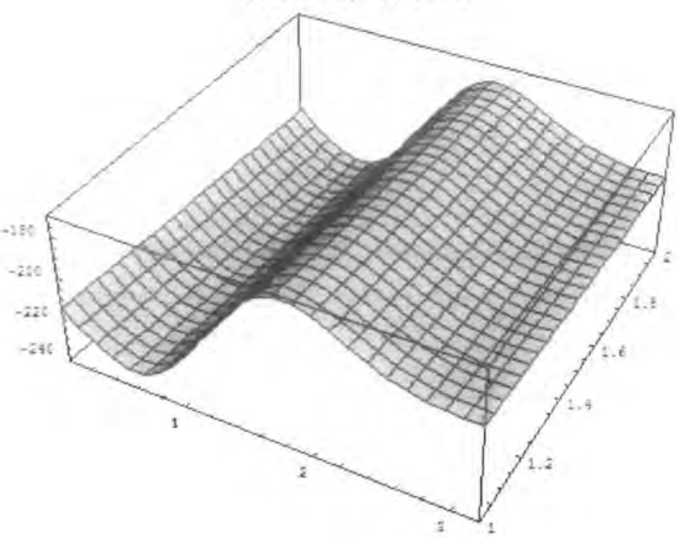


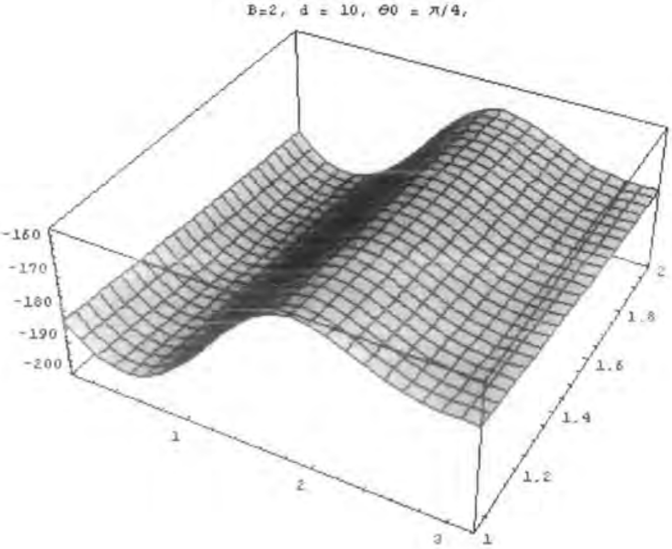
M=0.1, d = 10, θ0 = π/4,



M=0.5, d = 10, θ0 = π/4,







4.6 Conclusion

We have studied the problem of a point source diffraction of an acoustic wave by a slit satisfying Myers' condition. While using Fourier transform technique, the boundary value problem is reduced to the functional Wiener-Hopf equation whose solution is obtained by considering the factorization of the kernel. The Myers' condition give rise to a corrective term $\left(\frac{BM\cos^2\theta}{\sin\theta(1-M^2)}\right)$ in the (third term of the kernel) solution. If this corrective term is ignored, the results of [135] can be achieved which shows that our results vary from [135] by a multiplicative factor. The total field is shown to be the sum of the fields produced by the two edges of the slit and a field caused due to the interaction of the two edges. This solution will contribute to the analysis of noise reduction. We have taken the source lying far from the slit whereas the reflected sound is measured at a point far from the slit to know, how successfully the barrier reduces the sound transmission despite the presence of the slit and as to how the absorption of the barrier makes its presence felt. The above results also take care of acoustic diffraction from a slit with a rigid barrier, in a moving fluid, which can be obtained by putting $\beta = 0$. Besides, the result for the still fluid can be found by putting $M = 0$. Further, a numerical solution is obtained for the problem showing the effect of sundry parameters (for various values of Mach number M and the absorbing parameter B against the velocity potential). The graphs show a clear variation of the velocity potential against these parameters. It is found that if a line source is removed to a far-off distance i.e., infinity, the graphical results of plane wave situation [135] can be recovered. The Myers' conditions give better attenuation results for the separated and inter-active diffracted fields as compared to Ingards' conditions. This route of solution of diffraction problem is more rigorous and involves tedious calculations. The mathematical importance of the above solved problem rests upon the fact that we have to find four unknown functions $\left(\rho_{\pm}(\alpha), \overline{\Psi}_{\pm}(\alpha)\right)$ against the case of strip geometry where we have to find the two unknown functions. The above solved boundary value problem (a three part boundary value problem) is a very substantial one in the diffraction theory, obtained through nerve-testing mathematical stages. Two diffracted fields i.e., one from each edge and the other from the interaction of one edge upon the other were obtained. The above solved problem of a slit in a moving fluid will help understand acoustic diffraction and will go a step further to complete the discussion for the slit geometry. To the best of author's knowledge, the above mentioned problem has not yet been solved by using the W-H technique.

Chapter 5

SOUND DUE TO AN IMPULSIVE LINE SOURCE

In this chapter, we shall discuss the problem of diffraction caused due to an impulse line source by an absorbing half plane, in the presence of a subsonic flow, satisfying Myers' impedance condition [133]. The problem of acoustic diffraction by an absorbing half plane in a moving fluid using Myers' condition was discussed by Ahmad [46]. He considered the diffraction of sound waves by a semi-infinite absorbing half plane, when the whole system was in a moving fluid. In [46], the time dependence was considered to be harmonic in nature and was suppressed throughout the analysis. In this problem, while we have taken into account the time dependence throughout, we apply the temporal Fourier transform to obtain the transform function in the transformed plane using the Wiener-Hopf technique [13] and the method of modified stationary phase [12]. When the transform function is available, an inverse temporal Fourier transform can be applied to obtain the results in the time domain. The time-dependence of field is introduced by a delta function with temporal and spatial Fourier transform. In line with the solution for diffracted field, asymptotic solutions are sought for spatial integrals in far-field approximation. The method to obtain the solution is firstly the transformed plane which gives the transfer function. The impulse response is then calculated by convoluting this function with appropriate time-dependent function after taking inverse Fourier transform, for example, Tolstoy [149], Lakhtakia [150,151] and Sun [152]. Here, inverse Fourier transform over the time variable is calculated through Convolution theorem which also provides the space-time diffracted field data. We have shown as to how the frequency of incident wave is affected by the amplitude of the diffracted field in different limiting positions. Also, the effects of different parameters on the field can be seen through the graphs.

Transient nature of the field is an important area in the theory of acoustic diffraction and provides a more complete picture of the wave phenomenon. The effect of the transient nature of the field has been taken into account by many scientist like Rienstra [117], Lakhtakia et.al. [153–155], Ayub et.al [74] and Asghar et.al. [119]. Fourier or Laplace transform route from time-harmonic regime may be used for transient solutions; however, a direct analysis of transient problems is simpler and easier to adopt, at times.

Numerous authors have undertaken research into scattering of sound and electromagnetic waves by half plane, e.g., Sommerfeld [3] obtained valid solution of plane diffraction waves from half plane using image waves. Hohmann [156] considered the cylindrical inhomogeneity buried in a conductive half space with line source excitation. Boersma and Lee [40] studied the electric line source diffraction by a perfectly conducting half plane. Sanyal and Bhattacharyya [157] attained a uniform asymptotic expansion of the Maliuzhinetz's exact solution for the plane wave and line source illuminations by a half plane with two face impedances, by using Vander Warden's method. Rawlins et al [43] considered the line source diffraction by an acoustically penetrable or an electromagnetically dielectric half plane. Asghar et al discussed line source and point source diffraction by three half planes in a moving fluid [44]. Ahmad [46] studied the line source diffraction of acoustic waves by an absorbing half plane using Myres' conditions. Hussain [158] analyzed the problem of line source diffraction of electromagnetic waves by a perfectly conducting half plane in a homogeneous bi-isotropic medium. Using uniform geometrical diffraction, further advancement on half plane solution with ideal boundaries was rendered by Pathak and Kouyoumjian [55]. Williams [56] also referred to diffraction of waves by half planes with no ideal boundary conditions in reference to surfaces with identical point reacting impedance in infinite product. Then, Rawlins [57] found closed form solution for diffraction of plane wave by rigid-soft half plane. The new physical applications enabled the study of scattering by half plane surfaces with more complicated boundary conditions. For example, application of absorbent liners in aero engine exhausts may be attributed to impedance surfaces using Wiener-Hopf technique.

5.1 Formulation of the Problem

Consider a small amplitude sound wave on a main stream moving with a velocity U parallel to the $x-axis$. A semi infinite absorbing half plane is assumed to occupy the position $y=0$, $x\geq 0$. The equations of motion are linearized and the effect of viscosity, thermal conductivity and gravity are neglected. The fluid is assumed to have a constant density (incompressible) and sound speed c. We assume that the plane satisfies the Myers' impedance condition

$$u_n=\left[\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right]\frac{g}{|\nabla_\alpha|},\tag{5.1}$$

where

$$\frac{\partial g}{\partial t}=-\frac{\widetilde{p}}{z_a}\left|\nabla_\alpha\right|,$$

and u_n is the normal derivative of the perturbation velocity at a point on the surface of the semi-infinite half plane, $\widetilde{p}$ is the surface pressure, Z_a is the acoustic impedance of the surface and $\mathbf{n}$ a normal vector pointing from the surface into the fluid.

The perturbation velocity $\mathbf{u}$ of the irrotational sound wave can be written in terms of the velocity potential ϕ as $\mathbf{u} = \nabla\phi$ and the resulting pressure p of the sound field is given by

$$\tilde{p} = -\rho_0 \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \phi, \tag{5.2}$$

where ρ_0 is density of undisturbed stream. The mathematical form of the problem may be expressed in terms of the equations satisfied by $\phi(x, y, t)$ as follows

$$\left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - \left(\frac{1}{c} \frac{\partial}{\partial x} + M \frac{\partial}{\partial x} \right)^2 \phi \right] = \delta(x - x_0)\delta(y - y_0)\delta(t), \tag{5.3}$$

and subject to the following boundary conditions in time domain

$$\left[\frac{\partial^2 \phi}{\partial y \partial t} \mp \beta M \frac{\partial^2 \phi}{\partial x \partial t} \pm \beta M^2 c \frac{\partial^2 \phi}{\partial x^2} \mp \frac{\beta}{c} \frac{\partial^2 \phi}{\partial t^2} \right] = 0 \qquad x < 0, \tag{5.4}$$

$$\left. \begin{array}{c} \frac{\partial}{\partial y}\phi(x, 0^+, t) = \frac{\partial}{\partial y}\phi(x, 0^-, t) \\ \phi(x, 0^+, t) = \phi(x, 0^-, t) \end{array} \right\} \qquad x < 0, \tag{5.5}$$

In above equations $k = \frac{\omega}{c}$ is the wave number, $\beta = \frac{\rho_0 c}{Z_a}$ is the specific complex admittance and, $M = \frac{U}{c}$ is the Mach number. It is assumed that the flow is subsonic, i.e., $|M| < 1$, and $\operatorname{Re}\beta > 0$, which is a necessary condition for an absorbing surface [114]. More details can be found in [46].

5.2 Temporal Transform of the Problem

We define a temporal Fourier transform and its inverse by

$$\left\{ \begin{array}{c} \chi(x, y, \omega) = \int\limits_{-\infty}^{\infty} \phi(x, y, t) e^{i\omega t} dt, \\ \phi(x, y, t) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \chi(x, y, w) e^{-i\omega t} d\omega, \end{array} \right. \tag{5.6}$$

where ω is the temporal frequency. We transform Eqs. (5.3) to (5.5) in frequency domain by using Eq. (5.6) , and obtain

$$\left[(1-M^2)\frac{\partial^2}{\partial x^2}+2ikM\frac{\partial}{\partial x}+\frac{\partial^2}{\partial y^2}+k^2\right]\chi(x,y,\omega)=\delta(x-x_0)\delta(y-y_0), \tag{5.7}$$

$$\left[\frac{\partial}{\partial y}\mp 2\beta M\frac{\partial}{\partial x}\pm ik\beta\mp\frac{i\beta M^2}{k}\frac{\partial^2}{\partial x^2}\right]\chi(x,0^{\pm},\omega)=0 \qquad x\geq 0, \tag{5.8}$$

$$\left.\begin{array}{c}\frac{\partial}{\partial y}\chi(x,0^+,\omega)=\frac{\partial}{\partial y}\chi(x,0^-,\omega)\\ \chi(x,0^+,\omega)=\chi(x,0^-,\omega)\end{array}\right] \qquad x<0, \tag{5.9}$$

with

$$\delta(t)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}e^{-i\omega t}d\omega. \tag{5.10}$$

We observe that the mathematical problem expressed in Eqs. (5.7) to (5.9) is the same as discussed by Ahmad [46] except that in our problem $k=\frac{\omega}{c}$ is not a constant; rather, it is a function of ω. Thus, without going into details, we mention the results only, i.e.,

$$\chi(x,y,\omega)=\frac{\exp\left[-iKM(X-X_0)\right]}{4\pi i\sqrt{(1-M^2)}}\int\limits_{-\infty}^{\infty}\frac{e^{i\nu(X-X_0)+i\kappa(Y-Y_0)}}{\kappa}d\nu$$

$$+\frac{\exp\left[-iKM(X-X_0)\right]}{8\pi^2\sqrt{(1-M^2)}}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}G(\nu,\xi,\omega)e^{i\nu X+i\sqrt{(K^2-\nu^2)}|Y|}\,e^{-i\xi X+i\sqrt{(K^2-\xi^2)}|Y_0|}d\xi d\nu. \tag{5.11}$$

where

$$G(\nu,\xi,\omega)=\frac{B\left[K\left(1+M^2\right)+2\xi M+\frac{M^2\xi^2}{(1-M^2)K}\right]-\sqrt{(K-\nu)}\sqrt{(K+\xi)}sgn(Y)sgn(Y_0)}{L_+(\nu)L_-(\xi)(\xi-\nu)\sqrt{(K^2-\nu^2)}\sqrt{(K^2-\xi^2)}}. \tag{5.12}$$

where $\kappa=\sqrt{(K^2-v^2)}$ is the wave number and v is the Fourier transform variable. Also

$$\kappa=\kappa_+(v)\kappa_-(v)=\sqrt{K+v}\sqrt{K-v},$$

where $\kappa_+(\upsilon)$ is regular for $\operatorname{Im}\upsilon > -\operatorname{Im}K$, i.e., upper half plane and $\kappa_-(\upsilon)$ is regular for $\operatorname{Im}\upsilon < \operatorname{Im}K$, i.e., lower half plane. Let

$$\chi(x, y, \omega) = I_1' + I_2', \tag{5.13}$$

where

$$I_1' = \int_{-\infty}^{\infty} \frac{e^{i\nu(X-X_0)+i\kappa(Y-Y_0)}}{\kappa} d\nu, \tag{5.14}$$

and

$$I_2' = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\upsilon, \xi, \omega)e^{i\nu X+i\sqrt{(K^2-\nu^2)}|Y|}\ e^{-i\xi X+i\sqrt{(K^2-\xi^2)}|Y_0|}d\xi d\nu. \tag{5.15}$$

In order to calculate the total field $\phi(x, y, t)$, we need to find out the inverse temporal Fourier transform of the above integrals. Let us first consider I_1' which can also be written in the form

$$I_1' = \frac{\exp\left[-iKMR'\cos\Theta'\right]}{4\pi\sqrt{(1-M^2)}} \int_{-\infty}^{\infty} e^{-iKR'\cosh\lambda}d\lambda, \tag{5.16}$$

where

$$X - X_0 = R'\cos\Theta', \quad |Y - Y_0| = R'\sin\Theta', \quad \nu = K\cos(\Theta' + i\lambda).$$

Taking the inverse temporal Fourier transform and noting that K is a function of ω, Eq. (5.16) can be written as

$$I_1 = \frac{1}{8\pi^2\sqrt{(1-M^2)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-iKMR'\cos\Theta'+iKR'\cosh\lambda}\ e^{-i\omega t}d\lambda d\omega,$$

using

$$k = \sqrt{(1-M^2)}K \qquad \text{and} \qquad k = \frac{\omega}{c},$$

we get

$$I_1 = \frac{c}{4\pi Q}\int_{-\infty}^{\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega[t+\frac{MR'\cos\Theta'}{Q}-\frac{R'\cosh\lambda}{Q}]}d\omega d\lambda,$$

where

$$Q = c\sqrt{(1-M^2)}.$$

We know that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}d\omega = \delta(t).$$

Thus, using this property of the $\delta-$function, we obtain

$$I_1 = \frac{c}{4\pi Q}\int_{-\infty}^{\infty} \delta(t + \frac{MR'\cos\Theta'}{Q} - \frac{R'\cosh\lambda}{Q})d\lambda.$$

Letting $\frac{R'\cosh\lambda}{Q} = \eta$ in the above integral, we get

$$I_1 = \frac{c}{4\pi Q}\int_{-\infty}^{\infty} \frac{\delta(t'-\eta)}{\sqrt{(\eta^2 - \frac{R'^2}{Q^2})}}d\eta, \tag{5.17}$$

where $t' = t + \frac{MR'\cos\Theta'}{Q}$. The integral appearing in Eq. (5.17) can now be calculated as

$$I_1 = \frac{c}{4\pi Q}\frac{H(t'-\eta)}{\sqrt{(\eta^2 - \frac{R'^2}{Q^2})}}, \tag{5.18}$$

where $H(t'-\eta)$ is the usual Heaviside function.

Before finding the inverse temporal Fourier transform of I_2', we calculate the double integral appearing in the expression for I_2'. To do so, we introduce the polar coordinates

$$X = R\cos\Theta, \qquad |Y| = R\sin\Theta,$$

$$X_0 = R_0\cos\Theta_0, \qquad |Y_0| = R_0\sin\Theta_0,$$

and the transformation $\xi = -K\cos(\Theta_0 + ip)$ which changes the contour of integration over ξ into a hyperbola through the point $\xi = -K\cos\Theta_0$ where $(0 < \Theta_0 < \pi,\ -\infty < \tau < \infty)$. Similarly, by the change of variable $\nu = K\cos(\Theta + iq)$, the contour of integration can be converted from ν into a hyperbola through the point $\nu = K\cos\Theta$. Thus, omitting the details of calculations, we obtain

$$I_2' = \frac{-i[B\{(1+M^2) - 2M\cos\Theta_0 + \frac{M^2\cos^2\Theta_0}{(1-M^2)}\} - 2\sin\frac{\Theta}{2}\sin\frac{\Theta_0}{2}]e^{iKM(X-X_0)+i\kappa(R+R_0)}}{16\pi K\sqrt{RR_0}\sqrt{(1-M^2)}L_+(K\cos\Theta)L_-(-K\cos\Theta_0)(\cos\Theta+\cos\Theta_0)}, \tag{5.19}$$

where

$$R = r\left(\sqrt{\frac{1-M^2\sin^2\theta}{1-M^2}}\right), \quad \cos\Theta = \frac{\cos\theta}{\sqrt{1-M^2\sin^2\theta}} \quad \text{and} \quad \Theta \neq \pi - \Theta$$

Now taking the inverse temporal Fourier transform of Eq. (5.19), we have

$$I_2 = -\frac{icA'}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\frac{i\omega}{Q}(R+R_0-MR'\cos\Theta')}}{\omega}e^{-i\omega t}d\omega, \tag{5.20}$$

where

$$A' = \frac{[B\{((1+M^2) - 2M\cos\Theta_0 + \frac{M^2\cos^2\Theta_0}{(1-M^2)})\} - 2\sin\frac{\Theta}{2}\sin\frac{\Theta_0}{2}]}{16\pi\sqrt{RR_0}L_+(K\cos\Theta)L_-(-K\cos\Theta_0)(\cos\Theta+\cos\Theta_0)}. \tag{5.21}$$

Let us take $g(\omega) = \frac{1}{\omega}$, $f(\omega) = e^{\frac{i\omega}{Q}(R+R_0-MR'\cos\Theta')}$ in Eq. (5.20) and using the convolution theorem, we can write

$$I_2 = -icA'F(t) * G(t), \tag{5.22}$$

where

$$F(t) = \frac{1}{2\pi}\int_{i\tau-\infty}^{i\tau+\infty} e^{\frac{i\omega}{Q}(R+R_0-MR'\cos\Theta')}e^{-i\omega t}d\omega,$$

$$G(t) = \frac{1}{2\pi}\int_{i\tau-\infty}^{i\tau+\infty}\frac{e^{-i\omega t}}{\omega}d\omega,$$

where τ lies in the region of analyticity such that $-\operatorname{Im}(K) < \tau < \operatorname{Im}(K)$. The asterisk in equation (5.22) denotes convolution in the time domain. For $\tau > 0$, we can close the contour of integration in the lower half plan. Knowing that ω has a small positive imaginary part, for $\tau > 0$, we get

$$F(t) = \delta(t - \frac{1}{Q}(R + R_0 - MR'\cos\Theta')),$$

$$G(t) = -i.$$

Hence,

$$I_2 = -cA' \int_{-\infty}^{\infty} \delta(t' - \frac{1}{Q}(R + R_0))dt,$$

or

$$I_2 = -2cA'H(t' - \frac{1}{Q}(R + R_0)). \tag{5.23}$$

Making use of Eqs. (5.6), (5.11) (5.13) ,(5.18) and (5.23) , we get

$$\phi(x, y, t) = \frac{c}{4\pi Q} \frac{H(t' - \eta)}{\sqrt{(\eta^2 - \frac{R'^2}{Q^2})}} - 2cA'H(t' - \frac{1}{Q}(R + R_0)), \tag{5.24}$$

where A' is given by Eq. (5.19).

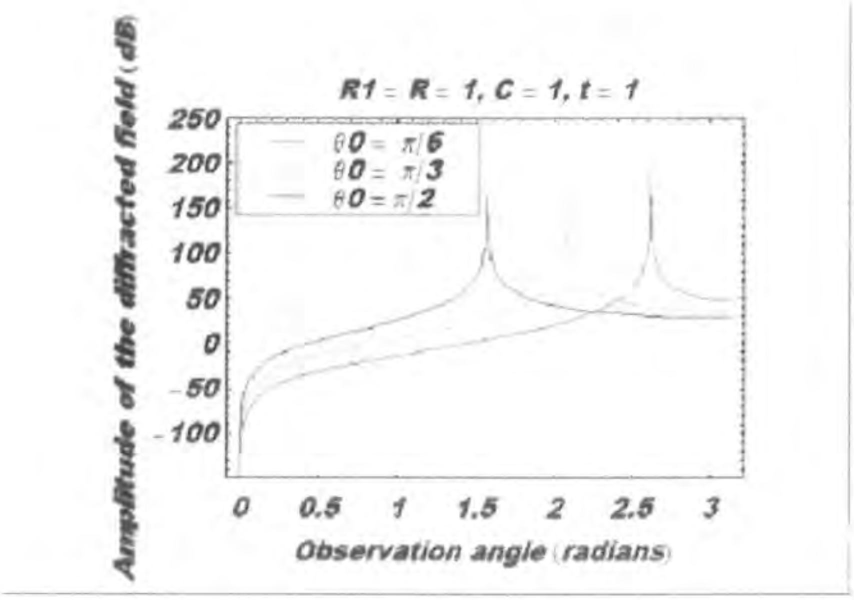
5.3 Graphical Results

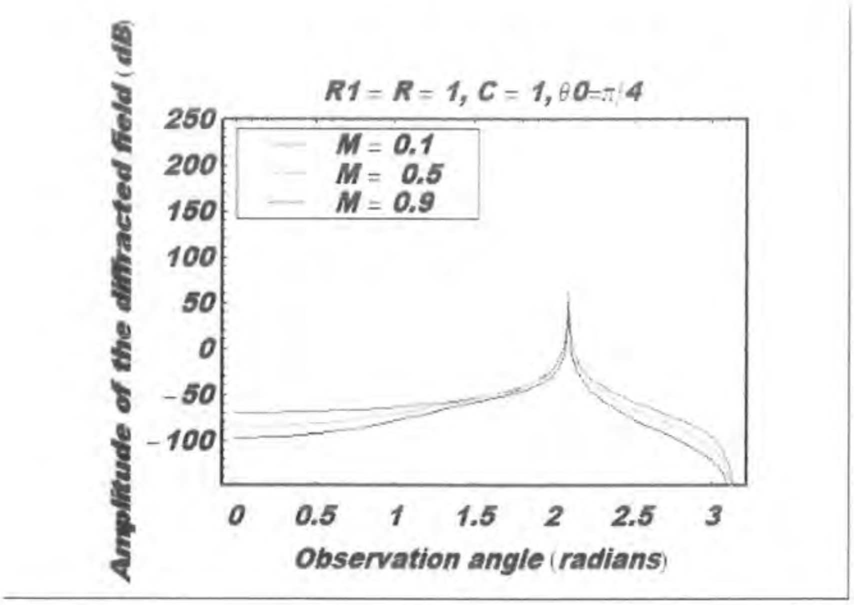
A computer programme MATHEMATICA has been used for the graphical plotting of the diffracted field in the time domain. The main features of the graphical results are as follows:

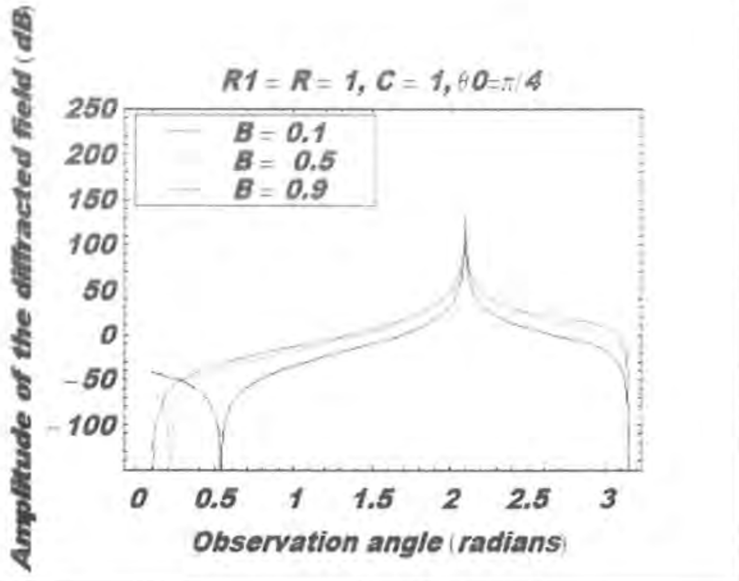
(a) In Fig. (1), the amplitude of the diffracted field is plotted against observation angle for different values of the incident angle by fixing all other parameters. It is observed that by increasing the incident wave angle, the amplitude of the diffracted field decreases.

(b) In Fig. (2), the effect of Mach number M can be seen. By increasing the Mach number, the amplitude of the diffracted field decreases, i.e., the sound intensity decreases.

(c) In Fig. (3), the amplitude of the diffracted field is plotted against observation angle for different values of the absorbing parameter B, by fixing all other parameters. It is observed that by increasing the absorbing parameter, the amplitude of the diffracted field decreases, i.e., the sound intensity decreases.







5.4 Conclusion

We have obtained an improved form of the diffracted field due to an impulsive line source by an absorbing half plane in a moving fluid by considering the time dependence. The first term in Eq. (5.22) represents the field at the observation point directly coming from the line source, whereas the second term corresponds to the diffracted field from the edge of the half plane. This field starts reaching the point $(x.y)$ after the time lapse $t' > \frac{1}{Q}(R + R_0)$. We note that the strength of the field dies down as $1/\sqrt{R_0}$. The results for the still air can be obtained by putting $M = 0$.

Chapter 6

DIFFRACTION OF AN IMPULSIVE LINE SOURCE WITH WAKE

In this chapter, we shall discuss the problem of diffraction due to an impulse line source by an absorbing half plane with wake using Myres' impedance condition [133] in the presence of a subsonic fluid flow. In the process, we shall examine the effect of Kutta-Joukowski condition by introducing the wake (trailing edge) attached to the half plane. The time dependence of the field requires a temporal Fourier transform in addition to the spatial Fourier transform. We apply the temporal Fourier transform to obtain the transform function in the transformed plane using the Wiener-Hopf technique [13] and the modified method of stationary phase [12]. When the transform function is available, an inverse temporal Fourier transform can be applied to obtain the results in time domain. The expressions for acoustic field for the trailing edge are obtained, i.e., a wake is attached to the absorbing half plane. A Kutta-Joukowski condition is also imposed in order to find a unique solution. Normally, the effect of the Kutta-Joukowski condition is to produce a beam of sound in neighborhood of the wake and to scatter a field elsewhere which is approximately the one given by Ffowcs-Williams and Hall [99]. The solution for the leading edge situation can be obtained if the wake, and consequently a Kutta-Joukowski edge condition, is ignored. This can also be seen from the numerical results.

The research work of various authors into scattering of sound and electromagnetic waves by half plane have been discussed in the previous chapter. Jones [38]studied the problem of line source diffraction of acoustic waves by a hard half plane attached to a wake in still air as well as when the medium is convective. He showed that wake acts as a convenient transmission channel for carrying intense sound away from the source. The problem was further extended to the point source excitation by Balasubramanyum [110] and diffraction by a cylindrical impulse by Rienstra [117] . Rawlins [114] discussed the diffraction of a cylindrical acoustic wave by an absorbing half-plane in a moving fluid in the presence of a wake. The nature of and basis for a Kutta-condition in unsteady flow has been discussed by Crighton [101] in detail.

Transient nature of the field is an important area in the theory of acoustic diffraction which we have discussed in Chapter Five. Effects of moving medium was first correctly given by Miles [159] and Ribner [160] for a plane interface of relative motion. The steady state (time harmonic) and initial value (impulsive source) situations have also been considered by Crighton et al. [101]. Ingard [121] discussed the effect of flow on boundary conditions at a plane impedance surface. Later on, Myers [133] discussed the diffraction of cylindrical acoustic waves, by a semi-infinite absorbing plane, which was generalization of the Ingard's condition. Now, Myers' condition [133] is the accepted form of the boundary condition for impedance walls with flow. Recently, Ahmad [46] has discussed the problem of acoustic diffraction by an absorbing half plane in a moving fluid using Myers' condition.

Formulation of the Problem

We consider the diffraction of an acoustic wave incident on the half plane occupying a space $y = 0$, $x \leq 0$ so that the stream is at zero incidence. The form of the plane wave in the moving fluid is produced by considering an impulsive line source at (x_0, y_0) and is of strength $\delta(t)$ (where δ denotes the Dirac delta function) that also radiates waves. A wake occupies $y = 0$, $x < 0$ with the velocity of the moving fluid parallel to the x-axis and of magnitude $U > 0$. The equation of motion is linearized and the effect of viscosity, thermal conductivity and gravity are neglected. The fluid is assumed to have a constant density (incompressible) and sound speed c. We assume that the plane satisfies the Myres' impedance condition [133].

$$u_n = \left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right]\frac{g}{|\nabla_a|}, \tag{6.1}$$

where the function g is related to the surface pressure $\widetilde{p}$ such that

$$\frac{\partial g}{\partial t} = -\frac{\widetilde{p}}{z_a}|\nabla_a|, \tag{6.2}$$

and u_n is the normal derivative of the perturbation velocity at a point on the surface of the semi-infinite half plane, Z_a is the acoustic impedance of the surface and $\mathbf{n}$ a normal vector pointing from the surface into the fluid.

The perturbation velocity $\mathbf{u}$ of the irrotational sound wave can be written in terms of the velocity potential ϕ as $\mathbf{u} = \nabla\phi$ and the resulting pressure p of the sound field is given by

$$\widetilde{p} = -\rho_0\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\phi, \tag{6.3}$$

where ρ_0 is the density of the undisturbed stream. The line source is considered parallel to the edge at the point (x_0, y_0). The governing convective wave equation with boundary conditions is given by

$$\left[\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}-\left(\frac{1}{c}\frac{\partial}{\partial t}+M\frac{\partial}{\partial x}\right)^2\varphi\right]=\delta(x-x_0)\delta(y-y_0)\delta(t), \tag{6.4}$$

The Problem in Frequency Domain

Let us transform the problem in frequency domain with the help of temporal Fourier transform by

$$\begin{cases} \chi(x,y,\omega)=\int\limits_{-\infty}^{\infty}\phi(x,y,t)e^{i\omega t}dt, \\ \phi(x,y,t)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\chi(x,y,\omega)e^{-i\omega t}d\omega, \end{cases} \tag{6.5}$$

where ω is the temporal frequency. We transform equations (6.1) and (6.4) by using Eq. (6.5) and obtain

$$\left[(1-M^2)\frac{\partial^2}{\partial x^2}+2ikM\frac{\partial}{\partial x}+\frac{\partial^2}{\partial y^2}+k^2\right]\chi(x,y,\omega)=\delta(x-x_0)\delta(y-y_0), \tag{6.6}$$

$$\left[\frac{\partial}{\partial y}\mp 2\beta M\frac{\partial}{\partial x}\pm ik\beta\mp\frac{i\beta M^2}{k}\frac{\partial^2}{\partial x^2}\right]\chi(x,0^{\pm},\omega)=0, \qquad x<0, \tag{6.7}$$

with

$$\delta(t)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}e^{-i\omega t}d\omega, \tag{6.8}$$

$$\frac{\partial}{\partial y}\chi(x,0^+,\omega)=\frac{\partial}{\partial y}\chi(x,0^-,\omega), \qquad x>0, \tag{6.9}$$

$$(-ik+M\frac{\partial}{\partial x})\chi(x,0^+,\omega)=(-ik+M\frac{\partial}{\partial x})\chi(x,0^-,\omega) \tag{6.10}$$

where $\beta=\frac{\rho_0 c}{Z_a}$ is the specific complex admittance, $k=\frac{\omega}{c}$ is the wave number and $k=k_1+ik_2$ has a small imaginary part to ensure the regularity of the Fourier transform integrals and $M=\frac{U}{c}$ (c is the velocity of sound) is the Mach number. We assume that the flow is subsonic, i.e., $|M|<1$, (for a leading edge situation $-1<M\leq 0$ and for a

trailing edge situation $0 < M < 1$) and $\mathrm{Re}\,\beta > 0$, which is a necessary condition for an absorbing surface. Also $\beta = 0$ corresponds to the rigid barrier and $\beta = \infty$ corresponds to the pressure release barrier. The trailing edge situation gives rise to a field which is singular at the origin for the trailing edge situation. Therefore, the Kutta-Joukowski condition is imposed to obtain a unique solution of the problem. The boundary condition (6.10) for a continuous pressure with massless wake as already discussed in [57] can be written in the alternative form

$$\chi(x, 0^+, \omega) - \chi(x, 0^-, \omega) = \lambda e^{\frac{ik}{M}x} \qquad x > 0, \tag{6.11}$$

$$\frac{\partial}{\partial y}\chi(x, 0^+, \omega) = \frac{\partial}{\partial y}\chi(x, 0^-, \omega), \tag{6.12}$$

In Eq. (6.11) the discontinuity in the field is due to imposition of wake which involves a parameter λ. This λ will be determined by the requirement that the velocity at the trailing edge should be finite which requires the imposition of Kutta-Joukowski edge condition. Also $\lambda = 0$ corresponds to the leading edge situation, i.e., no wake. Initially, we shall impose the edge condition

$$\phi = O(1), \quad \text{and} \quad \frac{\partial \phi}{\partial r} = O(\frac{1}{\sqrt{x}}),$$

combined with the condition that the diffracted field is outgoing at infinity. Physically, we can consider the flow of an incompressible fluid past the edge of a sheet on which the normal velocity is zero. We also need conditions to limit the singularities at the edge $(0, 0)$. In absence of mean flow, a solution exists with $\phi = O(\sqrt{r})$ as $r \to 0$ (radiation conditions or Sommerfeld conditions), and to this solution may be added any eigen solution of the problem. The eigen solutions, however, are all more singular than this and it is; therefore, convenient to choose the solution with $\phi = O(\sqrt{r})$. The better reason for taking this solution can also be obtained by taking into account the effect of viscosity to the half plane.

Let us introduce the following real substitutions in Eqs. (6.6), (6.7), (6.9), (6.10), (6.11) and (6.12)

$$x=\sqrt{1-M^2}X,\ x_0=\sqrt{1-M^2}X_0,\ y=Y,\ y_0=Y_0,\ \beta=\sqrt{1-M^2}B,\ k=\sqrt{1-M^2}K,$$

and

$$\chi(x,y,\omega)=\psi(X,Y,\omega)e^{-iKMX}, \tag{6.13}$$

to get

$$\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}+K^2\right)\psi(X,Y,\omega)=\frac{\delta(X-X_0)\delta(Y-Y_0)}{\sqrt{1-M^2}}e^{iKMX_0}, \tag{6.14}$$

$$\left[\frac{\partial}{\partial Y}\mp 2BM\frac{\partial}{\partial X}\pm iKB(1+M^2)\mp\frac{iBM^2}{(1-M^2)K}\frac{\partial^2}{\partial X^2}\right]\psi(X,0^{\pm},\omega)=0\,, \qquad x<0 \tag{6.15}$$

and

$$\left.\begin{array}{r}\frac{\partial}{\partial Y}\psi(X,0^+,\omega)=\frac{\partial}{\partial Y}\psi(X,0^-,\omega)\\ \psi(X,0^+,\omega)-\psi(X,0^-,\omega)=\lambda e^{\frac{iK}{M}X}\end{array}\right\}\qquad x>0. \tag{6.16}$$

The total field $\psi(X,Y,\omega)$ may be expressed as a sum of the incident and scattered field as follows :

$$\psi(X,Y,\omega)=\Psi(X,Y,\omega)+\Psi_i(X,Y,\omega), \tag{6.17}$$

where $\Psi_i(X,Y,\omega)$ is the incidence field (corresponding to the inhomogeneous equation) and $\Psi(X,Y,\omega)$ is the diffracted field (corresponding to the homogeneous equation), so that $\Psi_i(X,Y,\omega)$ satisfies

$$\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}+K^2\right)\Psi_i(X,Y,\omega)=\frac{\delta(X-X_0)\delta(Y-Y_0)}{\sqrt{1-M^2}}e^{iKMX_0} \tag{6.18}$$

and $\Psi(X, Y, \omega)$ satisfies

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + K^2\right) \Psi\left(X, Y, \omega\right) = 0. \tag{6.19}$$

By the Green's function method, the solution of Eq. (6.18) can be obtained as

$$\Psi_i(X, Y, \omega) = \frac{a}{4i} H_0^1(KR) = \frac{a}{4\pi i} \int\limits_{-\infty}^{\infty} \frac{1}{\kappa} e^{i[\alpha(X - X_0) + \kappa|Y - Y_0|]} d\alpha, \tag{6.20}$$

where $a = \frac{e^{iKMX_0}}{\sqrt{1-M^2}}$, $R = \sqrt{(X - X_0)^2 + (Y - Y_0)^2}$ and $\kappa = \sqrt{K^2 - \alpha^2}$. Here, K, is the wave number and α is the Fourier transform variable.

Let us introduce the Fourier transform over the variable X as

$$\overline{\Psi}(\alpha, Y, \omega) = \int\limits_{-\infty}^{\infty} \Psi(X, Y, \omega) e^{-i\alpha X} dX, \tag{6.21}$$

and

$$\Psi(X, Y, \omega) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \overline{\Psi}(\alpha, Y, \omega) e^{i\alpha X} d\alpha. \tag{6.22}$$

We cater for the two part boundary condition on $Y = 0$, split $\overline{\Psi}(\alpha, Y, \omega)$ as

$$\overline{\Psi}(\alpha, Y, \omega) = \overline{\Psi}_-(\alpha, Y, \omega) + \overline{\Psi}_+(\alpha, Y, \omega), \tag{6.23}$$

where

$$\overline{\Psi}_-(\alpha, Y, \omega) = \int\limits_{-\infty}^{0} \Psi(X, Y, \omega) e^{-i\alpha x},$$

and

$$\overline{\Psi}_+(\alpha, Y, \omega) = \int\limits_{0}^{\infty} \Psi(X, Y, \omega) e^{-i\alpha x}$$

Here $\overline{\Psi}_-(\alpha, Y, \omega)$ is regular for $\text{Im}\alpha < \text{Im}\, K$ and $\overline{\Psi}_+(\alpha, Y, \omega)$ is regular for $\text{Im}\alpha > -\text{Im}\, K$.

