Some Fixed Point Theorems for Contractive Type Mappings

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Chapter 1

Preliminaries

The purpose of this chapter is to present the basic notions and results which are further considered in the next chapters of this work. allowing us to present the results of this thesis.

1.1 Normed Algebra

A Banach algebra. named after Stefan Banach, is an associative algebra A over the real or complex numbers, that at the same time is also a Banach space; i.e. normed and complete. The algebra multiplication and the Banach space norm are required to be related by the following inequality

$$\|x.y\| \leq \|x\| . \|y\| \quad \forall\ x, y \in A$$

If, in the above, we relax the Banach space to normed space the analogous structure is called a *normed algebra*.

Example 1.1.1. If X is a topological space then the set of complex-valued, continuous functions on X is an algebra over $\mathbb{C}$, denoted by $C(X)$, with the algebraic operations defined pointwise; i.e.

if $f, g \in C\ (X)$ and $\alpha \in \mathbb{C}$ then

$$(f + \alpha g)(x) = f(x) + \alpha g(x) \ and \ (fg)(x) = f(x)g(x) \qquad \forall x \in X.$$

1.2 Functional inequalities

Functional equations are equations for unknown functions instead of unknown numbers. These equations were studied by Augustin Louis Cauchy, and since then they have formed the cornerstone of the theory. The study of functional equations stability originated from a question of Ulam 1940 concerning the stability of group homomorphisms as follows:

Let G be a group endowed with a metric d. Given an $\varepsilon > 0$, does there exist a $k > 0$ such that, for every function $f : G \to G$ satisfying the inequality

$$d(f(x.y), f(x).f(y)) < \varepsilon, \forall x, y \in G,$$

there exists an automorphism a of G with

$$d(f(x), a(x)) < k\varepsilon, \forall x \in G,$$

In 1941 Hyers gave an affirmative answer to the question of Ulam for Cauchy equations in Banach spaces.

Let E_1 and E_2 be Banach spaces and let $f : E_1 \to E_2$ be a mapping such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in E_1$ and $\delta > 0$; that is, f is δ-additive. Then there exists a unique additive $T : E_1 \to E_2$, which satisfies

$$\|f(x) - T(x)\| \leq \delta, \forall x \in E_1$$

Gilányi showed that if f satisfies the functional inequality

$$\left\|2f(x) + 2f(y) - f\left(xy^{-1}\right)\right\| \leq \|f(xy)\|,$$

then f satisfies the *Jordan-von Neumann* functional equation

$$2f(x) + 2f(y) = f(xy) + f\left(xy^{-1}\right).$$

Hyers Theorem was generalized by Aoki [3] for additive mappings and by Rassias [76] for linear mappings, by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [40] by replacing the unbounded Cauchy difference by a general control function, in the spirit of the Rassias approach. See ([26], [56], [71]) for more information on functional equations and their stability.

1.3 Fuzzy sets

Most of our traditional tools for formal modeling, reasoning, and computing are crisp, deterministic and precise in character. Crisp means dichotomous: that is, yes-or-no type rather than more-or-less type. In traditional dual logic, for instance, a statement can be true or false—and nothing in between. In set theory an element can either belong to a set or not. In optimization a solution can be feasible or not. Precision assumes that the parameters of a model represent exactly the real system that has been modeled. This generally also implies that the model is unequivocal; that is, it contains no ambiguities. Certainty eventually indicates that we assume the structures and parameters of the model to be definitely known, and that there are no doubts about their values or their occurrence.

Fuzzy set theory was initiated by Zadeha in the early 1960s. However, the term *ensemble flou* (a posteriori the french counterpart of *fuzzy set*) was coined by Menger in 1951. Manger explicitly used a "max-product" transitive fuzzy relation, but with a probalistic interpretation. On a semantic level Zadeh's theory is more closely related to Black's work on vagueness, where "consistency profiles" (the ancestors of fuzzy membership functions) "characterized vague symbols."

Let X be an arbitrary set. A fuzzy set A over X is defined by a function μ_A,

$$\mu_A : X \to [0, 1],$$

where μ_A is called a *membership function* of A, and the value $\mu_A(x)$ is called the *grade of membership* of x in X. The value represents the degree of x belonging to the fuzzy set X. It is clear that A is completely determined by the set,

$$A = \{(x, \mu_A(x)) : x \in X\}$$

Frequently we will write $A(x)$ instead of $\mu_A(x)$.

The *α-level set* of A is denoted by $[A]_\alpha$, and defined as follows:

$$\begin{aligned} [A]_\alpha &= \{x : A(x) \geq \alpha\} \ \ if \ \ \alpha \in (0, 1], \\ [A]_0 &= \overline{\{x : A(x) > 0\}}, \end{aligned}$$

where $\overline{B}$ denotes the closure of the set B.

Let (X) be the collection of all fuzzy sets in a metric space X. For $A, B \in (X)$, $A \subset B$ means $A(x) \leq B(x)$ for each $x \in X$.

Definition 1.3.1. Let X be an arbitrary set and (Y, d) be a metric space . A mapping G is called a *fuzzy mapping* if G is a mapping from X into (Y).

A fuzzy mapping G is a fuzzy subset on $X \times Y$ with membership function $G(x)(y)$. The function $G(x)(y)$ is the grade of membership of y in $G(x)$. For convenience, we denote the α-level set of $G(x)$ by $[Gx]_\alpha$ instead of $[G(x)]_\alpha$.

Definition 1.3.2. Let G, H be fuzzy mappings from X into (X). A point z in X is called an α-fuzzy fixed point of H if $z \in [Hz]_\alpha$. The point z is called a common α-fuzzy fixed point of G and H if $z \in [Gz]_\alpha \cap [Hz]_\alpha$.

1.4 Weakly contractive fuzzy mappings

Let (X, d) be a metric space, $B(X)$ and $CB(X)$ be the sets of all nonempty bounded and closed bounded subsets of X, respectively. For $P, Q \in B(X)$ we define

$$\delta(P, Q) = \sup\{d(p, q) : p \in P, q \in Q\},$$

$$D(P, Q) = \inf\{d(p, q) : p \in P, q \in Q\}.$$

If $P = \{p\}$, we write $\delta(P, Q) = \delta(p, Q)$ and if $Q = \{q\}$, then $\delta(p, Q) = d(p, q)$.

For P, Q, R in $B(X)$ one can easily prove the following properties.

$$\begin{aligned}
\delta(P, Q) &= \delta(Q, P) \geq 0, \\
\delta(P, Q) &\leq \delta(P, R) + \delta(R, Q), \\
\delta(P, P) &= \sup\{d(p, r) : p, r \in P\} = diam\ P, \\
\delta(P, Q) &= 0, \Rightarrow P = Q = \{p\}.
\end{aligned}$$

Let $\{A_n\}$ be a sequence in $B(X)$. Then the sequence $\{A_n\}$ converges to A if and only if (i) $a \in A$ implies that $a_n \to a$ for some sequence $\{a_n\}$ with $a_n \in A_n$ for $n \in N$, and (ii) for each $\varepsilon > 0$, there exist $n, m \in N$ with $n > m$ such that $A_n \subseteq A_\varepsilon = \{x \in X : d(x, a) < \varepsilon$ *for some* $a \in A\}$.

See [42].

The following results will be useful in the proof of our results in upcoming chapters.

Lemma 1.4.1. [37] Let $\{A_n\}$ and $\{B_n\}$ be sequences in $B(X)$ and (X, d) be a complete metric space. If $A_n \to A \in B(X)$ and $B_n \to B \in B(X)$, then $\delta(A_n, B_n) \to \delta(A, B)$.

Lemma 1.4.2. [51] Let (X, d) be a complete metric space. If $\{A_n\}$ is a sequence of nonempty bounded subsets in (X, d) and, if $\delta(A_n, y) \to 0$ for some $y \in X$, then $A_n \to \{y\}$.

Theorem 1.4.3. [80] Let(X, d) be a complete metric space and T be a φ-weak contraction on X; that is, for each $x, y \in X$, there exists a function $\varphi : [0, \infty) \to [0, \infty)$ such that φ is positive on $(0, \infty)$, $\varphi(0) = 0$, and

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) \tag{1.4.1}$$

Also if φ is a continuous and nondecreasing function, then T has a unique fixed point.

A weakly contractive mapping is a map satisfying the inequality (1.4.1) and was first defined by Alber and Guerre-Delabriere [1]. For more results on these mappings; see [6], [7], [37], [66], [91] and the related references therein. Zhang and Song [94] gave the following theorem.

Theorem 1.4.4. [94] Let (X, d) be a complete metric space and $T, S : X \to X$ be two mappings such that, for each $x, y \in X$,

$$d(Tx, Sy) \leq m(x, y) - \varphi(m(x, y)),$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function with $\varphi(t) > 0$ for $t > 0$, $\varphi(0) = 0$, and

$$m(x, y) = max\left\{d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2}[d(y, Tx) + d(x, Sy)]\right\}$$

Then there exists a unique point $u \in X$ such that $u = Tu = Su$.

1.5 Fuzzy Hilbert space

It was Katsaras [54], who, while studying fuzzy topological vector spaces, was the first to introduce the idea of a fuzzy norm on a linear space in 1984. Later, many other mathematicians like Felbin [33], Cheng & Mordeson [23], Bag & Samanta [10], etc. introduced definitions of fuzzy normed linear spaces using different approach. A large number of papers have been published in fuzzy normed linear spaces. For reference see [11], [12], [13], [14], [15], [16], [34], [35]. On the other hand studies on fuzzy inner product spaces are relatively recent and few works have been done in fuzzy inner product spaces. Biswas [17], El-Abyad & Hamouly [30] were among the first to give a meaningful definition of a fuzzy inner product space and associated fuzzy norm function. Later on, Kohli & Kumar [58] modified the definition of inner product space as introduced by Biswas.

Now we discuss some definitions and results obtained by the above mentioned authors.

Definition 1.5.1. Let U be a real linear space. A fuzzy subset N of $U \times R$ (R is the set of real numbers) is called a fuzzy norm on U if , $\forall\ x, u \in U$ and $c \in F$, the following conditions are satisfied:

$(N1) : \forall t \in R\ , t \leq 0; N(x,t) = 0,$

$(N2) : (\forall t \in R, t > 0; N(x,t) = 1)$ *iff* $x = 0,$

$(N3) : \forall t \in R, t > 0; N(x,t) = N(x, \frac{t}{|c|})\ if\ c \neq 0,$

$(N4) : \forall s, t \in R, x, u \in U,$

$$N(x+u, s+t) \geq min\{N(x,s), N(u,t)\}$$

$(N5) : N(x,.)$ is a nondecreasing function of R and $\lim_{t\to\infty} N(x,t) = 1.$

The pair (U, N) will be referred to as a fuzzy normed linear space.

Example 1.5.2. Let $(X, \|.\|)$ be a normed space. We define

$$N(x,t) = \begin{cases} \frac{t}{t+\|x\|}, t > 0,\ x \in X \\ 0, t \leq 0,\ x \in X \end{cases}$$

Then (X, N) is a fuzzy normed linear space.

Theorem 1.5.3. Let (U, N) be a fuzzy normed linear space. Also suppose that,

$(N6)$: $\forall t > 0, N(x,t) > 0$ implies $x = 0$.

Define $\|x\|_\alpha = \wedge\{t > 0 : N(x,t) \geq \alpha\}$ $, \alpha \in (0,1)$.

Then $\{\|.\|_\alpha : \alpha \in (0,1)\}$is an ascending family of norms on U and they are called α-norms on U corresponding to the fuzzy norm N on U.

Theorem 1.5.4. Let (U, N) be a fuzzy normed linear space satisfying $(N6)$. Along with this assumption,

$(N7)$: for $x \neq 0$, $N(x, \cdot)$ is a continuous function of $\mathbb{R}$.

Let $\|x\|_\alpha = \wedge \{t > 0 : N(x,t) \geq \alpha\}, \alpha \in (0,1)$ and $N' : U \times \mathbb{R} \to [0,1]$ be a function defined by

$$N'(x,t) = \begin{cases} \wedge \{\alpha \in (0,1) : \|x\|_\alpha \leq t\}, & \text{if } (x,t) \neq (0,0) \\ 0 & \text{if } (x,t) = (0,0). \end{cases}$$

Then (i) $\{\|.\|_\alpha : \alpha \in (0,1)\}$ is an ascending family of norms on U.

(ii) N' is a fuzzy norm on U.

(iii) $N' = N$

Definition 1.5.5. A mapping $* : [0,1] \times [0,1] \to [0,1]$ is a t-norm if $*$ satisfies the following conditions:

(1) $*$ is associative and commutative,

(2) $a * 1 = 1 \ \forall a \in [0,1]$,

(3) $a * b \leq c * d$ such that $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$.

Examples of t-norms are $a * b = \min\{a, b\}$, $a * b = ab$ and $a * b = \max\{a + b - 1, 0\}$.

Definition 1.5.6. A fuzzy inner product space is a triplet $(X, F, *)$, where X is a real vector space, $*$ is a t-norm and F is a fuzzy set on $X^2 \times R$ satisfying the following conditions for every $x, y, z \in X$ and $t, s \in R$,

$(F1)$ $F(x, y, 0) = 0$,

$(F2)$ $F(x, y, t) = F(y, x, t)$,

$(F3)$ $F(x, x, t) = 1$, for all $t > 0$, if and only if $x = 0$,

$(F4)$ For any real number $c \in R$ and $t \neq 0$,

$$F(cx, y, t) = \begin{cases} F(x, y, t/c) \text{ for } c > 0, \\ H(t) \text{ for } c = 0, \\ 1 - F(x, y, t/c) \text{ for } c < 0. \end{cases}$$

where

$$H(t) = \begin{cases} 1 \text{ if } t > 0, \\ 0 \text{ if } t \leq 0. \end{cases}$$

$(F5)$ $F(x,z,t) * F(y,z,s) \leq F(x+y,z,t+s)$, for all $t, s > 0$,

$(F6)$ $\lim_{t\to\infty} F(x,y,t) = 1$.

Example 1.5.7. Let $(X, < ..., >)$ be an ordinary inner product space. We define a mapping $F : X^2 \times R \to [0,1]$ as follows

$$F(x,y,t) = \begin{cases} t^{1/2}/\left(t^{1/2} + |\langle x,y\rangle|^{1/2}\right) \text{ for } t > 0, \\ 0 \text{ for } t = 0, \\ |\langle x,y\rangle|^{1/2} / \left((-t^{1/2}) + |\langle x,y\rangle|^{1/2}\right). \end{cases}$$

$(X, F, *)$ is a fuzzy inner product space, where $*$ is an arbitrary t-norm.

Chapter 2

Additive ρ-functional inequalities in matrix normed spaces

In this chapter, we prove the Hyers-Ulam stability of the additive ρ-functional inequalities in complex matrix normed spaces and investigate some other additive ρ-functional equations associated with these inequalities.

2.1 Hyers-Ulam stability of the additive ρ-functional inequality in matrix normed spaces.

In 1996, Isac and Rassias were the first to provide applications of stability theory of functional inequalities for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors.

Gilányi [42], Fechner [32] and Park et al. [69] proved the Hyers-Ulam stability of additive functional inequalities. Kim et al. [56] solved the additive ρ-functional inequalities in complex normed spaces and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in complex Banach spaces.

