Generalizations of Soft Expert Sets and Their Applications in Decision Analysis

QUAID-I-AZAM UNIVERSITY
ISLAMABAD

Ph.D. Thesis

By

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Department of Mathematics
Quaid-i-Azam University
Islamabad, Pakistan
2017

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A THESIS SUBMITTED IN THE PARTIAL FULFILMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

Supervised By

Dr. Muhammad Aslam

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Quaid-i-Azam University
Islamabad, Pakistan
2017

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No part of this thesis has been submitted anywhere else for any other degree. This thesis is submitted to the Dept. of Mathematics Quaid-i-Azam University Islamabad (Name of Department of the University) in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Field of Mathematics (Subject Name)

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Generalizations of Soft Expert Sets and Their Applications in Decision Analysis

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CERTIFICATE

A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIRMENT FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

We accept this dissertation as conforming to the required standard.

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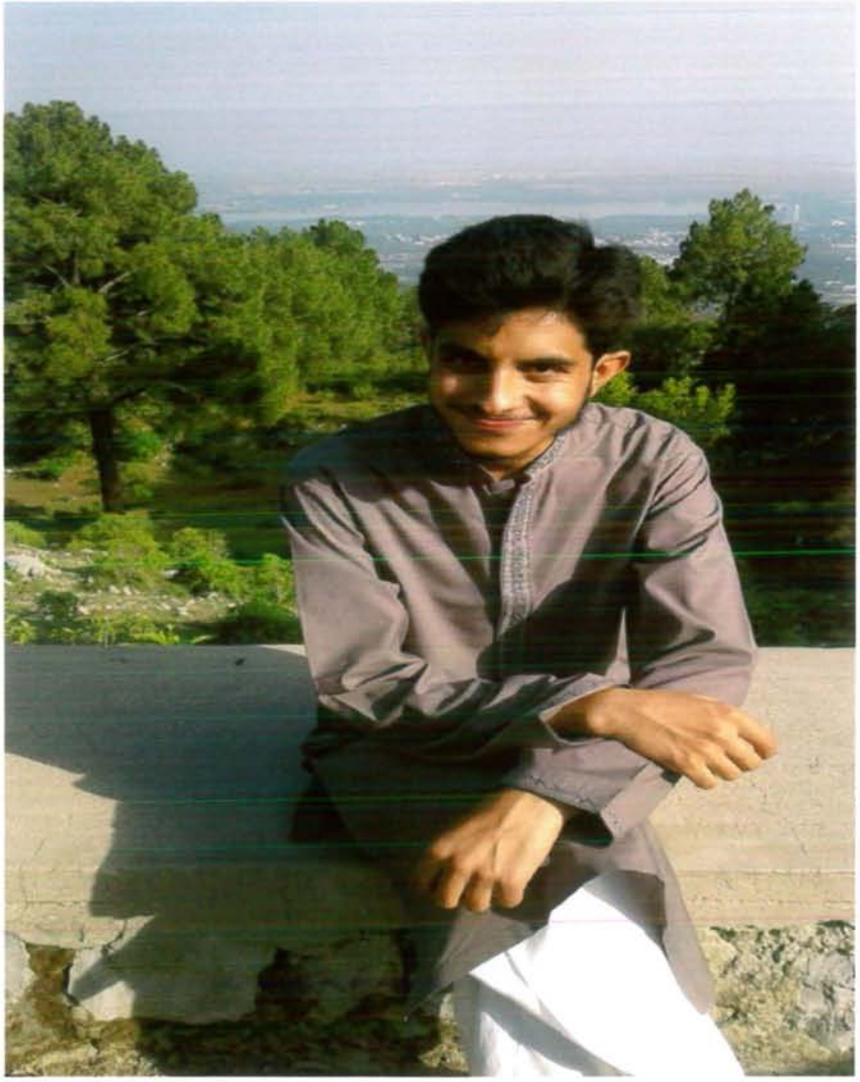
Dedicated

To my beloved brother

Khanan Qayyum (Late)

(09 November 1995 to 07 August 2016)



May God have mercy and forgive him

Contents

Acknowledgement

In the name of **Almighty Allah**, the most Merciful and Compassionate, who enabled me to complete this thesis. I offer my humblest and sincerest words of thanks to **Holy Prophet Muhammad (Peace Be Upon Him)** Who is, forever, a torch of guidance and knowledge for humanity.

I wish to express my profound gratitude to my former supervisor, **Professor Qaiser Mushtaq**, whose guidance and keen interest enabled me to accomplish this work. I would never have been able to do it up to the standard otherwise.

I would like to express deep sense of gratitude and appreciation to my supervisor **Dr. Muhammad Aslam,** during my studies at Quaid-i-Azam University, Islamabad, for his suggestions. This dissertation would not have been possible without the kind support and creative abilities of my supervisor. In spite of his extremely busy schedule he always uses to take his precious time for me. I wish to express my heartfelt thanks and deep gratitude to **Professor Muhammad Shabir** for his selfless support.

My thanks also go to all the staff and faculty members at the Department of Mathematics, Quaid-i-Azam University, Islamabad.

I am grateful to my research fellows, **Tanzeela Shaheen** and **Salma Shaheen**, for their sincere help and cooperation.

I acknowledge with sincere thanks the help, the prayers, the encouragement and moral support extended by parents **Abdul Qayyum**, **Gulshan Bibi**, brothers

Irfan Qayyum, **Rehan Qayyum** , **Khanan Qayyum** (late) and sister **Ansa Qayyum**.

I wish to thank my entire extended family for providing a loving environment for me. My uncles and aunts were particularly supportive.

I wish to express my heartfelt thanks to my chachu **Ghulam Abbas** and chachi **Nusrat Abbas** for their encouragement and moral support during my studies.

My heartiest acknowledgement are for my colleagues, juniors, seniors and friends who is one way or the other helped me in completing my thesis.

I would like to thank everybody who was important to the successful realization of this thesis, as well as expressing my apology that i could not mention personally one by one.

Finally, I would like to place on record my gratitude to my teacher sir **Ejaz**.

Afshan Qayyum

Introduction

Researchers and mathematicians all over the world developed important analytical skills and problem-solving strategies to assess a broad range of issues in commerce, science and arts. But the most challenging issues were related to the problems which were more qualitative rather than quantitative in nature. Thus, the need to handle uncertain situations and vagueness in practical as well as theoretical problems led the researchers to the development of theories like fuzzy set theory. Many studies show that this theory may represent an important theoretical and practical tool to tackle uncertainty. In 1965, Zadeh initiated fuzzy sets. Fuzzy sets deal with possibilistic uncertainty connected with imprecision of states, perceptions and preferences. Zadeh extended the concept of fuzzy sets by interval valued fuzzy sets in 1975. Concept of intuitionistic fuzzy sets was introduced by Atanassov in 1983. To develop a model that is enriched with parameters, soft set theory was initiated by Molodtsov in 1999. It attracted the attention of many researchers as the theory proved its worth in many dimensions like medicine and decision analysis. Maji et al. discussed decision making problems through soft sets and fuzzy soft sets. Maji et al. defined the operations of union and intersection on soft sets. To analyze decision making problems, hesitant fuzzy set theory also proves pretty worthwhile. It was presented by Torra and Narukawa as a generalization of fuzzy set theory. Jun et al. introduced a new notion of cubic sets in 2011 by using a fuzzy sets and an interval-valued fuzzy sets. In 2011, Alkhazaleh et al. defined the concept of soft expert sets where the user can know the opinion of all the experts in one model.

In this thesis, we introduce a generalization of soft expert sets defined by Alkhazaleh et al. which may be called graded soft expert (GSE) sets. We give three generalizations of soft expert sets named as graded soft expert sets, cubic soft expert sets and interval-valued intuitionistic fuzzy soft expert sets. Joint application of soft expert sets and other theories may result in a fruitful way in multi-criteria decision making. We also propose matrix algebra by using these generalizations. In each generalization, we propose an algorithm in decision analysis.

Chapter-wise study

The present work in this thesis consists of seven chapters. Concluding remarks and future work of each chapter are presented at the end of each contribution chapter. The first chapter gives a general introduction of the research work where the motivation and objectives are defined. In second chapter, some basic concepts of fuzzy sets, interval-valued fuzzy sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets, soft sets, soft expert sets, hesitant fuzzy sets, cubic sets and soft matrices are given with some of their properties and operations, which will be helpful for understanding the rest of the thesis.

Chapter three is a generalization of hesitant fuzzy sets. It is also a modified form of the soft expert sets introduced by Alkhazaleh and Salleh. Hesitant fuzzy sets play a vital role in decision analysis. With respect to a given set of criteria some decision makers have to decide among various alternatives. Although it has proved to be a landmark in evaluating informations, yet there are certain deficiencies in the structure. To be more specific, there is no standard inclusion measure to compare two hesitant fuzzy sets. The most significant among them was proposed by Xia and Xu [72]. But then, containment of two hesitant fuzzy elements in each other does not imply their equality. Also, in decision analysis with the aid of hesitant fuzzy sets, relative importance of the decision makers according to their area of expertise is ignored completely which may be misleading in some situations. These sort of issues have been resolved in this work by using graded soft expert (GSE) sets. The concept of the graded soft expert sets are defined and their basic operations such as complement, union and intersection are given. Some examples for these concepts, basic properties of the operations are also given. On the other hand, an algorithm along with the application of graded soft expert sets in decision making problem is illustrated at the end.

In chapter four, as a generalization of soft expert sets the concept of cubic soft expert sets have been introduced. In cubic soft expert sets we basically presented opinions of experts in cubic sets. Cubic sets consist of fuzzy sets and interval valued fuzzy sets. The aim of cubic soft expert sets is to present opinions of expert in the form of interval valued fuzzy set as well as a fuzzy set. In some cases experts give their opinions for present time period in the form of fuzzy sets and for future time period opinion may be represented in the form of interval valued fuzzy sets. In this type of structure we will easily aggregate the opinions of experts for present time as well as for future time. The cubic soft sets are primarily concerned with generalizing the soft sets by using fuzzy sets and interval valued fuzzy sets. We introduce the concept of cubic soft expert sets ($CSESs$) which can be considered as a generalization of both soft expert and cubic soft expert sets. The notions of internal cubic soft expert sets ($ICSESs$), external cubic soft expert sets ($ECSESs$), $P-order$, $P-union$, $P-intersection$, $P-AND$, $P-OR$ and $R-order$, $R-union$, $R-intersection$, $R-AND$, $R-OR$ have been defined for cubic soft expert sets ($CSESs$). We also investigate structural properties of these operations on cubic soft expert sets ($CSESs$). It has also been proved that cubic soft expert sets ($CSESs$) satisfy commutative, associative, De Morgan's, distributive, idempotent and absorption laws. At the end, an application of cubic soft expert sets theory in decision making is given with an algorithm and worked out example is provided for decision making with cubic soft expert sets.

In chapter five, we introduced some new operations on cubic soft expert sets ($CSESs$). Jun et al. only defined basic operation of inclusion, union and intersection in [34]. These new operations were not defined earlier for cubic sets but in this chapter we have defined addition and product of two cubic soft expert sets, scalar product, power of cubic soft expert set, score and accuracy function of $CSESs$. The purpose of defining score function and accuracy function is that we can determine the ranking of $CSESs$ which help us in some aggregation operators. Some aggregation operator on $CSESs$ have been introduced. By using these operators we can choose best alternative. Since fuzzy sets and interval-valued fuzzy sets play a fundamental role in decision analysis therefore, the aim of this chapter is to determine the most preferable choice among all possible choices, when data is in cubic set form. Finally, an example has been shown to highlight the procedure of the proposed algorithms.

In chapter six, In this chapter we intend to introduce interval valued intuitionistic fuzzy soft expert set ($IVFSE$ set) and certain operations on it. These include $IVIFSE$ weighted average operator, ordered weighted average operator, generalized ordered weighted average operator, ordered weighted arithmetic operator, fusion weighted average operator and generalized fusion weighted average operator. An algorithm of multicriteria decision making has been developed by using these operators and applied on practical decision making problem.

In chapter seven, we have initiated the concept of Graded soft expert matrices ($GSEMs$), cubic soft expert matrices ($CSEMs$) and interval-valued intuitionistic soft expert matrices ($IVIFSEMs$). The aim of this work is to handle a big data in easy way. We can easily aggregate the opinion of two experts point-wise. We gave some types and properties of these matrices. Two matrices are not commutative in general in ordinary matrices algebra. But there is a very interesting result that $GSEMs$, $CSEMs$ and $IVIFSEMs$ satisfy commutative law with respect to product. Also De Morgan's laws hold with respect to the product over addition and vice versa.

Chapter 1

Introduction

1.1 General Introduction

The classical set theory, also called crisp set theory, serves as one of the fundamental concepts in Mathematics. However, only a limited number of traditional methods of modelling and computing can be dealt with the help of crisp set theory. In practice, most of the problems in fields such as economics, engineering, environmental sciences, medical sciences and social sciences involve information sets which are vague rather than precise. Due to vagueness and uncertainties in these domains traditional methods cannot be applied here. Mathematicians develop important analytical skills and problem-solving strategies to assess a broad range of some issues in commerce, science and the arts. Mathematical models and simulations, and the interpretation of their results, are being called on increasingly in global decisions, as business, politics and management all become more quantitative in their methods. The application of mathematics is also in demand in the social sciences, particularly economics, where mathematical tools are used to formulate models of the complex interactions in an economic system. In several problems stochastic methods are widely used for uncertainty assessment in future performance. The probabilistic approach has always been considered the most important, but it has often been shown that it can involve problems that may be difficult to handle. Many studies show that fuzzy numbers may represent an important theoretical and practical tool to tackle uncertainty. In 1965, Zadeh initiated Fuzzy sets [83]. Fuzzy sets deal with possibilistic uncertainty, connected with imprecision of states, perceptions and preferences. Zadeh extended the concept of fuzzy sets by Interval valued fuzzy sets [84]. Interval-valued fuzzy sets have been used in medicine [37]. Klir discussed fuzzy sets, uncertainty and information in [39]. Turken discussed interval valued fuzzy sets in detail [67, 68, 69]. Atanassov introduced the concept of intuitionistic fuzzy sets [8]. The intuitionistic fuzzy sets can represent three states of the support, opposition, and neutrality simultaneously. Thus, the intuitionistic fuzzy sets may represent information more abundant and flexible than the fuzzy sets when uncertainty such as hesitancy degree is involved and hereby seems to be suitable for dealing with natural attributes of physical phenomena in complex management situations. He also introduced the notion of interval valued intuitionistic fuzzy sets by combining interval valued fuzzy sets and intuitionistic fuzzy sets [9]. Dubois studied on the combination of uncertain or imprecise pieces of information in ruled based systems [19, 20]. Luhandjula used compensatory operators in fuzzy linear programming with multiple objectives [44]. Gau and Buehrer proposed the concept of vague sets [27]. Further, Burillo and Bustince showed that concept of vague sets coincide with intuitionistic fuzzy sets in 1996 [10]. Soft set theory is a mathematical theory dealing with uncertainty was introduced by Molodstov in 1999 [52]. It attracted the attention of many researchers as the theory was well equipped with parameters. The soft set theory has been applied to many different fields. Molodtsov applied this theory to several directions [53]. Molodtsov has been given soft sets technique and its applications [54]. Vague soft sets and their properties have been discussed in [71]. Yang et al. discussed the combination of interval-valued intuitionistic fuzzy sets and soft sets in [77].

Maji et al. discussed decision making problems through soft sets and fuzzy soft sets [46, 48]. Maji et al. defined the operations of union and intersection on soft sets [47]. Ali et al. improved those operations which were based on the selection of parameters in particular [4]. Ali et al. examined soft sets algebraically using these new operations [5]. Sezgin and Atagun proved certain De Morgan's laws for soft sets theory and extended theoretical aspect of operations on soft sets [59]. They also discussed soft groups and normalistic soft groups [60]. Chen et al. and Ali studied parametrization reduction of soft sets and discussed its application in decision analysis [15, 6]. Jiang et al. discussed interval valued fuzzy soft sets and their properties in [30]. Feng et al. extended soft sets to soft rough sets [25] and Shabir et al. improved the structure by introducing modified soft rough sets [61]. Further extensions can be seen in [3, 26, 49, 50]. Maji also defined fuzzy soft sets theory and some properties of fuzzy soft sets [45]. Cagman studied fuzzy soft sets theory and its application [13]. Pei et al. [56] and Chen et al. [15] improved Maji's work. Li has been given an approach to fuzzy multi-attribute decision making under uncertainty in [42]. Szmidt has been discussed a consensus reaching process under intuitionistic fuzzy preference relation [63]. He also used intuitionistic fuzzy sets in group decision making [62]. Jun et al. has been developed soft BCK/BCI-algebras, soft p-ideals of soft BCI-algebras and applications of soft sets in ideal theory of BCK/BCI-algebras [31, 33, 32]. Gorzalczany discussed a method of inference in approximate reasoning based on interval valued fuzzy sets

[28]. Xu has been given methods for aggregating interval valued intuitionistic fuzzy information and their application to decision making [73]. Soft set theory has been applied to decision making problems [12, 23, 38]. Acar has been discussed soft sets and soft rings [1]. Aktas has been studied some algebraic applications of soft sets [3]. Lee defined bipolar-valued fuzzy sets and their operations in [40]. He also compared interval-valued fuzzy sets, intuitionistic fuzzy sets and bipolar-valued fuzzy sets in [41].

The requirement for information combination strategies is increasing in several fields of human knowledge. Aggregation is a basic concern for all kinds of knowledge based systems, from image processing to decision making, from pattern recognition to machine learning. Generally, we can say that aggregation has for purpose the synchronous utilization of different pieces of information (provided by several sources) in order to come to a conclusion or a decision. Several research groups are directly interested in finding solutions, among them the multi-criteria community, the sensor fusion community, the decision-making community, the data mining community, etc. and each of these groups use or propose some methodologies in order to perform an intelligent aggregation, as for instance the use of rules, the use of neuronal networks, the use of fusion specific techniques, the use of probability theory and fuzzy set theory, etc. But all these approaches are based on some numerical aggregation operator. Dombi defined the aggregated operator in [18]. Fuzzy multicriteria decision making is discussed in [29, 35, 57]. The ordered weighted geometric averaging operator is introduced by Xu [73]. Yager introduced the ordered weighted averaging operator [81]. In 1988, Yager provided a parameterized family of aggregation operators which have been used in many applications [79]. Yager provided a generalization of OWA operator by combining it with the generalized mean operator [22]. This combination leads to a class of operators which is called as the generalized ordered weighted averaging ($GOWA$) operators [82].

In the context of decision making analysis, Alkhazaleh and Salleh introduced the concept of soft expert sets [7]. This structure can be considered as a generalization of soft sets in which experts and their opinions have been added to make decision analysis more easy to handle. Jun et al. introduced the concept of cubic sets in 2012 by using fuzzy and interval valued fuzzy sets [34]. Khan et al. discussed the generalized version of Jun's cubic sets in semigroups [36]. Muhiuddin and Al-roqi introduced the concept of cubic soft sets with applications in BCI/BCK-algebras [55].

To analyze decision making problems, hesitant fuzzy set theory also proves pretty worthwhile. It was presented by Torra and Narukawa as a generalization of fuzzy set theory [66, 65]. Motivation behind this theory was the degree of hesitancy while making a decision. They introduced some basic operations and also discussed briefly its role in decision making analysis. Yang et al. extended hesitant fuzzy sets to hesitant fuzzy rough sets and also discussed operational laws in hesitant fuzzy sets [78]. Xia et al., Meng et al. and Tan et al. developed a series of aggregation operators for hesitant fuzzy information and discussed their application in decision making problems [70], [51] and [64]. Xu and Xia proposed a variety of distance and similarity measures on hesitant fuzzy sets [72].

1.2 Research Motivation and Objectives

Soft expert sets can be considered as a generalization of soft sets in which experts and their opinions have been added to make decision analysis easier to handle. Soft expert sets has the advantage over the existing theories that it gives expert's opinion for each parameter separately. Joint application of soft expert sets and other theories may result in a fruitful way in multi-criteria decision making. These include fuzzy soft expert sets and its applications, fuzzy parameterized soft expert sets, fuzzy parameterized fuzzy soft expert sets, application of generalized vague soft expert sets in decision making and possibility fuzzy soft expert sets. Generalized fuzzy soft expert sets is a combination of fuzzy soft expert sets and generalized fuzzy soft sets.

The objectives of research are:

1) In this work, we shall redefine and revise soft expert sets defined by Alkhazaleh et al. which may be called as graded soft expert (GSE) sets. We shall develop an algorithm of decision making with the aid of GSE sets. We'll develop the relationship of GSE sets with hesitant fuzzy sets. This will lead us to the generalization of many results which were valid for hesitant fuzzy sets.

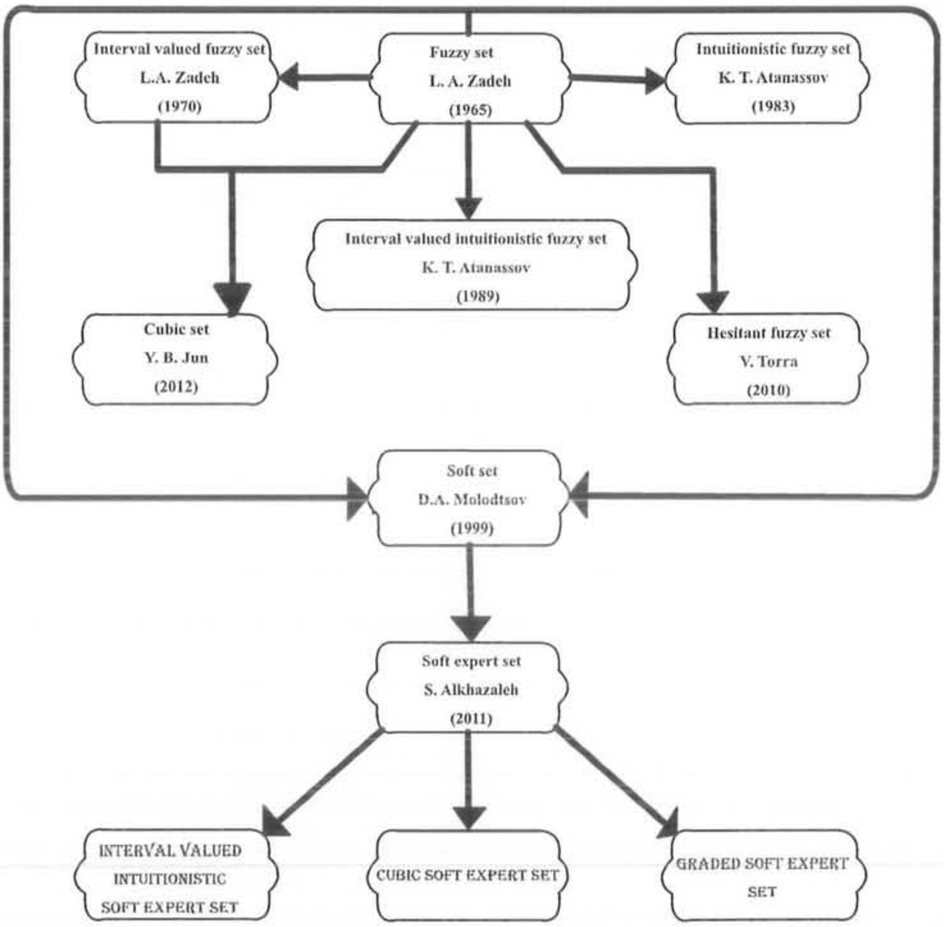
2) We shall give the concept of cubic soft expert sets ($CSESs$). We shall also consider the problem of combining soft expert sets with other theories like intuitionistic fuzzy sets, interval-valued fuzzy sets etc. We shall also develop different algorithms to support our theories in multi-criteria decision making problems.

3) We shall give some new operations on cubic soft expert sets. By using these operations we will define some aggregation operators which will help us in multicriteria decision making problem.

4) We shall also give the concept of interval-valued intuitionistic fuzzy soft expert sets ($IVIFSESs$). We shall define some operations and some aggregation operators on it. After that we shall develop algorithm for multicriteria decision making problem.

5) We shall give the concept of graded soft expert matrices, cubic soft expert matrices and interval-valued intuitionistic fuzzy soft expert matrices.

1.3 Contribution Diagram



Chapter 2

Literature Review

In this chapter, we recall some definitions related to fuzzy set, interval valued fuzzy set, intuitionistic fuzzy set, intuitionistic fuzzy soft set, interval valued intuitionistic fuzzy sets.

2.1 Basic Definitions

Definition 2.1.1 *[34] A fuzzy subset in a set U is defined to be a function $\lambda : U \to I$ where $I = [0, 1]$. The collection of all fuzzy sets in a set U is denoted by I^U. For any $\lambda, \mu \in I^U$ define a relation $\leq$ on I^U as follows:$\lambda \leq \mu \iff \lambda(u) \leq \mu(u)\ \forall u \in U$. The join $(\vee)$ and meet $(\wedge)$ of λ and μ are defined by*

$$(\lambda \vee \mu)(u) = \sup\{\lambda(u), \mu(u)\},$$
$$(\lambda \wedge \mu)(u) = \inf\{\lambda(u), \mu(u)\}$$

respectively, for all $u \in U$. The complement of λ, denoted by λ^c, is defined by

$$\lambda^c(u) = 1 - \lambda(u)\ \forall u \in U$$

For a family $\{\lambda_i | i \in \Lambda\}$ of fuzzy sets in U, we define the join $(\vee)$ and meet $(\wedge)$ operations as follows:

$$(\bigvee_{i \in \mathrm{u}} \lambda_i)(u) = sup\{\lambda_i(u) \mid i \in \Lambda\},$$
$$(\bigwedge_{i \in \mathrm{u}} \lambda_i)(u) = inf\{\lambda_i(u) \mid i \in \Lambda\}.$$

respectively, for all $u \in U$.

Definition 2.1.2 *[83] The fuzzy subsets of U, denoted by $\bar{0}$ and $\bar{1}$, which map every element of U onto 0 and 1 respectively, are called the empty fuzzy set or null fuzzy subset and the whole fuzzy subset of U respectively.*

Definition 2.1.3 *[28] Let U be a non-empty set. A function $A : U \rightarrow Int([0,1])$ is called an interval-valued fuzzy set, where $Int([0,1])$ stands for the set of all closed sub intervals of $[0,1]$, the set of all interval-valued fuzzy sets on U is denoted by $[I]^U$. For every $A \in [I]^U$ and $u \in U$, $A(u) = [A^-(u), A^+(u)]$ is called the degree of membership of an element u to A, where $A^- : U \rightarrow I$ and $A^+ : U \rightarrow I$ are fuzzy sets in U which are called lower fuzzy set and upper fuzzy set in U respectively. For simplicity, we denote $A = [A^-, A^+]$. For every $A, B \in [I]^U$, the complement of A is denoted by $A^c = [1 - A^+, 1 + A^-]$.*

Definition 2.1.4 *[34] Let $A = [A^-, A^+]$, and $B = [B^-, B^+]$ be two interval valued fuzzy sets in U. Then inf and sup of A and B are defined as follows:*

$$\inf\{A(u), B(u)\} = [\inf\{A^-(u), B^-(u)\}, \inf\{A^+(u), B^+(u)\}]$$

$$\sup\{A(u), B(u)\} = [\sup\{A^-(u), B^-(u)\}, \sup\{A^+(u), B^+(u)\}]$$

Definition 2.1.5 *[84] Let $A = [A^-, A^+]$, and $B = [B^-, B^+]$ be two interval valued fuzzy sets in U. Then, we define "$\preceq$", and "$\succeq$", as*

$$A(u) \preceq B(u) \text{ if and only if } A^-(u) \leq B^-(u) \text{ and } A^+(u) \leq B^+(u)$$

Similarly,

$$A(u) \succeq B(u) \text{ if and only if } A^-(u) \geq B^-(u) \text{ and } A^+(u) \geq B^+(u)$$

for all $u \in U$. For every $A, B \in [I]^U$, we define $A \subseteq B$ if and only if $A(u) \preceq B(u)$ for all $u \in U$.

Definition 2.1.6 *[8] Let U be a non empty set of the universe. If there are two mappings on U,*

$$\mu_{\widetilde{A}} : U \rightarrow [0,1]$$

$u \longmapsto \mu_{\widetilde{A}}(u)$ and

$$\nu_{\widetilde{A}} : U \rightarrow [0,1]$$

$u \longmapsto \nu_{\widetilde{A}}(u)$ satisfying the condition $\mu_{\widetilde{A}}(u) + \nu_{\widetilde{A}}(u) \leq 1$. Intuitionistic fuzzy set on the universal set U is denoted and defined as $\widetilde{A} = \{< u, \mu_{\widetilde{A}}(u), \nu_{\widetilde{A}}(u) >: u \in U\}$. $\mu_{\widetilde{A}}(u)$ and $\nu_{\widetilde{A}}(u)$ are called the membership degree and nonmembership degree of an element u belonging to $\widetilde{A} \subseteq U$ respectively. The set of all intuitionistic fuzzy sets on the universal set U is denoted by $F(U)$.

Definition 2.1.7 *[30] Consider U and E as a universe set and a set of parameters respectively . Let $A \subseteq E$. A pair (β, A) is an intuitionistic fuzzy soft set over U, where β is a mapping given by $\beta : A \rightarrow F(U)$. For any parameter $a \in A$, $\beta(a)$ is an intuitionistic fuzzy subset of U and it is called intuitionistic fuzzy value set of parameter a. $\beta(a)$ can be written as an intuitionistic fuzzy set*

$$\beta(a) = \{< u, \mu_{\beta(a)}(u), \nu_{\beta(a)}(u) >: u \in U\}$$

where $\mu_{\beta(a)}(u)$ and $\nu_{\beta(a)}(u)$ are the membership and non-membership functions respectively. If for all $u \in U$, $\mu_{\beta(a)}(u) = 1- \nu_{\beta(a)}(u)$ then $\beta(a)$ will degenerate to a standard fuzzy set and then (β, A) will degenerate to a traditional fuzzy soft set.

Definition 2.1.8 *[9] Let $U \neq \phi$ be a set of the universe. $\mu_{\widetilde{A}}$ and $\nu_{\widetilde{A}}$ determine an interval-valued intuitionistic fuzzy $(IVIF)$ set $\widetilde{A}$ on X if the two interval-valued mappings*

$$\mu_{\widetilde{A}} : U \rightarrow I_{[0,1]}$$

$u \longmapsto \mu_{\widetilde{A}}(u)$ *and*

$$\nu_{\widetilde{A}} : U \rightarrow I_{[0,1]}$$

$u \longmapsto \nu_{\widetilde{A}}(u)$ *satisfy the following condition:* $0 \leq sup\{\mu_{\widetilde{A}}(u)\} + sup\{\nu_{\widetilde{A}}(u)\} \leq 1$. *The interval-valued intuitionistic fuzzy set is denoted as*

$$\widetilde{A} = \{< u, \mu_{\widetilde{A}}(u), \nu_{\widetilde{A}}(u) >: u \in U\},$$

where $\mu_{\widetilde{A}}(u)$ and $\nu_{\widetilde{A}}(u)$ are called the interval valued membership degree and non membership degree of an element u belonging to $\widetilde{A}$ respectively. The set of the $IVIF$ sets on the universe set U is denoted by $\widetilde{\mathfrak{F}}_I(U)$. The interval valued intuitionistic fuzzy set $\widetilde{A}$ can be expressed in the interval valued format as follows

$$\widetilde{A} = \{< u, [\mu^-_{\widetilde{A}}(u), \mu^+_{\widetilde{A}}(u)], [\nu^-_{\widetilde{A}}(u), \nu^+_{\widetilde{A}}(u)] >: u \in U\},$$

where $\mu^-_{\widetilde{A}}(u), \mu^+_{\widetilde{A}}(u), \nu^-_{\widetilde{A}}(u), \nu^+_{\widetilde{A}}(u) \in [0, 1]$ *and* $\mu^+_{\widetilde{A}}(u) + \nu^+_{\widetilde{A}}(u) \leq 1$.

Definition 2.1.9 *[30] The interval-valued hesitancy degree (or intuitionistic fuzzy index) of an element u belonging to the interval-valued intuitionistic fuzzy set $\widetilde{A}$ is denoted and defined as follows:*

$$\pi_{\widetilde{A}}(u) = [1 - \mu^+_{\widetilde{A}}(u) - \nu^+_{\widetilde{A}}(u), 1 - \mu^-_{\widetilde{A}}(u) - \nu^-_{\widetilde{A}}(u)].$$

In situations where more than one expert opinion is necessary, Alkhazaleh [7] introduced soft expert sets and claimed that if we want to take the opinion of more than one experts, we need some operations such as union, intersection, and so forth. This causes a problem with the user, especially with those who use questionnaires in their work and studies. So in this model, the user can know the opinion of all experts in one model without using any operations.

In this section, we give some basic concepts related to soft sets, soft expert sets and hesitant fuzzy sets. These will be required in the later sections.

2.2 Soft Sets

Let U be a non-empty set representing the universe set and $P(U)$ denotes the power set of U. Let E be the set of parameters and A, B be non-empty subsets of E.

Definition 2.2.1 *[52] A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$. Soft set is basically a parameterized family of subsets of the set U. Thus, soft set can be considered as a parameterized family of subsets of the universe U. For $e \in A$, $F(e)$ gives the set of e-approximate elements of the soft set (F, A).*

Definition 2.2.2 *[47] For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B), denoted by $(F, A)\tilde{\subset}(G, B)$, if*

(1) $A \subseteq B$ and

(2) $F(e) \subseteq G(e)$ for all $e \in A$.

Definition 2.2.3 *[47] Two soft sets (F, A) and (G, B) over a common universe U are said to be soft equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).*

Definition 2.2.4 *[4] Let U be an initial universe set, E be the set of parameters, and $A \subseteq E$.*

(a) (F, A) is called a relative null soft set (with respect to the parameter set A), denoted by $\emptyset_A$, if $F(e) = \emptyset$ for all $e \in A$.

(b) (G, A) is called a relative whole soft set (with respect to the parameter set A), denoted by $A_{\mathcal{U}}$, if $G(e) = U$ for all $e \in A$.

Remark 2.2.5 *If relative null soft set is taken over E, it is called null soft set over U and is denoted by $\emptyset_E$. In a similar way, relative whole soft set with respect to the set of parameters E is called the absolute soft set over U and is denoted by E_U.*

Empty soft set over U, denoted by $\emptyset_\emptyset$, is a unique soft set over U with an empty parameter set.

The operations of union and intersection on soft sets have been defined as below.

Definition 2.2.6 *[4] (1) Extended union of two soft sets (F, A) and (G, B) over the common universe U is the soft set (H, C), where $C = A \cup B$ and for all $e \in C$,*

$$H(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B \\ G(e) & \text{if } e \in B \setminus A \\ F(e) \cup G(e) & \text{if } e \in A \cap B \end{cases}$$

We write $(F, A) \cup_{\varepsilon} (G, B) = (H, C)$.

(2) Let (F, A) and (G, B) be two soft sets over the same universe U, such that $A \cap B \neq \emptyset$. The restricted union of (F, A) and (G, B) is denoted by $(F, A) \cup_{\mathcal{R}} (G, B)$ and is defined as $(F, A) \cup_{\mathcal{R}} (G, B) = (H, C)$, where $C = A \cap B$ and for all $e \in C$, $H(e) = F(e) \cup G(e)$. If $A \cap B = \emptyset$, then $(F, A) \cup_{\mathcal{R}} (G, B) = \emptyset_{\emptyset}$.

Definition 2.2.7 *[4] (1) The extended intersection of two soft sets (F, A) and (G, B) over a common universe U, is the soft set (H, C) where $C = A \cup B$ and for all $e \in C$,*

$$H(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B \\ G(e) & \text{if } e \in B \setminus A \\ F(e) \cap G(e) & \text{if } e \in A \cap B \end{cases}$$

We write $(F, A) \cap_{\varepsilon} (G, B) = (H, C)$.

(2) Let (F, A) and (G, B) be two soft sets over the same universe U such that $A \cap B \neq \emptyset$. The restricted intersection of (F, A) and (G, B) is denoted by $(F, A) \cap_{\mathcal{R}} (G, B)$ and is defined as $(F, A) \cap_{\mathcal{R}} (G, B) = (H, A \cap B)$ where $H(e) = F(e) \cap G(e)$ for all $e \in A \cap B$. If $A \cap B = \emptyset$ then $(F, A) \cap_{\mathcal{R}} (G, B) = \emptyset_{\emptyset}$.

2.3 Soft Expert Sets

Now, we give some basic concepts related to soft expert sets. All the definitions related to soft expert sets have been taken from [7].

Let U be a universe set, E be a set of parameters, X be a set of experts and O be the set of opinions. Let A be a non-empty subset of Z, where $Z = E \times X \times O$. With these notations Alkhazaleh [7] defined soft expert set as stated below:

Definition 2.3.1 *A pair (F, A) is called a soft expert set over U, where F is a mapping given by $F : A \to P(U)$. Thus, a soft expert set can be considered as a soft set in which parameter set is replaced with Cartesian product of set of parameters, set of experts and set of opinions.*

Definition 2.3.2 *For two soft expert sets (F, A) and (G, B) over U, (F, A) is called a soft expert subset of (G, B) if*

1) $A \subseteq B$ *and*

2) $F(a) \subseteq G(a)$ *for all $a \in A$. In that case (G, B) will be called soft expert superset of (F, A).*

Definition 2.3.3 *Two soft expert sets (F, A) and (G, B) over U are said to be equal if (F, A) is a soft expert subset of (G, B) and (G, B) is a soft expert subset of (F, A).*

Definition 2.3.4 *Let $E = \{e_1, e_2, ..., e_n\}$ be a set of parameters. The NOT set of E denoted by $\rceil E$ is defined by $\rceil E = \{\rceil e_1, \rceil e_2, ..., \rceil e_n\}$ where $\rceil e_i$ represents 'not e_i' for all i.*

Definition 2.3.5 *The complement of a soft expert set (F, A) is denoted and defined as $(F, A)^c = (F^c, ' IA)$ where $F^c : ' IA \longrightarrow P(U)$ is a mapping given by*

$$F^c(a) = U - F(' IA) \quad \text{for all } a \in' IA.$$

Definition 2.3.6 *The union of two soft expert sets (F, A) and (G, B) over U, denoted by $(F, A)\ \tilde{U}\ (G, B)$, is a soft expert set (H, C), where $C = A \cup B$ and for all $a \in C$,*

$$H(a) = \begin{cases} F(a) & \text{if } a \in A - B \\ G(a) & \text{if } a \in B - A \\ F(a) \cup G(a) & \text{if } a \in A \cap B. \end{cases}$$

Proposition 2.3.7 *If (F, A) , (G, B) , and (H, C) are three soft expert sets over U, then*

1) $(F, A)\ \tilde{\cup}\ (G, B) = (G, B)\ \tilde{\cup}\ (F, A)$,

2) $(F, A)\ \tilde{\cup}\ ((G, B)\ \tilde{\cup}\ (H, C)) = ((F, A)\ \tilde{\cup}\ (G, B))\ \tilde{\cup}\ (H, C)$.

Definition 2.3.8 *The intersection of two soft expert sets (F, A) and (G, B) over U, denoted by $(F, A)\ \tilde{\cap}\ (G, B)$, is a soft expert set (H, C) where $C = A \cup B$ and for all $a \in C$,*

$$H(a) = \begin{cases} F(a) & \text{if } a \in A - B \\ G(a) & \text{if } a \in B - A \\ F(a) \cap G(a) & \text{if } a \in A \cap B. \end{cases}$$

Proposition 2.3.9 *If (F, A) , (G, B) , and (H, C) are three soft expert sets over U, then*

1) $(F, A)\ \tilde{\cap}\ (G, B) = (G, B)\ \tilde{\cap}\ (F, A)$,

2) $(F, A)\ \tilde{\cap}\ ((G, B)\ \tilde{\cap}\ (H, C)) = ((F, A)\ \tilde{\cap}\ (G, B))\ \tilde{\cap}\ (H, C)$.