We transform Eq. (6.19) by Fourier transform to get

$$\frac{d^2}{dY^2}\overline{\Psi}(\alpha, Y, \omega) + \kappa^2\overline{\Psi}(\alpha, Y, \omega) = 0, \tag{6.24}$$

and $\alpha-$plane is cut such that $\operatorname{Im} k > 0$, (for bounded solution). The solution satisfying radiation condition is given by

$$\overline{\Psi}(\alpha, Y, \omega) = \begin{cases} B_1(\alpha)e^{i\kappa Y} & \text{if } Y \geq 0 \\ B_2(\alpha)e^{-i\kappa Y} & \text{if } Y < 0. \end{cases} \tag{6.25}$$

The Fourier transform of the boundary conditions as given by Eqs. (6.15) and (6.16) takes the following form

$$\overline{\Psi}'_{-}(\alpha, 0^+, \omega) + iB\left\{K\left(1 + M^2\right) + 2\alpha M + \frac{\alpha^2 M^2}{(1 - M^2)K}\right\}\overline{\Psi}_{-}(\alpha, 0^+, \omega)$$

$$= -iB\left\{K\left(1 + M^2\right) + 2\alpha M + \frac{\alpha^2 M^2}{(1 - M^2)K}\right\}\overline{\Psi}_i(\alpha, 0, \omega) - \overline{\Psi}'_i(\alpha, 0, \omega), \tag{6.26}$$

and

$$\overline{\Psi}'_{-}(\alpha, 0^-, \omega) - iB\left\{K\left(1 + M^2\right) + 2\alpha M + \frac{\alpha^2 M^2}{(1 - M^2)K}\right\}\overline{\Psi}_{-}(\alpha, 0^-, \omega)$$

$$= iB\left\{K\left(1 + M^2\right) + 2\alpha M + \frac{\alpha^2 M^2}{(1 - M^2)K}\right\}\overline{\Psi}_i(\alpha, 0, \omega) - \overline{\Psi}'_i(\alpha, 0, \omega). \tag{6.27}$$

Also

$$\overline{\Psi}_{+}(\alpha, 0^+, \omega) - \overline{\Psi}_{+}(\alpha, 0^-, \omega) = \frac{-i\lambda}{(\alpha - \frac{K}{M})}, \tag{6.28}$$

and

$$\overline{\Psi}'_{+}(\alpha, 0^+, \omega) = \overline{\Psi}'_{+}(\alpha, 0^-, \omega) = \overline{\Psi}'_{+}(\alpha, 0, \omega). \tag{6.29}$$

With the help of Eqs. (6.25) to (6.29), we get

$$B_1(\alpha) = J_-(\alpha, 0, \omega) + \frac{J'_-(\alpha, 0, \omega)}{i\kappa} - \frac{i\lambda}{2(\alpha - \frac{K}{M})}, \tag{6.30}$$

$$B_2(\alpha) = -J_-(\alpha, 0, \omega) + \frac{J'_-(\alpha, 0, \omega)}{i\kappa} + \frac{i\lambda}{2(\alpha - \frac{K}{M})}, \tag{6.31}$$

where

$$J_-(\alpha, 0, \omega) = \frac{1}{2}[\overline{\Psi}_-(\alpha, 0^+, \omega) - \overline{\Psi}_-(\alpha, 0^-, \omega)], \tag{6.32}$$

$$J'_-(\alpha, 0, \omega) = \frac{1}{2}[\overline{\Psi}'_-(\alpha, 0^+, \omega) - \overline{\Psi}'_-(\alpha, 0^-, \omega)]. \tag{6.33}$$

From Eqs. (6.25) and (6.30) to (6.33), we have

$$\overline{\Psi}'_-(\alpha, 0^+, \omega) + \overline{\Psi}'_+(\alpha, 0, \omega) = i\kappa[\overline{\Psi}_-(\alpha, 0^+, \omega) + \overline{\Psi}_+(\alpha, 0^+, \omega)], \tag{6.34}$$

$$\overline{\Psi}'_-(\alpha, 0^-, \omega) + \overline{\Psi}'_+(\alpha, 0, \omega) = -i\kappa[\overline{\Psi}_-(\alpha, 0^-, \omega) + \overline{\Psi}_+(\alpha, 0^-, \omega)]. \tag{6.35}$$

Eliminating $\overline{\Psi}'_-(\alpha, 0^+, \omega)$ from Eqs. (6.26) and (6.34), and eliminating $\overline{\Psi}'_-(\alpha, 0^-, \omega)$ from Eqs. (6.27) and (6.35), we get

$$\overline{\Psi}'_+(\alpha, 0, \omega) - \left[iB\left\{K\left(1+M^2\right) + 2\alpha M + \frac{\alpha^2 M^2}{(1-M^2)K}\right\} + i\kappa\right]\overline{\Psi}_-(\alpha, 0^+, \omega)$$

$$-iB\left\{K\left(1+M^2\right) + 2\alpha M + \frac{\alpha^2 M^2}{(1-M^2)K}\right\}\overline{\Psi}_i(\alpha, 0, \omega)$$

$$= i\kappa\overline{\Psi}_+(\alpha, 0^+, \omega) + \overline{\Psi}'_i(\alpha, 0, \omega), \tag{6.36}$$

and

$$\overline{\Psi}'_+(\alpha, 0, \omega) + \left[iB\left\{K\left(1+M^2\right) + 2\alpha M + \frac{\alpha^2 M^2}{(1-M^2)K}\right\} + i\kappa\right]\overline{\Psi}_-(\alpha, 0^-, \omega)$$

$$+iB\left\{K\left(1+M^2\right)+2\alpha M+\frac{\alpha^2M^2}{\left(1-M^2\right)K}\right\}\overline{\Psi}_i(\alpha,0,\omega)$$

$$=-i\kappa\overline{\Psi}_+(\alpha,0^-,\omega)+\overline{\Psi}_i'(\alpha,0,\omega). \tag{6.37}$$

Addition of Eqs. (6.36) and (6.37) results in

$$i\kappa L(\alpha)J_-(\alpha,0,\omega)-\overline{\Psi}_+'(\alpha,0,\omega)+\overline{\Psi}_i'(\alpha,0,\omega)+\frac{K\lambda}{2(\alpha-\frac{K}{M})}=0. \tag{6.38}$$

Similarly, eliminating $\overline{\Psi}_-(\alpha,0^+,\omega)$ from Eqs. (6.26) and (6.34), and eliminating $\overline{\Psi}_-(\alpha,0^-,\omega)$ from Eqs. (6.27) and (6.35) and then subtracting the resulting equations, we get

$$\frac{L(\alpha)J_-'(\alpha,0,\omega)}{B\left[K\left(1+M^2\right)+2\alpha M+\frac{\alpha^2M^2}{(1-M^2)K})\right]}-\frac{i\overline{\Psi}_+(\alpha,0^+,\omega)}{2}-\frac{i\overline{\Psi}_+(\alpha,0^-,\omega)}{2}+i\overline{\Psi}_i(\alpha,0,\omega)=0, \tag{6.39}$$

where

$$L(\alpha)=1+\frac{B}{\kappa}\left[K\left(1+M^2\right)+2\alpha M+\frac{\alpha^2M^2}{\left(1-M^2\right)K}\right]. \tag{6.40}$$

Eqs. (6.38) and (6.39) are the standard Wiener-Hopf equations. Let us proceed to find the solution for these equations.

Solution of the Wiener-Hopf Equations

Let us write

$$L(\alpha)=L_+(\alpha)L_-(\alpha) \tag{6.41}$$

and

$$\kappa=\kappa_+(\alpha)\kappa_-(\alpha)=\sqrt{K+\alpha}\sqrt{K-\alpha}, \tag{6.42}$$

where $L_+(\alpha)$ and $\kappa_+(\alpha)$ are regular for $\operatorname{Im}\alpha>-\operatorname{Im}K$, i.e., upper half plane and $L_-(\alpha)$ and $\kappa_-(\alpha)$ are regular for $\operatorname{Im}\alpha<\operatorname{Im}K$, i.e., lower half plane. Making use of Eqs. (6.41) and (6.42) in Eq. (6.38), we get

$$iJ_-(\alpha,0,\omega)L_-(\alpha)\sqrt{K-\alpha}+\frac{\overline{\Psi}'_i(\alpha,0,\omega)}{L_+(\alpha)\sqrt{K+\alpha}}+\frac{\lambda\sqrt{K-\alpha}}{2L_+(\alpha)(\alpha-\frac{K}{M})}=\frac{\overline{\Psi}'_+(\alpha,0,\omega)}{L_+(\alpha)\sqrt{K+\alpha}}, \quad (6.43)$$

Whereas in Eq. (6.43), the first term on the left hand side is regular in the lower half plane and the term on the right hand side is regular in the upper half plane, for the other two terms whose genders are not known, can be written as [13]

$$\frac{\overline{\Psi}'_i(\alpha,0,\omega)}{L_+(\alpha)\sqrt{K+\alpha}}=T_+(\alpha)+T_-(\alpha), \quad (6.44)$$

and

$$\frac{\lambda\sqrt{K-\alpha}}{2L_+(\alpha)(\alpha-\frac{K}{M})}=F_+(\alpha)+F_-(\alpha), \quad (6.45)$$

These decompositions cannot be performed by inspection and it is necessary to use the general theorem B of [13]. Now, invoking Eqs. (6.44) to (6.45) in Eq. (6.43), we get

$$iJ_-(\alpha,0,\omega)\sqrt{K-\alpha}L_-(\alpha)+T_-(\alpha)+F_-(\alpha)=\frac{\overline{\Psi}'_+(\alpha,0,\omega)}{L_+(\alpha)\sqrt{K+\alpha}}-T_+(\alpha)-F_+(\alpha), \quad (6.46)$$

where

$$T_+(\alpha)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\overline{\Psi}'_i(\xi,0,\omega)}{L_+(\xi)\sqrt{K+\xi}(\xi-\alpha)}d\xi, \quad (6.47)$$

$$T_-(\alpha)=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\overline{\Psi}'_i(\xi,0,\omega)}{L_+(\xi)\sqrt{K+\xi}(\xi-\alpha)}d\xi, \quad (6.48)$$

$$F_+(\alpha)=\frac{\lambda}{2(\alpha-\frac{K}{M})}\left[\frac{\sqrt{K-\alpha}}{L_+(\alpha)}-\frac{\sqrt{K-\frac{K}{M}}}{L_+(\frac{K}{M})}\right], \quad (6.49)$$

$$F_-(\alpha)=\frac{\lambda}{2(\alpha-\frac{K}{M})}\left[\frac{\sqrt{K-\frac{K}{M}}}{L_+(\frac{K}{M})}\right]. \quad (6.50)$$

We have equated the terms with negative sign on the left hand side and terms with positive sign on the right hand side of Eq. (6.46). Let $J(\alpha)$ be a function equal to both sides of Eq. (6.46), which are regular in lower and upper half planes respectively. We use analytical continuation so that the definition of $J(\alpha)$ can be extended throughout the complex α plane. We examine the asymptotic behaviour of Eq. (6.46) to ascertain the form of $J(\alpha)$ as $\alpha \to \infty$. It is noted that $|L_{\pm}(\alpha)|\,\tilde{}\,O(1)$, [46] as $|\alpha| \to \infty$ and with the help of edge condition, it is found that $J_{-}(\alpha, 0, \omega)$ should be at least of $O(|\alpha|^{-\frac{1}{2}})$ as $|\alpha| \to \infty$. So, using the extended form of Liouville's theorem [13], we see that $J(\alpha)\,\tilde{}\,O(|\alpha|^{-\frac{1}{2}})$ and so a polynomial that represents $J(\alpha)$ can only be a constant equal to zero, i.e.,

$$iJ_{-}(\alpha, 0, \omega)\sqrt{K-\alpha}L_{-}(\alpha) + T_{-}(\alpha) + F_{-}(\alpha) = 0.$$

By using Eqs. (6.48) and (6.50) in above equation, we have

$$J_{-}(\alpha, 0, \omega) = \frac{-1}{2\pi L_{-}(\alpha)\sqrt{K-\alpha}} \int_{-\infty}^{\infty} \frac{\overline{\Psi}_i'(\xi, 0, \omega)}{L_{+}(\xi)\sqrt{K+\xi}(\xi-\alpha)} d\xi$$

$$+ \frac{i\lambda\sqrt{K-\frac{K}{M}}}{2L_{-}(\alpha)\sqrt{K-\alpha}L_{+}(\frac{K}{M})(\alpha-\frac{K}{M})}. \tag{6.51}$$

Similarly, by adopting the same procedure as in the case of Eq. (6.38), we can write for Eq. (6.39) as follows

$$J_{-}'(\alpha, 0, \omega) = \frac{1}{2\pi L_{-}(\alpha)} \int_{-\infty}^{\infty} B\left[K\left(1+M^2\right) + 2\xi M + \frac{\xi^2 M^2}{(1-M^2)K}\right] \frac{\overline{\Psi}_i(\xi, 0, \omega)}{L_{+}(\xi)(\xi-\alpha)} d\xi. \tag{6.52}$$

Invoking Eqs. (6.51) and (6.52) in Eqs. (6.30) and (6.31), respectively, and then making use of Eq. (6.20) in the resulting equation, we get

$$\left.\begin{array}{l} B_1(\alpha) \\ \\ B_2(\alpha) \end{array}\right\} = \frac{-a}{4\pi L_-(\alpha)\kappa}\int_{-\infty}^{\infty}\frac{B\left[K\left(1+M^2\right)+2\xi M+\frac{\xi^2M^2}{(1-M^2)K}\right]}{L_+(\xi)(\xi-\alpha)\sqrt{K^2-\xi^2}}e^{-i\xi X_0+i\sqrt{K^2-\xi^2}|Y_0|}\,d\xi$$

$$\pm\frac{a}{4\pi L_-(\alpha)\sqrt{K-\alpha}}\int_{-\infty}^{\infty}\frac{e^{-i\xi X_0+i\sqrt{K^2-\xi^2}|Y_0|}sgn\,|Y_0|}{L_+(\xi)\sqrt{K+\xi}(\xi-\alpha)}d\xi$$

$$\pm\frac{i\lambda}{2(\alpha-\frac{K}{M})}\left[\frac{\sqrt{K-\frac{K}{M}}-L_-(\alpha)\sqrt{K-\alpha}L_+(\frac{K}{M})}{L_-(\alpha)\sqrt{K-\alpha}L_+(\frac{K}{M})}\right]. \tag{6.53}$$

In order to ensure a unique mathematical solution, we must impose Kutta-Joukowski edge condition which requires that the velocity should be finite at the origin, which in fact means that in the above expression, the term of $O(|\alpha|^{-\frac{1}{2}+\delta})$ as $|\alpha|\to\infty$ must vanish and $|L_\pm(\alpha)|^{-}O(|\alpha|^{\pm\delta})$ as $|\alpha|\to\infty$, $0\leq\delta\leq\frac{1}{2}$. Hence, in order apply the Kutta-Joukowski condition we choose λ as

$$\lambda=\frac{aL_+(\frac{K}{M})sgn\,|Y_0|}{2\pi i\left[\sqrt{K-\frac{K}{M}}-L_+(\frac{K}{M})\sqrt{K-\alpha}L_-(\alpha)\right]}\int_{-\infty}^{\infty}\frac{e^{-i\xi X_0+i\sqrt{K^2-\xi^2}|Y_0|}}{L_+(\xi)\sqrt{K+\xi}}d\xi \quad \text{as } |\xi|\to\infty. \tag{6.54}$$

The integrand in the above integral expression is exponentially bounded as $|\xi|\to\infty$. By, invoking Eq. (6.53) with (6.54) in Eq. (6.25) and taking the inverse Fourier transform of the resulting equation, we get

$$\Psi(X,Y,\omega)=\left[\frac{-a}{8\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{B\left[K\left(1+M^2\right)+2\xi M+\frac{\xi^2M^2}{(1-M^2)K}\right]\mp\sqrt{K+\alpha}\sqrt{K-\xi}\;sgn\,|Y_0|}{L_-(\alpha)L_+(\xi)(\xi-\alpha)\sqrt{K^2-\alpha^2}\sqrt{K^2-\xi^2}}\right.$$

$$\left.-\frac{\sqrt{K-\xi}\sqrt{K+\alpha}\;sgn\,|YY_0|}{(\alpha-\frac{K}{M})L_-(\alpha)L_+(\xi)\sqrt{K^2-\alpha^2}\sqrt{K^2-\xi^2}}\right]e^{-i\xi X_0+i\sqrt{K^2-\xi^2}|Y_0|}\,e^{i\kappa|Y|+i\alpha X}d\xi d\alpha \tag{6.55}$$

where the path of integration is indented below the pole $\alpha = \frac{K}{M}$ for $\operatorname{Im} k = 0$ and with the help of Eqs. (6.14) and (6.55), we get

$$\chi(x, y, \omega) = \frac{\exp\left[-iKM(X - X_0)\right]}{4\pi i\sqrt{(1 - M^2)}} \int_{-\infty}^{\infty} \frac{e^{i\alpha(X - X_0) + i\kappa(Y - Y_0)}}{\kappa} d\alpha$$

$$+\frac{e^{iKM(X_0 - X)} sgn\,|YY_0|}{8\pi^2\sqrt{1 - M^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\alpha, \xi, \omega) e^{-i\xi X_0 + i\sqrt{K^2 - \xi^2}|Y_0|}\, e^{i\kappa|Y| + i\alpha X} d\xi d\alpha$$

$$-\frac{e^{iKM(X_0 - X)}}{8\pi^2\sqrt{1 - M^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\alpha, \xi, \omega) e^{-i\xi X_0 + i\sqrt{K^2 - \xi^2}|Y_0|}\, e^{i\kappa|Y| + i\alpha X} d\xi d\alpha, \tag{6.56}$$

where

$$F(\alpha, \xi, \omega) = \frac{B\left[K(1 + M^2) + 2\xi M + \frac{\xi^2 M^2}{(1 - M^2)K}\right] - \sqrt{K + \alpha}\sqrt{K - \xi}\; sgn\,|Y_0|}{L_-(\alpha)L_+(\xi)(\xi - \alpha)\sqrt{K^2 - \alpha^2}\sqrt{K^2 - \xi^2}}, \tag{6.57}$$

and

$$G(\alpha, \xi, \omega) = \frac{\sqrt{K - \xi}\sqrt{K + \alpha}}{(\alpha - \frac{K}{M})L_-(\alpha)L_+(\xi)\sqrt{K^2 - \alpha^2}\sqrt{K^2 - \xi^2}}. \tag{6.58}$$

It can be seen that if the second term in Eq. (6.56) which is carrying the effect of wake in it, is ignored, the resulting equation is very much similar to that of Eq. (32), where no vertex sheet (wake) is attached to the absorbing half plane [38] (leading edge situation). A natural simplification of the problem is obtained by considering the disturbance to be simple harmonic in time t having frequency ω. The time dependence is described by the factor $\exp(-i\omega t)$. We shall solve the harmonic problem as if the frequency ω is pure imaginary and then obtain the solution for real ω by analytic continuation with respect to ω. In order to calculate the total field $\phi(x, y, t)$, including the details in appendix A, we finally get

$$\phi(x, y, t) = \frac{c}{4\pi Q}\frac{H(t' - \varrho)}{\sqrt{(\varrho^2 - \frac{R'^2}{Q^2})}} - 2cF_1 H(t' - \frac{1}{Q}(R + R_0)) + 2QG_1 H(t - \frac{1}{Q}(R + R_0)), \tag{6.59}$$

It is interesting to note that, the Eq. (6.59) representing the total field with trailing edge situation in the transient nature. The first term in the expression (6.59) represents the field at the observation point when the wave is directly coming from the line source while the second term corresponds to the diffracted field from the edge of the half plane and the third term includes the effect of wake.

Graphical Results

The graphical plotting of the diffracted field given by the second and third term of expression (6.59) is obtained when the source is fixed in one position and the affect of different parameters is observed for the diffracted field. The absorbing parameter B is to be taken such that $\operatorname{Re} B > 0$, which is the necessary condition for an absorbing surface. Since the results are being plotted for the trailing edge situation therefore the Mach number is to be taken such that $0 < M < 1$, for a subsonic flow, which also indicates that the stream flow is from left to right. The following two situations are considered:

1. When the source is fixed in one position (for all values of Mach number), relative to the absorbing barrier, ($\Theta_0 = \frac{\pi}{4}$, M and Θ are allowed to vary).

2. When the source is fixed in one position (for all values of absorbing parameter), relative to the absorbing barrier, ($\Theta_0 = \frac{\pi}{4}$, B and Θ are allowed to vary).

It is observed that the field in the region $0 < \Theta < \pi$, is most affected by the changes in M, B and K. Since there is no field before the source is activated therefore the value of t is taken to be positive in all the cases. The observations are given below:

(a). Fig. (1) is plotted for the diffracted field (trailing edge situation) against the observation angle for different values of the Mach number M by fixing all other parameters while Fig. (3) is plotted for the leading edge situation (no wake) in transient nature just for the comparison. Fig. (5) is plotted for the diffracted field given in [6.59]. As we are considering the subsonic flow i.e., $U < c$, and by increasing the Mach number by fixing all other parameters ($K = 1$, $B = 0.5$, $R = 2.$), the velocity of the fluid is coming closer to the velocity of the sound. The modulus of the velocity potential function is proportional to the amplitude of the perturbation sound pressure and; therefore, gives a measure of sound intensity. Now, It can be seen from the Fig. (1), which is plotted for the trailing situation that the amplitude of the field increases initially with some fluctuations due to the singularities, while in figs. $(3, 5)$, the amplitude decreases by increasing M i.e., diffracted sound intensity was less for leading edge situation than for the trailing edge situation. On physical grounds, one would expect the opposite to be the case. The wake would be having a shielding effect. Since the edge condition employed in the diffraction theory was the normal edge condition, that is why, this theoretical contradiction occurred. This requires that the sound energy is bounded in the finite region around the edge of the half plane which gave rise to field more singular at the origin for the trailing edge situation than for the leading edge situation. The normal edge condition used in the diffraction theory can only be regarded to model the leading edge situation satisfactorily.

(b) In a similar way, the Figs. $(2, 4, 6)$ are plotted for the diffracted field against the observation angle for different values of absorbing parameter B, by fixing all other parameters ($K = 1$, $M = \pm 0.5$, $R = 2,$). By increasing the absorbing parameter B, the amplitude of the diffracted wave will be decreased and consequently the amplitude of the velocity potential will decrease which is also observed in Figs. $(2, 4, 6)$. These figures also show that the sound attenuation increases when the absorbing parameter increases. In particular, the presence of wake considerably reduces sound intensity in the shadow region as compared to the leading edge situation. From fig. (6) it is observed that the absorbing parameter does not make remarkable change in the amplitude of the field when compared it with fig. (2) which is plotted for the trailing edge situation.

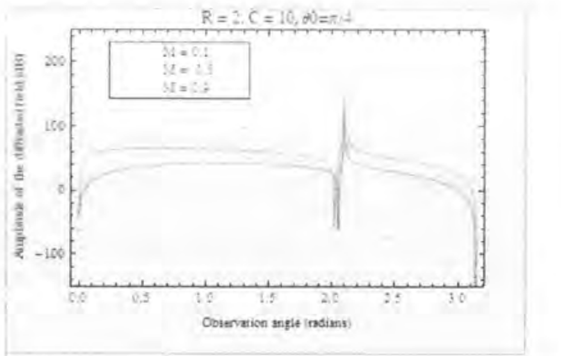


Fig. 6.1: Amplitude of the diffracted field plotted against the observation angle for different values of the Mach number M.

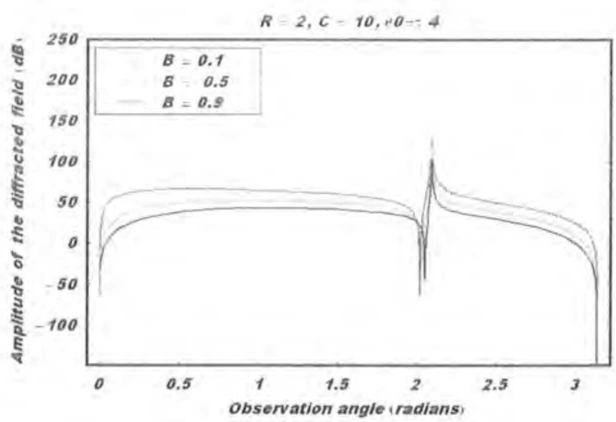


Fig. 6.2: Amplitude of the diffracted field plotted against the observation angle for different values of the admittance parameter B.

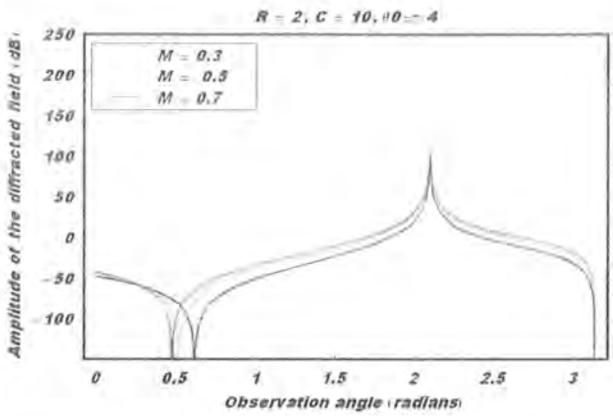


Fig. 6.3: Amplitude of the diffracted field plotted against the observation angle for different values of the Mach number M.

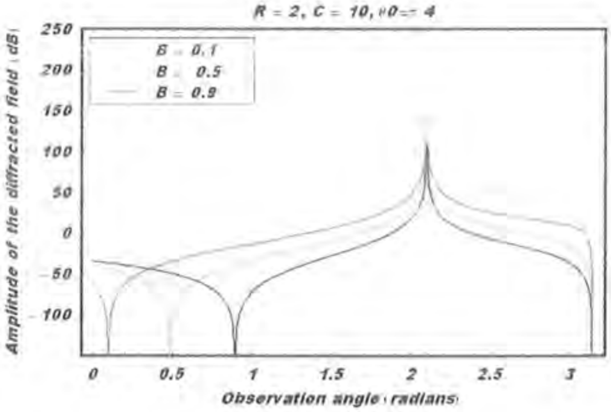


Fig. 6.4: Amplitude of the diffracted field plotted against the observation angle for different values of the admittance parameter B.

It is of interest to consider the half plane as a noise barrier which has effectiveness as a sound barrier, for example an engine above a wing and thus to examine the effects of flow to the sound level in the shadow region. It is shown that the magnitude of the sound diffracted into the shadow region is reduced by the presence of flow which shows that the trailing edge situation is most efficient for reducing the noise in this region.

Conclusion

It is observed that the presence of the Kutta-Joukowski condition does not have much influence on the diffracted field away from the diffracting plane and produces a much stronger field near the wake than elsewhere even when the source is not near the edge. If the transient nature of the field and the presence of wake is ignored in Eq. (6.59), the resulting field becomes that of leading edge situation [38], which is well supported by the graphical results presented in the next section. Also, it is observed that the diffracted field starts reaching the point (x, y) after the time lapse $t' > \frac{1}{Q}(R + R_0)$ and $t > \frac{1}{Q}(R + R_0)$ and the strength of the field dies down as $\frac{1}{\sqrt{R_0}}$. The graphical and the analytical comparison of leading edge situation and trailing edge situation has been made and discussed in detail in the previous section. It is observed that the absorbing half plane with wake gives a more generalized model in diffraction theory and more situations can be discussed as a special case of this problem by choosing the suitable parameters. The problem with more practical applications is one of an absorbing strip in a moving fluid with trailing edge situation. This can be model for an aeroplane wing and has the advantage of being cheaper than to construct a strip with faces entirely coated in absorbent materials. Finally, this work with Myers' impedance condition in contrast to the Ingard's condition, will offer useful theoretical comparisons in conjunction with experimental results. This should lead to a choice as to which boundary condition is to be used in practice.

We can obtain no wake (leading edge) situation by taking $\lambda = 0$ and a field for a rigid barrier by putting $\beta = 0$. Also, the results for the still fluid can be recovered by putting $M = 0$.

Chapter 7

INTERMEDIATE RANGE DIFFRACTION BY A HALF PLANE WITH WAKE

The aim of the present chapter is to analyze the diffraction of waves, in intermediate zone, due to a line source by an absorbing half plane with a wake attached to it, in a moving fluid using Myres condition [133]. It is also to examine the effect of the Kutta-Joukowski condition by introducing the wake (trailing edge) attached to the half plane. As mentioned in 'Abstract', it is a fact that as per available record, no attempt so far has been made to calculate the diffracted field at an intermediate range, from an absorbing half plane with a wake attached to it using Myres' condition.

The solution to the problem, in the intermediate zone, is obtained by using Greens' function method, Fourier transform, Wiener-hopf technique [13] and a modified version of stationary phase method [12]. It is observed that the far field solution may be recovered by shifting the source from intermediate range to the far field position. It is also observed that the field produced by the Kutta-Joukowski condition will be substantially in excess of the field when this condition is absent. The solution for the leading edge situation can be obtained if the wake, and consequently a Kutta-Joukowski edge condition is ignored.

7.1 Formulation of the Problem

We consider the diffraction of an acoustic wave incident on the half plane occupying a space $y = 0, \ x \leq 0$. The line source is located at (x_0, y_0) and the system is placed in a fluid moving with subsonic velocity U parallel to the x-axis. The time dependence is considered to be of harmonic type $e^{-i\omega t}$ (ω is the angular frequency) and is suppressed throughout the process. The plane is assumed to be satisfying the Myers' condition [133]

$$u_n = \frac{-\widetilde{p}}{Z_a} + \frac{U}{i\omega Z_a}\frac{\partial \widetilde{p}}{\partial x}, \tag{7.1}$$

where u_n is the normal derivative of the perturbation velocity, $\widetilde{p}$ is the surface pressure, Z_a is the acoustic impedance of the surface and $-\mathbf{n}$ a normal pointing from the fluid into the surface. The perturbation velocity $\mathbf{u}$ of the irrotational sound wave can be written in terms of the velocity potential Φ as $\mathbf{u} = \nabla\Phi$. The resulting pressure $\widetilde{p}$ of the sound field can be written as

$$\widetilde{p} = -\rho_0\left(-iw + U\frac{\partial}{\partial x}\right)\Phi(x, y). \tag{7.2}$$

where ρ_0 is the density of the undisturbed stream.

The governing convective wave equation with boundary conditions is given by

$$\left[(1 - M^2)\frac{\partial^2}{\partial x^2} + 2ikM\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} + k^2\right]\phi(x, y) = \delta(x - x_0)\delta(y - y_0) \tag{7.3}$$

and

$$\left[\frac{\partial}{\partial y} \mp 2\beta M\frac{\partial}{\partial x} \pm ik\beta \mp \frac{i\beta M^2}{k}\frac{\partial^2}{\partial x^2}\right]\phi(x, 0^{\pm}) = 0 \quad x < 0, \tag{7.4}$$

$$\left.\begin{aligned} \frac{\partial}{\partial y}\phi(x,0^+) &= \frac{\partial}{\partial y}\phi(x,0^-) \\ \left[ik + M\frac{\partial}{\partial x}\right]\phi(x,0^+) &= \left[ik + M\frac{\partial}{\partial x}\right]\phi(x,0^-) \end{aligned}\right\} \quad x > 0. \tag{7.5}$$

In Eqs. (7.4), the quantity 0^+ refers to the situation that $y \to 0$ through positive $y - axis$ and the quantity 0^- refers to the situation that $y \to 0$ through negative $y - axis$. For analytic ease, we shall assume that the wave number $k = k_1 + ik_2$ has a small positive imaginary part to ensure the regularity of the Fourier transform integrals and k_2 is the loss factor of the medium. Here, $\beta = \frac{\rho_0 c}{Z_a}$ is the specific complex admittance and $M = \frac{U}{c}$ (c is the velocity of sound) is the Mach number. It is assumed that the flow is subsonic, i.e., $|M| < 1$ and $\mathrm{Re}\,\beta > 0$, which is a necessary condition for an absorbing surface [114]. We mark that $\beta = 0$ corresponds to the rigid barrier and $\beta = \infty$ corresponds to the pressure release barrier. Let us assume that the flow is subsonic, i.e., $|M| < 1$ (for a leading edge situation $-1 < M \leq 0$ and for a trailing edge situation $0 < M < 1$) and $\mathrm{Re}\,\beta > 0$, which is a necessary condition for an absorbing surface. The trailing edge situation gives rise to the complication of a wake attached to the absorbing half plane. The usual edge conditions give rise to a field which is singular at the origin for the trailing edge situation. Therefore, the Kutta-Joukowski condition is imposed to obtain a unique solution to the problem. The boundary condition (7.5) is obtained due to pressure continuity and can be written in the alternative form as

$$\left.\begin{aligned} \frac{\partial}{\partial y}\phi(x,0^+) &= \frac{\partial}{\partial y}\phi(x,0^-) \\ \phi(x,0^+) - \phi(x,0^-) &= \lambda e^{\frac{ik}{M}x} \end{aligned}\right\} \quad x > 0. \tag{7.6}$$

In Eq. (7.6) the discontinuity in the field is due to imposition of wake which involves a parameter λ. This λ is a constant which can be determined by means of Kutta-Joukowski condition and $\lambda = 0$ corresponds to leading edge situation, i.e., no wake.

Let us introduce the following real substitutions in Eqs. (7.3) – (7.6)

$$x = \sqrt{1-M^2}X,\ x_0 = \sqrt{1-M^2}X_0,\ y = Y,\ y_0 = Y_0,\ \beta = \sqrt{1-M^2}B,\ k = \sqrt{1-M^2}K, \tag{7.7}$$

$$\phi(x,y) = \psi(X,Y)e^{-iKMX}, \tag{7.8}$$

and get

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + K^2\right)\psi(X,Y) = \frac{\delta(X-X_0)\delta(Y-Y_0)}{\sqrt{1-M^2}}e^{iKMX_0}, \tag{7.9}$$

$$\left[\frac{\partial}{\partial Y} \mp 2BM\frac{\partial}{\partial X} \pm iKB(1+M^2) \mp \frac{iBM^2}{(1-M^2)K}\frac{\partial^2}{\partial X^2}\right]\psi(X,0^{\pm}) = 0, \quad x<0, \tag{7.10}$$

$$\left.\begin{aligned} \tfrac{\partial}{\partial Y}\psi(X,0^+) &= \tfrac{\partial}{\partial Y}\psi(X,0^-) \\ \psi(X,0^+) - \psi(X,0^-) &= \lambda e^{\frac{iK}{M}X} \end{aligned}\right\} \quad x>0. \tag{7.11}$$

The total field $\psi(X,Y)$ may be expressed as sum of the incident and scattered field as follows :

$$\psi(X,Y) = \Psi(X,Y) + \Psi_i(X,Y), \tag{7.12}$$

where $\Psi_i(X,Y)$ is the incidence field and $\Psi(X,Y)$ is the diffracted field, so that $\Psi_i(X,Y)$ satisfies

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + K^2\right)\Psi_i(X,Y) = \frac{\delta(X-X_0)\delta(Y-Y_0)}{\sqrt{1-M^2}}e^{-iKMX_0}, \tag{7.13}$$

and $\Psi(X,Y)$ satisfies

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + K^2\right)\Psi(X,Y) = 0. \tag{7.14}$$

We define a spatial Fourier transform and its inverse over the variable x by

$$\left.\begin{aligned} \overline{\Psi}(\alpha,Y) &= \int_{-\infty}^{\infty}\Psi(X,Y)e^{-i\alpha X}dX, \\ \Psi(X,Y) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\Psi}(\alpha,Y)e^{i\alpha X}d\alpha, \end{aligned}\right\} \tag{7.15}$$

and to cater for the two part boundary condition, we write

$$\overline{\Psi}(\alpha, Y) = \overline{\Psi}_-(\alpha, Y) + \overline{\Psi}_+(\alpha, Y), \tag{7.16}$$

where

$$\overline{\Psi}_-(\alpha, Y) = \int_{-\infty}^{0} \Psi(X, Y) e^{-i\alpha x}$$

and

$$\overline{\Psi}_+(\alpha, Y) = \int_{0}^{\infty} \Psi(X, Y) e^{-i\alpha x},$$

so that $\overline{\Psi}_-(\alpha, Y, \omega)$ is regular for $\mathrm{Im}\alpha < \mathrm{Im}\, K$ and $\overline{\Psi}_+(\alpha, Y, \omega)$ is regular for $\mathrm{Im}\alpha > -\mathrm{Im}\, K$.

By the Green's function method, the solution of Eq. (7.13) can be obtained as

$$\Psi_i(X, Y) = \frac{a}{4i} H_0^1(KR) = \frac{a}{4\pi i} \int_{-\infty}^{\infty} \frac{1}{\kappa} e^{i[\alpha(X - X_0) + \kappa|Y - Y_0|]} d\alpha, \tag{7.17}$$

where $a = \frac{e^{iKMX_0}}{\sqrt{1-M^2}}$, $R = \sqrt{(X - X_0)^2 + (Y - Y_0)^2}$, $\kappa = \sqrt{K^2 - \alpha^2}$, K is the wave number and α is the Fourier transform variable.

Transforming Eq. (7.14) by Fourier transform, we get

$$\frac{d^2}{dY^2} \overline{\Psi}(\alpha, Y) + \kappa^2 \overline{\Psi}(\alpha, Y) = 0 \tag{7.18}$$

and its solution satisfying radiation condition is given by

$$\overline{\Psi}(\alpha, Y) = \begin{cases} B_1(\alpha) e^{i\kappa Y} & \text{if } Y \geq 0 \\ B_2(\alpha) e^{-i\kappa Y} & \text{if } Y < 0. \end{cases} \tag{7.19}$$

The Fourier transform of the boundary conditions as given by Eqs. (7.10) and (7.11) takes the following form

$$\overline{\Psi}'_{-}(\alpha,0^{+})+iB\left\{K\left(1+M^{2}\right)+2\alpha M+\frac{\alpha^{2}M^{2}}{\left(1-M^{2}\right)K}\right\}\overline{\Psi}_{-}(\alpha,0^{+})$$

$$=-iB\left\{K\left(1+M^{2}\right)+2\alpha M+\frac{\alpha^{2}M^{2}}{\left(1-M^{2}\right)K}\right\}\overline{\Psi}_{i}(\alpha,0)-\overline{\Psi}'_{i}(\alpha,0), \tag{7.20}$$

and

$$\overline{\Psi}'_{-}(\alpha,0^{-})-iB\left\{K\left(1+M^{2}\right)+2\alpha M+\frac{\alpha^{2}M^{2}}{\left(1-M^{2}\right)K}\right\}\overline{\Psi}_{-}(\alpha,0^{-})$$

$$=iB\left\{K\left(1+M^{2}\right)+2\alpha M+\frac{\alpha^{2}M^{2}}{\left(1-M^{2}\right)K}\right\}\overline{\Psi}_{i}(\alpha,0)-\overline{\Psi}'_{i}(\alpha,0). \tag{7.21}$$

Further

$$\overline{\Psi}_{+}(\alpha,0^{+})-\overline{\Psi}_{+}(\alpha,0^{-})=\frac{-i\lambda}{(\alpha-\frac{K}{M})} \tag{7.22}$$

and

$$\overline{\Psi}'_{+}(\alpha,0^{+})=\overline{\Psi}'_{+}(\alpha,0^{-})=\overline{\Psi}'_{+}(\alpha,0). \tag{7.23}$$

With the help of Eqs. (7.16) and (7.19) – (7.23), we get

$$B_{1}(\alpha)=J_{-}(\alpha,0)+\frac{J'_{-}(\alpha,0)}{i\kappa}-\frac{i\lambda}{2(\alpha-\frac{K}{M})}, \tag{7.24}$$

$$B_{2}(\alpha)=-J_{-}(\alpha,0)+\frac{J'_{-}(\alpha,0,)}{i\kappa}+\frac{i\lambda}{2(\alpha-\frac{K}{M})}, \tag{7.25}$$

where

$$J_{-}(\alpha,0)=\frac{1}{2}[\overline{\Psi}_{-}(\alpha,0^{+})-\overline{\Psi}_{-}(\alpha,0^{-})], \tag{7.26}$$

$$J'_{-}(\alpha,0)=\frac{1}{2}[\overline{\Psi}'_{-}(\alpha,0^{+})-\overline{\Psi}'_{-}(\alpha,0^{-})]. \tag{7.27}$$

From Eqs. (7.16), (7.19), (7.26) and (7.27) and (7.25), we have

$$\overline{\Psi}'_{-}(\alpha,0^{+})+\overline{\Psi}'_{+}(\alpha,0)=i\kappa[\overline{\Psi}_{-}(\alpha,0^{+})+\overline{\Psi}_{+}(\alpha,0^{+})], \tag{7.28}$$

$$\overline{\Psi}'_-(\alpha,0^-) + \overline{\Psi}'_+(\alpha,0) = -i\kappa[\overline{\Psi}_-(\alpha,0^-) + \overline{\Psi}_+(\alpha,0^-)]. \tag{7.29}$$

Eliminating $\overline{\Psi}'_-(\alpha,0^+)$ from Eqs. (7.20) and (7.28), and eliminating $\overline{\Psi}'_-(\alpha,0^-)$ from Eqs. (7.21) and (7.29), we get

$$\overline{\Psi}'_+(\alpha,0) - \left[iB\left\{K\left(1+M^2\right) + 2\alpha M + \frac{\alpha^2 M^2}{(1-M^2)K}\right\} + i\kappa\right]\overline{\Psi}_-(\alpha,0^+)$$

$$-iB\left\{K\left(1+M^2\right) + 2\alpha M + \frac{\alpha^2 M^2}{(1-M^2)K}\right\}\overline{\Psi}_i(\alpha,0)$$

$$= i\kappa\overline{\Psi}_+(\alpha,0^+) + \overline{\Psi}'_i(\alpha,0) \tag{7.30}$$

and

$$\overline{\Psi}'_+(\alpha,0) + \left[iB\left\{K\left(1+M^2\right) + 2\alpha M + \frac{\alpha^2 M^2}{(1-M^2)K}\right\} + i\kappa\right]\overline{\Psi}_-(\alpha,0^-)$$

$$+iB\left\{K\left(1+M^2\right) + 2\alpha M + \frac{\alpha^2 M^2}{(1-M^2)K}\right\}\overline{\Psi}_i(\alpha,0)$$

$$= -i\kappa\overline{\Psi}_+(\alpha,0^-) + \overline{\Psi}'_i(\alpha,0). \tag{7.31}$$

The addition of Eqs. (7.30) and (7.31), results in

$$i\kappa L(\alpha)J_-(\alpha,0) - \overline{\Psi}'_+(\alpha,0) + \overline{\Psi}'_i(\alpha,0) + \frac{K\lambda}{2(\alpha - \frac{K}{M})} = 0. \tag{7.32}$$

Similarly, eliminating $\overline{\Psi}_-(\alpha,0^+)$ from Eqs. (7.20) and (7.28), and eliminating $\overline{\Psi}_-(\alpha,0^-)$ from Eqs. (7.21) and (7.29), we get

$$\overline{\Psi}'_-(\alpha,0^+) + \frac{1}{i\kappa}iB\left\{K\left(1+M^2\right) + 2\alpha M + \frac{\alpha^2 M^2}{(1-M^2)K}\right\}\overline{\Psi}'_-(\alpha,0^+)$$

$$+\frac{1}{i\kappa}iB\left\{K\left(1+M^2\right) + 2\alpha M + \frac{\alpha^2 M^2}{(1-M^2)K}\right\}\overline{\Psi}'_+(\alpha,0)$$

$$-iB\left\{K\left(1+M^2\right) + 2\alpha M + \frac{\alpha^2 M^2}{(1-M^2)K}\right\}\overline{\Psi}_+(\alpha,0^+)$$

$$+iB\left\{K\left(1+M^2\right)+2\alpha M+\frac{\alpha^2M^2}{(1-M^2)K}\right\}\overline{\Psi}_i(\alpha,0,\omega)+\overline{\Psi}_i'(\alpha,0)=0 \tag{7.33}$$

and

$$\overline{\Psi}_-'(\alpha,0^-)+\frac{1}{i\kappa}iB\left\{K\left(1+M^2\right)+2\alpha M+\frac{\alpha^2M^2}{(1-M^2)K}\right\}\overline{\Psi}_-'(\alpha,0^-)$$

$$+\frac{1}{i\kappa}iB\left\{K\left(1+M^2\right)+2\alpha M+\frac{\alpha^2M^2}{(1-M^2)K}\right\}\overline{\Psi}_+'(\alpha,0)$$

$$+iB\left\{K\left(1+M^2\right)+2\alpha M+\frac{\alpha^2M^2}{(1-M^2)K}\right\}\overline{\Psi}_+(\alpha,0^-)+iB\left\{K\left(1+M^2\right)+2\alpha M\right.$$

$$\left.+\frac{\alpha^2M^2}{(1-M^2)K}\right\}\overline{\Psi}_i(\alpha,0)+\overline{\Psi}_i'(\alpha,0)=0. \tag{7.34}$$

The subtraction of Eqs. (7.33) and (7.35) gives

$$\frac{2L(\alpha)J_-'(\alpha,0)}{B\left[K\left(1+M^2\right)+2\alpha M+\frac{\alpha^2M^2}{(1-M^2)K}\right]}-i\overline{\Psi}_+(\alpha,0^+)-i\overline{\Psi}_+(\alpha,0^-)+2i\overline{\Psi}_i(\alpha,0)=0, \tag{7.35}$$

where

$$L(\alpha)=1+\frac{B}{\kappa}\left[K\left(1+M^2\right)+2\alpha M+\frac{\alpha^2M^2}{(1-M^2)K}\right]. \tag{7.36}$$

Eqs. (7.32) and (7.35) are the standard Wiener-Hopf ($W-H$) equations. We proceed to find their solutions in the next section.