In [56], Kim et al. introduced and investigated the following additive ρ-functional inequality,

$$\begin{aligned}
&\|f(x+y+z)-f(x)-f(y)-f(z)\| \\
&\quad\le \left\|\rho\left(2f\left(\frac{x+y}{2}+z\right)-f(x)-f(y)-2f(z)\right)\right\|,
\end{aligned} \tag{2.1.1}$$

In this section we prove the above inequality in complex matrix normed spaces.

Throughout this section $(X,\{\|\cdot\|_n\})$ and $(Y,\{\|\cdot\|_n\})$ are a matrix normed space and a matrix Banach space, respectively. Let ρ be a fixed complex number with $|\rho|<1$. The following lemma will be helpful in the proof of our main results.

Lemma 2.1.1. ([70]) Let $(X,\{\|.\|_n\})$ be a matrix normed space.

(1) $\| E_{kl}\otimes x \|_n=\| x \|$ for $x\in X$.

(2) $\| x_{kl} \|\le\| [x_{ij}] \|_n\le \sum_{i,j=1}^{n} \| x_{ij} \|$ for $[x_{ij}]\in M_n(X)$.

(3) $\lim_{n\to\infty} x_n = x$ iff $\lim_{n\to\infty} x_{nij}=x_{ij}$ for $x_n=[x_{nij}], x=[x_{ij}]\in M_k(X)$.

Theorem 2.1.2. Let $r>1$ and be nonnegative real numbers, and let $f:X\to Y$ be a mapping such that

$$\begin{aligned}
&\|f_n([x_{ij}+y_{ij}+z_{ij}])-f_n([x_{ij}])-f_n([y_{ij}])-f_n([z_{ij}])\|_n \\
&\quad\le \left\|\rho\left(2f_n\left(\frac{[x_{ij}]+[y_{ij}]}{2}+[z_{ij}]\right)-f_n([x_{ij}])-f_n([y_{ij}])-2f_n([z_{ij}])\right)\right\|_n \\
&\qquad+\sum_{i,j=1}^{n}\theta\left(\| x_{ij} \|^r+\| y_{ij} \|^r+\| z_{ij} \|^r\right)
\end{aligned} \tag{2.1.2}$$

for all $x=[x_{ij}], y=[y_{ij}], z=[z_{ij}]\in M_n(X)$.

Then there exists a unique additive mapping $h:X\to Y$

such that

$$\|f_n([x_{ij}])-h_n([x_{ij}])\|_n\le\sum_{i,j=1}^{n}\frac{2\theta}{2^r-2}\| x_{ij} \|^r \tag{2.1.3}$$

for all $x=[x_{ij}]\in M_n(X)$.

Proof. Let $n=1$ in (2.1.2). Then we obtain

$$\begin{aligned}
&\|f(a+b+c)-f(a)-f(b)-f(c)\| \\
&\quad\le\left\|\rho\left(2f\left(\frac{a+b}{2}+c\right)-f(a)-f(b)-2f(c)\right)\right\| \\
&\quad+\theta\left(\|a\|^r+\|b\|^r+\|c\|^r\right)
\end{aligned}$$

for all $a, b, c \in X$.

By [56, Theorem 2.3], there is a unique additive mapping $h : X \to Y$ such that

$$\|f(a) - h(a)\| \leq \frac{2\theta}{2^r - 2}\|a\|^r$$

for all $a \in X$.

By Lemma 2.1.1,

$$\|f_n([x_{ij}]) - h_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{2\theta}{2^r - 2} \parallel x_{ij} \parallel^r$$

for all $x = [x_{ij}] \in M_n(X)$. Thus the mapping $h : X \to Y$ is the unique additive mapping satisfying (2.1.3). □

In relation to completion, Theorem 2.1.2 can be reformulated as follows.

Theorem 2.1.3. Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (2.1.2). Then there exists a unique additive mapping $h : X \to Y$ such that

$$\|f_n([x_{ij}]) - h_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{2\theta}{2 - 2^r} \parallel x_{ij} \parallel^r \tag{2.1.4}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. By [56, Theorem 2.4], there is a unique additive mapping $h : X \to Y$ such that

$$\|f(a) - h(a)\| \leq \frac{2\theta}{2 - 2^r}\|a\|^r$$

for all $a \in X$.

By Lemma 2.1.1,

$$\|f_n([x_{ij}]) - h_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{2\theta}{2 - 2^r} \parallel x_{ij} \parallel^r$$

for all $x = [x_{ij}] \in M_n(X)$. Thus the mapping $h : X \to Y$ is a unique additive mapping satisfying (2.1.4). □

By the triangle inequality, we have

$$\begin{aligned}
&\|f_n([x_{ij} + y_{ij} + z_{ij}]) - f_n([x_{ij}]) - f_n([y_{ij}]) - f_n([z_{ij}])\|_n \\
&\quad - \left\|\rho\left(2f_n\left(\frac{[x_{ij}] + [y_{ij}]}{2} + [z_{ij}]\right) - f_n([x_{ij}]) - f_n([y_{ij}]) - 2f_n([z_{ij}])\right)\right\|_n \\
\leq\ & \|f_n([x_{ij} + y_{ij} + z_{ij}]) - f_n([x_{ij}]) - f_n([y_{ij}]) - f_n([z_{ij}]) \\
&\quad - \rho\left(2f_n\left(\frac{[x_{ij}] + [y_{ij}]}{2} + [z_{ij}]\right) - f_n([x_{ij}]) - f_n([y_{ij}]) - 2f_n([z_{ij}])\right)\right\|_n .
\end{aligned}$$

The following corollaries illustrate that how we can obtain the Hyers-Ulam stability results for the additive ρ-functional equations associated with the additive-ρ-functional inequality (2.1.1) in complex matrix Banach spaces.

Corollary 2.1.4. Let $r > 1$ be nonnegative real number, and let $f : X \to Y$ be a mapping such that

$$\begin{aligned}&\left\|2f_n\left(\frac{[x_{ij}]+[y_{ij}]}{2}+[z_{ij}]\right)-f_n([x_{ij}])-f_n([y_{ij}])-2f_n([z_{ij}])\right. \qquad (2.1.5)\\ &\left.-\rho\left(2f_n\left(\frac{[x_{ij}]+[y_{ij}]+[z_{ij}]}{2}\right)-f_n([x_{ij}])-f_n([y_{ij}])-f_n([z_{ij}])\right)\right\|_n\\ &\leq \sum_{i,j=1}^{n}\theta(\| x_{ij}\|^r+\| y_{ij}\|^r+\| z_{ij}\|^r)\end{aligned}$$

for all $x=[x_{ij}], y=[y_{ij}], z=[z_{ij}]\in M_n(X)$. Then there exists a unique additive mapping $h : X \to Y$ satisfying (2.1.3).

Corollary 2.1.5. Let $r < 1$ and be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (2.1.5). Then there exists a unique additive mapping $h : X \to Y$ satisfying (2.1.4).

Remark 2.1.6. If ρ is a real number such that $-1 < \rho < 1$ and Y is a real Banach space, then all of the assertions in this section remain valid.

A similar construction can be applied to derive a Hyers-Ulam stability of the following additive ρ-functional inequality

$$\begin{aligned}&\left\|2f\left(\frac{x+y}{2}+z\right)-f(x)-f(y)-2f(z)\right\|\\ &\leq \|\rho(f(x+y+z)-f(x)-f(y)-f(z))\|\end{aligned}$$

in complex matrix Banach spaces.

2.2 Results for Hyers-Ulam stability of the additive ρ-functional inequality in matrix normed spaces

In this section, we prove the following Hyers-Ulam stability of the additive ρ-functional inequality in complex matrix normed spaces.

$$\|2f\left(\frac{x+y}{2}+z\right)-f(x)-f(y)-2f(z)\| \tag{2.2.1}$$
$$\leq \left\|\rho\left(2f\left(\frac{x+y}{2}+z\right)-f(x)-f(y)-f(z)\right)\right\|,$$

where ρ be a fixed complex number with $|\rho|<1$

Theorem 2.2.1. Let $r>1$ and θ be nonnegative real numbers, and let $f:X\to Y$ be a mapping such that

$$\left\|2f_n\left(\frac{[x_{ij}]+[y_{ij}]}{2}+[z_{ij}]\right)-f_n([x_{ij}])-f_n([y_{ij}])-2f_n([z_{ij}])\right\|_n \tag{2.2.2}$$
$$\leq \left\|\rho\left(2f_n\left(\frac{[x_{ij}]+[y_{ij}]+[z_{ij}]}{2}\right)-f_n([x_{ij}])-f_n([y_{ij}])-f_n([z_{ij}])\right)\right\|_n$$
$$+\sum_{i,j=1}^{n}\theta\left(\|x_{ij}\|^r+\|y_{ij}\|^r+\|z_{ij}\|^r\right)$$

for all $x=[x_{ij}], y=[y_{ij}], z=[z_{ij}]\in M_n(X)$. Then there exists a unique additive mapping $h:X\to Y$ satisfying

$$\|f_n([x_{ij}])-h_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n}\frac{2^{r-1}\theta}{(1-|\rho|)(2^r-2)}\ \|\ x_{ij}\ \|^r \tag{2.2.3}$$

Proof. Let $n=1$ in (2.2.2). Then we get

$$\left\|2f\left(\frac{a+b}{2}+c\right)-f(a)-f(b)-2f(c)\right\|$$
$$\leq \left\|\rho\left(2f\left(\frac{a+b+c}{2}\right)-f(a)-f(b)-f(c)\right)\right\|+\theta\left(\|a\|^r+\|b\|^r+\|c\|^r\right)$$

for all $a,b,c\in X$.

By [56, Theorem 3.3], there is a unique additive mapping $h:X\to Y$ such that

$$\|f(a)-h(a)\|\leq\frac{2^{r-1}\theta}{(1-|\rho|)(2^r-2)}\|a\|^r$$

for all $a\in X$.

By Lemma 2.1.1,

$$\|f_n([x_{ij}])-h_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n}\frac{2^{r-1}\theta}{(1-|\rho|)(2^r-2)}\ \|\ x_{ij}\ \|^r$$

for all $x=[x_{ij}]\in M_n(X)$. Thus the mapping $h:X\to Y$ is a unique additive mapping satisfying (2.2.3). □

Using the same idea of proof we get the following useful result.

Theorem 2.2.2. Let $r < 1$ and θ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying (2.2.2).

Then there exists a unique mapping $h : X \to Y$ such that

$$\|f_n([x_{ij}]) - h_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{2^r\theta}{(1-|\rho|)(2-2^r)} \parallel x_{ij} \parallel^r \tag{2.2.4}$$

for all $x = [x_{ij}] \in M_n(X)$

Proof. By [56, Theorem 3.4], there is a unique additive mapping $h : X \to Y$ such that

$$\|f(a) - h(a)\| \leq \frac{2^r\theta}{(1-|\rho|)(2-2^r)}\|a\|^r$$

for all $a \in X$.

By Lemma 2.1.1,

$$\|f_n([x_{ij}]) - h_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{2^r\theta}{(1-|\rho|)(2-2^r)} \parallel x_{ij} \parallel^r$$

for all $x = [x_{ij}] \in M_n(X)$. Thus the mapping $h : X \to Y$ is a unique additive mapping satisfying (2.2.4). □

By the triangle inequality, we have

$$\begin{aligned} & \left\|2f_n\left(\frac{[x_{ij}]+[y_{ij}]}{2}+[z_{ij}]\right) - f_n([x_{ij}]) - f_n([y_{ij}]) - 2f_n([z_{ij}])\right\|_n \\ & \quad - \left\|\rho\left(2f_n\left(\frac{[x_{ij}]+[y_{ij}]+[z_{ij}]}{2}\right) - f_n([x_{ij}]) - f_n([y_{ij}]) - f_n([z_{ij}])\right)\right\|_n \\ \leq & \left\|2f_n\left(\frac{[x_{ij}]+[y_{ij}]}{2}+[z_{ij}]\right) - f_n([x_{ij}]) - f_n([y_{ij}]) - 2f_n([z_{ij}])\right. \\ & \quad \left. -\rho\left(2f_n\left(\frac{[x_{ij}]+[y_{ij}]+[z_{ij}]}{2}\right) - f_n([x_{ij}]) - f_n([y_{ij}]) - f_n([z_{ij}])\right)\right\|_n . \end{aligned}$$

We obtain the Hyers-Ulam stability results for the additive ρ-functional equations associated with the additive-ρ-functional inequality (2.2.1) in complex matrix Banach spaces, in the following corollaries.

Corollary 2.2.3. Let $r > 1$ and be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that

$$\begin{aligned} &\left\| 2f_n\left(\frac{[x_{ij}]+[y_{ij}]}{2}+[z_{ij}]\right) - f_n([x_{ij}]) - f_n([y_{ij}]) - 2f_n([z_{ij}]) \right. \\ &\left. -\rho\left(2f_n\left(\frac{[x_{ij}]+[y_{ij}]+[z_{ij}]}{2}\right) - f_n([x_{ij}]) - f_n([y_{ij}]) - f_n([z_{ij}])\right)\right\|_n \\ \le &\sum_{i,j=1}^{n} \theta\left(\| x_{ij} \|^r + \| y_{ij} \|^r + \| z_{ij} \|^r\right) \end{aligned} \tag{2.2.5}$$

for all $x = [x_{ij}], y = [y_{ij}], z = [z_{ij}] \in M_n(X)$. Then there exists a unique additive mapping $h : X \to Y$ satisfying (2.2.2).

Corollary 2.2.4. Let $r < 1$ and be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (2.2.5). Then there exists a unique additive mapping $h : X \to Y$ satisfying (2.2.4).

The above results allow us to give the proof of Hyers-Ulam stability of the following additive ρ-functional inequality,

$$\begin{aligned} &\left\| 2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z) \right\| \\ &\qquad \le \left\| \rho\left(2f\left(\frac{x+y}{2}+z\right) - f(x) - f(y) - 2f(z)\right)\right\| \end{aligned}$$

in complex matrix Banach spaces.

2.3 Conclusion

The primary goal has been to introduce and investigate the additive ρ-functional inequalities in matrix noremd spaces therefore we generalized/extended some results of Kim et al [56] and provided partial improvement to the main results. Moreover, we apply similar construction to derive Hyer-Ulam stability associated with these ρ-functional inequalities. Corollaries and remarks indicate the novelty of the results.

Chapter 3

Matrix generalized (θ, ϕ)-derivation on a matrix Banach algebra

In this chapter the concept of Hyers-Ulam stability of matrix generalized (θ, ϕ)-derivation on matrix Banach algebra is introduced.

3.1 (θ, ϕ)-derivation on a matrix Banach algebra

Choonkil Park and Dong Yun Shin [72] give the concept of generalized (θ, ϕ)-derivations on Banach algebras, and prove the Cauchy-Rassias stability of generalized (θ, ϕ)-derivations on Banach algebras. We extend their work to a matrix Banach algebra.

Throughout this section, let $(X, \{\|\cdot\|_n\})$ be a matrix Banach algebra.

Definition 3.1.1. Let A be a Banach algebra. By a derivation on A, we mean a linear mapping $D : A \to A$, which satisfies $D(ab) = aD(b) + D(a)b$ for all a and b in A.

Definition 3.1.2. Let $\theta, \phi : X \to X$ be additive mappings.

An additive mapping $D : X \to X$ is called a *matrix (θ, ϕ)-derivation* on X if $D_n([x_{ij}][y_{ij}]) = D_n([x_{ij}])\theta_n([y_{ij}]) + \phi_n([x_{ij}])D_n([y_{ij}])$ holds for all $[x_{ij}], [y_{ij}] \in M_n(X)$.