Proposition 2.3.10 *If (F, A) , (G, B) , and (H, C) are three soft expert sets over U, then*

$$1)\ (F,A)\ \widetilde{\cup}\ ((G,B)\ \widetilde{\cap}\ (H,C)) = ((F,A)\ \widetilde{\cup}\ (G,B))\ \widetilde{\cap}\ ((F,A)\ \widetilde{\cup}\ (H,C))\ ,$$
$$2)\ (F,A)\ \widetilde{\cap}\ ((G,B)\ \widetilde{\cup}\ (H,C)) = ((F,A)\ \widetilde{\cap}\ (G,B))\ \widetilde{\cup}\ ((F,A)\ \widetilde{\cap}\ (H,C))\ .$$

Definition 2.3.11 *If $Z = E \times X \times \{1\}$ in Definition 2.3.1, then (F, A) is called agree soft expert set over U and it is denoted by $(F, A)_1$.*

Definition 2.3.12 *If $Z = E \times X \times \{0\}$ in Definition 2.3.1, then (F, A) is called disagree soft expert set over U and it is denoted by $(F, A)_0$.*

Proposition 2.3.13 *If (F, A) is a soft expert set over U, then*

$$1)\ ((F,A)^c)^c = (F,A),$$
$$2)\ (F,A)_1^c = (F,A)_0,$$
$$3)\ (F,A)_0^c = (F,A)_1.$$

Definition 2.3.14 *If (F, A) and (G, B) are two soft expert sets over U then (F, A) AND (G, B) denoted by $(F, A) \wedge (G, B)$, is defined by*

$$(F,A) \wedge (G,B) = (H, A \times B),$$

where $H(a, b) = F(a) \cap G(b)$, for all $(a, b) \in A \times B$.

Definition 2.3.15 *If (F, A) and (G, B) are two soft expert sets then (F, A) OR (G, B) denoted by $(F, A) \vee (G, B)$, is defined by*

$$(F,A) \vee (G,B) = (O, A \times B).$$

where $O(a, b) = F(a) \cup G(b)$, for all $(a, b) \in A \times B$.

Proposition 2.3.16 *If (F, A) and (G, B) are two soft expert sets over U, then*

$$1)\ ((F,A) \wedge (G,B))^c = (F,A)c \vee (G,B)^c,$$
$$2)\ ((F,A) \vee (G,B))^c = (F,A)^c \wedge (G,B)^c.$$

Proposition 2.3.17 *If (F, A) , (G, B) , and (H, C) are three soft expert sets over U, then*

$$1)(F,A) \wedge ((G,B) \wedge (H,C)) = ((F,A) \wedge (G,B)) \wedge (H,C),$$
$$2)(F,A) \vee ((G,B) \vee (H,C)) = ((F,A) \vee (G,B)) \vee (H,C),$$
$$3)(F,A) \vee ((G,B) \wedge (H,C)) = ((F,A) \vee (G,B)) \wedge ((F,A) \vee (H,C)),$$
$$4)(F,A) \wedge ((G,B) \vee (H,C)) = ((F,A) \wedge (G,B)) \vee ((F,A) \wedge (H,C)).$$

2.4 Hesitant Fuzzy Sets

Definition 2.4.1 *[66] Let X be a fixed set. A hesitant fuzzy set (HFS) on X is in terms of a function that when applied to X returns a subset of $[0,1]$.*

Thus, if h is a hesitant fuzzy set on X, then $h(x)$ $(x \in X)$, being a subset of $[0,1]$, gives the possible degrees of membership. For any $x \in X$, $h(x)$ is called a hesitant fuzzy element.

Remark 2.4.2 *Torra [66] defined lower and upper bounds for a hesitant fuzzy element as below:*
lower bound: $h^-(x) = \min\{\gamma : \gamma \in h(x)\}$
upper bound: $h^+(x) = \max\{\gamma : \gamma \in h(x)\}$

Basic operations on hesitant fuzzy sets are given below.

Definition 2.4.3 *For hesitant fuzzy sets h, h_1 and h_2 on X, the following operations have been defined:*

1) Containment [78]: h_1 is contained in h_2, denoted by $h_1 \preceq h_2$, if and only if $h_1^-(x) \leq h_2^-(x)$ and $h_1^+(x) \leq h_2^+(x)$ for all $x \in X$;

2) Union [66]: union of h_1 and h_2, denoted by $h_1 \Cup h_2$, is defined for any $x \in X$ as $(h_1 \Cup h_2)(x) = \{h \in h_1(x) \cup h_2(x) : h \geq \max\{h_1^-(x), h_2^-(x)\}$;

3) Intersection [66]: intersection of h_1 and h_2, denoted by $h_1 \Cap h_2$, is defined for any $x \in X$ as $(h_1 \Cap h_2)(x) = \{h \in h_1(x) \cup h_2(x) : h \leq \min\{h_1^+(x), h_2^+(x)\}$;

4) Complement [66]: complement of h is denoted by h^c and is defined for any $x \in X$ as $h^c(x) = \bigcup_{\gamma \in h(x)} \{1-\gamma\}$.

Operational laws investigated by Yang et al. [78] are stated in the next theorem.

Theorem 2.4.4 *For hesitant fuzzy sets h_1, h_2 and h_3 on X, following properties hold:*

1) Idempotent: $h_1 \Cup h_1 = h_1$, $h_1 \Cap h_1 = h_1$;

2) Commutative: $h_1 \Cup h_2 = h_2 \Cup h_1$, $h_1 \Cap h_2 = h_2 \Cap h_1$;

3) Associative: $h_1 \Cup (h_2 \Cup h_3) = (h_1 \Cup h_2) \Cup h_3$, $h_1 \Cap (h_2 \Cap h_3) = (h_1 \Cap h_2) \Cap h_3$;

4) Distributive: $h_1 \Cup (h_2 \Cap h_3) = (h_1 \Cup h_2) \Cap (h_1 \Cup h_3)$, $h_1 \Cap (h_2 \Cup h_3) = (h_1 \Cap h_2) \Cup (h_1 \Cap h_3)$;

5) De Morgan's laws: $(h_1 \Cup h_2)^c = h_1^c \Cap h_2^c$, $(h_1 \Cap h_2)^c = h_1^c \Cup h_2^c$;

6) Double negation: $(h^c)^c = h$.

2.5 Cubic Sets

Now, we give some basic concepts related to cubic sets. All the definitions related to cubic sets have been taken from [34].

Definition 2.5.1 *Let U be a non-empty set. By a cubic set in U, we mean a structure $\alpha = \{< u, A(u), \lambda(u) >: u \in U\}$ in which A is an interval valued fuzzy set in U (briefly, IVF set) and λ is a fuzzy set in U. A cubic set $\alpha = \{< u, A(u), \lambda(u) >: u \in U\}$ is simply denoted by $\alpha =< A, \lambda >$. A cubic set $\alpha =< A, \lambda >$ in which for all $u \in U$, $A(u) = 0$ and $\lambda(u) = 1$ (respectively $A(u) = 1$ and $\lambda(u) = 0$) for all $u \in U$ is denoted by $\ddot{0}$ (respectively $\ddot{1}$). A cubic set $\beta =< B, \mu >$, in which $B(u) = 0$ and $\mu(u) = 0$ (respectively $B(u) = 1$ and $\mu(u) = 1$) is denoted by $\ddot{\ddot{0}}$ (respectively $\ddot{\ddot{1}}$). The collection of all cubic sets in U is denoted by $CP(U)$.*

Definition 2.5.2 *A cubic set $\alpha = \{< u, A(u), \lambda(u) >: u \in U\}$ is said to be an internal cubic set (ICS) if $A^-(u) \leq \lambda(u) \leq A^+(u)$ for all $u \in U$.*

Definition 2.5.3 *A cubic set $\alpha = \{< u, A(u), \lambda(u) >: u \in U\}$ is said to be an external cubic set (ECS) if $\lambda(u) \notin (A^-(u), A^+(u))$ for all $u \in U$.*

Theorem 2.5.4 *Let $\alpha = \{< u, A(u), \lambda(u) >: u \in U\}$ be a cubic set in U which is not an ECS. Then there exists a $u \in U$ such that $\lambda(u) \in ((A^-(u), A^+(u))$.*

Theorem 2.5.5 *Let $\alpha =< A, \lambda >$ be a cubic set in U. If α is both an ICS and an ECS, then*

for all $u \in U$ $(\lambda(u) \in (U(A) \cup L(A))$,
where $U(A) = \{A^+(u) \mid u \in U\}$ and $L(A) = \{A^-(u) \mid u \in U\}$.

Remark 2.5.6 *Every intuitionistic fuzzy set $A = \{\lambda(u), \mu(u) | u \in U\}$ in U is considered as a cubic set in U.*

Definition 2.5.7 *Let $\alpha =< A, \lambda >$ and $\beta =< B, \mu >$ be cubic sets in U. Then we define*

1) $\alpha = \beta$ if and only if $A = B$ and $\lambda = \mu$. (Equality)
2) $\alpha \subseteq_P \beta$ if and only if $A \subseteq B$ and $\lambda \leq \mu$. ($P-order$)
3) $\alpha \subseteq_R \beta$ if and only if $A \subseteq B$ and $\lambda \geq \mu$. ($R-order$)

Definition 2.5.8 *For any $\alpha_i =< A_i(u), \lambda_i(u) : u \in U > i \in \Lambda$, we define*

1) $\underset{i\in\Lambda}{\cup_P}\alpha_i = \{u, (\underset{i\in\Lambda}{\cup} A_i)(u), (\underset{i\in\Lambda}{\vee}\lambda_i)(u) : u \in U\}$ ($P-union$)
2) $\underset{i\in\Lambda}{\cap_P}\alpha_i = \{u, (\underset{i\in\Lambda}{\cap} A_i)(u), (\underset{i\in\Lambda}{\wedge}\lambda_i)(u) : u \in U\}$ ($P-intersection$)
3) $\underset{i\in\Lambda}{\cup_R}\alpha_i = \{u, (\underset{i\in\Lambda}{\cup} A_i)(u), (\underset{i\in\Lambda}{\wedge}\lambda_i)(u) : u \in U\}$ ($R-union$)
4) $\underset{i\in\Lambda}{\cap_R}\alpha_i = \{u, (\underset{i\in\Lambda}{\cap} A_i)(u), (\underset{i\in\Lambda}{\vee}\lambda_i)(u) : u \in U\}$ ($R-intersection$)

Definition 2.5.9 *The complement of cubic set* $\alpha =< A, \lambda >$, *is defined as*

$$\alpha^c =< u,\ A^c(u), 1 - \lambda(u) : u \in U > .$$

Proposition 2.5.10 *For any cubic sets* $\alpha =< A, \lambda >$, $\beta =< B, \mu >$, $\gamma =< C, \nu >$ *and* $\delta =< D, \upsilon >$, *we have*

1) *If* $\alpha \subseteq_P \beta$ *and* $\beta \subseteq_P \gamma$, *then* $\alpha \subseteq_P \gamma$,
2) *If* $\alpha \subseteq_P \beta$ *then,* $\beta^c \subseteq_P \alpha^c$,
3) *If* $\alpha \subseteq_P \beta$ *and* $\alpha \subseteq_P \gamma$, *then* $\alpha \subseteq_P \beta \cap_P \gamma$,
4) *If* $\alpha \subseteq_P \beta$ *and* $\gamma \subseteq_P \beta$, *then* $\alpha \cup_P \gamma \subseteq_P \beta$,
5) *If* $\alpha \subseteq_P \beta$ *and* $\gamma \subseteq_P \delta$, *then* $\alpha \cup_P \gamma \subseteq_P \beta \cup_P \delta$ *and* $\alpha \cap_P \gamma \subseteq_P \beta \cap_P \delta$,
6) *If* $\alpha \subseteq_R \beta$ *and* $\beta \subseteq_R \gamma$, *then* $\alpha \subseteq_R \gamma$,
7) *If* $\alpha \subseteq_R \beta$, *then* $\beta^c \subseteq_R \alpha^c$,
8) *If* $\alpha \subseteq_R \beta$ *and* $\alpha \subseteq_R \gamma$, *then* $\alpha \subseteq_R \beta \cap_R \gamma$,
9) *If* $\alpha \subseteq_R \beta$ *and* $\gamma \subseteq_R \beta$, *then* $\alpha \cup_R \gamma \subseteq_R \beta$,
10) *If* $\alpha \subseteq_R \beta$ *and* $\gamma \subseteq_R \delta$, *then* $\alpha \cup_R \gamma \subseteq_R \beta \cup_R \delta$ *and* $\alpha \cap_R \gamma \subseteq_R \beta \cap_R \delta$.

Theorem 2.5.11 *Let* $\alpha =< A, \lambda >$, $\beta =< B, \mu >$ *be two ECSs in U, such that* $\alpha^* =< A, \mu >$, $\beta^* =< B, \lambda >$ *are ICSs in U. Then* $\alpha \cup_P \beta$ *is an ICS in U.*

Theorem 2.5.12 *Let* $\alpha =< A, \lambda >$, $\beta =< B, \mu >$ *be two ECSs in U, such that* $\alpha^* =< A, \mu >$, $\beta^* =< B, \lambda >$ *are ICSs in U. Then* $\alpha \cap_P \beta$ *is an ICS in U.*

Theorem 2.5.13 *Let* $\alpha =< A, \lambda >$, $\beta =< B, \mu >$ *be two ECSs in U, such that* $\alpha^* =< A, \mu >$, $\beta^* =< B, \lambda >$ *are ECSs in U. Then* $\alpha \cap_P \beta$ *is an ECS in U.*

2.6 Soft Matrices

The material presented in this section is taken from [11]. We give the definitions and types of soft matrices and some related results.

Definition 2.6.1 *Let U be an initial universe,* $P(U)$ *be the power set of U; E be the set of all parameters and* $A \subseteq E$. *A soft set* (f_A, E) *on the universe U is defined by the set of ordered pairs*

$$(f_A, E) = \{(e, f_A(e)) : e \in E,\ f_A(e) \in P(U)\},$$

where $f_A : E \to P(U)$ *such that* $f_A(e)=\varnothing$ *if* $e \notin A$.

Here, f_A *is called an approximate function of the soft set* (f_A, E). *The set* $f_A(e)$ *is called e-approximate value set or e-approximate set which consists of related objects of the parameter* $e \in E$.

Definition 2.6.2 *Let (f_A, E) be a soft set over U. Then a subset of $U \times E$ is uniquely defined by*

$$R_A = \{(u, e):\ e \in A,\ u \in f_A(e)\}$$

which is called a relation form of (f_A, E). The characteristic function of R_A is written as

$$\chi_{R_A} : U \times E \to [0,1],\ \chi_{R_A}(u,e) = \begin{cases} 1, & (u,e) \in R_A \\ 0, & (u,e) \notin R_A \end{cases}$$

Definition 2.6.3 *If $U = \{u_1, u_2, \ , u_m\}$, $E = \{e_1, e_2, ..., e_n\}$ and $A \subseteq E$, then the R_A can be presented by a table as in the following form*

R_A	e_1	e_2	...	e_n
u_1	$\chi_{R_A}(u_1, e_1)$	$\chi_{R_A}(u_1, e_2)$	...	$\chi_{R_A}(u_1, e_n)$
u_2	$\chi_{R_A}(u_2, e_1)$	$\chi_{R_A}(u_2, e_2)$	...	$\chi_{R_A}(u_2, e_n)$
...	...	...	...	...
u_m	$\chi_{R_A}(u_m, e_1)$	$\chi_{R_A}(u_m, e_2)$	...	$\chi_{R_A}(u_m, e_n)$

Table 2.6.1

If $a_{ij} = \chi_{R_A}(u_i, e_j)$, then we can define a matrix

$$[a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}.$$

which is called an $m \times n$ soft matrix of the soft set (f_A, E) over U.
The set of all $m \times n$ soft matrices over U will be denoted by $SM_{m\times n}$.

Example 2.6.4 *Assume that $U = \{u_1, u_2, u_3, u_4, u_5\}$ is a universal set and $E = \{e_1, e_2, e_3, e_4\}$ is a set of all parameters. If $A = \{e_2, e_3, e_4\}$ and $f_A(e_2) = \{u_2, u_4\}$, $f_A(e_3) = \varnothing$, $f_A(e_4) = U$, then we write a soft set $(f_A, E) = \{(e_2, \{u_2, u_4\}), (e_4, U)\}$ and the relation form of (f_A, E) is written as*

$$R_A = \{(u_2, e_2), (u_4, e_2), (u_1, e_4), (u_2, e_4), (u_3, e_4), (u_4, e_4), (u_5, e_4)\}.$$

Hence, the soft matrix $[a_{ij}]$ is written by

$$[a_{ij}] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Definition 2.6.5 *Let* $[a_{ij}] \in SM_{m\times n}$. *Then* $[a_{ij}]$ *is called*

a) a zero soft matrix, denoted by $[0]$, *if* $a_{ij} = 0$ *for all* i *and* j.
b) an A-universal soft matrix, denoted by $[\widetilde{a_{ij}}]$, *if* $a_{ij} = 1$ *for all* $j \in I_A = \{j : e_j \in A)$ *and* i.
c) a universal soft matrix, denoted by $[1]$, *if* $a_{ij} = 1$ *for all* i *and* j.

Definition 2.6.6 *Let* $[a_{ij}], [b_{ij}] \in SM_{m\times n}$. *Then*

a) $[a_{ij}]$ *is a soft submatrix of* $[b_{ij}]$, *denoted by* $[a_{ij}] \subseteq [b_{ij}]$ *if* $a_{ij} \leq b_{ij}$ *for all* i *and* j.
b) $[a_{ij}]$ *is a proper soft submatrix of* $[b_{ij}]$, *denoted by* $[a_{ij}] \subset [b_{ij}]$ *if* $a_{ij} \leq b_{ij}$ *for atleast one term* $a_{ij} < b_{ij}$ *for all* i *and* j.
c) $[a_{ij}]$ *and* $[b_{ij}]$ *are soft equal matrices, denoted by* $[a_{ij}] = [b_{ij}]$ *if* $a_{ij} = b_{ij}$ *for all* i *and* j.

Definition 2.6.7 *Let* $[a_{ij}], [b_{ij}] \in SM_{m\times n}$. *Then the soft matrix* $[c_{ij}]$ *is called*

(a) union of $[a_{ij}]$ *and* $[b_{ij}]$, *denoted* $[a_{ij}]\widetilde{\cup}[b_{ij}] = [c_{ij}]$ *if* $c_{ij} = \max\{a_{ij}, b_{ij}\}$ *for all* i *and* j.
b) intersection of $[a_{ij}]$ *and* $[b_{ij}]$, *denoted* $[a_{ij}]\widetilde{\cap}[b_{ij}] = [c_{ij}]$ *if* $c_{ij} = \min\{a_{ij}, b_{ij}\}$ *for all* i *and* j.
c) complement of $[a_{ij}]$, *is denoted by* $[a_{ij}]^o = [c_{ij}]$, *if* $c_{ij} = 1 - a_{ij}$ *for all* i *and* j.

Definition 2.6.8 *Let* $[a_{ij}], [b_{ij}] \in SM_{m\times n}$. *Then* $[a_{ij}]$ *and* $[b_{ij}]$*are disjoint, if*

$$[a_{ij}]\widetilde{\cap}[b_{ij}] = [0],$$

for all i *and* j.

Proposition 2.6.9 *Let* $[a_{ij}] \in SM_{m\times n}$. *Then*

$$1)\ [[a_{ij}]^o]^o = [a_{ij}],$$
$$2)\ [0]^o = [1].$$

Proposition 2.6.10 *Let* $[a_{ij}], [b_{ij}] \in SM_{m\times n}$. *Then*

1) $[a_{ij}] \subseteq [1]$,
2) $[0] \subseteq [a_{ij}]$,
3) $[a_{ij}] \subseteq [a_{ij}]$,
4) *if* $[a_{ij}] \subseteq [b_{ij}]$ *and* $[b_{ij}] \subseteq [c_{ij}]$ *then* $[a_{ij}] \subseteq [c_{ij}]$.

Proposition 2.6.11 *Let* $[a_{ij}], [b_{ij}], [c_{ij}] \in SM_{m\times n}$. *Then*

1) *if* $[a_{ij}] = [b_{ij}]$ *and* $[b_{ij}] = [c_{ij}]$ *if and only if* $[a_{ij}] = [c_{ij}]$,
2) *if* $[a_{ij}] = [b_{ij}]$ *and* $[b_{ij}] = [a_{ij}]$ *if and only if* $[a_{ij}] = [b_{ij}]$.

Proposition 2.6.12 *Let $[a_{ij}]$, $[b_{ij}], [c_{ij}] \in SM_{m\times n}$. Then*

1) $[a_{ij}]\widetilde{\cup}[a_{ij}] = [a_{ij}]$,
2) $[a_{ij}]\widetilde{\cup}[0] = [a_{ij}]$,
3) $[a_{ij}]\widetilde{\cup}[1] = [1]$,
4) $[a_{ij}]\widetilde{\cup}[a_{ij}]^o = [1]$,
5) $[a_{ij}]\widetilde{\cup}[b_{ij}] = [b_{ij}]\widetilde{\cup}[a_{ij}]$,
6) $[a_{ij}]\widetilde{\cup}([b_{ij}]\widetilde{\cup}[c_{ij}]) = ([a_{ij}]\widetilde{\cup}[b_{ij}])\widetilde{\cup}[c_{ij}]$.

Proposition 2.6.13 *Let $[a_{ij}]$, $[b_{ij}], [c_{ij}] \in SM_{m\times n}$. Then*

1) $[a_{ij}]\widetilde{\cap}[a_{ij}] = [a_{ij}]$,
2) $[a_{ij}]\widetilde{\cap}[0] = [0]$,
3) $[a_{ij}]\widetilde{\cap}[1] = [a_{ij}]$,
4) $[a_{ij}]\widetilde{\cap}[a_{ij}]^o = [0]$,
5) $[a_{ij}]\widetilde{\cap}[b_{ij}] = [b_{ij}]\widetilde{\cap}[a_{ij}]$,
6) $[a_{ij}]\widetilde{\cap}([b_{ij}]\widetilde{\cap}[c_{ij}]) = ([a_{ij}]\widetilde{\cap}[b_{ij}])\widetilde{\cap}[c_{ij}]$.

Proposition 2.6.14 *Let $[a_{ij}]$, $[b_{ij}], [c_{ij}] \in SM_{m\times n}$. Then*

1) $([a_{ij}]\widetilde{\cap}[b_{ij}])^o = [a_{ij}]^o\widetilde{\cup}[b_{ij}]^o$,
2) $([a_{ij}]\widetilde{\cup}[b_{ij}])^o = [a_{ij}]^o\widetilde{\cap}[b_{ij}]^o$,
3) $[a_{ij}]\widetilde{\cup}([b_{ij}]\widetilde{\cap}[c_{ij}]) = ([a_{ij}]\widetilde{\cup}[b_{ij}])\widetilde{\cap}([a_{ij}]\widetilde{\cup}[c_{ij}]$,
4) $[a_{ij}]\widetilde{\cap}([b_{ij}]\widetilde{\cup}[c_{ij}]) = ([a_{ij}]\widetilde{\cap}[b_{ij}])\widetilde{\cup}([a_{ij}]\widetilde{\cap}[c_{ij}]$.

Definition 2.6.15 *Let $[a_{ij}]$, $[b_{ik}] \in SM_{m\times n}$. Then And-product of $[a_{ij}]$ and $[b_{ik}]$ is defined as*

$$\curlywedge : SM_{m\times n} \times SM_{m\times n} \to SM_{m\times n^2},\ [a_{ij}] \wedge [b_{ik}] = [c_{ip}],$$

where $c_{ip} = \min\{a_{ij}, b_{ik}\}$ such that $p = n(j-1)+k$.

Definition 2.6.16 *Let $[a_{ij}]$, $[b_{ik}] \in SM_{m\times n}$. Then Or-product of $[a_{ij}]$ and $[b_{ik}]$ is defined as*

$$\curlyvee : SM_{m\times n} \times SM_{m\times n} \to SM_{m\times n^2},\ [a_{ij}] \vee [b_{ik}] = [c_{ip}],$$

where $c_{ip} = \max\{a_{ij}, b_{ik}\}$ such that $p = n(j-1)+k$.

Definition 2.6.17 *Let $[a_{ij}]$, $[b_{ik}] \in SM_{m\times n}$. Then And-Not-product of $[a_{ij}]$ and $[b_{ik}]$ is defined as*

$$\overline{\wedge} : SM_{m\times n} \times SM_{m\times n} \to SM_{m\times n^2},\ [a_{ij}]\ \overline{\wedge}\ [b_{ik}] = [c_{ip}],$$

where $c_{ip} = \min\{a_{ij}, 1-b_{ik}\}$ such that $p = n(j-1)+k$.

Definition 2.6.18 *Let $[a_{ij}]$, $[b_{ik}] \in SM_{m\times n}$. Then Or-Not-product of $[a_{ij}]$ and $[b_{ik}]$ is defined as*

$$\veebar : SM_{m\times n} \times SM_{m\times n} \to SM_{m\times n^2}, \ [a_{ij}] \veebar [b_{ik}] = [c_{ip}],$$

where $c_{ip} = \max\{a_{ij}, 1 - b_{ik}\}$ such that $p = n(j-1) + k$.

Proposition 2.6.19 *Let $[a_{ij}]$, $[b_{ik}] \in SM_{m\times n}$. Then the following De Morgan's types of results are true.*

$$1)\ ([a_{ij}] \wedge [b_{ik}])^o = ([a_{ij}])^o \vee ([b_{ik}])^o,$$
$$2)\ ([a_{ij}] \vee [b_{ik}])^o = ([a_{ij}])^o \wedge ([b_{ik}])^o,$$
$$3)\ ([a_{ij}] \veebar [b_{ik}])^o = ([a_{ij}])^o \barwedge ([b_{ik}])^o,$$
$$4)\ ([a_{ij}] \barwedge [b_{ik}])^o = ([a_{ij}])^o \veebar ([b_{ik}])^o.$$

Chapter 3

Graded Soft Expert Sets

3.1 Introduction

Liang and Liu introduced hesitant fuzzy sets into decision theoretic rough sets and explored their decision mechanism [43]. Zhang and Wu investigated the deviation of the priority weights from hesitant multiplicative preference relations in group decision-making environments [85]. Although this theory proved to be valuable in the context of decision analysis, yet there are some deficiencies in it. No standard inclusion measure has yet been developed. In its application in decision analysis, experts' individual weightage has totally been ignored. To overcome these problems, we introduce graded soft expert (GSE) sets which can be treated as a generalization of hesitant fuzzy sets. This structure is a modified form of soft expert sets but its structural and operational approach is totally different. We mainly focussed to fill the gaps in hesitant fuzzy set theory. In Section 3.2, graded soft expert sets (GSE) have been introduced. Some basic operations have been defined and related laws have been proved. Section 3.3 has been devoted to the study of decision making problems with the aid of GSE sets. At the end, Section 3.4 contains some concluding remarks.

3.2 Graded Soft Expert Sets (GSE Sets) Versus Hesitant Fuzzy Sets and Soft Expert Sets

In this section soft expert set defined by Alkhazaleh and Salleh [7] has been redefined and revised which may be called as graded soft expert set. In order to strengthen the structure, its basic operations have been redefined in a more fruitful manner. Several laws and related results have also been investigated some of which does not hold in hesitant fuzzy sets.

Hesitant fuzzy sets are basically introduced to handle decision making problems in which there are several alternatives and decision makers. But in the definition of hesitant fuzzy sets, alternatives and decision makers have not been specified. This may lead to the wrong use and interpretation of the set. Also, if we take x_1, x_2, and x_3 as three alternatives and hesitant fuzzy set h represents a particular criteria then for each i $(i = 1, 2, ..., n)$ $h(x_i)$ represents opinions of various decision makers in which there is no space to highlight individual decision maker's opinions separately. For that purpose different techniques and algorithms were introduced which makes the decision making problems somehow difficult to handle. One of them is to assign weights to the opinions. But again since opinions of the decision makers have been collected in a set without specifying their individual decisions, it is not possible to give more weightage to a particular decision maker. It may be possible by introducing a complex algorithm.

To avoid such type of situations, GSE set can prove its worth. In GSE set, each alternative (or attribute) and decision maker have been specified separately. Formally it is stated as below:

Definition 3.2.1 *Let U be a finite universe set containing n alternatives, E; a set of criteria and X; a set of experts (or decision makers). Let O be a set of opinions with a given preference relation $\precsim$ among the opinions. A graded soft expert set (abbreviated as GSE set) (F, A, Y) is characterized by a mapping $F : A \times Y \longrightarrow P(U \times O)$ defined for every $e \in A$ and $p \in Y$ by $F(e,p) = \{(u_i, o_i) : i \in I\}$, where $I = \{1, 2, 3, ..., n\}$, $A \subseteq E$, $Y \subseteq X$ and $P(U \times O)$ denotes the power set of $U \times O$. Here the set of opinions O contains graded values of the given parameters i.e. the values $o_1, o_2, ..., o_n$ can be graded as $o_1 \precsim o_2 \precsim ... \precsim o_n$ which means that o_n is the most preferred value while o_1 is the least preferred one and so forth.*

The above definition states that for a given criteria e the decision maker p gives the opinion o_i for each alternative u_i $(i = 1, 2, ..., n)$. As an example of the preference relation in the above definition consider the set of opinions $O = \{$excellent, very good, good, poor, very poor$\}$. It is obvious that "excellent" is preferred over "very good" which is preferred over "good" which is preferred over "poor" and the least preferred one is "very poor". For simplicity we can fuzzify these values according to their grading and preference, that is, the opinions can be assigned values from the interval $[0, 1]$ based on their preference. For U, in the above mentioned set O of opinions, "excellent" is the most preferred opinion, so it can be assigned value 1 from the interval $[0, 1]$ while "very poor" is the least preferred opinion, so it can be assigned the value 0. Rest of the opinions will be assigned values between 0 to 1.

In the rest of the chapter, the set of opinions O will be taken as a subset of $[0, 1]$.

Example 3.2.2 *Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be a set of wheat types (alternatives), $E = \{e_1$ =moister content, e_2 =protein content, e_3 =milling quality, e_4 =baking quality$\}$ be a set of criteria, $X = \{a, b, c\}$ be a set of experts and $O = \{0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$ be the set of possible grades for the given parameters.*

Suppose that a farmer has distributed a questionnaire to the team of experts to judge the quality of wheat types on the basis of given criteria. Decision of experts in the form of graded soft expert set $F : A \times X \longrightarrow P(U \times O)$ is given below:

$F(e_1, a) = \{(u_1, 0.5), (u_2, 0.1), (u_3, 0.7), (u_4, 0.9), (u_5, 0.2)\}$, $F(e_1, b) = \{(u_1,0.5), (u_2,0.2), (u_3, 0.7), (u_4, 0.3), (u_5, 0.4)\}$, $F(e_1, c) = \{(u_1, 0.4), (u_2, 0.3), (u_3, 0.3), (u_4, 0.6), (u_5, 0.7)\}$, $F(e_2, a) = \{(u_1, 0.9), (u_2, 0.3), (u_3, 0.2), (u_4, 0.3), (u_5, 0.6)\}$, $F(e_2, b) = \{(u_1, 0.8), (u_2, 0.9), (u_3, 0.4), (u_4, 0.1), (u_5, 0.4)\}$, $F(e_2, c) = \{(u_1, 0.7), (u_2, 0.0), (u_3, 0.3), (u_4, 0.3), (u_5, 0.6)\}$, $F(e_3, a) = \{(u_1, 0.5), (u_2, 0.5), (u_3, 0.9), (u_4, 0.7), (u_5, 0.2)\}$, $F(e_3, b) = \{(u_1, 0.4), (u_2, 0.4), (u_3, 0.8), (u_4, 0.2), (u_5, 0.3)\}$, $F(e_3, c) = \{(u_1, 0.4), (u_2, 0.4), (u_3, 0.9), (u_4, 0.7), (u_5, 0.2)\}$, $F(e_4, a) = \{(u_1, 0.6), (u_2, 0.7), (u_3, 0.5), (u_4, 0.9), (u_5, 0.7)\}$, $F(e_4, b) = \{(u_1, 0.5), (u_2, 0.8), (u_3, 0.4), (u_4, 0.6), (u_5, 0.3)\}$, $F(e_4, c) = \{(u_1, 0.3), (u_2, 0.9), (u_3, 0.5), (u_4, 0.0), (u_5, 0.6)\}$.

In soft set theory, basic concept is parametrization of objects in a given universe set. The various operations thus defined on soft sets depend upon the e-approximate elements of a given set for all attributes e. Since soft expert set does not only depend upon the various parameters involved but also on the opinion of experts, which is basically the main purpose of introducing soft expert sets, the operations on soft expert sets should consider these opinions as well. In the rest of the section, we define operations on GSE sets taking into consideration the respective opinions as well.

In particular, we can see that the operation of complement on soft expert set defined in [7] takes into consideration the objects of universe and their respective attributes only ignoring their respective opinions. As in U 3.9 of [7] we can see that the complement of $F(e_1, p, 1) = \{u_3\}$ is given as $F^c(Ie_1, p, 1) = \{u_1, u_2, u_4\}$ which means that according to the expert 'p' only the object u_3 has attribute e_1 and its complement states that according to the same opinion of expert 'p' the objects u_1, u_2 and u_4 do not have attribute e_1. This idea can work if we are taking only two opinions (agree 1, disagree 0). If we consider more than two opinions (as in GSE sets) the idea may not work. In the same above case, if we take $F(e_1, p, 0.3) = \{u_3\}$ and $F^c(Ie_1, p, 0.3) = \{u_1, u_2, u_4\}$ then the objects not having attribute 'e_1' in the same degree 0.3 as the objects having that attribute does not sound accurate. Thus, for more than two opinions we define complement of GSE set as follows:

Definition 3.2.3 *The complement of a GSE set (F, A, Y), denoted by $(F, A, Y)^c$, is defined as $(F, A, Y)^c = (F^c, A^c, Y)$ where $F^c : A^c \times Y \rightarrow P(U \times O^c)$ is a mapping* *given as*

$$F^c(e^c, p) = \{(u_i\ , o_i^c) : i \in I\},$$

whenever

$$F(e, p) = \{(u_i, o_i) : i \in I\} \text{ and } o_i^c = 1 - o_i.$$

Example 3.2.4 *Consider U 3.2.2. Then*

$F^c(e_1^c, a) = \{(u_1, 0.5), (u_2, 0.9), (u_3, 0.3), (u_4, 0.1), (u_5, 0.8)\}$, $F^c(e_1^c, b) = \{(u_1,0.5), (u_2,0.8), (u_3, 0.3), (u_4, 0.7), (u_5, 0.6)\}$, $F^c(e_1^c, c) = \{(u_1, 0.6), (u_2, 0.7), (u_3, 0.7), (u_4, 0.4), (u_5, 0.3)\}$, $F^c(e_2^c, a) = \{(u_1, 0.1), (u_2, 0.7), (u_3, 0.8), (u_4, 0.7), (u_5, 0.4)\}$, $F^c(e_2^c, b) = \{(u_1, 0.2), (u_2, 0.1), (u_3, 0.6), (u_4, 0.9), (u_5, 0.6)\}$, $F^c(e_2^c, c) = \{(u_1, 0.3), (u_2, 1.0), (u_3, 0.7), (u_4, 0.7), (u_5, 0.4)\}$, $F^c(e_3^c, a) = \{(u_1, 0.5), (u_2, 0.5), (u_3, 0.1), (u_4, 0.3, (u_5, 0.8)\}$, $F^c(e_3^c, b) = \{(u_1, 0.6), (u_2, 0.6), (u_3, 0.2), (u_4, 0.8), (u_5, 0.7)\}$, $F^c(e_3^c, c) = \{(u_1, 0.6), (u_2, 0.6), (u_3, 0.1), (u_4, 0.3), (u_5, 0.8)\}$, $F^c(e_4^c, a) = \{(u_1, 0.4), (u_2, 0.3), (u_3, 0.5), (u_4, 0.1), (u_5, 0.3)\}$, $F^c(e_4^c, b) = \{(u_1, 0.5), (u_2, 0.2), (u_3, 0.6), (u_4, 0.4), (u_5, 0.7)\}$, $F^c(e_4^c, c) = \{(u_1, 0.7), (u_2, 0.1), (u_3, 0.5), (u_4, 1.0), (u_5, 0.4)\}$.

Definition 3.2.5 *The union of two GSE sets (F, A, Y) and (G, B, Z) over U, denoted by $(F, A, Y) \cup (G, B, Z)$, is a GSE set (H, C, X) where $C = A \cup B$, $X = Y \cup Z$ and for all $e \in C$ and $p \in X$;*

$$H(e, p) = \begin{cases} \{(u_i,\ \max\{o_i, o_i'\}) : i \in I\} & \text{if } (e, p) \in (A \cap B, Y \cap Z) \\ \{(u_i, o_i) : i \in I\} & \text{if } (e, p) \in (A, Y) \setminus (B, Z) \\ \{(u_i, o_i') : i \in I\} & \text{if } (e, p) \in (B, Z) \setminus (A, Y) \end{cases}$$

whenever $F(e, p) = \{(u_i, o_i) : i \in I\}$ and $G(e, p) = \{(u_i, o_i') : i \in I\}$.

Example 3.2.6 *Let $A \times Y = \{(e_1, a), (e_1, b), (e_2, a), (e_2, b), (e_2, c), (e_3, a), (e_3, b), (e_4, a), (e_4, b), (e_4, c)\}$,*

$B \times Z = \{(e_1, a), (e_1, b), (e_1, c), (e_2, a), (e_2, c), (e_3, b), (e_3, c), (e_4, a), (e_4, b)\}$.

Let two GSE sets (F, A, Y) and (G, B, Z) over U are given by

$F(e_1, a) = \{(u_1, 0.5), (u_2, 0.1), (u_3, 0.7)\}$, $F(e_1, b) = \{(u_1,0.5), (u_2,0.2), (u_3, 0.7)\}$, $F(e_2, a) = \{(u_1, 0.9), (u_2, 0.3), (u_3, 0.2)\}$, $F(e_2, b) = \{(u_1, 0.8), (u_2, 0.9), (u_3, 0.4)\}$, $F(e_2, c) = \{(u_1, 0.7), (u_2, 0.0), (u_3, 0.3)\}$, $F(e_3, a) = \{(u_1, 0.5), (u_2, 0.5), (u_3, 0.9)\}$, $F(e_3, b) = \{(u_1, 0.4), (u_2, 0.4), (u_3, 0.8)\}$, $F(e_4, a) = \{(u_1, 0.6), (u_2, 0.7), (u_3, 0.5)\}$, $F(e_4, b) = \{(u_1, 0.5), (u_2, 0.8), (u_3, 0.4)\}$, $F(e_4, c) = \{(u_1, 0.3), (u_2, 0.9), (u_3, 0.5)\}$.

$G(e_1, a) = \{(u_1, 0.8), (u_2, 0.3), (u_3, 0.4)\}$, $G(e_1, b) = \{(u_1,0.3, (u_2,0.7), (u_3, 0.9)\}$, $G(e_1, c) = \{(u_1, 0.4), (u_2, 0.3), (u_3, 0.3)\}$, $G(e_2, a) = \{(u_1, 0.5), (u_2, 0.2), (u_3, 0.8)\}$, $G(e_2, c) = \{(u_1, 0.6), (u_2, 0.8), (u_3, 0.1)\}$, $G(e_3, b) = \{(u_1, 0.6), (u_2, 0.4), (u_3, 0.2)\}$, $G(e_3, c) = \{(u_1, 0.4), (u_2, 0.4), (u_3, 0.9)\}$, $G(e_4, a) = \{(u_1, 0.6), (u_2, 0.6), (u_3, 0.8)\}$, $G(e_4, b) = \{(u_1, 0.6), (u_2, 0.4), (u_3, 0.2)\}$.