7.2 Solution of the W-H Equations

Let us write

$$L(\alpha)=L_+(\alpha)L_-(\alpha) \tag{7.37}$$

and

$$\kappa=\sqrt{K+\alpha}\sqrt{K-\alpha}\,. \tag{7.38}$$

Making use of Eqs. (7.37) and (7.38) in Eq. (7.32), we get

$$iJ_-(\alpha,0)L_-(\alpha)\sqrt{K-\alpha}+\frac{\overline{\Psi}'_i(\alpha,0)}{L_+(\alpha)\sqrt{K+\alpha}}+\frac{\lambda\sqrt{K-\alpha}}{2L_+(\alpha)(\alpha-\frac{K}{M})}=\frac{\overline{\Psi}'_+(\alpha,0)}{L_+(\alpha)\sqrt{K+\alpha}}, \quad (7.39)$$

whereas in Eq. (7.39), the first term on the left hand side is regular in the lower half plane and the term on the right hand side is regular in the upper half plane. For the other two terms, let us write

$$\frac{\overline{\Psi}'_i(\alpha,0)}{L_+(\alpha)\sqrt{K+\alpha}}=T_+(\alpha)+T_-(\alpha), \quad (7.40)$$

where

$$T_+(\alpha)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\overline{\Psi}'_i(\xi,0)}{L_+(\xi)\sqrt{K+\xi}(\xi-\alpha)}d\xi, \quad (7.41)$$

$$T_-(\alpha)=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\overline{\Psi}'_i(\xi,0)}{L_+(\xi)\sqrt{K+\xi}(\xi-\alpha)}d\xi \quad (7.42)$$

and

$$\frac{\lambda\sqrt{K-v}}{2L_+(\alpha)(\alpha-\frac{K}{M})}=F_+(\alpha)+F_-(\alpha), \quad (7.43)$$

where

$$F_+(\alpha)=\frac{\lambda}{2(\alpha-\frac{K}{M})}\left[\frac{\sqrt{K-\alpha}}{L_+(\alpha)}-\frac{\sqrt{K-\frac{K}{M}}}{L_+(\frac{K}{M})}\right], \quad (7.44)$$

$$F_-(\alpha)=\frac{\lambda}{2(\alpha-\frac{K}{M})}\left[\frac{\sqrt{K-\frac{K}{M}}}{L_+(\frac{K}{M})}\right]. \quad (7.45)$$

Now, we use Eqs. (7.40) and (7.43) in Eq. (7.39), and equate the terms with negative sign on the left hand side and terms with positive sign on the right hand side, and then use the extended form of Liouville's theorem to get

$$iJ_-(\alpha,0)\sqrt{K-\alpha}L_-(\alpha)+T_-(\alpha)+F_-(\alpha)=0, \quad (7.46)$$

and by using Eq. (7.42) and (7.45), we get

$$J_{-}(\alpha,0)=\frac{-1}{2\pi L_{-}(\alpha)\sqrt{K-\alpha}}\int_{-\infty}^{\infty}\frac{\overline{\Psi}_{i}'(\xi,0)}{L_{+}(\xi)\sqrt{K+\xi}(\xi-\alpha)}d\xi+\frac{i\lambda\sqrt{K-\frac{K}{M}}}{2L_{-}(\alpha)\sqrt{K-\alpha}L_{+}(\frac{K}{M})(\alpha-\frac{K}{M})}. \tag{7.47}$$

Similarly, adopting the same procedure as in the case of Eq. (7.32), we can write Eq. (7.35) as follows

$$J_{-}'(\alpha,0)=\frac{1}{2\pi L_{-}(\alpha)}\int_{-\infty}^{\infty}B\left[K\left(1+M^{2}\right)+2\xi M+\frac{\xi^{2}M^{2}}{\left(1-M^{2}\right)K}\right]\frac{\overline{\Psi}_{i}(\xi,0)}{L_{+}(\xi)(\xi-\alpha)}d\xi. \tag{7.48}$$

Making use of Eqs. (7.47) – (7.48) in Eqs. (7.24) – (7.25) respectively, we obtain

$$\left.\begin{array}{c}B_{1}(\alpha)\\ \\ B_{2}(\alpha)\end{array}\right\}=\frac{1}{2\pi L_{-}(\alpha)i\kappa}\int_{-\infty}^{\infty}B\left[K\left(1+M^{2}\right)+2\xi M+\frac{\xi^{2}M^{2}}{\left(1-M^{2}\right)K}\right]\frac{\overline{\Psi}_{i}(\xi,0)}{L_{+}(\xi)(\xi-\alpha)}d\xi$$

$$\mp\frac{1}{2\pi L_{-}(\alpha)\sqrt{K-\alpha}}\int_{-\infty}^{\infty}\frac{\overline{\Psi}_{i}'(\xi,0)}{L_{+}(\xi)\sqrt{K+\xi}(\xi-\alpha)}d\xi$$

$$\pm\frac{i\lambda}{2(\alpha-\frac{K}{M})}\left[\frac{\sqrt{K-\frac{K}{M}}-L_{-}(\alpha)\sqrt{K-\alpha}L_{+}(\frac{K}{M})}{L_{-}(\alpha)\sqrt{K-\alpha}L_{+}(\frac{K}{M})}\right], \tag{7.49}$$

With the help of Eq. (7.17), we can write the above equation as

$$\left.\begin{array}{c}B_{1}(\alpha)\\ \\ B_{2}(\alpha)\end{array}\right\}=\frac{-a}{4\pi L_{-}(\alpha)\kappa}\int_{-\infty}^{\infty}\frac{B\left[K\left(1+M^{2}\right)+2\xi M+\frac{\xi^{2}M^{2}}{(1-M^{2})K}\right]}{L_{+}(\xi)(\xi-\alpha)\sqrt{K^{2}-\xi^{2}}}e^{-i\xi X_{0}+i\sqrt{K^{2}-\xi^{2}}|Y_{0}|}\,d\xi$$

$$\pm\frac{a}{4\pi L_{-}(\alpha)\sqrt{K-\alpha}}\int_{-\infty}^{\infty}\frac{e^{-i\xi X_{0}+i\sqrt{K^{2}-\xi^{2}}|Y_{0}|}sgn\,|Y_{0}|}{L_{+}(\xi)\sqrt{K+\xi}(\xi-\alpha)}d\xi$$

$$\pm\frac{i\lambda}{2(\alpha-\frac{K}{M})}\left[\frac{\sqrt{K-\frac{K}{M}}-L_{-}(\alpha)\sqrt{K-\alpha}L_{+}(\frac{K}{M})}{L_{-}(\alpha)\sqrt{K-\alpha}L_{+}(\frac{K}{M})}\right]. \tag{7.50}$$

In order to ensure a unique mathematical solution, we must impose Kutta-Joukowski edge condition which requires that the velocity should be finite at the origin. Hence, in order that the Kutta-Joukowski condition be satisfied, we choose λ as

$$\lambda = \frac{aL_+(\frac{K}{M}) sgn\,|Y_0|}{2\pi i\left[\sqrt{K-\frac{K}{M}} - L_+(\frac{K}{M})\sqrt{K-\alpha}L_-(\alpha)\right]} \int\limits_{-\infty}^{\infty} \frac{e^{-i\xi X_0 + i\sqrt{K^2-\xi^2}|Y_0|}}{L_+(\xi)\sqrt{K+\xi}} d\xi \text{ as } |\xi| \to \infty. \tag{7.51}$$

Using the value of λ in Eq. (7.50), we get

$$\left.\begin{array}{l} B_1(\alpha) \\ \\ B_2(\alpha) \end{array}\right\} = \frac{-a}{4\pi} \int\limits_{-\infty}^{\infty} \frac{B\left[K(1+M^2) + 2\xi M + \frac{\xi^2 M^2}{(1-M^2)K}\right]}{L_-(\alpha)L_+(\xi)(\xi-\alpha)\sqrt{K^2-\alpha^2}\sqrt{K^2-\xi^2}} e^{-i\xi X_0 + i\sqrt{K^2-\xi^2}|Y_0|}\, d\xi$$

$$\pm \frac{a}{4\pi} \int\limits_{-\infty}^{\infty} \frac{\sqrt{K+\alpha}\sqrt{K-\xi}\; sgn\,|Y_0|}{L_-(\alpha)L_+(\xi)(\xi-\alpha)\sqrt{K^2-\alpha^2}\sqrt{K^2-\xi^2}} e^{-i\xi X_0 + i\sqrt{K^2-\xi^2}|Y_0|}\, d\xi$$

$$\pm \frac{a\; sgn\,|Y_0|}{4\pi(\alpha - \frac{K}{M})L_-(\alpha)\sqrt{K-\alpha}} \int\limits_{-\infty}^{\infty} \frac{e^{-i\xi X_0 + i\sqrt{K^2-\xi^2}|Y_0|}}{L_+(\xi)\sqrt{K+\xi}} d\xi. \tag{7.52}$$

We use this Eq. (7.52) in Eq. (7.19) to obtain

$$\overline{\Psi}(\alpha, Y) = \left[\frac{-a}{4\pi} \int\limits_{-\infty}^{\infty} \frac{B\left[K(1+M^2) + 2\xi M + \frac{\xi^2 M^2}{(1-M^2)K}\right]}{L_-(\alpha)L_+(\xi)(\xi-\alpha)\sqrt{K^2-\alpha^2}\sqrt{K^2-\xi^2}} e^{-i\xi X_0 + i\sqrt{K^2-\xi^2}|Y_0|}\, d\xi\right.$$

$$\pm \frac{a}{4\pi} \int\limits_{-\infty}^{\infty} \frac{\sqrt{K+\alpha}\sqrt{K-\xi}\; sgn\,|Y_0|}{L_-(\alpha)L_+(\xi)(\xi-\alpha)\sqrt{K^2-\alpha^2}\sqrt{K^2-\xi^2}} e^{-i\xi X_0 + i\sqrt{K^2-\xi^2}|Y_0|}\, d\xi$$

$$\left.\pm \frac{a\; sgn\,|Y_0|}{4\pi} \int\limits_{-\infty}^{\infty} \frac{e^{-i\xi X_0 + i\sqrt{K^2-\xi^2}|Y_0|}}{(\alpha - \frac{K}{M})L_-(\alpha)\sqrt{K-\alpha}L_+(\xi)\sqrt{K+\xi}} d\xi\right] e^{i\kappa|Y|}. \tag{7.53}$$

We take the inverse Fourier transform of Eq, (7.53) and get

$$\Psi(X,Y) = \left[\frac{-a}{8\pi^2} \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{B\left[K(1+M^2) + 2\xi M + \frac{\xi^2 M^2}{(1-M^2)K}\right] \mp \sqrt{K+\alpha}\sqrt{K-\xi}\; sgn\,|Y_0|}{L_-(\alpha)L_+(\xi)(\xi-\alpha)\sqrt{K^2-\alpha^2}\sqrt{K^2-\xi^2}}\right.$$

$$-\frac{\sqrt{K-\xi}\sqrt{K+\alpha}\ sgn\,|YY_0|}{(\alpha-\frac{K}{M})L_-(\alpha)L_+(\xi)\sqrt{K^2-\alpha^2}\sqrt{K^2-\xi^2}}\Bigg]e^{-i\xi X_0+i\sqrt{K^2-\xi^2}|Y_0|}\,e^{i\kappa|Y|+i\alpha X}d\xi d\alpha \quad (7.54)$$

and with the help of Eqs. (7.9) and (7.12), we get

$$\phi(x,y)=\frac{\exp\left[-iKM(X-X_0)\right]}{4\pi i\sqrt{(1-M^2)}}\int\limits_{-\infty}^{\infty}\frac{e^{i\alpha(X-X_0)+i\kappa(Y-Y_0)}}{\kappa}d\alpha$$

$$+\frac{e^{iKM(X_0-X)}sgn\,|YY_0|}{8\pi^2\sqrt{1-M^2}}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}G(\alpha,\xi)e^{-i\xi X_0+i\sqrt{K^2-\xi^2}|Y_0|}\,e^{i\kappa|Y|+i\alpha X}d\xi d\alpha$$

$$-\frac{e^{iKM(X_0-X)}}{8\pi^2\sqrt{1-M^2}}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}F(\alpha,\xi)e^{-i\xi X_0+i\sqrt{K^2-\xi^2}|Y_0|}\,e^{i\kappa|Y|+i\alpha X}d\xi d\alpha, \quad (7.55)$$

where

$$F(\alpha,\xi)=\frac{B\left[K\left(1+M^2\right)+2\xi M+\frac{\xi^2M^2}{(1-M^2)K}\right]-\sqrt{K+\alpha}\sqrt{K-\xi}\ sgn\,|Y_0|}{L_-(\alpha)L_+(\xi)(\xi-\alpha)\sqrt{K^2-\alpha^2}\sqrt{K^2-\xi^2}} \quad (7.56)$$

and

$$G(\alpha,\xi)=\frac{\sqrt{K-\xi}\sqrt{K+\alpha}}{(\alpha-\frac{K}{M})L_-(\alpha)L_+(\xi)\sqrt{K^2-\alpha^2}\sqrt{K^2-\xi^2}}. \quad (7.57)$$

7.3 Intermediate Range Solution

In the calculations of the integrals of Ahmad [46], the terms of $O\left(1/\left(KR_0\right)^{\frac{1}{2}}\right)$ are retained and higher order terms neglected. If we consider intermediate range approximation in terms of source position, we need to retain the next terms of $O\left(1/\left(KR_0\right)^{\frac{3}{2}}\right)$ in the expansion of the Hankel function. For the the intermediate range solution, let us write Eq. (7.55) as

$$\phi(x,y)=\frac{\exp\left[-iKM(X-X_0)\right]}{4\pi i\sqrt{(1-M^2)}}I_1++\frac{e^{iKM(X_0-X)}sgn\,|YY_0|}{8\pi^2\sqrt{1-M^2}}I_2-\frac{e^{iKM(X_0-X)}}{8\pi^2\sqrt{1-M^2}}I_3, \quad (7.58)$$

where

$$I_1 = H_0^{(1)}(KR) = \int_{-\infty}^{\infty} \frac{e^{i\alpha(X-X_0)+i\kappa(Y-Y_0)}}{\kappa} d\alpha, \tag{7.59}$$

and its solution for the intermediate range (which is accomplished in appendix B) is given as

$$I_1 = \frac{e^{iKMX_0}\ e^{iK[X\cos(\theta_0+iq)+\kappa Y\sin(\theta_0+iq)]}\ e^{iKR-i\frac{\pi}{4}}}{2i\sqrt{1-M^2}\sqrt{2\pi KR_0}}(1-\frac{i}{8KR_0}). \tag{7.60}$$

and

$$I_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\alpha,\xi)e^{-i\xi X_0+i\sqrt{K^2-\xi^2}|Y_0|}\ e^{i\kappa|Y|+i\alpha X}d\xi d\alpha \tag{7.61}$$

and its solution for the intermediate range (as in appendix C) is given as

$$I_2 = \frac{i}{2}(2A_1+A_2)\alpha_s\left[-i\alpha_t G(\alpha_p,\alpha_m)A_3 - \frac{i}{2}\begin{bmatrix} 2i\alpha_p G_{q_2}(\alpha_p,\alpha_m) \\ +\alpha_t\left\{G(\alpha_p,\alpha_m)+G_{q_2q_2}(\alpha_p,\alpha_m)\right\}\end{bmatrix}A_4\right]$$

$$-\alpha_m A_2\left[-i\alpha_t G_{q_1}(\alpha_p,\alpha_m)A_3 - \frac{i}{2}\begin{bmatrix} 2i\alpha_p G_{q_1q_2}(\alpha_p,\alpha_m) \\ +\alpha_t\left\{G_{q_1}(\alpha_p,\alpha_m)+G_{q_1q_2q_2}(\alpha_p,\alpha_m)\right\}\end{bmatrix}A_4\right]$$

$$+\frac{i\alpha_s}{2}A_2\left[-i\alpha_t G_{q_1q_1}(\alpha_p,\alpha_m)A_3 - \frac{i}{2}\begin{bmatrix} 2i\alpha_p G_{q_1q_1q_2}(\alpha_p,\alpha_m) \\ +\alpha_t\left\{G_{q_1q_1}(\alpha_p,\alpha_m)+G_{q_1q_1q_2q_2}(\alpha_p,\alpha_m)\right\}\end{bmatrix}A_4\right]. \tag{7.62}$$

and similarly

$$I_3 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\alpha,\xi)e^{-i\xi X_0+\sqrt{K^2-\xi^2}|Y_0|}e^{i\alpha X+i\kappa|Y_0|}d\xi d\alpha. \tag{7.63}$$

and its solution for the intermediate range (which is given in appendix C) is given as

$$I_3 = \frac{e^{-iKM(X-X_0)}}{8\pi^2\sqrt{1-M^2}\sqrt{K^2-\alpha_p^2}}\left(\frac{2\pi}{KR_0}\right)^{\frac{1}{2}}\left[\frac{F(\alpha_p)}{(\alpha_m-\alpha_p)}e^{i\left(KR_0+\frac{\pi}{4}\right)}\right.$$

$$+\left(\frac{1}{KR_0}\right)\left\{\frac{F_1(\alpha_p)}{(\alpha_m-\alpha_p)}+\frac{F_2(\alpha_p)}{(\alpha_m-\alpha_p)^2}+\frac{F_3(\alpha_p)}{(\alpha_m-\alpha_p)^3}\right\}e^{i\left(KR_0-\frac{\pi}{4}\right)}\Bigg]$$

$$\times(\alpha_t)\sqrt{\frac{2\pi}{KR}}e^{i\left(KR+\frac{\pi}{4}\right)}. \quad (7.64)$$

We can also write Eq. (7.64) as

$$I_3=\frac{e^{-iKM(X-X_0)}}{8\pi^2\sqrt{1-M^2}\sqrt{K^2-\alpha_p^2}}\left(\frac{2\pi}{KR_0}\right)^{\frac{1}{2}}e^{iKR}$$

$$\times\left\{\begin{array}{c}-i\pi^{\frac{1}{2}}F(\alpha_p)\,e^{iKR_0}\\ -\left(\frac{1}{KR_0}\right)\left\{\pi^{\frac{1}{2}}F_1(\alpha_p)\,e^{i\frac{\pi}{4}}+\frac{\pi^{\frac{1}{2}}F_2(\alpha_p)}{(\alpha_m-\alpha_p)}e^{i\frac{\pi}{4}}+\frac{\pi^{\frac{1}{2}}F_3(\alpha_p)}{(\alpha_m-\alpha_p)^2}e^{i\frac{\pi}{4}}\right\}\\ \times e^{i\left(KR_0-\frac{\pi}{4}\right)}\end{array}\right\}\times F_r\left(b\eta^{\frac{1}{2}}\right), \quad (7.65)$$

where $F_r\left(b\eta^{\frac{1}{2}}\right)$ is Fresnel function, given by

$$F_r\left(b\eta^{\frac{1}{2}}\right)=e^{-i\eta b^2}\int_{b\eta^{\frac{1}{2}}}^{+\infty}e^{-ih^2}dh \quad (7.66)$$

and

$$b=(\alpha_m-\alpha_p). \quad (7.67)$$

With the help of Eqs. (7.60), (7.62) and (7.65), we can write Eq. (7.58),i.e., the complete solution as

$$\phi(x,y)=\frac{e^{iKMX_0}\;e^{iK[X\cos(\theta_0+iq)+\kappa Y\sin(\theta_0+iq)]}\;e^{iKR-i\frac{\pi}{4}}}{2i\sqrt{1-M^2}\sqrt{2\pi KR_0}}(1-\frac{i}{8KR_0})$$

$$+\frac{e^{iKM(X_0-X)}sgn\,|YY_0|}{8\pi^2\sqrt{1-M^2}}\left[\frac{i}{2}(2A_1+A_2)\alpha_s\left\{\begin{array}{c}-i\alpha_tG(\alpha_p,\alpha_m)A_3-\\ \frac{i}{2}\left(\begin{array}{c}2i\alpha_pG_{q_2}(\alpha_p,\alpha_m)\\ +\alpha_t\left\{G(\alpha_p,\alpha_m)+G_{q_2q_2}(\alpha_p,\alpha_m)\right\}\end{array}\right)\end{array}\right\}A_4\right.$$

$$-\alpha_m A_2 \left\{ \frac{i}{2} \begin{pmatrix} -i\alpha_t G_{q_1}(\alpha_p, \alpha_m) A_3 - \\ 2i\alpha_p G_{q_1 q_2}(\alpha_p, \alpha_m) \\ +\alpha_t \left\{ G_{q_1}(\alpha_p, \alpha_m) + G_{q_1 q_2 q_2}(\alpha_p, \alpha_m) \right\} \end{pmatrix} A_4 \right\}$$

$$+\frac{i\alpha_s}{2} A_2 \left\{ -i\alpha_t G_{q_1 q_1}(\alpha_p, \alpha_m) A_3 - \frac{i}{2} \begin{pmatrix} 2i\alpha_p G_{q_1 q_1 q_2}(\alpha_p, \alpha_m) \\ +\alpha_t \left\{ G_{q_1 q_1}(\alpha_p, \alpha_m) + G_{q_1 q_1 q_2 q_2}(\alpha_p, \alpha_m) \right\} \end{pmatrix} A_4 \right\} \Bigg]$$

$$-\left[\frac{e^{-iKM(X-X_0)}}{8\pi^2 \sqrt{1-M^2} \sqrt{K^2 - \alpha_p^2}} \left(\frac{2\pi}{KR_0} \right)^{\frac{1}{2}} e^{iKR} \right.$$

$$\left. \times \left\{ \begin{array}{c} -i\pi^{\frac{1}{2}} F(\alpha_p) e^{iKR_0} \\ -\left(\frac{1}{KR_0}\right) \left\{ \pi^{\frac{1}{2}} F_1(\alpha_p) e^{i\frac{\pi}{4}} + \frac{\pi^{\frac{1}{2}} F_2(\alpha_p)}{(\alpha_m - \alpha_p)} e^{i\frac{\pi}{4}} + \frac{\pi^{\frac{1}{2}} F_3(\alpha_p)}{(\alpha_m - \alpha_p)^2} e^{i\frac{\pi}{4}} \right\} \\ \times e^{i\left(KR_0 - \frac{\pi}{4}\right)} \end{array} \right\} \times F_r\left(b\eta^{\frac{1}{2}}\right) \right]. \quad (7.68)$$

7.4 Conclusion

1. It is observed that for a line source near the edge of an absorbing half plane, the field caused by the Kutta-Joukowski condition will be larger than the field caused in its absence. The wake has an effect of producing a stronger field when the source is near the edge. The leading edge solution can be recovered if wake and consequently Kutta-Joukowski condition is ignored.

2. It can be seen that the solution for plane wave diffraction and line source diffraction differ by a factor

$$\frac{e^{-iKM(X-X_0)}}{4i\sqrt{1-M^2}} \left(\frac{2}{\pi KR_0} \right)^{\frac{1}{2}} e^{i\left(KR_0 - \frac{\pi}{4}\right)},$$

which appears due to presence of line source but for intermediate range solution, the plane wave and the intermediate solutions differ by a multiplicative factor

$$\frac{e^{-iKM(X-X_0)}}{4i\sqrt{1-M^2}} \left(\frac{2}{\pi KR_0} \right)^{\frac{1}{2}} e^{i\left(KR_0 - \frac{\pi}{4}\right)} \left(1 - \frac{i}{8KR_0}\right).$$

3. It is observed that the diffracted area in far field is of order $O(\frac{1}{\sqrt{KR_0}})$,while in the intermediate field, it is of order $O(\frac{1}{\sqrt[3]{KR_0}})$. The term containing $(\frac{1}{\sqrt[3]{KR_0}})$ gives rise to an additional term in the diffracted field which produces a stronger field where the diffraction pattern will also be more visible and clear.

4. The diffracted field for both the far field approximations and the intermediate approximations differ by a factor

$$\frac{e^{-iKM(X-X_0)}}{8\pi^2\sqrt{1-M^2}\sqrt{K^2-\alpha_s^2}}\left(\frac{2\pi}{KR_0}\right)^{\frac{1}{2}}e^{iKR}$$
$$\left[-i\pi^{\frac{1}{2}}F\left(\alpha_p\right)e^{iKR_0}-\left(\frac{1}{KR_0}\right)\left\{\pi^{\frac{1}{2}}F_1\left(\alpha_s\right)e^{i\frac{\pi}{4}}+\frac{\pi^{\frac{1}{2}}F_2\left(\alpha_s\right)}{\left(\alpha_p-\alpha_s\right)}e^{i\frac{\pi}{4}}+\frac{\pi^{\frac{1}{2}}F_3\left(\alpha_s\right)}{\left(\alpha_p-\alpha_s\right)^2}e^{i\frac{\pi}{4}}\right\}e^{i\left(KR_0-\frac{\pi}{4}\right)}\right]$$
$$\times e^{-\eta b^2}F\left(b\eta^{\frac{1}{2}}\right).$$

5. The leading edge solution can be recovered if wake is ignored and R is kept very large such that the terms of order $O(\frac{1}{\sqrt[3]{KR}})$ are neglected in Eq. (7.68) when the solution reduces to that of the far field approximations [46].

Chapter 8

DIFFRACTION OF SOUND WAVES BY A FINITE BARRIER IN AN INTERMEDIATE ZONE

In this chapter, we discuss the diffraction of an acoustic wave from a finite absorbing barrier at an intermediate range by using Myers' impedance boundary condition. As mentioned in 'Abstract', it is a fact that as per available record, no attempt so far has been made to calculate the diffracted field at an intermediate range, from an absorbing finite barrier using Myres' condition. This is the first attempt in this direction and may lead to a variety of research in the diffraction phenomenon.

In calculation of the integrals [134] , the terms of $\left[\frac{1}{(KR_0)^{\frac{1}{2}}}\right]$ are retained. If we consider intermediate range approximation in terms of source position, we need to retain the next terms of $\left[\frac{1}{(KR_0)^{\frac{3}{2}}}\right]$ in the expansion of the Hankel function. With consideration of these terms, the difficulty that arises is the solution of integrals occurring in inverse Fourier transform. These integrals are normally difficult to handle because of the presence of branch points and are only amenable to solution using asymptotic approximations. The analytic solution of these integrals is obtained by using a modified version of the stationary phase method and the field for an intermediate range in terms of the source position is calculated. The integral transforms, Wiener-Hopf technique [13] and asymptotic methods [12] are used to calculate the diffracted field. It is found that the two edges of the finite barrier give rise to two diffracted fields, i.e. one from each edge and the second from an interaction field. The results for the half plane [46] can be recovered by taking an appropriate limit.

8.1 Formulation of the Problem

We consider the diffraction of an acoustic wave incident on the finite absorbing plane occupying a space $y = 0, -l \leq x \leq 0$. The mathematical form of the problem may be expressed in terms of the equations satisfied by $\Phi(x, y)$ as follows:

$$\left[(1 - M^2)\frac{\partial^2}{\partial x^2} + 2ikM\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} + k^2\right]\Phi(x, y) = \delta(x - x_0)\delta(y - y_0), \tag{8.1}$$

subject to the following boundary conditions

$$\left[\frac{\partial}{\partial y} \mp 2\beta M\frac{\partial}{\partial x} \pm ik\beta \mp \frac{i\beta M^2}{k}\frac{\partial^2}{\partial x^2}\right]\Phi(x, 0^\pm) = 0 \quad -l < x < 0, \tag{8.2}$$

$$\frac{\partial}{\partial y}\Phi(x, 0^+) = \frac{\partial}{\partial y}\Phi(x, 0^-) \quad -\infty < x < -l, \; x > 0,$$

$$\Phi(x, 0^+) = \Phi(x, 0^-) \quad -\infty < x < -l, \; x > 0. \tag{8.3}$$

In above equations, $k = \frac{\omega}{c}$ is the wave number, $\beta = \frac{\rho_0 c}{Z_a}$ is the specific complex admittance and $M = \frac{U}{c}$ (c being velocity of sound) is the Mach number. It is assumed that the flow is subsonic, i.e., $|M| < 1$, and $\operatorname{Re}\beta > 0$, a necessary condition for an absorbing surface.

The formulation of the problem is the same as discussed in [78]. Therefore, without going into details, we mention the results only. The solution for the incident wave is given in integral form as

$$\Psi_0(X,Y) = \frac{a}{4\pi i}\int_{-\infty}^{\infty} \frac{1}{\kappa} e^{i[\alpha(X-X_0)+\kappa|Y-Y_0|]} d\alpha, \tag{8.4}$$

where $a = \frac{e^{iKMX_0}}{\sqrt{1-M^2}}$, $R = \sqrt{(X-X_0)^2+(Y-Y_0)^2}$, $\kappa = \sqrt{K^2-\alpha^2}$. Here, K is the wave number, and α is the Fourier transform variable. We are to present the incident wave solution in an intermediate zone, and for that, we use the following transformation

$$X_0 = R_0\cos\theta_0, |Y_0| = R_0\sin\theta_0 \text{ and } \alpha = -K\cos(\theta_0+iq), \ (0<\theta_0<\pi, -\infty<q<\infty).$$

in the integral appearing in Eq. (8.4), and get

$$\Psi_0(X,Y) = \frac{ae^{iK[X\cos(\theta_0+iq)+\kappa Y\sin(\theta_0+iq)]}}{4\pi i}\int_{-\infty}^{\infty} e^{iKR_0\cosh q} dq,$$

or

$$\Psi_0(X,Y) = \frac{ae^{iK[X\cos(\theta_0+iq)+\kappa Y\sin(\theta_0+iq)]}}{4\pi i}\int_{-\infty}^{\infty} e^{\frac{iKR_0q^2}{2}(1+\frac{q^2}{12}+\ldots)} dq,$$

by rotating $q = r\ e^{\frac{i\pi}{4}}$ in above equation, we obtain

$$\Psi_0(X,Y) = \frac{ae^{iK[X\cos(\theta_0+iq)+\kappa Y\sin(\theta_0+iq)]}\ e^{iKR+i\frac{\pi}{4}}}{2\pi i}\int_{0}^{\infty} e^{-\frac{KR_0r^2}{2}(1+\frac{ir^2}{12}+\ldots)} dr.$$

Now letting $\frac{KR_0r^2}{2} = \tau$ in above integral, we get

$$\Psi_0(X,Y) = \frac{ae^{iK[X\cos(\theta_0+iq)+\kappa Y\sin(\theta_0+iq)]}\ e^{iKR-i\frac{\pi}{4}}}{2\pi i\sqrt{2KR_0}}\int_{0}^{\infty} e^{-\tau}(\tau)^{-\frac{1}{2}}(1-\frac{i\tau^2}{6KR_0})d\tau,$$

or

$$\Psi_0(X,Y) = \frac{ae^{iK[X\cos(\theta_0+iq)+\kappa Y\sin(\theta_0+iq)]}\ e^{iKR-i\frac{\pi}{4}}}{2\pi i\sqrt{2KR_0}}(\sqrt{\pi}-\frac{i\sqrt{\pi}}{8KR_0}),$$

or

$$\Psi_0(X,Y) = \frac{e^{iKMX_0}\, e^{iK[X\cos(\theta_0+iq)+\kappa Y\sin(\theta_0+iq)]}\, e^{iKR-i\frac{\pi}{4}}}{2i\sqrt{1-M^2}\sqrt{2\pi KR_0}}(1-\frac{i}{8KR_0}). \tag{8.5}$$

Eq. (8.5) gives the incident wave solution in the intermediate zone. To find the diffracted field for the source (which is not very far from edge of the barrier) in the intermediate zone, we use Eq. (97) of [78] and get

$$\Psi(X,Y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left. \begin{array}{c} A_1(\alpha) \\ \\ A_2(\alpha) \end{array} \right\} \frac{e^{i\kappa|Y|-i\alpha X}}{i\kappa} d\alpha, \tag{8.6}$$

where

$$\left. \begin{array}{c} A_1(\alpha) \\ \\ A_2(\alpha) \end{array} \right\} = \frac{bsgn(Y)}{\sqrt{2\pi}L(\alpha)} \left[(S_+(\alpha)G_1(\alpha) + T(\alpha)S_+(\alpha)C_1) \right.$$

$$+e^{-i\alpha l}(S_-(\alpha)G_2(-\alpha) + T(-\alpha)S_-(\alpha)C_2)$$

$$\left. - \left[\frac{1-e^{-il(\alpha-K\cos\theta_0)}}{(\alpha-K\cos\theta_0)} \right] \right]$$

$$-\frac{B\left[(2\alpha M + K(1+M^2) + \frac{M^2\alpha^2}{(1-M^2)K})\right]}{\sqrt{2\pi}L(\alpha)} \left[\frac{L_+(\alpha)G_1'(\alpha)}{\left[(2\alpha MB + KB(1+M^2) + \frac{BM^2\alpha^2}{(1-M^2)K})\right]} \right.$$

$$+\frac{T(\alpha)L_+(\alpha)C_1'}{\left[(2\alpha MB + KB(1+M^2) + \frac{BM^2\alpha^2}{(1-M^2)K})\right]}$$

$$+e^{-i\alpha l}\left[\frac{L_-(\alpha)G_2'(-\alpha)}{\left[(-2\alpha MB + KB(1+M^2) + \frac{BM^2\alpha^2}{(1-M^2)K})\right]} \right.$$

$$\left. \left. +\frac{T(-\alpha)L_-(\alpha)C_2'}{\left[(-2\alpha MB + KB(1+M^2) + \frac{BM^2\alpha^2}{(1-M^2)K})\right]} \right] + \left[\frac{1-e^{-il(\alpha-K\cos\theta_0)}}{(\alpha-K\cos\theta)} \right] \right], \tag{8.7}$$

Here, $A_1(\alpha)$ corresponds to $Y > 0$ and $A_2(\alpha)$ corresponds to $Y < 0$.