An additive mapping $U : X \to X$ is called a *matrix generalized (θ, ϕ)-derivation* on X if there exists a matrix (θ, ϕ)-derivation $D : X \to X$ such that $U_n([x_{ij}][y_{ij}]) = U_n([x_{ij}])\theta_n([y_{ij}]) + \phi_n([x_{ij}])D_n([y_{ij}])$ holds for all $[x_{ij}], [y_{ij}] \in M_n(X)$.

Theorem 3.1.3. Let $f, g, h, u : X \to X$ be mappings with $f_n([0_{ij}]) = g_n([0_{ij}]) = h_n([0_{ij}]) = u_n([0_{ij}]) = [0_{ij}]$ for which there exists a function $\varphi : X \times X \to [0, \infty)$ such that

$$\widetilde{\varphi_n}([x_{ij}], [y_{ij}]) := \sum_{i,j=1}^{n} (\sum_{r=0}^{\infty} \frac{1}{2^r} \varphi(2^r x_{ij}, 2^r y_{ij})) \qquad < \infty. \tag{3.1.1}$$

$$\|f_n([x_{ij}] + [y_{ij}]) - f_n([x_{ij}]) - f_n([y_{ij}])\|_n \le \sum_{i,j=1}^{n} \varphi(x_{ij}, y_{ij}), \tag{3.1.2}$$

$$\|g_n([x_{ij}] + [y_{ij}]) - g_n([x_{ij}]) - g_n([y_{ij}])\|_n \le \sum_{i,j=1}^{n} \varphi(x_{ij}, y_{ij}), \tag{3.1.3}$$

$$\|h_n([x_{ij}] + [y_{ij}]) - h_n([x_{ij}]) - h_n([y_{ij}])\|_n \le \sum_{i,j=1}^{n} \varphi(x_{ij}, y_{ij}), \tag{3.1.4}$$

$$\|u_n([x_{ij}] + [y_{ij}]) - u_n([x_{ij}]) - u_n([y_{ij}])\|_n \le \sum_{i,j=1}^{n} \varphi(x_{ij}, y_{ij}), \tag{3.1.5}$$

$$\|f_n([x_{ij}][y_{ij}]) - f_n([x_{ij}])g_n([y_{ij}]) - h_n([x_{ij}])f_n([y_{ij}])\|_n \le \sum_{i,j=1}^{n} \varphi(x_{ij}, y_{ij}), \tag{3.1.6}$$

$$\|u_n([x_{ij}][y_{ij}]) - u_n([x_{ij}])g_n([y_{ij}]) - h_n([x_{ij}])f_n([y_{ij}])\|_n \le \sum_{i,j=1}^{n} \varphi(x_{ij}, y_{ij}). \tag{3.1.7}$$

for all $[x_{ij}], [y_{ij}] \in M_n(X)$.

Then there exist unique additive mappings $D, \theta, \phi, U : X \to X$ such that

$$\|f_n([x_{ij}]) - D_n([x_{ij}])\|_n \le \tfrac{1}{2}\widetilde{\varphi_n}([x_{ij}], [x_{ij}]), \tag{3.1.8}$$

$$\|g_n([x_{ij}]) - \theta_n([x_{ij}])\|_n \le \tfrac{1}{2}\widetilde{\varphi_n}([x_{ij}], [x_{ij}]), \tag{3.1.9}$$

$$\|h_n([x_{ij}]) - \phi_n([x_{ij}])\|_n \le \tfrac{1}{2}\widetilde{\varphi_n}([x_{ij}], [x_{ij}]), \tag{3.1.10}$$

$$\|u_n([x_{ij}]) - U_n([x_{ij}])\|_n \le \tfrac{1}{2}\widetilde{\varphi_n}([x_{ij}], [x_{ij}]). \tag{3.1.11}$$

for all $[x_{ij}] \in M_n(X)$.

Moreover, $D : X \to X$ is a matrix (θ, ϕ)-derivation on X, and $U : X \to X$ is a matrix generalized (θ, ϕ)-derivation on X.

Proof. Putting $n = 1$ in (3.1.2), we have

$$\|f(x + y) - f(x) - f(y)\| \le \varphi(x, y)$$

for all $x, y \in X$

By the Găvruta's theorem [40], there exists a unique additive mapping $D : X \to X$ satisfying

$$\|f(x) - D(x)\| \le \frac{1}{2}\widetilde{\varphi}(x, x)$$

for all $x, y \in X$. Varying the values of n, we have

$$\|f_n([x_{ij}]) - D_n([x_{ij}])\|_n \le \frac{1}{2}\widetilde{\varphi_n}([x_{ij}], [x_{ij}]).$$

for all $[x_{ij}], [y_{ij}] \in M_n(X)$.

Similarly, for (3.1.2)-(3.1.5) there exist θ, ϕ and U satisfying (3.1.8)-(3.1.11).

The additive mappings $D, \theta, \phi, U : X \to X$ are given by

$$D(x) = \lim_{l\to\infty} \frac{1}{2^l} f(2^l x), \tag{3.1.12}$$

$$\theta(x) = \lim_{l\to\infty} \frac{1}{2^l} g(2^l x), \tag{3.1.13}$$

$$\phi(x) = \lim_{l\to\infty} \frac{1}{2^l} h(2^l x), \tag{3.1.14}$$

$$U(x) = \lim_{l\to\infty} \frac{1}{2^l} u(2^l x), \tag{3.1.15}$$

for all $x \in X$.

It follows from (3.1.6) that

$$\|f_n([x_{ij}][y_{ij}]) - f_n([x_{ij}])g_n([y_{ij}]) - h_n([x_{ij}])f_n([y_{ij}])\|_n \le \sum_{i,j=1}^{n} \varphi_n([x_{ij}], [y_{ij}])$$

for all $[x_{ij}], [y_{ij}] \in M_n(X)$.

Putting $n = 1$, we get

$$\|f(xy) - f(x)g(y) - h(x)f(y)\| \le \varphi(x, y)$$

for all $x, y \in X$. So

$$\frac{1}{2^{2l}}\|f(2^{2l}xy) - f(2^l x)g(2^l y) - h(2^l x)f(2^l y)\| \le \frac{1}{2^{2l}}\varphi(2^l x, 2^l y) \le \frac{1}{2^l}\varphi(2^l x, 2^l y),$$

$$\frac{1}{2^{2l}}\|f_n(2^{2l}[x_{ij}][y_{ij}]) - f_n(2^l[x_{ij}])g_n(2^l[y_{ij}]) - h_n(2^l[x_{ij}])f_n(2^l[y_{ij}])\|_n$$

$$\le \sum_{i,j=1}^{n} \frac{1}{2^l}\varphi(2^l x_{ij}, 2^l y_{ij}).$$

which tends to zero as $l \to \infty$ for all $x, y \in X$ and $[x_{ij}], [y_{ij}] \in M_n(X)$ by (3.1.1). By (3.1.12)- (3.1.14)

$$D_n([x_{ij}][y_{ij}]) = D_n([x_{ij}])\theta_n([y_{ij}]) + \phi_n([x_{ij}])D_n([y_{ij}])$$

for all $[x_{ij}], [y_{ij}] \in M_n(X)$. So the additive mapping $D : X \to X$ is a matrix (θ, ϕ)-derivation on X.

Similarly, by using (3.1.7), we can show that the additive mapping $U : X \to X$ is a matrix generalized (θ, ϕ)-derivation on X. □

Corollary 3.1.4. Let $f, g, h, u : X \to X$ be mappings with $f_n([0_{ij}]) = g_n([0_{ij}]) = h_n([0_{ij}]) = u_n([0_{ij}]) = [0_{ij}]$ for which there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\begin{aligned}
\|f_n([x_{ij}] + [y_{ij}]) - f_n([x_{ij}]) - f_n([y_{ij}])\|_n &\leq \sum_{i,j=1}^{n} \epsilon(\|x_{ij}\|^p + \|y_{ij}\|^p), \\
\|g_n([x_{ij}] + [y_{ij}]) - g_n([x_{ij}]) - g_n([y_{ij}])\|_n &\leq \sum_{i,j=1}^{n} \epsilon(\|x_{ij}\|^p + \|y_{ij}\|^p), \\
\|h_n([x_{ij}] + [y_{ij}]) - h_n([x_{ij}]) - h_n([y_{ij}])\|_n &\leq \sum_{i,j=1}^{n} \epsilon(\|x_{ij}\|^p + \|y_{ij}\|^p), \\
\|u_n([x_{ij}] + [y_{ij}]) - u_n([x_{ij}]) - u_n([y_{ij}])\|_n &\leq \sum_{i,j=1}^{n} \epsilon(\|x_{ij}\|^p + \|y_{ij}\|^p),
\end{aligned}$$

$$\begin{aligned}
\|f_n([x_{ij}][y_{ij}]) - f_n([x_{ij}])g_n([y_{ij}]) - h_n([x_{ij}])f_n([y_{ij}])\|_n &\leq \sum_{i,j=1}^{n} \epsilon(\|x_{ij}\|^p + \|y_{ij}\|^p), \\
\|u_n([x_{ij}][y_{ij}]) - u_n([x_{ij}])g_n([y_{ij}]) - h_n([x_{ij}])f_n([y_{ij}])\|_n &\leq \sum_{i,j=1}^{n} \epsilon(\|x_{ij}\|^p + \|y_{ij}\|^p).
\end{aligned}$$

for all $[x_{ij}], [y_{ij}] \in M_n(X)$.

Then there exist unique additive mappings $D, \theta, \phi, U : X \to X$ such that

$$\begin{aligned}
\|f_n([x_{ij}]) - D_n([x_{ij}])\|_n &\leq \sum_{i,j=1}^{n} \frac{2\epsilon}{2 - 2^p}\|x_{ij}\|^p, \\
\|g_n([x_{ij}]) - \theta_n([x_{ij}])\|_n &\leq \sum_{i,j=1}^{n} \frac{2\epsilon}{2 - 2^p}\|x_{ij}\|^p, \\
\|h_n([x_{ij}]) - \phi_n([x_{ij}])\|_n &\leq \sum_{i,j=1}^{n} \frac{2\epsilon}{2 - 2^p}\|x_{ij}\|^p, \\
\|u_n([x_{ij}]) - U_n([x_{ij}])\|_n &\leq \sum_{i,j=1}^{n} \frac{2\epsilon}{2 - 2^p}\|x_{ij}\|^p.
\end{aligned}$$

for all $[x_{ij}] \in M_n(X)$.

Moreover, $D : X \to X$ is a matrix (θ, ϕ)-derivation on X, and $U : X \to X$ is a matrix generalized (θ, ϕ)-derivation on X.

Proof. Defining $\varphi(x, y) = \epsilon(\|x\|^p + \|y\|^p)$, and applying Theorem 3.1.3, we get the desired result. □

Corollary 3.1.5. Let $\theta, \phi : X \to X$ be additive mappings. Let $f, u : X \to X$ be mappings with $f_n([0_{ij}]) = u_n([0_{ij}]) = [0_{ij}]$ for which there exist a function $\varphi : X \times X \to [0, \infty)$ satisfying (3.1.1), (3.1.2) and (3.1.5) such that

$$\|f_n([x_{ij}][y_{ij}]) - f_n([x_{ij}])\theta_n([y_{ij}]) - \phi_n([x_{ij}])f_n([y_{ij}])\|_n \le \sum_{i,j=1}^{n} \varphi(x_{ij}, y_{ij}) \tag{3.1.16}$$

$$\|u_n([x_{ij}][y_{ij}]) - u_n([x_{ij}])\theta_n([y_{ij}]) - \phi_n([x_{ij}])f_n([y_{ij}])\|_n \le \sum_{i,j=1}^{n} \varphi(x_{ij}, y_{ij}) \tag{3.1.17}$$

for all $[x_{ij}], [y_{ij}] \in M_n(X)$.

Then there exist a unique matrix (θ, ϕ)-derivation $D : X \to X$ satisfying (3.1.8), and there exist a unique matrix generalized (θ, ϕ)-derivation $U : X \to X$ satisfying (3.1.11).

Proof. Letting $\theta = g$ and $\varphi = h$ in the statement of Theorem 3.1.3, we get the result. □

Theorem 3.1.6. Let $f, g, h, u : X \to X$ be mappings with $f_n([0_{ij}]) = g_n([0_{ij}]) = h_n([0_{ij}]) = u_n([0_{ij}]) = [0_{ij}]$ for which there exists a function $\varphi : X \times X \to [0, \infty)$ satisfying (3.1.6), (3.1.7) and

$$\widetilde{\varphi_n}([x_{ij}], [y_{ij}]) := \sum_{i,j=1}^{n} \sum_{r=0}^{\infty} \frac{1}{3^r} \varphi(3^r x_{ij}, 3^r y_{ij}) \qquad < \infty, \tag{3.1.18}$$

$$\left\|2f_n\left(\frac{[x_{ij}] + [y_{ij}]}{2}\right) - f_n([x_{ij}]) - f_n([y_{ij}])\right\|_n \le \sum_{i,j=1}^{n} \varphi(x_{ij}, y_{ij}), \tag{3.1.19}$$

$$\left\|2g_n\left(\frac{[x_{ij}] + [y_{ij}]}{2}\right) - g_n([x_{ij}]) - g_n([y_{ij}])\right\|_n \le \sum_{i,j=1}^{n} \varphi(x_{ij}, y_{ij}), \tag{3.1.20}$$

$$\left\|2h_n\left(\frac{[x_{ij}] + [y_{ij}]}{2}\right) - h_n([x_{ij}]) - h_n([y_{ij}])\right\|_n \le \sum_{i,j=1}^{n} \varphi(x_{ij}, y_{ij}), \tag{3.1.21}$$

$$\left\|2u_n\left(\frac{[x_{ij}] + [y_{ij}]}{2}\right) - u_n([x_{ij}]) - u_n([y_{ij}])\right\|_n \le \sum_{i,j=1}^{n} \varphi(x_{ij}, y_{ij}). \tag{3.1.22}$$

for all $[x_{ij}], [y_{ij}] \in M_n(X)$

Then there exist unique additive mappings $D, \theta, \phi, U : X \to X$ such that

$$\|f_n([x_{ij}]) - D_n([x_{ij}])\|_n \le \frac{1}{3}\big(\widetilde{\varphi_n}([x_{ij}], -[x_{ij}]) + \widetilde{\varphi}(-[x_{ij}], 3[x_{ij}])\big), \tag{3.1.23}$$

$$\|g_n([x_{ij}]) - \theta_n([x_{ij}])\|_n \le \frac{1}{3}\big(\widetilde{\varphi_n}([x_{ij}], -[x_{ij}]) + \widetilde{\varphi}(-[x_{ij}], 3[x_{ij}])\big), \tag{3.1.24}$$

$$\|h_n([x_{ij}]) - \varphi_n([x_{ij}])\|_n \le \frac{1}{3}\big(\widetilde{\varphi_n}([x_{ij}], -[x_{ij}]) + \widetilde{\varphi}(-[x_{ij}], 3[x_{ij}])\big), \tag{3.1.25}$$

$$\|u_n([x_{ij}]) - U_n([x_{ij}])\|_n \le \frac{1}{3}\big(\widetilde{\varphi_n}([x_{ij}], -[x_{ij}]) + \widetilde{\varphi}(-[x_{ij}], 3[x_{ij}])\big). \tag{3.1.26}$$

for all $[x_{ij}] \in M_n(X)$.

Moreover, $D : X \to X$ is a matrix (θ, ϕ)-derivation on X, and $U : X \to X$ is a matrix generalized (θ, ϕ)-derivation on X.