Hence $(F, A, Y) \cup (G, B, Z) = (H, C, X)$ *is given as*

$H(e_1, a) = \{(u_1, 0.8), (u_2, 0.3), (u_3, 0.7)\}$, $H(e_1, b) = \{(u_1, 0.5), (u_2, 0.7), (u_3, 0.9)\}$, $H(e_1, c) = \{(u_1, 0.4), (u_2, 0.3), (u_3, 0.3)\}$, $H(e_2, a) = \{(u_1, 0.9), (u_2, 0.3), (u_3, 0.8)\}$, $H(e_2, b) = \{(u_1, 0.8), (u_2, 0.9), (u_3, 0.4)\}$, $H(e_2, c) = \{(u_1, 0.7), (u_2, 0.8), (u_3, 0.3)\}$, $H(e_3, a) = \{(u_1, 0.5), (u_2, 0.5), (u_3, 0.9)\}$, $H(e_3, b) = \{(u_1, 0.6), (u_2, 0.4), (u_3, 0.8)\}$, $H(e_3, c) = \{(u_1, 0.4), (u_2, 0.4), (u_3, 0.9)\}$, $H(e_4, a) = \{(u_1, 0.6), (u_2, 0.7), (u_3, 0.8)\}$, $H(e_4, b) = \{(u_1, 0.6), (u_2, 0.8), (u_3, 0.4)\}$, $H(e_4, c) = \{(u_1, 0.3), (u_2, 0.9), (u_3, 0.5)\}$.

Definition 3.2.7 *The intersection of two GSE sets* (F, A, Y) *and* (G, B, Z) *over* U, *denoted by* $(F, A, Y) \cap (G, B, Z)$, *is a GSE set* (H, C, X) *where* $C = A \cap B$, $X = Y \cap Z$ *and for all* $e \in C$ *and* $p \in X$;

$$H(e, p) = \{(u_i, \min\{o_i, o'_i\}) : i \in I\},$$

whenever $F(e, p) = \{(u_i, o_i) : i \in I\}$ *and* $G(e, p) = \{(u_i, o'_i) : i \in I\}$.

Example 3.2.8 *Consider U 3.2.6.* $(F, A, Y) \cap (G, B, Z) = (H, C, X)$ *is given as:*

$H(e_1, a) = \{(u_1, 0.5), (u_2, 0.1), (u_3, 0.4)\}$, $H(e_1, b) = \{(u_1, 0.3), (u_2, 0.2), (u_3, 0.7)\}$, $H(e_2, a) = \{(u_1, 0.5), (u_2, 0.2), (u_3, 0.2)\}$, $H(e_2, c) = \{(u_1, 0.6), (u_2, 0.0), (u_3, 0.1)\}$, $H(e_3, b) = \{(u_1, 0.4), (u_2, 0.4), (u_3, 0.2)\}$, $H(e_4, a) = \{(u_1, 0.6), (u_2, 0.6), (u_3, 0.5)\}$, $H(e_4, b) = \{(u_1, 0.5), (u_2, 0.4), (u_3, 0.2)\}$.

In classical set, the hierarchy is characterized through set containment. But, in case of other generalizations of classical set like fuzzy set, soft set or hesitant fuzzy set, it is characterized through different ways. Alkhazaleh and Salleh [7] defined soft expert subset by using the classical set containment approach in which grading of opinions is not considered. Taking into consideration the opinions of experts, we define the notion of subset for graded soft expert sets in a more generalized way as below:

Definition 3.2.9 *For a GSE set* (F, A, Y) *over* U *and for any* $e, e' \in A$, $p, p' \in Y$, *if*

$$F(e, p) = \{(u_i, o_i) : i \in I\} \text{ and } F(e', p') = \{(u_i, o'_i) : i \in I\},$$

then $F(e, p)$ *is said to be contained in* $F(e', p')$ *(or equivalently* $F(e, p)$ *is subset of* $F(e', p')$, *denoted by* $F(e, p) \subseteq F(e', p')$, *if*

$$o_i \leq o'_i \text{ for each } i \in \{1, 2, 3, ..., n\}.$$

The above condition states that the degree of each alternative in $F(e, p)$ is less than the corresponding degree in $F(e', p')$.

Example 3.2.10 *In U 3.2.2,* $F(e_1,b) = \{(u_1,0.2), (u_2,0.5), (u_3,0.4), (u_4,0.5), (u_5,0.6)\} \subseteq F(e_4,c) = \{(u_1,0.3), (u_2,0.8), (u_3,0.5), (u_4,0.5), (u_5,0.6)\}$ *because opinion for each* u_i *in* $F(e_1,b)$ *is less than or equal to its corresponding value in* $F(e_4,c)$.

Definition 3.2.11 *For two GSE sets* (F,A,Y) *and* (G,B,Z) *over* U, (F,A,Y) *is called subset of* (G,B,Z), *denoted by* $(F,A,Y) \subseteq (G,B,Z)$, *if*

1) $A \subseteq B$,
2) $Y \subseteq Z$,
3) $F(e,p) \subseteq G(e,p)$ *for all* $e \epsilon A$, $p \epsilon Y$.

In this case (G,B,Z) *is called a superset of* (F,A,Y) *denoted by* $(G,B,Z) \supseteq (F,A,Y)$.

Example 3.2.12 *Let* $B \times Z = \{(e_1,a), (e_1,b), (e_2,a), (e_2,b), (e_2,c), (e_3,a), (e_3,b), (e_4,a), (e_4,b), (e_4,c)\}$, $A \times Y = \{(e_1,a), (e_3,b), (e_4,a), (e_4,b)\}$.

Let two GSE sets (G,B,Z) *and* (F,A,Y) *over* U *are given by*

$G(e_1,a) = \{(u_1,0.5), (u_2,0.1), (u_3,0.7)\}$, $G(e_1,b) = \{(u_1,0.5), (u_2,0.2), (u_3,0.7)\}$, $G(e_2,a) = \{(u_1,0.9), (u_2,0.3), (u_3,0.2)\}$, $G(e_2,b) = \{(u_1,0.8), (u_2,0.9), (u_3,0.4)\}$, $G(e_2,c) = \{(u_1,0.7), (u_2,0.0), (u_3,0.3)\}$, $G(e_3,a) = \{(u_1,0.5), (u_2,0.5), (u_3,0.9)\}$, $G(e_3,b) = \{(u_1,0.4), (u_2,0.4), (u_3,0.8)\}$, $G(e_4,a) = \{(u_1,0.6), (u_2,0.7), (u_3,0.5)\}$, $G(e_4,b) = \{(u_1,0.5), (u_2,0.8), (u_3,0.4)\}$, $G(e_4,c) = \{(u_1,0.3), (u_2,0.9), (u_3,0.5)\}$.

$F(e_1,a) = \{(u_1,0.3), (u_2,0.1), (u_3,0.4)\}$, $F(e_3,b) = \{(u_1,0.2), (u_2,0.1), (u_3,0.2)\}$, $F(e_4,a) = \{(u_1,0.3), (u_2,0.6), (u_3,0.2)\}$, $F(e_4,b) = \{(u_1,0.4), (u_2,0.7), (u_3,0.2)\}$.

Clearly $A \subseteq B$, $Y \subseteq Z$, $F(e,p) \subseteq G(e,p)$ *for all* $e \epsilon A$, $p \epsilon Y$. *Hence* $(F,A,Y) \subseteq (G,B,Z)$.

By Definition 3.2.11, we can see that the comparison of two GSE sets is pointwise which means that the values of the two GSE sets are compared for each pair of values separately. In case of soft expert sets, containment as defined in [7], is a global property which ignores individual opinions completely. Also, in that case two soft expert sets can be compared but there is no way to compare their respective values separately.

Definition 3.2.13 *Two GSE sets* (F,A,Y) *and* (G,B,Z) *over* U *are said to be equal, denoted by* $(F,A,Y) = (G,B,Z)$, *if* $A = B$, $Y = Z$ *and* $F(e,p) = G(e,p)$ *for all* $e \in A(=B)$, $p \in Y(=Z)$.

Proposition 3.2.14 *For two GSE sets* (F,A,Y) *and* (G,B,Z) *over* U, *if* $(F,A,Y) \subseteq (G,B,Z)$ *and* $(G,B,Z) \subseteq (F,A,Y)$, *then* $(F,A,Y) = (G,B,Z)$.

Proof. It can easily be proved using Definitions 3.2.13 and 3.2.11. ■

This is one of the most significant results for GSE sets. The inclusion here is based on graded values or opinions as in hesitant fuzzy sets but the above result does not hold in case of hesitant fuzzy sets. To overcome this shortcoming many inclusion measures and criteria have been developed. Hesitant equality has also been introduced. But all these attempts were more or less useless in practical implementations.

Xia and Xu [70] defined the score function of hesitant fuzzy element h, that is, $s(h) = \left(\sum_{\gamma \in h} \gamma \right) / \#h$, where $s(\cdot)$ is the score function and $\#h$ is the number of elements in h. This score function serves as a measure to compare two hesitant fuzzy sets. Following the same technique, we define score function for a GSE set as below:

Definition 3.2.15 *For a given GSE set (F, A, Y) over $U = \{u_1, u_2, ..., u_n\}$, where A contains m criteria, score function for any u_i $(i = 1, 2, ..., n)$ with respect to the opinions of an expert $p \in Y$ is denoted by $s(u_i, p)$ and is defined as*

$$s(u_i, p) = \left(\sum_{j=1}^{m} o_j \right) / m,$$

where o_1, $o_2, ..., o_m$ are the respective opinions of the expert p for the alternative u_i with respect to the criteria $e_1, e_2, ..., e_n$.

Theorem 3.2.16 *For any two GSE sets (F, A, Y) and (G, B, Z) over U, we have*

$$1)\ (F, A, Y) \cap (G, B, Z) \subseteq (F, A, Y),\ (G, B, Z);$$
$$2)\ (F, A, Y),\ (G, B, Z) \subseteq (F, A, Y) \cup (G, B, Z).$$

Proof. 1) For any GSE sets (F, A, Y) and (G, B, Z), let $(F, A, Y) \cap (G, B, Z) = (H, A \cap B, Y \cap Z)$. Since $A \cap B \subseteq A, B$ and $Y \cap Z \subseteq Y, Z$ and for any $e \in A \cap B$, $p \in Y \cap Z$ using Definition 3.2.7 we have

$$H(e, p) \quad = \{(u_i, \min\{o_i, o_i'\}) : i \in I\},$$

where $F(e, p) = \{(u_i, o_i) : i \in I\}$ and $G(e, p) = \{(u_i, o_i') : i \in I\}$. Thus, by Definition 3.2.9, $H(e, p) \subseteq \{(u_i, o_i) : i \in I\} = F(e, p)$ and $H(e, p) \subseteq \{(u_i, o_i') : i \in I\} = G(e, p)$. This shows that $(F, A, Y) \cap (G, B, Z) \subseteq (F, A, Y)$, (G, B, Z).

2) Let $(F, A, Y) \cup (G, B, Z) = (J, A \cup B, Y \cup Z)$. Since $A \subseteq A \cup B$ and $Y \subseteq Y \cup Z$, for any $e \in A$, $p \in Y$, using Definition 3.2.5, we have

$$J(e, p) = \begin{cases} \{(u_i, \max\{o_i, o_i'\}) : i \in I\} & \text{if } (e, p) \in (A \cap B, Y \cap Z) \\ \{(u_i, o_i) : i \in I\} & \text{if } (e, p) \in (A, Y) \setminus (B, Y) \end{cases}$$

In both the cases, using Definition 3.2.9, we have $F(e, p) \subseteq J(e, p)$. Similarly, $G(e, p) \subseteq J(e, p)$. Thus, (F, A, Y), $(G, B, Z) \subseteq (F, A, Y) \cup (G, B, Z)$. ■

Theorem 3.2.17 *Let U be the universe set. For all GSE sets (F, A, Y), (G, B, Z) and (H, C, X) over U, the following properties hold:*

1) Idempotent: $(F, A, Y) \cap (F, A, Y) = (F, A, Y)$, $(F, A, Y) \cup (F, A, Y) = (F, A, Y)$;

2) Commutative: $(F, A, Y) \cap (G, B, Z) = (G, B, Z) \cap (F, A, Y)$, $(F, A, Y) \cup (G, B, Z) = (G, B, Z) \cup (F, A, Y)$;

3) Associative: $(F, A, Y) \cap ((G, B, Z) \cap (H, C, X)) = ((F, A, Y) \cap (G, B, Z)) \cap (H, C, X)$,

$(F, A, Y) \cup ((G, B, Z) \cup (H, C, X)) = ((F, A, Y) \cup (G, B, Z)) \cup (H, C, X)$;

4) Distributive: $(F, A, Y) \cap ((G, B, Z) \cup (H, C, X)) = ((F, A, Y) \cap (G, B, Z)) \cup ((F, A, Y) \cap (H, C, X))$,

$(F, A, Y) \cup ((G, B, Z) \cap (H, C, X)) = ((F, A, Y) \cup (G, B, Z)) \cap ((F, A, Y) \cup (H, C, X))$;

5) De Morgan's laws: $((F, A, Y) \cap (G, B, Z))^c = (F, A, Y)^c \cup (G, B, Z)^c$, $((F, A, Y) \cup (G, B, Z))^c = (F, A, Y)^c \cap (G, B, Z)^c$;

6) Double negation law: $((F, A, Y)^c)^c = (F, A, Y)$.

Proof. 1) By Definition 3.2.7, for any GSE set $(F, A, Y) \cap (F, A, Y) = (F, A \cap A, Y \cap Y) = (F, A, Y)$. Since for any $e \in A \cap A = A$ and $p \in Y \cap Y = Y$ by similar definition for any $F(e, p) \in (F, A, Y)$ we have $F(e, p) = \{(u_i, o_i) : i \in I\} = \{(u_i, \min\{o_i, o_i\}) : i \in I\} = F(e, p) \cap F(e, p) \in (F, A, Y) \cap (F, A, Y)$. Hence $(F, A, Y) \subseteq (F, A, Y) \cap (F, A, Y)$.

Conversely by similar definition for any $F(e, p) \in (F, A, Y) \cap (F, A, Y) = \{(u_i, \min\{o_i, o_i\}) : i \in I\} = \{(u_i, o_i) : i \in I\}$. Hence $(F, A, Y) \cap (F, A, Y) \subseteq (F, A, Y)$. So $(F, A, Y) \cap (F, A, Y) = (F, A, Y)$.

2) By Definition 3.2.7, for any GSE sets (F, A, Y) and (G, B, Z), $(F, A, Y) \cap (G, B, Z) = (H, A \cap B, Y \cap Z)$. Since for any $e \in A \cap B = B \cap A$ and $p \in Y \cap Z = Z \cap Y$. let $H(e, p) \in (F, A, Y) \cap (G, B, Z)$ by Definition 3.2.7 we have $H(e, p) = \{(u_i, \min\{o_i, o'_i\}) : i \in I\} = \{(u_i, \min\{o'_i, o_i\}) : i \in I\} = G(e, p) \cap F(e, p)$ where $F(e, p) = \{(u_i, o_i) : i \in I\}$ and $G(e, p) = \{(u_i, o'_i) : i \in I\}$. Thus $(F, A, Y) \cap (G, B, Z) \subseteq (G, B, Z) \cap (F, A, Y)$.

Conversely let $H(e, p) \in (G, B, Z) \cap (F, A, Y)$ by similar definition we have $H(e, p) = \{(u_i, \min\{o'_i, o_i\}) : i \in I\} = \{(u_i, \min\{o_i, o'_i\}) : i \in I\} = F(e, p) \cap G(e, p)$ where $F(e, p) = \{(u_i, o_i) : i \in I\}$ and $G(e, p) = \{(u_i, o'_i) : i \in I\}$. Thus $(G, B, Z) \cap (F, A, Y) \subseteq (F, A, Y) \cap (G, B, Z)$. Hence $(F, A, Y) \cap (G, B, Z) = (G, B, Z) \cap (F, A, Y)$.

3) By Definition 3.2.7, for any GSE sets (F, A, Y), (G, B, Z) and (H, C, X), let $(F, A, Y) \cap ((G, B, Z) \cap (H, C, X)) = (I, A \cap (B \cap C), Y \cap (Z \cap X))$. Since $A \cap (B \cap C) = (A \cap B) \cap C$ and $Y \cap (Z \cap X) = (Y \cap Z) \cap X$ and for any $e \in A \cap (B \cap C) = (A \cap B) \cap C$, $p \in Y \cap (Z \cap X) = (Y \cap Z) \cap X$, let $I(e, p) \in (F, A, Y) \cap ((G, B, Z) \cap (H, C, X))$ by similar definition we have $I(e, p) = \{(u_i, \min\{o_i, (o'_i, o''_i)\}) : i \in I\} = \{(u_i, \min\{(o_i, o'_i),$

$o_i''\}) : i \in I\} = ((F(e,p) \cap G(e,p)) \cap H(e,p) \in ((F,A,Y) \cap (G,B,Z)) \cap (H,C,X)$. where $F(e,p) = \{(u_i, o_i) : i \in I\}$, $G(e,p) = \{(u_i, o_i') : i \in I\}$ and $H(e,p) = \{(u_i, o_i'') : i \in I\}$. Thus $(F,A,Y) \cap ((G,B,Z) \cap (H,C,X)) \subseteq ((F,A,Y) \cap (G,B,Z)) \cap (H,C,X)$.

Conversely let $I(e,p) \in ((F,A,Y) \cap (G,B,Z)) \cap (H,C,X)$ by similar definition we have $I(e,p) = \{(u_i, \min\{(o_i, o_i'), o_i''\}) : i \in I\} = \{(u_i, \min\{o_i, (o_i', o_i'')\}) : i \in I\} =$

$F(e,p) \cap (G(e,p) \cap H(e,p)) \in (F,A,Y) \cap ((G,B,Z)) \cap (H,C,X))$. where $F(e,p) = \{(u_i, o_i) : i \in I\}$, $G(e,p) = \{(u_i, o_i') : i \in I\}$ and $H(e,p) = \{(u_i, o_i'') : i \in I\}$. Thus $((F,A,Y) \cap ((G,B,Z)) \cap (H,C,X) \subseteq (F,A,Y) \cap ((G,B,Z)) \cap (H,C,X))$.

Hence $(F,A,Y) \cap ((G,B,Z) \cap (H,C,X)) = ((F,A,Y) \cap (G,B,Z)) \cap (H,C,X)$.

4) By Definitions 3.2.7 and 6.2.13, for any GSE sets (F,A,Y), (G,B,Z) and (H,C,X), let $(F,A,Y) \cap ((G,B,Z) \cup (H,C,X)) = (I, A \cap (B \cup C), Y \cap (Z \cup X))$. Since for any $e \in A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $p \in Y \cap (Z \cup X) = (Y \cap Z) \cup (Y \cap X)$, let $I(e,p) \in (F,A,Y) \cap ((G,B,Z) \cup (H,C,X))$ by similar definitions we have $I(e,p) = \{(u_i, \min\{o_i, \max\{o_i', o_i''\}\}) : i \in I\} = \{(u_i, \max\{\min\{o_i, o_i'\}, \min\{o_i, o_i''\}\}) : i \in I\} = (F(e,p) \cap G(e,p)) \cup (F(e,p) \cap H(e,p)) \in ((F,A,Y) \cap (G,B,Z)) \cup ((F,A,Y) \cap (H,C,X))$

where $F(e,p) = \{(u_i, o_i) : i \in I\}$, $G(e,p) = \{(u_i, o_i') : i \in I\}$ and $H(e,p) = \{(u_i, o_i'') : i \in I\}$. Thus $(F,A,Y) \cap ((G,B,Z) \cup (H,C,X)) \subseteq ((F,A,Y) \cap (G,B,Z)) \cup ((F,A,Y) \cap (H,C,X))$.

Conversely, let $I(e,p) \in ((F,A,Y) \cap (G,B,Z)) \cup ((F,A,Y) \cap (H,C,X))$ by similar definitions we have $I(e,p) = \{(u_i, \max\{\min\{o_i, o_i'\}, \min\{o_i, o_i''\}\}) : i \in I\} = \{(u_i, \min\{o_i, \max\{o_i', o_i''\}\}) : i \in I\} = F(e,p) \cap ((G(e,p) \cup H(e,p)) \in (F,A,Y) \cap ((G,B,Z) \cup (H,C,X))$. where $F(e,p) = \{(u_i, o_i) : i \in I\}$, $G(e,p) = \{(u_i, o_i') : i \in I\}$ and $H(e,p) = \{(u_i, o_i'') : i \in I\}$. Thus $((F,A,Y) \cap (G,B,Z)) \cup ((F,A,Y) \cap (H,C,X)) \subseteq (F,A,Y) \cap ((G,B,Z) \cup (H,C,X))$.

Hence $(F,A,Y) \cap ((G,B,Z) \cup (H,C,X)) = ((F,A,Y) \cap (G,B,Z)) \cup ((F,A,Y) \cap (H,C,X))$.

Rest of the parts can be proved in a similar way. ■

In general, absorption laws do not hold for hesitant fuzzy sets. But these laws hold in case of GSE set as can be seen in the next result.

Theorem 3.2.18 *For any two GSE sets (F,A,Y) and (G,B,Z) over U, the following absorption laws hold:*

$$1)\ (F,A,Y) \cap ((F,A,Y) \cup (G,B,Z)) = (F,A,Y),$$

$$2)\ (F,A,Y) \cup ((F,A,Y) \cap (G,B,Z)) = (F,A,Y).$$

Proof. 1) By Definitions 3.2.5 and 3.2.7 we have $(F,A,Y) \cap ((F,A,Y) \cup (G,B,Z)) =$

$(H, A\cap(A\cup B), Y\cap(Y\cup Z)) = (H, A, Y)$ such that for any $e \in A$ and $p \in Y$ we have

$$H(e,p) = \begin{cases} F(e,p)\cap(F(e,p)\cup G(e,p)) & \text{if } (e,p)\in(A\cap B, Y\cap Z) \\ F(e,p)\cap(F(e,p) & \text{if } (e,p)\in(A,Y)\setminus(B,Z) \end{cases}$$

In the first case when $(e,p) \in (A\cap B, Y\cap Z)$, $F(e,p) = \{(u_i, o_i) : i \in I\}$ and $G(e,p) = \{(u_i, o_i') : i \in I\}$, using Definitions 3.2.5, 3.2.7 and 3.2.9 we get

$$\begin{aligned} F(e,p)\cap(F(e,p)\cup G(e,p)) &= \{(u_i,o_i) : i\in I\}\cap(\{(u_i,o_i) : i\in I\}\cup\{(u_i,o_i') : i\in I\}) \\ &= \{(u_i,o_i) : i\in I\}\cap\{(u_i,\max\{o_i,o_i'\}) : i\in I\} \\ &= \{(u_i,\min\{o_i,\max\{o_i,o_i'\}\}) : i\in I\} \\ &\subseteq \{(u_i,o_i) : i\in I\} = F(e,p) \\ &\subseteq \{(u_i,\max\{o_i,\min\{o_i,o_i'\}\}) : i\in I\} \\ &= \{(u_i,\min\{o_i,\max\{o_i,o_i'\}\}) : i\in I\} \\ &= F(e,p)\cap(F(e,p)\cup G(e,p)). \end{aligned}$$

The above arguments gives us our required result for the first case.

In the second case when $(e,p) \in (A,Y)\setminus(B,Z)$, using Definition 3.2.5, we have

$$(F,A,Y)\cap((F,A,Y)\cup(G,B,Z)) = (F,A,Y)\cap(F,A,Y) = (F,A,Y)$$

which is our required result for this case as well. Thus, in both the cases we have

$$(F,A,Y)\cap((F,A,Y)\cup(G,B,Z)) = (F,A,Y).$$

2) This can be proved in a similar way. ■

3.3 Decision Making with the Aid of GSE Sets

Decision making problems have extensively been studied using hesitant fuzzy sets in which there are several experts who have to decide among various alternatives. For that purpose, the most common approach is to aggregate the opinions first for each criteria and alternative. Then, alternatives are ranked by aggregating the average criteria.

As already mentioned, the experts' individual opinions have been ignored while modelling decisions by hesitant fuzzy sets. Experts may have different expertise regarding different criteria. To overcome this shortcoming, GSE sets can be used to give due weightage to the opinions of experts individually.

In this section, we develop an algorithm with the aid of GSE sets for decision analysis in which experts will be given weightage according to their area of expertise. Let $\{u_1, u_2, ..., u_n\}$ be a finite set of n alternatives and $E = \{e_1, e_2, ..., e_m\}$ be a set of

m criteria. Further, we take X as set of experts and O as set of possible opinions. Our goal is to decide among the various alternatives subject to expert's opinion regarding given criteria. This is a decision making problem. To handle such type of problems by using GSE sets, we propose following algorithmic steps:

Step 1: Utilize the evaluations of experts in the form of GSE sets to determine the opinions regarding given alternatives and criteria.

Step 2: Find weighted average of opinions for each pair (u_i, e_j) $(i = 1, 2, ..., n,$ $j = 1, 2, ..., m)$ by assigning suitable weights to the experts according to their area of expertise.

Step 3: Using Definition 3.2.15, calculate the scores $s(u_i)$ of u_i $(i = 1, 2, ..., n)$ considering the aggregate values of experts in step 2.

Step 4: Rank all the alternatives according to $s(u_i)$ in descending order.

Step 5: End.

Example 3.3.1 *A person wants to start a small business with low capital. He is considering five different business;* u_1 *is computer and mobile repair business,* u_2 *is baby sitting and child care business,* u_3 *is dairy products business,* u_4 *is real estate agency business and* u_5 *is artist freelance business. Let us denote the set of these business types (alternatives) by* U.

Let $Q = \{e_1$ *=High profit,* e_2 *=Market area,* e_3 *=Revenue and profitability,* e_4 *=ownership and taxes*$\}$ *be the set of criteria. Let* $Y = \{a, b, c\}$ *be the set of experts. Expert* a *is selected for acknowledged expertise in evaluating* e_1 *and* e_4*, expert* b *in evaluating* e_1*,* e_2 *and* e_3*, and expert* c *in evaluating* e_2*,* e_3 *and* e_4*. Also we take* $O = \{0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$ *as the set of possible opinions of experts regarding risk factor.*

Step 1: *Utilize the evaluations of experts in the form of GSE sets for the given problem. For ease of calculation, these can also be written in tabular form as in Tables 3.3.1, 3.3.2 and 3.3.3.*

$F(e_1, a) = \{(u_1, 0.3), (u_2, 0.4), (u_3, 0.2), (u_4, 0.5), (u_5, 0.8)\}$, $F(e_1, b) = \{(u_1, 0.2),$ $(u_2, 0.5), (u_3, 0.4), (u_4, 0.5), (u_5, 0.6)\}$, $F(e_1, c) = \{(u_1, 0.4), (u_2, 0.5), (u_3, 0.3), (u_4, 0.6),$ $(u_5, 0.7)\}$, $F(e_2, a) = \{(u_1, 0.9), (u_2, 0.0), (u_3, 0.2), (u_4, 0.3), (u_5, 0.6)\}$, $F(e_2, b) =$ $\{(u_1, 0.8), (u_2, 0.1), (u_3, 0.4), (u_4, 0.1), (u_5, 0.4)\}$, $F(e_2, c) = \{(u_1, 0.7), (u_2, 0.3), (u_3, 0.3),$ $(u_4, 0.3), (u_5, 0.5)\}$, $F(e_3, a) = \{(u_1, 0.5), (u_2, 0.3), (u_3, 0.9), (u_4, 0.7), (u_5, 0.2)\}$, $F(e_3, b) =$ $\{(u_1, 0.4), (u_2, 0.4), (u_3, 0.7), (u_4, 0.5), (u_5, 0.3)\}$, $F(e_3, c) = \{(u_1, 0.5), (u_2, 0.3), (u_3, 0.9),$ $(u_4, 0.7), (u_5, 0.2)\}$, $F(e_4, a) = \{(u_1, 0.6), (u_2, 0.8), (u_3, 0.5), (u_4, 0.7), (u_5, 0.6)\}$, $F(e_4, b) =$ $\{(u_1, 0.5), (u_2, 0.6), (u_3, 0.4), (u_4, 0.6), (u_5, 0.3)\}$, $F(e_4, c) = \{(u_1, 0.3), (u_2, 0.8), (u_3, 0.5),$

$(u_4, 0.5), (u_5, 0.6)\}$.

	(e_1, a)	(e_2, a)	(e_3, a)	(e_4, a)
u_1	0.3	0.9	0.5	0.6
u_2	0.4	0.0	0.3	0.8
u_3	0.2	0.2	0.9	0.5
u_4	0.5	0.3	0.7	0.7
u_5	0.8	0.6	0.2	0.6

Table 3.3.1. ***Opinions of expert a***

	(e_1, b)	(e_2, b)	(e_3, b)	(e_4, b)
u_1	0.2	0.8	0.4	0.5
u_2	0.5	0.1	0.4	0.6
u_3	0.4	0.4	0.7	0.4
u_4	0.5	0.1	0.5	0.6
u_5	0.6	0.4	0.3	0.3

Table 3.3.2. ***Opinions of expert b***

	(e_1, c)	(e_2, c)	(e_3, c)	(e_4, c)
u_1	0.4	0.7	0.5	0.3
u_2	0.5	0.3	0.3	0.8
u_3	0.3	0.3	0.9	0.5
u_4	0.6	0.3	0.7	0.5
u_5	0.7	0.5	0.2	0.6

Table 3.3.3. ***Opinions of expert c***

Step 2: *Find weighted average of opinions for each pair* (u_i, e_j) $(i = 1, 2, 3, 4, 5,$ $j = 1, 2, 3, 4)$ *by assigning weight 2 to expert a for* e_1 *and* e_4 *and 1 for* e_2 *and* e_3. *Similarly, assign weight 2 to expert b each for* e_1, e_2 *and* e_3 *and 1 for* e_4 *and assign weight 2 to expert c each for* e_2, e_3 *and* e_4 *and 1 for* e_1. *Thus, opinions of experts have been aggregated in this step and results have been displayed in Table (3.3.4).*

	e_1	e_2	e_3	e_4
u_1	0.28	0.78	0.46	0.46
u_2	0.46	0.16	0.34	0.76
u_3	0.30	0.32	0.82	0.48
u_4	0.52	0.22	0.62	0.60
u_5	0.70	0.48	0.24	0.54

Table 3.3.4.

For U, for the pair (u_1, e_1) weighted average has been calculated as:

$[2(0.3) + 2(0.2) + 1(0.4)]/(2 + 2 + 1) = 0.28.$

Rest of the entries can be calculated in a similar way.

Step 3: *Using Definition 3.2.15, for aggregated experts' opinions instead of individual values, calculate scores $s(u_i)$ $(i = 1, 2, 3, 4, 5)$ to get:*

$s(u_1) = 0.495$, $s(u_2) = 0.43$, $s(u_3) = 0.48$, $s(u_4) = 0.49$, $s(u_5) = 0.49$.

Step 4: *Rank all the business types u_i $(i = 1, 2, 3, 4, 5)$ in accordance with their scores $s(u_i)$ to get the preference relation $u_2 \succ u_3 \succ u_4 \approx u_5 \succ u_1$ (alternative with lowest overall risk factor is the most preferred one while the one with highest overall risk factor is least preferred). Thus, the most appropriate business is u_2.*

3.4 Conclusion and Future Work

In this chapter, GSE set has been discussed which can be treated as a generalization of hesitant fuzzy set. Some basic operations associated with the structure have been defined and analyzed. For comparison purpose, notions of 'subset' and 'score' have also been defined. Some important results have been proved which fail to hold in case of hesitant fuzzy sets. For U, the notion of containment in hesitant fuzzy sets is an open problem. One of the most widely used measure of containment was given by Xia and Xu [70]. But in that case inclusion of two hesitant fuzzy elements in each other does not imply their equality. This issue can be resolved by using the proposed structure. In addition, a decision making algorithm with the aid of GSE set is developed. There are so many techniques to solve decision making problems through hesitant fuzzy sets. But the suggested technique has an advantage over the existing methods that it considers relative importance of the experts according to their area of expertise. A practical risk decision making U is presented to reveal significance of the algorithm. As future work we aim to study and define appropriate aggregation operators, distance and similarity measures for GSE sets.

Chapter 4

Cubic Soft Expert Sets and their Applications in Decision Making

4.1 Introduction

In this chapter we define cubic soft expert sets ($CSESs$) by using fuzzy sets and interval valued fuzzy sets as opinion of experts. Corresponding to each attribute every expert gives his expertise in the relevant field through fuzzy sets and interval valued fuzzy sets. There are so many methods to solve decision making problems in various fields but this technique has the advantage over the existing ones in that the decision makers may take decision on the basis of different conditions such as climate condition, time period condition and geographical conditions. We define internal, external $CSESs$, $P-order$, $P-union$, $P-intersection$, $P-AND$, $P-OR$ and $R-order$, $R-union$, $R-intersection$, $R-AND$ and $R-OR$. We also investigate the properties of these operations on $CSESs$. $CSESs$ satisfy commutative, associative, De Morgan's, distributive, idempotent and absorption laws. We derive the conditions for $P-OR$, $P-AND$ of two internal cubic soft expert ($ICSE$) sets to be internal cubic soft expert set. We also give the conditions for the $P-OR$, $R-OR$ and $R-AND$ of two external cubic soft expert ($ECSE$) sets to be an external cubic soft expert set. We provide conditions for the $R-AND$ and $P-AND$ of two cubic soft expert sets to be an internal cubic soft expert ($ICSE$) set and an external cubic soft expert ($ECSE$) set. At the end, an algorithm has been presented to support our structure in decision analysis.

4.2 Cubic Soft Expert Sets

In this section we define the concept of cubic soft expert sets, give their types and definitions of their basic operations namely, $P-order$, $R-order$, $P-containment$, $R-containment$, $P-union$, $P-intersection$, $R-union$, $R-intersection$, complement, $PAND$, $P-OR$, $R-AND$ and $R-OR$. Several laws and related results have also been investigated.

Definition 4.2.1 *Let U be a finite universe set containing n alternatives, E; a set of criteria and X; a set of experts (or decision makers). A pair (β, E, X) is called a cubic soft expert set over U if and only if $\beta : E \times X \longrightarrow CP(U)$ is a mapping into the set of all cubic sets in U. Cubic soft expert set is denoted and defined as*

$$(\beta, E, X) = \{\beta(e, x) = \{(u, A_{(e,x)}(u), \lambda_{(e,x)}(u)) : u \in U, (e, x) \in E \times X\},$$

where $A_{(e,x)}(u)$ is an interval valued fuzzy set and $\lambda_{(e,x)}(u)$ is a fuzzy set.

Example 4.2.2 *Let $U = \{u_1, u_2, u_3\}$ be the set of countries, $E = \{e_1$ =Physiological natality, e_2 = Potential mortality$\}$ be the set of factors affecting population , $X = \{x_1, x_2\}$ be the set of experts. Let $E \times X = \{(e_1, x_1), (e_1, x_2), (e_2, x_1), (e_2, x_2)\}$. Then the cubic soft expert set (β, E, X) in U is given by*

$\beta(e_1, x_1) = \{(u_1, [0.07, 0.09], 0.09), (u_2, [0.06, 0.08], 0.02), (u_3, [0.03, 0.06], 0.04)\}$,
$\beta(e_2, x_1) = \{(u_1, [0.03, 0.05], 0.06), (u_2, [0.05, 0.06], 0.03), (u_3, [0.07, 0.08], 0.05)\}$,
$\beta(e_1, x_2) = \{(u_1, [0.05, 0.08], 0.08), (u_2, [0.06, 0.09], 0.07), (u_3, [0.05, 0.08], 0.06)\}$,
$\beta(e_2, x_2) = \{(u_1, [0.07, 0.09], 0.02), (u_2, [0.05, 0.08], 0.08), (u_3, [0.04, 0.07], 0.04)\}$.

In above example interval valued fuzzy set indicates the experts opinion for future time period and fuzzy set indicates the experts opinion for present time period under the different circumstances related to the given problem.

Definition 4.2.3 *A cubic soft expert set is said to be an internal cubic soft expert $(ICSE)$ set if $A^-_{(e,x)}(u) \leq \lambda_{(e,x)}(u) \leq A^+_{(e,x)}(u)$ for all $(e, x) \in E \times X$ and for all $u \in U$.*

Example 4.2.4 *Let $U = \{u_1, u_2, u_3\}$ be the initial universe, $E = \{e_1, e_2\}$ be the set of attributes, $X = \{x_1, x_2\}$ be the set of experts. Then the cubic set $(\beta, E, X) = \{\beta(e, x) = \{(u,\ A_{(e,x)}(u),\ \lambda_{(e,x)}(u)); u \in U, (e, x) \in (E \times X)\}$ in U is an internal cubic soft expert set.*

$\beta(e_1, x_1) = \{(u_1, [0.5, 0.8], 0.7), (u_2, [0.6, 0.9], 0.8), (u_3, [0.4, 0.7], 0.5)\}$,
$\beta(e_2, x_1) = \{(u_1, [0.4, 0.7], 0.6), (u_2, [0.7, 0.9], 0.8), (u_3, [0.3, 0.5], 0.4)\}$,
$\beta(e_1, x_2) = \{(u_1, [0.4, 0.8], 0.5), (u_2, [0.6, 0.9], 0.8), (u_3, [0.4, 0.6], 0.5)\}$,
$\beta(e_2, x_2) = \{(u_1, [0.3, 0.8], 0.4), (u_2, [0.6, 0.9], 0.7), (u_3, [0.5, 0.7], 0.6)\}$.

Definition 4.2.5 *A cubic soft expert set is said to be an external cubic soft expert (ECSE) set, if* $\lambda_{(e,x)}(u) \notin \left]A^-_{(e,x)}(u), A^+_{(e,x)}(u)\right[$ *for all* $(e,x) \in E \times X$ *and for all* $u \in U$.

Example 4.2.6 *Let* $U = \{u_1, u_2, u_3\}$ *be the initial universe,* $E = \{e_1, e_2\}$ *be the set of attributes and* $X = \{x_1, x_2\}$ *be the set of experts. Then the cubic set* $(\beta, E, X) = \beta(e,x) = \{(u,\ A_{(e,x)}(u),\ \lambda_{(e,x)}(u)); u \in U, (e,x) \in E \times X\}$ *in* U *is an external cubic soft expert set.*

$\beta(e_1, x_1) = \{(u_1, [0.5, 0.8], 0.3), (u_2, [0.6, 0.9], 0.5), (u_3, [0.4, 0.7], 0.2)\}$,
$\beta(e_2, x_1) = \{(u_1, [0.4, 0.7], 0.4), (u_2, [0.5, 0.9], 0.9), (u_3, [0.3, 0.5], 0.8)\}$,
$\beta(e_1, x_2) = \{(u_1, [0.4, 0.8], 0.9), (u_2, [0.7, 0.9], 0.6), (u_3, [0.5, 0.7], 0.8)\}$,
$\beta(e_2, x_2) = \{(u_1, [0.3, 0.8], 0.2), (u_2, [0.6, 0.9], 0.4), (u_3, [0.4, 0.6], 0.7)\}$.

4.3 Operations on Cubic Soft Expert Sets ($CSESs$)

Some operations on cubic soft expert sets have been discussed in below.