8.2 Intermediate Zone Solution

To find out the diffracted wave solution in an intermediate zone, we introduce the following transformation

$$X = R\cos\theta, |Y| = R\sin\theta \quad \text{and} \quad \alpha = -K\cos(\theta + it), \ (0 < \theta < \pi, \ -\infty < t < \infty).$$

Thus Eq. (8.6) reduces to

$$\Psi(X,Y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left. \begin{array}{c} A_1(-K\cos(\theta+it)) \\ A_2(-K\cos(\theta+it)) \end{array} \right\} e^{iKR\cosh t} dt, \tag{8.8}$$

Writing the Maclaurin series of $A_1(-K\cos(\theta+it))$ and $A_2(-K\cos(\theta+it))$ in the above integral, we get

$$\Psi(X,Y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left. \begin{array}{c} A_1(-K\cos\theta) + tA_1'(-K\cos\theta) + \frac{t^2}{2!}A_1''(-K\cos\theta) + ... \\ A_2(-K\cos\theta) + tA_2'(-K\cos(\theta)) + \frac{t^2}{2!}A_2''(-K\cos\theta) + ... \end{array} \right\} e^{iKR\cosh t} dt,$$

The second term in the above integral will vanish. Therefore, the integral reduces to

$$\Psi(X,Y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left. \begin{array}{c} A_1(-K\cos\theta) + \frac{t^2}{2!}A_1''(-K\cos\theta) + ... \\ A_2(-K\cos\theta) + \frac{t^2}{2!}A_2''(-K\cos\theta) + ... \end{array} \right\} e^{iKR\cosh t} dt, \tag{8.9}$$

where $A_1(-K\cos\theta)$, $A_1''(-K\cos\theta)$, $A_2(-K\cos\theta)$, and $A_2''(-K\cos\theta)$ can be calculated from Eq. (8.7) as follows

$$\left. \begin{array}{c} A_1(-K\cos\theta) \\ A_2(-K\cos\theta) \end{array} \right\} = \frac{bsgn(Y)}{\sqrt{2\pi}L(\alpha_m)} \left[(S_+(\alpha_m)Q_1) + e^{iKl\cos\theta}S_-(\alpha_m)Q_4 + \left(\frac{1-e^{iKlm}}{Km}\right)\right]$$

$$-\frac{Q_{12}L_+(\alpha_m)}{\sqrt{2\pi}L(\alpha_m)} \left[\frac{G_1'(\alpha_m)}{Q_9} + \frac{T(\alpha_m)C_1'}{Q_{12}} + \frac{L_-(\alpha_m)e^{ilK\cos\theta}Q_8}{BQ_9} + \left(\frac{1-e^{iKlm}}{m}\right)\right] \tag{8.10}$$

$$\left.\begin{array}{l} A_1'(-K\cos\theta) \\ A_2'(-K\cos\theta) \end{array}\right\} = \frac{bb_1 sgn(Y)}{\sqrt{2\pi}L^2(\alpha_m)}\left[\left[L(\alpha_m)\{S_+'(\alpha_m)Q_1 + S_+(\alpha_m)Q_2\}\right.\right.$$

$$-L'(\alpha_m)S_+(\alpha_m)Q_3] + \{L'(\alpha_m)S_-(\alpha_m)Q_4 + L(\alpha_m)Q_5\}\, e^{ilK\cos\theta}$$

$$\left.+\frac{(1-e^{iKlm})L'(\alpha_m)}{Km} + \frac{e^{iKlm}L(\alpha_m)}{Km^2}\{1 - iKlm\}\right]$$

$$-\frac{b_1}{\sqrt{2\pi}L_-^2(\alpha_m)}[L_-(\alpha_m)Q_6 + L_-'(\alpha_m)Q_7]$$

$$-\frac{b_1 B^2 Q_8\ e^{il\alpha_p}}{\sqrt{2\pi}L_+^2(\alpha_m)BKQ_9}\left[4MQ_{10}L_+(\alpha_m) - K^2Q_{11}\{L_+'(\alpha_m) + ilL_+(\alpha_m)\}\right]$$

$$+\frac{b_1Q_{12}Q_{13}e^{il\alpha_p}}{\sqrt{2\pi}L_+(\alpha_m)Q_9} - \frac{b_1Be^{iKlm}}{\sqrt{2\pi}L(\alpha_m)Km}\left[ilKQ_{12} + M\left\{2 - \frac{M\cos\theta}{(1-M^2)}\right\}\right]$$

$$+\frac{b_1BM\{2 - \frac{M\cos\theta}{(1-M^2)}\}}{\sqrt{2\pi}L(\alpha_m)Km} - \frac{b_1BQ_{12}\{Km-1\}(1-e^{iKlm})L_+(\alpha_m)}{\sqrt{2\pi}L^2(\alpha_m)Km^2}, \tag{8.11}$$

and

$$\left.\begin{array}{l} A_1''(-K\cos\theta) \\ A_2''(-K\cos\theta) \end{array}\right\} = \frac{bK[2K\sin^2\theta L'(\alpha_m) - i\cos\theta L(\alpha_m)]sgn(Y)}{\sqrt{2\pi}L^3(\alpha_m)}\left[L(\alpha_m)\{S_+'(\alpha_m)Q_1 + S_+(\alpha_m)Q_2\}\right.$$

$$-L_-'(\alpha_m)\left\{S_+(\alpha_m)Q_3 + S_-(\alpha_m)e^{il\alpha_p}Q_4\right\} - L(\alpha_m)Q_{14}e^{il\alpha_p}$$

$$\left.+\frac{L'(\alpha_m) - (L'(\alpha_m) + ilL(\alpha_m))e^{iKlm}}{Km} + \frac{L(\alpha_m)e^{iKlm}}{K^2m^2}\right]$$

$$+\frac{bb_1^2}{\sqrt{2\pi}L^2(\alpha_m)}\left[L(\alpha_m)\{S_+''(\alpha_m)Q_1 + S_+'(\alpha_m)Q_2\}\right.$$

$$L(\alpha_m)[S_+'(\alpha_m)\{Q_2 + Q_{15}\}S_+(\alpha_m)Q_{16}] + L'(\alpha_m)C_1\{S_+'(\alpha_m) + S_+(\alpha_m)\}\{T(\alpha_m) - T'(\alpha_m)\}$$

$$-L''(\alpha_m)S_+(\alpha_m)Q_3 + \left[\left\{L''(\alpha_m)S_-(\alpha_m) + L'(\alpha_m)S_-'(\alpha_m)\right\}Q_4 + 2ilL(\alpha_m)Q_{17}\right.$$

$$-S_-(\alpha_m)G_2(\alpha_p)\{l^2L(\alpha_m) + 2ilL'(\alpha_m)\} + 2S_-'(\alpha_m)Q_{18} + L(\alpha_m)Q_{19}$$

$$L'(\alpha_m)S_-(\alpha_m)C_2\{T(\alpha_m) - T'(\alpha_m)\}]\, e^{iKlm}$$

$$+\frac{Km[L''(\alpha_m)+\{ilL'(\alpha_m)-L''(\alpha_m)\}e^{il\alpha_p}]-L'(\alpha_m)(1-e^{il\alpha_p})}{K^2m^2}$$

$$+\frac{[Km\{L'(\alpha_m)-ilL(\alpha_m)\}-2L(\alpha_m)]\,(1-iKlm)e^{iKlm}}{K^3m^3}-\frac{2ilL(\alpha_m)e^{iKlm}}{K^2m^2}\Bigg]$$

$$+\frac{2K^2\sin^2\theta L'^2(\alpha_m)Q_{20}}{\sqrt{2\pi}L^3(\alpha_m)}-\frac{K\cos\theta}{\sqrt{2\pi}L_-^2(\alpha_m)}\{L_-(\alpha_m)Q_{21}+L_-'(\alpha_m)Q_{20}\}$$

$$-\frac{K\sin^2\theta}{\sqrt{2\pi}L_-^2(\alpha_m)}\{L_-(\alpha_m)Q_{22}+L_-''(\alpha_m)Q_{20}\}$$

$$\frac{e^{il\alpha_p}}{\sqrt{2\pi}L_-^2(\alpha_m)Q_9}\{\left(ilK\sin^2\theta-\cos\theta\right)Q_8+K\sin^2\theta Q_{23}\}[4B^2MQ_{10}L_+(\alpha_m)$$

$$-B^2K^2Q_{11}\{L'(\alpha_m)+ilL(\alpha_m)\}]$$

$$\frac{K\sin^2\theta e^{il\alpha_p}}{\sqrt{2\pi}L_+^2(\alpha_m)Q_9}\left[\begin{array}{c}\frac{2KB^2Q_8}{L_+(\alpha_m)Q_9}\{L_+'(\alpha_m)-BML_+(\alpha_m)(1+\frac{M}{(1-M^2)})\}[4MQ_{10}L_+(\alpha_m)\\ -K^2Q_{11}\{L'(\alpha_m)+ilL(\alpha_m)\}]\end{array}\right.$$

$$-4B^2MQ_8[KQ_{10}L'(\alpha_m)-iL(\alpha_m)\{\frac{M^2}{(1-M^2)\,K}-2MK^2\cos\theta\}]$$

$$-KQ_8[2K\{L'(\alpha_m)+ilL(\alpha_m)\}\{\tfrac{(3-M^2)}{(1-M^2)}+\tfrac{M^2}{(1-M^2)^2}\}2M^2B^2\cos\theta$$

$$-C\{L''(\alpha_m)+ilL'(\alpha_m)\}]$$

$$+KQ_{23}[2BMQ_{10}L_+(\alpha_m)-K^2Q_{11}\{L'(\alpha_m)+ilL(\alpha_m)\}]]$$

$$+\frac{e^{iKlm}}{\sqrt{2\pi}L_+^2(\alpha_m)Km^2}\Bigg[KmL(\alpha_m)[\{ilQ_{12}+2BM-\frac{BM^2\cos\theta}{(1-M^2)}\}\{\cos\theta-iKl\sin\theta\}$$

$$+BK\sin^2\theta\{\frac{M^2}{(1-M^2)\,K}+il(2M-\frac{M^2\cos\theta}{(1-M^2)})\}]$$

$$-BK\sin^2\theta\{ilQ_{12}+2BM-\frac{BM^2\cos\theta}{(1-M^2)}\}\{KmL'(\alpha_m)-L(\alpha_m)\}\Bigg]$$

$$+\frac{2BM-\frac{BM^2\cos\theta}{(1-M^2)}}{\sqrt{2\pi}L^2(\alpha_m)Km^2}\left[K\sin^2\theta\{KmL'(\alpha_m)-L(\alpha_m)\}-KmL(\alpha_m)\cos\theta\right]$$

$$-\frac{BM^2\sin^2\theta}{\sqrt{2\pi}\,(1-M^2)\,L(\alpha_m)m}$$

$$\frac{1}{\sqrt{2\pi}\,(1-M^2)\,L^4(\alpha_m)K^3m^4}\left[\begin{array}{c}\Big\{BMK\sin^2\theta\{2-\frac{M\cos\theta}{(1-M^2)}\}+Q_{12}\cos\theta\Big\}\\ \{KmL_+'(\alpha_m)-L_+(\alpha_m)\}(1-e^{iKlm})\end{array}\right.$$

$$+KQ_{12}\sin^2\theta\{KmL_+'(\alpha_m)-L_+(\alpha_m)\}\left[e^{iKlm}+(1-e^{iKlm})\{KmL_+''(\alpha_m)-2L_+'(\alpha_m)\}\right. \tag{8.12}$$

where

$$b_1=iK\sin\theta,\ \alpha_m=-K\cos\theta,\quad \alpha_p=K\cos\theta,\ m=\cos\theta+\cos\theta_0,$$

$$Q_1=G_1(\alpha_m)+T(\alpha_m)C_1,$$

$$Q_2=G_1'(\alpha_m)+T(\alpha_m)C_1,$$

$$Q_3=G_1(\alpha_m)+T'(\alpha_m)C_1,$$

$$Q_4=G_2(\alpha_p)+T(\alpha_p)C_2,$$

$$Q_5=-ilS_-(\alpha_m)G_2(\alpha_p)+S_-'(\alpha_m)G_2(\alpha_p)-S_-(\alpha_m)G_2'(\alpha_p),$$

$$Q_6=G_3'(\alpha_m)+T'(\alpha_m)C_3,$$

$$Q_7=G_3(\alpha_m)+T(\alpha_m)C_3,$$

$$Q_8=G_4(\alpha_p)+T(\alpha_p)C_4,$$

$$Q_9=2M\cos\theta+\left(1+M^2\right)+\frac{M^2\cos^2\theta}{(1-M^2)},$$

$$Q_{10}=2MK^2\cos^2\theta-\frac{2M^2\cos\theta}{(1-M^2)\,K}+K\left(1+M^2\right),$$

$$Q_{11}=\frac{2\,(3-M^2)}{(1-M^2)}M^2\cos^2\theta+\frac{M^4\cos^4\theta}{(1-M^2)^2}+\left(1+M^2\right)^2,$$

$$Q_{12}=-2M\cos\theta+\left(1+M^2\right)+\frac{M^2\cos^2\theta}{(1-M^2)},$$

$$Q_{13}=G_4'(\alpha_p)+T'(\alpha_p)C_4,$$

$$Q_{14} = ilS_-(\alpha_m)G_2(\alpha_p) - S'_-(\alpha_m)G_2(\alpha_p) + S_-(\alpha_m)G'_2(\alpha_p),$$

$$Q_{15} = G'_1(\alpha_m) + T'(\alpha_m)C_1,$$

$$Q_{16} = G''_1(\alpha_m) - T'(\alpha_m)C_1,$$

$$Q_{17} = S_-(\alpha_m)G'_2(\alpha_p) - S'_-(\alpha_m)G_2(\alpha_p)$$

$$Q_{18} = L'(\alpha_m)G_2(\alpha_p) - L(\alpha_m)G'_2(\alpha_p)$$

$$Q_{19} = S_-(\alpha_m)G''_2(\alpha_p) - S''_-(\alpha_m)G_2(\alpha_p)$$

$$Q_{20} = G_3(\alpha_m) + T(\alpha_m)C_3,$$

$$Q_{21} = G'_3(\alpha_m) + T'(\alpha_m)C_3,$$

$$Q_{22} = G''_3(\alpha_m) + T''(\alpha_m)C_3,$$

$$Q_{23} = G'_4(\alpha_p) + T'(\alpha_p)C_4,$$

$$Q_{24} = -\frac{2M^2\cos\theta}{(1-M^2)K} + K\left(1+M^2\right) + 2MK^2\cos^2\theta$$

Thus the diffracted field in the intermediate zone is given by

$$\Psi(X,Y) = \left.\begin{matrix} A_1(-K\cos\theta) \\ A_2(-K\cos\theta) \end{matrix}\right] \sqrt{\frac{2\pi}{KR}} e^{iKR+i\frac{\pi}{4}} + \left.\begin{matrix} A''_1(-K\cos\theta) \\ A''_2(-K\cos\theta) \end{matrix}\right] \sqrt{\frac{2\pi}{(KR)^3}} e^{iKR-i\frac{\pi}{4}}, \quad (8.13)$$

where $A_1(-K\cos\theta)$, $A_2(-K\cos\theta)$, $A''_1(-K\cos\theta)$ and $A''_2(-K\cos\theta)$ are given by Eqs. (8.10) and (8.12) respectively. Thus, the total field in the intermediate range is given by

$$\Phi(x,y) = \frac{e^{iKMX_0}\, e^{iK[X\cos\theta_0+\kappa Y\sin\theta_0]}\, e^{iKR-i\frac{\pi}{4}}}{2i\sqrt{1-M^2}\sqrt{2\pi KR_0}}\left(1 - \frac{i}{8KR_0}\right)$$

$$\pm \left. \begin{matrix} A_1(-K\cos\theta) \\ A_2(-K\cos\theta) \end{matrix} \right] \sqrt{\frac{2\pi}{KR}} e^{iKR + i\frac{\pi}{4}} + \left. \begin{matrix} A_1''(-K\cos\theta) \\ A_2''(-K\cos\theta) \end{matrix} \right] \sqrt{\frac{2\pi}{(KR)^3}} e^{iKR - i\frac{\pi}{4}}, \tag{8.14}$$

8.3 Numerical Discussion

The graph are being plotted for the diffracted field against the observation angle for different values of different parameters θ_0, B and M. The observations are given below:

(a) From figure (8.1), we can see that for the same values of incident angle θ_0, the amplitude is higher when position of line source is closer as compared to far off position of line source. When the incident angle θ_0 is increased, the amplitude of the separated field decreases. Also, in case of far field approximation [78], the frequency of the field remain the same when the wave number is kept constant but it is decreasing in the present case.

(b) Figures (8.2) shows that the amplitude of the diffracted field in intermediate zone is higher as compared to far field zone solution as discussed in [78]; and also, it decreases with the increase in absorbing parameter β.

(c) Figure (8.3) is plotted for the diffracted field against observation angle for different values of Mach number M. It is observed that the behavior of diffracted field does not change as compared with [78]; but overall, the amplitude of the diffracted field increases when the position of line source is shifted from far zone to intermediate zone.

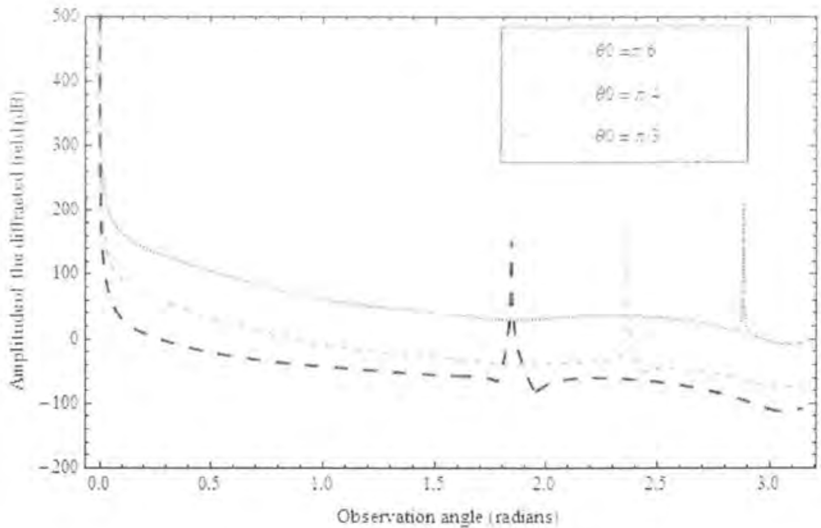


Fig. 8.1: Amplitude of the diffracted field against the observation angle for different values of incident angle θ_0 with $K = 1$, $M = 0.5$ and $\beta = 0.5$.

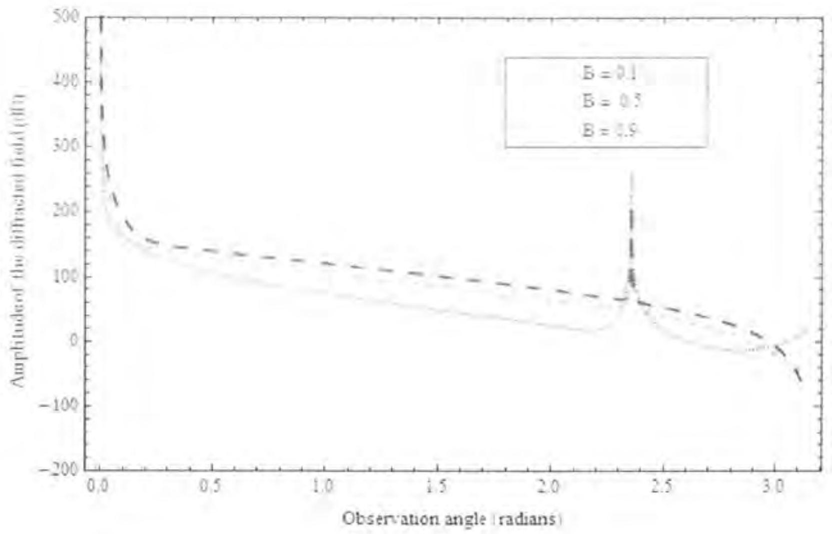


Fig. 8.2: Amplitude of the diffracted field against the observation angle for different values of absorbing parameter with $K = 1$, $M = 0.5$ and $\theta_0 = \frac{\pi}{4}$.

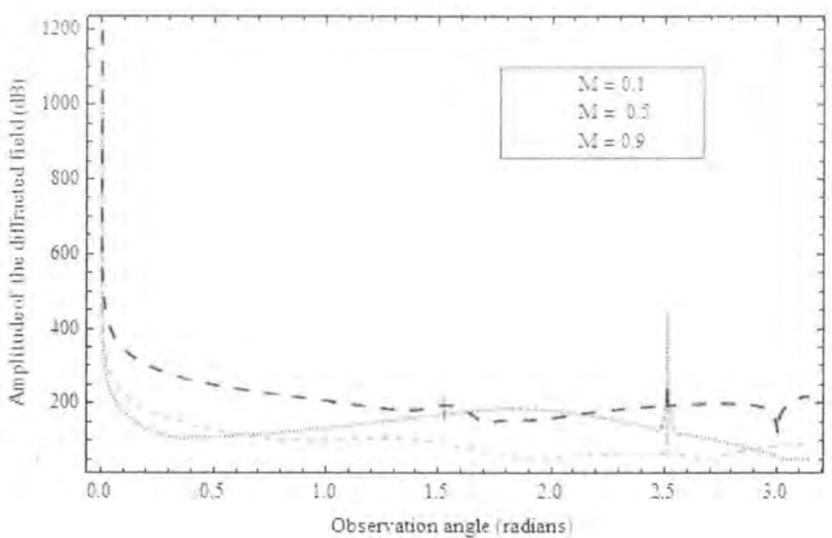


Fig. 8.3: Amplitude of the diffracted field against the observation angle for different values of mach number M with $K = 1$, $\beta = 0.5$ and $\theta_0 = \frac{\pi}{4}$.

8.4 Conclusion

(a) As discussed earlier, the main purpose of considering the intermediate zone solution is that we can visualize the diffracted pattern more optically than in case of far off position.

(b) It can be seen that the far field solution and the intermediate zone solution differ by a multiplicative factor.

(c) It is observed that the diffracted field in the far field is of order $O(\frac{1}{\sqrt{KR}})$,while in the intermediate field, it is of order $O(\frac{1}{\sqrt[3]{KR}})$. The term containing $(\frac{1}{\sqrt[3]{KR}})$ gives rise to an additional term in the diffracted field which produces a stronger field and also the diffraction pattern will be more visible and clear.

(d) The diffracted field for both the far field approximations and the intermediate field approximations differ by a factor

$$\frac{A_1''(-K\cos\theta)}{A_2''(-K\cos\theta)}\sqrt{\frac{2\pi}{(KR)^3}}e^{iKR-i\frac{\pi}{4}}.$$

(e) If R is very large such that the terms of order $O(\frac{1}{\sqrt[3]{KR}})$ are neglected in Eq. (8.14), the solution reduces to that of the far field approximation and if $R \to \infty$, then the solution obtained is that of plane wave. Also, if the strip length l is taken to be infinity, the results obtained are that of half plane [46].

Chapter 9

CONCLUSION

In Chapters 3 & 4, the line/point source diffraction of acoustic waves by the slit in a moving fluid using Myers' conditions is pondered upon. While using Fourier transform technique, the boundary value problem is reduced to the functional Wiener-Hopf equation and the solution is obtained by considering the factorization of the kernel. The Myers' conditions give rise to a corrective term in the solution. If this corrective term is ignored, the results of [135] can be achieved which shows that our results vary by a multiplicative factor. Further, a numerical solution is obtained to the problem showing the effects of sundry parameters (for various values of Mach number M and the absorbing parameter B against the velocity potential) . The graphs show a clear variation of the velocity potential against these parameters. It is found that if a line/point source is moved to a far-off distance, i.e., infinity, the graphical results of plane wave situation [135] can be recovered. The Myers' conditions give better attenuation results for the separated and inter-active diffracted fields as compared to Ingards' conditions.

The solutions obtained will contribute to the analysis of noise reduction. We have taken the source lying far from the slit and the reflected sound is measured at a point far from the slit to know, how successfully the barrier reduces the sound transmission despite presence of the slit and how the absorption of the barrier makes its presence felt.

The above results also take care of acoustic diffraction from a slit with a rigid barrier, in a moving fluid, which can be obtained by putting $\beta = 0$. Also, the result for the still fluid can be found by putting $M = 0$. The above solved problem of a slit in a moving fluid will help understand acoustic diffraction and will go a step further to complete the discussion for the slit geometry. To the best of author's knowledge, the above mentioned problem has not been solved so far by using the Myres' conditions.

In chapter five, we have obtained an improved form of the diffracted field due to an impulsive line source by an absorbing half plane in a moving fluid by considering time dependence. The solution represents the fields at the observation point directly coming from the line source, and the diffracted field from the edge of the half plane. We observe that the field starts reaching the point $(x.y)$ after the time lapse $t' > \frac{1}{Q}(R + R_0)$ and the strength of the field dies down as $1/\sqrt{R_0}$.

In chapter six, it is observed that the absorbing half plane with wake gives a more generalized model in diffraction theory and more situations can be discussed as a special case of this problem by choosing the suitable parameters. The problem with more practical applications is one of an absorbing strip in a moving fluid with trailing edge situation. This can be a model for an aeroplane wing and has the advantage of being cheaper than to construct a strip with faces entirely coated in absorbent materials. Finally, this work, which is carried with Myers' impedance condition in contrast to the results of Ingard's condition, will offer useful theoretical comparisons in conjunction with experimental results. It should then lead to a choice as to which boundary conditions to be used in practice. We can obtain no wake (leading edge) situation by taking $\lambda = 0$ and a field for a rigid barrier by putting $\beta = 0$. Also, the results for the still fluid can be recovered by putting $M = 0$.

In chapter seven, it is observed that for a line source near the edge of an absorbing half plane, the field caused by the Kutta-Joukowski condition will be larger than the field caused in its absence. The wake has an effect of producing a stronger field when the source is near the edge. The leading edge solution can be recovered if wake and consequently Kutta-Joukowski condition is ignored. It can be seen that the solution for plane wave diffraction and line source diffraction differ by a factor which appears due to the presence of line source, but for the intermediate range solution, the plane wave and the intermediate solutions differ by a multiplicative factor. It is further observed that the diffracted field in far field is of order $O(\frac{1}{\sqrt{KR_0}})$, while in the intermediate field, it is of order $O(\frac{1}{\sqrt[3]{KR_0}})$. The term containing $(\frac{1}{\sqrt[3]{KR_0}})$ gives rise to an additional term in the diffracted field which produces a stronger field where the diffraction pattern will be more visible and clear. The diffracted field for both the far field approximations and the intermediate approximations differ by a factor. The leading edge solution can be recovered if wake is ignored and R is kept very large such that the terms of order $O(\frac{1}{\sqrt[3]{KR}})$ are neglected in Eq. (7.68) and the solution reduces to that of the far field approximations [46].

In chapter eight, as discussed earlier, it is shown that the main purpose of considering the intermediate zone solution is that we should visualize the diffracted pattern more optically than in case of far off position. It can be seen that the far field solution and the intermediate zone solution differ by a multiplicative factor. The diffracted field for both the far field approximations and the intermediate field approximations differ by a factor. If R is very large such that the terms of order $O(\frac{1}{\sqrt[3]{KR}})$ are neglected in Eq. (8.14) then the solution reduces to that of the far field approximations and if $R \to \infty$, the solution obtained is that of plane wave. Also if the strip length l is taken to be infinity, the results obtained are that of half plane [46].

Chapter 10

APPENDICES

1. Appendix A

In order to calculate the integrals in Eq. (6.56), we let

$$\chi(x, y, \omega) = \widehat{I}_1 + \widehat{I}_2 - \widehat{I}_3 \tag{A.1}$$

where

$$\widehat{I}_1 = \frac{\exp\left[-iKM(X - X_0)\right]}{4\pi i\sqrt{(1-M^2)}} \int_{-\infty}^{\infty} \frac{e^{i\alpha(X-X_0)+i\kappa(Y-Y_0)}}{\kappa} d\alpha, \tag{A.2}$$

$$\widehat{I}_2 = \frac{e^{iKM(X_0-X)} sgn\,|YY_0|}{8\pi^2\sqrt{1-M^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\alpha,\xi,\omega) e^{-i\xi X_0 + i\sqrt{K^2-\xi^2}|Y_0|}\; e^{i\kappa|Y| + i\alpha X} d\xi d\alpha \tag{A.3}$$

and

$$\widehat{I}_3 = \frac{e^{iKM(X_0-X)}}{8\pi^2\sqrt{1-M^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\alpha,\xi,\omega) e^{-i\xi X_0 + i\sqrt{K^2-\xi^2}|Y_0|}\; e^{i\kappa|Y| + i\alpha X} d\xi d\alpha. \tag{A.4}$$

Now, we consider Eq. $(A.2)$ first, which can also be written as

$$\widehat{I}_1 = \frac{\exp\left[-iKMR'\cos\Theta'\right]}{4\pi\sqrt{(1-M^2)}} \int_{-\infty}^{\infty} e^{-iKR'\cosh\varsigma} d\varsigma, \tag{A.5}$$

where

$$X - X_0 = R' \cos\Theta', \quad |Y - Y_0| = R' \sin\Theta', \quad \alpha = K\cos(\Theta' + i\varsigma).$$

Taking the inverse temporal Fourier transform and noting that K is a function of ω, Eq. $(A.5)$ can be written as

$$I_1 = \frac{1}{8\pi^2\sqrt{(1-M^2)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-iKMR'\cos\Theta' + iKR'\cosh\varsigma}\, e^{-i\omega t} d\varsigma d\omega$$

or

$$I_1 = \frac{c}{4\pi Q}\int_{-\infty}^{\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega[t + \frac{MR'\cos\Theta'}{Q} - \frac{R'\cosh\varsigma}{Q}]} d\omega d\varsigma,$$

where

$$Q = c\sqrt{(1-M^2)}.$$

Now, we know that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t} d\omega = \delta(t),$$

and using this property of the $\delta-$function, we obtain

$$I_1 = \frac{c}{4\pi Q}\int_{-\infty}^{\infty} \delta(t + \frac{MR'\cos\Theta'}{Q} - \frac{R'\cosh\varsigma}{Q}) d\varsigma.$$

We let $\frac{R'\cosh\varsigma}{Q} = \varrho$ in the above integral to get

$$I_1 = \frac{c}{4\pi Q}\int_{-\infty}^{\infty} \frac{\delta(t' - \varrho)}{\sqrt{(\varrho^2 - \frac{R'^2}{Q^2})}} d\varrho, \tag{A.6}$$

where $t' = t + \frac{MR'\cos\Theta'}{Q}$. The integral appearing in Eq. $(A-6)$ can be calculated as

$$I_1 = \frac{c}{4\pi Q}\frac{H(t' - \varrho)}{\sqrt{(\varrho^2 - \frac{R'^2}{Q^2})}}, \tag{A.7}$$

where $H(t' - \varrho)$ is the usual Heaviside function.

Now, before finding the inverse temporal Fourier transform of $\widehat{I}_2$, we calculate the double integral appearing in that expression, i.e., Eq. $(A.3)$. To do so, we introduce the polar coordinates

$$\begin{aligned} X &= R\cos\Theta, \qquad |Y| = R\sin\Theta, \\ X_0 &= R_0\cos\Theta_0, \qquad |Y_0| = R_0\sin\Theta_0, \end{aligned}$$

and use the transformation $\xi = -K\cos(\Theta_0 + ip)$ which changes the contour of integration over ξ into a hyperbola through the point $\xi = -K\cos\Theta_0$. Similarly, by the change of variable $\alpha = K\cos(\Theta + iq)$, the contour of integration can be converted from α into a hyperbola through the point $\alpha = K\cos\Theta$. Thus, omitting the details of calculations, we obtain

$$\widehat{I}_2 = \frac{-i[B\{(1+M^2) - 2M\cos\Theta_0 + \frac{M^2\cos^2\Theta_0}{(1-M^2)}\} - 2\sin\frac{\Theta}{2}\sin\frac{\Theta_0}{2}]e^{iKM(X-X_0)+i\kappa(R+R_0)}}{16\pi K\sqrt{RR_0}\sqrt{(1-M^2)}L_+(K\cos\Theta)L_-(-K\cos\Theta_0)(\cos\Theta + \cos\Theta_0)}, \tag{A.8}$$

where

$$R = r\left(\sqrt{\frac{1 - M^2\sin^2\theta}{1 - M^2}}\right), \qquad \cos\Theta = \frac{\cos\theta}{\sqrt{1 - M^2\sin^2\theta}} \qquad \text{and} \quad \Theta \neq \pi - \Theta.$$

Now, taking the inverse temporal Fourier transform of Eq. $(A.8)$, we have

$$I_2 = -\frac{icF_1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\frac{i\omega}{Q}(R+R_0-MR'\cos\Theta')}}{\omega}e^{-i\omega t}d\omega, \tag{A.9}$$

where

$$F_1 = \frac{[B\{(1+M^2) - 2M\cos\Theta_0 + \frac{M^2\cos^2\Theta_0}{(1-M^2)}\} - 2\sin\frac{\Theta}{2}\sin\frac{\Theta_0}{2}]}{16\pi\sqrt{RR_0}L_+(K\cos\Theta)L_-(-K\cos\Theta_0)(\cos\Theta + \cos\Theta_0)}. \tag{A.10}$$

Note that the explicit form of the functions $L_{\pm}(\alpha)$ do not involve ω.

Let us take $u(w) = \frac{1}{w}$, $v(w) = e^{\frac{iw}{Q}(R+R_0-MR'\cos\Theta')}$ in Eq. $(A.9)$ and using the convolution theorem, we can write

$$I_2 = -icF_1 V(t) * U(t), \tag{A.11}$$

where

$$V(t) = \frac{1}{2\pi} \int_{i\tau-\infty}^{i\tau+\infty} e^{\frac{i\omega}{Q}(R+R_0-MR'\cos\Theta')} e^{-i\omega t} d\omega,$$

and

$$U(t) = \frac{1}{2\pi} \int_{i\tau-\infty}^{i\tau+\infty} \frac{e^{-i\omega t}}{\omega} d\omega,$$

where τ lies in the region of analyticity such that $-\operatorname{Im}(K) < \tau < \operatorname{Im}(K)$ and the asterisk in equation $(A.11)$ denotes convolution in the time domain. For $\tau > 0$, we can close the contour of integration in the lower half plane. Knowing that ω has a small positive imaginary part for $\tau > 0$, we get

$$V(t) = \delta(t - \frac{1}{Q}(R + R_0 - MR'\cos\Theta'))$$

and

$$U(t) = -i.$$

Hence,

$$I_2 = -2cF_1 H(t' - \frac{1}{Q}(R + R_0)). \tag{A.12}$$

Similarly, by adopting the same procedure as in the case of $\widehat{I}_2$, we can solve $\widehat{I}_3$ and get

$$I_3 = -2QG_1 H(t - \frac{1}{Q}(R + R_0)), \tag{A.13}$$

where

$$G_1 = \frac{\sqrt{1+\cos\Theta_0}\sqrt{1+\cos\Theta}}{(\frac{1}{M}-\cos\Theta)L_-(K\cos\Theta)L_+(-K\cos\Theta_0)}. \tag{A.14}$$

So, using Eqs. (A.7), (A.12) and (A.13) in Eq. (A.1), we obtain

$$\phi(x,y,t) = \frac{c}{4\pi Q}\frac{H(t'-\varrho)}{\sqrt{(\varrho^2-\frac{R'^2}{Q^2})}} - 2cF_1H(t'-\frac{1}{Q}(R+R_0)) + 2QG_1H(t-\frac{1}{Q}(R+R_0)),$$

which is Eq. (6.59).

2. Appendix B

In this appendix, we give the detail solution of I_1 of Eq.(7.59) for an intermediate range. We use the following transformation

$$\begin{aligned} X_0 &= R_0\cos\theta_0, |Y_0| = R_0\sin\theta_0 \\ \text{and}\quad \alpha &= -K\cos(\theta_0 - iq),\ (0<\theta_0<\pi, -\infty<q<\infty), \end{aligned} \tag{B.1}$$

in Eq. (7.59), which change the contour of integration over α into a hyperbola which passes through the point $\alpha = -K\cos\theta_0$. So, we get

$$I_1 = \frac{ae^{iK[X\cos(\theta_0+iq)+\kappa Y\sin(\theta_0+iq)]}}{4\pi i}\int_{-\infty}^{\infty} e^{iKR_0\cosh q}dq, \tag{B.2}$$

and after using the expansion of $\cosh q$, we get

$$I_1 = \frac{ae^{iK[X\cos(\theta_0+iq)+\kappa Y\sin(\theta_0+iq)]}}{4\pi i}\int_{-\infty}^{\infty} e^{\frac{iKR_0q^2}{2}(1+\frac{q^2}{12}+\dots)}dq. \tag{B.3}$$

We rotate the above equation by $q = r\,e^{\frac{i\pi}{4}}$ to obtain

$$I_1 = \frac{ae^{iK[X\cos(\theta_0+iq)+\kappa Y\sin(\theta_0+iq)]}\,e^{iKR+i\frac{\pi}{4}}}{2\pi i}\int_{0}^{\infty} e^{-\frac{KR_0r^2}{2}(1+\frac{ir^2}{12}+\dots)}dr, \tag{B.4}$$

and now we put $\frac{KR_0 r^2}{2} = \tau$ in the above integral, so that

$$I_1 = \frac{a e^{iK[X\cos(\theta_0+iq)+\kappa Y\sin(\theta_0+iq)]}\, e^{iKR-i\frac{\pi}{4}}}{2\pi i\sqrt{2KR_0}} \int_0^\infty e^{-\tau}(\tau)^{-\frac{1}{2}}(1-\frac{i\tau^2}{6KR_0})d\tau. \tag{B.5}$$

We use the Gamma function in Eq. $(B.5)$ and get

$$I_1 = \frac{a e^{iK[X\cos(\theta_0+iq)+\kappa Y\sin(\theta_0+iq)]}\, e^{iKR-i\frac{\pi}{4}}}{2\pi i\sqrt{2KR_0}}(\sqrt{\pi}-\frac{i\sqrt{\pi}}{8KR_0}), \tag{B.6}$$

or

$$I_1 = \frac{e^{iKMX_0}\, e^{iK[X\cos(\theta_0+iq)+\kappa Y\sin(\theta_0+iq)]}\, e^{iKR-i\frac{\pi}{4}}}{2i\sqrt{1-M^2}\sqrt{2\pi KR_0}}(1-\frac{i}{8KR_0}). \tag{B.7}$$

Eq. $(B.7)$ gives the incident wave solution in the intermediate range which is same as Eq.(7.60).

3. **Appendix C**

1. In this appendix, we give the detail solutions of I_2 and I_3 of Eqs. (7.61) and (7.63) for an intermediate range. We first solve I_2, i.e.,

$$I_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\alpha,\xi)e^{-i\xi X_0+i\sqrt{K^2-\xi^2}|Y_0|}\, e^{i\kappa|Y|+i\alpha X}d\xi d\alpha, \tag{C.1}$$

and we can write the value of $G(\alpha,\xi)$ from Eq. (7.57) as follows

$$G(\alpha,\xi) = \frac{1}{L_+(\xi)L_-(\alpha)(\alpha-\frac{K}{M})\sqrt{K+\xi}\sqrt{K-\alpha}}. \tag{C.2}$$

We introduce the following transformations

$$\begin{aligned} X_0 &= R_0\cos\theta_0, |Y_0| = R_0\sin\theta_0 \\ \text{and} \quad \xi &= -K\cos(\theta_0+iq_1),\ (0<\theta_0<\pi,\ -\infty<q_1<\infty), \end{aligned} \tag{C.3}$$

which change the contour of integration over ξ into a hyperbola passing through the points $\xi = -K\cos\theta_0$ and we write

$$I_2 = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} G(\alpha, -K\cos(\theta_0 + iq_1))iK\sin(\theta_0 + iq_1)\, e^{iKR_0\cosh q_1} dq_1\right] e^{i\alpha X + i\kappa|Y|} d\alpha. \tag{C.4}$$

In Eq. $(C.4)$, for convenience, we take

$$I^{/} = \int_{-\infty}^{\infty} iKG(\alpha, -K\cos(\theta_0 + iq_1))\sin(\theta_0 + iq_1)\, e^{iKR_0\cosh q_1} dq_1, \tag{C.5}$$

where

$$G(\alpha, -K\cos(\theta_0 + iq_1) = \frac{1}{L_+(-K\cos(\theta_0 + iq_1))L_-(\alpha)(\alpha - \frac{K}{M})\sqrt{K(1 - \cos(\theta_0 + iq_1))}\sqrt{K - \alpha} \tag{C.6}$$

We use the Maclaurin's expansion for the integrand appearing in Eq. $(C.5)$, and we note that $q_1 = 0$ is a saddle point. Also, for convenience, we write

$$\alpha_s = K\sin\theta_0,\ \alpha_t = K\sin\theta \qquad \alpha_m = -K\cos\theta_0,\ \alpha_p = K\cos\theta \tag{C.7}$$

so that Eq.$(C.5)$ takes the form

$$I^{/} = i\alpha_s G(\alpha, \alpha_m)\int_{-\infty}^{\infty} e^{iKR_0\cosh q_1} dq_1$$

$$+\frac{i}{2}\left[-2i\alpha_s G_{q_1}(\alpha, \alpha_m) + \alpha_s\left\{G(\alpha, \alpha_m) + G_{q_1q_1}(\alpha, \alpha_m)\right\}\right]\int_{-\infty}^{\infty} q_1^2 e^{iKR_0\cosh q_1} dq_1 \tag{C.8}$$

Now, we use Eq. $(C.7)$, and find its derivative w.r.t. q_1 as

$$G_{q_1}(\alpha, \alpha_m) = \frac{2\alpha_s L_+^{/}(\alpha_m)}{\sqrt{1 - \cos(\theta_0 + iq_1)}L_+^2(\alpha_m)}$$

$$+\frac{1}{2i\sqrt{K-\alpha}(\alpha-\frac{K}{M})L_-(\alpha)\sqrt{K}}\left[\frac{\sin(\theta_0+iq_1)}{L_+(\alpha_m)(1-\cos(\theta_0+iq_1))^{\frac{3}{2}}}\right] \tag{C.9}$$

and the second derivative of Eq. $(C.7)$ w.r.t. q_1 is given as

$$\begin{aligned} G_{q_1q_1} &= \frac{1}{2\sqrt{K}L_+^2(\alpha_m)\sqrt{(1-\cos\theta_0)}\sqrt{K-\alpha}(\alpha-\frac{K}{M})L_-(\alpha)} \\ &\left[\left(\cos^2\theta_0+2\cos\theta_0-3\right)\frac{L_+(\alpha_m)}{(1-\cos\theta_0)^2}\right. \end{aligned}$$

$$\left.+\left(2\cos\theta_0-\frac{3\sin^2\theta_0}{1-\cos\theta_0}\right)KL_+^{/}(\alpha_m)+2\alpha_s^2\left\{L_+^{//}(\alpha_m)-\frac{2L_+^{/\,2}(\alpha_m)}{L_+(\alpha_m)}\right\}\right]. \tag{C.10}$$

We substitute the values of Eqs. $(C.9)$ and $(C.10)$ in Eq. $(C.6)$ and get

$$I^{/}=i\alpha_sG(\alpha,\alpha_m)\sqrt{\frac{2\pi}{KR_0}}e^{i\left(KR_0+\frac{\pi}{4}\right)}$$

$$\frac{i}{2}\left[-2i\alpha_mG_{q_1}(\alpha,\alpha_m)+\alpha_s\left(G(\alpha,\alpha_m)+G_{q_1q_1}(\alpha,\alpha_m)\right)\right]\sqrt{\frac{2\pi}{(KR_0)^3}}e^{i\left(KR_0-\frac{\pi}{4}\right)} \tag{C.11}$$

We use the above expression in Eq. $(C.4)$, and

$$\begin{aligned} I_2 &= i\alpha_s\sqrt{\frac{2\pi}{KR_0}}e^{i\left(KR_0+\frac{\pi}{4}\right)}\int_{-\infty}^{\infty}G(\alpha,\alpha_m)e^{i\alpha X+i\kappa|Y|}d\alpha \\ &+\sqrt{\frac{2\pi}{(KR_0)^3}}\alpha_me^{i\left(KR_0-\frac{\pi}{4}\right)}\int_{-\infty}^{\infty}G_{q_1}(\alpha,\alpha_m)e^{i\alpha X+i\kappa|Y|}d\alpha \end{aligned}$$

$$\begin{aligned} &+\frac{i\alpha_s}{2}e^{i\left(KR_0-\frac{\pi}{4}\right)}\sqrt{\frac{2\pi}{(KR_0)^3}}\int_{-\infty}^{\infty}G(\alpha,\alpha_m)e^{i\alpha X+i\kappa|Y|}d\alpha \\ &+\frac{i\alpha_s}{2}e^{i\left(KR_0-\frac{\pi}{4}\right)}\sqrt{\frac{2\pi}{(KR_0)^3}}\int_{-\infty}^{\infty}G_{q_1q_1}(\alpha,\alpha_m)e^{i\alpha X+i\kappa|Y|}d\alpha, \end{aligned} \tag{C.12}$$

or

$$I_2 = \frac{i\alpha_s}{2}(2A_1 + A_2)\int_{-\infty}^{\infty} G(\alpha,\alpha_m)e^{i\alpha X + i\kappa|Y|}d\alpha + \alpha_m A_2 \int_{-\infty}^{\infty} G_{q_1}(\alpha,\alpha_m)e^{i\alpha X + i\kappa|Y|}d\alpha$$

$$+\frac{i\alpha_s}{2}A_2\int_{-\infty}^{\infty} G_{q_1q_1}(\alpha,\alpha_m)e^{i\alpha X + i\kappa|Y|}d\alpha, \qquad (C.13)$$

where

$$A_1 = \sqrt{\frac{2\pi}{KR_0}}e^{i\left(KR_0+\frac{\pi}{4}\right)}, \qquad A_2 = \sqrt{\frac{2\pi}{(KR_0)^3}}e^{i\left(KR_0-\frac{\pi}{4}\right)} \qquad (C.14)$$

Now, we write Eq. $(C.13)$ in the following form

$$I_2 = \frac{i}{2}(2A_1 + A_2)\alpha_s J_1 - \alpha_m A_2 J_2 + \frac{i\alpha_s}{2}A_2 J_3, \qquad (C.15)$$

where

$$J_1 = \int_{-\infty}^{\infty} G e^{i\alpha X + i\kappa|Y|}d\alpha, \qquad (C.16)$$

$$J_2 = \int_{-\infty}^{\infty} G_{q_1} e^{i\alpha X + i\kappa|Y|}d\alpha \qquad (C.17)$$

and

$$J_3 = \int_{-\infty}^{\infty} G_{q_1q_1} e^{i\alpha X + i\kappa|Y|}d\alpha. \qquad (C.18)$$

Now, we calculate the values of the integrals appearing in Eqs.$(C.16)$, $(C.17)$ and $(C.18)$ in the intermediate range. First, we write the Maclaurin's series for G and find their required derivatives as follows, i.e., from Eqs. $(C.7)$, $(C.9)$ and $(C.10)$, we have

$$G(\alpha,\alpha_m) = \frac{B_1}{L_-(\alpha)(\alpha - \frac{K}{M})\sqrt{K-\alpha}} \qquad (C.19)$$

$$G_{q_1}(\alpha,\alpha_m) = \frac{B_2}{L_-(\alpha)(\alpha - \frac{K}{M})\sqrt{K-\alpha}} \qquad (C.20)$$

$$G_{q_1q_1}(\alpha,\alpha_m) = \frac{B_3}{L_-(\alpha)(\alpha - \frac{K}{M})\sqrt{K-\alpha}} \tag{C.21}$$

with

$$B_1 = \frac{1}{L_+(\alpha_m)\sqrt{K-\alpha_m}}, \tag{C.22}$$

$$B_2 = \frac{-i\alpha_s\left[L_+(\alpha_m) + 2(K+\alpha_m)L_+^{/}(\alpha_m)\right]}{2(K+\alpha_m)^{\frac{3}{2}}L_+^2(\alpha_m)}, \tag{C.23}$$

and

$$B_3 = \frac{i}{L_+^2(\alpha_m)\sqrt{(1-\cos\theta_0)}}\left[\left(\cos^2\theta_0 + 2\cos\theta_0 - 3\right)\frac{L_+(\alpha_m)}{(1-\cos\theta_0)^2}\right]$$
$$+\left(2\cos\theta_0 - \frac{3\sin^2\theta_0}{1-\cos\theta_0}\right)\frac{KL_+^{/}(\alpha_m)}{\sqrt{(1-\cos\theta_0)}} + \frac{2\alpha_s^2}{\sqrt{(1-\cos\theta_0)}}\left(L_+^{//}(\alpha_m) - \frac{2L_+^{/\,2}(\alpha_m)}{L_+(\alpha_m)}\right). \tag{C.24}$$