Proof. Putting $n = 1$ in (3.1.19), we get

$$\|2f(\frac{x+y}{2}) - f(x) - f(y)\| \le \varphi(x, y)$$

for all $x, y \in X$. By the Jun and Lee's theorem [52, Theorem 1], there exist a unique additive mapping $D : X \to X$

$$\|f(x) - D(x)\| \le \frac{1}{3}\big(\widetilde{\varphi}(x, -x) + \widetilde{\varphi}(-x, 3x)\big)$$

for all $x \in X$. Varing the values of n, we have

$$\|f_n([x_{ij}]) - D_n([x_{ij}])\|_n \le \frac{1}{3}\big(\widetilde{\varphi_n}([x_{ij}], -[x_{ij}]) + \widetilde{\varphi_n}(-[x_{ij}], 3[x_{ij}])\big)$$

for all $[x_{ij}], [y_{ij}] \in M_n(X)$.

Similarly, there exist θ, ϕ and U satisfying (3.1.23)-(3.1.26).

The additive mappings $D, \theta, \phi, U : X \to X$ are given by

$$D(x) = \lim_{l\to\infty} \frac{1}{3^l} f(3^l x), \tag{3.1.27}$$

$$\theta(x) = \lim_{l\to\infty} \frac{1}{3^l} g(3^l x), \tag{3.1.28}$$

$$\phi(x) = \lim_{l\to\infty} \frac{1}{3^l} h(3^l x), \tag{3.1.29}$$

$$U(x) = \lim_{l\to\infty} \frac{1}{3^l} u(3^l x), \tag{3.1.30}$$

for all $x \in X$.

It follows from (3.1.6) that

$$\|f_n([x_{ij}][y_{ij}]) - f_n([x_{ij}])g_n([y_{ij}]) - h_n([x_{ij}])f_n([y_{ij}])\|_n \leq \sum_{i,j=1}^{n} \varphi(x_{ij}, y_{ij})$$

for all $[x_{ij}], [y_{ij}] \in M_n(X)$.

Putting $n = 1$ in the above inequality, we get

$$\|f(xy) - f(x)g(y) - h(x)f(y)\| \leq \varphi(x, y)$$

for all $x, y \in X$. So

$$\begin{aligned}
&\frac{1}{3^{2l}}\|f(3^{2l}xy) - f(3^l x)g(3^l y) - h(3^l x)f(3^l y)\| \leq \frac{1}{3^{2l}}\varphi(3^l x, 3^l y) \leq \frac{1}{3^l}\varphi(3^l x, 3^l y), \\
&\frac{1}{3^{2l}}\|f_n(3^{2l}[x_{ij}][y_{ij}]) - f_n(3^l[x_{ij}])g_n(3^l[y_{ij}]) - h_n(3^l[x_{ij}])f_n(3^l[y_{ij}])\|_n \\
&\leq \sum_{i,j=1}^{n} \frac{1}{3^{2l}}\varphi(3^l x, 3^l y) \leq \sum_{i,j=1}^{n} \frac{1}{3^l}\varphi(3^l x, 3^l y),
\end{aligned}$$

which tends to zero as $l \to \infty$ for all $x, y \in X$ and $[x_{ij}], [y_{ij}] \in M_n(X)$ by (3.1.18). By (3.1.27)-(3.1.30),

$$D_n([x_{ij}][y_{ij}]) = D_n([x_{ij}])\theta_n([y_{ij}]) + \phi_n([x_{ij}])D_n([x_{ij}])$$

for all $[x_{ij}], [y_{ij}] \in M_n(X)$.

Hence the additive mapping $D : X \to X$ is a matrix (θ, ϕ)-derivation on X.

Similarly, by using (3.1.7) we can show that the additive mapping U is a matrix generalized (θ, ϕ)-derivation on X. □

Corollary 3.1.7. Let $f, g, h, u : X \to X$ be mappings with $f_n([0_{ij}]) = g_n([0_{ij}]) = h_n([0_{ij}]) = u_n([0_{ij}]) = [0_{ij}]$ for which there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\begin{aligned}
\|2f_n(\frac{[x_{ij}] + [y_{ij}]}{2}) - f_n([x_{ij}]) - f_n([y_{ij}])\|_n &\leq \sum_{i,j=1}^{n} \epsilon(\|x_{ij}\|^p + \|x_{ij}\|^p), \\
\|2g_n(\frac{[x_{ij}] + [y_{ij}]}{2}) - g_n([x_{ij}]) - g_n([y_{ij}])\|_n &\leq \sum_{i,j=1}^{n} \epsilon(\|x_{ij}\|^p + \|y_{ij}\|^p), \\
\|2h_n(\frac{[x_{ij}] + [y_{ij}]}{2}) - h_n([x_{ij}]) - h_n([y_{ij}])\|_n &\leq \sum_{i,j=1}^{n} \epsilon(\|x_{ij}\|^p + \|y_{ij}\|^p), \\
\|2u_n(\frac{[x_{ij}] + [y_{ij}]}{2}) - u_n([x_{ij}]) - u_n([y_{ij}])\|_n &\leq \sum_{i,j=1}^{n} \epsilon(\|x_{ij}\|^p + \|y_{ij}\|^p).
\end{aligned}$$

$$\|f_n([x_{ij}][y_{ij}]) - f_n([x_{ij}])g_n([y_{ij}]) - h_n([x_{ij}])f_n([y_{ij}])\|_n \leq \sum_{i,j=1}^{n} \epsilon(\|x_{ij}\|^p + \|y_{ij}\|^p),$$
$$\|u_n([x_{ij}][y_{ij}]) - u_n([x_{ij}])g_n([y_{ij}]) - h_n([x_{ij}])f_n([y_{ij}])\|_n \leq \sum_{i,j=1}^{n} \epsilon(\|x_{ij}\|^p + \|y_{ij}\|^p).$$

for all $[x_{ij}], [y_{ij}] \in M_n(X)$.

Then there exist unique additive mappings $D, \theta, \phi, U : X \to X$ such that

$$\|f_n([x_{ij}]) - D_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{3+3^p}{3-3^p}\epsilon\|x_{ij}\|^p,$$
$$\|g([x_{ij}]) - \theta([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{3+3^p}{3-3^p}\epsilon\|[x_{ij}]\|^p,$$
$$\|h_n([x_{ij}]) - \phi_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{3+3^p}{3-3^p}\epsilon\|x_{ij}\|^p,$$
$$\|u_n([x_{ij}]) - U_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{3+3^p}{3-3^p}\epsilon\|x_{ij}\|^p.$$

for all $[x_{ij}] \in M_n(X)$.

Moreover, $D : X \to X$ is a matrix (θ, ϕ)-derivation on X, and $U : X \to X$ is a matrix generalized (θ, ϕ)-derivation on X.

Proof. Defining $\varphi(x, y) = \epsilon(\|x\|^p + \|y\|^p)$, and applying Theorem 3.1.5, we get the desired result. □

Corollary 3.1.8. Let $\theta, \phi : X \to X$ be additive mappings. Let $f, u : X \to X$ be mappings with $f_n([0_{ij}]) = u_n([0_{ij}]) = [0_{ij}]$ for which there exist a function $\varphi : X \times X \to [0, \infty)$ satisfying (3.1.18),(3.1.19),(3.1.22),(3.1.16) and (3.1.17). Then there exist a unique matrix (θ, ϕ)-derivation $D : X \to X$ satisfying (3.2.23), and there exist a unique matrix generalized (θ, ϕ)-derivation $U : X \to X$ satisfying (3.1.26).

Proof. Letting $\theta = g$ and $\phi = h$ in the statement of Theorem 3.1.5, we get the result. □

Theorem 3.1.9. Let $f, g, h, u : X \to X$ be mappings with $f_n([0_{ij}]) = g_n([0_{ij}]) = h_n([0_{ij}]) = u_n([0_{ij}]) = [0_{ij}]$ for which there exist a function $\varphi : X \times X \to [0, \infty)$ satisfying (3.1.19)-(3.1.22), (3.1.6) and (3.1.7) such that

$$\widetilde{\varphi_n}([x_{ij}], [y_{ij}]) := \sum_{i,j=1}^{n} (\sum_{r=0}^{\infty} 3^{2r} \varphi(\frac{x_{ij}}{3^r}, \frac{y_{ij}}{3^r}) < \infty, \qquad (3.1.31)$$

for all $[x_{ij}], [y_{ij}] \in M_n(X)$.

Then there exist unique additive mappings $D, \theta, \phi, U : X \to X$ such that

$$\|f_n([x_{ij}]) - D_n([x_{ij}])\|_n \le \widetilde{\varphi}(\frac{[x_{ij}]}{3}, -\frac{[x_{ij}]}{3}) + \widetilde{\varphi}(-\frac{[x_{ij}]}{3}, [x_{ij}]), \quad (3.1.32)$$

$$\|g_n([x_{ij}]) - \theta_n([x_{ij}])\|_n \le \widetilde{\varphi}(\frac{[x_{ij}]}{3}, -\frac{[x_{ij}]}{3}) + \widetilde{\varphi}(-\frac{[x_{ij}]}{3}, [x_{ij}]), \quad (3.1.33)$$

$$\|h_n([x_{ij}]) - \phi_n([x_{ij}])\|_n \le \widetilde{\varphi}(\frac{[x_{ij}]}{3}, -\frac{[x_{ij}]}{3}) + \widetilde{\varphi}(-\frac{[x_{ij}]}{3}, [x_{ij}]), \quad (3.1.34)$$

$$\|u_n([x_{ij}]) - U_n([x_{ij}])\|_n \le \widetilde{\varphi}(\frac{[x_{ij}]}{3}, -\frac{[x_{ij}]}{3}) + \widetilde{\varphi}(-\frac{[x_{ij}]}{3}, [x_{ij}])). \quad (3.1.35)$$

for all $[x_{ij}] \in M_n(X)$.

Moreover, $D : X \to X$ is a matrix (θ, ϕ)-derivation on X, and $U : X \to X$ is a matrix generalized (θ, ϕ)-derivation on X.

Proof. By the Jun and Lee's theorem [52, Theorem 7], we can show that there exist unique additive mappings $D, \theta, \phi, U : B \to B$ satisfying (3.1.32)–(3.1.35).

The additive mappings $D, \theta, \phi, U : X \to X$ are given by

$$\begin{aligned} D(x) &= \lim_{l\to\infty} 3^l f(\frac{x}{3^l}), \\ \theta(x) &= \lim_{l\to\infty} 3^l g(\frac{x}{3^l}), \\ \phi(x) &= \lim_{l\to\infty} 3^l h(\frac{x}{3^l}), \\ U(x) &= \lim_{l\to\infty} 3^l u(\frac{x}{3^l}). \end{aligned}$$

for all $x \in X$.

It follows from (3.1.6) that

$$\|f_n([x_{ij}][y_{ij}]) - f_n([x_{ij}])g_n([y_{ij}]) - h_n([x_{ij}])f_n([y_{ij}])\|_n \le \varphi([x_{ij}], [y_{ij}])$$

for all $[x_{ij}], [y_{ij}] \in M_n(X)$.

Putting $n = 1$ in the above inequality, we get

$$\|f(xy) - f(x)g(y) - h(x)f(y)\| \le \varphi(x, y)$$

for all $x, y \in X$. So

$$3^{2l}\|f(\frac{xy}{3^{2l}}) - f(\frac{x}{3^l})g(\frac{y}{3^l}) - h(\frac{x}{3^l})f(\frac{y}{3^l})\| \le 3^{2l}\varphi(\frac{x}{3^l}, \frac{y}{3^l}).$$

$$3^{2l}\|f_n(\frac{[x_{ij}][y_{ij}]}{3^{2l}}) - f_n(\frac{[x_{ij}]}{3^l})g_n(\frac{[y_{ij}]}{3^l}) - h_n(\frac{[x_{ij}]}{3^l})f_n(\frac{[y_{ij}]}{3^l})\|_n$$
$$\le \sum_{i,j=1}^{n} 3^{2l}\varphi(\frac{x_{ij}}{3^l}, \frac{y_{ij}}{3^l}),$$

which tends to zero as $l \to \infty$ for all $x, y \in X$ and $[x_{ij}], [y_{ij}] \in M_n(X)$ by (3.1.31). By (3.1.36)-(3.1.39),

$$D_n([x_{ij}][y_{ij}]) = D_n([x_{ij}])\theta_n([y_{ij}]) + \phi_n([x_{ij}])D_n([y_{ij}])$$

for all $[x_{ij}], [y_{ij}] \in M_n(X)$.

So the additive mapping $D : X \to X$ is a matrix (θ, ϕ)-derivation on X.

Similarly, by using (3.1.7), we can show that the additive mapping $U : X \to X$ is a matrix generalized (θ, ϕ)-derivation on X. □

Corollary 3.1.10. Let $f, g, h, u : X \to X$ be mappings with $f_n([0_{ij}]) = g_n([0_{ij}]) = h_n([0_{ij}]) = u_n([0_{ij}]) = [0_{ij}]$ for which there exist constants $\epsilon \ge 0$ and $p \in (2, \infty)$ such that

$$\begin{aligned}
\|2f_n(\frac{[x_{ij}]+[y_{ij}]}{2}) - f_n([x_{ij}]) - f_n([y_{ij}])\|_n &\le \sum_{i,j=1}^{n} \epsilon(\|x_{ij}\|^p + \|x_{ij}\|^p),\\
\|2g_n(\frac{[x_{ij}]+[y_{ij}]}{2}) - g_n([x_{ij}]) - g_n([y_{ij}])\|_n &\le \sum_{i,j=1}^{n} \epsilon(\|x_{ij}\|^p + \|y_{ij}\|^p),\\
\|2h_n(\frac{[x_{ij}]+[y_{ij}]}{2}) - h_n([x_{ij}]) - h_n([y_{ij}])\|_n &\le \sum_{i,j=1}^{n} \epsilon(\|x_{ij}\|^p + \|y_{ij}\|^p),\\
\|2u_n(\frac{[x_{ij}]+[y_{ij}]}{2}) - u_n([x_{ij}]) - u_n([y_{ij}])\|_n &\le \sum_{i,j=1}^{n} \epsilon(\|x_{ij}\|^p + \|y_{ij}\|^p),\\
\|f_n([x_{ij}][y_{ij}]) - f_n([x_{ij}])g_n([y_{ij}]) - h_n([x_{ij}])f_n([y_{ij}])\|_n &\le \sum_{i,j=1}^{n} \epsilon(\|x_{ij}\|^p + \|y_{ij}\|^p),\\
\|u_n([x_{ij}][y_{ij}]) - u_n([x_{ij}])g_n([y_{ij}]) - h_n([x_{ij}])f_n([y_{ij}])\|_n &\le \sum_{i,j=1}^{n} \epsilon(\|x_{ij}\|^p + \|y_{ij}\|^p).
\end{aligned}$$

for all $[x_{ij}], [y_{ij}] \in M_n(X)$.