Definition 4.3.1 *Let* (β, E, X) *be a* $CSES$ *over* U. *For any* $e_1,\ e_2 \in E,\ x_1,\ x_2 \in X$ *if* $\beta(e_1, x_1) = \{(u,\ A_{1(e_1,x_1)}(u),\ \lambda_{1(e_1,x_1)}(u)) : u \in U\}$ *and* $\beta(e_2, x_2) = \{(u,\ A_{2(e_2,x_2)}(u),\ \lambda_{2(e_2,x_2)}(u)) : u \in U\}$. *Then* $P-order$, *denoted by* $\beta(e_1, x_1) \subseteq_P \beta(e_2, x_2)$, *is defined as below:*

$$1)\ A_{1(e_1,x_1)}(u) \preceq A_{2(e_2,x_2)}(u),\ \forall u \in U,$$
$$2)\ \lambda_{1(e_1,x_1)}(u) \leq \lambda_{2(e_2,x_2)}(u),\ \forall u \in U.$$

Example 4.3.2 *In Example 4.2.6,* $\beta(e_2,\ x_2) = \{(u_1, [0.3, 0.8], 0.2), (u_2, [0.6, 0.9], 0.4), (u_3, [0.4, 0.6], 0.7)\} \subseteq_P \beta(e_1,\ x_2) = \{(u_1,\ [0.4, 0.8], 0.9),\ (u_2,\ [0.7, 0.9],\ 0.6),\ (u_3,\ [0.5, 0.7], 0.8)\}$.

Clearly conditions 1) and 2) of Definition 4.3.1 hold.

Definition 4.3.3 *A* $CSES$ (β_1, E_1, X_1) *over* U *is said to be* $P-order$ *contained in another* $CSES$ (β_2, E_2, X_2) *over* U, *denoted by* $(\beta_1, E_1, X_1) \subseteq_P (\beta_2, E_2, X_2)$, *if the following conditions are satisfied:*

$$1)\ E_1 \subseteq E_2,$$
$$2)\ X_1 \subseteq X_2,$$
$$3)\ \beta_1(e,x) \subseteq_P \beta_2(e,x) \text{ for all } e \epsilon E_1,\ x \epsilon X_1.$$

Example 4.3.4 *Let* $(E_1 \times X_1) = \{(e_1, x_1),\ (e_2, x_1),\ (e_1, x_2),\ (e_2, x_2)\}$, $(E_2 \times X_2) = \{(e_1, x_1),\ (e_2, x_1)\}$. *Let* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *be two* $CSESs$ *over* U *defined as below:*

$\beta_1(e_1, x_1) = \{(u_1, [0.5, 0.8], 0.7), (u_2, [0.6, 0.9], 0.8), (u_3, [0.4, 0.7], 0.5)\}$,

$\beta_1(e_2, x_1) = \{(u_1, [0.4, 0.7], 0.6), (u_2, [0.7, 0.9], 0.8), (u_3, [0.3, 0.5], 0.4)\}$,
$\beta_1(e_1, x_2) = \{(u_1, [0.4, 0.8], 0.5), (u_2, [0.6, 0.9], 0.8), (u_3, [0.4, 0.6], 0.5)\}$,
$\beta_1(e_2, x_2) = \{(u_1, [0.3, 0.8], 0.4), (u_2, [0.6, 0.9], 0.7), (u_3, [0.5, 0.7], 0.6)\}$.
$\beta_2(e_1, x_1) = \{(u_1, [0.2, 0.5], 0.6), (u_2, [0.5, 0.7], 0.3), (u_3, [0.1, 0.4], 0.3)\}$,
$\beta_2(e_2, x_1) = \{(u_1, [0.2, 0.5], 0.1), (u_2, [0.5, 0.9], 0.6), (u_3, [0.2, 0.4], 0.4)\}$.

Clearly conditions 1), 2) and 3) of Definition 4.3.3 hold. So, $(\beta_2, E_2, X_2) \subseteq_P (\beta_1, E_1, X_1)$.

Definition 4.3.5 *Let* (β, E, X) *be a CSES over* U *for any* $e_1, e_2 \in E$, $x_1, x_2 \in X$. *If* $\beta(e_1, x_1) = \{(u, A_{1(e_1,x_1)}(u), \lambda_{1_{(e_1,x_1)}}(u)) : u \in U\}$ *and* $\beta(e_2, x_2) = \{(u, A_{2(e_2,x_2)}(u), \lambda_{2_{(e_2,x_2)}}(u)) : u \in U\}$ *then the* $R-$ *order denoted by* $\beta(e_1, x_1) \subseteq_R \beta(e_2, x_2)$, *is defined as below:*

$$1)\ A_{1(e_1,x_1)}(u) \preceq A_{2(e_2,x_2)}(u),\ \forall u \in U,$$
$$2)\ \lambda_{1_{(e_1,x_1)}}(u) \geq \lambda_{2_{(e_2,x_2)}}(u),\ \forall u \in U.$$

Example 4.3.6 *In Example 4.2.6,* $\beta(e_2, x_1) = \{(u_1, [0.4, 0.7], 0.4), (u_2, [0.5, 0.9], 0.9), (u_3, [0.3, 0.5], 0.8)\} \subseteq_R \beta(e_1, x_1) = \{(u_1, [0.5, 0.8], 0.3), (u_2, [0.6, 0.9], 0.5), (u_3, [0.4, 0.7], 0.2)\}$.

Clearly conditions 1) and 2) of Definition 4.3.5 hold.

Definition 4.3.7 *A CSES* (β_1, E_1, X_1) *over* U *is said to be* $R-$ *order contained in another CSES* (β_2, E_2, X_2) *over* U, *denoted by* $(\beta_1, E_1, X_1) \subseteq_R (\beta_2, E_2, X_2)$, *if the following conditions are satisfied:*

$$1)\ E_1 \subseteq E_2,$$
$$2)\ X_1 \subseteq X_2,$$
$$3)\ \beta_1(e, x) \subseteq_R \beta_2(e, x) \text{ for all } e \epsilon E_1,\ x \epsilon X_1.$$

Example 4.3.8 *Let* $(E_1 \times X_1) = \{(e_1, x_1), (e_2, x_1), (e_1, x_2), (e_2, x_2)\}, (E_2 \times X_2) = \{(e_1, x_1), (e_2, x_1)\}$. *Let* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *be two CSESs over* U *defined as below.*

$\beta_1(e_1, x_1) = \{(u_1, [0.5, 0.8], 0.6), (u_2, [0.6, 0.9], 0.3), (u_3, [0.4, 0.7], 0.3)\}$,
$\beta_1(e_2, x_1) = \{(u_1, [0.4, 0.7], 0.1), (u_2, [0.7, 0.9], 0.6), (u_3, [0.3, 0.5], 0.4)\}$,
$\beta_1(e_1, x_2) = \{(u_1, [0.4, 0.8], 0.5), (u_2, [0.6, 0.9], 0.8), (u_3, [0.4, 0.6], 0.5)\}$,
$\beta_1(e_2, x_2) = \{(u_1, [0.3, 0.8], 0.4), (u_2, [0.6, 0.9], 0.7), (u_3, [0.5, 0.7], 0.6)\}$.
$\beta_2(e_1, x_1) = \{(u_1, [0.2, 0.5], 0.7), (u_2, [0.5, 0.7], 0.8), (u_3, [0.1, 0.4], 0.5)\}$,
$\beta_2(e_2, x_1) = \{(u_1, [0.2, 0.5], 0.6), (u_2, [0.5, 0.9], 0.8), (u_3, [0.2, 0.4], 0.4)\}$.

Clearly conditions 1), 2) and 3) of Definition 4.3.7 hold. So, $(\beta_2, E_2, X_2) \subseteq_R (\beta_1, E_1, X_1)$.

Definition 4.3.9 *Two CSESs* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *over* U, *are equal, denoted by* $(\beta_1, E_1, X_1) = (\beta_2, E_2, X_2)$, *if*

1) $E_1 = E_2$,
2) $X_1 = X_2$,
3) $\beta_1(e, x) = \beta_2(e, x)$ (*that is* $A_{1(e,x)}(u) = A_{2(e,x)}(u)$ *and* $\lambda_{1_{(e,x)}}(u) = \lambda_{2_{(e,x)}}(u)$) *for all* $e \in E_1 = E_2$, $x \in X_1 = X_2$.

Corollary 4.3.10 *For any two CSESs* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *over* U;

1) *If* $(\beta_1, E_1, X_1) \subseteq_P (\beta_2, E_2, X_2)$ *and* $(\beta_2, E_2, X_2) \subseteq_P (\beta_1, E_1, X_1)$, *then* $(\beta_1, E_1, X_1) = (\beta_2, E_2, X_2)$,
2) *If* $(\beta_1, E_1, X_1) \subseteq_R (\beta_2, E_2, X_2)$ *and* $(\beta_2, E_2, X_2) \subseteq_R (\beta_1, E_1, X_1)$, *then* $(\beta_1, E_1, X_1) = (\beta_2, E_2, X_2)$.

Definition 4.3.11 *The* $P-$*union of two CSESs* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *over* U *is denoted by* $(\beta_3, F, Y) = (\beta_1, E_1, X_1) \cup_P (\beta_2, E_2, X_2)$ *where* $F = E_1 \cup E_2$, $Y = X_1 \cup X_2$ *and for all* $g \in F$ *and* $z \in Y$, *it is defined as:*

$$\beta_3(g, z) = \begin{cases} \{(u,\ A_{1(g,z)}(u),\ \lambda_{1(g,z)}(u))\} & \text{if } (g, z) \in (E_1 \times X_1) \setminus (E_2 \times X_2) \\ \{(u,\ A_{2(g,z)}(u),\ \lambda_{2(g,z)}(u))\} & \text{if } (g, z) \in (E_2 \times X_2) \setminus (E_1 \times X_1) \\ \{(u,\ \sup\{A_{1(g,z)}(u),\ A_{2(g,z)}(u)\} & \text{if } (g, z) \in (E_1 \cap E_2 \times X_1 \cap X_2) \\ ,\ \sup\{\lambda_{1(g,z)}(u),\ \lambda_{2(g,z)}(u)\})\}, & \end{cases}$$

whenever $\beta_1(g, z) = \{(u,\ A_{1(g,z)}(u),\ \lambda_{1(g,z)}(u)) : u \in U\}$ *and* $\beta_2(g, z) = \{(u,\ A_{2(g,z)}(u),\ \lambda_{2(g,z)}(u)) : u \in U\}$.

Example 4.3.12 *Consider Example 4.3.8. Let* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *be two CSESs over* U *defined as below:*

$\beta_1(e_1, x_1) = \{(u_1, [0.5, 0.8], 0.6), (u_2, [0.6, 0.9], 0.3), (u_3, [0.4, 0.7], 0.3)\}$,
$\beta_1(e_2, x_1) = \{(u_1, [0.4, 0.7], 0.1), (u_2, [0.7, 0.9], 0.6), (u_3, [0.3, 0.5], 0.4)\}$,
$\beta_1(e_1, x_2) = \{(u_1, [0.4, 0.8], 0.5), (u_2, [0.6, 0.9], 0.8), (u_3, [0.4, 0.6], 0.5)\}$,
$\beta_1(e_2, x_2) = \{(u_1, [0.3, 0.8], 0.4), (u_2, [0.6, 0.9], 0.7), (u_3, [0.5, 0.7], 0.6)\}$.
$\beta_2(e_1, x_1) = \{(u_1, [0.2, 0.5], 0.7), (u_2, [0.5, 0.7], 0.8), (u_3, [0.1, 0.4], 0.5)\}$,
$\beta_2(e_2, x_1) = \{(u_1, [0.2, 0.5], 0.6), (u_2, [0.5, 0.9], 0.8), (u_3, [0.2, 0.4], 0.4)\}$.
Therefore, $(\beta_3, F, Y) = (\beta_1, E_1, X_1) \cup_P (\beta_2, E_2, X_2)$ *is given below:*
$\beta_3(e_1, x_1) = \{(u_1, [0.5, 0.8], 0.7), (u_2, [0.6, 0.9], 0.8), (u_3, [0.4, 0.7], 0.5)\}$,
$\beta_3(e_2, x_1) = \{(u_1, [0.4, 0.7], 0.6), (u_2, [0.7, 0.9], 0.8), (u_3, [0.3, 0.5], 0.4)\}$,
$\beta_3(e_1, x_2) = \{(u_1, [0.4, 0.8], 0.5), (u_2, [0.6, 0.9], 0.8), (u_3, [0.4, 0.6], 0.5)\}$,
$\beta_3(e_2, x_2) = \{(u_1, [0.3, 0.8], 0.4), (u_2, [0.6, 0.9], 0.7), (u_3, [0.5, 0.7], 0.6)\}$.

Definition 4.3.13 *The $P-$intersection of two CSESs (β_1, E_1, X_1) and (β_2, E_2, X_2) over U is denoted by $(\beta_3, F, Y) = (\beta_1, E_1, X_1) \cap_P (\beta_2, E_2, X_2)$ where $F = E_1 \cap E_2$, $Y = X_1 \cap X_2$ and for all $g \in F$ and $z \in Y$, it is defined as:*

$$\beta_3(g, z) = \{(u, \ \inf\{A_{1(g,z)}(u), \ A_{2(g,z)}(u)\}, \ \inf\{\lambda_{1(g,z)}(u), \ \lambda_{2(g,z)}(u)\})\},$$

whenever $\beta_1(g, z) = \{(u, \ A_{1(g,z)}(u), \ \lambda_{1(g,z)}(u)) : u \in U\}$ and $\beta_2(g, z) = \{(u, \ A_{2(g,z)}(u), \ \lambda_{2(g,z)}(u)) : u \in U\}$.

Example 4.3.14 *Consider Example 4.3.8. Let (β_1, E_1, X_1) and (β_2, E_2, X_2) be two CSESs over U defined as below:*

$\beta_1(e_1, x_1) = \{(u_1, [0.5, 0.8], 0.6), (u_2, [0.6, 0.9], 0.3), (u_3, [0.4, 0.7], 0.3)\}$,
$\beta_1(e_2, x_1) = \{(u_1, [0.4, 0.7], 0.1), (u_2, [0.7, 0.9], 0.6), (u_3, [0.3, 0.5], 0.4)\}$,
$\beta_1(e_1, x_2) = \{(u_1, [0.4, 0.8], 0.5), (u_2, [0.6, 0.9], 0.8), (u_3, [0.4, 0.6], 0.5)\}$,
$\beta_1(e_2, x_2) = \{(u_1, [0.3, 0.8], 0.4), (u_2, [0.6, 0.9], 0.7), (u_3, [0.5, 0.7], 0.6)\}$.
$\beta_2(e_1, x_1) = \{(u_1, [0.2, 0.5], 0.7), (u_2, [0.5, 0.7], 0.8), (u_3, [0.1, 0.4], 0.5)\}$,
$\beta_2(e_2, x_1) = \{(u_1, [0.2, 0.5], 0.6), (u_2, [0.5, 0.9], 0.8), (u_3, [0.2, 0.4], 0.4)\}$.
Therefore, $(\beta_3, F, Y) = (\beta_1, E_1, X_1) \cap_P (\beta_2, E_2, X_2)$ is given below:
$\beta_3(e_1, x_1) = \{(u_1, [0.2, 0.5], 0.6), (u_2, [0.5, 0.7], 0.3), (u_3, [0.1, 0.4], 0.3)\}$,
$\beta_3(e_2, x_1) = \{(u_1, [0.2, 0.5], 0.1), (u_2, [0.5, 0.9], 0.6), (u_3, [0.2, 0.4], 0.4)\}$.

Definition 4.3.15 *The $R-$union of two CSESs (β_1, E_1, X_1) and (β_2, E_2, X_2) over U is denoted by $(\beta_3, F, Y) = (\beta_1, E_1, X_1) \cup_R (\beta_2, E_2, X_2)$ where F=$E_1 \cup E_2$, $Y = X_1 \cup X_2$ and for all $g \in F$ and $z \in Y$, it is defined as:*

$$\beta_3(g, z) = \begin{cases} \{(u, \ A_{1(g,z)}(u), \ \lambda_{1(g,z)}(u))\} & \text{if } (g, z) \in (E_1 \times X_1) \setminus (E_2 \times X_2) \\ \{(u, \ A_{2(g,z)}(u), \ \lambda_{2(g,z)}(u))\} & \text{if } (g, z) \in (E_2 \times X_2) \setminus (E_1 \times X_1) \\ \{(u, \ \sup\{A_{1(g,z)}(u), \ A_{2(g,z)}(u)\} & \text{if } (g, z) \in (E_1 \cap E_2 \times X_1 \cap X_2) \\ , \ \inf\{\lambda_{1(g,z)}(u), \ \lambda_{2(g,z)}(u)\})\}, & \end{cases}$$

whenever $\beta_1(g, z) = \{(u, \ A_{1(g,z)}(u), \ \lambda_{1(g,z)}(u)) : u \in U\}$ and $\beta_2(g, z) = \{(u, \ A_{2(g,z)}(u), \ \lambda_{2(g,z)}(u)) : u \in U\}$.

Example 4.3.16 *Consider Example 4.3.4. Let (β_1, E_1, X_1) and (β_2, E_2, X_2) be two CSESs over U defined as below:*

$\beta_1(e_1, x_1) = \{(u_1, [0.5, 0.8], 0.7), (u_2, [0.6, 0.9], 0.8), (u_3, [0.4, 0.7], 0.5)\}$,
$\beta_1(e_2, x_1) = \{(u_1, [0.4, 0.7], 0.6), (u_2, [0.7, 0.9], 0.8), (u_3, [0.3, 0.5], 0.4)\}$,
$\beta_1(e_1, x_2) = \{(u_1, [0.4, 0.8], 0.5), (u_2, [0.6, 0.9], 0.8), (u_3, [0.4, 0.6], 0.5)\}$,
$\beta_1(e_2, x_2) = \{(u_1, [0.3, 0.8], 0.4), (u_2, [0.6, 0.9], 0.7), (u_3, [0.5, 0.7], 0.6)\}$.
$\beta_2(e_1, x_1) = \{(u_1, [0.2, 0.5], 0.6), (u_2, [0.5, 0.7], 0.3), (u_3, [0.1, 0.4], 0.3)\}$,
$\beta_2(e_2, x_1) = \{(u_1, [0.2, 0.5], 0.1), (u_2, [0.5, 0.9], 0.6), (u_3, [0.2, 0.4], 0.4)\}$.

Therefore, $(\beta_3, F, Y) = (\beta_1, E_1, X_1) \cup_R (\beta_2, E_2, X_2)$ *is given below:*
$\beta_3(e_1, x_1) = \{(u_1, [0.5, 0.8], 0.6), (u_2, [0.6, 0.9], 0.3), (u_3, [0.4, 0.7], 0.3)\}$,
$\beta_3(e_2, x_1) = \{(u_1, [0.4, 0.7], 0.1), (u_2, [0.7, 0.9], 0.6), (u_3, [0.3, 0.5], 0.4)\}$,
$\beta_3(e_1, x_2) = \{(u_1, [0.4, 0.8], 0.5), (u_2, [0.6, 0.9], 0.8), (u_3, [0.4, 0.6], 0.5)\}$,
$\beta_3(e_2, x_2) = \{(u_1, [0.3, 0.8], 0.4), (u_2, [0.6, 0.9], 0.7), (u_3, [0.5, 0.7], 0.6)\}$.

Definition 4.3.17 *The* $R-$*intersection of two CSESs* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *over* U *is denoted by* $(\beta_3, F, Y) = (\beta_1, E_1, X_1) \cap_R (\beta_2, E_2, X_2)$ *where* $F = E_1 \cap E_2$, $Y = X_1 \cap X_2$ *and for all* $g \in F$ *and* $z \in Y$, *it is defined as:*

$$\beta_3(g, z) = \{(u,\ \inf\{A_{1(g,z)}(u),\ A_{2(g,z)}(u)\},\ \sup\{\lambda_{1(g,z)}(u),\ \lambda_{2(g,z)}(u)\})\},$$

whenever $\beta_1(g, z) = \{(u,\ A_{1(g,z)}(u),\ \lambda_{1(g,z)}(u)) : u \in U\}$ *and* $\beta_2(g, z) = \{(u,\ A_{2(g,z)}(u),\ \lambda_{2(g,z)}(u)) : u \in U\}$.

Example 4.3.18 *Consider Example 4.3.4. Let* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *be two CSESs over* U *defined as below:*
$\beta_1(e_1, x_1) = \{(u_1, [0.5, 0.8], 0.7), (u_2, [0.6, 0.9], 0.8), (u_3, [0.4, 0.7], 0.5)\}$,
$\beta_1(e_2, x_1) = \{(u_1, [0.4, 0.7], 0.6), (u_2, [0.7, 0.9], 0.8), (u_3, [0.3, 0.5], 0.4)\}$,
$\beta_1(e_1, x_2) = \{(u_1, [0.4, 0.8], 0.5), (u_2, [0.6, 0.9], 0.8), (u_3, [0.4, 0.6], 0.5)\}$,
$\beta_1(e_2, x_2) = \{(u_1, [0.3, 0.8], 0.4), (u_2, [0.6, 0.9], 0.7), (u_3, [0.5, 0.7], 0.6)\}$.
$\beta_2(e_1, x_1) = \{(u_1, [0.2, 0.5], 0.6), (u_2, [0.5, 0.7], 0.3), (u_3, [0.1, 0.4], 0.3)\}$,
$\beta_2(e_2, x_1) = \{(u_1, [0.2, 0.5], 0.1), (u_2, [0.5, 0.9], 0.6), (u_3, [0.2, 0.4], 0.4)\}$.
Therefore, $(\beta_3, F, Y) = (\beta_1, E_1, X_1) \cap_R (\beta_2, E_2, X_2)$ *is given below:*
$\beta_3(e_1, x_1) = \{(u_1, [0.2, 0.5], 0.7), (u_2, [0.5, 0.7], 0.8), (u_3, [0.1, 0.4], 0.5)\}$,
$\beta_3(e_2, x_1) = \{(u_1, [0.2, 0.5], 0.6), (u_2, [0.5, 0.9], 0.8), (u_3, [0.2, 0.4], 0.4)\}$.

Definition 4.3.19 *The complement of a CSES* (β, E, X) *is denoted and defined as* $(\beta, E, X)^c = (\beta^c, E^c, X)$ *where* $\beta^c : E^c \times X \longrightarrow CP(U)$ *is a mapping given as*

$$\beta^c(e^c, x) = \{(u,\ A^c_{(e,x)}(u),\ \lambda^c_{(e,x)}(u)) : u \in U,\ (e^c, x) \in E^c \times X\},$$

where $A^c_{(e,x)}(u) = [1 - A^+_{(e,x)}(u), 1 - A^-_{(e,x)}(u)]$ *and* $\lambda^c_{(e,x)}(u) = 1 - \lambda_{(e,x)}(u)$ *whenever* $\beta(e, x) = \{(u,\ A_{(e,x)}(u),\ \lambda_{(e,x)}(u)) : u \in U\}$.

Example 4.3.20 *Consider Example 4.2.6. The complement of CSES is given as follows:*
$\beta^c(e_1^c, x_1) = \{(u_1,\ [0.2, 0.5],\ 0.7),\ (u_2,\ [0.1, 0.4],\ 0.5),\ (u_3,\ [0.3, 0.6],\ 0.8)\}$,
$\beta^c(e_2^c, x_1) = \{(u_1, [0.3, 0.6], 0.6), (u_2, [0.1, 0.5], 0.1), (u_3, [0.5, 0.7], 0.2)\}$,
$\beta^c(e_1^c, x_2) = \{(u_1, [0.2, 0.6], 0.1), (u_2, [0.1, 0.3], 0.4), (u_3, [0.3, 0.5], 0.2)\}$,
$\beta^c(e_2^c, x_2) = \{(u_1, [0.2, 0.7], 0.8), (u_2, [0.1, 0.4], 0.6), (u_3, [0.4, 0.6], 0.3)\}$.

4.4 Properties of Cubic Soft Expert Sets ($CSESs$)

In below we discuss some properties of $CSESs$.

Proposition 4.4.1 *For any* $CSESs$ $(\beta_1, E_1, X_1), (\beta_2, E_2, X_2), (\beta_3, E_3, X_3)$ *and* (β_4, E_4, X_4) *over* U*, we have*

1) *If* $(\beta_1, E_1, X_1) \subseteq_P (\beta_2, E_2, X_2)$ *and* $(\beta_2, E_2, X_2) \subseteq_P (\beta_3, E_3, X_3)$*,*
then $(\beta_1, E_1, X_1) \subseteq_P (\beta_3, E_3, X_3)$.

2) *If* $(\beta_1, E_1, X_1) \subseteq_P (\beta_2, E_2, X_2)$*, then* $(\beta_2, E_2, X_2)^c \subseteq_P (\beta_1, E_1, X_1)^c$.

3) *If* $(\beta_1, E_1, X_1) \subseteq_P (\beta_2, E_2, X_2))$ *and* $(\beta_1, E_1, X_1) \subseteq_P (\beta_3, E_3, X_3)$*,*
then $(\beta_1, E_1, X_1) \subseteq_P (\beta_2, E_2, X_2)) \cap_P (\beta_3, E_3, X_3)$.

4) *If* $(\beta_1, E_1, X_1) \subseteq_P (\beta_2, E_2, X_2)$ *and* $(\beta_3, E_3, X_3) \subseteq_P (\beta_2, E_2, X_2)$*,*
then $(\beta_1, E_1, X_1) \cup_P (\beta_3, E_3, X_3) \subseteq_P (\beta_2, E_2, X_2)$.

5) *If* $(\beta_1, E_1, X_1) \subseteq_P (\beta_2, E_2, X_2)$ *and* $(\beta_3, E_3, X_3) \subseteq_P (\beta_4, E_4, X_4)$*,*
then $(\beta_1, E_1 X_1) \cup_P (\beta_3, E_3, X_3) \subseteq_P (\beta_2, E_2, X_2) \cup_P (\beta_4, E_4, X_4)$ *and*
$(\beta_1, E_1, X_1) \cap_P (\beta_3, E_3, X_3) \subseteq_P (\beta_2, E_2, X_2) \cap_P (\beta_4, E_4, X_4)$.

All the above results also holds for $R-$ *order.*

Proof. These can easily be proved by using Definitions 4.3.11, 4.3.13, 4.3.15, 4.3.17 and 4.3.19. ■

Theorem 4.4.2 *For any* $CSESs$ $(\beta_1, E_1, X_1), (\beta_2, E_2, X_2), (\beta_3, E_3, X_3)$ *and* (β_4, E_4, X_4) *over* U *the following properties hold.*

1) ***Idempotent*** $(\beta_1, E_1, X_1) \cup_P (\beta_1, E_1, X_1) = (\beta_1, E_1, X_1) = (\beta_1, E_1, X_1) \cap_P (\beta_1, E_1, X_1)$,

$(\beta_1, E_1, X_1) \cup_R (\beta_1, E_1, X_1) = (\beta_1, E_1, X_1) = (\beta_1, E_1, X_1) \cap_R (\beta_1, E_1, X_1)$.

2) ***Commutative*** $(\beta_1, E_1, X_1) \cup_P (\beta_2, (E_2, X_2) = (\beta_2, E_2, X_2) \cup_P (\beta_1, E_1, X_1)$,

$(\beta_1, E_1, X_1) \cup_R (\beta_2, E_2, X_2) = (\beta_2, E_2, X_2) \cup_R (\beta_1, E_1, X_1)$,

3) ***Associative*** $((\beta_1, E_1, X_1) \cup_P (\beta_2, E_2, X_2)) \cup_P (\beta_3, E_3, X_3) = (\beta_1, E_1, X_1) \cup_P ((\beta_2, E_2, X_2) \cup_P (\beta_3, E_3, X_3))$,

$((\beta_1, E_1, X_1) \cup_R (\beta_2, E_2, X_2)) \cup_R (\beta_3, E_3, X_3) = (\beta_1, E_1, X_1) \cup_R ((\beta_2, E_2, X_2) \cup_R (\beta_3, E_3, X_3))$.

4) ***Distributive*** $(\beta_1, E_1, X_1) \cup_P ((\beta_2, E_2, X_2) \cap_P (\beta_3, E_3, X_3)) = ((\beta_1, E_1, X_1) \cup_P (\beta_2, E_2, X_2)) \cap_P ((\beta_1, E_1, X_1) \cup_P (\beta_3, E_3, X_3))$,

$(\beta_1, E_1, X_1) \cap_P ((\beta_2, E_2, X_2) \cup_P (\beta_3, E_3, X_3)) = ((\beta_1, E_1, X_1) \cap_P (\beta_2, E_2, X_2)) \cup_P ((\beta_1, E_1, X_1) \cap_P (\beta_3, E_3, X_3))$,

$(\beta_1, E_1, X_1) \cup_R ((\beta_2, E_2, X_2) \cap_R (\beta_3, E_3, X_3)) = ((\beta_1, E_1, X_1) \cup_R (\beta_2, E_2, X_2)) \cap_R ((\beta_1, E_1, X_1) \cup_R (\beta_3, E_3, X_3))$,

$(\beta_1, E_1, X_1) \cap_R ((\beta_2, E_2, X_2) \cup_R (\beta_3, E_3, X_3)) = ((\beta_1, E_1, X_1) \cap_R (\beta_2, E_2, X_2)) \cup_R ((\beta_1, E_1, X_1) \cap_R (\beta_3, E_3, X_3))$.

5) ***De Morgan's laws*** $((\beta_1, E_1, X_1)) \cup_P (\beta_2, E_2, X_2))^c = (\beta_1, E_1, X_1)^c \cap_P (\beta_2, E_2, X_2)^c$,
$((\beta_1, E_1, X_1) \cap_P (\beta_2,, E_2, X_2))^c = (\beta_1, E_1, X_1)^c \cup_P (\beta_2, E_2, X_2)^c$,
$((\beta_1, E_1, X_1) \cup_R (\beta_2, E_2, X_2))^c = (\beta_1, E_1, X_1)^c \cap_R (\beta_2, E_2, X_2)^c$,
$((\beta_1, E_1, X_1) \cap_R (\beta_2, E_2, X_2))^c = (\beta_1, E_1, X_1)^c \cup_R (\beta_2, E_2, X_2)^c$.

6) ***Involution law*** $((\beta_1, E_1, X_1)^c)^c = (\beta_1, E_1, X_1)$.

Proof. These properties can be verified using Definitions 4.3.11, 4.3.13, 4.3.15, 4.3.17 and 4.3.19. ■

Proposition 4.4.3 *For any two CSES (β_1, E_1, X_1) and (β_2, E_2, X_2) over U the following are equivalent.*

$$1)\ (\beta_1, E_1, X_1) \subseteq_P (\beta_2, E_2, X_2),$$
$$2) (\beta_1, E_1, X_1) \cap_P (\beta_2, E_2, X_2) = (\beta_1, E_1, X_1),$$
$$3) (\beta_1, E_1, X_1) \cup_P (\beta_2, E_2, X_2) = (\beta_2, E_2, X_2).$$

Proof. 1) $\Longrightarrow$ 2) By Definition 4.3.13, we have $(\beta_1, E_1, X_1) \cap_P (\beta_2, E_2, X_2) = (\beta_1 \cap_P \beta_2,\ E_1 \cap E_2,\ X_1 \cap X_2) = (\beta_1 \cap_P \beta_2,\ E_1, X_1)$ as $E_1 \subseteq E_2$ and $X_1 \subseteq X_2$ by hypothesis. Now, for any $(e, x) \in E_1 \times X_1$, since $\beta_1(e, x) \subseteq_P \beta_2(e, x)$, Definition 4.3.1 implies that $A_{1(e,x)}(u) \preceq A_{2(e,x)}(u)$ and $\lambda_{1(e,x)}(u) \leq \lambda_{2(e,x)}(u)$ for any $u \in U$, where $\beta_1(e, x) = \{(u,\ A_{1(e,x)}(u),\ \lambda_{1(e,x)}(u)) : u \in U\}$. By Definition 2.1.5, we have $A^-_{1(e,x)}(u) \leq A^-_{2(e,x)}(u)$ and $A^+_{1(e,x)}(u) \leq A^+_{2(e,x)}(u)$. Thus $\inf\{A_{1(e,x)}(u),\ A_{2(e,x)}(u)\} = [\inf\{A^-_{1(e,x)}(u) \leq A^-_{2(e,x)}(u)\},\ \inf\{A^+_{1(e,x)}(u) \leq A^+_{2(e,x)}(u)\}] = [A^-_{1(e,x)}(u) \leq A^+_{1(e,x)}(u)]$ and $\inf\{\lambda_{1(e,x)}(u),\ \lambda_{2(e,x)}(u)\} = \lambda_{1(e,x)}(u)$. By using Definition 4.3.13, $\beta_1(e, x) \cap_P \beta_2(e, x) = \{\ (u,\ \inf\ \{A_{1(e,x)}(u),\ A_{2(e,x)}(u))\ \}\ ,\ \inf\ \{\lambda_{1(e,x)}(u),\ \lambda_{2(e,x)}(u)\}) : u \in U\} = \{(u,\ A_{1(e,x)}(u),\ \lambda_{1(e,x)}(u))\}\ ) : u \in U\} = \beta_1(e, x)$. Hence $\beta_1(e, x) \cap_P \beta_2(e, x) = \beta_1(e, x)$.

2) $\Longrightarrow$ 3) By Definition 4.3.11, we have $(\beta_1, E_1, X_1) \cup_P (\beta_2, E_2, X_2) = (\beta_1 \cup_P \beta_2,\ E_1 \cup E_2,\ X_1 \cup X_2) = (\beta_1 \cup_P \beta_2,\ E_2,\ X_2)$ as $E_1 \cap E_2 = E_1$ and $X_1 \cap X_2 = X_1$ by hypothesis. Now for any $(e, x) \in E_1 \times X_1$, since $\beta_1(e, x) \cap_P \beta_2(e, x) = \beta_1(e, x)$, by Definition 4.3.13, we have $\inf\{A_{1(e,x)}(u),\ A_{2(e,x)}(u)\} = A_{1(e,x)}(u)$ and $\inf\{\lambda_{1(e,x)}(u),\ \lambda_{2(e,x)}(u)\} = \lambda_{1(e,x)}(u)$. This implies that $\sup\{A_{1(e,x)}(u), A_{2(e,x)}(u)\} = A_{2(e,x)}(u)$ and $\sup\{\lambda_{1(e,x)}(u), \lambda_{2(e,x)}(u)\} = \lambda_{2(e,x)}(u)$. Thus, we have $\beta_1(e, x) \cup_P \beta_2(e, x) = \{< (u,\ \sup\{A_{1(e,x)}(u),\ A_{2(e,x)}(u)\},\ \sup\{\lambda_{1(e,x)}(u),\ \lambda_{2(e,x)}(u)\}) : u \in U\} = \{(u, A_{2(e,x)}(u),\ \lambda_{2(e,x)}(u)\}) : u \in U\} = \beta_2(e, x)$. Hence, $\beta_1(e, x) \cup_P \beta_2(e, x) = \beta_2(e, x)$.

3) $\Longrightarrow$ 1) By hypothesis, we have $(\beta_1, E_1, X_1) \cup_P (\beta_2, E_2, X_2) = (\beta_1 \cup_P \beta_2, E_1 \cup E_2, X_1 \cup X_2) = (\beta_1 \cup_P \beta_2, E_2, X_2)$ as $E_1 \cup E_2 = E_2$ and $X_1 \cup X_2 = X_2 \Longrightarrow E_1 \subseteq E_2$ and $X_1 \subseteq X_2$.

Also, $\beta_1(e,x)\cup_P\beta_2(e,x) = \{(u, \sup\{A_{1(e,x)}(u), A_{2(e,x)}(u)\}, \sup\{\lambda_{1(e,x)}(u), \lambda_{2(e,x)}(u)\})$ $: u \in U\} = \{(u, A_{2(e,x)}(u), \lambda_{2(e,x)}(u)) : u \in U\} = \beta_2(e,x)$. This implies that $A_{1(e,x)}(u) \preceq A_{2(e,x)}(u)$ and $\lambda_{1(e,x)}(u) \leq \lambda_{2(e,x)}(u)$ for any $u \in U$. Hence $(\beta_1, E_1, X_1) \subseteq_P (\beta_2, E_2, X_2)$. ■

Corollary 4.4.4 *If we take $X_1 = X_2 = X$ in the above proposition, then the following are equivalent.*

$$1)(\beta_1, E_1, X) \subseteq_P (\beta_2, E_2, X),$$
$$2)(\beta_1, E_1, X) \cap_P (\beta_2, E_2, X) = (\beta_1, E_1, X),$$
$$3)(\beta_1, E_1, X) \cup_P (\beta_2, E_2, X) = (\beta_2, E_2, X),$$
$$4)(\beta_2, E_2, X)^c \subseteq_P (\beta_1, E_1, X)^c.$$

Definition 4.4.5 *Let $\{\mathcal{L}_i\}_{i\in\Im} = \{(\beta_i, E_i, X_i)\}_{i\in\Im}$ be a family of CSESs over U, where $\beta_i(e,x) = \{(u, A_{i(e,x)}(u), \lambda_{i(e,x)}(u)) : u \in U$, for any $e \in E_i$, $x \in X_i\}$. Then $P-union$, $P-intersection$, $R-union$ and $R-intersection$ are defined as below:*

1) $\underset{i\in\Im}{\cup_P}\{\mathcal{L}_i\} = \{(u, (\underset{i\in\Im}{\sup} A_{i(e,x)})(u), (\bigvee_{i\in\Im} \lambda_{i(e,x)})(u)) : u \in U\}$.
2) $\underset{i\in\Im}{\cap_P}\{\mathcal{L}_i\} = \{(u, (\inf_{i\in\Im} A_{i(e,x)})(u), (\bigwedge_{i\in\Im} \lambda_{i(e,x)})(u)) : u \in U\}$.
3) $\underset{i\in\Im}{\cup_R}\{\mathcal{L}_i\} = \{(u, (\underset{i\in\Im}{\sup} A_{i(e,x)})(u), (\bigwedge_{i\in\Im} \lambda_{i(e,x)})(u)) : u \in U\}$.
4) $\underset{i\in\Im}{\cap_R}\{\mathcal{L}_i\} = \{(u, (\inf_{i\in\Im} A_{i(e,x)})(u), (\bigvee_{i\in\Im} \lambda_{i(e,x)})(u)) : u \in U\}$.

Theorem 4.4.6 *Let $\{\mathcal{L}_i\}_{i\in\Im} = \{(\beta_i, E_i, X_i)\}_{i\in\Im}$ be a family of ICSESs over U, where $\beta_i(e,x) = \{(u, A_{i(e,x)}(u), \lambda_{i(e,x)}(u)) : u \in U$, for any $e \in E_i$, $x \in X_i\}$. Then the $\underset{i\in\Im}{\cup_P}\{\mathcal{L}_i\}$ and $\underset{i\in\Im}{\cap_P}\{\mathcal{L}_i\}$ are ICSESs over U.*

Proof. As $\{\mathcal{L}_i\}_{i\in\Im}$ be a family of $ICSESs$ over U. Then, $A^-_{i(e,x)}(u) \leq \lambda_{i(e,x)}(u)$ $\leq A^+_{i(e,x)}(u)$ for each $i \in \Im$. This implies that $(\underset{i\in\Im}{\sup} A_{i(e,x)})^-(u) \leq (\bigvee_{i\in\Im} \lambda_{i(e,x)})(u) \leq$ $(\underset{i\in\Im}{\sup} A_{i(e,x)})^+(u)$ and $(\inf_{i\in\Im} A_{i(e,x)})^-(u) \leq (\bigwedge_{i\in\Im} \lambda_{i(e,x)})(u) \leq (\inf_{i\in\Im} A_{i(e,x)})^+(u)$. Hence $\underset{i\in\Im}{\cup_P}\{\mathcal{L}_i\}$ and $\underset{i\in\Im}{\cap_P}\{\mathcal{L}_i\}$ are $ICSESs$ over U. ■

Theorem 4.4.7 *Let (β_1, E_1, X_1) and (β_2, E_2, X_2) be two ICSESs over U, where $\beta_1(e,x) = \{(u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U\}$ for any $(e,x) \in E_1 \times X_1$ and $\beta_2(f,y) = \{(u, A_{2(f,y)}(u), \lambda_{2(f,y)}(u)) : u \in U\}$ for any $(f,y) \in E_2 \times X_2$. Then the $P-union$ of (β_1, E_1, X_1) and (β_2, E_2, X_2) is also an ICSES.*

Proof. Since (β_1, E_1, X_1) and (β_2, E_2, X_2) are $ICSESs$ over U so $A^-_{1(g,z)}(u) \leq$ $\lambda_{1(g,z)}(u) \leq A^+_{1(g,z)}(u)$ for all $u \in U$ and $A^-_{2(g,z)}(u) \leq \lambda_{2(g,z)}(u) \leq A^+_{2(g,z)}(u)$ for all $u \in U$.