Now making the following substitutions in Eqs. $(C.16)$, $(C.17)$ and $(C.18)$,

$$X = R\cos\theta, |Y| = R\sin\theta \text{ and } \alpha = K\cos(\theta + iq_2), \ (0 < \theta < \pi, \ -\infty < q_2 < \infty). \tag{C.25}$$

we get

$$J_1 = -iK\int_{-\infty}^{\infty} G(K\cos(\theta+iq_2), -\alpha_m)\sin(\theta+iq_2)\, e^{iKR\cosh q_2}dq_2, \tag{C.26}$$

$$J_2 = -iK\int_{-\infty}^{\infty} G_{q_1}(K\cos(\theta+iq_2), -\alpha_m)\sin(\theta+iq_2)\, e^{iKR\cosh q_2}dq_2, \tag{C.27}$$

$$J_3 = -iK\int_{-\infty}^{\infty} G_{q_1q_1}(K\cos(\theta+iq_2), -\alpha_m)\sin(\theta+iq_2)\, e^{iKR\cosh q_2}dq_2, \tag{C.28}$$

We write the Maclaurin's expansion of the integrand appearing in Eqs.$(C.26)$ to

$(C.28)$ and get

$$J_1 = -i\alpha_t G(\alpha_p, \alpha_m) \int_{-\infty}^{\infty} e^{iKR\cosh q_2} dq_2 - \frac{i}{2} \begin{bmatrix} 2i\alpha_p G_{q_2}(\alpha_p, \alpha_m) \\ +\alpha_t \left\{ G(\alpha_p, \alpha_m) + G_{q_2 q_2}(\alpha_p, \alpha_m) \right\} \end{bmatrix}$$

$$\times \int_{-\infty}^{\infty} q_2^2 e^{iKR\cosh q_2} dq_2, \tag{C.29}$$

$$J_2 = -i\alpha_t G_{q_1}(\alpha_p, \alpha_m) \int_{-\infty}^{\infty} e^{iKR\cosh q_2} dq_2 - \frac{i}{2} \begin{bmatrix} 2i\alpha_p G_{q_1 q_2}(\alpha_p, \alpha_m) \\ +\alpha_t \left\{ G_{q_1}(\alpha_p, \alpha_m) + G_{q_1 q_2 q_2}(\alpha_p, \alpha_m) \right\} \end{bmatrix}$$

$$\times \int_{-\infty}^{\infty} q_2^2 e^{iKR\cosh q_2} dq_2, \tag{C.30}$$

and

$$J_3 = -i\alpha_t G_{q_1 q_1}(\alpha_p, \alpha_m) \int_{-\infty}^{\infty} e^{iKR\cosh q_2} dq_2$$

$$-\frac{i}{2} \begin{bmatrix} 2i\alpha_p G_{q_1 q_1 q_2}(\alpha_p, \alpha_m) \\ +\alpha_t \left\{ G_{q_1 q_1}(\alpha_p, \alpha_m) + G_{q_1 q_1 q_2 q_2}(\alpha_p, \alpha_m) \right\} \end{bmatrix}$$

$$\times \int_{-\infty}^{\infty} q_2^2 e^{iKR\cosh q_2} dq_2, \tag{C.31}$$

Now, at $q_2 = 0$,we have

$$G(\alpha_p, \alpha_m) = \frac{B_1}{K\sqrt{K - \alpha_p} L_-(\alpha_p)(\cos\theta - \frac{1}{M})}, \tag{C.32}$$

$$G_{q_2}(\alpha_p, \alpha_m) = iB_1 \left[\frac{-\alpha_t}{KL_-(\alpha_p)(\cos\theta - \frac{1}{M})(K - \alpha_p)^{\frac{3}{2}}} + \frac{KL'_-(\alpha_p)}{\sqrt{K - \alpha_p}(\cos\theta - \frac{1}{M})L_-^2(\alpha_p)} \right]$$

$$+\frac{\sqrt{K}B_1}{\sqrt{K-\alpha_p}(\cos\theta-\frac{1}{M})^2L_-(\alpha_p)} \tag{C.33}$$

$$G_{q_2q_2}(\alpha_p,\alpha_m)=\frac{B_1}{\sqrt{K}\,(K-\alpha_p)\,(\cos\theta-\frac{1}{M})^2L_-^2(\alpha_p)}\left[\frac{L_-(\alpha_p)(\cos\theta-\frac{1}{M})\alpha_p}{\sqrt{K}\sqrt{K-\alpha_p}}\right.$$

$$\left.-\frac{(\cos\theta-\frac{1}{M})\alpha_t^2L_-(\alpha_p)}{\sqrt{K}(K-\alpha_p)^{\frac{3}{2}}}+\frac{\sqrt{K}L_-(\alpha_p)}{\sqrt{K-\alpha_p}}\right]+\frac{B_1(K)^{\frac{3}{2}}(\cos\theta-\frac{1}{M})L_-^{/}(\alpha_p)}{\sqrt{K-\alpha_p}}-\frac{iB_1\alpha_tL_-(\alpha_p)}{2\sqrt{K}\sqrt{K-\alpha_p}}$$

$$+\frac{2iB_1\alpha_t\sqrt{1-\cos\theta}L_-(\alpha_p)}{K(\cos\theta-\frac{1}{M})}+iB_1\sqrt{K}\alpha_t\sqrt{K-\alpha_p}L_-^{/}(\alpha_p)$$
$$+\frac{iB_1\alpha_t(\cos\theta-\frac{1}{M})^2\sqrt{K-\alpha_p}L_-^{//}(\alpha_p)}{\sqrt{K}L_-^2(\alpha_p)}$$

$$+\frac{iB_1\sqrt{K}\alpha_t(\cos\theta-\frac{1}{M})L_-^{/}(\alpha_p)}{2\sqrt{K-\alpha_p}}-\frac{iB_1\alpha_t}{\sqrt{K}}\sqrt{K-\alpha_p}L_-^{/}(\alpha_p)$$
$$-\frac{2iB_1\alpha_t(\cos\theta-\frac{1}{M})\sqrt{K-\alpha_p}L_-^{/}(\alpha_p)}{\sqrt{K}L_-(\alpha_p)}, \tag{C.34}$$

Similarly, from Eq. $(C.32)$ we can have the required values of the derivatives, i.e., the first derivative of Eq. $(C.32)$ w.r.t. q_1 as

$$G_{q_1}(\alpha_p,\alpha_m)=\frac{B_2}{K^{\frac{3}{2}}\sqrt{(1-\cos\theta)}L_-(\alpha_p)(\cos\theta-\frac{1}{M})}, \tag{C.35}$$

and the derivative of Eq.$(C.35)$ w.r.t. q_2 as

$$G_{q_1q_2}(\alpha_p,\alpha_m)=iB_2\left[\frac{-\alpha_t}{KL_-(\alpha_p)(\cos\theta-\frac{1}{M})(K-\alpha_p)^{\frac{3}{2}}}+\frac{KL_-^{/}(\alpha_p)}{\sqrt{K-\alpha_p}(\cos\theta-\frac{1}{M})L_-^2(\alpha_p)}\right]$$

$$+\frac{\sqrt{K}B_2}{\sqrt{K-\alpha_p}(\cos\theta-\frac{1}{M})^2L_-(\alpha_p)} \tag{C.36}$$

and derivative of Eq.(C.36) w.r.t. q_2 as

$$G_{q_1q_2q_2}(\alpha_p,\alpha_m)=\frac{B_2}{\sqrt{K}\,(K-\alpha_p)\,(\cos\theta-\frac{1}{M})^2L_-^2(\alpha_p)}\left[\frac{L_-(\alpha_p)(\cos\theta-\frac{1}{M})\alpha_p}{\sqrt{K}\sqrt{K-\alpha_p}}\right.$$

$$\left.-\frac{(\cos\theta-\frac{1}{M})\alpha_t^2L_-(\alpha_p)}{\sqrt{K}(K-\alpha_p)^{\frac{3}{2}}}+\frac{\sqrt{K}L_-(\alpha_p)}{\sqrt{K-\alpha_p}}\right]+\frac{B_2(K)^{\frac{3}{2}}(\cos\theta-\frac{1}{M})L_-^{/}(\alpha_p)}{\sqrt{K-\alpha_p}}-\frac{iB_1\alpha_tL_-(\alpha_p)}{2\sqrt{K}\sqrt{K-\alpha_p}}$$

$$+\frac{2iB_2\alpha_t\sqrt{1-\cos\theta}L_-(\alpha_p)}{K(\cos\theta-\frac{1}{M})}+iB_1\sqrt{K}\alpha_t\sqrt{K-\alpha_p}L_-^{/}(\alpha_p)+\frac{iB_2\alpha_t(\cos\theta-\frac{1}{M})^2\sqrt{K-\alpha_p}L_-^{//}(\alpha_p)}{\sqrt{K}L_-^2(\alpha_p)}$$

$$+\frac{iB_2\sqrt{K}\alpha_t(\cos\theta-\frac{1}{M})L_-^{/}(\alpha_p)}{2\sqrt{K-\alpha_p}}-\frac{iB_2\alpha_t}{\sqrt{K}}\sqrt{K-\alpha_p}L_-^{/}(\alpha_p)-\frac{2iB_2\alpha_t(\cos\theta-\frac{1}{M})\sqrt{K-\alpha_p}L_-^{/}(\alpha_p)}{\sqrt{K}L_-(\alpha_p)}. \tag{C.37}$$

Also the derivative of Eq. (C.35) w.r.t. q_1 is given as

$$G_{q_1q_1}(\alpha_p,\alpha_m)=\frac{B_3}{K^{\frac{3}{2}}\sqrt{(1-\cos\theta)}L_-(\alpha_p)(\cos\theta-\frac{1}{M})} \tag{C.38}$$

and the derivative of Eq. (C.38) w.r.t. q_2 is given as

$$G_{q_1q_1q_2}(\alpha_p,\alpha_m)=iB_3\left[\frac{-\alpha_t}{KL_-(\alpha_p)(\cos\theta-\frac{1}{M})(K-\alpha_p)^{\frac{3}{2}}}+\frac{KL_-^{/}(\alpha_p)}{\sqrt{K-\alpha_p}(\cos\theta-\frac{1}{M})L_-^2(\alpha_p)}\right]$$

$$+\frac{\sqrt{K}B_3}{\sqrt{K-\alpha_p}(\cos\theta-\frac{1}{M})^2L_-(\alpha_p)}, \tag{C.39}$$

and the derivative of Eq. (C.36) w.r.t. q_2 as

$$G_{q_1q_2q_2}(\alpha_p,\alpha_m)=\frac{B_3}{\sqrt{K}\,(K-\alpha_p)\,(\cos\theta-\frac{1}{M})^2L_-^2(\alpha_p)}\left[\frac{L_-(\alpha_p)(\cos\theta-\frac{1}{M})\alpha_p}{\sqrt{K}\sqrt{K-\alpha_p}}\right.$$

$$\left.-\frac{(\cos\theta-\frac{1}{M})\alpha_t^2L_-(\alpha_p)}{\sqrt{K}(K-\alpha_p)^{\frac{3}{2}}}+\frac{\sqrt{K}L_-(\alpha_p)}{\sqrt{K-\alpha_p}}\right]+\frac{B_3(K)^{\frac{3}{2}}(\cos\theta-\frac{1}{M})L_-^{/}(\alpha_p)}{\sqrt{K-\alpha_p}}-\frac{iB_3\alpha_tL_-(\alpha_p)}{2\sqrt{K}\sqrt{K-\alpha_p}}$$

$$+\frac{2iB_3\alpha_t\sqrt{1-\cos\theta}L_-(\alpha_p)}{K(\cos\theta-\frac{1}{M})}+iB_3\sqrt{K}\alpha_t\sqrt{K-\alpha_p}L_-^{/}(\alpha_p)$$
$$+\frac{iB_3\alpha_t(\cos\theta-\frac{1}{M})^2\sqrt{K-\alpha_p}L_-^{//}(\alpha_p)}{\sqrt{K}L_-^2(\alpha_p)}$$

$$+\frac{iB_3\sqrt{K}\alpha_t(\cos\theta-\frac{1}{M})L_-^{/}(\alpha_p)}{2\sqrt{K-\alpha_p}}-\frac{iB_3\alpha_t}{\sqrt{K}}\sqrt{K-\alpha_p}L_-^{/}(\alpha_p)$$
$$-\frac{2iB_3\alpha_t(\cos\theta-\frac{1}{M})\sqrt{K-\alpha_p}L_-^{/}(\alpha_p)}{\sqrt{K}L_-(\alpha_p)}. \tag{C.40}$$

We use Eqs. $(C.32)$ to $(C.40)$ in Eqs. $(C.29)$ to $(C.31)$ and get

$$J_1=-i\alpha_tG(\alpha_p,\alpha_m)A_3-\frac{i}{2}\begin{bmatrix}2i\alpha_pG_{q_2}(\alpha_p,\alpha_m)\\+\alpha_t\left\{G(\alpha_p,\alpha_m)+G_{q_2q_2}(\alpha_p,\alpha_m)\right\}\end{bmatrix}A_4, \tag{C.41}$$

$$J_2=-i\alpha_tG_{q_1}(\alpha_p,\alpha_m)A_3-\frac{i}{2}\begin{bmatrix}2i\alpha_pG_{q_1q_2}(\alpha_p,\alpha_m)\\+\alpha_t\left\{G_{q_1}(\alpha_p,\alpha_m)+G_{q_1q_2q_2}(\alpha_p,\alpha_m)\right\}\end{bmatrix}A_4 \tag{C.42}$$

and

$$J_3=-i\alpha_tG_{q_1q_1}(\alpha_p,\alpha_m)A_3-\frac{i}{2}\begin{bmatrix}2i\alpha_pG_{q_1q_1q_2}(\alpha_p,\alpha_m)\\+\alpha_t\left\{G_{q_1q_1}(\alpha_p,\alpha_m)+G_{q_1q_1q_2q_2}(\alpha_p,\alpha_m)\right\}\end{bmatrix}A_4, \tag{C.43}$$

where

$$A_3=\sqrt{\frac{2\pi}{KR}}e^{i\left(KR+\frac{\pi}{4}\right)}, \quad \text{and} \quad A_4=\sqrt{\frac{2\pi}{(KR)^3}}e^{i\left(KR-\frac{\pi}{4}\right)}. \tag{C.44}$$

Now, we use the values of Eqs. $(C.41)$ to $(C.43)$ in Eq. $(C.15)$ and obtain

$$I_2=\frac{i}{2}(2A_1+A_2)\alpha_s\left[-i\alpha_tG(\alpha_p,\alpha_m)A_3-\frac{i}{2}\begin{bmatrix}2i\alpha_pG_{q_2}(\alpha_p,\alpha_m)\\+\alpha_t\left\{G(\alpha_p,\alpha_m)+G_{q_2q_2}(\alpha_p,\alpha_m)\right\}\end{bmatrix}A_4\right]$$

$$-\alpha_m A_2 \left[-i\alpha_t G_{q_1}(\alpha_p, \alpha_m) A_3 - \frac{i}{2} \left[\begin{array}{c} 2i\alpha_p G_{q_1q_2}(\alpha_p, \alpha_m) \\ +\alpha_t \left\{ G_{q_1}(\alpha_p, \alpha_m) + G_{q_1q_2q_2}(\alpha_p, \alpha_m) \right\} \end{array} \right] A_4 \right]$$

$$+\frac{i\alpha_s}{2} A_2 \left[-i\alpha_t G_{q_1q_1}(\alpha_p, \alpha_m) A_3 - \frac{i}{2} \left[\begin{array}{c} 2i\alpha_p G_{q_1q_1q_2}(\alpha_p, \alpha_m) \\ +\alpha_t \left\{ G_{q_1q_1}(\alpha_p, \alpha_m) + G_{q_1q_1q_2q_2}(\alpha_p, \alpha_m) \right\} \end{array} \right] A_4 \right] \tag{C.45}$$

Similarly, we can calculate I_3 as given by

$$I_3 = \frac{e^{-iKM(X-X_0)}}{8\pi^2\sqrt{1-M^2}\sqrt{K^2-\alpha^2}} \left(\frac{2\pi}{KR_0} \right)^{\frac{1}{2}} \left[\frac{F(\alpha_p)}{(\alpha_m - \alpha_p)} e^{i\left(KR_0 + \frac{\pi}{4}\right)} \right.$$

$$\left. + \left(\frac{1}{KR_0} \right) \left\{ \frac{F_1(\alpha_p)}{(\alpha_m - \alpha_p)} + \frac{F_2(\alpha_p)}{(\alpha_m - \alpha_p)^2} + \frac{F_3(\alpha_p)}{(\alpha_m - \alpha_p)^3} \right\} e^{i\left(KR_0 - \frac{\pi}{4}\right)} \right]$$

$$\times (\alpha_t) \sqrt{\frac{2\pi}{KR}} e^{i\left(KR + \frac{\pi}{4}\right)}, \tag{C.46}$$

where

$$F(\alpha_p) = \left[\begin{array}{c} \left[\begin{array}{c} BK(1+M^2) + 2BM\alpha_m + \frac{BM^2\alpha_m^2}{K(1-M^2)} \\ -\sqrt{K-\alpha_p}\sqrt{K+\alpha_m} sgn(Y) \end{array} \right] \\ \times \left[L_+(\alpha_p) L_-(\alpha_m) \right]^{-1} \end{array} \right], \tag{C.47}$$

$$F_1(\alpha_p) = \frac{1}{2} \left[\left(\begin{array}{c} -BM\alpha_m + \frac{BM^2\alpha_m^2}{K(1-M^2)} \\ -\frac{1}{8\sqrt{2}}\sqrt{K-\alpha_p}\sqrt{K} \sin^{-3}\frac{\theta_0}{2} \sin^2\theta_0 sgn(Y) \end{array} \right) (L_-(\alpha_m)) \right.$$

$$+ \left(\begin{array}{c} 4BM\alpha_s^2 - 4M^2Bi\sin^2\theta_0\alpha_m \\ -\frac{1}{\sqrt{2}}\sqrt{(1-\frac{\alpha_p}{K})} \sin^{-1}\frac{\theta_0}{2}\alpha_s^2 sgn(Y) \end{array} \right) (L'_-(\alpha_m))$$

$$+ \left(\begin{array}{c} -2BK(1+M^2)\alpha_s^2 - 4BM\alpha_s^2\alpha_m \\ -\frac{2BM^2\alpha_m^2\alpha_s^2}{K(1-M^2)} + 2^{\frac{3}{2}}\alpha_s^2\sqrt{(1-\frac{\alpha_p}{K})} \sin\frac{\theta_0}{2} sgn(Y) \end{array} \right)$$

$$\times \left(L_-^{2\prime}(\alpha_m) / L_-(\alpha_m) \right)$$

$$+\left(\begin{array}{c} BK\left(1+M^2\right)\alpha_s^2+2MB\alpha_s^2\alpha_m \\ +\frac{M^2B\alpha_s^2\alpha_m^2}{K(1-M^2)}-\sqrt{2}\sqrt{(1-\frac{\alpha_p}{K})}\sin\frac{\theta_0}{2}\alpha_s^2 sgn\left(Y\right) \end{array}\right)\times\left(L''_-\left(\alpha_m\right)\right)$$

$$\left.+\left(\begin{array}{c} -BK\left(1+M^2\right)\alpha_m+2MKB\alpha_m \\ -\frac{BM^2\alpha_m^3}{K(1-M^2)}+\sqrt{2}\sqrt{K-\alpha_p}K^{\frac{1}{2}}\sin\frac{\theta_0}{2}\alpha_m sgn\left(Y\right) \end{array}\right)\left(L'_-\left(\alpha_m\right)\right)\right]$$

$$\times\left[L_+\left(\alpha_p\right)L_-^2\left(\alpha_m\right)\right]^{-1}, \tag{C.48}$$

$$F_2\left(\alpha_p\right)=\frac{1}{2}\left[\left(\begin{array}{c} 4BM\alpha_s^2-\frac{4iM^2B\alpha_s^2\alpha_m}{K(1-M^2)} \\ -\frac{1}{\sqrt{2}}\sqrt{(1-\frac{\alpha_p}{K})}\sin^{-1}\frac{\theta_0}{2}\alpha_s^2 sgn\left(Y\right) \end{array}\right)\right.$$

$$+\left(\begin{array}{c} -2B\left(1+M^2\right)\alpha_s^2-4MB\alpha_s^2\alpha_m \\ -\frac{2BM^2\alpha_s^2\alpha_m^2}{K(1-M^2)}+2^{\frac{3}{2}}\sqrt{K(K-\alpha_p)}\sin\frac{\theta_0}{2}\alpha_s^2 sgn\left(Y\right) \end{array}\right)$$

$$\times\left(L'_-\left(\alpha_m\right)/L_-\left(\alpha_m\right)\right)$$

$$\left.+\left(\begin{array}{c} -BK\left(1+M^2\right)\alpha_m-2BM\alpha_m^2 \\ +\frac{BM^2\alpha_m^3}{K(1-M^2)}-\sqrt{2}\sqrt{K-\alpha_p}K^{\frac{3}{2}}\sin\frac{\theta_0}{2}sgn\left(Y\right) \end{array}\right)\right]$$

$$\times\left[L_+\left(\alpha_p\right)L_-\left(\alpha_m\right)\right]^{-1} \tag{C.49}$$

and

$$F_3\left(\alpha_p\right)=\frac{1}{2}\left[\begin{array}{c} -2BK\left(1+M^2\right)\alpha_s^2-4BM\alpha_s^2\alpha_m \\ +\frac{2BM^2\alpha_s^2\alpha_m}{(1-M^2)}+2^{\frac{3}{2}}\sqrt{(1-\frac{\alpha_p}{K})}\sin\frac{\theta_0}{2}\alpha_s^2 sgn\left(Y\right) \end{array}\right]$$

$$\times\left[L_+\left(\alpha_p\right)L_-\left(\alpha_m\right)\right]^{-1}. \tag{C.50}$$

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Line-source diffraction by a slit in a moving fluid

M. Ayub, A. Naeem, and R. Nawaz

Abstract: The diffraction of a cylindrical acoustic wave from a slit in a moving fluid using Myers condition (J. Sound Vib. **71**, 429 (1980)) is investigated, and an improved form of the analytical solution for the diffracted field is presented. The problem is solved analytically using an integral transform, Wiener–Hopf technique, and the modified method of stationary phase. The mathematical results are well supported by graphical discussion showing how the absorbing parameter and Mach number affect the amplitude of the velocity potential.

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Résumé : Nous étudions la diffraction d'une onde acoustique cylindrique par une fente dans un fluide en mouvement en utilisant la condition de Myers (J. Sound Vib. **71**, 429 (1980)). La condition de Myers est maintenant la forme acceptée des conditions limites pour des barrières absorbantes dans un fluide en mouvement et donne une forme correcte au champ acoustique. Le problème est solutionné analytiquement en utilisant une transformation intégrale, la technique de Wiener–Hopf et la méthode modifiée de la phase stationnaire. Le champ diffracté que nous obtenons est la somme des champs produits par les deux bords de la fente et d'un champ d'interaction.

[Traduit par la Rédaction]

1. Introduction

The noise reduction by barriers is a common method of reducing noise pollution in heavily built-up areas. Noise from traffic, heavy construction machinery, large transformers, or plants can be shielded by a barrier that intercepts the line of sight from source to receiver [1, 2]. An ideal barrier should be a good attenuator of sound and economical at the same time. Such barriers have an absorbing lining on the surfaces and satisfy absorbing boundary condition. The discussion of acoustic diffraction in the presence of moving fluid is desirable in cases of noise radiated from aero-engines and for noise inside wind tunnels. Related studies about noise reduction by barriers can be found in [3–6].

Various investigations have been carried out to study the classical problems of line-source diffraction of electromagnetic and acoustic waves by various types of half-planes. For example, the line-source diffraction of electromagnetic and acoustic waves by a perfectly conducting half-plane was investigated by Jones [7]. The problem of line-source diffraction of acoustic waves by a hard half-plane attached to a wake, in still air as well as when the medium is convective, was studied by Jones [8]. Rawlins considered the line-source diffraction of acoustic waves by an absorbing barrier [9]. The introduction of line source changes the incident field, and the solution method requires a careful analysis when calculating the diffracted field.

The problem of diffraction from a slit or strip has been addressed by many scientists. Asghar et al. [10, 11], Asghar and Hayat [12, 13], Ayub et al. [14], and Birbir and Büyükaksoy [15] have also contributed to the field. Asyestas and Kleinman [16] have done a lot of work on it. The Wiener–Hopf technique has been used to solve the problem of diffraction from a slit or strip by Jones [7] and Nobel [17]. The diffracted rays are produced by incident rays, which hit edges, corners, etc. Ordinary geometrical optics fail to account for this diffraction phenomenon, while the geometrical theory of diffraction by Keller [18, 19] takes care of this problem. The WKB method [7, 20] is also used to deal with these diffraction problems. The assumption that the slit is large with respect to the wavelength is used to find the diffracted field from the interaction between the edges. This assumption is also used to approximate several integrals asymptotically. The geometrical theory of diffraction was used by Karp and Keller [21] to calculate the interactive term for diffraction from a slit in a perfectly rigid barrier. Therefore, it is worthwhile to consider the diffraction of an acoustic wave from a slit.

Approximate boundary conditions, such as impedance boundary conditions [22–24], are also useful for computational purposes in acoustic and electromagnetics with respect to noise abatement and the diffraction by barriers satisfying various boundary conditions [25–30]. The Myers' condition is now the accepted form of the boundary condition for absorbing barriers in a moving fluid and gives a correct form of the acoustic field. This condition allows a straightforward manipulation of the condition into a form that is more convenient to apply than the Ingard condition. It permits a simple generalization describing a passive boundary on which the normal acoustic is known. At any such surface that generates an acoustic field by small deformation or undergoes small deformation in response to an acoustic field, the acoustic field particle displacement in the direction normal to the undeformed boundary must equal the displacement of the boundary itself in the same direction. In the Ingard condition, only those surfaces are considered whose

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M. Ayub,[1] A. Naeem, and R. Nawaz. Department of Mathematics, Quaid-i-Azam University, 45320, Islamabad 44000, Pakistan.

[1]Corresponding author (e-mail: mayub59@yahoo.com).

undeformed shape is planar, and it is applicable to only those base flows that are everywhere parallel to those planes. But the extra term appearing in Myers' condition carries more general surface shapes or base flows, which can be overlooked when using the physical state [31].

Keeping in mind the work mentioned above, we have studied the problem of line-source diffraction of acoustic waves by a slit in a moving fluid using Myers' condition [31]. It is a new diffraction problem, which is not addressed in the existing literature. Graphical results are also presented to observe the effects of various parameters on the velocity potential. We have drawn the graphs, and we obtain a corrective term in the solution of our problem that was altogether missing in [32]. The graphs give a clear picture of the variation of the velocity potential for various parameters in different cases. The graphs also verify our mathematical calculations. It is found that the two edges of the plane give rise to two diffracted fields (one from each edge) and an interaction field (double diffraction of the two edges). The asymptotic analysis of the resulting integrals is only carried far enough to permit the calculations of the diffracted fields far from the slit. The solution method consists of Fourier transform, Wiener–Hopf technique [17, 20], and the asymptotic expansion method [33, 34]. The appendix at the end of the work gives more detail of the calculations that were used in the main body of the paper.

2. Formulation of the problem

We consider the diffraction of an acoustic wave due to a line source from a slit occupying a space $y = 0$, $p \le x \le q$. The line source is located at (x_0, y_0), and the system is placed in a fluid moving with subsonic velocity U parallel to the x-axis. The time dependence is considered to be of harmonic type $e^{-i\omega t}$ (ω is the angular frequency) and is suppressed throughout the manuscript. The plane is assumed to satisfy Myers' condition [31].

$$u_n = -\frac{\tilde{p}}{Z_a} + \frac{U}{i\omega Z_a}\partial\frac{\tilde{p}}{\partial x} \tag{1}$$

where u_n is the normal derivative of the perturbation velocity, $\tilde{p}$ is the surface pressure, Z_a is the acoustic impedance of the surface, and $-\boldsymbol{n}$ is a normal pointing from the fluid into the surface. The perturbation velocity $\boldsymbol{u}$ of the irrotational sound wave can be written in terms of the velocity potential Φ as $\boldsymbol{u} = \nabla\Phi$. The resulting pressure $\tilde{p}$ of the sound field can be written as,

$$\tilde{p} = -\rho_0\left(-i\omega + U\frac{\partial}{\partial x}\right)\Phi(x, y) \tag{2}$$

where ρ_0 is the density of the undisturbed stream. The geometry of the problem is shown in Fig. 1.

The wave equation satisfied by the total velocity potential Φ in the presence of the line source is given by,

$$\left[(1 - M^2)\frac{\partial^2}{\partial x^2} + 2ikM\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} + k^2\right]\Phi(x, y) = \delta(x - x_0)\delta(y - y_0) \tag{3}$$

and at $y = 0$, (1) and (2) lead to the following boundary conditions,

$$\left(\frac{\partial}{\partial y} \mp 2\beta M\frac{\partial}{\partial x} \pm ik\beta \mp \frac{i\beta M^2}{k}\frac{\partial^2}{\partial x^2}\right)\Phi(x, 0^{\pm}) = 0 \quad \text{for} -\infty < x < p, \quad q < x < \infty \tag{4}$$

where $k = \omega/c$ is the wave number, $\beta = (\rho_0 c)/Z_a$ is the specific complex admittance, and $M = U/c$ (c is the velocity of sound) is the Mach number. $k = k_1 + ik_2$ has a small positive imaginary part to ensure the regularity of the Fourier transform integrals. It is assumed that the flow is subsonic, i.e., $|M| < 1$ and $\mathrm{Re}\ \beta > 0$, which is a necessary condition for an absorbing surface [9]. We remark that $\beta = 0$ corresponds to the rigid barrier and $\beta = \infty$ corresponds to the pressure release barrier.

Also, the potential Φ and its derivative $\partial\Phi/\partial y$ are continuos on the slit, i.e.,

$$\begin{aligned}\frac{\partial\Phi}{\partial y}(x, 0^+) &= \frac{\partial\Phi}{\partial y}(x, 0^-), \qquad p \le x \le q \\ \Phi(x, 0^+) &= \Phi(x, 0^-), \qquad p \le x \le q\end{aligned} \tag{5}$$

Also, as $r \to \infty$, if Φ represents an out-going wave at infinity and if the time factor is taken as $e^{-i\omega t}$ then,

$$(r)^{1/2}\left(\frac{\partial\Phi}{\partial r} - ik\Phi\right) \to 0 \tag{6}$$

is the radiation condition [17, Sect. 1.5, p. 31]. The edge field behavior at $x \to p^-$ and $x \to q^+$ will be governed by edge conditions [17, Sect. 2.6, p. 75], i.e.,

$$\begin{aligned}\Phi(x, 0) &= O(1) \\ \frac{\partial\Phi(x, 0)}{\partial r} &= O\left(\frac{1}{\sqrt{x}}\right)\end{aligned} \tag{7}$$

Physically, we can consider the flow of an incompressible fluid past the edge of a sheet on which the normal velocity is zero.

For subsonic flow, we can make the following real substitutions,

$$\begin{aligned}&x = \sqrt{1 - M^2}X, \qquad x_0 = \sqrt{1 - M^2}X_0, \qquad y = Y, \\ &y_0 = Y_0, \qquad \beta = \sqrt{1 - M^2}B, \qquad k = \sqrt{1 - M^2}K\end{aligned} \tag{8}$$

with

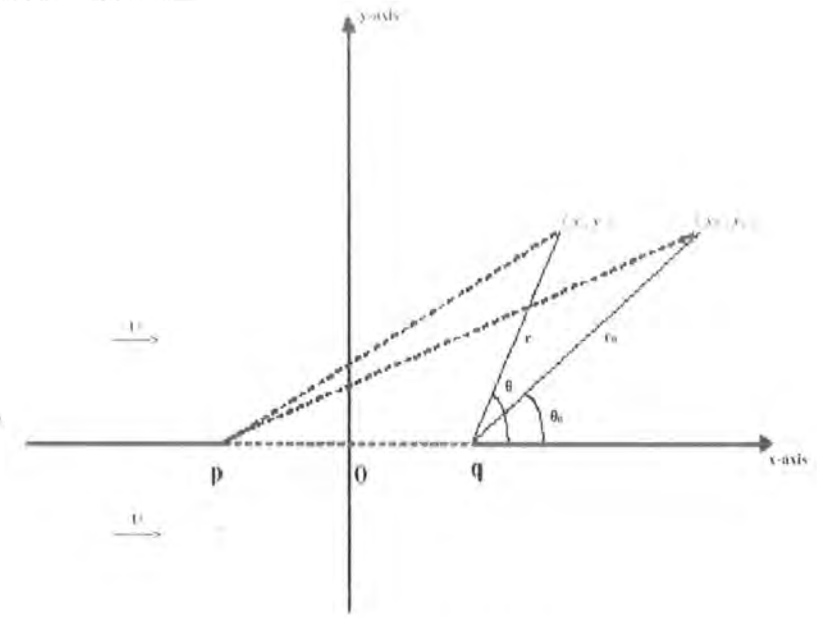
$$\Phi(x, y) = \psi_t(X, Y)e^{-iKMX} \tag{9}$$

Now using the relations (8) and (9), (3) to (5) can be written as:

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + K^2\right)\psi_t(X, Y) = \frac{\delta(X - X_0)\delta(Y - Y_0)}{\sqrt{1 - M^2}}e^{-iKMX_0} \tag{10}$$

$$\left[\frac{\partial}{\partial Y} \mp 2BM\frac{\partial}{\partial X} \pm iKB(1 + M^2) \mp \frac{iBM^2}{(1 - M^2)K}\frac{\partial^2}{\partial x^2}\right]\psi_t(X, 0^{\pm}) = 0 \quad \text{for} -\infty < X < p, \qquad q < X < \infty \tag{11}$$

Fig. 1. Geometry of the diffraction problem.



and

$$\frac{\partial\psi_t}{\partial Y}(X,0^+)=\frac{\partial\psi_t}{\partial Y}(X,0^-),\qquad p\le X\le q$$
$$\psi_t(X,0^+)=\psi_t(X,0^-),\qquad p\le X\le q \tag{12}$$

Let us split the total field $\psi_t\ (X,\ Y)$,

$$\psi_t(X,Y)=\begin{cases}\Psi_i(X,Y)+\Psi_r(X,Y)+\Psi(X,Y), & Y\ge 0\\ \Psi(X,Y), & Y\le 0\end{cases} \tag{13}$$

where $\Psi_i\ (X,\ Y)$ is the incident field (corresponding to the inhomogeneous equation), $\Psi\ (X,\ Y)$ is the diffracted field (corresponding to the homogenous equation), and $\Psi_r\ (X,\ Y)$ is the reflected field. Thus, $\Psi_i\ (X,\ Y)$ and $\Psi\ (X,\ Y)$ satisfy the following equations:

$$\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}+K^2\right)\Psi_i(X,Y)=\frac{\delta(X-X_0)\delta(Y-Y_0)}{\sqrt{1-M^2}}e^{-iKMX_0} \tag{14}$$

$$\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}+K^2\right)\Psi(X,Y)=0 \tag{15}$$

The solution of (14), obtained by Green's function method, is

$$\Psi_i(X,Y)=\frac{a}{4i}H_0^1(KR)=\frac{a}{4\pi i}\int_{-\infty}^{\infty}\frac{1}{\kappa}e^{-i(\alpha(X-X_0)+\kappa|Y-Y_0|)}\,\mathrm{d}\alpha \tag{16}$$

where

$$a=\frac{e^{iKMX_0}}{\sqrt{1-M^2}},\qquad R=\sqrt{(X-X_0)^2+(Y-Y_0)^2},$$
$$\kappa=\sqrt{K^2-\alpha^2}.$$

K is the wave number, and α is the Fourier transform variable. Now, by making the substitution $X_0 = R_0\cos\theta_0$, $Y_0 = R_0\sin\theta_0$, $0<\theta_0<\pi$, in (16), and letting $K R_0 \to \infty$, one can write

$$\Psi_i(X,Y)=be^{-iKX\cos\theta_0-iKY\sin\theta_0} \tag{17}$$

and

$$\Psi_r(X,Y)=be^{-iKX\cos\theta_0+iKY\sin\theta_0} \tag{18}$$

where the asymptotic form of the Hankel function is used to derive (16) and

$$b=\frac{a}{4i}\left(\frac{2}{\pi KR_0}\right)^{1/2}e^{i[KR_0-(\pi/4)]} \tag{19}$$

3. Wiener–Hopf equations

The spatial Fourier transform over the variable X is defined as,

$$\overline{\Psi}(\alpha,Y)=\int_{-\infty}^{\infty}\Psi(X,Y)e^{i\alpha X}\,\mathrm{d}X \tag{20}$$

To accommodate three-part boundary conditions on $Y = 0$, we split $\overline{\Psi}(\alpha, Y)$ as,

$$\overline{\Psi}(\alpha,Y)=\overline{\Psi}_+(\alpha,Y)e^{i\alpha q}+\overline{\Psi}_1(\alpha,Y)+\overline{\Psi}_-(\alpha,Y)e^{i\alpha p} \tag{21}$$

where

$$\overline{\Psi}_-(\alpha,Y)=\int_{-\infty}^{p}\Psi(X,Y)e^{i\alpha(X-p)}\,\mathrm{d}X \tag{22}$$

$$\overline{\Psi}_1(\alpha,Y)=\int_{p}^{q}\Psi(X,Y)e^{i\alpha X}\,\mathrm{d}X \tag{23}$$

$$\overline{\Psi}_-(\alpha, Y) = \int_q^{\infty} \Psi(X, Y) e^{i\alpha(X-q)}\, \mathrm{d}X \tag{24}$$

Here $\overline{\Psi}_-(\alpha, Y)$ is regular for Im $\alpha <$ Im K, and $\overline{\Psi}_+(\alpha, Y)$ is regular for Im $\alpha >$ –Im K, while $\overline{\Psi}_1(\alpha, Y)$ is an integral function and is analytic in the common region –Im $K < \alpha <$ Im K.