Then there exist unique additive mappings $D, \theta, \phi, U : X \to X$ such that

$$\begin{aligned}
\|f_n([x_{ij}]) - D_n([x_{ij}])\|_n &\leq \sum_{i,j=1}^{n} \frac{3^p+3}{3^p-3}\epsilon\|x_{ij}\|^p, \\
\|g_n([x_{ij}]) - \theta_n([x_{ij}])\|_n &\leq \sum_{i,j=1}^{n} \frac{3^p+3}{3^p-3}\epsilon\|x_{ij}\|^p, \\
\|h_n([x_{ij}]) - \phi_n([x_{ij}])\|_n &\leq \sum_{i,j=1}^{n} \frac{3^p+3}{3^p-3}\epsilon\|x_{ij}\|^p, \\
\|u_n([x_{ij}]) - U_n([x_{ij}])\|_n &\leq \sum_{i,j=1}^{n} \frac{3^p+3}{3^p-3}\epsilon\|x_{ij}\|^p
\end{aligned}$$

for all $[x_{ij}] \in M_n(X)$.

Moreover, $D : X \to X$ is a matrix (θ, ϕ)-derivation on X, and $U : X \to X$ is a matrix generalized (θ, ϕ)-derivation on X.

Proof. Defining $\varphi(x, y) = \epsilon(\|x\|^p + \|y\|^p)$, and applying Theorem 3.1.8, we get the desired result. □

Corollary 3.1.11. Let $\theta, \phi : X \to X$ be additive mappings. Let $f, u : X \to X$ be mappings with $f_n([0_{ij}]) = u_n([0_{ij}]) = [0_{ij}]$ for which there exists a function $\varphi : X \times X \to [0, \infty)$ satisfying (3.1.31), (3.1.19), (3.1.22), (3.1.16) and (3.1.17). Then there exists a unique matrix (θ, ϕ)-derivation $D : X \to X$ satisfying (3.1.32), and there exists a unique matrix generalized (θ, ϕ)-derivation $U : X \to X$ satisfying (3.1.35).

Proof. Letting $\theta = g$ and $\phi = h$ in the statement of Theorem 3.1.8, we get the result. □

3.2 Conclusion

The purpose of this study is stipulated with the generalisation of the concept of (θ, ϕ)-derivation on Banach algebras given by Choonkil Park and Dong Yun Shin [72]. We established the existence and uniqueness of solutions to functions in the form of additive mappings. This approach is particularly associated with the work of Jun and Lee [52], although it was Găvruta [40] who indroduced this idea. We generalized the stability of approximately additive mappings in the spirit of Hyers, Ulam and Rassias.

Chapter 4

Generalized φ-weak contractive fuzzy mappings

Existence theorems of fixed points have been established for mappings defined on various types of spaces and satisfying different types of contractive inequalities. The notion of fuzzy sets was introduced by Zadeh [95] in 1965. Following this initial result, Weiss [92] and Butnariu [21] studied on the characterization of several notion in the sense of fuzzy numbers. Heilpern [46] introduced the fuzzy mapping and further he established the fuzzy Banach contraction principle on a complete metric space. Subsequently several other researchers studied the existence of fixed points and common fixed points of fuzzy mappings satisfying a contractive type condition on a metric space (see [?], [2], [4], [9], [18], [61], [74], [93]).

In this chapter we prove the existence and uniqueness of a (common) fixed point of generalized φ-weak contractive fuzzy mappings on complete metric spaces. We present some examples to illustrate the obtained results.

4.1 Fixed point theorems for fuzzy φ-weak contractive.

Throughout this section (X, d) be a complete metric space.

$$\begin{aligned} \xi^X &= \{A : A \text{ is the subset of } X\}, \\ B(\xi^X) &= \{A \in \xi^X : A \text{ is nonempty bounded}\}, \\ CB(\xi^X) &= \{A \in \xi^X : A \text{ is nonempty closed and bounded}\}. \end{aligned}$$

(X) be the collection of all fuzzy sets in a metric space X.

Theorem 4.1.1. Let (X, d) be a complete metric space and $S, T : X \to (X)$ and, for $x \in X$, there exist $\alpha_S(x), \alpha_T(x) \in (0, 1]$ such that $[Sx]_{\alpha_S(x)}, [Tx]_{\alpha_T(x)} \in B(\xi^X)$, such that for all $x, y \in X$,

$$\delta\left([Sx]_{\alpha_S(x)}, [Ty]_{\alpha_T(y)}\right) \leq M(x, y) - \varphi(M(x, y)) \tag{4.1.1}$$

where, $\varphi : [0, \infty) \to [0, \infty)$ is a lower semicontinous function with $\varphi(t) > 0$ for $t \in (0, \infty)$ and $\varphi(0) = 0$, $[Sx]_{\alpha_S(x)}$ and $[Ty]_{\alpha_T(y)}$ are level sets of Sx and Ty, defined in the section (1.3.2).

$$M(x, y) = \max \left\{ \begin{array}{c} d(x, y), D(x, [Sx]_{\alpha_S(x)}), D(y, [Ty]_{\alpha_T(y)}), \\ \frac{1}{2}\left[D(y, [Sx]_{\alpha_S(x)}) + D(x, [Ty]_{\alpha_T(y)})\right] \end{array} \right\} \tag{4.1.2}$$

Then there exists a unique $z \in [Sx]_{\alpha_S(x)}$ and $z \in [Tx]_{\alpha_T(x)}$.

Proof. Take $a_0 \in X$. According to the given condition, there exists an $\alpha(a_0) \in (0, 1]$ such that $[Sa_0]_{\alpha(a_0)} \in CB(\xi^X)$.

Let us denote $\alpha(x_0)$ by α_1 and set $a_1 \in [Sa_0]_{\alpha(a_0)}$. For this a_1 there exists an $\alpha_2 \in (0, 1]$ such that, $[Ta_1]_{\alpha_2} \in CB(\xi^X)$. Iteratively, a sequence $\{a_n\}$ in X is constructed so that

$$\begin{aligned} a_{2k+1} &\in [Sa_{2k}]_{\alpha_{2k+1}}, \\ a_{2k+2} &\in [Ta_{2k+1}]_{\alpha_{2k+2}} \end{aligned}$$

It is clear that if $M(a_n, a_{n+1}) = 0$, then the proof is complete.

Consequently, throughout the proof, it is assumed that

$$M(a_n, a_{n+1}) > 0 \text{ for all } n \geq 0. \tag{4.1.3}$$

We shall prove that

$$d(a_{2n+1}, a_{2n+2}) \leq d(a_{2n}, a_{2n+1}) \ for \ all \ n \geq 0. \tag{4.1.4}$$

Suppose, on the contrary, that there exists an $\check{n} \geq 0$ such that

$$d(a_{2\check{n}+1}, a_{2\check{n}+2}) > d(a_{2\check{n}}, a_{2\check{n}+1}),$$

which yields the inequality

$$M(a_{2\check{n}}, a_{2\check{n}+1}) \leq d(a_{2\check{n}+1}, a_{2\check{n}+2}).$$

Regarding (4.1.1), one can derive that

$$\begin{aligned} d(a_{2\check{n}_0+1}, a_{2\check{n}+2}) &\leq \delta([Sa_{2\check{n}}]_{\alpha(a_{2\check{n}})}, [Ta_{2\check{n}+1}]_{\alpha(a_{2\check{n}+1})}) \\ &\leq M(a_{2n}, a_{2n+1}) - \varphi(M(a_{2\check{n}}, a_{2\check{n}+1})) \\ &\leq d(a_{2\check{n}}, a_{2\check{n}+1}) - \varphi(M(a_{2\check{n}}, a_{2\check{n}+1})). \end{aligned}$$

Consequently, $\varphi(M(a_{2\check{n}}, a_{2\check{n}+1})) = 0$ is obtained and so $M(a_{2\check{n}}, a_{2\check{n}+1}) = 0$.

This contradicts the observation (4.1.3). Hence the inequality (4.1.4) is satisfied.

In an analogous way, one can conclude that

$$d(a_{2n+2}, a_{2n+3}) \leq d(a_{2n+1}, a_{2n+2}) \ for \ all \ n \geq 0. \tag{4.1.5}$$

By combining (4.1.4) and (4.1.5), we get that

$$d(a_{n+1}, a_{n+2}) \leq d(a_n, a_{n+1}) \ for \ all \ n \geq 0.$$

Hence It is proved that the sequence $\{d(a_n, a_{n+1})\}$ is non-increasing and bounded below. Since (X, d) is complete, there exists an $l \geq 0$ such that

$$\lim_{n \to \infty} d(a_n, a_{n+1}) = l. \tag{4.1.6}$$

Due to the hypothesis, it is observed that

$$\begin{aligned} & d(a_{2n}, a_{2n+1}) \leq M(a_{2n}, a_{2n+1}) \\ & = max \left\{ \begin{array}{c} d(a_{2n}, a_{2n+1}), D(a_{2n}, [Sa_{2n}]_{\alpha(a_{2n})}), D(a_{2n+1}, [Ta_{2n+1}]_{\alpha(a_{2n+1})}), \\ \frac{1}{2}\left[D(a_{2n+1}, [Sa_{2n}]_{\alpha(a_{2n})}) + D(a_{2n}, [Ta_{2n+1}]_{\alpha(a_{2n+1})})\right] \end{array} \right\} \\ & \leq max \left\{ d(a_{2n}, a_{2n+1}), d(a_{2n+1}, a_{2n+2}), \frac{1}{2}[d(a_{2n}, a_{2n+1}) + d(a_{2n+1}, a_{2n+2})] \right\}. \end{aligned}$$

Thus,

$$l \leq \lim_{n\to\infty} M(a_{2n}, a_{2n+1}) \leq l.$$

Hence we get

$$\lim_{n\to\infty} M(a_{2n}, a_{2n+1}) = l. \tag{4.1.7}$$

Analogously,

$$\lim_{n\to\infty} M(a_{2n+1}, a_{2n+2}) = l. \tag{4.1.8}$$

By combining (4.1.6), (4.1.7) and (4.1.8), the following inequality is established

$$\lim_{n\to\infty} d(a_n, a_{n+1}) = \lim_{n\to\infty} M(a_n, a_{n+1}) = l.$$

By the lower semi-continuity of φ,

$$\varphi(l) \leq \liminf_{n\to\infty} \varphi M(a_n, a_{n+1})).$$

Now we claim that $l = 0$. From (4.1.1),

$$\begin{aligned} d(a_{2n+1}, a_{2n+2}) &\leq \delta([Sa_{2n}]_{\alpha(a_{2n})}, [Ta_{2n+1}]_{\alpha(a_{2n+1})}) \\ &\leq M(a_{2n}, a_{2n+1}) - \varphi(M(a_{2n}, a_{2n+1}) \end{aligned}$$

By letting the upper limit as $n \to \infty$ in the above inequality above, we obtain

$$\begin{aligned} l &\leq l - \liminf_{n\to\infty} \varphi(M(a_{2n}, a_{2n+1})) \\ &\leq l - \varphi(l), \end{aligned}$$

that is, $\varphi(l) = 0$.
Regarding the property of φ, finally get $l = 0$.

As a next step, we shall show that $\{a_n\}$ is Cauchy. For this purpose, it is sufficient to show that $\{a_{2n}\}$ is Cauchy.
Suppose on contrary, that $\{a_{2n}\}$ is not Cauchy. Then there is an $\epsilon > 0$ such that for an even integer $2k$ there exist even integers $2m(k) > 2n(k) > 2k$ such that

$$d(a_{2n(k)}, a_{2m(k)}) > \epsilon. \tag{4.1.9}$$

For every even integer $2k$, let $2m(k)$ be the least positive integer exceeding $2n(k)$ satisfying (4.1.9), and such that

$$d(a_{2n(k)}, a_{2m(k)-2}) < \epsilon. \tag{4.1.10}$$

Now

$$\begin{aligned}\epsilon &\leq d(a_{2n(k)}, a_{2m(k)}) \\ &\leq d(a_{2n(k)}, a_{2m(k)-2}) + d(a_{2m(k)-2}, a_{2m(k)-1}) \\ &\quad + d(a_{2m(k)-1}, a_{2m(k)}).\end{aligned}$$

By (4.1.9) and (4.1.10),

$$\lim_{k\to\infty} d(a_{2n(k)}, a_{2m(k)}) = \epsilon. \tag{4.1.11}$$

Using the triangle inequality, we have

$$\left|d(a_{2n(k)}, a_{2m(k)-1}) - d(a_{2n(k)}, a_{2m(k)})\right| < d(a_{2m(k)-1}, a_{2m(k)}).$$

By (4.1.11),

$$d(a_{2n(k)}, a_{2m(k)-1}) = \epsilon. \tag{4.1.12}$$

Now by (4.1.2),

$$\begin{aligned}
&d(a_{2n(k)}, a_{2m(k)-1}) \leq M(a_{2n(k)}, a_{2m(k)-1}) \\
&= \max\left\{\begin{array}{c} d(a_{2n(k)}, a_{2m(k)-1}), D(a_{2n(k)}, [Sa_{2n(k)}]_{\alpha(a_{2n(k)})}), \\ D(a_{2m(k)-1}, [Ta_{2m(k)-1}]_{\alpha(a_{2m(k)-1})}), \\ \frac{1}{2}D(a_{2m(k)-1}, [Sa_{2n(k)}]_{\alpha\left(a_{2n(k)}\right)}) + D(a_{2n(k)}, [Ta_{2m(k)-1}]_{\alpha\left(a_{2m(k)-1}\right)})\} \end{array}\right\} \\
&\leq \max\left\{\begin{array}{c} d(a_{2n(k)}, a_{2m(k)-1}), d(a_{2n(k)}, a_{2n(k)+1}), d(a_{2m(k)-1}, a_{2m(k)}) \\ \frac{1}{2}[d(a_{2m(k)-1}, a_{2n(k)+1} + d(a_{2n(k)}, a_{2m(k)})] \end{array}\right\} \\
&\leq \max\left\{\begin{array}{c} d(a_{2n(k)}, a_{2m(k)-1}), d(a_{2n(k)}, a_{2n(k)+1}), d(a_{2m(k)-1}, a_{2m(k)}) \\ \frac{1}{2}\left[[d(a_{2m(k)-1}, a_{2n(k)}) + d(a_{2n(k)}, a_{2n(k)+1}) + d(a_{2n(k)}, a_{2m(k)})\right] \end{array}\right\}.
\end{aligned}$$

By letting $k \to \infty$ in the above inequality and taking (4.1.11) and (4.1.12) into account, we conclude that

$$\varepsilon \leq \lim_{k\to\infty} M(x_{2n(k)}, x_{2m(k)-1}) \leq \varepsilon.$$

Consequently,

$$\lim_{k\to\infty} M(x_{2n(k)}, x_{2m(k)-1}) = \varepsilon.$$

By the lower semi-continuity of φ, we derive that

$$\varphi(\varepsilon) \leq \lim_{k\to\infty}\inf \varphi(M(x_{2n(k)}, x_{2m(k)-1})).$$

Now by (4.1.1),

$$\begin{aligned}
&d(x_{2n(k)}, x_{2m(k)})\\
&\leq d(x_{2n(k)}, x_{2n(k)+1}) + \delta([Sx_{2n(k)}]_{\alpha(x_{2n(k)})}, [Tx_{2m(k)-1}]_{\alpha(x_{2m(k)-1})})\\
&\leq d(x_{2n(k)}, x_{2n(k)+1}) + M(x_{2n(k)}, x_{2m(k)-1}) - \varphi(M(x_{2n(k)}, x_{2m(k)-1})).
\end{aligned}$$

Letting the upper limit $k \to \infty$ in the above inequality,

$$\begin{aligned}
\epsilon &\leq \epsilon - \lim_{k\to\infty}\inf \varphi(M(a_{2n(k)}, a_{2m(k)-1}))\\
&\leq \epsilon - \varphi(\epsilon),
\end{aligned}$$

which is a contradiction.