Then we have $\sup\{A^-_{1(g,z)}(u), A^-_{2(g,z)}(u)\} \leq (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) \leq \sup\{A^+_{1(g,z)}(u), A^+_{2(g,z)}(u)\}$ for all $u \in U$ and $(g,z) \in (E_1 \cup E_2 \times X_1 \cup X_2)$. By Definition 4.3.11, we have $(\beta_3, F, Y) = (\beta_1, E_1, X_1) \cup_P (\beta_2, E_2, X_2)$ where $F = E_1 \cup E_2$ and $Y = X_1 \cup X_2$ and for any $g \in F$ and $z \in Y$.

$$\beta_3(g,z) = \begin{cases} \{(u, A_{1(g,z)}(u), \lambda_{1(g,z)}(u))\} & \text{if } (g,z) \in (E_1 \times X_1) \setminus (E_2 \times X_2) \\ \{(u, A_{2(g,z)}(u), \lambda_{2(g,z)}(u))\} & \text{if } (g,z) \in (E_2 \times X_2) \setminus (E_1 \times X_1) \\ \{(u, \sup\{A_{1(g,z)}(u), A_{2(g,z)}(u)\} & \text{if } (g,z) \in (E_1 \cap E_2 \times X_1 \cap X_2) \\ , \sup\{\lambda_{1(g,z)}(u), \lambda_{2(g,z)}(u)\})\}, & \end{cases}$$

if $(g,z) \in (E_1 \cap E_2 \times X_1 \cap X_2)$, then $\beta_3(g,z) = \{(u, \sup\{A_{1(g,z)}(u), A_{2(g,z)}(u)\}, (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u)) : u \in U\}$. Thus $(\beta_1, E_1, X_1) \cup_P (\beta_2, E_2, X_2))$ is an $ICSE$ set. If $(g,z) \in (E_1 \times X_1) \setminus (E_2 \times X_2)$ or if $(g,z) \in (E_2 \times X_2) \setminus (E_1 \times X_1)$, then the result is trivial. Hence $(\beta_1, E_1, X_1) \cup_P (\beta_2, E_2, X_2)$ is an $ICSES$ over U. ■

Theorem 4.4.8 *Let (β_1, E_1, X_1) and (β_2, E_2, X_2) be two ICSESs over U, where $\beta_1(e,x) = \{(u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U\}$ for any $(e,x) \in E_1 \times X_1$ and $\beta_2(f,y) = \{(u, A_{2(f,y)}(u), \lambda_{2(f,y)}(u)) : u \in U\}$ for any $(f,y) \in E_2 \times X_2$. Then the $P-intersection$ of (β_1, E_1, X_1) and (β_2, E_2, X_2) is also an ICSES.*

Proof. Since (β_1, E_1, X_1) and (β_2, E_2, X_2) are $ICSESs$ over U so $A^-_{1(g,z)}(u) \leq \lambda_{1(g,z)}(u) \leq A^+_{1(g,z)}(u)$ for all $u \in U$ and $A^-_{2(g,z)}(u) \leq \lambda_{2(g,z)}(u) \leq A^+_{2(g,z)}(u)$ for all $u \in U$. Then we have $\inf\{A^-_{1(g,z)}(u), A^-_{2(g,z)}(u)\} \leq (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) \leq \inf\{A^+_{1(g,z)}(u), A^+_{2(g,z)}(u)\}$ for all $u \in U$ and $(g,z) \in (E_1 \cap E_2 \times X_1 \cap X_2)$. By Definition 4.3.13 we have $(\beta_3, F, Y) = (\beta_1, E_1, X_1) \cap_P (\beta_2, E_2, X_2)$ where $F = E_1 \cap E_2$ and $Y = X_1 \cap X_2$ and for any $g \in F$ and $z \in Y$.

$$\beta_3(g,z) = \begin{cases} \{(u, \inf\{A_{1(g,z)}(u), A_{2(g,z)}(u)\} & \text{if } (g,z) \in (E_1 \cap E_2 \times X_1 \cap X_2) \\ , \inf\{\lambda_{1(g,z)}(u), \lambda_{2(g,z)}(u)\})\}. & \end{cases}$$

Thus $(\beta_1, E_1, X_1) \cap_P (\beta_2, E_2, X_2))$ is an $ICSES$ over U. ■

Theorem 4.4.9 *Let (β_1, E_1, X_1) and (β_2, E_2, X_2) be two ECSESs over U, where $\beta_1(e,x) = \{(u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U\}$ for any $(e,x) \in E_1 \times X_1$ and $\beta_2(f,y) = \{(u, A_{2(f,y)}(u), \lambda_{2(f,y)}(u)) : u \in U\}$ for any $(f,y) \in E_2 \times X$ such that*

$$(\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) \in \begin{cases} \inf\{\sup\{A^+_{1(g,z)}(u), A^-_{2(g,z)}(u)\}, \sup\{A^-_{1(g,z)}(u), A^+_{2(g,z)}(u)\}\}, \\ \sup\{\inf\{A^+_{1(g,z)}(u), A^-_{2(g,z)}(u)\}, \inf\{A^-_{1(g,z)}(u), A^+_{2(g,z)}(u)\}\}, \end{cases}$$

for all $(g,z) \in (E_1 \cap E_2 \times X_1 \cap X_2)$ and $u \in U$. Then the $P-intersection$ of (β_1, E_1, X_1) and (β_2, E_2, X_2) is also an ECSES over U.

Proof. By Definition 4.3.13, we have $(\beta_3, E_3, X_3) = (\beta_1, E_1, X_1) \cap_P (\beta_2, E_2, X_2)$ where

$$\beta_3(g,z) = \begin{cases} \{(u, \sup\{A_{1(g,z)}(u), A_{2(g,z)}(u)\} & \text{if } (g,z) \in (E_1 \cap E_2 \times X_1 \cap X_2) \\ , \inf\{\lambda_{1(g,z)}(u), \lambda_{2(g,z)}(u)\})\}, & \end{cases}$$

if $(g,z) \in (E_1 \cap E_2 \times X_1 \cap X_2)$, take $\hbar = \inf\{\sup\{A^+_{1(g,z)}(u), A^-_{2(g,z)}(u)\}, \sup\{A^-_{1(g,z)}(u), A^+_{2(g,z)}(u)\}\}$ and $\Re = \sup\ \{\inf\ \{A^+_{1(g,z)}(u), A^-_{2(g,z)}(u)\}, \inf\ \{A^-_{1(g,z)}(u), A^+_{2(g,z)}(u)\}\}$. Then $\hbar$ is one of $A^-_{1(g,z)}(u)$, $A^-_{2(g,z)}(u)$, $A^+_{1(g,z)}(u)$, $A^+_{2(g,z)}(u)$. We only consider $\hbar = A^-_{1(g,z)}(u)$ or $A^+_{1(g,z)}(u)$ because remaining cases are similar to this one. If $\hbar = A^-_{1(g,z)}(u)$ then $A^-_{2(g,z)}(u) \leq A^+_{2(g,z)}(u) \leq A^-_{1(g,z)}(u) \leq A^+_{1(g,z)}(u)$ and so $\Re = A^+_{2(g,z)}(u)$.Thus $A^-_{2(g,z)}(u) = (\inf(\{A_{1(g,z)}, A_{2(g,z)}\})^-(u) \leq (\inf\{A_{1(g,z)}, A_{2(g,z)}\})^+(u)$
$= A^+_{2(g,z)}(u) = \Re < (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u)$. Hence $(\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) \notin ((\inf\{A_{1(g,z)}, A_{2(g,z)}\}^-(u), \inf\{A_{1(g,z)}, A_{2(g,z)}\}^+(u))$. if $\hbar = A^+_{1(g,z)}(u)$ then $A^-_{2(g,z)}(u) \leq A^+_{1(g,z)}(u) \leq A^+_{2(g,z)}(u)$ so $\Re = \sup\{A^-_{1(g,z)}(u), A^-_{2(g,z)}(u)\}$. Assume $\Re = A^-_{1(g,z)}(u)$, then we have $A^-_{2(g,yz)}(u) \leq A^-_{1(g,z)}(u) < (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) \leq A^+_{1(g,z)}(u) \leq A^+_{2(g,z)}(u)$. So we can write $A^-_{2(g,z)}(u) \leq A^-_{1(g,z)}(u) < (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) < A^+_{1(g,z)}(u) \leq A^+_{2(g,z)}(u)$ or $A^-_{2(g,z)}(u) \leq A^-_{1(g,z)}(u) < (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) = A^+_{1(g,z)}(u) \leq A^+_{2(g,z)}(u)$.

The case $A^-_{2(g,z)}(u) \leq A^-_{1(g,z)}(u) < (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) < A^+_{1(g,z)}(u) \leq A^+_{2(g,z)}(u)$ which contradicts the fact that (β_1, E_1, X_1) and (β_2, E_2, X_2) are $ECSESs$. For the case $A^-_{2(g,z)}(u) \leq A^-_{1(g,z)}(u) < (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) = A^+_{1(g,z)}(u) \leq A^+_{2(g,z)}(u)$ we have $(\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) \notin ((\inf\{A_{1(g,z)}, A_{2(g,z)}\})^-(u), (\inf\{A_{1(g,z)}, A_{2(g,z)}\})^+(u))$ because $(\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) = A^+_{1(g,z)}(u) = (\inf\{A_{1(g,z)}, A_{2(g,z)}\})^+(u)$. Again assume that $\Re = A^-_{2(g,z)}(u)$, then we have $A^-_{1(g,z)}(u) \leq A^-_{2(g,z)}(u) < (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) \leq A^+_{1(g,z)}(u) \leq A^+_{2(g,z)}(u)$. We can write $A^-_{1(g,z)}(u) \leq A^-_{2(g,z)}(u) < (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) < A^+_{1(g,z)}(u) \leq A^+_{2(g,z)}(u)$ or $A^-_{1(g,z)}(u) \leq A^-_{2(g,z)}(u) < (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) = A^+_{1(g,z)}(u) \leq A^+_{2(g,z)}(u)$. For the case $A^-_{1(g,z)}(u) \leq A^-_{2(g,z)}(u) < (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) < A^+_{1(g,z)}(u) \leq A^+_{2(g,z)}(u)$ which contradict the fact that (β_1, E_1, X_1) and (β_2, E_2, X_2) are $ECSESs$. For the case $A^-_{1(g,z)}(u) \leq A^-_{2(g,z)}(u) < (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) = A^+_{1(g,z)}(u) \leq A^+_{2(g,z)}(u)$ we have $(\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) \notin (\inf\{A_{1(g,z)}, A_{2(g,z)}\})^-(u), (\inf\{A_{1(g,z)}, A_{2(g,z)}\})^+(u)$ because $(\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) = A^+_{1(g,z)}(u) = (\inf\{A_{1(g,z)}, A_{2(g,z)}\})^+(u)$. Hence $(\beta_1, E_1, X_1) \cap_P (\beta_2, E_2, X_2)$ is an $ECSES$ over U. ■

The following example yields that $R - union$ of two $ICSESs$ need not to be an $ICSES$.

Example 4.4.10 *Let* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *are two* $ICSESs$ *over* U, *where* $\beta_1(e,x) = \{(u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U\}$ *for any* $(e,x) \in E_1 \times X_1$ *and* $\beta_2(f,y) = \{(u, A_{2(f,y)}(u), \lambda_{2(f,y)}(u)) : u \in U\}$ *for any* $(f,y) \in E_2 \times X_2$ *in which* $A_{1(g,z)}(u) = [0.5, 0.8]$ $\lambda_{1(g,z)}(u) = 0.6$ *and* $A_{2(g,z)}(u) = [0.2, 0.5]$,$\lambda_{2(g,z)}(u) = 0.3$. *Now by Definition*

4.3.15, we have $A_{3(g,z)}(u) = [0.5, 0.8]$, $\lambda_{3(g,z)}(u) = 0.3$ *Hence* $R - union$ *is not an* $ICSES$ *because* $\lambda_{3(g,z)}(u) \notin [A^-_{3(g,z)}(u), A^+_{3(g,z)}(u)]$.

The following theorem gives the condition that $R - union$ of two $ICSESs$ is also an $ICSES$.

Theorem 4.4.11 *Let* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *are two* $ICSESs$ *over* U, *where* $\beta_1(e, x) = \{(u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U\}$ *for any* $(e, x) \in E_1 \times X_1$ *and* $\beta_2(f, y) = \{(u, A_{2(f,y)}(u), \lambda_{2(f,y)}(u)) : u \in U\}$ *for any* $(f, y) \in E_2 \times X_2$ *such that* $\sup\{A^-_{1(g,z)}(u), A^-_{2(g,z)}(u)\} \leq (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u)$ *for all* $u \in U$ *and* $(g, z) \in (E_1 \cap E_2 \times X_1 \cap X_2)$. *Then the* $R - union$ *of* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *is also an* $ICSES$ *over* U.

Proof. By Definition 4.3.15, we have $(\beta_3, F, Y) = (\beta_1, E_1, X_1) \cup_R (\beta_2, E_2, X_2)$ where $F = E_1 \cup E_2$ and $Y = X_1 \cup X_2$ and for any $g \in F$ and $z \in Y$.

$$\beta_3(g, z) = \begin{cases} \{(u, A_{1(g,z)}(u), \lambda_{1(g,z)}(u))\} & \text{if } (g, z) \in (E_1 \times X_1) \setminus (E_2 \times X_2) \\ \{(u, A_{2(g,z)}(u), \lambda_{2(g,z)}(u))\} & \text{if } (g, z) \in (E_2 \times X_2) \setminus (E_1 \times X_1) \\ \{(u, \sup\{A_{1(g,z)}(u), A_{2(g,z)}(u)\} & \text{if } (g, z) \in (E_1 \cap E_2 \times X_1 \cap X_2) \\ , \inf\{\lambda_{1(g,z)}(u), \lambda_{2(g,z)}(u)\})\}. & \end{cases}$$

If $(g, z) \in (E_1 \times X_1) \setminus (E_2 \times X_2)$ or $(g, z) \in (E_2 \times X_2) \setminus (E_1 \times X_1)$, then the result holds trivially. If $(g, z) \in (E_1 \cap E_2 \times X_1 \cap X_2)$, then

$\beta_3(g, z) = \{(u, \sup\{A_{1(g,z)}(u), A_{2(g,z)}(u)\}, \inf\{\lambda_{1(g,z)}(u), \lambda_{2(g,z)}(u)\})\}$. Since (β_1, E_1, X_1) and (β_2, E_2, X_2) are $ICSESs$ over U. So we have $A^-_{1(g,z)}(u) \leq \lambda_{1(g,z)}(u) \leq ^+_{1(g,z)}(u)$ for all $u \in U$ and $A^-_{2(g,z)}(u) \leq \lambda_{2(g,z)}(u) \leq A^+_{2(g,z)}(u)$ for all $u \in U$. Also $\sup\{A^-_{1(g,z)}(u), A^-_{2(g,z)}(u)\} \leq (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) \leq \sup\{A^+_{1(g,z)}(u), A^+_{2(g,z)}(u)\}$ for all $u \in U$ and $(g, z) \in (E_1 \cap E_2, X_1 \cap X_2)$. Hence, $(\beta_1, E_1, X_1) \cup_R (\beta_2, E_2, X_2)$ is an $ICSES$ over U. ■

The following example yields that $R - union$ of two $ICSESs$ need not be an $ECSES$.

Example 4.4.12 *Let* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *be two* $ICSESs$ *over* U, *where* $\beta_1(e, x) = \{(u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U\}$ *for any* $(e, x) \in E_1 \times X_1$ *and* $\beta_2(f, y) = \{(u, A_{2(f,y)}(u), \lambda_{2(f,y)}(u)) : u \in U\}$ *for any* $(f, y) \in E_2 \times X_2$ *in which* $A_{1(g,z)}(u) = [0.5, 0.8]$ $\lambda_{1(g,z)}(u) = 0.6$ *and* $A_{2(g,z)}(u) = [0.2, 0.9]$,$\lambda_{2(g,z)}(u) = 0.8$. *Now by Definition 4.3.15, we have* $A_{3(g,z)}(u) = [0.5, 0.9]$, $\lambda_{3(g,z)}(u) = 0.6$ *Hence* $R - union$ *is not an* $ECSES$ *because* $\lambda_{3(g,z)}(u) \in [A^-_{3(g,z)}(u), A^+_{3(g,z)}(u)]$.

The following theorem gives the condition that $R - union$ of two $ICSESs$ is also an $ECSES$.

Theorem 4.4.13 *Let* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *be two* $ICSESs$ *over* U, *where* $\beta_1(e, x) = \{(u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U\}$ *for any* $(e, x) \in E_1 \times X_1$ *and* $\beta_2(f, y) = \{(u, A_{2(f,y)}(u), \lambda_{2(f,y)}(u)) : u \in U\}$ *for any* $(f, y) \in E_2 \times X_2$ *such that* $\sup\{A^-_{1(g,z)}(u), A^-_{2(g,z)}(u)\} \geq (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u)$ *for all* $u \in U$ *and* $(g, z) \in (E_1 \cap E_2 \times X_1 \cap X_2)$. *Then the* $R-union$ *of* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *is an* $ECSES$.

Proof. Straightforward by using Definition 4.3.15. ∎

The followingt example yields that $R-intersection$ of two $ICSESs$ need not be an $ICSES$.

Example 4.4.14 *Let* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *be two* $ICSESs$ *over* U, *where* $\beta_1(e, x) = \{(u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U\}$ *for any* $(e, x) \in E_1 \times X_1$ *and* $\beta_2(f, y) = \{(u, A_{2(f,y)}(u), \lambda_{2(f,y)}(u)) : u \in U\}$ *for any* $(f, y) \in E_2 \times X_2$ *in which* $A_{1(g,z)}(u) = [0.2, 0.5]$, $\lambda_{1(g,z)}(u) = 0.4$ *and* $A_{2(g,z)}(u) = [0.5, 0.8]$,$\lambda_{2(g,z)}(u) = 0.7$. *Now by Definition 4.3.17, we have* $A_{3(g,z)}(u) = [0.2, 0.5]$, $\lambda_{3(g,z)}(u) = 0.7$ *Hence* $R-intersection$ *is not an* $ICSES$ *because* $\lambda_{3(g,z)}(u) \notin [A^-_{3(g,z)}(u), A^+_{3(g,z)}(u))]$.

The following theorem gives the condition that $R-intersection$ of two $ICSESs$ is also an $ICSES$.

Theorem 4.4.15 *Let* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *be two* $ICSESs$ *over* U, *where* $\beta_1(e, x) = \{(u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U\}$ *for any* $(e, x) \in E_1 \times X_1$ *and* $\beta_2(f, y) = \{(u, A_{2(f,y)}(u), \lambda_{2(f,y)}(u)) : u \in U\}$ *for any* $(f, y) \in E_2 \times X_2$ *such that* $\inf\{A^+_{1(g,z)}(u), A^+_{2(g,z)}(u)\} \geq (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u)$ *for all* $u \in U$ *and* $(g, z) \in (E_1 \cap E_2 \times X_1 \cap X_2)$. *Then the* $R-intersection$ *of* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *is also an* $ICSES$ *over* U.

Proof. By definition 4.3.17, we have $(\beta_3, E_3, X_3) = (\beta_1, E_1, X_1) \cap_R (\beta_2, E_2, X_2)$ where $E_3 = E_1 \cap E_2$ and $X_3 = X_1 \cap X_2$, $g \in E_3$ and $z \in X_3$.

$$\beta_3(g, z) = \{(u, \inf\{A_{1(g,z)}(u), A_{2(g,z)}(u)\}, \sup\{\lambda_{1(g,z)}(u), \lambda_{2(g,z)}(u)\})\}.$$

Since (β_1, E_1, X_1) and (β_2, E_2, X_2) are $ICSESs$ over U. We have $A^-_{1(g,z)}(u) \leq \lambda_{1(g,z)}(u) \leq A^+_{1(g,z)}(u)$ for all $u \in U$ and $A^-_{2(g,z)}(u) \leq \lambda_{2(g,z)}(u) \leq A^+_{2(g,z)}(u)$ for all $u \in U$. Also $\inf\{A^-_{1(g,z)}(u), A^-_{2(g,z)}(u)\} \leq (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) \leq \inf\ \{A^+_{1(g,z)}(u), A^+_{2(g,z)}(u)\}$ for all $u \in U$ and $(g, z) \in (E_1 \cap E_2 \times X_1 \cap X_2)$. Hence, $(\beta_1, E_1, X_1) \cap_R (\beta_2, E_2, X_2)$ is an $ICSES$ over U. ∎

The following example yields that $R-intersection$ of two $ICSESs$ need not be an $ECSES$.

Example 4.4.16 *Let* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *are two* $ICSESs$ *over* U, *where* $\beta_1(e, x) = \{(u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U\}$ *for any* $(e, x) \in E_1 \times X_1$ *and* $\beta_2(f, y) =$

$\{(u, A_{2(f,y)}(u), \lambda_{2(f,y)}(u)) : u \in U\}$ *for any* $(f, y) \in E_2 \times X_2$ *in which* $A_{1(g,z)}(u) = [0.5, 0.7]$, $\lambda_{1(g,z)}(u) = 0.6$ *and* $A_{2(g,z)}(u) = [0.2, 0.9]$,$\lambda_{2(g,z)}(u) = 0.8$. *Now by Definition 4.3.17, we have* $A_{3(g,z)}(u) = [0.2, 0.9]$, $\lambda_{3(g,z)}(u) = 0.8$ *Hence* $R-$*intersection is not an* $ECSES$ *because* $\lambda_{3(g,z)}(u) \in [A^-_{3(g,z)}(u), A^+_{3(g,z)}(u))]$.

The following theorem gives the condition that $R-intersection$ of two $ICSESs$ is also an $ECSES$.

Theorem 4.4.17 *Let* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *be two* $ICSESs$ *over* U, *where* $\beta_1(e, x) = \{(u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U\}$ *for any* $(e, x) \in E_1 \times X_1$ *and* $\beta_2(f, y) = \{(u, A_{2(f,y)}(u), \lambda_{2(f,y)}(u)) : u \in U\}$ *for any* $(f, y) \in E_2 \times X_2$ *such that* $\inf\{A^+_{1(g,z)}(u), A^+_{2(g,z)}(u)\} \leq (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u)$ *for all* $u \in U$ *and* $(g, z) \in (E_1 \cap E_2 \times X_1 \cap X_2)$. *Then the* $R-intersection$ *of* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *is an* $ECSES$ *over* U.

Proof. By Definition 4.3.17, we have $(\beta_3, E_3, X_3) = (\beta_1, E_1, X_1) \cap_R (\beta_2, E_2, X_2)$ where $E_3 = E_1 \cap E_2$ and $X_3 = X_1 \cap X_2$, $g \in E_3$ and $z \in X_3$.

$$\beta_3(g, z) = \{(u, \inf\{A_{1(g,z)}(u), A_{2(g,z)}(u)\}, \sup\{\lambda_{1(g,z)}(u), \lambda_{2(g,z)}(u)\})\}.$$

Since (β_1, E_1, X_1) and (β_2, E_2, X_2) are $ICSESs$ over U. So we have $A^-_{1(g,z)}(u) \leq \lambda_{1(g,z)}(u) \leq A^+_{1(g,z)}(u)$ for all $u \in U$ and $A^-_{2(g,z)}(u) \leq \lambda_{2(g,z)}(u) \leq A^+_{2(g,z)}(u)$ for all $u \in U$. Given condition is that $\inf\{A^+_{1(g,z)}(u), A^+_{2(g,z)}(u)\} \leq (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u)$ for all $u \in U$ and $(g, z) \in (E_1 \cap E_2 \times X_1 \cap X_2)$. This implies that $(\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) \notin (\inf\{A^-_{1(g,z)}(u), A^-_{2(g,z)}(u)\}, \inf\{A^+_{1(g,z)}(u), A^+_{2(g,z)}(u)\})$. Hence $(\beta_1, E_1, X_1) \cap_R (\beta_2, E_2, X_2)$ is an $ECSE$ in U. ■

The following example shows that $R-union$ of two $ECSESs$ need not be an $ECSES$.

Example 4.4.18 *Let* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *be two* $ECSESs$ *over* U, *where* $\beta_1(e, x) = \{(u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U\}$ *for any* $(e, x) \in E_1 \times X_1$ *and* $\beta_2(f, y) = \{(u, A_{2(f,y)}(u), \lambda_{2(f,y)}(u)) : u \in U\}$ *for any* $(f, y) \in E_2 \times X_2$ *in which* $A_{1(g,z)}(u) = [0.4, 0.5]$, $\lambda_{1(g,z)}(u) = 0.6$ *and* $A_{2(g,z)}(u) = [0.3, 0.7]$, $\lambda_{2(g,z)}(u) = 0.7$. *Now by Definition 4.3.15, we have* $A_{3(g,z)}(u) = [0.4, 0.7]$, $\lambda_{3(g,z)}(u) = 0.6$. *Hence* $R-$*union is not* $ECSES$ *because* $\lambda_{3(g,z)}(u) \notin (A^-_{3(g,z)}(u), A^+_{3(g,z)}(u))$.

In the next theorem, we derive a condition for $R-union$ of two $ECSESs$ to be an $ECSES$.

Theorem 4.4.19 *Let* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *be two* $ECSESs$ *over* U, *where* $\beta_1(e, x) = \{(u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U\}$ *for any* $(e, x) \in E_1 \times X_1$ *and* $\beta_2(f, y) =$

$\{(u, A_{2(f,y)}(u), \lambda_{2(f,y)}(u)) : u \in U\}$ *for any* $(f,y) \in E_2 \times X_2$ *such that*

$$(\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) \in \begin{cases} \inf\{\sup\{A^+_{1(g,z)}(u), A^-_{2(g,z)}(u)\}, \sup\{A^-_{1(g,z)}(u), A^+_{2(g,z)}(u)\}\}, \\ \sup\{\inf\{A^+_{1(g,z)}(u), A^-_{2(g,z)}(u)\}, \inf\{A^-_{1(g,z)}(u), A^+_{2(g,z)}(u)\}\} \end{cases}$$

for all $(g,z) \in (E_1 \cap E_2 \times X_1 \cap X_2)$ *and* $u \in U$. *Then* $(\beta_1, E_1, X_1) \cup_R (\beta_2, E_2, X_2)$ *is also an* $ECSES$ *over* U.

Proof. By Definition 4.3.15, we have $(\beta_3, E_3, X_3) = (\beta_1, E_1, X_1) \cup_R (\beta_2, E_2, X_2)$ where

$$\beta_3(g,z) = \begin{cases} \{(u, A_{1(g,z)}(u), \lambda_{1(g,z)}(u))\} & \text{if } (g,z) \in (E_1 \times X_1) \setminus (E_2 \times X_2) \\ \{(u, A_{2(g,z)}(u), \lambda_{2(g,z)}(u))\} & \text{if } (g,z) \in (E_2 \times X_2) \setminus (E_1 \times X_1) \\ \{(u, \sup\{A_{1(g,z)}(u), A_{2(g,z)}(u)\} & \text{if } (g,z) \in (E_1 \cap E_2 \times X_1 \cap X_2) \\ , \inf\{\lambda_{1(g,z)}(u), \lambda_{2(g,z)}(u)\}\}\}, & \end{cases}$$

if $(g,z) \in (E_1 \cap E_2 \times X_1 \cap X_2)$, take $\hbar = \inf\{\sup\{A^+_{1(g,z)}(u), A^-_{2(g,z)}(u)\}, \sup\{A^-_{1(g,z)}(u), A^+_{2(g,z)}(u)\}\}$ and $\Re = \sup\ \{\inf\ \{A^+_{1(g,z)}(u),\ A^-_{2(g,z)}(u)\},\ \inf\ \{A^-_{1(g,z)}(u),\ A^+_{2(g,z)}(u)\}\}$. Then $\hbar$ is one of $A^+_{1(g,z)}(u)$, $A^-_{2(g,z)}(u)$, $A^-_{1(g,z)}(u)$, $A^+_{2(g,z)}(u)$ we only consider $\hbar = A^-_{2(g,z)}(u)$ or $A^+_{2(g,z)}(u)$ because remaining cases are similar to this one. If $\hbar = A^-_{2(g,z)}(u)$ then $A^-_{1(g,z)}(u) \leq A^+_{1(g,z)}(u) \leq A^-_{2(g,z)}(u) \leq A^+_{2(g,z)}(u)$ and so $\Re = A^+_{1(g,z)}(u)$. Thus $(\sup\{A_{1(g,z)},\ A_{2(g,z)}\})^-(u) = A^-_{2(g,z)}(u) = \hbar > (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u)$. Hence $(\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) \notin ((\sup\{A_{1(g,z)},\ A_{2(g,z)}\}^-(u), \sup\{A_{1(g,z)},\ A_{2(g,z)}\}^+(u))$. If $\hbar = A^+_{2(g,z)}(u)$ then $A^-_{1(g,z)}(u) \leq A^+_{2(g,z)}(u) \leq A^+_{1(g,z)}(u)$ so $\Re = \sup\{A^-_{1(g,z)}(u),\ A^-_{2(g,z)}(u)\}$. Assume $\Re = A^-_{1(g,z)}(u)$, then we have $A^-_{2(g,yz)}(u) \leq A^-_{1(g,z)}(u) \leq (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) < A^+_{2(g,z)}(u) \leq A^+_{1(g,z)}(u)$. So we can write $A^-_{2(g,z)}(u) \leq A^-_{1(g,z)}(u) < (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) < A^+_{2(g,z)}(u) \leq A^+_{1(g,z)}(u)$ or $A^-_{2(g,z)}(u) \leq A^-_{1(g,z)}(u) = (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) \leq A^+_{2(g,z)}(u) \leq A^+_{1(g,z)}(u)$.

The case $A^-_{2(g,z)}(u) \leq A^-_{1(g,z)}(u) < (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) < A^+_{2(g,z)}(u) \leq A^+_{1(g,z)}(u)$ which contradicts the fact that (β_1, E_1, X_1) and (β_2, E_2, X_2) are $ECSESs$. For the case $A^-_{2(g,z)}(u) < A^-_{1(g,z)}(u) = (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) \leq A^+_{2(g,z)}(u) \leq A^+_{1(g,z)}(u)$ we have $(\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) \notin ((\sup\{A_{1(g,z)}, A_{2(g,z)}\})^-(u), (\sup\{A_{1(g,z)}, A_{2(g,z)}\})^+(u))$ because $(\sup\ \{A_{1(g,z)},\ A_{2(g,z)}\})^-(u) = A^-_{1(g,z)}(u) = (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u)$. Again assume that $\Re = A^-_{2(g,z)}(u)$, then we have $A^-_{1(g,z)}(u) \leq A^-_{2(g,z)}(u) \leq (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) \leq A^+_{2(g,z)}(u) \leq A^+_{1(g,z)}(u)$. We can write $A^-_{1(g,z)}(u) \leq A^-_{2(g,z)}(u) < (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) < A^+_{2(g,z)}(u) \leq A^+_{1(g,z)}(u)$ or $A^-_{1(g,z)}(u) \leq A^-_{2(g,z)}(u) = (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) < A^+_{2(g,z)}(u) \leq A^+_{1(g,z)}(u)$. For the case $A^-_{1(g,z)}(u) \leq A^-_{2(g,z)}(u) < (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) < A^+_{2(g,z)}(u) \leq A^+_{1(g,z)}(u)$ which contradict the fact that (β_1, E_1, X_1) and (β_2, E_2, X_2) are $ECSESs$. For the case $A^-_{1(g,z)}(u) \leq A^-_{2(g,z)}(u) = (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) \leq A^+_{2(g,z)}(u) \leq A^+_{1(g,z)}(u)$ we have $(\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) \notin (\sup\{A_{1(g,z)},\ A_{2(g,z)}\})^-(u),\ (\sup\{A_{1(g,z)},\ A_{2(g,z)}\})^+(u)$ because

$(\sup\{A_{1(g,z)}, A_{2(g,z)}\})^{-}(u) = A^{-}_{2(g,z)}(u) = (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u)$. if $(g,z) \in (E_1 \times X_1) \setminus (E_2 \times X_2)$ or $(g,z) \in (E_2 \times X_2) \setminus (E_1 \times X_1)$. Then the result holds trivially. Hence $(\beta_1, E_1, X_1) \cup_R (\beta_2, E_2, X_2)$ is an $ECSES$ over U. ■

The following example shows that $R-intersection$ of two $ECSESs$ need not be an $ECSES$.

Example 4.4.20 *Let (β_1, E_1, X_1) and (β_2, E_2, X_2) be two ECSESs over U, where $\beta_1(e,x) = \{(u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U\}$ for any $(e,x) \in E_1 \times X_1$ and $\beta_2(f,y) = \{(u, A_{2(f,y)}(u), \lambda_{2(f,y)}(u)) : u \in U\}$ for any $(f,y) \in E_2 \times X_2$ in which $A_{1(g,z)}(u) = [0.5, 0.6]$, $\lambda_{1(g,z)}(u) = 0.4$ and $A_{2(g,z)}(u) = [0.3, 0.7]$, $\lambda_{2(g,z)}(u) = 0.3$. Now by Definition 4.3.17, we have $A_{3(g,z)}(u) = [0.3, 0.6]$, $\lambda_{3(g,z)}(u) = 0.4$. Hence $R-intersection$ is not an ECSES because $\lambda_{3(g,z)}(u) \notin (A^{-}_{3(g,z)}(u), A^{+}_{3(g,z)}(u)))$.*

In the next theorem, we derive a condition for $R-intersection$ of two $ECSESs$ to be an $ECSES$.

Theorem 4.4.21 *Let (β_1, E_1, X_1) and (β_2, E_2, X_2) be two ECSESs over U, where $\beta_1(e,x) = \{(u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U\}$ for any $(e,x) \in E_1 \times X_1$ and $\beta_2(f,y) = \{(u, A_{2(f,y)}(u), \lambda_{2(f,y)}(u)) : u \in U\}$ for any $(f,y) \in E_2 \times X_2$ such that*

$$(\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) \in \left\{ \begin{array}{l} \inf\{\sup\{A^{+}_{1(g,z)}(u), A^{-}_{2(g,z)}(u)\},\ \sup\{A^{-}_{1(g,z)}(u), A^{+}_{2(g,z)}(u)\}\}, \\ \sup\{\inf\{A^{+}_{1(g,z)}(u), A^{-}_{2(g,z)}(u)\},\ \inf\{A^{-}_{1(g,z)}(u), A^{+}_{2(g,z)}(u)\}\} \end{array} \right.$$

for all $(g,z) \in (E_1 \cap E_2 \times X_1 \cap X_2)$ and $u \in U$. Then $(\beta_1, E_1, X_1) \cap_R (\beta_2, E_2, X_2)$ is also an ECSES over U.

Proof. By Definition 4.3.17, we have $(\beta_3, E_3, X_3) = (\beta_1, E_1, X_1) \cap_R (\beta_2, E_2, X_2)$ where

$$\beta_3(g,z) = \left\{ \begin{array}{ll} \{(u, \sup\{A_{1(g,z)}(u), A_{2(g,z)}(u)\} & \text{if } (g,z) \in (E_1 \cap E_2 \times X_1 \cap X_2) \\ , \inf\{\lambda_{1(g,z)}(u), \lambda_{2(g,z)}(u)\})\}, & \end{array} \right.$$

if $(g,z) \in (E_1 \cap E_2 \times X_1 \cap X_2)$, take $\hbar = \inf\{\sup\{A^{+}_{1(g,z)}(u), A^{-}_{2(g,z)}(u)\}, \sup\{A^{-}_{1(g,z)}(u), A^{+}_{2(g,z)}(u)\}\}$ and $\Re = \sup\ \{\inf\ \{A^{+}_{1(g,z)}(u), A^{-}_{2(g,z)}(u)\}, \inf\ \{A^{-}_{1(g,z)}(u), A^{+}_{2(g,z)}(u)\}\}$. Then $\hbar$ is one of $A^{+}_{1(g,z)}(u)$, $A^{-}_{2(g,z)}(u)$, $A^{-}_{1(g,z)}(u)$, $A^{+}_{2(g,z)}(u)$. We only consider $\hbar = A^{-}_{2(g,z)}(u)$ or $A^{+}_{2(g,z)}(u)$ because remaining cases are similar to this one. If $\hbar = A^{-}_{2(g,z)}(u)$ then $A^{-}_{1(g,z)}(u) \leq A^{+}_{1(g,z)}(u) \leq A^{-}_{2(g,z)}(u) \leq A^{+}_{2(g,z)}(u)$ and so $\Re = A^{+}_{1(g,z)}(u)$. Thus $(\inf\{A_{1(g,z)}, A_{2(g,z)}\})^{+}(u) = A^{+}_{1(g,z)}(u) = \Re < (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u)$. Hence $(\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) \notin ((\inf\{A_{1(g,z)}, A_{2(g,z)}\}^{-}(u), \inf\{A_{1(g,z)}, A_{2(g,z)}\}^{+}(u))$. If $\hbar = A^{+}_{2(g,z)}(u)$ then $A^{-}_{1(g,z)}(u) \leq A^{+}_{2(g,z)}(u) \leq A^{+}_{1(g,z)}(u)$ so $\Re = \sup\{A^{-}_{1(g,z)}(u), A^{-}_{2(g,z)}(u)\}$. Assume $\Re = A^{-}_{1(g,z)}(u)$, then we have $A^{-}_{2(g,yz)}(u) \leq A^{-}_{1(g,z)}(u) < (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) <$

$A^{+}_{2(g,z)}(u) \leq A^{+}_{1(g,z)}(u)$. So we can write $A^{-}_{2(g,z)}(u) \leq A^{-}_{1(g,z)}(u) < (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) < A^{+}_{2(g,z)}(u) \leq A^{+}_{1(g,z)}(u)$ or $A^{-}_{2(g,z)}(u) \leq A^{-}_{1(g,z)}(u) < (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) = A^{+}_{2(g,z)}(u) \leq A^{+}_{1(g,z)}(u)$.

The case $A^{-}_{2(g,z)}(u) \leq A^{-}_{1(g,z)}(u) < (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) < A^{+}_{2(g,z)}(u) \leq A^{+}_{1(g,z)}(u)$ contradicts the fact that (β_1, E_1, X_1) and (β_2, E_2, X_2) are $ECSESs$. For the case $A^{-}_{2(g,z)}(u) \leq A^{-}_{1(g,z)}(u) < (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) = A^{+}_{2(g,z)}(u) \leq A^{+}_{1(g,z)}(u)$ we have $(\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) \notin ((\inf\{A_{1(g,z)}, A_{2(g,z)}\})^{-}(u), (\inf\{A_{1(g,z)}, A_{2(g,z)}\})^{+}(u))$ because $(\inf\{A_{1(g,z)}, A_{2(g,z)}\})^{+}(u) = A^{+}_{2(g,z)}(u) = (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u)$. Again assume that $\Re = A^{-}_{2(g,z)}(u)$, then we have $A^{-}_{1(g,z)}(u) \leq A^{-}_{2(g,z)}(u) < (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) \leq A^{+}_{2(g,z)}(u) \leq A^{+}_{1(g,z)}(u)$. We can write $A^{-}_{1(g,z)}(u) \leq A^{-}_{2(g,z)}(u) < (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) < A^{+}_{2(g,z)}(u) \leq A^{+}_{1(g,z)}(u)$ or $A^{-}_{1(g,z)}(u) \leq A^{-}_{2(g,z)}(u) < (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) = A^{+}_{2(g,z)}(u) \leq A^{+}_{1(g,z)}(u)$. The case $A^{-}_{1(g,z)}(u) \leq A^{-}_{2(g,z)}(u) < (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) < A^{+}_{2(g,z)}(u) \leq A^{+}_{1(g,z)}(u)$ contradicts the fact that (β_1, E_1, X_1) and (β_2, E_2, X_2) are $ECSESs$. For the case $A^{-}_{1(g,z)}(u) \leq A^{-}_{2(g,z)}(u) < (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) = A^{+}_{2(g,z)}(u) \leq A^{+}_{1(g,z)}(u)$, we have $(\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) \notin (\inf\{A_{1(g,z)}, A_{2(g,z)}\})^{-}(u), (\inf\{A_{1(g,z)}, A_{2(g,z)}\})^{+}(u)$ because $(\inf\{A_{1(g,z)}, A_{2(g,z)}\})^{+}(u) = A^{+}_{2(g,z)}(u) = (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u)$. Hence $(\beta_1, E_1, X_1) \cap_R (\beta_2, E_2, X_2)$ is an $ECSES$ over U. ■

The following example shows that the $P-union$ and $P-intersection$ of two $ECSESs$ need not to be an $ECSES$.