Now, taking the Fourier transform of (15), we obtain

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}Y^2} + \kappa^2\right)\overline{\Psi}(\alpha, Y) = 0 \tag{25}$$

and the α-plane is cut such that Im $\kappa > 0$, for Im $\alpha <$ Im K (for bounded solution) [17, Sect. 1.1, p. 2]. The solution for (21) satisfying radiation condition is given by,

$$\overline{\Psi}(\alpha, Y) = \begin{cases} A_1(\alpha) e^{i\kappa Y} & \text{if } Y \geq 0 \\ A_2(\alpha) e^{-i\kappa Y} & \text{if } Y < 0 \end{cases} \tag{26}$$

Taking the derivative with respect to "Y" i.e.,

$$\overline{\Psi}'(\alpha, Y) = \begin{cases} i\kappa A_1(\alpha) e^{i\kappa Y} & \text{if } Y \geq 0 \\ -i\kappa A_2(\alpha) e^{-i\kappa Y} & \text{if } Y < 0 \end{cases} \tag{27}$$

and using (26), we get

$$\overline{\Psi}'(\alpha, Y) = \begin{cases} i\kappa \overline{\Psi}(\alpha, Y) & \text{if } Y \geq 0 \\ -i\kappa \overline{\Psi}(\alpha, Y) & \text{if } Y < 0 \end{cases} \tag{28}$$

The Fourier transform of (11), when $X < p$, gives

$$\begin{aligned} \overline{\Psi}'_-(\alpha, 0^+) = {}& -iB\left[K(1+M^2) + 2\alpha M + \frac{\alpha^2 M^2}{K(1-M^2)}\right]\overline{\Psi}_-(\alpha, 0^+) \\ & - iB\left[K(1+M^2) + 2\alpha M + \frac{\alpha^2 M^2}{K(1-M^2)}\right] \\ & \times [\overline{\Psi}_{i-}(\alpha, 0) + \overline{\Psi}_{r-}(\alpha)] \end{aligned} \tag{29}$$

and

$$\overline{\Psi}'_-(\alpha, 0^-) = iB\left[K(1+M^2) + 2\alpha M + \frac{\alpha^2 M^2}{K(1-M^2)}\right] \times \overline{\Psi}_-(\alpha, 0^-) \tag{30}$$

and the Fourier transform of (9), when $X > q$, gives

$$\begin{aligned} \overline{\Psi}'_+(\alpha, 0^+) = {}& -iB\left[K(1+M^2) + 2\alpha M + \frac{\alpha^2 M^2}{K(1-M^2)}\right]\overline{\Psi}_+(\alpha, 0^+) \\ & - iB\left[K(1+M^2) + 2\alpha M + \frac{\alpha^2 M^2}{K(1-M^2)}\right] \\ & [\overline{\Psi}_{i+}(\alpha, 0) + \overline{\Psi}_{r+}(\alpha)] \end{aligned} \tag{31}$$

and

$$\overline{\Psi}'_+(\alpha, 0^-) = iB\left[K(1+M^2) + 2\alpha M + \frac{\alpha^2 M^2}{K(1-M^2)}\right] \times \overline{\Psi}_+(\alpha, 0^-) \tag{32}$$

where

$$\overline{\Psi}_{1-}(\alpha, 0) = b\int_{-\infty}^{p} \Psi_1(X, 0) e^{i\alpha(X-p)}\, \mathrm{d}X \tag{33}$$

$$\overline{\Psi}_{1+}(\alpha, 0) = b\int_{q}^{\infty} \Psi_1(X, 0) e^{i\alpha(X-q)}\, \mathrm{d}X \tag{34}$$

where

$$\overline{\Psi}'_1(\alpha, 0^-) = \overline{\Psi}'_1(\alpha, 0^-) = \overline{\Psi}'_1(\alpha) \tag{35}$$

$$\overline{\Psi}_1(\alpha, 0^-) - \overline{\Psi}_1(\alpha, 0^-) = -[\overline{\Psi}_{i1}(\alpha, 0) + \overline{\Psi}_{r1}(\alpha)] \tag{36}$$

Now, with the help of (21) and (28), we can write

$$\begin{aligned} & \overline{\Psi}'_+(\alpha, 0^+) e^{i\alpha q} + \overline{\Psi}'_1(\alpha, 0^+) + \overline{\Psi}'_-(\alpha, 0^+) e^{i\alpha p} \\ & \quad = i\kappa[\overline{\Psi}_+(\alpha, 0^+) e^{i\alpha q} + \overline{\Psi}_1(\alpha, 0^+) + \overline{\Psi}_-(\alpha, 0^+) e^{i\alpha p}] \end{aligned} \tag{37}$$

and

$$\begin{aligned} & \overline{\Psi}'_+(\alpha, 0^-) e^{i\alpha q} + \overline{\Psi}'_1(\alpha, 0^-) + \overline{\Psi}'_-(\alpha, 0^-) e^{i\alpha p} \\ & \quad = -i\kappa[\overline{\Psi}_+(\alpha, 0^-) e^{i\alpha q} + \overline{\Psi}_1(\alpha, 0^-) + \overline{\Psi}_-(\alpha, 0^-) e^{i\alpha p}] \end{aligned} \tag{38}$$

Making use of (29)–(32), (35), and (36) in (37) and (38) and then adding the resulting equations, we get

$$\begin{aligned} & -i\left\{B\left[K(1+M^2) + 2\alpha M + \frac{\alpha^2 M^2}{K(1-M^2)}\right] + \kappa\right\}\rho_+(\alpha) e^{i\alpha q} \\ & i\left\{B\left[K(1+M^2) + 2\alpha M + \frac{\alpha^2 M^2}{K(1-M^2)}\right] + \kappa\right\}\rho_-(\alpha) e^{i\alpha p} \\ & -\overline{\Psi}'_1(\alpha) = iB\left[K(1+M^2) + 2\alpha M + \frac{\alpha^2 M^2}{K(1-M^2)}\right] \\ & \quad [\overline{\Psi}_{i+}(\alpha) e^{i\alpha q} + \overline{\Psi}_{i-}(\alpha) e^{i\alpha p}] + \frac{i\kappa}{2}[\overline{\Psi}_1(\alpha, 0^+) - \overline{\Psi}_1(\alpha, 0^-)] \end{aligned} \tag{39}$$

where

$$\overline{\Psi}_\pm(\alpha, 0^+) - \overline{\Psi}_\pm(\alpha, 0^-) = 2\rho_\pm(\alpha) \tag{40}$$

and

$$\overline{\Psi}_{i\pm}(\alpha) = \frac{\pm ibe^{-iK\cos\theta_0 q}}{(\alpha - K\cos\theta_0)} \tag{41}$$

From (36), we have

$$\begin{aligned} \overline{\Psi}_1(\alpha, 0^+) - \overline{\Psi}_1(\alpha, 0^-) &= -[\overline{\Psi}_{i1}(\alpha) + \overline{\Psi}_{r1}(\alpha)] \\ &= -2\overline{\Psi}_{i1}(\alpha) = 2ibG(\alpha) \end{aligned} \tag{42}$$

where

$$G(\alpha) = \frac{[e^{i(\alpha - K\cos\theta_0)q} - e^{i(\alpha - K\cos\theta_0)p}]}{(\alpha - K\cos\theta_0)} \tag{43}$$

Now making use of (41) and (42) in (39), we get

$$\rho_+(\alpha) e^{i\alpha q} + \rho_-(\alpha) e^{i\alpha p} + \frac{i\overline{\Psi}'_1(\alpha)}{\kappa L(\alpha)} = -ibG(\alpha) \tag{44}$$

where

$$L_{+}(\alpha) = 1 + \frac{B}{\kappa_{\pm}(\alpha)}\left[K(1+M^{2}) + 2\alpha M + \frac{\alpha^{2}M^{2}}{(1-M^{2})K}\right] \quad (45)$$

Note that (44) is a standard Wiener–Hopf equation. The solution to this equation is found in the subsequent section.

4. Solution of the Wiener–Hopf equation

For the solution of (44), we make the following factorization,

$$\kappa L(\alpha) = S(\alpha) = S_{+}(\alpha)S_{-}(\alpha) \quad (46)$$

and

$$S_{\pm}(\alpha) = \kappa_{\pm}(\alpha)L_{\pm}(\alpha) \quad (47)$$

and $L_{+}(\alpha)$ and $\kappa_{+}(\alpha)$ are regular for Im $\alpha > -$Im K, i.e., upper half-plane and $L_{-}(\alpha)$ and $\kappa_{-}(\alpha)$ are regular for Im $\alpha >$ Im K, i.e., lower half-plane. $S_{+}(\alpha)$ is regular in the upper half-plane and $S_{-}(\alpha)$ is regular in the lower half-plane.

We write (44) as

$$\rho_{+}(\alpha)e^{i\alpha q} + \frac{i\overline{\Psi}_{1}'(\alpha)}{S_{+}(\alpha)S_{-}(\alpha)} + \rho_{-}(\alpha)e^{i\alpha p} = AG(\alpha) \quad (48)$$

where

$$G(\alpha) = \frac{\left[e^{i(\alpha-K_m)q} - e^{i(\alpha-K_m)p}\right]}{(\alpha-K_m)} \quad (49)$$

and for simplification,

$$A = -ib \qquad \text{and} \qquad K_m = K\cos\theta_0 \quad (50)$$

We multiply (48) by $S_{+}(\alpha)e^{-i\alpha q}$, include the value of $G(\alpha)$, add and subtract pole contributions to get,

$$S_{+}(\alpha)\rho_{+}(\alpha) + \frac{i\overline{\Psi}_{1}'(\alpha)}{S_{-}(\alpha)}e^{-i\alpha q} + S_{+}(\alpha)\rho_{-}(\alpha)e^{i\alpha(p-q)} = \frac{AS_{+}(\alpha)e^{-i\alpha q}}{(\alpha-K_m)}\left[\left(e^{i(\alpha-K_m)q} - e^{i(\alpha-K_m)p}\right)\right] \quad (51)$$

The first term on the left-hand side of the above equation is regular in the upper half-plane, and the terms, whose gender is not known, can be written as [17, Sect. 5.5, p. 198].

$$S_{+}(\alpha)\rho_{-}(\alpha)e^{i\alpha(p-q)} = U_{+}(\alpha) + U_{-}(\alpha) \quad (52)$$

and

$$\frac{AS_{+}(\alpha)e^{i\alpha(p-q)-iK_m p}}{(\alpha-K_m)} = V_{+}(\alpha) + V_{-}(\alpha) \quad (53)$$

Invoking (52) and (53) in (51), we obtain

$$S_{+}(\alpha)\rho_{+}(\alpha) - \frac{Ae^{-iK_m q}}{(\alpha-K_m)}\left[S_{+}(\alpha) - S_{+}(K_m)\right] + U_{+}(\alpha) + V_{+}(\alpha) = \frac{AS_{+}(K_m)e^{-iK_m q}}{(\alpha-K_m)} - \frac{i\overline{\Psi}_{1}'(\alpha)}{S_{-}(\alpha)}e^{-i\alpha q} - U_{-}(\alpha) - V_{-}(\alpha) \quad (54)$$

Now, multiplying (48) by $S_{-}(\alpha)e^{-ip}$ on both sides and putting in the value of $G(\alpha)$, we get

$$S_{-}(\alpha)\rho_{-}(\alpha) + \frac{AS_{-}(\alpha)e^{-iK_m p}}{(\alpha-K_m)} + R_{-}(\alpha) - P_{-}(\alpha) = -\frac{i\overline{\Psi}_{1}'(\alpha)e^{-i\alpha p}}{S_{+}(\alpha)} - R_{+}(\alpha) + P_{+}(\alpha) \quad (55)$$

where

$$S_{-}(\alpha)\rho_{+}(\alpha)e^{i\alpha(q-p)} = R_{+}(\alpha) + R_{-}(\alpha) \quad (56)$$

and

$$\frac{AS_{-}(\alpha)e^{i\alpha(q-p)-iK_m q}}{\alpha-K_m} = P_{+}(\alpha) + P_{-}(\alpha) \quad (57)$$

Let $J(\alpha)$ be a function equal to both sides of (43). We observe that left-hand side of (43) is regular in Im $\alpha > -$Im K, and the right-hand side is regular for Im$\alpha < K\cos\theta_0$, respectively. We use analytical continuation so that the definition of $J(\alpha)$ can be extended throughout the complex α-plane. We examine the asymptotic behaviour of (54) to ascertain the form of $J(\alpha)$ as $|\alpha| \to \infty$. We note that $|L_{\pm}(\alpha)| \sim O(1)$, [ref. 4, p. 11] as $|\alpha| \to \infty$, and with the help of the edge condition we find that $\rho_{+}(\alpha)$ and $\rho_{-}(\alpha)$ should be at least of $O(|\alpha|^{-1/2})$ as $|\alpha| \to \infty$. So, using the extended form of Liouville's theorem [ref. 17, p. 6 and 74], we see that $J(\alpha) \sim O(|\alpha|^{-1/2})$, and so a polynomial that represents $J(\alpha)$ can only be a constant equal to zero. Similarly, the same conclusion can be made for (55), therefore each side of (54) and (55) is equal to zero, i.e.,

$$S_{+}(\alpha)\rho_{+}(\alpha) - \frac{Ae^{-iK_m q}}{(\alpha-K_m)}\left[S_{+}(\alpha) - S_{+}(K_m)\right] + U_{+}(\alpha) + V_{+}(\alpha) = 0 \quad (58)$$

and

$$S_{-}(\alpha)\rho_{-}(\alpha) + \frac{AS_{-}(\alpha)e^{-iK_m p}}{(\alpha-K_m)} + R_{-}(\alpha) - P_{-}(\alpha) = 0 \quad (59)$$

The decomposition of (52), (53), (56), and (57) can be performed by inspection, and it is necessary to use the general theorem B of [17, Sect. 1.3], hence we have,

$$U_{+}(\alpha) = \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{S_{+}(\eta)\rho_{-}(\eta)e^{i\eta(p-q)}}{(\eta-\alpha)}\mathrm{d}\eta \quad (60)$$

$$V_{+}(\alpha) = \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{AS_{+}(\eta)e^{i\eta(p-q)-iK_m p}}{(\eta-K_m)(\eta-\alpha)}\mathrm{d}\eta \quad (61)$$

$$R_{-}(\alpha) = -\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id}\frac{S_{-}(\eta)\rho_{+}(\eta)e^{i\eta(q-p)}}{(\eta-\alpha)}\mathrm{d}\eta \quad (62)$$

$$P_{-}(\alpha) = -\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id}\frac{AS_{-}(\eta)e^{i\eta(q-p)-iK_m q}}{(\eta-K_m)(\eta-\alpha)}\mathrm{d}\eta \quad (63)$$

In these equations, $-K_2 < c < \tau < d < K_2\cos\theta_0$, where $\tau =$ Im α and $K_2 =$ Im K.

Now making use of (60) and (61) in (58) and using (62) and (63) in (59), we get

$$S_+(\alpha)\rho_+^*(\eta) + \frac{AS_+(\alpha)e^{-iK_m q}}{(\alpha - K_m)} + \frac{1}{2\pi i}\int_{\infty - ic}^{\infty - ic} \frac{S_-(\eta)\rho_-^*(\eta)e^{i\eta(p-q)}}{(\eta - \alpha)}\mathrm{d}\eta = 0 \quad (64)$$

and

$$S_-(\alpha)\rho_-^*(\eta) - \frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id} \frac{S_-(\eta)\rho_+^*(\eta)e^{i\eta(q-p)}}{(\eta - \alpha)}\mathrm{d}\eta = 0 \quad (65)$$

where $\tau > -a$ in both equations. Define

$$\rho_+(\alpha) - \frac{Ae^{-iK_m q}}{\alpha - K_m} = \rho_+^*(\eta) \quad (66)$$

and

$$\rho_-(\alpha) + \frac{Ae^{-iK_m p}}{\alpha - K_m} = \rho_-^*(\eta) \quad (67)$$

where * denotes that the expressions are regular in $\mathrm{Im}\alpha > -K_2\cos\theta$ except for simple poles at $\alpha = K_m$. The assumption, $0 < \theta_0 < \pi$, allows us to choose a so that $-K_2\cos\theta_0 < a < K_2\cos\theta_0$. Also, take $c = d = a$, replace η by $-\eta$, α by $-\alpha$ in (64) and (65), respectively, and use $S_+(-\alpha) = S_-(\alpha)$ to get

$$S_+(\alpha)\left[F_+(\alpha) - \frac{Ae^{-iK_m q}}{\alpha - K_m} + \lambda\frac{Ae^{-iK_m p}}{\alpha + K_m}\right] + \frac{\lambda}{2\pi i}\int_{-\infty+ia}^{\infty+ia} \frac{S_-(\eta)e^{i\eta(q-p)}}{(\eta + \alpha)}\left[F_+(\eta) - \frac{Ae^{-iK_m q}}{(\eta - K_m)} - \lambda\frac{Ae^{-iK_m p}}{(\eta + K_m)}\right]\mathrm{d}\eta + \frac{AS_+(K_m)e^{-iK_m q}}{(\eta - K_m)} = 0 \quad (68)$$

where

$$\rho_+^*(\alpha) \pm \rho_-^*(\alpha) = F_+^*(\alpha) \quad (69)$$

where $F_+^*(\alpha)$ will be set equal to ρ_+^* and ρ_-^* in turn, so that it can be calculated by using (66), (67), (69), and combining the results,

$$F_-^*(\alpha) = F_+(\alpha) - \frac{Ae^{-iK_m q}}{\alpha - K_m} - \lambda\frac{Ae^{-iK_m p}}{\alpha + K_m} \quad (70)$$

where

$$F_+(\alpha) = \rho_+(\alpha) - \lambda\rho_-(-\alpha) \quad (71)$$

and λ is a constant, which we shall later take as −1 or +1. $F_+(\alpha)$ is regular in $\tau > -K_2$. We should expect that $F_+(\alpha)$ will have a branch point at $\alpha = -k$. Equation (68), after calculating the integral (details are given in Appendix A), can be written as,

$$S_+(\alpha)F_+(\alpha) = A[G_1(\alpha) - \lambda G_2(\alpha)] - \lambda T(\alpha)F_+(K)L_-(K) \quad (72)$$

or

$$F_+(\alpha) = \frac{A}{S_+(\alpha)}[G_1(\alpha) - \lambda G_2(\alpha)] - \frac{A\lambda T(\alpha)}{S_+(\alpha)}\left[\frac{G_1(K) - \lambda G_2(K)}{S_+(K) + \lambda T(K)L_-(K)}\right] \quad (73)$$

where

$$G_1(\alpha) = P_1(\alpha)e^{-iK_m q} - R_1(\alpha)e^{-iK_m p} \quad (74)$$

$$G_2(\alpha) = P_2(\alpha)e^{-iK_m q} - R_2(\alpha)e^{-iK_m p} \quad (75)$$

and

$$P_{1,2}(\alpha) = \frac{S_+(\alpha) - S_+(K_m)}{\alpha \mp K_m} \quad (76)$$

and where $F_+(K)$ can be calculated from (72) by putting $\alpha = K$.

Now, using $\lambda = \mp 1$, in (70) and (73) and adding the resultant equations, we obtain

$$\rho_+(\alpha) = A\frac{G_1(\alpha)}{S_+(\alpha)} + \frac{AT(\alpha)}{S_+(\alpha)S_+(K)\left(1 - \frac{T^2(K)L_-^2(K)}{S_+^2(K)}\right)} \times \left[G_2(K) + \frac{G_1(K)T(K)L_-(K)}{S_+(K)}\right] \quad (77)$$

and by subtracting the same resultant equations with $\alpha = -\alpha$, we get $\rho_-(\alpha)$, or replacing G_1 by G_2, and G_2 by G_1, changing the sign of α in (72) and using $S_+(-\alpha) = S_-(\alpha)$, we get

$$\rho_-(\alpha) = \frac{AG_2(-\alpha)}{S_-(\alpha)} + \frac{AT(-\alpha)}{S_-(\alpha)S_+(K)\left[\left(1 - \frac{T^2(K)L_-^2(K)}{S_+^2(K)}\right)\right]} \times \left[G_1(K) + \frac{G_2(K)T(K)L_-(K)}{S_+(K)}\right] \quad (78)$$

In specific problems, it will be possible to simplify these formulae to some extent by neglecting some terms completely and using the asymptotic form of Whittikar functions in other terms. Using (40) in (77) and (78), respectively, we get

$$\overline{\Psi}_+(\alpha, 0^+) - \overline{\Psi}_+(\alpha, 0^-) = \frac{2A}{S_+(\alpha)}[G_1(\alpha) + C_1(K)T(\alpha)] \quad (79)$$

and

$$\overline{\Psi}_-(\alpha, 0^+) - \overline{\Psi}_-(\alpha, 0^-) = \frac{2A}{S_-(\alpha)}[G_2(-\alpha) + C_2(K)T(-\alpha)] \quad (80)$$

where

$$C_1(K) = \frac{1}{S_+(K)\left(1 - \frac{T^2(K)L_-^2(K)}{S_+^2(K)}\right)} \times \left[G_2(K) + \frac{G_1(K)T(K)L_-(K)}{S_+(K)}\right] \quad (81)$$

$$C_2(K) = \frac{1}{S_+(K)\left[\left(1 - \frac{T^2(K)L_-^2(K)}{S_+^2(K)}\right)\right]} \times \left[G_1(K) + \frac{G_2(K)T(K)L_-(K)}{S_+(K)}\right] \quad (82)$$

Using (21), (26), (28), (42), (79), (80), we get

$$A_1(\alpha) = -A_2(\alpha) = \frac{A}{S_+(\alpha)}[G_1(\alpha) + C_1(K)T(\alpha)]e^{i\alpha q} - AG(\alpha) + \frac{A}{S_-(\alpha)}[G_2(-\alpha) + C_2(K)T(-\alpha)]e^{i\alpha p} \quad (83)$$

where $A_1(\alpha)$ corresponds to $Y \geq 0$ and $A_2(\alpha)$ corresponds to $Y < 0$.

We substitute these values of $A_1(\alpha)$ and $A_2(\alpha)$ into (26) and then obtain $\Psi(X, Y)$ by taking the inverse Fourier transform as.

$$\Psi(X,Y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{A}{S_+(\alpha)}[G_1(\alpha) + C_1(K)T(\alpha)]e^{i\alpha q} - AG(\alpha) + \frac{A}{S_-(\alpha)}[G_2(-\alpha) + C_2(K)T(-\alpha)]e^{i\alpha p}\right\}e^{-i\alpha X + i\kappa|Y|}\,\mathrm{d}\alpha \quad (84)$$

Invoking (74), (75), and (76) in (84), we get

$$\Psi(X,Y) = \frac{A}{2\pi}\int_{-\infty}^{\infty}\left\{\left[\frac{e^{i(\alpha-K_m)q}S_+(K_m)}{S_+(\alpha)(\alpha-K_m)} + \frac{e^{-iK_m p}R_1(\alpha)e^{i\alpha q}}{S_+(\alpha)} - \frac{C_1(K)T(\alpha)e^{i\alpha q}}{S_+(\alpha)} - \frac{e^{i(\alpha-K_m)p}S_-(K_m)e^{-i\alpha X - i\kappa|Y|}}{S_-(\alpha)(\alpha-K_m)} + \frac{e^{-iK_m q}R_2(-\alpha)e^{i\alpha p}}{S_-(\alpha)} - \frac{C_2(K)T(-\alpha)e^{i\alpha p}}{S_-(\alpha)}\right]\right\}e^{-i\alpha X + i\kappa|Y|}\,\mathrm{d}\alpha \quad (85)$$

We can break up the field $\Psi(X, Y)$ into two parts

$$\Psi(X,Y) = \Psi^{\mathrm{sep}}(X,Y) + \Psi^{\mathrm{int}}(X,Y) \quad (86)$$

where

$$\Psi^{\mathrm{sep}}(X,Y) = \frac{ib}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i(\alpha-K_m)q}S_-(K_m)e^{-i\alpha X + i\kappa|Y|}}{S_-(\alpha)(\alpha-K_m)}\mathrm{d}\alpha - \frac{ib}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i(\alpha-K_m)p}S_-(K_m)e^{-i\alpha X + i\kappa|Y|}}{S_-(\alpha)(\alpha-K_m)}\mathrm{d}\alpha \quad (87)$$

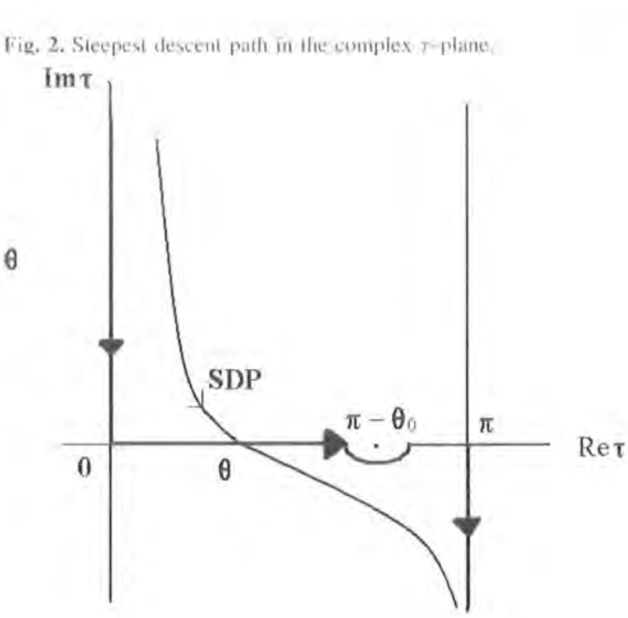
and

$$\Psi^{\mathrm{int}}(X,Y) = \frac{ib}{2\pi}\int_{-\infty}^{\infty}\left[\frac{e^{-iK_m p}R_1(\alpha)e^{i\alpha q}}{S_+(\alpha)} - \frac{C_1(K)T(\alpha)e^{i\alpha q}}{S_+(\alpha)} + \frac{e^{-iK_m q}R_2(-\alpha)e^{i\alpha p}}{S_-(\alpha)} - \frac{C_2(K)T(-\alpha)e^{i\alpha p}}{S_-(\alpha)}\right]e^{-i\alpha X + i\kappa|Y|}\,\mathrm{d}\alpha \quad (88)$$

5. Far-field approximation

The far field may now be calculated by evaluating the integrals in (87) and (88) asymptotically [24, 25]. For that, put $X = r\cos\theta$, $Y = r\sin\theta$, and deform the contour by the transformation $\alpha = -K\cos(\theta + i\tau)$, which changes the contour of integration over α into a hyperbole through the point $\alpha = -K\cos\theta$, where ($0 < \theta < \pi$, $-\infty < \tau < \infty$). The steepest descent path is shown in Fig. 2.

Fig. 2. Steepest descent path in the complex τ-plane.



$$\Psi^{\mathrm{sep}}(X,Y) = \frac{bK}{2\pi}\int_{-\infty}^{\infty}f_1[K\cos(\theta + i\tau)][\exp(-iKr\cosh\tau)] \times \sin(\theta + i\tau)\,\mathrm{d}\tau \quad (89)$$

where

$$f_1[K\cos(\theta + i\tau)] = \frac{K}{[K\cos(\theta + i\tau) - K_m]} \times \left[\frac{e^{i\{K\cos(\theta+i\tau)-K_m\}d}S_+(K_m)}{S_+[K\cos(\theta + i\tau)]} - \frac{e^{i\{K\cos(\theta+i\tau)-K_m\}d}S_-(K_m)}{S_-[K\cos(\theta + i\tau)]}\right] \quad (90)$$

Similarly, we have

$$\Psi^{\mathrm{int}}(X,Y) = \frac{bK}{2\pi}\int_{-\infty}^{\infty}f_2[K\cos(\theta + i\tau)]\exp(-iKr\cosh\tau) \times \sin(\theta + i\tau)\,\mathrm{d}\tau \quad (91)$$

where

$$f_2[K\cos(\theta + i\tau)] = \left[\frac{R_1[K\cos(\theta + i\tau)]e^{i\{K\cos(\theta+i\tau)-K_m\}d}}{S_+[K\cos(\theta + i\tau)]} - \frac{C_1(K)T[K\cos(\theta + i\tau)]e^{iK\cos(\theta+i\tau)d}}{S_+[K\cos(\theta + i\tau)]}\right] + \frac{R_2[-K\cos(\theta + i\tau)]e^{i\{K\cos(\theta+i\tau)-K_m\}d}}{S_-[K\cos(\theta + i\tau)]} - \frac{C_2(K)T[-K\cos(\theta + i\tau)]e^{iK\cos(\theta+i\tau)d}}{S_-[K\cos(\theta + i\tau)]}] \quad (92)$$

A modification of the method of stationary phase [34, p. 19] is required because the pole may come close to the stationary point. Hence, for large Kr, by using modified method of stationary phase, (89) and (91) become,

$$\Psi^{\text{sep}}(X, Y) = \frac{-ibK\sin\theta}{\sqrt{2\pi Kr}} f_1(K\cos\theta)\exp i\left(Kr - \frac{\pi}{4}\right) \tag{93}$$

and

$$\Psi^{\text{int}}(X, Y) = \frac{-ibK\sin\theta}{\sqrt{2\pi Kr}} f_2(K\cos\theta)\exp i\left(Kr - \frac{\pi}{4}\right) \tag{94}$$

where $f_1(K\cos\theta)$ and $f_2(K\cos\theta)$ are given by setting $t = 0$ in (90) and (92). Now, using (93) and (94) in (9) implies that.

$$\Phi^{\text{sep}}(x, y) = \frac{-ibK\sin\theta e^{-iKMX}}{\sqrt{2\pi Kr}} f_1(K\cos\theta)\exp i\left(Kr - \frac{\pi}{4}\right) \tag{95}$$

and

$$\Phi^{\text{int}}(x, y) = \frac{-ibK\sin\theta e^{-iKMX}}{\sqrt{2\pi Kr}} f_2(K\cos\theta)\exp i\left(Kr - \frac{\pi}{4}\right) \tag{96}$$

Here, $\Psi^{\text{sep}}(X, Y)$ consists of two parts, each representing the diffracted field produced by the edges at $x = p$ and $x = q$, respectively, as though the other edge was absent, while $\Psi^{\text{int}}(X, Y)$ gives the interaction of one edge upon the other (double diffraction of the two edges). It is noted that the extra term, which is also called the perturbation term

$$\left[\frac{BM\cos^2\theta}{\sin\theta(1 - M)^2}\right] \tag{97}$$

(in the kernel of the factors $f_1(K\cos\theta)$ and $f_2(K\cos\theta)$), appeared in the expressions (95) and (96) with some other minor changes, differ from(54) and (55) of [ref. 32, p. 531]. The significance of this extra term (corrected term), which appears because of Myers' condition, is also shown graphically in the next section.

Thus, using (9), (13), (16), (95), and (96), the total far field is given by,

$$\Phi(x, y) \sim \frac{a}{4i} H_0^1(KR)e^{-iKMX} - \frac{ibK\sin\theta e^{-iKMX}}{\sqrt{2\pi Kr}}\left[\exp i\left(Kr - \frac{\pi}{4}\right)\right] f_{1,2}(K\cos\theta) \tag{98}$$

6. Graphical results

A computer program, MATHEMATICA, has been used for the numerical evaluation and graphical plotting of the separated field given by (95) for $Kr = 10\pi$, angle of incidence $\pi/2$, and the absorbing parameter B is to be taken such that $\text{Re}B > 0$ for an absorbing surface and the Mach number $|M| < 1$ for a subsonic flow. A positive Mach number indicates that the stream flow is from left to right and negative Mach number indicates that the stream flow is from right to left. The following situations are considered:

1. When the source is fixed in one position (for all Mach numbers), relative to the slit, ($\theta_0 = 90°$, $60°$, M, and θ are allowed to vary).
2. When the source is fixed in one position, ($\theta_0 = 90°$, $60°$, B, and θ are allowed to vary).

One can see from Figs. 3–6 that the field in the region $0 < \theta < \pi$ is most affected by the changes in M, B, and K. The main features of the graphical results are as follows:

Fig. 3. Variation of the amplitude of the separated field versus observation angle θ, for different values of M, at $\theta_0 = \pi/2$, $K = 1$, $l = 10$.

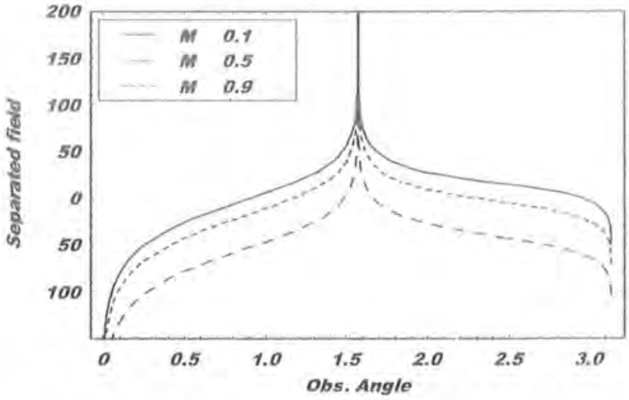


Fig. 4. Variation of the amplitude of the separated field versus observation angle θ, for different values of M, at $\theta_0 = \pi/3$, $K = 2$, $l = 10$.

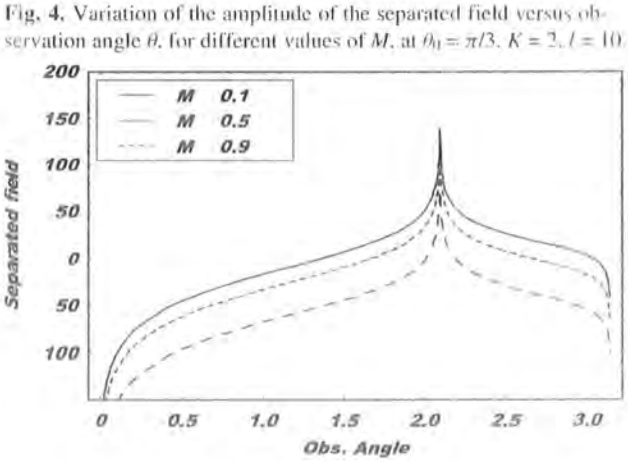
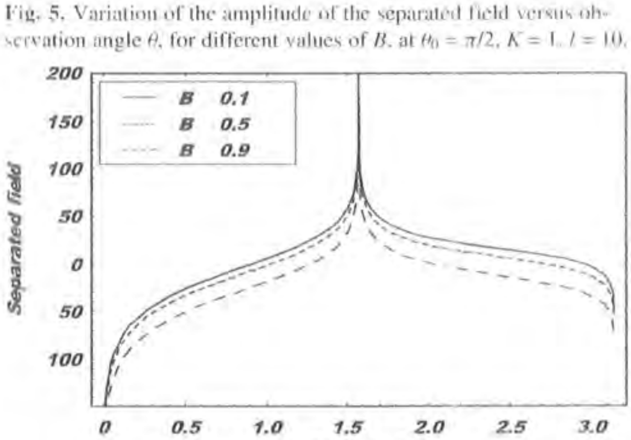


Fig. 5. Variation of the amplitude of the separated field versus observation angle θ, for different values of B, at $\theta_0 = \pi/2$, $K = 1$, $l = 10$.



(a) Since we are considering subsonic flow, i.e., $U < c$, by increasing the Mach number M, and by fixing all other parameters, respectively, the velocity of the fluid comes close to the velocity of sound. The modulus of the velocity potential function is proportional to the amplitude of the perturbation sound pressure and therefore gives a

Fig. 6. Variation of the amplitude of the separated field versus observation angle θ, for different values of B, at $\theta_0 = \pi/3$, $K = 2$, $l = 10$.

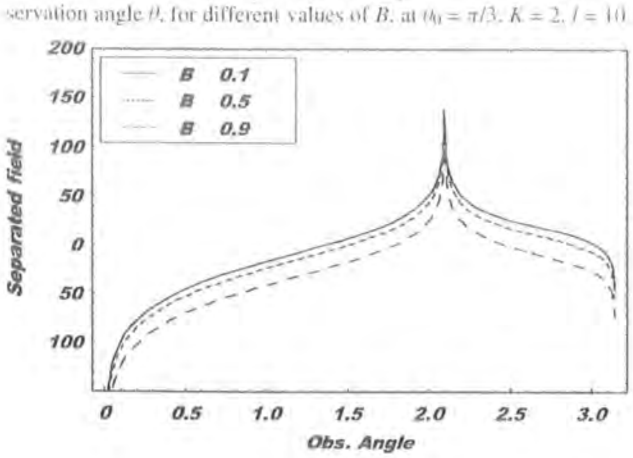


Fig. 7. Variation of the amplitude of the separated field versus observation angle θ, for different values of M, at $\theta_0 = \pi/2$, $K = 1$, $l = 10$.

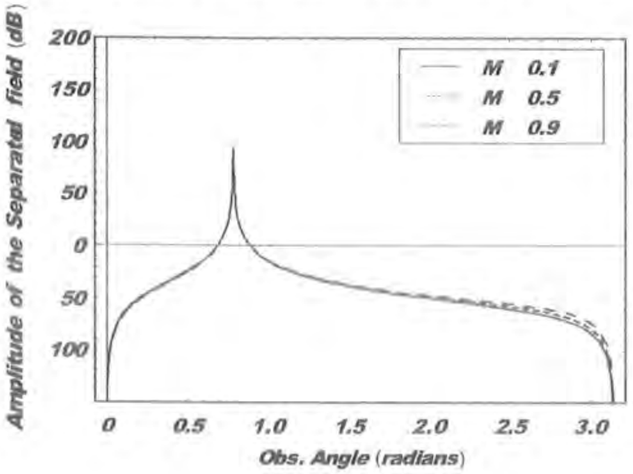


Fig. 8. Variation of the amplitude of the separated field versus observation angle θ, for different values of B, at $\theta_0 = \pi/3$, $K = 2$, $l = 10$.

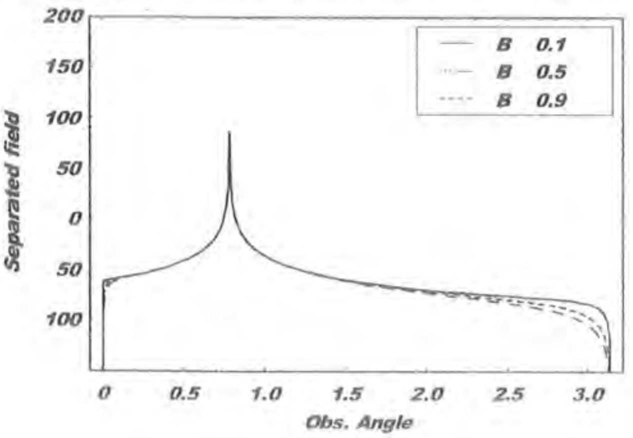


Fig. 9. Variation of the amplitude of the separated field versus observation angle θ, for different values of M, at $\theta_0 = \pi/2$, $K = 1$, $l = 0.5$.

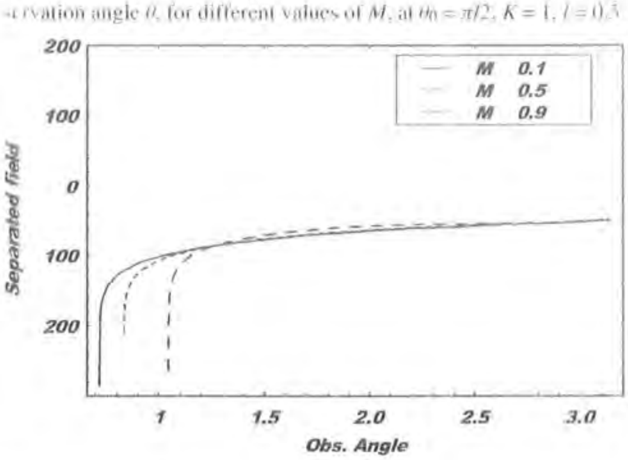
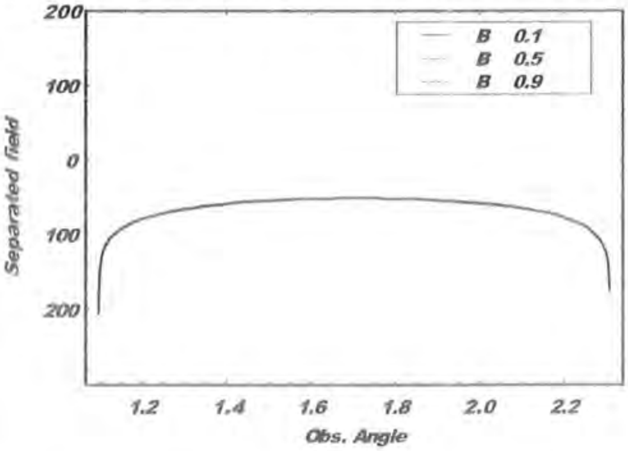


Fig. 10. Variation of the amplitude of the separated field versus observation angle θ, for different values of B, at $\theta_0 = \pi/3$, $K = 1$.



measure of sound intensity. We observe from Figs. 3 and 4 that the amplitude of the velocity potential decreases i.e., the sound intensity reduces. Thus, a clear variation of the velocity potential with respect to Mach number can be observed from the graphs.

(*b*) In Figs. 5 and 6, the effects of absorbing parameter B on the velocity potential is shown. It can be seen from the graphs that by increasing the absorbing parameter B, the amplitude of the velocity potential is decreased, and consequently the sound intensity is also reduced, by fixing all other parameters and taking $K = 1, 2$, respectively.

The graphical results shown below are plotted for the following two cases:

(1) Figures 7 and 8 are plotted with the perturbation term $(i\beta M^2/k)(\partial^2\varphi/\partial x^2)$ ignored in the boundary conditions, and consequently in the subsequent analysis the diffracted field expression is given by expressions (91) and (93).

(2) Figures 9 and 10 are plotted for the field given in [32] for Ingard conditions.

It is observed that the presence of an impedance condition has a remarkable effect on the diffracted field, which can be seen by comparing Figs. 3–6 with Figs. 7–10. A little explanation of Figs. 7–10 follows. Figures 7 and 8 are plotted with the perturbation term ignored. It is observed that in this case, the variation in the amplitude of the field is slower, compared with the present analysis for the different values of convection parameters. Figures 9 and 10 clearly depict that the field due to Ingard's condition remains almost unaffected. There is no deviation in the amplitude of the field. Therefore, we can say that Ingard's condition does not have a significant effect on the diffracted field, compared with Figs. 3–6, plotted for the Myers' impedance condition. When the perturbation term is ignored, we are left with Figs. 7 and 8. Hence, in a heroical manner, we can say that Figs. 3–6 are plotted for impedance boundary conditions, and when the perturbation term in the impedance boundary condition is neglected, we are left with Figs. 7 and 8, and when Ingard's conditions are used, we are left with Figs. 9 and 10. It can be seen througout the analysis that the field in the shadow region can be greater or smaller depending on the values of different parameters θ, M, and B in the range $0 < \theta < \pi$, when the absorbing lining is on the shadow face. It is interesting to see that the sound level is greater when the graphs are plotted for Myers' impedance condition and very much smaller for the other two cases i.e., when the perturbation term is taken into account Figs. 7 and 8 and when graphs are plotted for Ingard's conditions in Figs. 9 and 10. Thus, the introduction of Myers' condition in the slit problem presents a correct form of the field obtained in [32].

7. Conclusion

In this paper, we studied the problem of line-source diffraction of acoustic waves by a slit, satisfying Myers' condition, which gave rise to a corrective term,

$$\left[\frac{BM\cos^2\theta}{\sin\theta(1-M^2)}\right]$$

in the solution (third term of the kernel). Furthermore, an analytical and graphical comparison of different cases was presented for various values of Mach number M and the absorbing parameter B against the velocity potential. The graphs showed a clear variation of the velocity potential against these convection parameters. The importance of Myers' impedence condition over Ingard's condition is discussed in detail. The deviation in field showed a much clearer picture in the case of Myers' condition than with Ingard's condition. It is also observed in the graphs that by increasing the value of Mach number M and the absorbing prameter B, the amplitude of the field is decreased. As the present field contains the terms of order M^2 due to the introduction of Myers' condition, so obviously the amplitude of the field decreases more quicky, which supports well the importance of Myers' condition in the graphs. Therefore, Myers' condition is a better choice than Ingard's condition. Moreover, the results for a rigid barrier in still fluid [18] can be recovered by putting $M = B = 0$ in the expression (97).

The problem in a more practical application is one of an absorbing strip in a moving fluid with trailing edge. This can be a model for an aeroplane wing and has the advantage of being cheaper to construct than a strip with faces entirely coated in absorbent materials. Finally this work, in contrast with the results of Ingard's condition, will offer useful theoretical comparisons with experimental results. This should then lead to a decision ofwhich is the more appropriate boundary condition to use in practice.