Hence $\{a_{2n}\}$ is a Cauchy sequence. It follows from the completeness of X that there exists a $c \in X$ such that $a_n \to c$ as $n \to \infty$. Furthermore, $a_{2n} \to c$ and $a_{2n+1} \to c$.

We shall prove that $c \in [Sc]_{\alpha_S(c)}$.

$$\begin{aligned}
&D(c, [Sc]_{\alpha_S(c)}) \leq M(c, a_{2n-1})\\
&= \max\left\{\begin{array}{c} d(c, a_{2n-1}), D(c, [Sc]_{\alpha_S(c)}), D(a_{2n-1}, [Ta_{2n-1}]_{\alpha_T(a_{2n-1})}),\\ \frac{1}{2}[D(a_{2n-1}, [Sc]_{\alpha_S(c)}) + D(c, [Ta_{2n-1}]_{\alpha_T(a_{2n-1})})] \end{array}\right\}\\
&\leq \max\left\{\begin{array}{c} d(c, a_{2n-1}), D(c, [Sc]_{\alpha_S(c)}), d(a_{2n-1}, a_{2n})\\ \frac{1}{2}[D(a_{2n-1},, [Sc]_{\alpha_S(c)}) + d(c, a_{2n})] \end{array}\right\}
\end{aligned}$$

Letting $n \to \infty$, we have $\lim_{n\to\infty} M(c, a_{2n-1}) = D(c, [Sc]_{\alpha_S(c)})$.

From the lower semi-continuity of φ,

$$\varphi(D(c, [Sc]_{\alpha_S(c)})) \leq \lim_{n\to\infty} \varphi(M(c, a_{2n-1})). \tag{4.1.13}$$

On the other hand, from (4.1.1)

$$\begin{aligned}
\delta([Sc]_{\alpha_S(c)}, a_{2n}) &\leq \delta([Sc]_{\alpha_S(c)}, [Ta_{2n-1}]_{\alpha_T(a_{2n-1})})\\
&\leq M(c, a_{2n-1}) - \varphi(M(c, a_{2n-1})),
\end{aligned}$$

and, letting $n \to \infty$, we have

$$\delta(([Sc]_{\alpha_S(c)}, c) \leq D(c, ([Sc]_{\alpha_S(c)}) - \lim_{n\to\infty} \varphi(M(c, a_{2n-1})). \tag{4.1.14}$$

This shows that $\lim_{n\to\infty} \varphi(M(c, a_{2n-1})) = 0$.
From (4.2.13) $\varphi(D(c, [Sc]_{\alpha_S(c)})) = 0$; that is, $D(c, [Sc]_{\alpha_S(c)}) = 0$. This implies, from (4.1.14), that $\{c\} = [Sc]_{\alpha_S(c)}$.
Now from (4.1.2) it is easy to see that $M(c, c) = D(c, [Tc]_{\alpha_T(c)})$, and so, from (4.1.1) we have

$$\begin{aligned}
\delta(c, [Tc]_{\alpha_T(c)}) &\leq \delta([Sc]_{\alpha_S(c)}, [Tc]_{\alpha_T(c)}) \\
&\leq M(c, c) - \varphi(M(c, c)) \\
&= D(c, [Tc]_{\alpha_T(c)}) - \varphi(D(c, [Tc]_{\alpha_T(c)})).
\end{aligned}$$

Therefore, we have $c \in [Tc]_{\alpha_T(c)}$ and so $\{c\} = [Tc]_{\alpha_T(c)}$. As a consequence, $\{c\}=[Sc]_{\alpha_S(c)} = [Tc]_{\alpha_T(c)}$;that is, c is a common fixed point of S and T.
Lastly we will show that the common fixed point is unique for this ssume that a and b are two common fixed points of S and T. Then $a \in [Sa]_{\alpha_S(a)}$, $a \in [Ta]_{\alpha_T(a)}$ and $b \in [Sb]_{\alpha_S(b)}$, $b \in [Tb]_{\alpha_T(b)}$. Therefore, from (4.1.2) we have $M(a, b) \leq d(a, b)$ and so from (4.1.1)

$$\begin{aligned}
d(a, b) &\leq \delta([Sa]_{\alpha_S(a)}, [Tb]_{\alpha_T(b)}) \\
&\leq M(a, b) - \varphi(M(a, b)) \\
&\leq d(a, b) - \varphi(M(a, b)).
\end{aligned}$$

This shows that $M(a, b) = 0$ and so $a = b$. □

The following example substantiates the validity of our results over some pre-existing results in literature.

Example 4.1.2. Let $X = [0, 1]$, $d(a, b) = |a - b|$, when $a, b \in X$ and let $G, H : X \to (X)$ be fuzzy mappings defined as:

$$G(a)(t) = \begin{cases} 1 \ \ if \ 0 \leq t < \frac{a}{6} \\ \\ \frac{1}{2} \ \ if \ \frac{a}{6} \leq t \leq \frac{a}{4} \\ \\ \frac{1}{3} \ \ if \ \frac{a}{4} \leq t < \frac{a}{3} \\ \\ 0 \ \ if \ \frac{a}{3} \leq t < \infty \end{cases}$$

$$H(a)(t) = \begin{cases} 1 \ \ if \ 0 \leq t < \frac{a}{6} \\ \\ \frac{1}{4} \ \ if \ \frac{a}{6} \leq t \leq \frac{a}{3} \\ \\ \frac{1}{6} \ \ if \ \frac{a}{3} \leq t \leq \frac{a}{2} \\ \\ 0 \ \ if \ \frac{a}{2} < t < \infty \end{cases}$$

$$[Ga]_{\frac{1}{3}} = \left\{ t \in X : G(a)(t) \geq \frac{1}{3}) \right\} = \left[0, \frac{a}{3}\right),$$

$$[Ha]_{\frac{1}{4}} = \left\{ t \in X : H(a)(t) \geq \frac{1}{4}) \right\} = [0, \frac{a}{3}]$$

It is clear that $[Ga]_{\frac{1}{3}}$ and $[Ha]_{\frac{1}{4}}$ are nonempty bounded for all $a \in X$. We will show that the condition (2) of Theorem 4.1.1 is satisfied with $\varphi(t) = \frac{t}{2}$. Indeed, for all $a, b \in X$,

$$\begin{aligned} \delta([Ga]_{\frac{1}{3}}, [Hb]_{\frac{1}{4}}) &= \delta\left(\left[0, \frac{a}{3}\right), [0, \frac{b}{3}]\right) \\ &= \frac{b}{3} = \frac{1}{2}\frac{2b}{3} = \frac{1}{2}D\left(b, [0, \frac{b}{3}]\right) \\ &= \frac{1}{2}D\left(b, [Hb]_{\frac{1}{4}}\right) \leq \frac{1}{2}M(a,b) = M(a,b) - \frac{1}{2}M(a,b) \\ &= M(a,b) - \varphi(M(a,b)). \end{aligned}$$

All of the conditions of Theorem 4.1.1 are satisfied and so these mappings have a unique common fixed point in X.

Example 4.1.3. Let $X = [0, 1], d(a, b) = |a - b|$, where $a, b \in X, \lambda, \mu \in (0, 1]$ and let $G, H : X \to (X)$ be fuzzy mappings defined as:

if $a = 0$,

$$G(a)(t) = \begin{cases} 1 \;\; if \;\; t=0 \\ \frac{1}{2} \;\; if \; 0 < t \leq \frac{1}{100} \\ 0 \;\; t > \frac{1}{100} \end{cases} \qquad T(a)(t) = \begin{cases} 1 \;\; if \; t=0 \\ \frac{1}{3} \;\; if \; 0 < t \leq \frac{1}{150} \\ 0 \; t > \frac{1}{150} \end{cases}$$

if $a \neq 0$,

$$G(a)(t) = \begin{cases} \lambda \;\; if \; 0 \leq t < \frac{a}{16} \\ \frac{\lambda}{2} \;\; if \; \frac{a}{16} \leq t \leq \frac{a}{10} \\ \frac{\lambda}{3} \;\; if \; \frac{a}{10} \leq t < a \\ 0 \;\; if \; a \leq t < \infty \end{cases} \qquad T(x)(t) = \begin{cases} \mu \;\; if \; 0 \leq t < \frac{a}{16} \\ \frac{\mu}{4} \;\; if \; \frac{a}{16} \leq t \leq \frac{a}{10} \\ \frac{\mu}{10} \;\; if \; \frac{a}{10} \leq t < a \\ 0 \;\; if \; a \leq t < \infty \end{cases}$$

Note that

$$[G0]_{\lambda_S(0)} = [H0]_{\lambda_T(0)} = \{0\}, \text{ if } \lambda_G(0) = \lambda_H(0) = 1,$$

and for $a \neq 0$,

$$[Ga]_\lambda = \left[0, \frac{a}{16}\right) \text{ and } [Ha]_\mu = \left[0, \frac{a}{16}\right),$$
$$[Ga]_{\frac{\lambda}{2}} = \left[0, \frac{a}{10}\right] \text{ and } [Ha]_{\frac{\mu}{4}} = \left[0, \frac{a}{10}\right].$$

Since X is not linear and also $[Ga]_\lambda$ and $[Ha]_\lambda$ are not compact for each λ, all of the previous fixed point results [4], [21], [51], [61] for fuzzy mappings on complete linear metric spaces are not applicable. However, G and H satisfy the conditions of Theorem 4.1.1.

4.2 Conclusion

The previous fixed point results [4], [21], [51], [61] for fuzzy mappings on complete linear metric spaces provide a fixed point wihich is not unique. To obtain a common and unique fixed point we established a convergence theorem on fuzzy mappings satisfying a weak contraction. Moreover, examples invoke and elucidate the generality of main theorem.

Chapter 5

Fixed point theorems in fuzzy Hilbert spaces

In this chapter we define nonexpansive, nonspreading, hybrid and contractive mappings in the context of fuzzy Hilbert spaces and prove some fixed point theorems for such mappings. We show that some existing fixed point theorems can be obtained as consequences of our results.

5.1 Nonlinear mappings in fuzzy Hilbert spaces.

We begin this section by defining nonlinear mappings in the setting of fuzzy Hilbert spaces.

Let H be a fuzzy Hilbert space with inner product $\{\langle .,. \rangle_\alpha : \alpha \in (0,1)\}$, norm $\{\|.\|_\alpha : \alpha \in (0,1)\}$ and C be a nonempty subset of H. A mapping $T : C \to H$ is said to be *nonexpansive*, *nonspreading*, and *hybrid* if

$$\|Tx - Ty\|_\alpha \leq \|x - y\|_\alpha ,$$

$$2\|Tx - Ty\|_\alpha^2 \leq \|Tx - y\|_\alpha^2 + \|Ty - x\|_\alpha^2$$

and

$$3\|Tx - Ty\|_\alpha^2 \leq \|x - y\|_\alpha^2 + \|Tx - y\|_\alpha^2 + \|Ty - x\|_\alpha^2$$

for all $x, y \in C$, respectively.

A *firmly nonexpansive mapping* can be defined as $F : C \to H$ such that,

$$\|Fx - Fy\|_\alpha^2 \le \langle x - y, Fx - Fy\rangle_\alpha$$

for all $x, y \in C$.

A mapping T from C into H is said to be *widely generalized hybrid* if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that

$$\alpha \|Tx - Ty\|_\alpha^2 + \beta \|x - Ty\|_\alpha^2 + \gamma \|Tx - y\|_\alpha^2 + \delta \|x - y\|_\alpha^2$$
$$+max\left\{\varepsilon \|x - Tx\|_\alpha^2, \zeta \|y - Ty\|_\alpha^2\right\} \le 0$$

for all $x, y \in C$ and T is called $(\alpha, \beta, \gamma, \delta)$-*symmetric generalized hybrid* if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\alpha \|Tx - Ty\|_\alpha^2 + \beta(\|x - Ty\|_\alpha^2 + \|Tx - y\|_\alpha^2)$$
$$+\gamma \|x - y\|_\alpha^2 + \delta(\|x - Tx\|_\alpha^2 + \|y - Ty\|_\alpha^2) \le 0 \qquad (5.1.1)$$

for all $x, y \in C$.

If $\alpha = 1, \beta = \delta = 0$ and $\gamma = -1$ in (5.2.1), then the mapping T is *nonexpansive*. If $\alpha = 2, \beta = -1$ and $\gamma = \delta = 0$ in (5.2.1), then the mapping T is *nonspreading*. Furthermore, if $\alpha = 3, \beta = \gamma = -1$ and $\delta = 0$ in (5.1.1), then the mapping T is *hybrid*.

A mapping $T : C \to H$ is called a *widely r-strict pseudo-contraction* if there exists $r \in \mathbb{R}$ with $r < 1$ such that

$$\|Tx - Ty\|_\alpha^2 \le \|x - y\|_\alpha^2 + r \|(I - T)x - (I - T)y\|_\alpha^2, \quad \forall\, x, y \in C.$$

If $0 \le r < 1$, then T is a *strict pseudo-contraction* (see [20]). Furthermore, if $r = 0$, then T is nonexpansive. Conversely, let $S : C \to H$ be a nonexpansive mapping and define $T : C \to H$ by,

$$T = \frac{1}{1+n}S + \frac{n}{1+n}I \qquad \forall x \in C, n \in \mathbb{N}$$

Then T is a *widely (-n)-strict pseudo-contraction*.

From the definition of T, we conclude that

$$S = (1+n)T - nI$$

Since S is nonexpansive, than for any $x, y \in C$,

$$\|(1+n)Tx - nx - ((1+n)Ty - ny)\|_\alpha^2 \le \|x-y\|_\alpha^2$$

and hence

$$\|Tx - Ty\|_\alpha^2 \le \|x-y\|_\alpha^2 + n\|(I-T)x - (I-T)y\|_\alpha^2.$$

The strong and weak convergence of $\{x_n\}$ is denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively, where $x \in H$.

Let A be a nonempty subset of H. The closure of the convex hull of A is denoted by $\overline{co}A$ and $F(T)$ denotes the set of fixed points of T.

In a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ respectively, it is known that

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \tag{5.1.2}$$

for all $x, y \in H$ and $\alpha \in \mathbb{R}$ (see [87]). Furthermore, in a Hilbert space, we have that

$$2\langle x-y, z-w\rangle = \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2 \tag{5.1.3}$$

for all $x, y, z, w \in H$.

Using the Riesz theorem, the following result can be obtained, (see [63], [84], [85], [86]).

Lemma 5.1.1. Let H be a Hilbert space, $\{x_n\}$ a bounded sequence in H and let μ be a mean on l^∞. Then there exists a unique point $z_0 \in \overline{co}\{x_n | n \in N\}$ such that

$$\mu_n \langle x_n, y\rangle = \langle z_0, y\rangle, \forall y \in H.$$

5.2 Fixed point theorems for generalized hybrid mappings in fuzzy Hilbert spaces

This section begins with a result to find a fixed point for nonempty closed convex subset of a fuzzy Hilbert space.

Theorem 5.2.1. Let H be a fuzzy Hilbert space, C a nonempty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta)$-symmetric generalized hybrid mapping from C into itself such that the conditions (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta + \delta > 0$ and (3) $\delta \geq 0$ hold. Then T has a fixed point if and only if there exists a $z \in C$ such that $\{T^n z : n = 0, 1, ...\}$ is bounded. In particular, a fixed point of T is unique if $\alpha + 2\beta + \gamma > 0$ in condition (1).