Example 4.4.22 *Let (β_1, E_1, X_1) and (β_2, E_2, X_2) be two $ECSESs$ over U, where $\beta_1(e,x) = \{(u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U\}$ for any $(e,x) \in E_1 \times X_1$ and $\beta_2(f,y) = \{(u, A_{2(f,y)}(u), \lambda_{2(f,y)}(u)) : u \in U\}$ for any $(f,y) \in E_2 \times X_2$ in which $A_{1(g,z)}(u) = [0.5, 0.8]$, $\lambda_{1(g,z)}(u) = 0.2$ and $A_{2(g,z)}(u) = [0.1, 0.4]$, $\lambda_{2(g,z)}(u) = 0.7$. Now by Definition 4.3.11, we have $A_{3(g,z)}(u) = [0.5, 0.8]$, $\lambda_{3(g,z)}(u) = 0.7$ and by Definition 4.3.13 we have $A_{3(g,z)}(u) = [0.1, 0.4]$, $\lambda_{3(e,x)}(u) = 0.2$. Hence $P-union$ and $P-intersection$ both are not $ECSESs$ because $A^{-}_{3(g,z)}(u) \leq \lambda_{3(g,z)}(u) \leq A^{+}_{3(g,z)}(u)$.*

In the next theorems, we derive conditions for $P-union$ of two $ECSESs$ to be an $ECSES$.

Theorem 4.4.23 *Let (β_1, E_1, X_1) and (β_2, E_2, X_2) be two $ECSESs$ over U, where $\beta_1(e,x) = \{(u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U\}$ for any $(e,x) \in E_1 \times X_1$ and $\beta_2(f,y) = \{(u, A_{2(f,y)}(u), \lambda_{2(f,y)}(u)) : u \in U\}$ for any $(f,y) \in E_2 \times X_2$ such that*

$$(\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) \in \left\{ \begin{array}{c} \inf\{\sup\{A^{+}_{1(g,z)}(u), A^{-}_{2(g,z)}(u)\}, \sup\{A^{-}_{1(g,z)}(u), A^{+}_{2(g,z)}(u)\}\}, \\ \{\inf\{A^{+}_{1(g,z)}(u), A^{-}_{2(g,z)}(u)\}, \inf\{A^{-}_{1(g,z)}(u), A^{+}_{2(g,z)}(u)\}\} \end{array} \right.$$

for all $(g,z) \in (E_1 \cap E_2 \times X_1 \cap X_2)$ and $u \in U$. Then $(\beta_1, E_1, X_1) \cup_P (\beta_2, E_2, X_2)$ is an $ECSES$ over U.

Proof. By Definition 4.3.11, we have $(\beta_3, E_3, X_3) = (\beta_1, E_1, X_1) \cup_P (\beta_2, E_2, X_2)$ where

$$\beta_3(g,z) = \begin{cases} \{(u, A_{1(g,z)}(u), \lambda_{1(g,z)}(u))\} & \text{if } (g,z) \in (E_1 \times X_1) \setminus (E_2 \times X_2) \\ \{(u, A_{2(g,z)}(u), \lambda_{2(g,z)}(u) >)\} & \text{if } (g,z) \in (E_2 \times X_2) \setminus (E_1 \times X_1) \\ \{(u, \sup\{A_{1(g,z)}(u), A_{2(g,z)}(u)\} & \text{if } (g,z) \in (E_1 \cap E_2 \times X_1 \cap X_2) \\ , \sup\{\lambda_{1(g,z)}(u), \lambda_{2(g,z)}(u)\})\}, & \end{cases}$$

if $(g,z) \in (E_1 \cap E_2 \times X_1 \cap X_2)$, take $\hbar = \inf\{\sup\{A^+_{1(g,z)}(u), A^-_{2(g,z)}(u)\}, \sup\{A^-_{1(g,z)}(u), A^+_{2(g,z)}(u)\}\}$ and $\Re = \sup\{\inf\{A^+_{1(g,z)}(u), A^-_{2(g,z)}(u)\}, \inf\{A^-_{1(g,z)}(u), A^+_{2(g,z)}(u)\}\}$. Then $\hbar$ is one of $A^+_{1(g,z)}(u), A^-_{2(g,z)}(u), A^-_{1(g,z)}(u), A^+_{2(g,z)}(u)$. We only consider $\hbar = A^-_{2(g,z)}(u)$ or $A^+_{2(g,z)}(u)$ because the remaining cases are similar to this one. If $\hbar = A^-_{1(g,z)}(u)$ then $A^-_{2(g,z)}(u) \leq A^+_{2(g,z)}(u) \leq A^-_{1(g,z)}(u) \leq A^+_{1(g,z)}(u)$ and so $\Re = A^+_{2(g,z)}(u)$. Thus $(\sup\{A_{1(g,z)}, A_{2(g,z)}\})^-(u) = A^-_{1(g,z)}(u) = \hbar > (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u)$. Hence $(\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) \notin (\sup\{A_{1(g,z)}, A_{2(g,z)}\})^-(u), (\sup\{A_{1(g,z)}, A_{2(g,z)}\})^+(u)$.

If $\hbar = A^+_{1(g,z)}(u)$ then $A^-_{2(g,z)}(u) \leq A^+_{1(g,z)}(u) \leq A^+_{2(g,z)}(u)$ so $\Re = \sup\{A^-_{1(g,z)}(u), A^-_{2(g,z)}(u)\}$. Assume $\Re = A^-_{1(g,z)}(u)$, then we have $A^-_{2(g,yz)}(u) \leq A^-_{1(g,z)}(u) \leq (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) < A^+_{1(g,z)}(u) \leq A^+_{2(g,z)}(u)$. So we can write $A^-_{2(g,z)}(u) \leq A^-_{1(g,z)}(u) < (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) < A^+_{1(g,z)}(u) \leq A^+_{2(g,z)}(u)$ or $A^-_{2(g,z)}(u) \leq A^-_{1(g,z)}(u) = (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) \leq A^+_{1(g,z)}(u) \leq A^+_{2(g,z)}(u)$.

For the case $A^-_{2(g,z)}(u) \leq A^-_{1(g,z)}(u) < (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) < A^+_{1(g,z)}(u) \leq A^+_{2(g,z)}(u)$ which contradict the fact that (β_1, E_1, X_1) and (β_2, E_2, X_2) are $ECSESs$. For the case $A^-_{2(g,z)}(u) < A^-_{1(g,z)}(u) = (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) \leq A^+_{1(g,z)}(u) \leq A^+_{2(g,z)}(u)$ we have $(\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) \notin ((\sup\{A_{1(g,z)}, A_{2(g,z)})^-(u), (\sup\{A_{1(g,z)}, A_{2(g,z)}\})^+(u))$ because $(\sup\{A_{1(g,z)}, A_{2(g,z)}\})^-(u) = A^-_{1(g,z)}(u) = (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u)$. Again assume that $\Re = A^-_{2(g,z)}(u)$, then we have $A^-_{1(g,z)}(u) \leq A^-_{2(g,z)}(u) \leq (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) \leq A^+_{1(g,z)}(u) \leq A^+_{2(g,z)}(u)$. We can write $A^-_{1(g,z)}(u) \leq A^-_{2(g,z)}(u) < (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) < A^+_{1(g,z)}(u) \leq A^+_{2(g,z)}(u)$ or $A^-_{1(g,z)}(u) \leq A^-_{2(g,z)}(u) = (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) < A^+_{1(g,z)}(u) \leq A^+_{2(g,z)}(u)$. For the case $A^-_{1(g,z)}(u) \leq A^-_{2(g,z)}(u) < (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) < A^+_{1(g,z)}(u) \leq A^+_{2(g,z)}(u)$ which contradict the fact that (β_1, E_1, X_1) and (β_2, E_2, X_2) are $ECSESs$. For the case $A^-_{1(g,z)}(u) \leq A^-_{2(g,z)}(u) = (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) \leq A^+_{1(g,z)}(u) \leq A^+_{2(g,z)}(u)$ we have $(\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) \notin ((\sup\{A_{1(g,z)}, A_{2(g,z)}\})^-(u), (\sup\{A_{1(g,z)}, A_{2(g,z)}\})^+(u))$ because $(\sup\{A_{1(g,z)}, A_{2(g,z)}\})^-(u) = A^-_{2(g,z)}(u) = (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u)$. if $(g,z) \in (E_1 \times X_1) \setminus (E_2 \times X_2)$ or $(g,z) \in (E_2 \times X_2) \setminus (E_1 \times X_1)$ then the result holds trivially. Hence $(\beta_1, E_1, X_1) \cup_P (\beta_2, E_2, X_2)$ is an $ECSES$ over U. ■

Theorem 4.4.24 *Let* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *be two* $CSESs$ *over* U, *where* $\beta_1(e,x) = \{(u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U\}$ *for any* $(e,x) \in E_1 \times X_1$ *and* $\beta_2(f,y) =$

$\{(u, A_{2(f,y)}(u), \lambda_{2(f,y)}(u)) : u \in U\}$ *for any* $(f, y) \in E_2 \times X_2$ *such that*

$$(\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) = \begin{cases} \inf\{\sup\{A^+_{1(g,z)}(u), A^-_{2(g,z)}(u)\}, \sup\{A^-_{1(g,z)}(u), A^+_{2(g,z)}(u)\}\}, \\ \{\inf\{A^+_{1(g,z)}(u), A^-_{2(g,z)}(u)\}, \inf\{A^-_{1(g,z)}(u), A^+_{2(g,z)}(u)\}\}, \end{cases}$$

for all $(g, z) \in (E_1 \cap E_2 \times X_1 \cap X_2)$ *and* $u \in U$. *Then* $(\beta_1, E_1, X_1) \cap_P (\beta_2, E_2, X_2))$ *is both an ECSES and ICSES over U.*

Proof. Consider $(\beta_1, E_1, X_1) \cap_P (\beta_2, E_2, X_2) = (\beta_3, E_3, X_3)$ where $(E_3 \times X_3) = (E_1 \cap E_2 \times X_1 \cap X_2))$. Also $\beta_3(g, z) = \{(u, \inf\{A_{1(e,x)}(u), A_{2(f,y)}(u)\}, (\lambda_{1(e,x)} \vee \lambda_{2(f,y)})(u)) : u \in U\}$ for any $(g, z) \in (E_1 \cap E_2 \times X_1 \cap X_2)$. If $(g, z) \in (E_1 \cap E_2 \times X_1 \cap X_2)$, take $\hbar = \inf\{\sup\{A^+_{1(g,z)}(u), A^-_{2(g,z)}(u)\}, \sup\{A^-_{1(g,z)}(u), A^+_{2(g,z)}(u)\}\}$ and

$\Re = \sup\{\inf\{A^+_{1(g,z)}(u) , A^-_{2(g,z)}(u)\}, \inf\{A^-_{1(g,z)}(u), A^+_{2(g,z)}(u)\}\}$. Then $\hbar$ is one of $A^+_{1(g,z)}(u)$, $A^-_{2(g,z)}(u)$, $A^-_{1(g,z)}(u)$, $A^+_{2(g,z)}(u)$. We only consider $\hbar = A^-_{1(g,z)}(u)$ or $A^+_{1(g,z)}(u)$ because remaining cases are similar to this one. If $\hbar = A^-_{1(g,z)}(u)$ then $A^-_{2(g,z)}(u) \leq A^+_{2(g,z)}(u) \leq A^-_{1(g,z)}(u) \leq A^+_{1(g,z)}(u)$ and so $\Re = A^+_{2(g,z)}(u)$. This implies that $A^-_{1(g,z)}(u) = \hbar = (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) = \Re = A^+_{2(g,z)}(u)$. Thus $A^-_{2(g,z)}(u) \leq A^+_{2(g,z)}(u) = (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) = A^-_{1(g,z)}(u) \leq A^+_{1(g,z)}(u)$, which implies that $(\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) = A^+_{2(g,z)}(u) = (\inf\{A_{1(g,z)}, A_{(g,z)}\})^+(u)$. Hence $(\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) \notin ((\inf\{A_{1(g,z)}, A_{2(g,z)}\})^-(u), (\inf\{A_{1(g,z)}, A_{2(g,z)}\})^+(u))$ and $(\inf\{A_{1(g,z)}, A_{2(g,z)}\})^-(u) \leq (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) \leq (\inf\{A_{1(g,z)}, A_{2(g,z)}\})^+(u)$. If $\hbar = A^+_{1(g,z)}(u)$ then $A^-_{2(g,z)}(u) \leq A^+_{1(g,z)}(u) \leq A^+_{2(g,z)}(u)$ and $(\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) = A^+_{1(g,z)}(u) = (\inf\{A_{1(g,z)}, A_{2(g,z)}\})^+(u)$. Hence $(\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) \notin ((\inf\{A_{1(g,z)}, A_{2(g,z)}\})^-(u), (\inf\{A_{1(g,z)}, A_{2(g,z)}\})^+(u))$ and $(\inf\{A_{1(g,z)}, A_{2(g,z)}\})^-(u) \leq (\lambda_{1(g,z)} \wedge \lambda_{2(g,z)})(u) \leq (\inf\{A_{1(g,z)}, A_{2(g,z)}\})^+(u)$. Hence $(\beta_1, E_1, X_1) \cap_R (\beta_2, E_2, X_2)$ is both an *ECSE* set and *ICSES* over *U*. ■

Theorem 4.4.25 *Let* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *be two CSESs over U, where* $\beta_1(e, x) = \{(u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U\}$ *for any* $(e, x) \in E_1 \times X_1$ *and* $\beta_2(f, y) = \{(u, A_{2(f,y)}(u), \lambda_{2(f,y)}(u)) : u \in U\}$ *for any* $(f, y) \in E_2 \times X_2$ *such that*

$$(\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) = \begin{cases} \inf\{\sup\{A^+_{1(g,z)}(u), A^-_{2(g,z)}(u)\}, \sup\{A^-_{1(g,z)}(u), A^+_{2(g,z)}(u)\}\}, \\ \sup\{\inf\{A^+_{1(g,z)}(u), A^-_{2(g,z)}(u)\}, \inf\{A^-_{1(g,z)}(u), A^+_{2(g,z)}(u)\}\} \end{cases}$$

for all $(g, z) \in (E_1 \cap E_2 \times X_1 \cap X_2)$ *and* $u \in U$. *Then* $(\beta_1, E_1, X_1) \cap_R (\beta_2, E_2, X_2)$ *is both an ECSE set and ICSES over U.*

Proof. Consider $(\beta_1, (E_1 \times X_1)) \cap_R (\beta_2, (E_2 \times X_2)) = (\beta_3, (E_3 \times X_3))$ where $(E_3 \times X_3) = (E_1 \cap E_2 \times X_1 \cap X_2)$. Also $\beta_3(g, z) = \{(u, \inf\{A_{1(e,x)}(u), A_{2(f,y)}(u)\}, (\lambda_{1(e,x)} \vee \lambda_{2(f,y)})(u)) : u \in U\}$ for any $(g, z) \in (E_1 \cap E_2, X_1 \cap X_2)$. if $(g, z) \in (E_1 \cap E_2, X_1 \cap X_2)$, take $\hbar = \inf\{\sup\{A^+_{1(g,z)}(u), A^-_{2(g,z)}(u)\}, \sup\{A^-_{1(g,z)}(u), A^+_{2(g,z)}(u)\}\}$ and $\Re =$

$\sup\{\inf\{A^+_{1(g,z)}(u), A^-_{2(g,z)}(u)\}, \inf\{A^-_{1(g,z)}(u), A^+_{2(g,z)}(u)\}\}$. Then $\hbar$ is one of $A^+_{1(g,z)}(u)$, $A^-_{2(g,z)}(u)$, $A^-_{1(g,z)}(u)$, $A^+_{2(g,z)}(u)$. We only consider $\hbar = A^-_{1(g,z)}(u)$ or $A^+_{1(g,z)}(u)$ because the remaining cases are similar to this one. If $\hbar = A^-_{1(g,z)}(u)$ then $A^-_{2(g,z)}(u) \leq A^+_{2(g,z)}(u) \leq A^-_{1(g,z)}(u) \leq A^+_{1(g,z)}(u)$ and so $\Re = A^+_{2(g,z)}(u)$. This implies that $A^-_{1(g,z)}(u) = \hbar = (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) = \Re = A^+_{2(g,z)}(u)$. Thus $A^-_{2(g,z)}(u) \leq A^+_{2(g,z)}(u) = (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) = A^-_{1(g,z)}(u) \leq A^+_{1(g,z)}(u)$, which implies that $(\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) = A^+_{2(g,z)}(u) = (\inf\{A_{1(g,z)}, A_{(g,z)}\})^+(u)$. Hence $(\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) \notin ((\inf\{A_{1(g,z)}, A_{2(g,z)}\})^-(u), (\inf\{A_{1(g,z)}, A_{2(g,z)}\})^+(u))$ and $(\inf\{A_{1(g,z)}, A_{2(g,z)}\})^-(u) \leq (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) \leq (\inf\{A_{1(g,z)}, A_{2(g,z)}\})^+(u)$. If $\hbar = A^+_{1(g,z)}(u)$ then $A^-_{2(g,z)}(u) \leq A^+_{1(g,z)}(u) \leq A^+_{2(g,z)}(u)$ and $(\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) = A^+_{1(g,z)}(u) = (\inf\{A_{1(g,z)}, A_{2(g,z)}\})^+(u)$. Hence $(\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) \notin ((\inf\{A_{1(g,z)}, A_{2(g,z)}\})^-(u), (\inf\{A_{1(g,z)}, A_{2(g,z)}\})^+(u))$ and $(\inf\{A_{1(g,z)}, A_{2(g,z)}\})^-(u) \leq (\lambda_{1(g,z)} \vee \lambda_{2(g,z)})(u) \leq (\inf\{A_{1(g,z)}, A_{2(g,z)}\})^+(u)$. Hence $(\beta_1, E_1, X_1) \cap_R (\beta_2, E_2, X_2)$ is both an $ECSES$ and $ICSES$ over U. ■

Theorem 4.4.26 *For any two cubic soft expert sets (β_1, E_1, X_1) and (β_2, E_2, X_2), the following absorption laws hold*

1) $(\beta_1, E_1, X_1) \cup_P ((\beta_1, E_1, X_1) \cap_P (\beta_2, E_2, X_2)) = (\beta_1, E_1, X_1)$,
2) $(\beta_1, E_1, X_1) \cap_P ((\beta_1, E_1, X_1) \cup_P (\beta_2, E_2, X_2)) = (\beta_1, E_1, X_1)$,
3) $(\beta_1, E_1, X_1) \cup_R ((\beta_1, E_1, X_1) \cap_R (\beta_2, E_2, X_2)) = (\beta_1, E_1, X_1)$,
4) $(\beta_1, E_1, X_1) \cap_R ((\beta_1, E_1, X_1) \cup_R (\beta_2, E_2, X_2)) = (\beta_1, E_1, X_1)$.

Proof. 1) By Definitions 4.3.11 and 4.3.13 we have $(\beta_1, E_1, X_1) \cup_P ((\beta_1, E_1, X_1) \cap_P (\beta_2, E_2, X_2)) = (\beta_3, E_1 \cup_P (E_1 \cap_P E_2), X_1 \cup_P (X_1 \cap_P X_2)) = (\beta_3, E_1, X_1)$
such that for any $g \in E_1$ and $z \in X_1$, we have

$$\beta_3(g, z) = \beta_1(g, z) \cup_P ((\beta_1(g, z) \cap_P \beta_2(g, z)) \; if (g, z) \in E_1 \times X_1.$$

$\beta_1(g, z) \cup_P ((\beta_1(g, z) \cap_P \beta_2(g, z)) = \{(u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U, (e, x) \in E_1 \times X_1\} \cup_P \{\{(u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U, (e, x) \in E_1 \times X_1\} \cap_P \{(u, A_{2(f,y)}(u), \lambda_{2(f,y)}(u)) : u \in U, (f, y) \in E_2 \times X_2\}\} = \{(u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U, (e, x) \in E_1 \times X_1\} \cup_P \{(u, \inf\{A_{1(e,x)}(u), A_{2(f,y)}(u)\}, \inf\{\lambda_{1(e,x)}(u), \lambda_{2(f,y)}(u)\})\} = \{(u, \sup\{A_{1(e,x)}(u), \inf\{A_{1(e,x)}(u), A_{2(f,y)}(u)\}\}, \sup\{\lambda_{1(e,x)}(u), \inf\{\lambda_{1(e,x)}(u), \lambda_{2(f,y)}(u)\}\})\} \subseteq \{(u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U, (e, x) \in E_1 \times X_1\} = \beta_1(e, x) \subseteq \{(u, \inf\{A_{1(e,x)}(u), r\sup\{A_{1(e,x)}(u), A_{2(f,y)}(u)\}\}, \inf\{\lambda_{1(e,x)}(u), \sup\{\lambda_{1(e,x)}(u), \lambda_{2(f,y)}(u)\}\}) = \{(u, \sup\{A_{(e,x)}(u), \inf\{A_{(e,x)}(u), A_{(f,y)}(u)\}\}, \sup\{\lambda_{(e,x)}(u), \inf\{\lambda_{(e,x)}(u), \lambda_{(f,y)}(u)\}\})\} = \beta_1(e, x) \cup_P ((\beta_1(e, x) \cap_P \beta_2(f, y))$.

In the second case when $(g, z) \in (E_1 \times X_1) \setminus (E_2 \times X_2)$, using Definitions 4.3.11 and 4.3.13, we have $\beta_1(e, x) \cup_P ((\beta_1(e, x) \cap_P \beta_2(f, y)) = \beta_1(e, x) \cup_P \beta_1(e, x) = \beta_1(e, x)$ which is the required result for both the cases. Similarly, we can prove 2), 3) and 4).
■

Definition 4.4.27 *For two CSESs* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *over* U, $P-AND$ *is denoted and defined as*

$(\beta_1, E_1, X_1)\bigwedge_P(\beta_2, E_2, X_2) = (\beta_3, (E_1 \times E_2), (X_1 \times X_2)),$

where $\beta_3((e,f),(x,y)) = \beta_1(e,x) \cap_P \beta_2(f,y)$ *for all* $((e,f),(x,y)) \in ((E_1 \times E_2)\times (X_1 \times X_2)),$

whenever $\beta_1(e,x) = \{ (u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U\}$ *and* $\beta_2(f,y) = \{(u, A_{2(f,y)}(u), \lambda_{2(f,y)}(u)) : u \in U\}.$

Example 4.4.28 *Let* $U = \{u_1, u_2, u_3\}$ *be the initial universe,* $E = \{e_1, e_2\}$ *be the set of attributes,* $X = \{x_1, x_2\}$ *be the set of experts. Then the cubic set* (β_1, E, X) *over* U *is given below:*

$\beta_1(e_1, x_1) = \{(u_1, [0.5, 0.8], 0.7), (u_2, [0.6, 0.7], 0.8), (u_3, [0.4bb, 0.8], 0.5)\},$

$\beta_1(e_2, x_1) = \{(u_1, [0.2, 0.7], 0.6), (u_2, [0.7, 0.8], 0.5), (u_3, [0.2, 0.5], 0.4)\},$

$\beta_1(e_1, x_2) = \{(u_1, [0.4, 0.8], 0.5), (u_2, [0.4, 0.9], 0.8), (u_3, [0.4, 0.7], 0.5)\},$

$\beta_1(e_2, x_2) = \{(u_1, [0.3, 0.8], 0.4), (u_2, [0.2, 0.9], 0.7), (u_3, [0.3, 0.7], 0.6)\}.$

Let $U = \{u_1, u_2, u_3\}$ *be the initial universe,* $F = \{f_1, f_2\}$ *be the set of attributes and* $Y = \{y_1, y_2\}$ *be the set of experts. Then the cubic set* (β_2, F, Y) *over* U *is given below:*

$\beta_2(f_1, y_1) = \{(u_1, [0.5, 0.8], 0.4), (u_2, [0.6, 0.9], 0.9), (u_3, [0.4, 0.7], 0.8)\},$

$\beta_2(f_2, y_1) = \{(u_1, [0.4, 0.7], 0.3), (u_2, [0.7, 0.9], 0.8), (u_3, [0.3, 0.5], 0.6)\},$

$\beta_2(f_1, y_2) = \{(u_1, [0.5, 0.8], 0.9), (u_2, [0.7, 0.9], 0.6), (u_3, [0.5, 0.6], 0.7)\},$

$\beta_2(f_2, y_2) = \{(u_1, [0.3, 0.8], 0.2), (u_2, [0.6, 0.9], 0.4), (u_3, [0.2, 0.7], 0.8)\}.$

By using Definition 4.4.27 we have $(\beta_1, E, X)\bigwedge_P(\beta_2, F, Y) = (\beta_3, (E \times F), (X \times Y)),$

where $\beta_3((e,f),(x,y)) = \beta_1(e,x) \cap_P \beta_2(f,y)$ *for all* $((e,f),(x,y)) \in ((E_1 \times E_2)\times (X_1 \times X_2)).$

$\beta_3((e_1, f_1), (x_1, y_1)) = \{(u_1, [0.5, 0.8], 0.4), (u_2, [0.6, 0.7], 0.8), (u_3, [0.4, 0.7], 0.5)\},$

$\beta_3((e_2, f_2), (x_1, y_1)) = \{(u_1, [0.2, 0.7], 0.3), (u_2, [0.7, 0.8], 0.5), (u_3, [0.2, 0.5], 0.4)\},$

$\beta_3((e_1, f_1), (x_2, y_2)) = \{(u_1, [0.4, 0.8], 0.5), (u_2, [0.4, 0.9], 0.6), (u_3, [0.4, 0.6], 0.5)\},$

$\beta_3((e_2, f_2), (x_2, y_2)) = \{(u_1, [0.3, 0.8], 0.2), (u_2, [0.2, 0.9], 0.4), (u_3, [0.2, 0.7], 0.6)\}.$

Definition 4.4.29 *For two CSESs* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *over* U, $R-AND$ *is denoted and defined as*

$(\beta_1, E_1, X_1)\bigwedge_R(\beta_2, E_2, X_2) = (\beta_3, (E_1 \times E_2), (X_1 \times X_2)),$

where $\beta_3((e,f),(x,y)) = \beta_1(e,x) \cap_R \beta_2(f,y)$ *for all* $((e,f),(x,y)) \in ((E_1 \times E_2) \times (X_1 \times X_2)),$

whenever $\beta_1(e,x) = \{(u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U\}$ *and* $\beta_2(f,y) = \{(u, A_{2(f,y)}(u), \lambda_{2(f,y)}(u)) : u \in U\}.$

Example 4.4.30 *Consider Example 4.4.28, by using Definition 4.4.29, we have* $(\beta_1, E, X) \bigwedge_R (\beta_2, F, Y) = (\beta_3, (E \times F), (X \times Y))$,

where $\beta_3((e,f),(x,y)) = \beta_1(e,x) \cap_R \beta_2(f,y)$ *for all* $((e,f),(x,y)) \in ((E_1 \times E_2) \times (X_1 \times X_2))$.

$\beta_3((e_1,f_1),(x_1,y_1)) = \{(u_1,[0.5,0.8],0.7),(u_2,[0.6,0.7],0.9),(u_3,[0.4,0.7],0.8)\}$,

$\beta_3((e_2,f_2),(x_1,y_1)) = \{(u_1,[0.2,0.7],0.6),(u_2,[0.7,0.8],0.8),(u_3,[0.2,0.5],0.6)\}$,

$\beta_3((e_1,f_1),(x_2,y_2)) = \{(u_1,[0.4,0.8],0.9),(u_2,[0.4,0.9],0.8),(u_3,[0.4,0.6],0.7)\}$,

$\beta_3((e_2,f_2),(x_2,y_2)) = \{(u_1,[0.3,0.8],0.4),(u_2,[0.2,0.9],0.7),(u_3,[0.2,0.7],0.8)\}$.

Definition 4.4.31 *For two CSESs* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *over* U, $P-OR$ *is denoted and defined as*

$(\beta_1, E_1, X_1) \bigvee_P (\beta_2, E_2, X_2) = (\beta_3, (E_1 \times E_2), (X_1 \times X_2))$,

where $\beta_3((e,f),(x,y)) = \beta_1(e,x) \cup_P \beta_2(f,y)$ *for all* $((e,f),(x,y)) \in ((E_1 \times E_2) \times (X_1 \times X_2))$,

whenever $\beta_1(e,x) = \{(u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U\}$ *and* $\beta_2(f,y) = \{(u, A_{2(f,y)}(u), \lambda_{2(f,y)}(u)) : u \in U\}$.

Example 4.4.32 *Consider Example 4.4.28, by using Definition 4.4.31, we have* $(\beta_1, E, X) \bigvee_P (\beta_2, F, Y) = (\beta_3, (E \times F), (X \times Y))$,

where $\beta_3((e,f),(x,y)) = \beta_1(e,x) \cup_P \beta_2(f,y)$ *for all* $((e,f),(x,y)) \in ((E_1 \times E_2) \times (X_1 \times X_2))$.

$\beta_3((e_1,f_1),(x_1,y_1)) = \{(u_1,[0.5,0.8],0.7),(u_2,[0.6,0.9],0.9),(u_3,[0.4,0.8],0.8)\}$,

$\beta_3((e_2,f_2),(x_1,y_1)) = \{(u_1,[0.4,0.7],0.6),(u_2,[0.7,0.9],0.8),(u_3,[0.3,0.5],0.6)\}$,

$\beta_3((e_1,f_1),(x_2,y_2)) = \{(u_1,[0.5,0.8],0.9),(u_2,[0.7,0.9],0.8),(u_3,[0.5,0.7],0.7)\}$,

$\beta_3((e_2,f_2),(x_2,y_2)) = \{(u_1,[0.3,0.8],0.4),(u_2,[0.6,0.9],0.7),(u_3,[0.3,0.7],0.8)\}$.

Definition 4.4.33 *For two CSESs* (β_1, E_1, X_1) *and* (β_2, E_2, X_2) *over* U, $R-OR$ *is denoted and defined as*

$(\beta_1, E_1, X_1) \bigvee_R (\beta_2, E_2, X_2) = (\beta_3, (E_1 \times E_2), (X_1 \times X_2))$

where $\beta_3((e,f),(x,y)) = \beta_1(e,x) \cup_R \beta_2(f,y)$ *for all* $((e,f),(x,y)) \in ((E_1 \times E_2) \times (X_1 \times X_2))$,

whenever $\beta_1(e,x) = \{(u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U\}$ *and* $\beta_2(f,y) = \{(u, A_{2(f,y)}(u), \lambda_{2(f,y)}(u)) : u \in U\}$.

Example 4.4.34 *Consider Example 4.4.28 by using Definition 4.4.33 we have* $(\beta_1, E, X) \bigvee_R (\beta_2, F, Y) = (\beta_3, (E \times F), (X \times Y))$,

where $\beta_3((e,f),(x,y)) = \beta_1(e,x) \cup_R \beta_2(f,y)$ *for all* $((e,f),(x,y)) \in ((E_1 \times E_2) \times (X_1 \times X_2))$.

$\beta_3((e_1, f_1), (x_1, y_1)) = \{(u_1, [0.5, 0.8], 0.4), (u_2, [0.6, 0.9], 0.8), (u_3, [0.4, 0.8], 0.5)\},$
$\beta_3((e_2, f_2), (x_1, y_1)) = \{(u_1, [0.4, 0.7], 0.3), (u_2, [0.7, 0.9], 0.5), (u_3, [0.3, 0.5], 0.4)\},$
$\beta_3((e_1, f_1), (x_2, y_2)) = \{(u_1, [0.5, 0.8], 0.5), (u_2, [0.7, 0.9], 0.6), (u_3, [0.5, 0.7], 0.5)\},$
$\beta_3((e_2, f_2), (x_2, y_2)) = \{(u_1, [0.3, 0.8], 0.2), (u_2, [0.6, 0.9], 0.4), (u_3, [0.3, 0.7], 0.6)\}.$

Theorem 4.4.35 *Let (β_1, E_1, X_1) be a CSES over U. If (β_1, E_1, X_1) is an ICSES (ECSES) then $(\beta_1, E_1, X_1)^c$ ICSES (ECSES) respectively.*

Proof. Since (β_1, E_1, X_1) is an $ICSES$ ($ECSES$) over U, so for any $(e, x) \in (E_1 \times X_1)$ we have $\beta_1(e, x) = \{(u, A_{1(e,x)}(u), \lambda_{1(e,x)}(u)) : u \in U\}$. By Definition 4.2.3, we have $A^-_{1(e,x)}(u) \leq \lambda_{1(e,x)}(u) \leq A^+_{1(e,x)}(u)$. This implies that $1 - A^+_{1(e,x)}(u) \leq 1 - \lambda_{1(e,x)}(u) \leq 1 - A^-_{1(e,x)}(u)$. Hence $(\beta_1, E_1, X_1)^c$ is an $ICSE$ set. Also by U 4.2.5, we have $(\lambda_{1(e,x)}(u) \notin (A^-_{1(e,x)}(u), A^+_{1(e,x)}(u)))$ for all $u \in U$. This gives $1 - \lambda_{1(e,x)}(u) \notin (1 - A^+_{1(e,x)}(u), 1 - A^-_{1(e,x)}(u))$. Hence $(\beta_1, E_1, X_1)^c$ is an $ECSES$. ■

Definition 4.4.36 *Let $A_{(e,\, x_i)}, \lambda_{(e,\, x_i)} \in CSES$ over U, $1 \leq i \leq n$. The cubic soft expert weighted average quotient operator (CSEWAQO) is denoted and defined as*

$$Þ_{w_i}(A_{(e,x_i)}, \lambda_{(e,x_i)}) = ([\frac{\prod_{i=1}^{n}(1 + A^-_{(e,x_i)}(u))^{w_i} - \prod_{i=1}^{n}(1 - A^-_{(e,x_i)}(u))^{w_i}}{\prod_{i=1}^{n}(1 + A^-_{(e,x_i)}(u))^{w_i} + \prod_{i=1}^{n}(1 - A^-_{(e,x_i)}(u))^{w_i}}, \frac{\prod_{i=1}^{n}(1 + A^+_{(e,x_i)}(u))^{w_i} - \prod_{i=1}^{n}(1 - A^+_{(e,x_i)}(u))^{w_i}}{\prod_{i=1}^{n}(1 + A^+_{(e,x_i)}(u))^{w_i} + \prod_{i=1}^{n}(1 - A^+_{(e,x_i)}(u))^{w_i}}], \prod_{i=1}^{n}\{\lambda_{(e,x_i)}(u)\}^{w_i})$$

where $\{w_i\}_{i \in \{1,2,\ldots,n\}}$ are the weights of experts' opinions, where $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$.

Definition 4.4.37 *Let $\beta = ([A^-_{(e,x)}, A^+_{(e,x)}], \lambda_{(e,x)})$ be a CSE value. A score function $\check{S}$ of CSES value is defined as*

$$\check{S}(\beta) = \frac{A^-_{(e,x)} + A^+_{(e,x)} - \lambda_{(e,x)}}{3}$$

where $\check{S}(\beta) \in [-1, 1]$.

4.5 Decision Making Problem Based on Multicriteria Cubic Soft Expert Sets

Decision making problems have been studied using fuzzy soft sets. Now we are going to present the multicriteria cubic soft expert sets in decision making along with their weights and score function.

Step 1: Input the cubic soft expert set (β_1, E, X).

Step 2: Utilize the opinions of experts in the form of $CSESs$ to determine the opinions regarding the given criteria. Make a separate table for the opinions of each expert.

Step 3: Assign weights to the experts according to their expertise.

Step 4: Apply cubic soft expert weighted average quotient operator to each of the above tables and find the cubic soft expert weighted average corresponding to each attribute.

Step 5: Calculate $\underset{P}{\vee}\hat{U}_j$.

Step 6: Calculate the scores of each $\hat{U}_j$.

Step 7: Generate the non-increasing order of all the alternatives according to their scores.

Fuzzy soft set theoretic approach has been used in decision making problems by Roy et al. [58]. In this section, we give an application of $CSES$ theory in a decision making problem.

Example 4.5.1 *Let $U = \{u_1$ =Guinea , u_2 =Liberia, u_3 =Sierra leone, u_4 =Nigeria$\}$ be the set of countries, $E = \{e_1 = Diarrhea, e_2 = Severe\ Headache\ , e_3 = Explained\ bleeding, e_4 = Fever\ and\ Vomiting\}$ be the set of symptoms of Ebola patients , $X = \{x_1, x_2, x_3\}$ be the set of Physicians.*

Step 1:

$\beta_1(e_1, x_1) = \{(u_1, [0.4, 0.6], 0.8), (u_2, [0.1, 0.5], 0.3), (u_3, [0.6, 0.7], 0.5), (u_4, [0.1, 0.9], 0.8)\}$

$\beta_1(e_2, x_1) = \{(u_1, [0.3, 0.7], 0.4), (u_2, [0.7, 0.9], 0.8), (u_3, [0.3, 0.9], 0.5), (u_4, [0.4, 0.6], 0.5)\}$,

$\beta_1(e_3, x_1) = \{(u_1, [0.5, 0.6], 0.6), (u_2, [0.5, 0.7], 0.6), (u_3, [0.2, 0.6], 0.4), (u_4, [0.3, 0.5], 0.4)\}$,

$\beta_1(e_4, x_1) = \{(u_1, [0.3, 0.9], 0.5), (u_2, [0.2, 0.8], 0.6), (u_3, [0.5, 0.7], 0.9), (u_4, [0.4, 0.8], 0.7)\}$,

$\beta_1(e_1, x_2) = \{(u_1, [0.3, 0.6], 0.4), (u_2, [0.6, 0.9], 0.3), (u_5, [0.4, 0.7], 0.3), (u_5, [0.4, 0.6], 0.4)\}$,

$\beta_1(e_2, x_2) = \{(u_1, [0.7, 0.9], 0.2), (u_2, [0.5, 0.8], 0.8), (u_3, [0.4, 0.7], 0.4), (u_4, [0.6, 0.9], 0.7)\}$,

$\beta_1(e_3, x_2) = \{(u_1, [0.6, 0.8], 0.4), (u_2, [0.3, 0.7], 0.5), (u_3, [0.5, 0.9], 0.7), (u_4, [0.4, 0.8], 0.8)\}$,

$\beta_1(e_4, x_2) = \{(u_1, [0.5, 0.8], 0.8), (u_2, [0.8, 0.9], 0.4), (u_3, [0.7, 0.9], 0.8), (u_4, [0.5, 0.6], 0.6)\}$,

$\beta_1(e_1, x_3) = \{(u_1, [0.6, 0.8], 0.5), (u_2, [0.5, 0.7], 0.5), (u_3, [0.7, 0.8], 0.6), (u_4, [0.6, 0.9], 0.6)\}$,

$\beta_1(e_2, x_3) = \{(u_1, [0.2, 0.7], 0.5), (u_2, [0.3, 0.7], 0.5), (u_3, [0.7, 0.8], 0.9), (u_4, [0.2, 0.5], 0.3)\}$,

$\beta_1(e_3, x_3) = \{(u_1, [0.8, 0.9], 0.9), (u_2, [0.6, 0.7], 0.6), (u_3, [0.4, 0.8], 0.5), (u_4, [0.1, 0.6], 0.4)\}$,

$\beta_1(e_4, x_3) = \{(u_1, [0.1, 0.9], 0.4), (u_2, [0.2, 0.9], 0.8), (u_3, [0.5, 0.9], 0.6), (u_4, [0.4, 0.8], 0.5)\}$.