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Appendix A

Consider the integral appearing in (68) by letting

$$I = \int_{-\infty+ia}^{\infty+ia} \frac{S_-(\eta)e^{i\eta(q-p)}}{(\eta+\alpha)}\left[F_+(\eta) - \frac{Ae^{-iK_mq}}{(\eta-K_m)} - \lambda\frac{Ae^{-iK_mp}}{(\eta+K_m)}\right]\mathrm{d}\eta$$

or

$$I = I_1 - Ae^{-iK_mq}I_2 - \lambda Ae^{-iK_mp}I_3 \qquad \text{(A1)}$$

where

$$I_1 = \int_{-\infty+ia}^{\infty+ia} \frac{S_-(\eta)F_+(\eta)e^{i\eta l}}{(\eta+\alpha)}\mathrm{d}\eta \qquad \text{(A2)}$$

$$I_2 = \int_{-\infty+ia}^{\infty+ia} \frac{S_-(\eta)e^{i\eta l}}{(\eta+\alpha)(\eta-K_m)}\mathrm{d}\eta \qquad \text{(A3)}$$

$$I_3 = \int_{-\infty+ia}^{\infty+ia} \frac{S_-(\eta)e^{i\eta l}}{(\eta+\alpha)(\eta+K_m)}\mathrm{d}\eta \qquad \text{(A4)}$$

and

$$l = (q-p)$$

We observe that $F_+(\alpha)$ is regular in $\tau > -K_2$, so we can expect that $F_+(\alpha)$ will have a branch point at $\alpha = -K$. But for large l, it is sufficiently far from the point $\alpha = K$, which enables us to evaluate the above integrals in the asymptotic expansion [A1, Sect. 5.5, p. 201].

Now, using the procedure as in [A1, Sec. 5.5, P-199–201] we get

$$I_1 = \int_{-\infty+ia}^{\infty+ia} \frac{S_-(\eta)F_+(\eta)e^{i\eta l}}{(\eta+\alpha)}\mathrm{d}\eta \approx E_0W_0[-i(K+\alpha)l]F_+(K)L_-(K) \qquad \text{(A5)}$$

or

$$I_1 \approx 2\pi i T(\alpha)F_+(K)L_-(K) \qquad \text{(A6)}$$

where

$$T(\alpha) = \frac{E_0W_0[-i(K+\alpha)l]}{2\pi i}$$

$$E_0 = 2e^{i\pi/4}e^{iKl}l^{-1/2}h_0$$

$$h_0 = e^{i\pi/4}$$

and

$$W_0(z) = \int_0^{\infty} \frac{u^{-1/2}e^{-u}}{u+z}\mathrm{d}u = \frac{\sqrt{\pi}}{2}e^{z/2}z^{-1/4}W_{-1/4,1/4}(z)$$

where

$$z = -i(K+\alpha)l$$

and $W_{i,j}(z)$ is the Whittaker function.

Now, the other integrals can be calculated by splitting the integrands into partial fractions. Hence we have,

$$I_2 = \int_{-\infty+ia}^{\infty+ia} \frac{S_-(\eta)e^{i\eta l}}{(\eta+\alpha)(\eta-K_m)}\mathrm{d}\eta \approx 2\pi i\left[\frac{S_-(K_m)e^{iK_ml}}{(\alpha+K_m)} + R_2(\alpha)\right] \qquad \text{(A7)}$$

and

$$I_3 = \int_{-\infty+ia}^{\infty+ia} \frac{S_-(\eta)e^{i\eta l}}{(\eta+\alpha)(\eta+K_m)}\mathrm{d}\eta \approx 2\pi i R_1(\alpha) \qquad \text{(A8)}$$

where we have

$$R_{1,2}(\alpha) = \frac{E_0\{W_0[-i(K\pm K_m)l] - W_0[-i(K+\alpha)l]\}}{2\pi i(\alpha \mp K_m)} \qquad \text{(A9)}$$

and the upper and lower signs refer to R_1, R_2, respectively. Thus, (A1) takes the form.

$$I = 2\pi i\left[T(\alpha)F_+(K)L_-(K) - AR_2(\alpha)e^{-iK_mq} - \frac{AS_-(K_m)e^{-iK_mp}}{(\alpha+K_m)} - \lambda Ae^{-iK_mp}R_1(\alpha)\right]$$

or

$$\int_{-\infty+ia}^{\infty+ia} \frac{S_-(\eta)e^{i\eta(q-p)}}{(\eta+\alpha)}\left[F_+(\eta) - \frac{Ae^{-iK_mq}}{(\eta-K_m)} - \lambda\frac{Ae^{-iK_mp}}{(\eta+K_m)}\right]\mathrm{d}\eta = 2\pi i\left[T(\alpha)F_+(K)L_-(K) - AR_2(\alpha)e^{-iK_mq} - \frac{AS_-(K_m)e^{-iK_mp}}{(\alpha+K_m)} - \lambda Ae^{-iK_mp}R_1(\alpha)\right] \qquad \text{(A10)}$$

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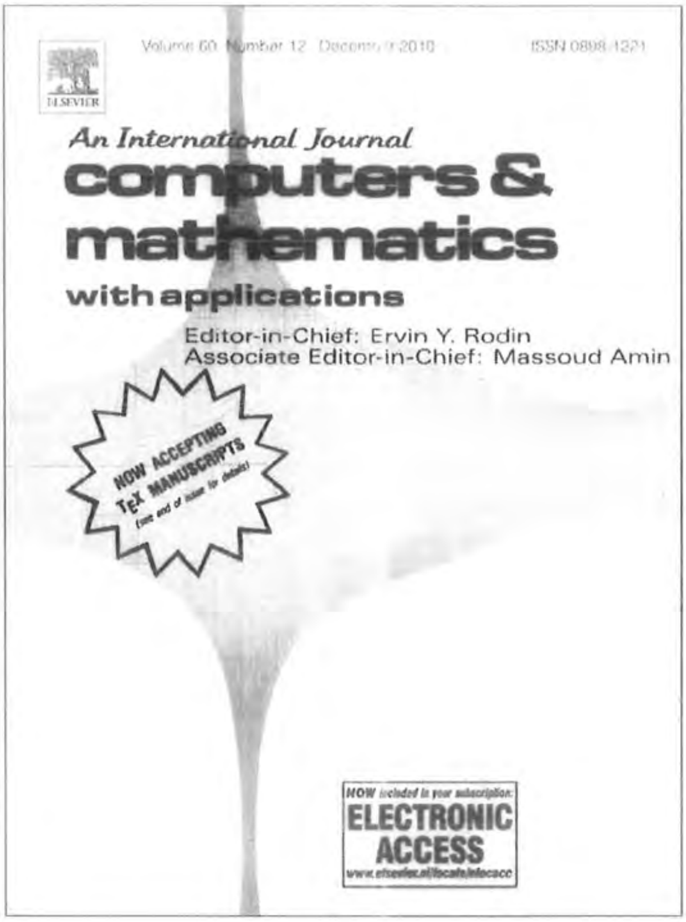
A1. J.B. Keller. J. Opt. Soc. Am. **52**, 116 (1962). doi:10.1364/JOSA.52.000116. PMID:14454948.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Sound due to an impulsive line source

M. Ayub [a,*], A. Naeem [a], Rab Nawaz [a,b]

[a] Department of Mathematics, Quaid-i-Azam University, 45320, Islamabad 44000, Pakistan

[b] Department of Humanities and Basic Sciences, HITEC University Taxila Cantt., Pakistan

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ABSTRACT

This paper deals with the problem of diffraction due to an impulse line source by an absorbing half plane, satisfying Myers' impedance condition (Myers, 1980 [13]) in the presence of a subsonic flow. The time dependence of the field requires a temporal Fourier transform in addition to the spatial Fourier transform. The spatial integral appearing in the solution for the diffracted field is solved asymptotically (Copson, 1967 [15]) in the far field approximation. Finally, a simple procedure is devised to calculate the inverse temporal Fourier transform.

1. Introduction

Transient nature of the field is an important area in the theory of acoustic diffraction and provides a more complete picture of the wave phenomenon under consideration. Recently, Barton and Rawlins [1] discussed diffraction by a half plane for both leading edge situations and trailing edge situations. Many scientists have taken into account the effect of the transient nature of the field, for example, Rienstra [2], Lakhtakia et al. [3–5], Ayub et al. [6], Asghar et al. [7,8] and Ahmad [9], to name a few. The problem of acoustic diffraction by an absorbing half plane in a moving fluid using Myers' condition was discussed by Ahmad [10]. He considered the diffraction of sound waves by a semi-infinite absorbing half plane, when the whole system was in a moving fluid. Fourier transform, Wiener–Hopf technique and asymptotic approximations were used to calculate the diffracted field. In [10], the time dependence was suppressed throughout the analysis, while, in this problem, we have taken into account the time dependence throughout. We apply the temporal Fourier transform to obtain the transform function in the transformed plane using the Wiener–Hopf technique [11] and the method of modified stationary phase [12]. When the transform function is available, an inverse temporal Fourier transform can be applied to obtain the results in the time domain. We have also shown how the frequency of an incident wave is affected by the amplitude of the diffracted field in different limiting positions. Also, the effects of different parameters on the field can be seen from the numerical results.

2. Formulation of the problem

Consider a small amplitude sound wave on a main stream moving with a velocity U parallel to the *x-axis*. A semi-infinite absorbing half plane is assumed to occupy the position $y = 0$, $x \geq 0$. The equations of motion are linearized and the effects of viscosity, thermal conductivity and gravity are neglected. The fluid is assumed to have a constant density (incompressible) and sound speed c. We assume that the plane satisfies Myers' impedance condition [13],

$$u_n = \left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right]\frac{g}{|\nabla_\alpha|}, \tag{a}$$

* Corresponding author.
E-mail address: mayub59@yahoo.com (M. Ayub).

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where

$$\frac{\partial g}{\partial t} = -\frac{p}{z_a}\left|\nabla_u\right|,$$

and u_n is the normal derivative of the perturbation velocity at a point on the surface of the semi-infinite half plane, p the surface pressure, Z_a the acoustic impedance of the surface and $\mathbf{n}$ a normal vector pointing from the surface into the fluid.

The perturbation velocity $\mathbf{u}$ of the irrotational sound wave can be written in terms of the velocity potential ϕ as $\mathbf{u} = \nabla\phi$ and the resulting pressure p of the sound field is given by

$$p = -\rho_0\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\phi, \tag{b}$$

where ρ_0 is the density of the undisturbed stream. The mathematical form of the problem may be expressed in terms of the equations satisfied by $\phi(x, y, t)$ as follows

$$\left[\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} - \left(\frac{1}{c}\frac{\partial}{\partial x} + M\frac{\partial}{\partial x}\right)^2\phi\right] = \delta(x - x_0)\delta(y - y_0)\delta(t), \tag{1}$$

and subject to the following boundary conditions in time domain

$$\left[\frac{\partial^2\phi}{\partial y\partial t} \mp \beta M\frac{\partial^2\phi}{\partial x\partial t} \pm \beta M^2 c\frac{\partial^2\phi}{\partial x^2} \mp \frac{\beta}{c}\frac{\partial^2\phi}{\partial t^2}\right] = 0 \quad x < 0, \tag{2}$$

$$\left.\begin{aligned}\frac{\partial}{\partial y}\phi(x, 0^+, t) &= \frac{\partial}{\partial y}\phi(x, 0^-, t)\\ \phi(x, 0^+, t) &= \phi(x, 0^-, t)\end{aligned}\right\} \quad x < 0. \tag{3}$$

In the above equations, $k = \frac{\omega}{c}$ is the wave number, $\beta = \frac{\rho_0 c}{Z_a}$ is the specific complex admittance, $M = \frac{U}{c}$ is the Mach number. It is assumed that the flow is subsonic, i.e., $|M| < 1$, and $\operatorname{Re}\beta > 0$, which is a necessary condition for an absorbing surface [14]. More details can be found in [10].

3. Temporal transform of the problem

We define a temporal Fourier transform and its inverse by

$$\left\{\begin{aligned}\chi(x, y, \omega) &= \int_{-\infty}^{\infty}\phi(x, y, t)e^{i\omega t}\,dt,\\ \phi(x, y, t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\chi(x, y, w)e^{-i\omega t}\,d\omega,\end{aligned}\right. \tag{4}$$

where ω is the temporal frequency. We transform Eqs. (1)–(3) in frequency domain by using Eq. (4), and obtain

$$\left[(1 - M^2)\frac{\partial^2}{\partial x^2} + 2ikM\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} + k^2\right]\chi(x, y, \omega) = \delta(x - x_0)\delta(y - y_0), \tag{5}$$

$$\left[\frac{\partial}{\partial y} \mp 2\beta M\frac{\partial}{\partial x} \pm ik\beta \mp \frac{i\beta M^2}{k}\frac{\partial^2}{\partial x^2}\right]\chi(x, 0^\pm, \omega) = 0 \quad x \geq 0, \tag{6}$$

$$\left.\begin{aligned}\frac{\partial}{\partial y}\chi(x, 0^+, \omega) &= \frac{\partial}{\partial y}\chi(x, 0^-, \omega)\\ \chi(x, 0^+, \omega) &= \chi(x, 0^-, \omega)\end{aligned}\right] \quad x < 0, \tag{7}$$

with

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega t}\,d\omega. \tag{8}$$

We observe that the mathematical problem expressed in Eqs. (5)–(7) is the same as discussed by Ahmad [10] except that in our problem $k = \frac{\omega}{c}$ is not a constant but it is a function of ω. Thus, without going into details, we mention the results only, i.e.,

$$\begin{aligned}\chi(x, y, \omega) = {}& \frac{\exp[-iKM(X - X_0)]}{4\pi i\sqrt{(1 - M^2)}}\int_{-\infty}^{\infty}\frac{e^{i\upsilon(X - X_0) + i\kappa(Y - Y_0)}}{\kappa}\,d\upsilon\\ &+ \frac{\exp[-iKM(X - X_0)]}{8\pi^2\sqrt{(1 - M^2)}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(\upsilon, \xi, \omega)e^{i\upsilon X + i\sqrt{(K^2 - \upsilon^2)}|Y|}\,e^{-i\xi X + i\sqrt{(K^2 - \xi^2)}|Y_0|}\,d\xi\,d\upsilon\end{aligned} \tag{9}$$

where

$$G(\nu,\xi,\omega)=\frac{B\left[K\left(1+M^2\right)+2\xi M+\frac{M^2\xi^2}{(1-M^2)K}\right]-\sqrt{(K-\nu)}\sqrt{(K+\xi)}\mathrm{sgn}(Y)\mathrm{sgn}(Y_0)}{L_+(\nu)L_-(\xi)(\xi-\nu)\sqrt{(K^2-\nu^2)}\sqrt{(K^2-\xi^2)}} \tag{10}$$

where $\kappa=\sqrt{(K^2-\nu^2)}$ is the wave number and ν is the Fourier transform variable. Also

$$\kappa=\kappa_+(\nu)\kappa_-(\nu)=\sqrt{K+\nu}\sqrt{K-\nu},$$

where $\kappa_+(\nu)$ is regular for Im $\nu>-$Im K, i.e., upper half plane and $\kappa_-(\nu)$ is regular for Im $\nu<$ Im K, i.e., lower half plane. Let

$$\chi(x,y,\omega)=I_1'+I_2', \tag{11}$$

where

$$I_1'=\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\nu(X-X_0)+\mathrm{i}\kappa(Y-Y_0)}}{\kappa}\mathrm{d}\nu, \tag{12}$$

and

$$I_2'=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(\nu,\xi,\omega)\mathrm{e}^{\mathrm{i}\nu X+\mathrm{i}\sqrt{(K^2-\nu^2)}|Y|}\,\mathrm{e}^{-\mathrm{i}\xi X+\mathrm{i}\sqrt{(K^2-\xi^2)}|Y_0|}\mathrm{d}\xi\,\mathrm{d}\nu. \tag{13}$$

In order to calculate the total field $\phi(x,y,t)$, we need to find out the inverse temporal Fourier transform of the above integrals. Let us first consider I_1' which can also be written in the form

$$I_1'=\frac{\exp\left[-\mathrm{i}KMR'\cos\Theta'\right]}{4\pi\sqrt{(1-M^2)}}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}KR'\cosh\lambda}\mathrm{d}\lambda, \tag{14}$$

where

$$X-X_0=R'\cos\Theta',\qquad |Y-Y_0|=R'\sin\Theta',\qquad \nu=K\cos(\Theta'+\mathrm{i}\lambda).$$

Taking the inverse temporal Fourier transform and noting that K is a function of ω, Eq. (14) can be written as

$$I_1=\frac{1}{8\pi^2\sqrt{(1-M^2)}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}KMR'\cos\Theta'+\mathrm{i}KR'\cosh\lambda}\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}\lambda\,\mathrm{d}\omega,$$

using

$$k=\sqrt{(1-M^2)}K\quad\text{and}\quad k=\frac{\omega}{c},$$

we get

$$I_1=\frac{c}{4\pi Q}\int_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\omega\left[t+\frac{MR'\cos\Theta'}{Q}-\frac{R'\cosh\lambda}{Q}\right]}\mathrm{d}\omega\mathrm{d}\lambda,$$

where

$$Q=c\sqrt{(1-M^2)}.$$

We know that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}\omega=\delta(t).$$

Thus, using this property of the δ-function, we obtain

$$I_1=\frac{c}{4\pi Q}\int_{-\infty}^{\infty}\delta\left(t+\frac{MR'\cos\Theta'}{Q}-\frac{R'\cosh\lambda}{Q}\right)\mathrm{d}\lambda.$$

letting $\frac{R'\cosh\lambda}{Q}=\eta$ in the above integral, we get

$$I_1=\frac{c}{4\pi Q}\int_{-\infty}^{\infty}\frac{\delta(t'-\eta)}{\sqrt{\left(\eta^2-\frac{R'^2}{Q^2}\right)}}\mathrm{d}\eta, \tag{15}$$

where $t' = t + \frac{MR'\cos\Theta'}{Q}$. The integral appearing in Eq. (15) can now be calculated as

$$I_1 = \frac{c}{4\pi Q}\frac{H(t'-\eta)}{\sqrt{\left(\eta^2 - \frac{R'^2}{Q^2}\right)}}, \tag{16}$$

where $H(t'-\eta)$ is the usual Heaviside function.

Before finding the inverse temporal Fourier transform of I_2', we calculate the double integral appearing in the expression for I_2'. To do so, we introduce the polar coordinates

$$X = R\cos\Theta, \qquad |Y| = R\sin\Theta,$$
$$X_0 = R_0\cos\Theta_0, \qquad |Y_0| = R_0\sin\Theta_0,$$

and the transformation $\xi = -K\cos(\Theta_0 + ip)$ which changes the contour of integration over ξ into a hyperbola through the point $\xi = -K\cos\Theta_0$ where $(0 < \Theta_0 < \pi, -\infty < \tau < \infty)$. Similarly, by the change of variable $\nu = K\cos(\Theta + iq)$, the contour of integration can be converted from ν into a hyperbola through the point $\nu = K\cos\Theta$. Thus, omitting the details of calculations using [15], we obtain

$$I_2' = \frac{-i\left[B\left\{(1+M^2) - 2M\cos\Theta_0 + \frac{M^2\cos^2\Theta_0}{(1-M^2)}\right\} - 2\sin\frac{\Theta}{2}\sin\frac{\Theta_0}{2}\right]e^{iKM(X-X_0)+i\kappa(R+R_0)}}{16\pi K\sqrt{RR_0}\sqrt{(1-M^2)}L_+(K\cos\Theta)L_-(-K\cos\Theta_0)(\cos\Theta + \cos\Theta_0)}, \tag{17}$$

where

$$R = r\left(\sqrt{\frac{1-M^2\sin^2\theta}{1-M^2}}\right), \qquad \cos\Theta = \frac{\cos\theta}{\sqrt{1-M^2\sin^2\theta}} \quad \text{and} \quad \Theta \neq \pi - \Theta.$$

Now taking the inverse temporal Fourier transform of Eq. (17), we have

$$I_2 = -\frac{icA'}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\frac{i\omega}{Q}(R+R_0-MR'\cos\Theta')}}{\omega}e^{-i\omega t}d\omega, \tag{18}$$

where

$$A' = \frac{\left[B\left\{\left((1+M^2) - 2M\cos\Theta_0 + \frac{M^2\cos^2\Theta_0}{(1-M^2)}\right)\right\} - 2\sin\frac{\Theta}{2}\sin\frac{\Theta_0}{2}\right]}{16\pi\sqrt{RR_0}L_+(K\cos\Theta)L_-(-K\cos\Theta_0)(\cos\Theta + \cos\Theta_0)}. \tag{19}$$

Let us take $g(\omega) = \frac{1}{\omega}, f(\omega) = e^{\frac{i\omega}{Q}(R+R_0-MR'\cos\Theta')}$ in Eq. (18) and using the convolution theorem, we can write

$$I_2 = -icA'F(t) * G(t), \tag{20}$$

where

$$F(t) = \frac{1}{2\pi}\int_{i\tau-\infty}^{i\tau+\infty} e^{\frac{i\omega}{Q}(R+R_0-MR'\cos\Theta')}e^{-i\omega t}d\omega,$$
$$G(t) = \frac{1}{2\pi}\int_{i\tau-\infty}^{i\tau+\infty}\frac{e^{-i\omega t}}{\omega}d\omega,$$

where τ lies in the region of analyticity such that $-\text{Im}(K) < \tau < \text{Im}(K)$. The asterisk in Eq. (20) denotes convolution in the time domain. For $\tau > 0$, we can close the contour of integration in the lower half plan. Knowing that ω has a small positive imaginary part, for $\tau > 0$, we get

$$F(t) = \delta\left(t - \frac{1}{Q}(R + R_0 - MR'\cos\Theta')\right),$$
$$G(t) = -i.$$

Hence

$$I_2 = -cA'\int_{-\infty}^{\infty}\delta\left(t' - \frac{1}{Q}(R+R_0)\right)dt,$$

or

$$I_2 = -2cA'H\left(t' - \frac{1}{Q}(R+R_0)\right). \tag{21}$$

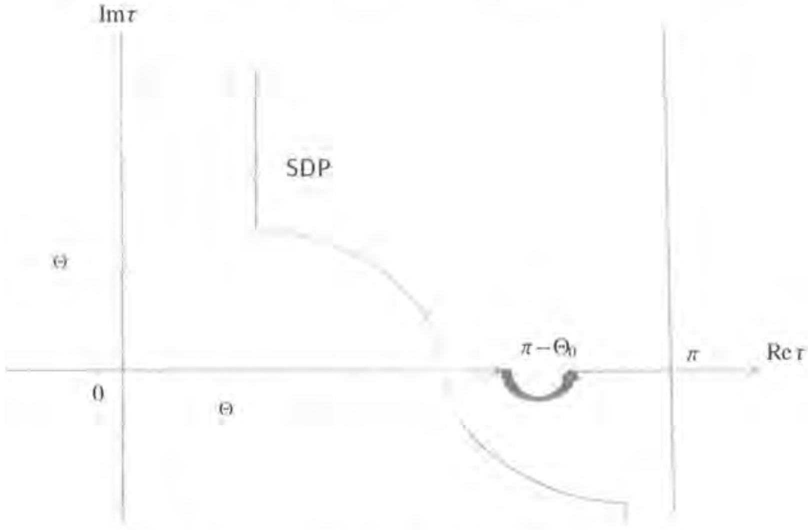

Fig. 1. Steepest descent path in the complex τ-plane.

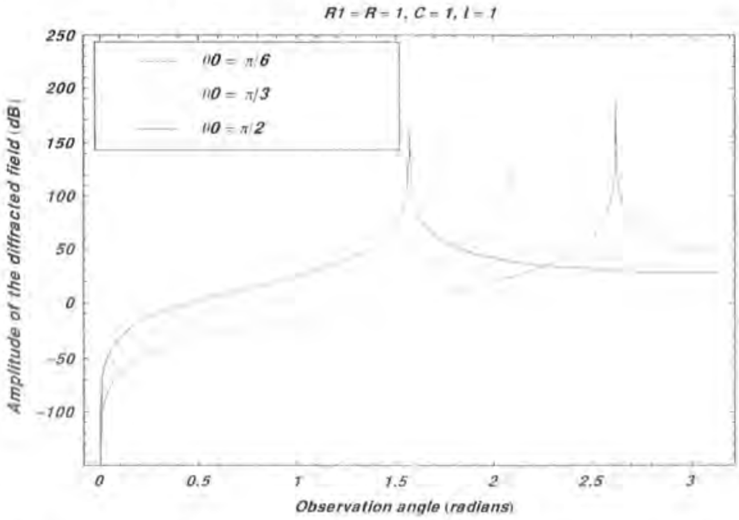

Fig. 2. Amplitude of the diffracted field plotted against the observation angle for different values of the incidence angle θ_0.

Making use of Eqs. (4), (9), (11), (16) and (21), we get

$$\phi(x,y,t) = \frac{c}{4\pi Q}\frac{H(t'-\eta)}{\sqrt{\left(\eta^2 - \frac{R'^2}{Q^2}\right)}} - 2cA'H\left(t' - \frac{1}{Q}(R+R_0)\right), \tag{22}$$

where A' is given by Eq. (19).

4. Graphical results

A computer program MATHEMATICA has been used for the graphical plotting of the diffracted field in the time domain (see Fig. 1). The main features of the graphical results are as follows.

(a) In Fig. 2, the amplitude of the diffracted field is plotted against observation angle for different values of the incident angle by fixing all other parameters. It is observed that by increasing the incident wave angle, the amplitude of the diffracted field decreases.
(b) In Fig. 3, the effect of Mach number M can be seen. By increasing the Mach number, the amplitude of the diffracted field decreases, i.e., the sound intensity decreases.
(c) In Fig. 4, the amplitude of the diffracted field is plotted against observation angle for different values of the absorbing parameter B, by fixing all other parameters. It is observed that by increasing the absorbing parameter, the amplitude of the diffracted field decreases, i.e., the sound intensity decreases.

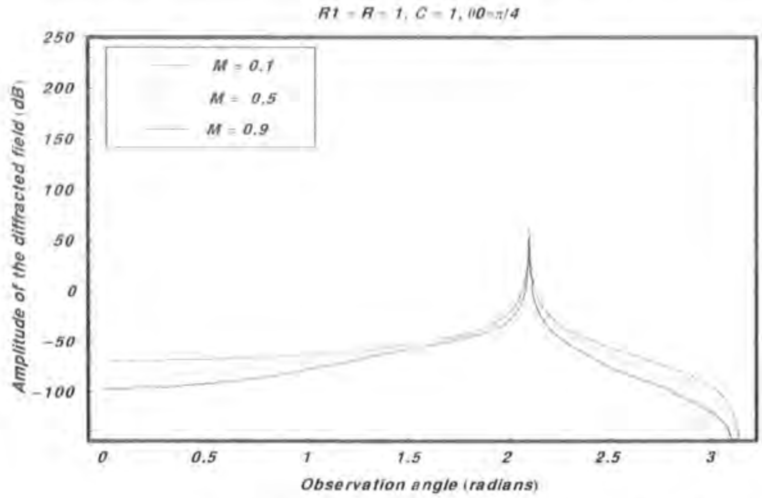

Fig. 3. Amplitude of the diffracted field plotted against the observation angle for different values of the Mach number M.

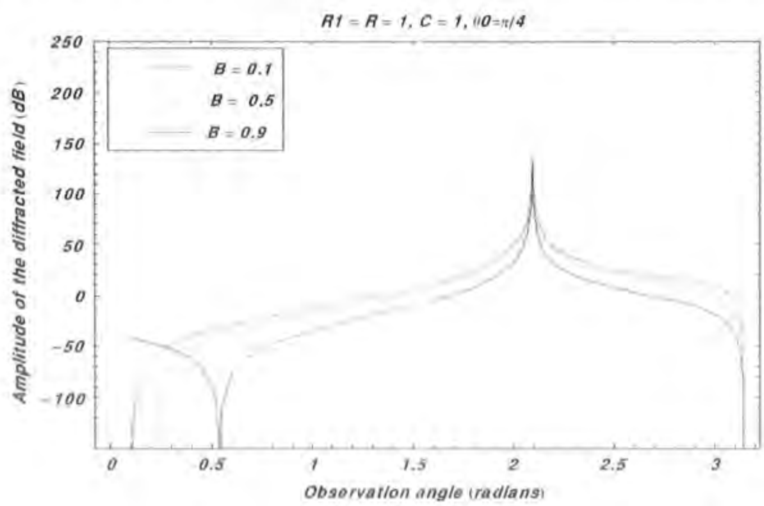

Fig. 4. Amplitude of the diffracted field plotted against the observation angle for different values of the admittance parameter B.

5. Conclusion

We have obtained an improved form of the diffracted field due to an impulsive line source by an absorbing half plane in a moving fluid by considering the time dependence. The first term in Eq. (22) represents the field at the observation point directly coming from the line source, whereas the second term corresponds to the diffracted field from the edge of the half plane. This field starts reaching the point $(x \cdot y)$ after the time lapse $t' > \frac{1}{\Omega}(R + R_0)$. We note that the strength of the field dies down as $1/\sqrt{R_0}$. The results for the still air can be obtained by putting $M = 0$.

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Diffraction of an impulsive line source with wake

M Ayub, A Naeem and Rab Nawaz

Department of Mathematics, Quaid-i-Azam University, 45320, Islamabad 44000, Pakistan

E-mail: mayub59@yahoo.com

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Abstract

The problem of diffraction due to an impulse line source by an absorbing half-plane with wake using Myres' impedance condition (Myers 1980 *J. Sound Vib.* **71** 429–34) in the presence of a subsonic fluid flow is studied. The time dependence of the field requires a temporal Fourier transform in addition to the spatial Fourier transform. The solution of the problem in the presence of wake is obtained by using Greens' function method, Fourier transform, the Wiener–Hopf technique and the modified stationary phase method. Expressions for the total far field for the trailing edge (wake present) situation are given. It is observed that the field produced by the Kutta–Joukowski condition will be substantially in excess of the field when this condition is ignored. Finally, a simple procedure is devised to calculate the inverse temporal Fourier transform. The solution for the leading edge situation can be obtained if the wake, and consequently a Kutta–Joukowski edge condition, is ignored. This can also be seen from the numerical results.

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(Some figures in this article are in colour only in the electronic version.)

1. Introduction

Barriers have become a common measure to reduce noise and hence protect the environment. Absorbing lining can be used to reduce the radiated sound intensity on the source side of the half-plane. Many attempts have been made to determine how effectively sound radiation is reduced by absorbing lining in the presence of fluid flow. It should be noted that radiated sound will be a complicated function of the Mach number and the absorptive properties of the surface scattering the sound waves. It is also worth mentioning that the use of an absorbing barrier, particularly for the reduction of traffic noise, has received much attention in recent years. Various investigations have been made to study the classical problems of line source diffraction of electromagnetic and acoustic waves by various types of strips, slits, half-planes and impedance surfaces [2]. For example, the line source diffraction of electromagnetic waves by a perfectly conducting strip and half-plane was investigated by Ayub *et al* [3] and Jones [4]. The problem of line source diffraction of acoustic waves by a hard half-plane attached to a wake in still air as well as when the medium is convective was studied by Jones [5]. Rawlins [6] considered the line source diffraction of acoustic waves by an absorbing barrier, line source diffraction by an acoustically penetrable or an electromagnetically dielectric half-plane whose width is small as compared to the incident wavelength [7] and line source diffraction of sound waves by an absorbent semi-infinite plane such that the two faces of the half-plane have different impedances [8]. The introduction of a line source changes the incident field, and the method of solution requires a careful analysis in calculating the diffracted field. Jones [5] used the wake condition to see the effects of the Kutta–Joukowski condition at the edge of the half-plane that is generating noise in the fluid at low Mach numbers. He showed that wake acts as a convenient transmission channel for carrying intense sound away from the source. This problem was further extended to point source excitation by Balasubramanyum [9] and to diffraction by a cylindrical impulse by Rienstra [10]. Rawlins [11] discussed the diffraction of a cylindrical acoustic wave by an absorbing half-plane in a moving fluid in the presence of a wake. The imposition of Kutta condition on unsteady perturbations to one of these mean flows represents the mechanism by which both the lift is changed and the amplitude and directivity of the sound field are modified. The nature of and basis for a Kutta condition in unsteady flow has been discussed by Crighton [12] in detail. The condition has recently been applied to unsteadiness in a variety of mean configurations that include trailing edge flows. Recently, Barton and Rawlins [13] discussed the diffraction by a half-plane for both the leading edge and the trailing edge situation.

The transient nature of the field is an important area in the theory of acoustic diffraction and provides a more complete picture of the wave phenomenon under consideration. Many scientists have taken into account the effect of the transient nature of the field: for example, Rienstra [10], Lakhtakia *et al* [14–16], Ayub *et al* [17], Asghar *et al* [18, 19] and Ahmad [20], to name a few. Rawlins [6] discussed the sound scattered by a semi-infinite absorbing plane due to a cylindrical acoustic wave, satisfying Ingard's condition [21] in a moving fluid. Later on, Asghar *et al* [22, 23] extended Rawlins' idea to calculate the diffraction of a spherical acoustic wave from an absorbing plane. Effects of a moving medium were first correctly given by Miles [24] and Ribner [25] for a plane interface of relative motion. The steady state (time harmonic) and initial value (impulsive source) situations have also been considered by Crighton and Leppington [26]. Ingard [21] discussed the effect of flow on boundary conditions at a plane impedance surface. Later on, Myers [1] discussed the diffraction of cylindrical acoustic waves by a semi-infinite absorbing plane, which was in fact a generalization of Ingard's condition. Now, Myers' condition [1] is the accepted form of the boundary condition for impedance walls with flow. Recently, Ahmad [27] has discussed the problem of acoustic diffraction by an absorbing half-plane in a moving fluid using Myers' condition. The aim of the present paper is to analyze the diffraction of waves due to an impulsive line source by an absorbing half-plane in a moving fluid and to examine the effect of the Kutta–Joukowski condition by introducing the wake (trailing edge) attached to the half-plane.

In the present work, expression are derived for the acoustic field for the trailing edge situations. The problem has the added complication of a wake attached to the absorbing half-plane. We apply the temporal Fourier transform to obtain the transform function in the transformed plane using the Wiener–Hopf technique [28] and the modified method of stationary phase [29]. When the transform function is available, an inverse temporal Fourier transform can be applied to obtain the results in the time domain. The expressions for the acoustic field for the trailing edge are obtained, i.e. a wake is attached to the absorbing half-plane. A Kutta–Joukowski condition is also imposed in order to obtain a unique mathematical solution, which also requires that the field is outgoing to infinity. Normally, the effect of the Kutta–Joukowski condition is to produce a beam of sound in the neighborhood of the wake and to scatter a field elsewhere that is approximately that given by Ffowcs-Williams and Hall [30]. Graphical plots of the modulus of the velocity potential for various values of the convection parameters are given.

2. Formulation of the problem

We consider the diffraction of an acoustic wave incident on the half-plane occupying a space $y=0$, $x\leqslant 0$ so that the stream is at zero incidence. The form of the plane wave in a moving fluid is produced by considering an impulsive line source at (x_0, y_0) and is of strength $\delta(t)$ (where δ denotes the Dirac delta function) that also radiates waves. A wake occupies $y=0$, $x<0$ with the velocity of the moving fluid parallel to the x-axis and of magnitude $U>0$. The equations of motion are linearized and the effects of viscosity, thermal conductivity and gravity are neglected. The fluid is assumed to have a constant density (incompressible) and sound speed c. We assume that the plane satisfies Myres' impedance condition [1],

$$u_n=\left[\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right]\frac{g}{|\nabla_\alpha|}, \qquad (1a)$$

where the function g is related to the surface pressure p such that

$$\frac{\partial g}{\partial t}=-\frac{p}{z_a}|\nabla_\alpha| \qquad (1b)$$

and u_n is the normal derivative of the perturbation velocity at a point on the surface of the semi-infinite half-plane. Z_a is the acoustic impedance of the surface and $\mathbf{n}$ is a normal vector pointing from the surface into the fluid.

The perturbation velocity $\mathbf{u}$ of the irrotational sound wave can be written in terms of the velocity potential ϕ as $\mathbf{u}=\nabla\phi$ and the resulting pressure p of the sound field is given by

$$p=-\rho_0\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\phi, \qquad (2)$$

where ρ_0 is the density of the undisturbed stream. The line source is considered parallel to the edge at the point (x_0, y_0). The governing convective wave equation with boundary conditions is given by

$$\left[\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}-\left(\frac{1}{c}\frac{\partial}{\partial t}+M\frac{\partial}{\partial x}\right)^2\phi\right]=\delta(x-x_0)\delta(y-y_0)\delta(t). \qquad (3)$$

3. The problem in the frequency domain

Let us transform the problem in the frequency domain with the help of temporal Fourier transform by

$$\begin{cases}\chi(x,y,\omega)=\displaystyle\int_{-\infty}^{\infty}\phi(x,y,t)\,e^{i\omega t}\,dt,\\ \phi(x,y,t)=\displaystyle\frac{1}{2\pi}\int_{-\infty}^{\infty}\chi(x,y,\omega)\,e^{-i\omega t}\,d\omega,\end{cases} \qquad (4)$$

where ω is the temporal frequency. We transform equations (1)–(3) by using equation (4) and obtain

$$\left[(1-M^2)\frac{\partial^2}{\partial x^2}+2ikM\frac{\partial}{\partial x}+\frac{\partial^2}{\partial y^2}+k^2\right]\chi(x,y,\omega)=\delta(x-x_0)\delta(y-y_0), \qquad (5)$$

$$\left[\frac{\partial}{\partial y}\mp 2\beta M\frac{\partial}{\partial x}\pm ik\beta\mp\frac{i\beta M^2}{k}\frac{\partial^2}{\partial x^2}\right]\times\chi(x,0^{\pm},\omega)=0,\quad x<0. \qquad (6)$$

with

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\, d\omega. \tag{7}$$

$$\frac{\partial}{\partial y}\chi(x, 0^+, \omega) = \frac{\partial}{\partial y}\chi(x, 0^-, \omega), \quad x > 0, \tag{8a}$$

$$\left(-ik + M\frac{\partial}{\partial x}\right)\chi(x, 0^+, \omega) = \left(-ik + M\frac{\partial}{\partial x}\right)\chi(x, 0^-, \omega), \quad x > 0, \tag{8b}$$

where $\beta = \frac{\rho_0 c}{Z_a}$ is the specific complex admittance, $k = \frac{\omega}{c}$ is the wave number and $k = k_1 + ik_2$ has a small imaginary part to ensure the regularity of the Fourier transform integrals and $M = \frac{U}{c}$ (c is the velocity of sound) is the Mach number. We assume that the flow is subsonic, i.e. $|M| < 1$ (for a leading edge situation $-1 < M \leqslant 0$ and for a trailing edge situation $0 < M < 1$) and $Re\beta > 0$, which is a necessary condition for an absorbing surface. Also $\beta = 0$ corresponds to the rigid barrier and $\beta = \infty$ corresponds to the pressure release barrier. The boundary condition (8b) for a continuous pressure with massless wake as already discussed in [5, 11] can be written in the alternative form

$$\chi(x, 0^+, \omega) - \chi(x, 0^-, \omega) = \lambda\, e^{(ik/M)x}, \quad x > 0, \tag{9a}$$

$$\frac{\partial}{\partial y}\chi(x, 0^+, \omega) = \frac{\partial}{\partial y}\chi(x, 0^-, \omega), \quad x > 0. \tag{9b}$$

In equation (9a), the discontinuity in the field is due to imposition of wake, which involves a parameter λ. This λ will be determined by the requirement that the velocity at the trailing edge should be finite, which requires the imposition of the Kutta–Joukowski edge condition. Also $\lambda = 0$ corresponds to the leading edge situation, i.e. no wake. Initially, we shall impose the edge condition

$$\phi = O(1) \quad \text{and} \quad \frac{\partial\phi}{\partial r} = O\left(\frac{1}{\sqrt{x}}\right),$$

combined with the condition that the diffracted field is outgoing to infinity. Physically, we can consider the flow of an incompressible fluid past the edge of a sheet on which the normal velocity is zero. We also need conditions to limit the singularities at the edge (0, 0). In the absence of mean flow, a solution exists with $\phi = O(\sqrt{r})$ as $r \to 0$ (in radiation conditions or Sommerfeld conditions), and to this solution may be added any eigen solution of the problem. The eigen solutions, however, are all more singular than this and it is therefore convenient to choose the solution with $\phi = O(\sqrt{r})$. A better reason for taking this solution can also be obtained by taking into account the effect of viscosity on the half-plane.

Let us introduce the following real substitutions in equations (5), (6), (8) and (9),

$$x = \sqrt{1-M^2}X, \quad x_0 = \sqrt{1-M^2}X_0, \quad y = Y, \quad y_0 = Y_0,$$
$$\beta = \sqrt{1-M^2}B, \quad k = \sqrt{1-M^2}K$$

and

$$\chi(x, y, \omega) = \psi(X, Y, \omega)\, e^{-iKMX}, \tag{10}$$

to obtain

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + K^2\right)\psi(X, Y, \omega) = \frac{\delta(X - X_0)\delta(Y - Y_0)}{\sqrt{1-M^2}}\, e^{iKMX_0}, \tag{11}$$

$$\left[\frac{\partial}{\partial Y} \mp 2BM\frac{\partial}{\partial X} \pm iKB(1+M^2) \mp \frac{iBM^2}{(1-M^2)K}\frac{\partial^2}{\partial X^2}\right] \times \psi(X, 0^\pm, \omega) = 0 \quad x < 0, \tag{12}$$

and

$$\left.\begin{aligned} &\frac{\partial}{\partial Y}\psi(X, 0^+, \omega) = \frac{\partial}{\partial Y}\psi(X, 0^-, \omega) \\ &\psi(X, 0^+, \omega) - \psi(X, 0^-, \omega) = \lambda\, e^{(iK/M)X} \end{aligned}\right\}, \quad x > 0. \tag{13}$$

The total field $\psi(X, Y, \omega)$ may be expressed as a sum of the incident and scattered fields as follows:

$$\psi(X, Y, \omega) = \Psi(X, Y, \omega) + \Psi_i(X, Y, \omega), \tag{14}$$

where $\Psi_i(X, Y, \omega)$ is the incidence field (corresponding to the inhomogeneous equation) and $\Psi(X, Y, \omega)$ is the diffracted field (corresponding to the homogeneous equation), so that $\Psi_i(X, Y, \omega)$ satisfies

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + K^2\right)\Psi_i(X, Y, \omega) = \frac{\delta(X - X_0)\delta(Y - Y_0)}{\sqrt{1-M^2}}\, e^{iKMX_0} \tag{15}$$

and $\Psi(X, Y, \omega)$ satisfies

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + K^2\right)\Psi(X, Y, \omega) = 0. \tag{16}$$

By the Green's function method the solution of equation (15) can be obtained as

$$\Psi_i(X, Y, \omega) = \frac{a}{4i}H_0^1(KR) = \frac{a}{4\pi i}\int_{-\infty}^{\infty}\frac{1}{\kappa}\, e^{i[\alpha(X - X_0) + \kappa|Y - Y_0|]}\, d\alpha, \tag{17}$$

where $a = \frac{e^{iKMX_0}}{\sqrt{1-M^2}}$, $R = \sqrt{(X - X_0)^2 + (Y - Y_0)^2}$ and $\kappa = \sqrt{K^2 - \alpha^2}$ is the wave number and α is the Fourier transform variable.