Proof. Suppose that T has a fixed point z. Then $\{T^n z : n = 0, 1, ...\} = \{z\}$ and hence $\{T^n z : n = 0, 1, ...\}$ is bounded. Conversely, suppose that there exists $z \in C$ such that $\{T^n z : n = 0, 1, ...\}$ is bounded. Since T is an $(\alpha, \beta, \gamma, \delta)$-symmetric generalized hybrid mapping of C into itself, we have that

$$\alpha \left\|Tx - T^{n+1}z\right\|_\alpha^2 + \beta(\left\|x - T^{n+1}z\right\|_\alpha^2 + \|Tx - T^n z\|_\alpha^2)$$

$$+\gamma \|x - T^n z\|_\alpha^2 + \delta(\|x - Tx\|_\alpha^2 + \left\|T^n z - T^{n+1}z\right\|_\alpha^2) \leq 0$$

for all $n \in \mathbb{N} \cup \{0\}$ and $x \in C$. Since $\{T^n z\}$ is bounded, we can apply a Banach limit μ to both sides of the inequality. Since $\mu_n \|Tx - T^n z\|_\alpha^2 = \mu_n \|Tx - T^{n+1}z\|_\alpha^2$ and $\mu_n \|x - T^{n+1}z\|_\alpha^2 = \mu_n \|x - T^n z\|_\alpha^2$, we have that

$$(\alpha + \beta)\mu_n \|Tx - T^n z\|_\alpha^2 + (\beta + \gamma)\mu_n \|x - T^n z\|_\alpha^2$$

$$+\delta(\|x - Tx\|_\alpha^2 + \mu_n \left\|T^n z - T^{n+1}z\right\|_\alpha^2) \leq 0.$$

Since

$$\mu_n \|Tx - T^n z\|_\alpha^2 = \|Tx - x\|_\alpha^2 + 2\mu_n \langle Tx - x, x - T^n z\rangle_\alpha + \mu_n \|x - T^n z\|_\alpha^2,$$

we have that

$$(\alpha + \beta + \delta) \|Tx - x\|_\alpha^2 + 2(\alpha + \beta)\mu_n \langle Tx - x, x - T^n z\rangle_\alpha$$

$$+(\alpha + 2\beta + \gamma)\mu_n \|x - T^n z\|_\alpha^2 + \delta\mu_n \left\|T^n z - T^{n+1}z\right\|_\alpha^2 \leq 0.$$

From (1) $\alpha + 2\beta + \gamma \geq 0$ and (3) $\delta \geq 0$, we have that

$$(\alpha + \beta + \delta) \|Tx - x\|_\alpha^2 + 2(\alpha + \beta)\mu_n \langle Tx - x, x - T^n z\rangle_\alpha \leq 0. \qquad (5.2.1)$$

Since there exists a $p \in H$ from Lemma 5.1.1 such that

$$\mu_n \langle y, T^n z\rangle_\alpha = \langle y, p\rangle_\alpha$$

for all $y \in H$, we have from (5.2.1) that

$$(\alpha + \beta + \delta) \|Tx - x\|_\alpha^2 + 2(\alpha + \beta) \langle Tx - x, x - p \rangle_\alpha \le 0. \tag{5.2.2}$$

Since C is closed and convex, we have that

$$p \in \overline{co}\{T^n x : n \in N\} \subset C.$$

Putting $x = p$, we obtain from (5.2.2) that

$$(\alpha + \beta + \delta) \|Tp - p\|_\alpha^2 \le 0. \tag{5.2.3}$$

We have from (5.2.2) $\alpha + \beta + \delta > 0$ that $\|Tp - p\|_\alpha^2 \le 0$. This implies that p is a fixed point of T.

Next suppose that $\alpha + 2\beta + \gamma > 0$. Let p_1 and p_2 be fixed points of T. Then we have that

$$\alpha \|Tp_1 - Tp_2\|_\alpha^2 + \beta(\|p_1 - Tp_2\|_\alpha^2 + \|Tp_1 - p_2\|_\alpha^2)$$
$$+\gamma \|p_1 - p_2\|_\alpha^2 + \delta(\|p_1 - Tp_1\|_\alpha^2 + \|p_2 - Tp_2\|_\alpha^2) \le 0$$

and hence$(\alpha + 2\beta + \gamma) \|p_1 - p_2\|_\alpha^2 \le 0$. We have from $\alpha + 2\beta + \gamma > 0$ that $p_1 = p_2$. Therefore the fixed point of T is unique. This completes the proof. □

A natural attempt to extend Theorem 5.2.1 would be to suppose that $\alpha + 2\beta + \gamma > 0$ for condition (1). However there is a positive result along these lines in the following theorem.

Theorem 5.2.2. Let H be a fuzzy Hilbert space, C a nonempty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta)$-symmetric generalized hybrid mapping from C into itself such that conditions (1) $\alpha + 2\beta + \gamma \ge 0$, (2) $\alpha + \beta + \delta > 0$ and (3) $\delta \ge 0$ hold. Then T has a fixed point. In particular, the fixed point of T is unique if $\alpha + 2\beta + \gamma > 0$ in (1).

A mapping T from C into H is called $(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid if there exist $\alpha, \beta, \gamma, \delta, \zeta \in \mathbb{R}$ such that

$$\alpha \|Tx - Ty\|_\alpha^2 + \beta(\|x - Ty\|_\alpha^2 + \|Tx - y\|_\alpha^2) + \gamma \|x - y\|_\alpha^2$$
$$+\delta(\|x - Tx\|_\alpha^2 + \|y - Ty\|_\alpha^2) + \zeta \|x - y - (Tx - Ty)\|_\alpha^2 \le 0 \tag{5.2.4}$$

for all $x, y \in C$.

Theorem 5.2.3. Let H be a fuzzy Hilbert space, C a nonempty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid mapping from C into itself such that the conditions (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta + \delta + \zeta > 0$ and (3) $\delta + \zeta \geq 0$ hold. Then T has a fixed point if and only if there exists a $z \in C$ such that $\{T^n z : n = 0, 1, ...\}$ is bounded. In particular, the fixed point of T is unique if $\alpha + 2\beta + \gamma > 0$ in (1).

Proof. Since $T : C \to C$ is an $(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid mapping, there exist $\alpha, \beta, \gamma, \delta, \zeta \in \mathbb{R}$ satisfying (5.2.4). We also have that

$$\|x - y - (Tx - Ty)\|_\alpha^2 = \|x - Tx\|_\alpha^2 + \|y - Ty\|_\alpha^2$$

$$- \|x - Ty\|_\alpha^2 - \|y - Tx\|_\alpha^2 + \|x - y\|_\alpha^2 + \|Tx - Ty\|_\alpha^2 \tag{5.2.5}$$

for all $x, y \in C$. Thus we obtain from (5.2.4) that

$$(\alpha + \zeta) \|Tx - Ty\|_\alpha^2 + (\beta - \zeta)(\|x - Ty\|_\alpha^2 + \|Tx - y\|_\alpha^2)$$

$$+(\gamma + \zeta) \|x - y\|_\alpha^2 + (\delta + \zeta)(\|x - Tx\|_\alpha^2 + \|y - Ty\|_\alpha^2) \leq 0. \tag{5.2.6}$$

The conditions (1) $\alpha + 2\beta + \gamma \geq 0$ and (2) $\alpha + \beta + \delta + \zeta > 0$ are equivalent to $(\alpha + \zeta) + 2(\beta - \zeta) + (\gamma + \zeta) \geq 0$ and $(\alpha + \zeta) + (\beta - \zeta) + (\delta + \zeta) > 0$, respectively. Furthermore, since (3) $\delta + \zeta \geq 0$ holds, we have the desired result from Theorem 5.2.1. □

As a direct consequence of Theorem 5.2.3, we obtain the following.

Theorem 5.2.4. Let H be a fuzzy Hilbert space, C a nonempty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid mapping from C into itself such that the conditions (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta + \delta + \zeta > 0$ and (3) $\delta + \zeta \geq 0$ hold. Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z : n = 0, 1, ...\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + 2\beta + \gamma > 0$ on the condition (1).

Making appropriate use of boundedness one can extend the above theorem as follows.

Theorem 5.2.5. Let H be a fuzzy Hilbert space, C a nonempty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \zeta)$-symmetric more generalized hybrid mapping from C into itself which satisfies the conditions (1) $\alpha+2\beta+\gamma\geq 0$, (2) $\alpha+\beta+\delta+\zeta>0$ and (3) there exists $\lambda\in[0,1)$ such that $(\alpha+\beta)\lambda+\delta+\zeta\geq 0$. Then T has a fixed point. In particular, the fixed point of T is unique if $\alpha+2\beta+\gamma>0$ in the condition (1).

Proof. Since $T:C\to C$ is an $(\alpha,\beta,\gamma,\delta,\zeta)$-symmetric more generalized hybrid mapping, we obtain that

$$\alpha\left\|Tx-T^{n+1}z\right\|_{\alpha}^{2}+\beta(\left\|x-T^{n+1}z\right\|_{\alpha}^{2}+\|Tx-T^{n}z\|_{\alpha}^{2})+\gamma\left\|x-T^{n}z\right\|_{\alpha}^{2}$$

$$+\delta(\|x-Tx\|_{\alpha}^{2}+\left\|T^{n}z-T^{n+1}z\right\|_{\alpha}^{2})+\zeta\left\|(x-Tx)-\left(T^{n}z-T^{n+1}z\right)\right\|_{\alpha}^{2}\leq 0$$

for all $n\in\mathbb{N}\cup\{0\}$ and all $x\in C$.

Let $\lambda\in[0,1)\cap\{\lambda:(\alpha+\beta)\lambda+\zeta+\eta\geq 0\}$ and define $S=(1-\lambda)T+\lambda I$. Since C is convex, S is a mapping from C into itself. Since C is bounded, $\{S^{n}z:n=0,1,...\}$ is bounded for all $z\in C$. Since $\lambda\neq 1$, we obtain that $F(S)=F(T)$. Moreover, from

$T = \frac{1}{1-\lambda}S - \frac{\lambda}{1-\lambda}I$ and (5.2.1), we have that

$$\begin{aligned}
& \alpha \left\| \left(\frac{1}{1-\lambda} Sx - \frac{\lambda}{1-\lambda} x \right) - \left(\frac{1}{1-\lambda} Sy - \frac{\lambda}{1-\lambda} y \right) \right\|_\alpha^2 \\
& + \beta \left\| x - \left(\frac{1}{1-\lambda} Sy - \frac{\lambda}{1-\lambda} y \right) \right\|_\alpha^2 + \beta \left\| \left(\frac{1}{1-\lambda} Sx - \frac{\lambda}{1-\lambda} x \right) - y \right\|_\alpha^2 + \gamma \left\| x - y \right\|_\alpha^2 \\
& + \delta \left\| x - \left(\frac{1}{1-\lambda} Sx - \frac{\lambda}{1-\lambda} x \right) \right\|_\alpha^2 + \delta \left\| y - \left(\frac{1}{1-\lambda} Sy - \frac{\lambda}{1-\lambda} y \right) \right\|_\alpha^2 \\
& + \zeta \left\| \left(x - \left(\frac{1}{1-\lambda} Sx - \frac{\lambda}{1-\lambda} x \right) \right) - \left(y - \left(\frac{1}{1-\lambda} Sy - \frac{\lambda}{1-\lambda} y \right) \right) \right\|_\alpha^2 \\
= \; & \alpha \left\| \left(\frac{1}{1-\lambda} (Sx - Sy) - \frac{\lambda}{1-\lambda} (x-y) \right) \right\|_\alpha^2 \\
& + \beta \left\| x - \left(\frac{1}{1-\lambda} (x - Sy) - \frac{\lambda}{1-\lambda} (x-y) \right) \right\|_\alpha^2 \\
& + \beta \left\| x - \left(\frac{1}{1-\lambda} (Sx - y) - \frac{\lambda}{1-\lambda} (x-y) \right) \right\|_\alpha^2 + \gamma \left\| x - y \right\|_\alpha^2 \\
& + \delta \left\| \frac{1}{1-\lambda} (x - Sx) \right\|_\alpha^2 + \delta \left\| \frac{1}{1-\lambda} (y - Sy) \right\|_\alpha^2 \\
& + \zeta \left\| \frac{1}{1-\lambda} (x - Sx) - \frac{1}{1-\lambda} (y - Sy) \right\|_\alpha^2 \\
= \; & \frac{\alpha}{1-\lambda} \left\| Sx - Sy \right\|_\alpha^2 + \frac{\beta}{1-\lambda} \left\| x - Sy \right\|_\alpha^2 \\
& + \frac{\beta}{1-\lambda} \left\| Sx - y \right\|_\alpha^2 + \left(-\frac{\lambda}{1-\lambda} (\alpha + 2\beta) + \gamma \right) \left\| x - y \right\|_\alpha^2 \\
& + \frac{\delta + \beta\lambda}{(1-\lambda)^2} \left\| x - Sx \right\|_\alpha^2 + \frac{\delta + \beta\lambda}{(1-\lambda)^2} \left\| y - Sy \right\|_\alpha^2 \\
& \qquad + \frac{\zeta + \alpha\lambda}{(1-\lambda)^2} \left\| (x - Sx) - (y - Sy) \right\|_\alpha^2 \leq 0
\end{aligned}$$

Therefore S is an $\left(\frac{\alpha}{1-\lambda}, \frac{\beta}{1-\lambda}, -\frac{\lambda}{1-\lambda} (\alpha + 2\beta) + \gamma, \frac{\delta+\beta\lambda}{(1-\lambda)^2}, \frac{\zeta+\alpha\lambda}{(1-\lambda)^2} \right)$-symmetric more generalized hybrid mapping. Furthermore, we obtain that

$$\frac{\alpha}{1-\lambda} + \frac{2\beta}{1-\lambda} + \frac{\gamma}{1-\lambda} - \frac{\lambda}{1-\lambda} (\alpha + 2\beta) + \gamma = \alpha + 2\beta + \gamma \geq 0,$$

$$\frac{\alpha}{1-\lambda} + \frac{\beta}{1-\lambda} + \frac{\delta + \beta\lambda}{(1-\lambda)^2} + \frac{\zeta + \alpha\lambda}{(1-\lambda)^2} = \frac{\alpha + \beta + \delta + \zeta}{(1-\lambda)^2} > 0,$$

$$\frac{\delta + \beta\lambda}{(1-\lambda)^2} + \frac{\zeta + \alpha\lambda}{(1-\lambda)^2} = \frac{(\alpha + \beta)\lambda + \delta + \zeta}{(1-\lambda)^2} \geq 0.$$

Therefore, by Theorem 5.3.4, $F(S) \neq \phi$.

Next, suppose that $\alpha + 2\beta + \gamma > 0$. Let p_1 and p_2 be fixed points of T. Then

$$\begin{aligned} &\alpha \|Tp_1 - Tp_2\|_\alpha^2 + \beta(\|p_1 - Tp_2\|_\alpha^2 + \|Tp_1 - p_2\|_\alpha^2) + \gamma \|p_1 - p_2\|_\alpha^2 \\ &\delta(\|p_1 - Tp_1\|_\alpha^2 + \|p_2 - Tp_2\|_\alpha^2) + \zeta \|(p_1 - Tp_1) + (p_2 - Tp_2)\|_\alpha^2 \\ &= (\alpha + 2\beta + \gamma) \|p_1 - p_2\|_\alpha^2 \leq 0 \end{aligned}$$

and hence $p_1 = p_2$. Therefore the fixed point of T is unique. □

As an illustration of the use of a bounded closed convex subset we finish this section by giving an important fixed point theorem in its final form. The theorem was formulated for the case $\beta + \delta = 0$ in Theorem 5.2.5.