Step 2:

	u_1	u_2	u_3	u_4
(e_1, x_1)	$([0.4, 0.6], 0.8)$	$([0.1, 0.5], 0.3)$	$([0.6, 0.7], 0.5)$	$([0.1, 0.9], 0.8)$
(e_2, x_1)	$([0.3, 0.7], 0.4)$	$([0.7, 0.9], 0.8)$	$([0.3, 0.9], 0.5)$	$([0.4, 0.6], 0.5)$
(e_3, x_1)	$([0.5, 0.6], 0.6)$	$([0.5, 0.7], 0.6)$	$([0.2, 0.6], 0.4)$	$([0.3, 0.5], 0.4)$
(e_4, x_1)	$([0.3, 0.9], 0.5)$	$([0.5, 0.7], 0.6)$	$([0.5, 0.7], 0.9)$	$([0.4, 0.8], 0.7)$

Table 4.5.1. Opinion of expert x_1

	u_1	u_2	u_3	u_4
(e_1, x_2)	$([0.3, 0.6], 0.4)$	$([0.6, 0.9], 0.3)$	$([0.4, 0.7], 0.3)$	$([0.4, 0.6], 0.4)$
(e_2, x_2)	$([0.7, 0.9], 0.2)$	$([0.5, 0.8], 0.8)$	$([0.4, 0.7], 0.4)$	$([0.6, 0.9], 0.7)$
(e_3, x_2)	$([0.6, 0.8], 0.4)$	$([0.3, 0.7], 0.5)$	$([0.5, 0.9], 0.7)$	$([0.4, 0.8], 0.8)$
(e_4, x_2)	$([0.5, 0.8], 0.8)$	$([0.8, 0.9], 0.4)$	$([0.7, 0.9], 0.8)$	$([0.5, 0.6], 0.6)$

Table 4.5.2. Opinion of expert x_2

	u_1	u_2	u_3	u_4
(e_1, x_3)	$([0.6, 0.8], 0.5)$	$([0.5, 0.7], 0.5)$	$([0.7, 0.8], 0.6)$	$([0.6, 0.9], 0.6)$
(e_2, x_3)	$([0.2, 0.7], 0.5)$	$([0.3, 0.7], 0.5)$	$([0.7, 0.8], 0.9)$	$([0.2, 0.5], 0.3)$
(e_3, x_3)	$([0.8, 0.9], 0.9)$	$([0.6, 0.7], 0.6)$	$([0.4, 0.8], 0.5)$	$([0.1, 0.6], 0.4)$
(e_4, x_3)	$([0.1, 0.9], 0.4)$	$([0.2, 0.9], 0.8)$	$([0.5, 0.9], 0.6)$	$([0.4, 0.8], 0.5)$

Table 4.5.3. Opinion of expert x_3

Step 3: $W = (0.36, 0.21, 0.43)^t$ where weight 0.36 is assigned to the expert x_1, weight 0.21 is assigned to the expert x_2 and weight 0.43 is assigned to the expert x_3.

Step 4: The cubic soft expert weighted average for each attribute have been calculated in Table 4.5.4.

	u_1	u_2	u_3	u_4
e_1	$([0.47, 0.70], 0.56)$	$([0.39, 0.70], 0.37)$	$([0.61, 0.74], 0.48)$	$([0.40, 0.86], 0.61)$
e_2	$([0.36, 0.75], 0.38)$	$([0.50, 0.81], 0.65)$	$([0.51, 0.83], 0.61)$	$([0.36, 0.66], 0.43)$
e_3	$([0.67, 0.80], 0.65)$	$([0.50, 0.70], 0.57)$	$([0.35, 0.77], 0.49)$	$([0.23, 0.62], 0.46)$
e_4	$([0.26, 0.88], 0.50)$	$([0.47, 0.85], 0.62)$	$([0.55, 0.85], 0.73)$	$([0.42, 0.76], 0.58)$

Table 4.5.4. CSE weighted averages

Step 5: Calculate the $P - union$ of 1st, 2nd, 3rd and 4th columns of the above Table by using Definition 4.4.31. So we have

$\hat{U}_1 = \overset{4}{\underset{j=1}{\vee}}\{(e_j, u_1)\} = ([0.67, 0.88], 0.65)$

$\hat{U}_2 = \overset{4}{\underset{j=1}{\vee}}\{(e_j, u_2)\} = ([0.50, 0.85], 0.65)$

$\hat{U}_3 = \overset{4}{\underset{j=1}{\vee}}\{(e_j, u_3)\} = ([0.61, 0.85], 0.73)$

$\hat{U}_4 = \overset{4}{\underset{j=1}{\vee}}\{(e_j, u_4)\} = ([0.42, 0.86], 0.61)$.

Step 6: Now calculate the score of the above $CSES$ elements by using Definition 4.4.37.

$\check{S}(\hat{U}_1) = 0.30$

$\check{S}(\hat{U}_2) = 0.23$

$\check{S}(\hat{U}_3) = 0.24$

$\check{S}(\hat{U}_4) = 0.22$

Step 7: Generate the non-decreasing order of the score of $CSES$ set values.

Corresponding to $P - union$ we have the following order:

$u_1 > u_3 > u_2 > u_4$.

In the above U, we want to check which country is more affected by Ebola. Hence Guinea is more effected by Ebola.

4.6 Conclusion and Future work

In this chapter, $CSES$ has been discussed which can be used in decision analysis. Some basic operations have been defined for $CSES$. Several properties have been investigated. We derive different conditions for different operations of two $ICSESs$ ($ECSESs$) to be an $ICSESs$ ($ECSESs$). There are so many methods to solve decision making problems in various fields but this technique is more suitable because in decision analysis there are some problems in which decision makers take decision on the basis of different conditions such as climate condition, time period condition and geographical conditions. If a decision maker wants to take a decision in some problems on the basis of such conditions then this structure is very useful. At the end, an algorithm has been presented along with an illustrative U. In future we aim to study $TOPSIS$ for group decision making with $CSES$ also we want to define different aggregation operators similarity and distance measures and distances and similarity degrees between $CSESs$.

Chapter 5

Some New Operations on Cubic Soft Expert Sets $(CSESs)$

5.1 Introduction

Cubic sets are basically a combination of fuzzy sets and interval valued fuzzy sets. Cubic sets was defined by Jun et al. [34]. Jun et al. defined basic operations of inclusion, union and intersection. There are certain operations which were not defined in this paper for U, addition, multiplication of two cubic sets, power and scalar product of cubic sets etc. In this chapter we have introduced some new operations such as addition and multiplication of two $CSESs$, product of a $CSESs$ with real number $k > 0$, power of $CSESs$, score and accuracy function of $CSESs$. The purpose of defining score function and accuracy function is that we can determine the ranking of $CSESs$ which helps us in some aggregation operators. Some aggregation operators on $CSESs$ have been introduced. Fuzzy sets and interval value fuzzy sets play a fundamental role in decision analysis. Similarly, $CSESs$ also gives fruitful results in decision making. Therefore, the aim of this chapter is to determine the most preferable choice among all possible choices, when data is in cubic set form. At the end, an algorithm has been presented. Finally, an U has been presented to highlight the applicability of the proposed algorithm.

5.2 Preliminaries

Definition 5.2.1 *[17] The simplest and most common way to aggregate is to use a simple arithmetic mean (also know as the average). Mathematically we have :*

$$M(x_1, x_2, ..., x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i = \sum_{i=1}^{n}\frac{1}{n}.x_i$$

The average is often used since it is simple and satisfies the properties of monotonicity, continuity, symmetry, associativity, idempotence and stability for linear transformations.

But it has neither absorbent nor neutral element and has no behavioral properties.

Definition 5.2.2 *[17] The weighted mean is a classical extension which allows placing weights on the arguments. But we loose the property of symmetry. It is expressed mathematically by :*

$$M_{w_1,\dots,w_n}(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n}(w_i x_{i)},$$

where $w_i \geq 0$ *and* $\sum_{i=1}^{n} w_i$.

Definition 5.2.3 *[21, 22] A very notable particular case, studied in detail by Dujmovic and by Dyckhoff corresponds to the function* f *is defined by* $f : x \to x^a$. *We obtain then a quasi arithmetic mean of the form :*

$$M(x_1, x_2, \dots, x_n) = [\frac{1}{n}\sum_{i=1}^{n} x_i^a]^{\frac{1}{a}} = [\sum_{i=1}^{n}(\frac{1}{n}x_i^a)]^{\frac{1}{a}} \tag{5.1}$$

It generalizes a group of common means, only by changing the value of a. *When* $a = 1$ *we obtain the arithmetic mean, when* $a \to 0$, *equation 5.1 tends to the geometric mean, when* $a = 2$ *we obtain the quadratic mean or the Euclidean mean and for* $a = -1$ *we obtain the harmonic mean.*

Definition 5.2.4 *[79] The ordered weighted averaging operators* (OWA) *were originally introduced by Yager in [55] to provide a means for aggregating scores associated with the satisfaction of multiple criteria, which unifies in one operator the conjunctive and disjunctive behavior:*

$$OWA(x_1, x_2, \dots, x_n) = \sum_{j=1}^{n} w_j x_{\sigma(j)},$$

where σ *is a permutation that orders the elements*

$$x_{\sigma(1)} \leq x_{\sigma(2)} \leq \dots \leq x_{\sigma(n)}$$

The weights are all non negative and their sum equals 1, that is, $w_i \geq 0$ *and* $\sum_{i=1}^{n} w_i$.

The OWA *operators provide a parameterized family of aggregation operators, which include many of the well-known operators such as the maximum, the minimum, the k-order statistics, the median and the arithmetic mean. In order to obtain these particular operators we should simply choose particular weights (see Table 5.2.1).*

	OWA
Minimum	$\begin{cases} w_1 = 1 & \\ w_i = 0 & \text{if } i \neq 1 \end{cases}$
Maximum	$\begin{cases} w_n = 1 & \\ w_i = 0 & \text{if } i \neq n \end{cases}$
Median	$\begin{cases} w_{\frac{n+1}{2}} = 1 & \text{if } n \text{ is odd} \\ w_{\frac{n}{2}} = \frac{1}{2} \text{ and } w_{\frac{n}{2}+1} = \frac{1}{2} & \text{if } n \text{ is even} \\ w_i = 0 & \text{else} \end{cases}$
Arithmetic mean	$w_i = \frac{1}{n}$ *for all* i

Table 5.2.1

The ordered weighted averaging operators are commutative, monotone, idempotent, they are stable for positive linear transformations and they have a compensatory behavior.

Definition 5.2.5 *[82] A mapping* $M : I^n \rightarrow I$ *is called a generalized ordered weighted aggregation* ($GOWA$) *operator of dimension* n *if*

$$M(x_1, ..., x_n) = (\sum_{j=1}^{n} w_j b_j^{\lambda})^{\frac{1}{\lambda}}$$

where, $\{w_j\}_{j\in\{1,2,...,n\}}$ *is a collection of weights satisfying* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$. λ *is a parameter such that* $\lambda \in [-\infty, \infty]$; b_j *is the jth largest among the* a_i.

The important special case is when $w_j = \frac{1}{n}$. *In this case*

$$M(x_1, ..., x_n) = (\sum_{j=1}^{n} \frac{1}{n} b_j^{\lambda})^{\frac{1}{\lambda}}$$

This is the generalized mean operator discussed by Dyckhoff and Pedrycz (1984). We note these are also mean operators: they are symmetric, monotonic and bounded.

5.3 Some New Operations on $CSESs$

Definition 5.3.1 *Let* U *be a finite universe set containing* n *alternatives,* E *be a set of criteria and* X *be a set of experts (or decision makers). A pair* (β, E, X) *is called a cubic soft expert set over* U *if and only if* $\beta : E \times X \longrightarrow CP(U)$ *is a mapping into the set of all cubic sets in* U. *Cubic soft expert set is denoted and defined as*

$$(\beta, E, X) = \{\beta(e, x) = \{(u, A_{(e,x)}(u), \lambda_{(e,x)}(u)) : u \in U, (e, x) \in E \times X\}.$$

where $A_{(e,x)}(u)$ is an interval valued fuzzy set and $\lambda_{(e,x)}(u)$ is a fuzzy set. Here decision makers give their opinions in the form of cubic set.

The collection of all cubic soft expert sets $CSESs$ is denoted as $\widehat{\beta}$.

Definition 5.3.2 *Let $(\beta_1, E_1, X_1) = \{\beta(e,x) = \{(u, A_{1(e,x)}(u), \lambda_{1_{(e,x)}}(u)) : u \in U, (e,x) \in E_1 \times X_1\}$ and $(\beta_2, E_2, X_2) = \{\beta(f,y) = \{(u, A_{2(f,y)}(u), \lambda_{2_{(f,y)}}(u)) : u \in U, (f,y) \in E_2 \times X_2\}$ be two $CSESs$ over U. Addition of two $CSESs$ (β_1, E_1, X_1) and (β_2, E_2, X_2) is denoted and defined as follows*

$$(\beta_1, E_1, X_1) \oplus (\beta_2, E_2, X_2) = \left\{ \begin{array}{l} (u, [A^-_{1(e,x)} + A^-_{2(f,y)} - A^-_{1(e,x)}A^-_{2(f,y)}, \\ A^+_{1(e,x)} + A^+_{2(f,y)} - A^+_{1(e,x)}A^+_{2(f,y)}], \\ \lambda_{1(e,x)} + \lambda_{2(f,y)} - \lambda_{1(e,x)}\lambda_{2(f,y)}) \end{array} \right\}$$

Example 5.3.3 *Consider two $CSESs$ (β_1, E_1, X_1) and (β_2, E_2, X_2) over U.*
$\beta_1(e_1, x_1) = \{(u_1, [0.5, 0.8], 0.7), (u_2, [0.6, 0.9], 0.8), (u_3, [0.4, 0.7], 0.5)\}$,
$\beta_2(f_1, y_1) = \{(u_1, [0.2, 0.7], 0.3), (u_2, [0.3, 0.5], 0.5), (u_3, [0.5, 0.9], 0.2)\}$.
Therefore $(\beta_1, E_1, X_1) \oplus (\beta_2, E_2, X_2) = \{(u_1, [0.60, 0.94], 0.79), (u_2, [0.72, 0.95], 0.90), (u_3, [0.70, 0.97], 0.60)\}$.

Definition 5.3.4 *Let (β_1, E_1, X_1) and (β_2, E_2, X_2) be two $CSESs$ over U. Product of two $CSESs$ (β_1, E_1, X_1) and (β_2, E_2, X_2) is denoted and defined as follows*

$$(\beta_1, E_1, X_1)(\beta_2, E_2, X_2) = \{(u, [A^-_{1(e,x)}A^-_{2(f,y)}, A^+_{1(e,x)}A^+_{2(f,y)}], \lambda_{1(e,x)}\lambda_{2(f,y)})\}.$$

Example 5.3.5 *Consider Example 5.3.3* $(\beta_1, E_1, X_1)(\beta_2, E_2, X_2) = \{(u_1, [0.10, 0.56], 0.21),$
$(u_2, [0.18, 0.45], 0.40), (u_3, [0.20, 0.63], 0.10)\}$.

Definition 5.3.6 *Let (β, E, X) be a $CSES$ over U. Product of $CSES$ (β, E, X) with real number $k > 0$ is denoted and defined as follows*

$$k(\beta, E, X) = \{(u, [1 - (1 - A^-_{(e,x)})^k, 1 - (1 - A^+_{(e,x)})^k], 1 - (1 - \lambda_{(e,x)})^k)\}.$$

Example 5.3.7 *Consider* $\beta(f_1, y_1) = \{(u_1, [0.2, 0.7], 0.3), (u_2, [0.3, 0.5], 0.5), (u_3, [0.5, 0.9], 0.2)\}$, $k = 5$. *Product of $CSES$ with scalar is given by*
$5(\beta, E, X) = \{(u_1, [0.672, 0.997], 0.831), (u_2, [0.831, 0.968], 0.968), (u_3, [0.968, 0.999], 0.672)\}$.

Definition 5.3.8 *Let (β, E, X) be a $CSES$ over U. Power of $CSES$ (β, E, X) with real number $k > 0$ is denoted and defined as follows*

$$(\beta, E, X)^k = \{(u, [(A^-_{(e,x)})^k, (A^+_{(e,x)})^k], (\lambda_{(e,x)})^k)\}.$$

Example 5.3.9 *Consider Example 5.3.7. Power of CSES with scalar is given by*
$(\beta, E, X)^5 = \{(u_1, [0.0003, 0.1680], 0.0024), (u_2, [0.0024, 0.0312], 0.0312),$
$(u_3, [0.0312, 0.5904], 0.0003)\}$.

Definition 5.3.10 *Let* $\beta = ([A^-_{(e,x)}, A^+_{(e,x)}], \lambda_{(e,x)})$ *be a CSE value. A score function* $\breve{S}$ *of CSE value is defined as*

$$\breve{S}(\beta) = \frac{A^-_{(e,x)} + A^+_{(e,x)} - \lambda_{(e,x)}}{3}$$

where $\breve{S}(\beta) \in [-1, 1]$.

Definition 5.3.11 *Let* $\beta = ([A^-_{(e,x)}, A^+_{(e,x)}], \lambda_{(e,x)})$ *be a CSE value. Accuracy function* $\breve{A}$ *of CSE value is defined as*

$$\breve{A}(\beta) = \frac{A^-_{(e,x)} + A^+_{(e,x)} + \lambda_{(e,x)}}{3}$$

where $\breve{A}(\beta) \in [0, 1]$.

Definition 5.3.12 *Let* $\beta_1 = ([A^-_{1(e,x)}, A^+_{1(e,x)}], \lambda_{1(e,x)})$ *and* $\beta_2 = ([A^-_{2(e,x)}, A^+_{2(e,x)}], \lambda_{2(e,x)})$ *are two CSE values.* $\breve{S}(\beta_1)$ *and* $\breve{S}(\beta_2)$ *are scores of* β_1 *and* β_2 *respectively, let* $\breve{A}(\beta_1)$ *and* $\breve{A}(\beta_2)$ *are accuracies of* β_1 *and* β_2 *respectively.*

$$\begin{aligned}
&\text{If } \breve{S}(\beta_1) < \breve{S}(\beta_2) \text{ then } \beta_1 < \beta_2 \\
&\text{If } \breve{S}(\beta_1) > \breve{S}(\beta_2) \text{ then } \beta_1 > \beta_2 \\
&\text{If } \breve{S}(\beta_1) = \breve{S}(\beta_2) \text{ then } \beta_1 = \beta_2 \\
&\text{If } \breve{A}(\beta_1) < \breve{A}(\beta_2) \text{ then } \beta_1 < \beta_2 \\
&\text{If } \breve{A}(\beta_1) > \breve{A}(\beta_2) \text{ then } \beta_1 > \beta_2 \\
&\text{If } \breve{A}(\beta_1) = \breve{A}(\beta_2) \text{ then } \beta_1 = \beta_2.
\end{aligned}$$

5.4 Some Aggregation Operators on Cubic Soft Expert Sets

Definition 5.4.1 *Let* $\beta_i = ([A^-_i, A^+_i], \lambda_i)$ $(i = 1, 2, ...n)$ *be CSES sets. A mapping* $\tau^{GEO}_w : \widehat{\beta}^n_i \to \widehat{\beta}_i$ *is called cubic soft expert weighted geometric operator if its satisfies*

$$\tau^{GEO}_w(\beta_1, \beta_2, ..., \beta_n) = \prod_{i=1}^{n} \beta_i^{w_i},$$

where $w_i = (w_1, w_2, ..., w_n)^t$ *is a weight vector of* β_i *satisfying the conditions* $w_i \in [0, 1]$ *and* $\sum_{i=1}^{n} w_i = 1$.

Theorem 5.4.2 *Suppose that $\beta_i = ([A_i^-, A_i^+], \lambda_i)$ $(i = 1, 2, ...n)$ are $CSES$ sets. By using cubic soft expert weighted geometric operator ($CSEWGO$) aggregation result is also $CSES$.*

$$\tau_w^{GEO}(\beta_1, \beta_2, ..., \beta_n) = ([\prod_{i=1}^{n}(A_i^-)^{w_i}, \prod_{i=1}^{n}(A_i^+)^{w_i}], \prod_{i=1}^{n}(\lambda_i)^{w_i}),$$

where $w_i = (w_1, w_2, ..., w_n)^t$ is a weight vector of β_i satisfying the conditions $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$.

Proof. It can be easily proved by using Definitions 5.3.4, 5.3.8, and 5.4.1. ■

Definition 5.4.3 *Let $\beta_i = ([A_i^-, A_i^+], \lambda_i)$ $(i = 1, 2, ...n)$ be $CSES$ sets. A mapping $\tau_w^{\beta_i} : \widehat{\beta}_i^n \rightarrow \widehat{\beta}_i$ is called cubic soft expert weighted average operator if its satisfies*

$$\tau_w^{\beta_i}(\beta_1, \beta_2, ..., \beta_n) = \bigoplus_{i=1}^{n} w_i\beta_i,$$

where $w_i = (w_1, w_2, ..., w_n)^t$ is a weight vector of β_i satisfying the conditions $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$.

Theorem 5.4.4 *Suppose that $\beta_i = ([A_i^-, A_i^+], \lambda_i)$ $(i = 1, 2, ...n)$ are $CSES$ sets. By using cubic soft expert weighted average operator ($CSEWAO$) aggregation result is also $CSES$.*

$$\tau_w^{\beta_i}(\beta_1, \beta_2, ..., \beta_n) = ([1 - \prod_{i=1}^{n}(1 - A_i^-)^{w_i}, 1 - \prod_{i=1}^{n}(1 - A_i^+)^{w_i}], 1 - \prod_{i=1}^{n}(1 - \lambda_i)^{w_i}),$$

where $w_i = (w_1, w_2, ..., w_n)^t$ is a weight vector of β_i satisfying the conditions $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$.

Proof. By using Definition 5.4.3 we have $\tau_w^{\beta_i}(\beta_1, \beta_2, ..., \beta_n) = \bigoplus_{i=1}^{n} w_i\beta_i$. Now we using mathematical induction, for $n = 1$, we have $\tau_w^{\beta_i}(\beta_1) = w_1\beta_1 = ([1 - (1 - A_1^-)^{w_1}, 1 - (1 - A_1^+)^{w_1}], 1 - (1 - \lambda_1)^{w_1})$ by using Definition 5.3.6. For $n = 2$, we have $\tau_w^{\beta_i}(\beta_1, \beta_2) = \bigoplus_{i=1}^{2} w_i\beta_i = w_1\beta_1 \oplus w_2\beta_2 = ([1 - (1 - A_1^-)^{w_1}, 1 - (1 - A_1^+)^{w_1}], 1 - (1 - \lambda_1)^{w_1}) \oplus ([1 - (1 - A_2^-)^{w_2}, 1 - (1 - A_2^+)^{w_2}], 1 - (1 - \lambda_2)^{w_2}) = ([1 - (1 - A_1^-)^{w_1}(1 - A_2^-)^{w_2}, 1 - (1 - A_1^+)^{w_1}(1 - A_2^+)^{w_2}], 1 - (1 - \lambda_1)^{w_1}(1 - \lambda_2)^{w_2}) = ([1 - \prod_{i=1}^{2}(1 - A_i^-)^{w_i}, 1 - \prod_{i=1}^{2}(1 - A_i^+)^{w_i}], 1 - \prod_{i=1}^{2}(1 - \lambda_i)^{w_i})$.

By using Definitions 5.3.6 and 5.3.2. Suppose it is true for $n = k$, $\tau_w^{\beta_i}(\beta_1, \beta_2, ..., \beta_k) = \bigoplus_{i=1}^{k} w_i\beta_i = ([1 - \prod_{i=1}^{k}(1 - A_i^-)^{w_i}, 1 - \prod_{i=1}^{k}(1 - A_i^+)^{w_i}], 1 - \prod_{i=1}^{k}(1 - \lambda_i)^{w_i})$.

Now we prove it for $n = k+1$, $\tau_w^{\beta_i}(\beta_1, \beta_2, ..., \beta_k, \beta_{k+1}) = \overset{k+1}{\underset{i=1}{\oplus}} w_i\beta_i = w_1\beta_1 \oplus w_2\beta_2 \oplus, ..., \oplus w_k\beta_k \oplus w_{k+1}\beta_{k+1} = \overset{k}{\underset{i=1}{\oplus}} w_i\beta_i \oplus w_{k+1}\beta_{k+1} = ([1-\prod_{i=1}^{k}(1-A_i^-)^{w_i}, 1-\prod_{i=1}^{k}(1-A_i^+)^{w_i}], 1-\prod_{i=1}^{k}(1-\lambda_i)^{w_i}) \oplus ([1-(1-A_{k+1}^-)^{w_{k+1}}, 1-(1-A_{k+1}^+)^{w_{k+1}}], 1-(1-\lambda_{k+1})^{w_{k+1}}) = ([1-\prod_{i=1}^{k+1}(1-A_i^-)^{w_i}, 1-\prod_{i=1}^{k+1}(1-A_i^+)^{w_i}], 1-\prod_{i=1}^{k+1}(1-\lambda_i)^{w_i})$, by using Definition 5.3.2. Hence it is true for $n = k+1$. So $\tau_w^{\beta_i}(\beta_1, \beta_2, ..., \beta_n) = ([1-\prod_{i=1}^{n}(1-A_i^-)^{w_i}, 1-\prod_{i=1}^{n}(1-A_i^+)^{w_i}], 1-\prod_{i=1}^{n}(1-\lambda_i)^{w_i})$. ■

Example 5.4.5 *Let there are five experts who give their opinions corresponding to different attributes in the form of cubic set* $\beta_1 = ([0.5, 0.8], 0.7)$, $\beta_2 = ([0.2, 0.6], 0.7)$, $\beta_3 = ([0.1, 0.5], 0.8)$,$\beta_4 = ([0.3, 0.9], 0.6)$, $\beta_5 = ([0.1, 0.6], 0.5)$. *The weight vector corresponding to experts is* $w = (0.32, 0.14, 0.20, 0.16, 0.18)^t$. *The aggregated result of these experts can be computed by using Theorem 5.4.4 as follows:*

$$\begin{aligned}\tau_w^{\beta_i}(\beta_1, \beta_2, ..., \beta_5) &= ([1-\prod_{i=1}^{5}(1-A_i^-)^{w_i}, 1-\prod_{i=1}^{5}(1-A_i^+)^{w_i}], 1-\prod_{i=1}^{5}(1-\lambda_i)^{w_i}) \\ &= ([1-(1-0.5)^{0.32}(1-0.2)^{0.14}(1-0.1)^{0.20}(1-0.3)^{0.16}(1-0.1)^{0.18}, \\ &\quad 1-(1-0.8)^{0.32}(1-0.6)^{0.14}(1-0.5)^{0.20}(1-0.9)^{0.16}(1-0.6)^{0.18}], \\ &\quad 1-(1-0.7)^{0.32}(1-0.7)^{0.14}(1-0.8)^{0.20}(1-0.6)^{0.16}(1-0.5)^{0.18}) \\ &= ([0.29542, 0.73160], 0.68244).\end{aligned}$$

Definition 5.4.6 *Let* $\beta_i = ([A_i^-, A_i^+], \lambda_i)$, $(i = 1, 2, ...n)$ *be* $CSES$ *sets. A mapping* $\tau_{\varpi}^{O} : \widehat{\beta}_i^n \to \widehat{\beta}_i$ *is called cubic soft expert OWA operator if its satisfies the following:*

$$\tau_{\varpi}^{O}(\beta_1, \beta_2, ..., \beta_n) = \overset{n}{\underset{i=1}{\oplus}} \varpi_i\widetilde{\beta}_i,$$

where $\varpi = (\varpi_1, \varpi_2, ...\varpi_n)^t$ is a position weight vector associated with the mapping τ_{ϖ}^{O} satisfying $\varpi_k \in [0,1]$ and $\sum_{k=1}^{n} \varpi_k = 1$. $\widetilde{\beta}_k = ([\widetilde{A}_k^-, \widetilde{A}_k^+], \widetilde{\lambda}_k)$ is the k-th largest cubic soft expert set which can be determined by using score function or accuracy function.

Theorem 5.4.7 *Suppose that* $\beta_i = ([A_i^-, A_i^+], \lambda_i)$, $(i = 1, 2, ...n)$ *are* $CSES$ *sets. By using cubic soft expert OWA operator aggregation result is also* $CSES$.

$$\tau_{\varpi}^{O}(\beta_1, \beta_2, ..., \beta_n) = ([1-\prod_{k=1}^{n}(1-\widetilde{A}_k^-)^{\varpi_k}, 1-\prod_{k=1}^{n}(1-\widetilde{A}_k^+)^{\varpi_k}], 1-\prod_{k=1}^{n}(1-\widetilde{\lambda}_k)^{\varpi_k}),$$

where $\varpi = (\varpi_1, \varpi_2, ...\varpi_n)^t$ is a position weight vector associated with the mapping τ_{ϖ}^{O} satisfying $\varpi_k \in [0,1]$ and $\sum_{k=1}^{n} \varpi_k = 1$. $\beta_k = ([\widetilde{A}_k^-, \widetilde{A}_k^+], \widetilde{\lambda}_k)$ is the k-th largest cubic soft expert sets β_i which can be determined by using score function or accuracy function.

Proof. It can be proved by using mathematical induction and by using Definitions 5.3.2, 5.3.6, and 5.4.6. ■

Example 5.4.8 *Let there are five experts who give their opinions corresponding to different attributes in the form of a cubic set* $\beta_1 = ([0.5, 0.8], 0.7)$, $\beta_2 = ([0.2, 0.6], 0.7)$, $\beta_3 = ([0.1, 0.5], 0.8)$,$\beta_4 = ([0.3, 0.9], 0.6)$, $\beta_5 = ([0.1, 0.6], 0.5)$. *Assume that the associated weight vector of* β_i *is* $\varpi = (0.28, 0.16, 0.12, 0.24, 0.20)^t$. *The aggregated result of these expert can be computed by using Theorem 5.4.7 a follows:*

By using Defintion 5.3.11, scores of the $CSESs$ are $\check{S}(\beta_1) = 0.2$ $\check{S}(\beta_2) = 0.034$ $\check{S}(\beta_3) = -0.067$, $\check{S}(\beta_4) = 0.2$, $\check{S}(\beta_5) = 0.067$. It is clear that $\check{S}(\beta_1) = \check{S}(\beta_4) > \check{S}(\beta_5) > \check{S}(\beta_2) > \check{S}(\beta_3)$. Since the score of $\beta_1 = \beta_4$, now we calculate accuracies of β_1 and β_4. $\breve{A}(\beta_1) = 0.67$, $\breve{A}(\beta_4) = 0.6$. So we can establish the ranking order of $CSES$ β_i $(i = 1, 2, ..., 5)$ as follows

$$\beta_1 > \beta_4 > \beta_5 > \beta_2 > \beta_3.$$

Then, we have

$$\begin{aligned}
\widetilde{\beta}_1 &= \beta_1 = ([0.5, 0.8], 0.7) \\
\widetilde{\beta}_2 &= \beta_4 = ([0.3, 0.9], 0.6) \\
\widetilde{\beta}_3 &= \beta_5 = ([0.1, 0.6], 0.5) \\
\widetilde{\beta}_4 &= \beta_2 = ([0.2, 0.6], 0.7) \\
\widetilde{\beta}_5 &= \beta_3 = ([0.1, 0.5], 0.8)
\end{aligned}$$

By using Theorem 5.4.7, we have

$$\begin{aligned}
\tau_{\varpi}^{O}(\beta_1, \beta_2, ..., \beta_5) &= ([1 - \prod_{i=1}^{n}(1 - \widetilde{A}_i^-)^{\varpi_i}, 1 - \prod_{i=1}^{n}(1 - \widetilde{A}_i^+)^{\varpi_i}], 1 - \prod_{i=1}^{n}(1 - \widetilde{\lambda}_i)^{\varpi_i}) \\
&= ([1 - (1-0.5)^{0.28}(1-0.3)^{0.16}(1-0.1)^{0.12}(1-0.2)^{0.24}(1-0.1)^{0.20}, \\
&\quad 1 - (1-0.8)^{0.28}(1-0.9)^{0.16}(1-0.6)^{0.12}(1-0.6)^{0.24}(1-0.5)^{0.20}], \\
&\quad 1 - (1-0.7)^{0.28}(1-0.6)^{0.16}(1-0.5)^{0.12}(1-0.7)^{0.24}(1-0.8)^{0.20}) \\
&= ([0.2871, 0.7240], 0.6920).
\end{aligned}$$

Definition 5.4.9 *Let* $\beta_i = ([A_i^-, A_i^+], \lambda_i)$, $(i = 1, 2, ...n)$ *be $CSES$ sets. A mapping* $\tau_{\varpi}^{GO} : \widehat{\beta}_i^n \to \widehat{\beta}_i$ *is called cubic soft expert $GOWA$ operator if its satisfies the following:*

$$\tau_{\varpi}^{GO}(\beta_1, \beta_2, ..., \beta_n) = \sqrt[r]{\bigoplus_{k=1}^{n} \varpi_k \widetilde{\beta}_k^r},$$

where $\varpi = (\varpi_1, \varpi_2, ...\varpi_n)^t$ is a position weight vector associated with the mapping τ_{ϖ}^{GO} satisfying $\varpi_k \in [0, 1]$ and $\sum_{k=1}^{n} \varpi_k = 1$. $\widetilde{\beta}_k = ([\widetilde{A}_k^-, \widetilde{A}_k^+], \widetilde{\lambda}_k)$ is the k-th largest cubic soft expert set which can be determined by using score function or accuracy function.

Remark 5.4.10 *If $r = 1$ in Definition 5.4.9 then cubic soft expert $GOWA$ operator τ_{ϖ}^{GO} degenerates to cubic soft expert OWA operator τ_{ϖ}^{O}.*

Theorem 5.4.11 *Suppose that $\beta_i = ([A_i^-, A_i^+], \lambda_i)$, $(i = 1, 2, ...n)$ are $CSES$ sets. By using cubic soft expert $GOWA$ operator aggregation result is also $CSES$.*

$$\tau_{\varpi}^{GO}(\beta_1, \beta_2, ..., \beta_n) = \left(\begin{array}{c} [\sqrt[r]{1 - \prod_{k=1}^{n}(1 - \widetilde{A}_k^{r-})^{\varpi_k}}, \sqrt[r]{1 - \prod_{k=1}^{n}(1 - \widetilde{A}_k^{r+})^{\varpi_k}}], \\ \sqrt[r]{1 - \prod_{k=1}^{n}(1 - \widetilde{\lambda}_k^{r})^{\varpi_k}} \end{array} \right),$$

where $\varpi = (\varpi_1, \varpi_2, ...\varpi_n)^t$ is a position weight vector associated with the mapping τ_{ϖ}^{GO} satisfying $\varpi_k \in [0, 1]$ and $\sum_{k=1}^{n} \varpi_k = 1$, $r > 0$. $\widetilde{\beta}_k = ([\widetilde{A}_k^-, \widetilde{A}_k^+], \widetilde{\lambda}_k)$ is the k-th largest cubic soft expert set which can be determined by using score function or accuracy function.

Proof. It can be proved by using mathematical induction and by using Definitions [5.3.2, 5.3.6, 5.3.8 and 5.4.9]. ■

5.5 Multicriteria Decision Making of Cubic Soft Expert Sets with Cubic Soft Expert $GOWA$ Operator

In this section, we develop an algorithm with the aid of CSE sets for decision analysis in which experts will be given weightage to attributes according to their area of expertise. Let $U = \{u_1, u_2, ...u_n\}$ be the set of alternatives, $E = \{e_1, e_2, ..., e_l\}$ be the set of attributes and $X = \{x_1, x_2, ..., x_m\}$ be the set of experts. Further, we take opinion of experts in the form of CSE elements.

Step 1: Utilize the evaluations of experts in the form of CSE sets.

Step 2: Separate the opinion of each expert.

Step 3: Calculate the score of each entry corresponding to (e_l, u_n). Arrange these attributes according to their scores.

Step 4: Assign weights to each attribute.

Step 5: Aggregate the attributes by using cubic soft expert $GOWA$ operator.

Step 6: Calculate accuracies of all alternatives corresponding to different experts.

Step 7: Find the average of these alternatives.

Step 8: Arrange these alternatives in ascending order.

Step 9: Choose the best alternative.

Example 5.5.1 *let* $U = \{u_1, u_2, u_3, u_4\}$ *be the set of cars,* $E = \{e_1 = cheap,\ e_2 = expensive,\ e_3 =$ model *2010 and above,* $e_4 =$*Made in japan,* $e_5 =$*white color*$\}$ *be the set of attributes, and* $X = \{x_1, x_2, x_3\}$ *be the set of experts. Evaluation of these experts is represented in the form of* CSE *set. Mr. A want to choose best car with respect to the given set of the attributes.*

Step 1: Utilize the evaluations of experts in the form of CSE sets.

$\beta(e_1, x_1) = \{(u_1, [0.0025, 0.0876], 0.0034), (u_2, [0.0225, 0.0446], 0.0369), (u_3, [0.0245, 0.0546], 0.0345), (u_4, [0.0525, 0.0646], 0.0387)\}$,

$\beta(e_2, x_1) = \{(u_1, [0.0225, 0.0984], 0.0139), (u_2, [0.0289, 0.0646], 0.0276), (u_3, [0.0287, 0.0486], 0.0765), (u_4, [0.0238, 0.0446], 0.0987)\}$,

$\beta(e_3, x_1) = \{(u_1, [0.0350, 0.0498], 0.0265), (u_2, [0.0478, 0.0946], 0.0876), (u_3, [0.0476, 0.0876], 0.0876), (u_4, [0.0275, 0.0546], 0.0028)\}$,

$\beta(e_4, x_1) = \{(u_1, [0.0122, 0.0646], 0.0975), (u_2, [0.0376, 0.0846], 0.0987), (u_3, [0.0277, 0.0765], 0.0298), (u_4, [0.0229, 0.0346], 0.0834)\}$,

$\beta(e_5, x_1) = \{(u_1, [0.0765, 0.0946], 0.0256), (u_2, [0.0027, 0.0046], 0.0187), (u_3, [0.0377, 0.0974], 0.0098), (u_4, [0.0453, 0.0496], 0.0287)\}$,

$\beta(e_1, x_2) = \{(u_1, [0.0176, 0.0246], 0.0876), (u_2, [0.0176, 0.0746], 0.0027), (u_3, [0.0276, 0.0843], 0.0234), (u_4, [0.0035, 0.0046], 0.0209)\}$,

$\beta(e_2, x_2) = \{(u_1, [0.0987, 0.0994], 0.0267), (u_2, [0.0987, 0.0996], 0.0376), (u_3, [0.0228, 0.0777], 0.0072), (u_4, [0.0533, 0.0876], 0.0002)\}$,

$\beta(e_3, x_2) = \{(u_1, [0.0346, 0.0446], 0.0275), (u_2, [0.0176, 0.0246], 0.0987), (u_3, [0.0425, 0.0678], 0.0088), (u_4, [0.0345, 0.0987], 0.0134)\}$,

$\beta(e_4, x_2) = \{(u_1, [0.0543, 0.0746], 0.0987), (u_2, [0.0198, 0.0346], 0.0376), (u_3, [0.0345, 0.0698], 0.0376), (u_4, [0.0654, 0.0877], 0.0087)\}$,

$\beta(e_5, x_2) = \{(u_1, [0.0123, 0.0946], 0.0256), (u_2, [0.0196, 0.0646], 0.0986), (u_3, [0.0445, 0.0868], 0.0098), (u_4, [0.0234, 0.0843], 0.0018)\}$,

$\beta(e_1, x_3) = \{(u_1, [0.0543, 0.0846], 0.0097), (u_2, [0.0472, 0.0546], 0.0287), (u_3, [0.0222, 0.0446], 0.0287), (u_4, [0.0432, 0.0876], 0.0007)\}$,

$\beta(e_2, x_3) = \{(u_1, [0.0098, 0.0146], 0.0189), (u_2, [0.0675, 0.0746], 0.0376), (u_3, [0.0356, 0.0946], 0.0765), (u_4, [0.0156, 0.0276], 0.0087)\}$,

$\beta(e_3, x_3) = \{(u_1, [0.0090, 0.0346], 0.0109), (u_2, [0.0354, 0.0446], 0.0098), (u_3, [0.0234, 0.0746], 0.0970), (u_4, [0.0321, 0.0762], 0.0034)\}$,

$\beta(e_4, x_3) = \{(u_1, [0.0037, 0.0046], 0.0018), (u_2, [0.0375, 0.0846], 0.0187), (u_3, [0.0543, 0.0646], 0.0365), (u_4, [0.0254, 0.0876], 0.0098)\}$,

$\beta(e_5, x_3) = \{(u_1, [0.0656, 0.0696], 0.0186), (u_2, [0.0316, 0.0546], 0.0465), (u_3, [0.0541, 0.0648], 0.0876), (u_4, [0.0765, 0.0954], 0.0365)\}$.