Let us introduce the Fourier transform over the variable X as

$$\bar{\Psi}(\alpha, Y, \omega) = \int_{-\infty}^{\infty}\Psi(X, Y, \omega)\, e^{-i\alpha X}\, dX \tag{18a}$$

and

$$\Psi(X, Y, \omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{\Psi}(\alpha, Y, \omega)\, e^{i\alpha X}\, d\alpha. \tag{18b}$$

To cater for the two-part boundary condition on $Y = 0$, we split $\bar{\Psi}(\alpha, Y, \omega)$ as

$$\bar{\Psi}(\alpha, Y, \omega) = \bar{\Psi}_-(\alpha, Y, \omega) + \bar{\Psi}_+(\alpha, Y, \omega), \tag{19}$$

where

$$\bar{\Psi}_-(\alpha, Y, \omega) = \int_{-\infty}^{0} \Psi(X, Y, \omega)\, e^{-i\alpha x}$$

and

$$\bar{\Psi}_+(\alpha, Y, \omega) = \int_{0}^{\infty} \Psi(X, Y, \omega)\, e^{-i\alpha x}.$$

Here $\bar{\Psi}_-(\alpha, Y, \omega)$ is regular for $\mathrm{Im}\,\alpha < \mathrm{Im}\,K$ and $\bar{\Psi}_+(\alpha, Y, \omega)$ is regular for $\mathrm{Im}\,\alpha > -\mathrm{Im}\,K$.

We transform equation (16) by Fourier transform to obtain

$$\frac{d^2}{dY^2}\bar{\Psi}(\alpha, Y, \omega) + \kappa^2 \bar{\Psi}(\alpha, Y, \omega) = 0 \qquad (20)$$

and the α-plane is cut such that $\mathrm{Im}\,k > 0$ (for bounded solution). The solution satisfying the radiation condition is given by

$$\bar{\Psi}(\alpha, Y, \omega) = \begin{cases} B_1(\alpha)\, e^{i\kappa Y}, & \text{if} \quad Y \geqslant 0, \\ B_2(\alpha)\, e^{-i\kappa Y}, & \text{if} \quad Y < 0. \end{cases} \qquad (21)$$

The Fourier transform of the boundary conditions as given by equations (12) and (13) takes the following form:

$$\bar{\Psi}'_-(\alpha, 0^+, \omega) + iB\left\{K\left(1+M^2\right) + 2\alpha M + \frac{\alpha^2 M^2}{\left(1-M^2\right)K}\right\} \times \bar{\Psi}_-(\alpha, 0^+, \omega) = -iB\left\{K\left(1+M^2\right) + 2\alpha M + \frac{\alpha^2 M^2}{\left(1-M^2\right)K}\right\} \times \bar{\Psi}_i(\alpha, 0, \omega) - \bar{\Psi}'_i(\alpha, 0, \omega) \qquad (22a)$$

and

$$\bar{\Psi}'_-(\alpha, 0^-, \omega) - iB\left\{K\left(1+M^2\right) + 2\alpha M + \frac{\alpha^2 M^2}{\left(1-M^2\right)K}\right\} \times \bar{\Psi}_-(\alpha, 0^-, \omega) = iB\left\{K\left(1+M^2\right) + 2\alpha M + \frac{\alpha^2 M^2}{\left(1-M^2\right)K}\right\} \times \bar{\Psi}_i(\alpha, 0, \omega) - \bar{\Psi}'_i(\alpha, 0, \omega). \qquad (22b)$$

Also

$$\bar{\Psi}_+(\alpha, 0^+, \omega) - \bar{\Psi}_+(\alpha, 0^-, \omega) = \frac{-i\lambda}{(\alpha - \frac{K}{M})} \qquad (23)$$

and

$$\bar{\Psi}'_+(\alpha, 0^+, \omega) = \bar{\Psi}'_+(\alpha, 0^-, \omega) = \bar{\Psi}'_+(\alpha, 0, \omega). \qquad (24)$$

With the help of equations (21)–(24), we obtain

$$B_1(\alpha) = J_-(\alpha, 0, \omega) + \frac{J'_-(\alpha, 0, \omega)}{i\kappa} - \frac{i\lambda}{2(\alpha - \frac{K}{M})}, \qquad (25a)$$

$$B_2(\alpha) = -J_-(\alpha, 0, \omega) + \frac{J'_-(\alpha, 0, \omega)}{i\kappa} + \frac{i\lambda}{2(\alpha - \frac{K}{M})}, \qquad (25b)$$

where

$$J_-(\alpha, 0, \omega) = \tfrac{1}{2}[\bar{\Psi}_-(\alpha, 0^+, \omega) - \bar{\Psi}_-(\alpha, 0^-, \omega)], \qquad (26a)$$

$$J'_-(\alpha, 0, \omega) = \tfrac{1}{2}[\bar{\Psi}'_-(\alpha, 0^+, \omega) - \bar{\Psi}'_-(\alpha, 0^-, \omega)]. \qquad (26b)$$

From equations (21), (25a) and (26b), we have

$$\bar{\Psi}'_-(\alpha, 0^+, \omega) + \bar{\Psi}'_+(\alpha, 0, \omega) = i\kappa[\bar{\Psi}_-(\alpha, 0^+, \omega) + \bar{\Psi}_+(\alpha, 0^+, \omega)], \qquad (27a)$$

$$\bar{\Psi}'_-(\alpha, 0^-, \omega) + \bar{\Psi}'_+(\alpha, 0, \omega) = -i\kappa[\bar{\Psi}_-(\alpha, 0^-, \omega) + \bar{\Psi}_+(\alpha, 0^-, \omega)]. \qquad (27b)$$

Eliminating $\bar{\Psi}'_-(\alpha, 0^+, \omega)$ from equations (22a) and (27a) and eliminating $\bar{\Psi}'_-(\alpha, 0^-, \omega)$ from equations (22b) and (27b), we obtain

$$\bar{\Psi}'_+(\alpha, 0, \omega) - \left[iB\left\{K\left(1+M^2\right) + 2\alpha M + \frac{\alpha^2 M^2}{\left(1-M^2\right)K}\right\} + i\kappa\right] \times \bar{\Psi}_-(\alpha, 0^+, \omega) - iB\left\{K\left(1+M^2\right) + 2\alpha M + \frac{\alpha^2 M^2}{(1-M^2)K}\right\} \times \bar{\Psi}_i(\alpha, 0, \omega) = i\kappa\bar{\Psi}_+(\alpha, 0^+, \omega) + \bar{\Psi}'_i(\alpha, 0, \omega) \qquad (28a)$$

and

$$\bar{\Psi}'_+(\alpha, 0, \omega) + \left[iB\left\{K(1+M^2) + 2\alpha M + \frac{\alpha^2 M^2}{(1-M^2)K}\right\} + i\kappa\right] \times \bar{\Psi}_-(\alpha, 0^-, \omega) + iB\left\{K\left(1+M^2\right) + 2\alpha M + \frac{\alpha^2 M^2}{(1-M^2)K}\right\} \times \bar{\Psi}_i(\alpha, 0, \omega) = -i\kappa\bar{\Psi}_+(\alpha, 0^-, \omega) + \bar{\Psi}'_i(\alpha, 0, \omega). \qquad (28b)$$

Addition of equations (28a) and (28b) results in

$$i\kappa L(\alpha) J_-(\alpha, 0, \omega) - \bar{\Psi}'_+(\alpha, 0, \omega) + \bar{\Psi}'_i(\alpha, 0, \omega) + \frac{K\lambda}{2(\alpha - \frac{K}{M})} = 0. \qquad (29)$$

Similarly, eliminating $\bar{\Psi}_-(\alpha, 0^+, \omega)$ from equations (22a) and (27a) and eliminating $\bar{\Psi}_-(\alpha, 0^-, \omega)$ from equations (22b) and (27b) and then subtracting the resulting equations, we obtain

$$\frac{L(\alpha) J'_-(\alpha, 0, \omega)}{B\left[K\left(1+M^2\right) + 2\alpha M + \frac{\alpha^2 M^2}{\left(1-M^2\right)K}\right]} - \frac{i\bar{\Psi}_+(\alpha, 0^+, \omega)}{2} - \frac{i\bar{\Psi}_+(\alpha, 0^-, \omega)}{2} + i\bar{\Psi}_i(\alpha, 0, \omega) = 0, \qquad (30)$$

where

$$L(\alpha) = 1 + \frac{B}{\kappa}\left[K\left(1+M^2\right) + 2\alpha M + \frac{\alpha^2 M^2}{\left(1-M^2\right)K}\right]. \qquad (31)$$

Equations (29) and (30) are the standard Wiener–Hopf equations. Let us proceed to find the solution for these equations.

4. Solution of the Wiener–Hopf equations

Let us write

$$L(\alpha) = L_+(\alpha)L_-(\alpha) \tag{32a}$$

and

$$\kappa = \kappa_+(\alpha)\kappa_-(\alpha) = \sqrt{K+\alpha}\sqrt{K-\alpha}, \tag{32b}$$

where $L_+(\alpha)$ and $\kappa_+(\alpha)$ are regular for $\mathrm{Im}\,\alpha > -\mathrm{Im}K$, i.e. the upper half-plane and $L_-(\alpha)$ and $\kappa_-(\alpha)$ are regular for $\mathrm{Im}\,\alpha < \mathrm{Im}K$, i.e. the lower half-plane. Making use of equations (32a) and (32b) in equation (29), we obtain

$$\mathrm{i}J_-(\alpha,0,\omega)L_-(\alpha)\sqrt{K-\alpha} + \frac{\bar{\Psi}_{\mathrm{i}}'(\alpha,0,\omega)}{L_+(\alpha)\sqrt{K+\alpha}} + \frac{\lambda\sqrt{K-\alpha}}{2L_+(\alpha)(\alpha-\frac{K}{M})} = \frac{\bar{\Psi}_+'(\alpha,0,\omega)}{L_+(\alpha)\sqrt{K+\alpha}}, \tag{33}$$

whereas in equation (33), the first term on the left-hand side is regular in the lower half-plane and the term on the right-hand side is regular in the upper half-plane. The other two terms, whose genders are not known, can be written as [28]

$$\frac{\bar{\Psi}_{\mathrm{i}}'(\alpha,0,\omega)}{L_+(\alpha)\sqrt{K+\alpha}} = T_+(\alpha) + T_-(\alpha) \tag{34}$$

and

$$\frac{\lambda\sqrt{K-\alpha}}{2L_+(\alpha)(\alpha-\frac{K}{M})} = F_+(\alpha) + F_-(\alpha). \tag{35}$$

These decompositions cannot be performed by inspection and it is necessary to use the general theorem B of [28]. Now, invoking equations (34) and (35) in (33), we obtain

$$\mathrm{i}J_-(\alpha,0,\omega)\sqrt{K-\alpha}L_-(\alpha) + T_-(\alpha) + F_-(\alpha) = \frac{\bar{\Psi}_+'(\alpha,0,\omega)}{L_+(\alpha)\sqrt{K+\alpha}} - T_+(\alpha) - F_+(\alpha), \tag{36}$$

where

$$T_+(\alpha) = \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\bar{\Psi}_{\mathrm{i}}'(\xi,0,\omega)}{L_+(\xi)\sqrt{K+\xi}(\xi-\alpha)}\,\mathrm{d}\xi, \tag{37a}$$

$$T_-(\alpha) = -\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\bar{\Psi}_{\mathrm{i}}'(\xi,0,\omega)}{L_+(\xi)\sqrt{K+\xi}(\xi-\alpha)}\,\mathrm{d}\xi, \tag{37b}$$

$$F_+(\alpha) = \frac{\lambda}{2(\alpha-\frac{K}{M})}\left[\frac{\sqrt{K-\alpha}}{L_+(\alpha)} - \frac{\sqrt{K-\frac{K}{M}}}{L_+\left(\frac{K}{M}\right)}\right], \tag{37c}$$

$$F_-(\alpha) = \frac{\lambda}{2(\alpha-\frac{K}{M})}\left[\frac{\sqrt{K-\frac{K}{M}}}{L_+\left(\frac{K}{M}\right)}\right]. \tag{37d}$$

We have equated the terms with negative sign on the left-hand side and the terms with positive sign on the right-hand side of equation (36). Let $J(\alpha)$ be a function equal to both sides of equation (36), which are regular in the lower and upper half-planes, respectively. We use analytical continuation so that the definition of $J(\alpha)$ can be extended throughout the complex α plane. We examine the asymptotic behavior of equation (36) to ascertain the form of $J(\alpha)$ as $\alpha\to\infty$. It is noted that $|L_\pm(\alpha)|\sim O(1)$ [25] as $|\alpha|\to\infty$, and with the help of edge condition, it is found that $J_-(\alpha,0,\omega)$ should be at least of $O(|\alpha|^{-1/2})$ as $|\alpha|\to\infty$. So using the extended form of Liouville's theorem [28], we see that $J(\alpha)\sim O(|\alpha|^{-1/2})$ and so a polynomial that represents $J(\alpha)$ can only be a constant equal to zero, i.e.

$$\mathrm{i}J_-(\alpha,0,\omega)\sqrt{K-\alpha}L_-(\alpha) + T_-(\alpha) + F_-(\alpha) = 0.$$

By using equations (37b) and (37d) in the above equation, we have

$$J_-(\alpha,0,\omega) = \frac{-1}{2\pi L_-(\alpha)\sqrt{K-\alpha}} \times \int_{-\infty}^{\infty}\frac{\bar{\Psi}_{\mathrm{i}}'(\xi,0,\omega)}{L_+(\xi)\sqrt{K+\xi}(\xi-\alpha)}\,\mathrm{d}\xi + \frac{\mathrm{i}\lambda\sqrt{K-\frac{K}{M}}}{2L_-(\alpha)\sqrt{K-\alpha}L_+\left(\frac{K}{M}\right)\left(\alpha-\frac{K}{M}\right)}. \tag{38}$$

Similarly, by adopting the same procedure as in the case of equation (29), we can write for equation (30) as follows:

$$J_-'(\alpha,0,\omega) = \frac{1}{2\pi L_-(\alpha)} \times \int_{-\infty}^{\infty} B\left[K\left(1+M^2\right)+2\xi M+\frac{\xi^2M^2}{(1-M^2)K}\right] \times \frac{\bar{\Psi}_{\mathrm{i}}(\xi,0,\omega)}{L_+(\xi)(\xi-\alpha)}\,\mathrm{d}\xi. \tag{39}$$

Invoking equations (38) and (39) in (25a) and (25b), respectively, and then making use of equation (17) in the resulting equation, we obtain

$$\left.\begin{matrix}B_1(\alpha)\\B_2(\alpha)\end{matrix}\right\} = \frac{-a}{4\pi L_-(\alpha)\kappa} \int_{-\infty}^{\infty}\frac{B\left[K\left(1+M^2\right)+2\xi M+\frac{\xi^2M^2}{(1-M^2)K}\right]}{L_+(\xi)(\xi-\alpha)\sqrt{K^2-\xi^2}} \times \mathrm{e}^{-\mathrm{i}\xi X_0+\mathrm{i}\sqrt{K^2-\xi^2}|Y_0|}\,\mathrm{d}\xi$$
$$\pm\frac{a}{4\pi L_-(\alpha)\sqrt{K-\alpha}}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}\xi X_0+\mathrm{i}\sqrt{K^2-\xi^2}|Y_0|}\,\mathrm{sgn}\,|Y_0|}{L_+(\xi)\sqrt{K+\xi}(\xi-\alpha)}\,\mathrm{d}\xi \pm\frac{\mathrm{i}\lambda}{2(\alpha-\frac{K}{M})}\left[\frac{\sqrt{K-\frac{K}{M}}-L_-(\alpha)\sqrt{K-\alpha}L_+\left(\frac{K}{M}\right)}{L_-(\alpha)\sqrt{K-\alpha}L_+\left(\frac{K}{M}\right)}\right]. \tag{40}$$

In order to ensure a unique mathematical solution, we must impose the Kutta–Joukowski edge condition that requires that the velocity should be finite at the origin, which in effect means that in the above expression the term $O(|\alpha|^{-1/2+\delta})$ as $|\alpha|\to\infty$ must vanish; in [27] it is shown that $|L_\pm(\alpha)|\sim O(|\alpha|^{\pm\delta})$ as $|\alpha|\to\infty$, $0\leqslant\delta\leqslant\frac{1}{2}$. Hence, in order that the Kutta–Joukowski condition be satisfied, we have chosen λ as

$$\lambda = \frac{aL_+\left(\frac{K}{M}\right)\operatorname{sgn}|Y_0|}{2\pi \mathrm{i}\left[\sqrt{K-\frac{K}{M}} - L_+\left(\frac{K}{M}\right)\sqrt{K-\alpha}L_-(\alpha)\right]} \times \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}\xi X_0 + \mathrm{i}\sqrt{K^2-\xi^2}|Y_0|}}{L_+(\xi)\sqrt{K+\xi}}\,\mathrm{d}\xi \quad \text{as } |\xi| \to \infty. \tag{41}$$

The integrand in the above integral expression is exponentially bounded as $|\xi| \to \infty$, which is not difficult to show. Now invoking equation (40) with (41) in equation (21) and taking the inverse Fourier transform of the resulting equation, we obtain

$$\begin{aligned}\Psi(X, Y, \omega) &= \left[\frac{-a}{8\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B\left[K\left(1+M^2\right)+2\xi M+\frac{\xi^2 M^2}{\left(1-M^2\right)K}\right]\right. \\ &\quad \frac{\mp\sqrt{K+\alpha}\sqrt{K-\xi}\operatorname{sgn}|Y_0|}{L_-(\alpha)L_+(\xi)(\xi-\alpha)\sqrt{K^2-\alpha^2}\sqrt{K^2-\xi^2}} \\ &\quad \left. - \frac{\sqrt{K-\xi}\sqrt{K+\alpha}\operatorname{sgn}|YY_0|}{(\alpha-\frac{K}{M})L_-(\alpha)L_+(\xi)\sqrt{K^2-\alpha^2}\sqrt{K^2-\xi^2}}\right] \\ &\quad \times \mathrm{e}^{-\mathrm{i}\xi X_0+\mathrm{i}\sqrt{K^2-\xi^2}|Y_0|}\,\mathrm{e}^{\mathrm{i}\kappa|Y|+\mathrm{i}\alpha X}\,\mathrm{d}\xi\,\mathrm{d}\alpha,\end{aligned} \tag{42}$$

where the path of integration is indented below the pole $\alpha = \frac{K}{M}$ for $\operatorname{Im} k = 0$, and with the help of equations (11) and (42), we obtain

$$\begin{aligned}\chi(x, y, \omega) &= \frac{\exp\left[-\mathrm{i}KM(X-X_0)\right]}{4\pi\mathrm{i}\sqrt{(1-M^2)}}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\alpha(X-X_0)+\mathrm{i}\kappa(Y-Y_0)}}{\kappa}\,\mathrm{d}\alpha \\ &\quad + \frac{\mathrm{e}^{\mathrm{i}KM(X_0-X)}\operatorname{sgn}|YY_0|}{8\pi^2\sqrt{1-M^2}} \\ &\quad \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\alpha, \xi, \omega)\,\mathrm{e}^{-\mathrm{i}\xi X_0+\mathrm{i}\sqrt{K^2-\xi^2}|Y_0|}\mathrm{e}^{\mathrm{i}\kappa|Y|+\mathrm{i}\alpha X}\,\mathrm{d}\xi\,\mathrm{d}\alpha \\ &\quad - \frac{\mathrm{e}^{\mathrm{i}KM(X_0-X)}}{8\pi^2\sqrt{1-M^2}} \\ &\quad \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\alpha, \xi, \omega)\,\mathrm{e}^{-\mathrm{i}\xi X_0+\mathrm{i}\sqrt{K^2-\xi^2}|Y_0|}\mathrm{e}^{\mathrm{i}\kappa|Y|+\mathrm{i}\alpha X}\,\mathrm{d}\xi\,\mathrm{d}\alpha,\end{aligned} \tag{43}$$

where

$$F(\alpha, \xi, \omega) = \frac{B\left[K(1+M^2)+2\xi M+\frac{\xi^2M^2}{(1-M^2)K}\right]-\sqrt{K+\alpha}\sqrt{K-\xi}\operatorname{sgn}|Y_0|}{L_-(\alpha)L_+(\xi)(\xi-\alpha)\sqrt{K^2-\alpha^2}\sqrt{K^2-\xi^2}} \tag{44}$$

and

$$G(\alpha, \xi, \omega) = \frac{\sqrt{K-\xi}\sqrt{K+\alpha}}{(\alpha-\frac{K}{M})L_-(\alpha)L_+(\xi)\sqrt{K^2-\alpha^2}\sqrt{K^2-\xi^2}}. \tag{45}$$

It can be seen that if the second term in equation (43) that is carrying the effect of wake in it is ignored, the resulting equation is very much similar to that of equation (32), where no vertex sheet (wake) is attached to the absorbing half-plane [27] (leading edge situation). A natural simplification of the problem is obtained by considering the disturbance to be simple harmonic in time t having frequency ω. The time dependence is described by the factor $\exp(-\mathrm{i}\omega t)$. We shall solve the harmonic problem as if the frequency ω is pure imaginary and then obtain the solution for real ω by analytic continuation with respect to ω. In order to calculate the total field $\phi(x, y, t)$, including the details in appendix A, we finally obtain

$$\phi(x, y, t) = \frac{c}{4\pi Q}\frac{H(t'-\varrho)}{\sqrt{\left(\varrho^2-\frac{R'^2}{Q^2}\right)}} - 2cF_1H\left(t'-\frac{1}{Q}(R+R_0)\right) + 2QG_1H\left(t-\frac{1}{Q}(R+R_0)\right). \tag{46}$$

It is interesting to note that equation (46) represents the total field with the trailing edge situation in transient nature. The first term in expression (46) represents the field at the observation point directly coming from the line source, the second term corresponds to the diffracted field from the edge of the half-plane and the third term includes the effect of wake as pointed out throughout the analysis. If the transient nature of the field and the presence of wake are ignored in equation (46), the resulting field becomes that of the leading edge situation [27], which is also well supported by the graphical results presented in the next section. Also, it is observed that the diffracted field starts reaching the point (x, y) after the time lapse $t' > \frac{1}{Q}(R+R_0)$ and $t > \frac{1}{Q}(R+R_0)$ and the strength of the field dies down as $\frac{1}{\sqrt{R_0}}$.

5. Graphical results

A mathematical program MATHEMATICA has been used for the numerical evaluation and graphical plotting of the diffracted field given by the second and third terms of expression (46). The source is fixed in one position and the effect of different parameters is observed for the diffracted field. The absorbing parameter B is to be taken such that $ReB > 0$, which is the necessary condition for an absorbing surface. Since the results are being plotted for the trailing edge situation, the Mach number is to be taken such that $0 < M < 1$, for a subsonic flow, which also indicates that the stream flow is from left to right. The following two situations are considered:

1. when the source is fixed at one position (for all values of Mach number), relative to the absorbing barrier ($\Theta_0 = \frac{\pi}{4}$, M and Θ are allowed to vary); and
2. when the source is fixed at one position (for all values of the absorbing parameter), relative to the absorbing barrier ($\Theta_0 = \frac{\pi}{4}$, B and Θ are allowed to vary).

It is observed that the field in the region $0 < \Theta < \pi$ is most affected by the changes in M, B and K. Since there is

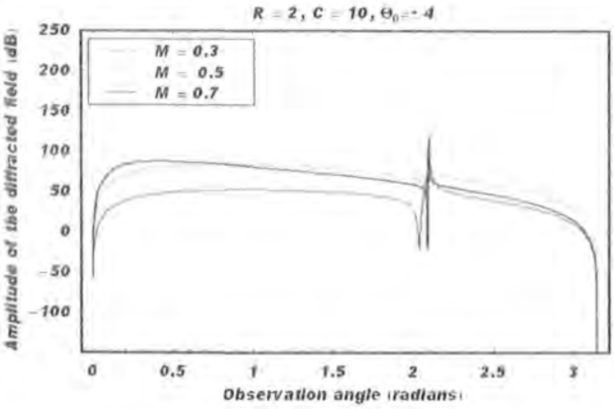

Figure 1. Amplitude of the diffracted field plotted against the observation angle for different values of the Mach number M.

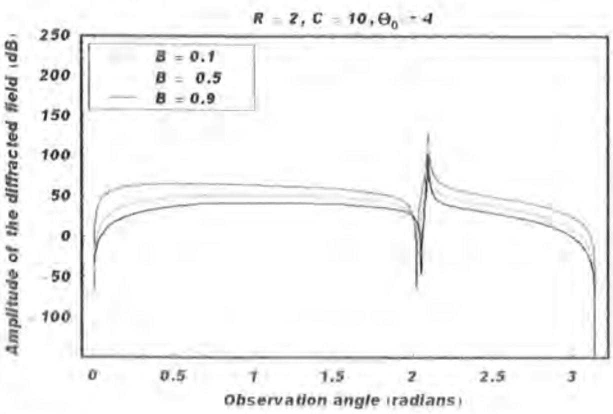

Figure 2. Amplitude of the diffracted field plotted against the observation angle for different values of the admittance parameter B.

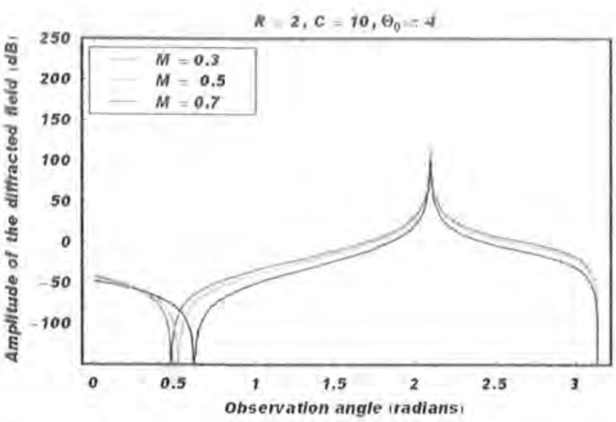

Figure 3. Amplitude of the diffracted field plotted against the observation angle for different values of the Mach number M.

no field before the source is activated, the value of t is taken to be positive in all cases. The observations are given below:

1. Figure 1 plots the diffracted field (trailing edge situation) against the observation angle for different values of the Mach number M, fixing all other parameters, while figure 3 plots the leading edge situation (no wake) in transient nature just for comparison. Figure 5 plots the diffracted field given in [27]. Since we are considering the

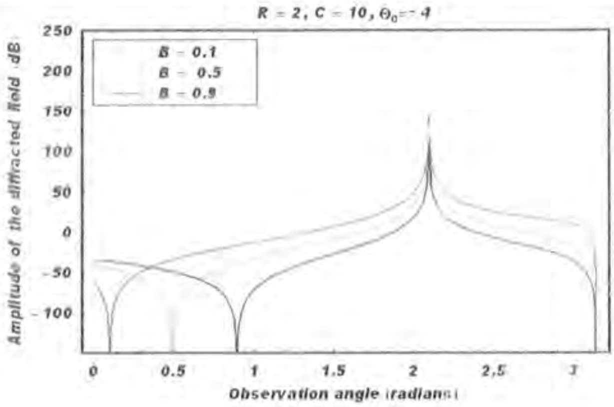

Figure 4. Amplitude of the diffracted field plotted against the observation angle for different values of the admittance parameter B.

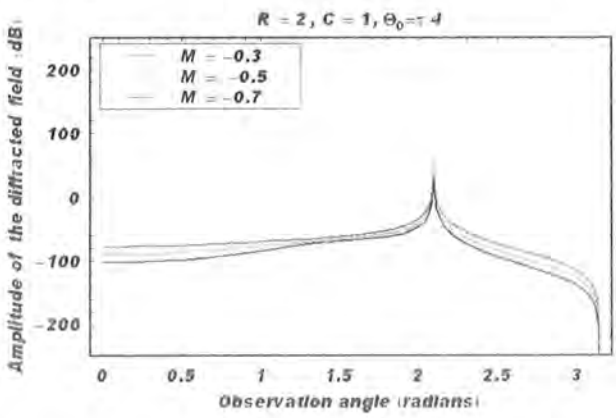

Figure 5. Amplitude of the diffracted field plotted against the observation angle for different values of the admittance parameter M.

subsonic flow, i.e. $U < c$, by increasing the Mach number while fixing all other parameters ($K = 1$, $B = 0.5$, $R = 2$), the velocity of the fluid comes closer to the velocity of sound. The modulus of the velocity potential function is proportional to the amplitude of the perturbation sound pressure and therefore gives a measure of sound intensity. Now, it can be seen from figure 1, which is plotted for the trailing situation, that the amplitude of the field increases initially with some fluctuation due to the singularities, while in figures 3 and 5, the amplitude decreases with increasing M, i.e. the diffracted sound intensity was less for the leading edge situation than for the trailing edge situation. On physical grounds, one would expect the opposite to be the case. The wake would be required to have a shielding effect. Since the edge condition employed in the diffraction theory was the normal edge condition, this theoretical contradiction occurred. This requires that the sound energy be bounded in the finite region around the edge of the half-plane that gave rise to the field, which was more singular at the origin for the trailing edge situation than for the leading edge situation. The normal edge condition used in the diffraction theory can only be regarded to model the leading edge situation satisfactorily.

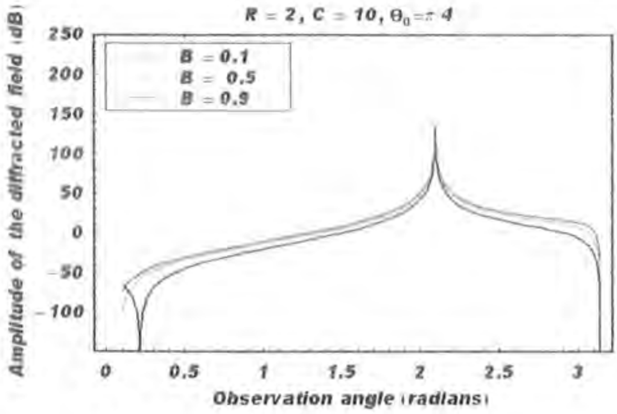

Figure 6. Amplitude of the diffracted field plotted against the observation angle for different values of the admittance parameter B.

2. In a similar way, figures 2, 4 and 6 plot the diffracted field against the observation angle for different values of absorbing parameter B, fixing all other parameters ($K = 1$, $M = \pm 0.5$, $R = 2$). By increasing the absorbing parameter B, the amplitude of the diffracted wave will decrease and consequently the amplitude of the velocity potential will decrease, which is also observed in figures 2, 4 and 6. These figures also show that sound attenuation increases as the absorbing parameter increases. In particular, the presence of wake reduces sound intensity in the shadow region considerably compared with the leading edge situation. From figure 6 it is observed that the absorbing parameter does not make remarkable changes in the amplitude of the field when compared with figure 2, which is plotted for the trailing edge situation.

It is of interest to consider the half-plane as a noise barrier that has effectiveness as a sound barrier, for example for an engine above a wing, and thus to examine the effects of flow on the sound level in the shadow region. It is shown that the magnitude of the sound diffracted into the shadow region is reduced by the presence of flow, which shows that the trailing edge situation is most efficient in reducing the noise in this region.

6. Conclusion

It is observed that the presence of the Kutta–Joukowski condition does not have much influence on the diffracted field away from the diffracting plane and produces a much stronger field near the wake than elsewhere even when the source is not near the edge. The graphical and analytical comparison of the leading edge situation and the trailing edge situation is made and discussed in detail in the previous section. It is observed that the absorbing half-plane with wake gives a more generalized model in diffraction theory and more situations can be discussed as a special case of this problem by choosing suitable parameters. The problem with more practical applications is one of an absorbing strip in a moving fluid with the trailing edge situation. This can be a model for an aeroplane wing and has the advantage of being cheaper to construct than a strip with faces entirely coated in absorbent materials. Finally, this work, which was carried out with Myers' impedence condition in contrast to the results of Ingard's condition, will offer useful theoretical comparisons in conjunction with experimental results. This should then lead to a decision on which is the more appropriate boundary condition to use in practice.

We can obtain the no wake (leading edge) situation by taking $\lambda = 0$ and a field for a rigid barrier by putting $\beta = 0$. Also the results for the still fluid can be recovered by putting $M = 0$.

Appendix

In order to calculate the integrals in equation (43), we let

$$\chi(x, y, w) = \hat{I}_1 + \hat{I}_2 - \hat{I}_3, \qquad (A.1)$$

where

$$\hat{I}_1 = \frac{\exp[-iKM(X - X_0)]}{4\pi i\sqrt{(1 - M^2)}} \int_{-\infty}^{\infty} \frac{e^{i\alpha(X-X_0)+i\kappa(Y-Y_0)}}{\kappa} d\alpha, \qquad (A.2)$$

$$\hat{I}_2 = \frac{e^{iKM(X_0-X)} \operatorname{sgn}|YY_0|}{8\pi^2\sqrt{1 - M^2}} \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\alpha, \xi, \omega) e^{-i\xi X_0 + i\sqrt{K^2-\xi^2}|Y_0|} e^{i\kappa|Y|+i\alpha X} d\xi\, d\alpha \qquad (A.3)$$

and

$$\hat{I}_3 = \frac{e^{iKM(X_0-X)}}{8\pi^2\sqrt{1 - M^2}} \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\alpha, \xi, \omega)\, e^{-i\xi X_0 + i\sqrt{K^2-\xi^2}|Y_0|} \times e^{i\kappa|Y|+i\alpha X} d\xi\, d\alpha. \qquad (A.4)$$

Now, we consider equation (A.2) first, which can also be written as

$$\hat{I}_1 = \frac{\exp[-iKMR'\cos\Theta']}{4\pi\sqrt{(1 - M^2)}} \int_{-\infty}^{\infty} e^{-iKR'\cosh\varsigma} d\varsigma, \qquad (A.5)$$

where

$$X - X_0 = R'\cos\Theta',$$
$$|Y - Y_0| = R'\sin\Theta',$$
$$\alpha = K\cos(\Theta' + i\varsigma).$$

Taking the inverse temporal Fourier transform and noting that K is a function of ω, equation (A.5) can be written as

$$I_1 = \frac{1}{8\pi^2\sqrt{(1 - M^2)}} \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-iKMR'\cos\Theta' + iKR'\cosh\varsigma} e^{-i\omega t} d\varsigma\, d\omega$$

or

$$I_1 = \frac{c}{4\pi Q}\int_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega[t+(MR'\cos\Theta'/Q)-(R'\cosh\varsigma/Q)]}\,\mathrm{d}\omega\,\mathrm{d}\varsigma,$$

where

$$Q = c\sqrt{(1-M^2)}$$

Now, we know that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega = \delta(t),$$

and using this property of the δ-function, we obtain

$$I_1 = \frac{c}{4\pi Q}\int_{-\infty}^{\infty}\delta\left(t+\frac{MR'\cos\Theta'}{Q}-\frac{R'\cosh\varsigma}{Q}\right)\mathrm{d}\varsigma.$$

We let $\frac{R'\cosh\varsigma}{Q} = \varrho$ in the above integral to obtain

$$I_1 = \frac{c}{4\pi Q}\int_{-\infty}^{\infty}\frac{\delta(t'-\varrho)}{\sqrt{(\varrho^2-\frac{R'^2}{Q^2})}}\,\mathrm{d}\varrho, \qquad \text{(A.6)}$$

where $t' = t+\frac{MR'\cos\Theta'}{Q}$. The integral appearing in equation (A.6) can be calculated as

$$I_1 = \frac{c}{4\pi Q}\frac{H(t'-\varrho)}{\sqrt{(\varrho^2-\frac{R'^2}{Q^2})}}, \qquad \text{(A.7)}$$

where $H(t'-\varrho)$ is the usual Heaviside function.

Now, before finding the inverse temporal Fourier transform of $\hat{I}_2$, we calculate the double integral appearing in that expression, i.e. equation (A.3). To do so, we introduce the polar coordinates

$$X = R\cos\Theta, \quad |Y| = R\sin\Theta,$$
$$X_0 = R_0\cos\Theta_0, \quad |Y_0| = R_0\sin\Theta_0,$$

and use the transformation $\xi = -K\cos(\Theta_0+\mathrm{i}p)$, which changes the contour of integration over ξ into a hyperbola through the point $\xi = -K\cos\Theta_0$. Similarly, by the change of variable $\alpha = K\cos(\Theta+\mathrm{i}q)$, the contour of integration can be converted from α into a hyperbola through the point $\alpha = K\cos\Theta$. Thus, omitting the details of calculations, we obtain

$$\hat{I}_2 = \frac{\left[\begin{array}{c}-\mathrm{i}[B\left\{(1+M^2)-2M\cos\Theta_0+\frac{M^2\cos^2\Theta_0}{(1-M^2)}\right\}\\ -2\sin\frac{\Theta}{2}\sin\frac{\Theta_0}{2}]\,\mathrm{e}^{\mathrm{i}KM(X-X_0)+\mathrm{i}\kappa(R+R_0)}\end{array}\right]}{\left[\begin{array}{c}16\pi K\sqrt{RR_0}\sqrt{(1-M^2)}L_+(K\cos\Theta)L_-\\ \times(-K\cos\Theta_0)(\cos\Theta+\cos\Theta_0)\end{array}\right]}. \qquad \text{(A.8)}$$

where

$$R = r\left(\sqrt{\frac{1-M^2\sin^2\theta}{1-M^2}}\right), \quad \cos\Theta = \frac{\cos\theta}{\sqrt{1-M^2\sin^2\theta}}$$
$$\text{and} \quad \Theta \neq \pi-\Theta.$$

Now, taking the inverse temporal Fourier transform of equation (A.8), we have

$$I_2 = -\frac{\mathrm{i}cF_1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{(\mathrm{i}\omega/Q)(R+R_0-MR'\cos\Theta')}}{\omega}\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega, \qquad \text{(A.9)}$$

where

$$F_1 = \frac{\left[B\left\{(1+M^2)-2M\cos\Theta_0+\frac{M^2\cos^2\Theta_0}{(1-M^2)}\right\}-2\sin\frac{\Theta}{2}\sin\frac{\Theta_0}{2}\right]}{16\pi\sqrt{RR_0}L_+(K\cos\Theta)L_-(-K\cos\Theta_0)(\cos\Theta+\cos\Theta_0)}. \qquad \text{(A.10)}$$

Note that the explicit form of the functions $L_\pm(\alpha)$ do not involve ω.

Let us take $u(w) = \frac{1}{w}$, $v(w) = \mathrm{e}^{(\mathrm{i}w/Q)(R+R_0-MR'\cos\Theta')}$ in equation (A.9) and using the convolution theorem, we can write

$$I_2 = -\mathrm{i}cF_1V(t)*U(t), \qquad \text{(A.11)}$$

where

$$V(t) = \frac{1}{2\pi}\int_{\mathrm{i}\tau-\infty}^{\mathrm{i}\tau+\infty}\mathrm{e}^{(\mathrm{i}\omega/Q)(R+R_0-MR'\cos\Theta')}\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega$$

and

$$U(t) = \frac{1}{2\pi}\int_{\mathrm{i}\tau-\infty}^{\mathrm{i}\tau+\infty}\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\omega}\,\mathrm{d}\omega,$$

where τ lies in the region of analyticity such that $-\mathrm{Im}(K) < \tau < \mathrm{Im}(K)$ and the asterisk in equation (A.11) denotes convolution in the time domain. For $\tau > 0$, we can close the contour of integration in the lower half-plane. Knowing that ω has a small positive imaginary part for $\tau > 0$, we obtain

$$V(t) = \delta\left(t-\frac{1}{Q}(R+R_0-MR'\cos\Theta')\right)$$

and

$$U(t) = -\mathrm{i}.$$

Hence

$$I_2 = -2cF_1H\left(t'-\frac{1}{Q}(R+R_0)\right). \qquad \text{(A.12)}$$

Similarly, by adopting the same procedure as in the case of $\hat{I}_2$, we can solve $\hat{I}_3$ and obtain

$$I_3 = -2QG_1H\left(t-\frac{1}{Q}(R+R_0)\right), \qquad \text{(A.13)}$$

where

$$G_1 = \frac{\sqrt{1+\cos\Theta_0}\sqrt{1+\cos\Theta}}{(\frac{1}{M}-\cos\Theta)L_-(K\cos\Theta)L_+(-K\cos\Theta_0)}. \qquad \text{(A.14)}$$

Hence, using equations (A.7), (A.12) and (A.13) in (A.1), we obtain

$$\phi(x,y,t) = \frac{c}{4\pi Q}\frac{H(t'-\varrho)}{\sqrt{(\varrho^2-\frac{R'^2}{Q^2})}} - 2cF_1H\left(t'-\frac{1}{Q}(R+R_0)\right)$$
$$+2QG_1H\left(t-\frac{1}{Q}(R+R_0)\right),$$

which is equation (46).

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