Theorem 5.2.6. Let H be a fuzzy Hilbert space, C a nonempty bounded closed convex subset of H and let T be an $(\alpha, -\beta, \gamma, \beta, \zeta)$-symmetric more generalized hybrid mapping from C into itself, i.e., there exist α, β, γ, $\zeta \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha \|Tx - Ty\|_\alpha^2 + \beta(\|x - Ty\|_\alpha^2 + \|Tx - y\|_\alpha^2) + \gamma \|x - y\|_\alpha^2 \\ &-\beta(\|x - Tx\|_\alpha^2 + \|y - Ty\|_\alpha^2) + \zeta \|x - y - (Tx - Ty)\|_\alpha^2 \leq 0 \end{aligned} \tag{5.2.7}$$

for all $x, y \in C$. Furthermore, suppose that T satisfies the conditions (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \zeta > 0$ and (3) there exists $\lambda \in [0, 1)$ such that $(\alpha + \beta)\lambda + \delta + \zeta \geq 0$. Then T has a fixed point. In particular, the fixed point of T is unique if $\alpha + 2\beta + \gamma > 0$ in the condition (1).

5.3 Consequences

In this section we prove well-known and new fixed point theorems in a fuzzy Hilbert space by using fixed point theorems obtained in the previous section.

The following result can be obtained from Theorem 5.2.3.

Theorem 5.3.1. Let H be a fuzzy Hilbert space, C a nonempty bounded closed convex subset of H and let T be a widely strict pseudo-contraction from C into itself; i.e., there exists an $r \in \mathbb{R}$ with $r < 1$ such that

$$\|Tx - Ty\|_\alpha^2 \leq \|x - y\|_\alpha^2 + r \|(I - T)x - (I - T)y\|_\alpha^2, \forall x, y \in C. \tag{5.3.1}$$

Then T has a fixed point in C.

Proof. We first assume that $r \leq 0$. We have from (5.2.8) that, for all $x, y \in C$,

$$\|Tx - Ty\|_\alpha^2 - \|x - y\|_\alpha^2 - r\,\|(I - T)x - (I - T)y\|_\alpha^2 \leq 0 \tag{5.3.2}$$

Then T is a $(1, 0, -1, 0, -r)$-symmetric more generalized hybrid mapping. Furthermore, (1) $\alpha + 2\beta + \gamma = 1 - 1 \geq 0$, (2) $\alpha + \beta + \delta + \zeta = 1 - r > 0$ and (3) $\delta + \zeta = -r \geq 0$ in Theorem 5.2.3 are satisfied. Thus T has a fixed point from Theorem 5.2.3. Assume that $0 \leq r < 1$ and define a mapping T as follows:

$$Sx = \lambda x + (1 - \lambda)Tx, \forall x \in C,$$

where $r \leq \lambda < 1$. Then S is a mapping from C into itself and $F(S) = F(T)$. From $Sx = \lambda x + (1 - \lambda)Tx$, we also have that

$$Tx = \frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x.$$

Thus we have

$$\begin{aligned}
0 &\geq \left\|\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y\right)\right\|_\alpha^2 \\
&\quad - \|x - y\|_\alpha^2 - r\left\|x - y - \left\{\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y\right)\right\}\right\|_\alpha^2 \\
&= \left\|\frac{1}{1-\lambda}(Sx - Sy) - \frac{\lambda}{1-\lambda}(x - y)\right\|_\alpha^2 \\
&\quad - \|x - y\|_\alpha^2 - r\left\|\frac{1}{1-\lambda}(x - y) - \frac{1}{1-\lambda}(Sx - Sy)\right\|_\alpha^2 \\
&= \frac{1}{1-\lambda}\|Sx - Sy\|_\alpha^2 - \frac{\lambda}{1-\lambda}\|x - y\|_\alpha^2 \\
&\quad + \frac{1}{1-\lambda}.\frac{\lambda}{1-\lambda}\|x - y - (Sx - Sy)\|_\alpha^2 - \|x - y\|_\alpha^2 \\
&\quad - \frac{r}{(1-\lambda)^2}\|x - y - (Sx - Sy)\|_\alpha^2 \\
&= \frac{1}{1-\lambda}\|Sx - Sy\|_\alpha^2 - \frac{1}{1-\lambda}\|x - y\|_\alpha^2 + \frac{\lambda - r}{(1-\lambda)^2}\|x - y - (Sx - Sy)\|_\alpha^2.
\end{aligned}$$

Then S is a $\left(\frac{1}{1-\lambda}, 0, -\frac{1}{1-\lambda}, 0, \frac{\lambda - r}{(1-\lambda)^2}\right)$)-symmetric more generalized hybrid. From

$$\begin{aligned}
\frac{1}{1-\lambda} + 2.0 - \frac{1}{1-\lambda} &= 0, \\
\frac{1}{1-\lambda} + \frac{\lambda - r}{(1-\lambda)^2} &> 0 \qquad \text{and} \\
\frac{\lambda - r}{(1-\lambda)^2} &\geq 0,
\end{aligned}$$

(1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta + \delta + \zeta > 0$ and (3) $\delta + \zeta \geq 0$ in Theorem 5.3.3 are satisfied.Thus S has a fixed point in C from Theorem 5.2.3 and hence T has a fixed point. This completes the proof. □

Let H be a fuzzy Hilbert space and let C be a nonempty subset of H. Let T be a mapping of C into H. For $u \in H$ and $s, t \in (0, 1)$, we define the following mapping:

$$Sx = tx + (1-t)(su + (1-s)Tx)$$

for all $x \in C$. We call such S a TWY mapping generated by u, T, s, t. Since $Sx = tx + s(1-t)u + (1-t)(1-s)Tx$, we have that for all $x, y \in C$,

$$\begin{aligned} \|Sx - Sy\|_\alpha^2 &= \|t(x-y) + (1-t)(1-s)(Tx - Ty)\|_\alpha^2 \qquad (5.3.3) \\ &= t^2 \|x-y\|_\alpha^2 + (1-t)^2(1-s)^2 \|Tx - Ty\|_\alpha^2 \\ &\quad +2t(1-t)(1-s)\langle x-y, Tx-Ty\rangle_\alpha \\ &= t^2 \|x-y\|_\alpha^2 + (1-t)^2(1-s)^2 \|Tx - Ty\|_\alpha^2 \\ &\quad +t(1-t)(1-s)(\|x-Ty\|_\alpha^2 + \|y-Tx\|_\alpha^2 \\ &\quad -\|x-Tx\|_\alpha^2 - \|y-Ty\|_\alpha^2) \\ &= t^2 \|x-y\|_\alpha^2 + (1-t)^2(1-s)^2 \|Tx - Ty\|_\alpha^2 \\ &\quad +t(1-t)(1-s)(\|x-Ty\|_\alpha^2 + \|y-Tx\|_\alpha^2) \\ &\quad -t(1-t)(1-s)(\|x-Tx\|_\alpha^2 - \|y-Ty\|_\alpha^2). \end{aligned}$$

Similarly, we have that

$$\begin{aligned} &\|x - Sy\|_\alpha^2 + \|y - Sx\|_\alpha^2 \qquad (5.3.4) \\ = \;& s(1-t)^2 \left(\|u-x\|_\alpha^2 + \|u-y\|_\alpha^2\right) \\ & -s(1-s)(1-t)^2(\|u-Tx\|_\alpha^2 + \|u-Ty\|_\alpha^2) \\ & -t(1-t)(1-s)(\|x-Tx\|_\alpha^2 + \|y-Ty\|_\alpha^2) \\ & +(1-t)(1-s)(\|x-Tx\|_\alpha^2 + \|y-Ty\|_\alpha^2) + 2t\|x-y\|_\alpha^2, \end{aligned}$$

and

$$\begin{aligned} & \|x - Sx\|_\alpha^2 + \|y - Sy\|_\alpha^2 \\ = \; & s(1-t)^2\left(\|u - x\|_\alpha^2 + \|u - y\|_\alpha^2\right) \\ & -s(1-s)(1-t)^2(\|u - Tx\|_\alpha^2 + \|u - Ty\|_\alpha^2) \\ & +s(1-s)(1-t)^2(\|x - Tx\|_\alpha^2 + \|y - Ty\|_\alpha^2). \end{aligned} \tag{5.3.5}$$

We also have that

$$\begin{aligned} & \|x - y - Sx - Sy\|_\alpha^2 \\ = \; & (1-s)(1-t)^2(\|x - Tx\|_\alpha^2 + \|y - Ty\|_\alpha^2) \\ & -(1-s)(1-t)^2(\|x - Ty\|_\alpha^2 + \|y - Tx\|_\alpha^2) \\ & +(1-t)^2\|x - y\|_\alpha^2 + (1-t)^2(1-s)^2\|Tx - Ty\|_\alpha^2. \end{aligned} \tag{5.3.6}$$

Using (5.3.4) and (5.3.5), we have that

$$\begin{aligned} & \|x - Sx\|_\alpha^2 + \|y - Sy\|_\alpha^2 - \|x - Sy\|_\alpha^2 - \|y - Sx\|_\alpha^2 \\ = \; & (1-s)(1-t)(\|x - Tx\|_\alpha^2 + \|y - Ty\|_\alpha^2 \\ & - \|x - Ty\|_\alpha^2 + \|y - Tx\|_\alpha^2 - 2t\|x - y\|_\alpha^2). \end{aligned} \tag{5.3.7}$$

We turn our attention to equations (5.3.3) and (5.3.7) to obtain the following theorem.

Theorem 5.3.2. Let H be a fuzzy Hilbert space, C a nonempty bounded closed convex subset of H and let T be a widely strict pseudo-contraction from C into itself; i.e., there exists an $r \in \mathbb{R}$ with $r < 1$ such that

$$\|Tx - Ty\|_\alpha^2 \le \|x - y\|_\alpha^2 + r\|(I-T)x - (I-T)y\|_\alpha^2 \quad \forall x, y \in C. \tag{5.3.8}$$

Let $u \in C$ and $s \in (0, 1)$. Define a mapping $U : C \to C$ as follows:

$$Ux = su + (1-s)Tx, \forall x \in C.$$

Then U has a unique fixed point in C.

Proof. Since T is a widely r-strict pseudo-contraction from C into itself, we have that, for all $x, y \in C$,

$$\|Tx - Ty\|_\alpha^2 - \|x - y\|_\alpha^2 - r\|(I-T)x - (I-T)y\|_\alpha^2 \le 0.$$

If $r \leq 0$, then T is a nonexpansive mapping. Therefore U is a contractive mapping. Using the fixed point theorem for contractive mappings, we have that U has a unique fixed point in C. Let $0 < r < 1$. Since

$$\begin{aligned} \|x-y-(Tx-Ty)\|_\alpha^2 &= \|x-Tx\|_\alpha^2 + \|y-Ty\|_\alpha^2 \\ &\quad - \|x-Ty\|_\alpha^2 - \|y-Tx\|_\alpha^2 + \|x-y\|_\alpha^2 + \|Tx-Ty\|_\alpha^2, \end{aligned}$$

we have that

$$(1-r)\|Tx-Ty\|_\alpha^2 - (1+r)\|x-y\|_\alpha^2$$
$$-r\|x-Tx\|_\alpha^2 + \|y-Ty\|_\alpha^2 - \|x-Ty\|_\alpha^2 - \|y-Tx\|_\alpha^2 \leq 0.$$

For u, T and $s, r \in (0,1)$, define a TWY mapping S as follows:

$$Sx = rx + (1-r)(su + (1-s)Tx), \forall x \in C.$$

Then we have from (5.3.3) that

$$\begin{aligned} &\frac{1}{(1-r)(1-s)^2}\|Sx-Sy\|_\alpha^2 - \frac{r^2}{(1-r)(1-s)^2}\|x-y\|_\alpha^2 \\ &+\frac{r}{(1-s)}\left(\|x-Tx\|_\alpha^2 + \|y-Ty\|_\alpha^2 - \|x-Ty\|_\alpha^2 - \|y-Tx\|_\alpha^2\right) \\ &-(1+r)\|x-y\|_\alpha^2 \\ &-r(\|x-Tx\|_\alpha^2 + \|y-Ty\|_\alpha^2 - \|x-Ty\|_\alpha^2 - \|y-Tx\|_\alpha^2) \leq 0. \end{aligned}$$

We have from (5.3.7) that

$$\begin{aligned} &\frac{1}{(1-r)(1-s)^2}\|Sx-Sy\|_\alpha^2 - \frac{r^2}{(1-r)(1-s)^2}\|x-y\|_\alpha^2 \\ &+\frac{r}{(1-r)(1-s)^2}\left(\|x-Sx\|_\alpha^2 + \|y-Sy\|_\alpha^2 - \|x-Sy\|_\alpha^2 - \|y-Sx\|_\alpha^2\right) \\ &+\frac{2r^2}{(1-r)(1-s)^2}\|x-y\|_\alpha^2 - -(1+r)\|x-y\|_\alpha^2 \\ &-\frac{r}{(1-r)(1-s)}\left(\|x-Sx\|_\alpha^2 + \|y-Sy\|_\alpha^2 - \|x-Sy\|_\alpha^2 - \|y-Sx\|_\alpha^2\right) \\ &\qquad -\frac{2r^2}{(1-r)(1-s)}\|x-y\|_\alpha^2 \leq 0 \end{aligned}$$

and hence

$$\begin{aligned} &\frac{1}{(1-r)(1-s)^2}\|Sx-Sy\|_\alpha^2 \\ &-\frac{rs}{(1-r)(1-s)^2}\left(\|x-Sy\|_\alpha^2 + \|y-Sx\|_\alpha^2\right) \\ &+\left(\frac{r^2}{(1-r)(1-s)^2} - \frac{1-s+r^2(1+s)}{(1-r)(1-s)}\right)\|x-y\|_\alpha^2 \end{aligned}$$

$$+\frac{rs}{(1-r)(1-s)^2}\left(\|x-Sx\|_\alpha^2+\|y-Sy\|_\alpha^2\right)\le 0.$$

For this inequality, we apply Theorem 5.2.2. We first obtain that

$$\begin{aligned}&\frac{1}{(1-r)(1-s)^2}-\frac{2rs}{(1-r)(1-s)^2}+\frac{r^2}{(1-r)(1-s)^2}-\frac{1-s+r^2(1+s)}{(1-r)(1-s)}\\ =\ &\frac{s(1+r)(2-s(1-r))}{(1-r)(1-s)^2}>0.\end{aligned}$$

Furthermore, we have that

$$\begin{aligned}&\frac{1}{(1-r)(1-s)^2}-\frac{rs}{(1-r)(1-s)^2}+\frac{rs}{(1-r)(1-s)^2}\\ =\ &\frac{1}{(1-r)(1-s)^2}>0.\end{aligned}$$

$$\frac{rs}{(1-r)(1-s)^2}\ge 0.$$

Thus S has a unique fixed point z in C from Theorem 1.5.4.. Since z is a fixed point of S, we have $z=rz+(1-r)(su+(1-s)Tz$. From $1-r\ \neq 0$, we have that

$$z=su+(1-s)Tz.$$

This completes the proof. □

5.4 Conclusion

We have defined nonlinear mappings in fuzzy Hilbert spaces and established fixed point theorems. Our results evoke the notions of weak contractions which substantiates the validity of results. As an application we established well-known and new fixed point theorems which not only give the unique solution but also define additive mappings.

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