Step 2:

x_1	u_1	u_2
e_1	$([0.0025, 0.0876], 0.0034)$	$([0.0225, 0.0446], 0.0369)$
e_2	$([0.0225, 0.0984], 0.0139)$	$([0.0289, 0.0646], 0.0276)$
e_3	$([0.0350, 0.0498], 0.0265)$	$([0.0478, 0.0946], 0.0876)$
e_4	$([0.0122, 0.0646], 0.0975)$	$([0.0376, 0.0846], 0.0987)$
e_5	$([0.0765, 0.0946], 0.0256)$	$([0.0027, 0.0046], 0.0187)$

x_1	u_3	u_4
e_1	$([0.0245, 0.0546], 0.0345)$	$([0.0525, 0.0646], 0.0387)$
e_2	$([0.0287, 0.0486], 0.0765)$	$([0.0238, 0.0446], 0.0987)$
e_3	$([0.0476, 0.0876], 0.0876)$	$([0.0275, 0.0546], 0.0028)$
e_4	$([0.0277, 0.0765], 0.0298)$	$([0.0229, 0.0346], 0.0834)$
e_5	$([0.0377, 0.0974], 0.0098)$	$([0.0453, 0.0496], 0.0287)$

Table 5.5.1 Opinion of expert x_1

x_2	u_1	u_2
e_1	$([0.0176, 0.0246], 0.0876)$	$([0.0176, 0.0746], 0.0027)$
e_2	$([0.0987, 0.0994], 0.0267)$	$([0.0987, 0.0996], 0.0376)$
e_3	$([0.0346, 0.0446], 0.0275)$	$([0.0176, 0.0246], 0.0987)$
e_4	$([0.0543, 0.0746], 0.0987)$	$([0.0198, 0.0346], 0.0376)$
e_5	$([0.0123, 0.0946], 0.0256)$	$([0.0196, 0.0646], 0.0986)$

x_2	u_3	u_4
e_1	$([0.0276, 0.0843], 0.0234)$	$([0.0035, 0.0046], 0.0209)$
e_2	$([0.0228, 0.0777], 0.0072)$	$([0.0533, 0.0876], 0.0002)$
e_3	$([0.0425, 0.0678], 0.0088)$	$([0.0345, 0.0987], 0.0134)$
e_4	$([0.0345, 0.0698], 0.0376)$	$([0.0654, 0.0877], 0.0087)$
e_5	$([0.0445, 0.0868], 0.0098)$	$([0.0234, 0.0843], 0.0018)$

Table 5.5.2 Opinion of expert x_2

x_3	u_1	u_2
e_1	$([0.0543, 0.0846], 0.0097)$	$([0.0472, 0.0546], 0.0287)$
e_2	$([0.0098, 0.0146], 0.0189)$	$([0.0675, 0.0746], 0.0376)$
e_3	$([0.0090, 0.0346], 0.0109)$	$([0.0354, 0.0446], 0.0098)$
e_4	$([0.0037, 0.0046], 0.0018)$	$([0.0375, 0.0846], 0.0187)$
e_5	$([0.0656, 0.0696], 0.0186)$	$([0.0316, 0.0546], 0.0465)$

x_3	u_3	u_4
e_1	$([0.0222, 0.0446], 0.0287)$	$([0.0432, 0.0876], 0.0007)$
e_2	$([0.0356, 0.0946], 0.0765)$	$([0.0156, 0.0276], 0.0087)$
e_3	$([0.0234, 0.0746], 0.0970)$	$([0.0321, 0.0762], 0.0034)$
e_4	$([0.0543, 0.0646], 0.0365)$	$([0.0254, 0.0876], 0.0098)$
e_5	$([0.0541, 0.0648], 0.0876)$	$([0.0765, 0.0954], 0.0365)$

Table 5.5.3 Opinion of expert x_3

Step 3: Calculate the scores of above attributes corresponding to different experts.

x_1	u_1	u_2	u_3	u_4
e_1	0.0289	0.0101	0.0149	0.0261
e_2	0.0357	0.0220	0.0003	−0.0101
e_3	0.0194	0.0183	0.0159	0.0264
e_4	−0.0069	0.0078	0.0248	−0.0086
e_5	0.0485	−0.0038	0.0418	0.0221

Table 5.5.4 Scores of expert x_1

$e_4 = \widetilde{e_5} < e_3 = \widetilde{e_4} < e_1 = \widetilde{e_3} < e_2 = \widetilde{e_2} < e_5 = \widetilde{e_1}$ for u_1,
$e_5 = \widetilde{e_5} < e_4 = \widetilde{e_4} < e_1 = \widetilde{e_3} < e_3 = \widetilde{e_2} < e_2 = \widetilde{e_1}$ for u_2,
$e_2 = \widetilde{e_5} < e_1 = \widetilde{e_4} < e_3 = \widetilde{e_3} < e_4 = \widetilde{e_2} < e_5 = \widetilde{e_1}$ for u_3,
$e_2 = \widetilde{e_5} < e_4 = \widetilde{e_4} < e_5 = \widetilde{e_3} < e_1 = \widetilde{e_2} < e_3 = \widetilde{e_1}$ for u_4.

Table 5.5.5 Arrange attributes according to their scores

x_1	u_1	u_2
$\widetilde{e_1}$	$([0.0765, 0.0946], 0.0256)$	$([0.0289, 0.0646], 0.0276)$
$\widetilde{e_2}$	$([0.0225, 0.0984], 0.0139)$	$([0.0478, 0.0946], 0.0876)$
$\widetilde{e_3}$	$([0.0025, 0.0876], 0.0034)$	$([0.0225, 0.0446], 0.0369)$
$\widetilde{e_4}$	$([0.0350, 0.0498], 0.0265)$	$([0.0376, 0.0846], 0.0987)$
$\widetilde{e_5}$	$([0.0122, 0.0646], 0.0975)$	$([0.0027, 0.0046], 0.0187)$

x_1	u_3	u_4
$\widetilde{e_1}$	$([0.0377, 0.0974], 0.0098)$	$([0.0275, 0.0546], 0.0028)$
$\widetilde{e_2}$	$([0.0277, 0.0765], 0.0298)$	$([0.0525, 0.0646], 0.0387)$
$\widetilde{e_3}$	$([0.0476, 0.0876], 0.0876)$	$([0.0453, 0.0496], 0.0287)$
$\widetilde{e_4}$	$([0.0245, 0.0546], 0.0345)$	$([0.0229, 0.0346], 0.0834)$
$\widetilde{e_5}$	$([0.0287, 0.0486], 0.0765)$	$([0.0238, 0.0446], 0.0987)$

Table 5.5.6 Arranged attributes

x_2	u_1	u_2	u_3	u_4
e_1	-0.0151	0.0298	0.0295	-0.0043
e_2	0.0571	0.0536	0.0311	0.0469
e_3	0.0172	-0.0188	0.0338	0.0399
e_4	0.0101	0.0056	0.0222	0.0481
e_5	0.0271	-0.0048	0.0405	0.0353

Table 5.5.7 Scores of expert x_2

$e_1 = \widetilde{e_5} < e_4 = \widetilde{e_4} < e_3 = \widetilde{e_3} < e_5 = \widetilde{e_2} < e_2 = \widetilde{e_1}$ for u_1,
$e_3 = \widetilde{e_5} < e_5 = \widetilde{e_4} < e_4 = \widetilde{e_3} < e_1 = \widetilde{e_2} < e_2 = \widetilde{e_1}$ for u_2,
$e_4 = \widetilde{e_5} < e_1 = \widetilde{e_4} < e_2 = \widetilde{e_3} < e_3 = \widetilde{e_2} < e_5 = \widetilde{e_1}$ for u_3,
$e_1 = \widetilde{e_5} < e_5 = \widetilde{e_4} < e_3 = \widetilde{e_3} < e_2 = \widetilde{e_2} < e_4 = \widetilde{e_1}$ for u_4.

Table 5.5.8 Arrange attributes according to their scores

x_2	u_1	u_2
$\widetilde{e_1}$	$([0.0987, 0.0994], 0.0267)$	$([0.0987, 0.0996], 0.0376)$
$\widetilde{e_2}$	$([0.0123, 0.0946], 0.0256)$	$([0.0176, 0.0746], 0.0027)$
$\widetilde{e_3}$	$([0.0346, 0.0446], 0.0275)$	$([0.0198, 0.0346], 0.0376)$
$\widetilde{e_4}$	$([0.0543, 0.0746], 0.0987)$	$([0.0196, 0.0646], 0.0986)$
$\widetilde{e_5}$	$([0.0176, 0.0246], 0.0876)$	$([0.0176, 0.0246], 0.0987)$

x_2	u_3	u_4
$\widetilde{e_1}$	$([0.0445, 0.0868], 0.0098)$	$([0.0654, 0.0877], 0.0087)$
$\widetilde{e_2}$	$([0.0425, 0.0678], 0.0088)$	$([0.0533, 0.0876], 0.0002)$
$\widetilde{e_3}$	$([0.0228, 0.0777], 0.0072)$	$([0.0345, 0.0987], 0.0134)$
$\widetilde{e_4}$	$([0.0276, 0.0843], 0.0234)$	$([0.0234, 0.0843], 0.0018)$
$\widetilde{e_5}$	$([0.0345, 0.0698], 0.0376)$	$([0.0035, 0.0046], 0.0209)$

Table 5.5.9 Arranged attributes

x_3	u_1	u_2	u_3	u_4
e_1	0.0431	0.0244	0.0127	0.0434
e_2	0.0018	0.0348	0.0179	0.0115
e_3	0.0109	0.0234	0.0003	0.0350
e_4	0.0022	0.0345	0.0275	0.0344
e_5	0.0389	0.0132	0.0104	0.0451

Table 5.5.10 Scores of expert x_3

$e_2 = \widetilde{e_5} < e_4 = \widetilde{e_4} < e_3 = \widetilde{e_3} < e_5 = \widetilde{e_2} < e_1 = \widetilde{e_1}$ for u_1,
$e_5 = \widetilde{e_5} < e_3 = \widetilde{e_4} < e_1 = \widetilde{e_3} < e_4 = \widetilde{e_2} < e_2 = \widetilde{e_1}$ for u_2,
$e_3 = \widetilde{e_5} < e_5 = \widetilde{e_4} < e_1 = \widetilde{e_3} < e_2 = \widetilde{e_2} < e_4 = \widetilde{e_1}$ for u_3,
$e_2 = \widetilde{e_5} < e_4 = \widetilde{e_4} < e_3 = \widetilde{e_3} < e_1 = \widetilde{e_2} < e_5 = \widetilde{e_1}$ for u_4.

Table 5.5.11 Arrange attributes according to their scores

x_3	u_1	u_2
$\tilde{e_1}$	([0.0543, 0.0846], 0.0097)	([0.0675, 0.0746], 0.0376)
$\tilde{e_2}$	([0.0656, 0.0696], 0.0186)	([0.0375, 0.0846], 0.0187)
$\tilde{e_3}$	([0.0090, 0.0346], 0.0109)	([0.0472, 0.0546], 0.0287)
$\tilde{e_4}$	([0.0037, 0.0046], 0.0018)	([0.0354, 0.0446], 0.0098)
$\tilde{e_5}$	([0.0098, 0.0146], 0.0189)	([0.0316, 0.0546], 0.0465)
x_3	u_3	u_4
$\tilde{e_1}$	([0.0543, 0.0646], 0.0365)	([0.0765, 0.0954], 0.0365)
$\tilde{e_2}$	([0.0356, 0.0946], 0.0765)	([0.0432, 0.0876], 0.0007)
$\tilde{e_3}$	([0.0222, 0.0446], 0.0287)	([0.0321, 0.0762], 0.0034)
$\tilde{e_4}$	([0.0541, 0.0648], 0.0876)	([0.0254, 0.0876], 0.0098)
$\tilde{e_5}$	([0.0234, 0.0746], 0.0970)	([0.0156, 0.0276], 0.0087)

Table 5.5.12 Arranged attributes

Step 4: Assign weights to each attributes $\varpi = (0.28, 0.25, 0.19, 0.16, 0.12)^t$

Step 5: Aggregate the attributes by using cubic soft expert $GOWA$ operator. By taking $r = 2$ we have

$$\tau_{\varpi}^{GO}(\beta_1, \beta_2, ..., \beta_n) = \left(\begin{array}{c} [\sqrt[2]{1 - \prod_{k=1}^{n}(1 - \widetilde{A}_k^{2-})^{\varpi_k}}, \sqrt[2]{1 - \prod_{k=1}^{n}(1 - \widetilde{A}_k^{2+})^{\varpi_k}}], \\ \sqrt[2]{1 - \prod_{k=1}^{n}(1 - \widetilde{\lambda}_k^{2})^{\varpi_k}} \end{array} \right),$$

u_1	u_2
([0.0445, 0.0853], 0.0386)	([0.0335, 0.0702], 0.0632)
u_3	u_4
([0.0348, 0.0796], 0.0510)	([0.0379, 0.0526], 0.0531)

Table 5.5.13 Aggregated values corresponding to expert x_1

u_1	u_2
([0.0592, 0.0796], 0.0547)	([0.0546, 0.0717], 0.0582)
u_3	u_4
([0.0370, 0.0782], 0.0177)	([0.0471, 0.0840], 0.0104)

Table 5.5.14 Aggregated values corresponding to expert x_2

u_1	u_2
([0.0439, 0.0589], 0.0133)	([0.0487, 0.0677], 0.0302)
u_3	u_4
([0.0308, 0.0715], 0.0659)	([0.0493, 0.0830], 0.0199)

Table 5.5.15 Aggregated values corresponding to expert x_3

The $GOWA$ operator has been applied to the CSE element corresponding to the pair (x_1, u_1) as below:

$$([\sqrt[2]{1-(1-0.0765^2)^{0.28}(1-0.0225^2)^{0.25}(1-0.0025^2)^{0.19}(1-0.0350^2)^{0.16}(1-0.0122^2)^{0.12}}$$
$$, \sqrt[2]{1-(1-0.0946^2)^{0.28}(1-0.0984^2)^{0.25}(1-0.0876^2)^{0.19}(1-0.0498^2)^{0.16}(1-0.0646^2)^{0.12}}],$$
$$\sqrt[2]{1-(1-0.0256^2)^{0.28}(1-0.0139^2)^{0.25}(1-0.0034^2)^{0.19}(1-0.0265^2)^{0.16}(1-0.0975^2)^{0.12}})$$
$$= ([0.0445, 0.0853], 0.0386)$$

Similarly rest of the entries can be calculated.

Step 6: Now we calculate scores of all alternatives corresponding to different experts.

	u_1	u_2	u_3	u_4
$x_1 =$	0.0304	0.0135	0.0211	0.0124

Table 5.5.16 Scores corresponding to expert x_1

	u_1	u_2	u_3	u_4
$x_2 =$	0.0280	0.0227	0.0325	0.0402

Table 5.5.17 Scores corresponding to expert x_2

	u_1	u_2	u_3	u_4
$x_3 =$	0.0298	0.0287	0.0121	0.0374

Table 5.5.18 Scores corresponding to expert x_3

Step 7:

u_1	u_2	u_3	u_4
0.0294	0.0216	0.0219	0.0300

Table 5.5.19 Average of alternatives

Step 8: Arrange these alternatives in ascending order.

$$u_2 < u_3 < u_1 < u_4$$

Step 9: Hence u_4 is the best. Mr. A chooses u_4.

5.6 Multicriteria Decision Making of Cubic Soft Expert Sets with Cubic Soft Expert OWA Operator

Consider Example 5.5.1. Proceed to step 4 as in the above example.

Step 5: Aggregate the attributes by using cubic soft expert OWA operator.

$$\tau_{\varpi}^{O}(\beta_1,\beta_2,...,\beta_n)=([1-\prod_{k=1}^{n}(1-\widetilde{A}_k^-)^{\varpi_k},1-\prod_{k=1}^{n}(1-\widetilde{A}_k^+)^{\varpi_k}],1-\prod_{k=1}^{n}(1-\widetilde{\lambda}_k)^{\varpi_k}).$$

u_1	u_2
$([0.0349, 0.0836], 0.0276)$	$([0.0307, 0.0647], 0.0552)$
u_3	u_4
$([0.0339, 0.0777], 0.0420)$	$([0.0360, 0.0518], 0.0416)$

Table 5.6.1 Aggregated values corresponding to expert x_1

u_1	u_2
$([0.0487, 0.0752], 0.0459)$	$([0.0417, 0.0667], 0.0466)$
u_3	u_4
$([0.0360, 0.0779], 0.0146)$	$([0.0425, 0.0796], 0.0078)$

Table 5.6.2 Aggregated values corresponding to expert x_2

u_1	u_2
$([0.0354, 0.0506], 0.0120)$	$([0.0467, 0.0662], 0.0278)$
u_3	u_4
$([0.0398, 0.0696], 0.0608)$	$([0.0445, 0.0806], 0.0137)$

Table 5.6.3 Aggregated values corresponding to expert x_3

Step 6: Now we calculate scores of all alternatives corresponding to different experts.

	u_1	u_2	u_3	u_4
$x_1 =$	0.0487	0.0502	0.0512	0.0431

Table 5.6.4 Scores corresponding to expert x_1

	u_1	u_2	u_3	u_4	
$x_2 =$	0.0566	0.0516	0.0428	0.0433	

Table 5.6.5 Scores corresponding to expert x_2

	u_1	u_2	u_3	u_4	
$x_1 =$	0.0326	0.0469	0.0567	0.0462	

Table 5.6.6 Scores corresponding to expert x_3

Step 7:

u_1	u_2	u_3	u_4
0.0459	0.0495	0.0502	0.0442

Table 5.6.7 Average of alternatives

Step 8: Arrange these alternatives in ascending order.

$$u_4 < u_1 < u_2 < u_3$$

Step 9: Hence u_3 is the best.

5.7 Multicriteria Decision Making of Cubic Soft Expert Sets with Cubic Soft Expert Weighted Average Operator

Let $U = \{u_1, u_2, ...u_n\}$ be the set of alternatives, $E = \{e_1, e_2, ..., e_l\}$ be the set of attributes and $X = \{x_1, x_2, ..., x_m\}$ be the set of experts.

Further, we take opinion of experts in the form of CSE elements. Algorithm for cubic soft expert weighted average operator is given below:

Step 1: Utilize the evaluations of experts in the form of CSE sets.

Step 2: Separate the opinion of each expert.

Step 3: Assign weights to each attribute.

Step 4: Aggregate the attributes by using cubic soft expert weighted average operator.

Step 5: Calculate accuracies of all alternatives corresponding to different experts.

Step 6: Find the average of these alternatives.

Step 7: Arrange these alternatives in ascending order.

Step 8: Choose best alternative.

Consider Example 5.5.1 proceed to **step 2** as in above example.

Step 3: Assign weights to each attributes $\varpi = (0.28, 0.25, 0.19, 0.16, 0.12)^t$

Step 4: Aggregate the attributes by using cubic soft expert weighted average operator

$$\tau_w^{\beta_i}(\beta_1, \beta_2, ..., \beta_n) = ([1 - \prod_{i=1}^{n}(1 - A_i^-)^{w_i}, 1 - \prod_{i=1}^{n}(1 - A_i^+)^{w_i}], 1 - \prod_{i=1}^{n}(1 - \lambda_i)^{w_i})$$

u_1	u_2
$([0.0138, 0.0778], 0.0155)$	$([0.0232, 0.0476], 0.0436)$
u_3	u_4
$([0.0310, 0.0656], 0.0422)$	$([0.0327, 0.0500], 0.0323)$

Table 5.7.1 Aggregated values corresponding to expert x_1

u_1	u_2
$([0.0353, 0.0548], 0.0459)$	$([0.0279, 0.0564], 0.0242)$
u_3	u_4
$([0.0313, 0.0771], 0.0140)$	$([0.0214, 0.0390], 0.0038)$

Table 5.7.2 Aggregated values corresponding to expert x_2

u_1	u_2
$([0.0167, 0.0282], 0.0096)$	$([0.0448, 0.0609], 0.0247)$
u_3	u_4
$([0.0324, 0.0658], 0.0549)$	$([0.0311, 0.0645], 0.0043)$

Table 5.7.3 Aggregated values of alternatives corresponding to expert x_3

Step 5: Accuracies of all alternatives corresponding to different experts.

	u_1	u_2	u_3	u_4
$x_1 =$	0.0357	0.0381	0.0462	0.0383

Table 5.7.4 Accuracies corresponding to expert x_1

$x_2 =$	u_1	u_2	u_3	u_4	
	0.0453	0.0361	0.0408	0.0214	

Table 5.7.5 Accuracies corresponding to expert x_2

$x_1 =$	u_1	u_2	u_3	u_4	
	0.0181	0.0434	0.0510	0.333	

Table 5.7.6 Accuracies corresponding to expert x_3

Step 6:

u_1	u_2	u_3	u_4
0.0330	0.0392	0.046	0.031

Table 5.7.7 Average of alternatives

Step 7: Arrange these alternatives in ascending order.

$$u_4 < u_1 < u_2 < u_3$$

Step 8: Hence u_3 is the best.

5.8 Multicriteria decision Making of Cubic Soft Expert Sets with Cubic Soft Expert Weighted Geometric Operator

Let $U = \{u_1, u_2, ...u_n\}$ be the set of alternatives, $E = \{e_1, e_2, ..., e_l\}$ be the set of attributes and $X = \{x_1, x_2, ..., x_m\}$ be the set of experts.

Further, we take opinion of experts in the form of CSE elements. Algorithm for cubic soft expert weighted average operator is given below:

Step 1: Utilize the evaluations of experts in the form of CSE sets.

Step 2: Separate the opinion of each expert.

Step 3: Assign weights to each attribute.

Step 4: Aggregate the attributes by using cubic soft expert weighted geometric operator.

Step 5: Calculate accuracies of all alternatives corresponding to different experts.

Step 6: Find the average of these alternatives.

Step 7: Arrange these alternatives in ascending order.

Step 8: Choose the best alternative.

Consider Example 5.5.1 proceed to **step 2** as in above example.

Step 3: Assign weights to each attributes $\varpi = (0.28, 0.25, 0.19, 0.16, 0.12)^t$

Step 4: Aggregate the attributes by using cubic soft expert weighted geometric operator.

$$\tau_w^{GEO}(\beta_1, \beta_2, ..., \beta_n) = ([\prod_{i=1}^{n}(A_i^-)^{w_i}, \prod_{i=1}^{n}(A_i^+)^{w_i}], \prod_{i=1}^{n}(\lambda_i)^{w_i})$$

u_1	u_2
$([0.0138, 0.0778], 0.0155)$	$([0.0232, 0.0476], 0.0436)$
u_3	u_4
$([0.0310, 0.0656], 0.0422)$	$([0.0327, 0.0500], 0.0323)$

Table 5.8.1 Aggregated values corresponding to expert x_1

u_1	u_2
$([0.0353, 0.0548], 0.0459)$	$([0.0279, 0.0564], 0.0242)$
u_3	u_4
$([0.0313, 0.0771], 0.0140)$	$([0.0214, 0.0390], 0.0038)$

Table 5.8.2 Aggregated values corresponding to expert x_2

u_1	u_2
$([0.0167, 0.0282], 0.0096)$	$([0.0448, 0.0609], 0.0247)$
u_3	u_4
$([0.0324, 0.0658], 0.0549)$	$([0.0311, 0.0645], 0.0043)$

Table 5.8.3 Aggregated values corresponding to expert x_3

Step 5: Accuracies of all alternatives corresponding to different experts.

	u_1	u_2	u_3	u_4
$x_1 =$	0.0357	0.0381	0.0462	0.0465

Table 5.8.4 Accuracies corresponding to expert x_1

	u_1	u_2	u_3	u_4
$x_2 =$	0.0453	0.0361	0.0408	0.0214

Table 5.8.5 Accuracies corresponding to expert x_2

$x_3 =$	u_1	u_2	u_3	u_4	
	0.0181	0.0434	0.0510	0.0333	

Table 5.8.6 Accuracies corresponding to expert x_3

Step 6:

u_1	u_2	u_3	u_4
0.0330	0.0392	0.046	0.0337

Table 5.8.7 Average of alternatives

Step 7: Choose the best alternative.

$$u_1 < u_4 < u_2 < u_3$$

Step 8: Hence u_3 is the best.

5.9 Conclusion and Future Work

Cubic sets are defined by Jun et al. [34]. Jun et al. defined basic operations of inclusion, union and intersection. There are certain operations which were not defined in the said paper. In this chapter we have introduced some new operations such as addition and multiplication of two $CSESs$, product of a $CSESs$ with real number $k > 0$, power of $CSESs$, score and accuracy function of $CSES$. The purpose of defining score function and accuracy function is that we can determine the ranking of $CSESs$ which helps us in aggregation. Some aggregation operators on $CSESs$ have introduced. Therefore, the aim of this chapter is to determine the most preferable choice among all possible choices, when data is presented in cubic set form. At the end, an algorithm has been presented along with an illustrative example. In this example we have used U of $GOWA$ operator for $CSESs$. In future we aim to study $TOPSIS$, AHP and ANP for group decision making with $CSESs$. We also aim at defining similarity and distance measures and distances and similarity degrees between $CSESs$.

Chapter 6

Aggregation Operators of Interval Valued Intuitionistic Fuzzy Soft Expert Sets ($IVIFSE$ $sets$)

6.1 Introduction

The ordered weighted geometric averaging operator was introduced by Xu [73]. Yager introduced the ordered weighted averaging operator [81]. Yager provides a parameterized family of aggregation operators which have been used in many applications in [79]. Yager provides a generalization of OWA operator by combining it with the generalized mean operator [22, 80]. This combination leads to a class of operators which is reffered to as the generalized ordered weighted averaging ($GOWA$) operators [82]. Li developed a new methodology for solving multiple attribute group decision making problems using intuitionistic fuzzy sets in which multiple attributes are explicitly considered [42]. Xu introduced different approaches to group decision making [74, 75, 76]. Szmidt proposed some solution concepts in group decision making with intuitionistic fuzzy preference relations, such as intuitionistic fuzzy core and consensus winner and also investigated the consensus-reaching process in group decision making based on individual intuitionistic fuzzy preference relations in [62, 63].

This chapter consists of definition of $IVIFSE$ $sets$, null $IVIFSE$ set, absolute $IVIFSE$ set and some of the operations such as containment of two elements of $IVIFSE$ $sets$, equality of two elements of $IVIFSE$ $sets$, subsets, equality of two $IVIFSE$ $sets$, complement of $IVIFSE$ set, addition, product, union and intersection of two *IVIFSE sets*, product of scalar with *IVIFSE set*, power of *IVIFSE set*, score and accuracy function of *IVIFSE sets*. Also some of the aggregation operators are presented. Further we introduce the multiple attribute decision making problem with *IVIFSE sets* by using *IVIFSE* ordered weighted arithmetic operator. An illustrative example is also presented.

6.2 Interval Valued Intuitionistic Fuzzy Soft Expert Sets (*IVIFSE sets*)

Definition 6.2.1 *Let U be the initial universe, $Ą$ be the set of attributes and $Ģ$ be the set of experts. Interval valued intuitionistic fuzzy soft expert set (IVIFSE set) is a triplet $(\xi, Ą, Ģ)$ which is characterized by a mapping $\xi : Ą \times Ģ \longrightarrow Ķ_I(U)$ where the set of the interval-valued intuitionistic fuzzy sets on the universe set U is denoted by $Ķ_I(U)$. For $b \in Ą$ and $p \in Ģ$ we define*

$$\xi(b,p) = \{< u,\ [\gamma^-_{(b,p)}(u), \gamma^+_{(b,p)}(u)], [\zeta^-_{(b,p)}(u), \zeta^+_{(b,p)}(u)] >: u \in U\}.$$

Example 6.2.2 *Suppose that there are four cars in the universe set $U = \{u_1, u_2, u_3, u_4\}$. $Ą = \{b_1 = cheap,\ b_2 = model,\ b_3 = \text{exp}ensive\}$ be the set of attributes and $Ģ = \{p_1, p_2\}$ be the set of experts. Then we can view the IVIFSE Set $(\xi, Ą, Ģ)$ as consisting of opinions of experts on the cars subject to the given attributes following collection of approximations:*

$\xi(b_1,p_1) = \{< u_1, [0.4,0.5],[0.2,0.4] >, < u_2, [0.1,0.5],[0.4,0.5] >, < u_3, [0.3,0.4],[0.4,0.5] >, < u_4, [0.2,0.4],[0.5,0.6] >\}$,

$\xi(b_2,p_1) = \{< u_1, [0.2,0.4],[0.5,0.6] >, < u_2, [0.2,0.4],[0.4,0.5] >, < u_3, [0.1,0.3],[0.3,0.5] >, < u_4, [0.3,0.4],[0.2,0.5] >\}$,

$\xi(b_3,p_1) = \{< u_1, [0.1,0.2],[0.3,0.5] >, < u_2, [0.3,0.5],[0.1,0.5] >, < u_3, [0.3,0.4],[0.2,0.4] >, < u_4, [0.0,0.2],[0.3,0.5] >\}$,

$\xi(b_1,p_2) = \{< u_1, [0.4,0.7],[0.2,0.3] >, < u_2, [0.2,0.5],[0.2,0.3] >, < u_3, [0.3,0.5],[0.4,0.5] >, < u_4, [0.1,0.3],[0.2,0.5] >\}$,

$\xi(b_2,p_2) = \{< u_1, [0.3,0.5],[0.0,0.3] >, < u_2, [0.1,0.4],[0.3,0.6] >, < u_3, [0.2,0.5],[0.3,0.4] >, < u_4, [0.2,0.3],[0.3,0.5] >\}$,

$\xi(b_3,p_2) = \{< u_1, [0.2,0.5],[0.0,0.3] >, < u_2, [0.1,0.3],[0.5,0.6] >, < u_3, [0.2,0.4],[0.1,0.4] >, < u_4, [0.2,0.4],[0.2,0.5] >\}$.

Definition 6.2.3 *The null IVIFSE set over U is denoted by $Þ$ and defined as*

$$[\gamma^-_{(b,p)}(u), \gamma^+_{(b,p)}(u)] = [1,1], [\zeta^-_{(b,p)}(u), \zeta^+_{(b,p)}(u)] = [0,0],$$

for all $b \in Ą$, $p \in Ģ$ and $u \in U$.

Definition 6.2.4 *The absolute IVIFSE set over U is denoted by Λ and defined as*

$$[\gamma^-_{(b,p)}(u), \gamma^+_{(b,p)}(u)] = [0,0], [\zeta^-_{(b,p)}(u), \zeta^+_{(b,p)}(u)] = [1,1],$$
$$\text{for all } b \in A,\ p \in G \text{ and } u \in U.$$

Definition 6.2.5 *For an IVIFSE set (ξ, A, G) over U and for any $b_1, b_2 \in A$, $p_1, p_2 \in G$, an element $\xi(b_1,p_1)$ is said to be contained in $\xi(b_2,p_2)$, denoted by $\xi(b_1,p_1) \subseteq \xi(b_2,p_2)$ if the following conditions are satisfied:*

$$1)\ \gamma^-_{(b_1,p_1)}(u) \le \gamma^-_{(b_2,p_2)}(u),\ \gamma^+_{(b_1,p_1)}(u) \le \gamma^+_{(b_2,p_2)}(u),$$
$$2)\ \zeta^-_{(b_1,p_1)}(u) \ge \zeta^-_{(b_2,p_2)}(u),\ \zeta^+_{(b_1,p_1)}(u) \ge \zeta^+_{(b_2,p_2)}(u),$$

where $\xi(b_1,p_1) = \{< u,\ [\gamma^-_{(b_1,p_1)}(u), \gamma^+_{(b_1,p_1)}(u)], [\zeta^-_{(b_1,p_1)}(u), \zeta^+_{(b_1,p_1)}(u)] >: u \in U\}$
and $\xi(b_2,p_2) = \{< u,\ [\gamma^-_{(b_2,p_2)}(u), \gamma^+_{(b_2,p_2)}(u)], [\zeta^-_{(b_2,p_2)}(u), \zeta^+_{(b_2,p_2)}(u)] >: u \in U\}$.

Example 6.2.6 *Consider* $\xi(b_1,p_1) = \{< u_1, [0.2,0.4], [0.2,0.4] >, < u_2, [0.1,0.3], [0.4,0.5] >, < u_3, [0.1,0.3], [0.3,0.5] >$, $\xi(b_2,p_2) = \{< u_1, [0.4,0.5], [0.3,0.5] >, < u_2, [0.2,0.4], [0.4,0.5] >, < u_3, [0.2,0.4], [0.4,0.5] >$. *Clearly* $\xi(b_1,p_1) \subseteq \xi(b_2,p_2)$.

Definition 6.2.7 *For an IVIFSE set (ξ, A, G) over U, for any $b_1, b_2 \in A$, $p_1, p_2 \in G$, an element $\xi(b_1,p_1)$ is said to be equal to $\xi(b_2,p_2)$, denoted by $\xi(b_1,p_1) = \xi(b_2,p_2)$ if following conditions are satisfied:*

$$1)\ \gamma^-_{(b_1,p_1)}(u) = \gamma^-_{(b_2,p_2)}(u),\ \gamma^+_{(b_1,p_1)}(u) = \gamma^+_{(b_2,p_2)}(u),$$
$$2)\ \zeta^-_{(b_1,p_1)}(u) = \zeta^-_{(b_2,p_2)}(u),\ \zeta^+_{(b_1,p_1)}(u) = \zeta^+_{(b_2,p_2)}(u).$$

Definition 6.2.8 *For two IVIFSE sets (ξ_1, A_1, G_1) and (ξ_2, A_2, G_2) over U, (ξ_1, A_1, G_1) is a subset of (ξ_2, A_2, G_2) if following conditions are satisfied:*

1) $A_1 \subseteq A_2$,
2) $G_1 \subseteq G_2$,
3) $\xi_1(b,p) \subseteq \xi_2(b,p)$ *for all* $b \in A_1$ *and* $p \in G_1$,

where $\xi_1(b,p) = \{< u,\ [\gamma^-_{1(b,p)}(u), \gamma^+_{1(b,p)}(u)], [\zeta^-_{1(b,p)}(u), \zeta^+_{1(b,p)}(u)] >: u \in U\}$
and $\xi_2(b,p) = \{< u,\ [\gamma^-_{2(b,p)}(u), \gamma^+_{2(b,p)}(u)], [\zeta^-_{2(b,p)}(u), \zeta^+_{2(b,p)}(u)] >: u \in U\}$.

Example 6.2.9 *Let* $U = \{u_1, u_2, u_3\}$. $A_1 = \{b_1,\ b_2\}$ $G_1 = \{p_1\}$ *then IVIFSE Set* (ξ_1, A_1, G_1) *is given by*

$\xi_1(b_1,\ p_1) = \{< u_1, [0.1,0.4], [0.2,0.4] >, < u_2, [0.1,0.3], [0.4,0.5] >, < u_3, [0.1,0.3], [0.3,0.5] >\}$,

$\xi_1(b_2,\ p_1) = \{< u_1,\ [0.2, 0.3],\ [0.4, 0.5] >,\ < u_2,\ [0.1, 0.2],\ [0.2, 0.4] >,\ < u_2,\ [0.2, 0.5],\ [0.1, 0.3] >\}$.

Also $A_2 = \{b_1,\ b_2,\ b_3\}$ $G_2 = \{p_1.p_2\}$ then $IVIFSE$ Set (ξ_2, A_2, G_2) is given by

$\xi_2(b_1,\ p_1) = \{< u_1,\ [0.2, 0.5],\ [0.3, 0.5] >,\ < u_2,\ [0.2, 0.4],\ [0.4, 0.5] >,\ < u_3,\ [0.2, 0.5],\ [0.4, 0.5] >\}$,

$\xi_2(b_2,\ p_1) = \{< u_1,\ [0.2, 0.4],\ [0.5, 0.6] >,\ < u_2,\ [0.3, 0.5],\ [0.4, 0.5] >,\ < u_2,\ [0.4, 0.5],\ [0.4, 0.5] >\}$,

$\xi_2(b_3,\ p_1) = \{< u_1,\ [0.1, 0.3],\ [0.2, 0.4] >,\ < u_2,\ [0.4, 0.5],\ [0.3, 0.4] >,\ < u_3,\ [0.2, 0.3],\ [0.3, 0.5] >\}$,

$\xi_2(b_1,\ p_2) = \{< u_1,\ [0.8, 0.9],\ [0.0, 0.1] >,\ < u_2,\ [0.5, 0.7],\ [0.2, 0.3] >,\ < u_2,\ [0.1, 0.3],\ [0.4, 0.5] >\}$,

$\xi_2(b_2,\ p_2) = \{< u_1,\ [0.1, 0.4],\ [0.2, 0.4] >,\ < u_2,\ [0.1, 0.3],\ [0.4, 0.5] >,\ < u_3,\ [0.1, 0.2],\ [0.4, 0.6] >\}$,

$\xi_2(b_3,\ p_2) = \{< u_1,\ [0.2, 0.4],\ [0.3, 0.5] >,\ < u_2,\ [0.5, 0.7],\ [0.2, 0.3] >,\ < u_2,\ [0.2, 0.5],\ [0.1, 0.3] >\}$.

Clearly $A_1 \subseteq A_2$, $G_1 \subseteq G_2$ and $\xi_1(b,p) \subseteq \xi_2(b,p)$ for all $b \in A_1$ and $p \in G_1$. Hence (ξ_1, A_1, G_1) is a subset of (ξ_2, A_2, G_2).

Definition 6.2.10 *For two $IVIFSE$ sets (ξ_1, A_1, G_1) and (ξ_2, A_2, G_2) over U, (ξ_1, A_1, G_1) is equal to (ξ_2, A_2, G_2) if following conditions are satisfied:*

1) $A_1 = A_2$,
2) $G_1 = G_2$,
3) $\xi_1(b,p) = \xi_2(b,p)$ *for all* $b \in A_1$ *and* $p \in G_1$.

Definition 6.2.11 *The complement of $IVIFSE$ set (ξ, A, G) is denoted by $(\xi, A, G)^c$, and is defined for all $b \in A$ and $p \in G$ as follows:*

$$(\xi,\ A,\ G)^c = \{< u,\ [\zeta^-_{(b^c,p)}(u), \zeta^+_{(b^c,p)}(u)], [\gamma^-_{(b^c,p)}(u), \gamma^+_{(b^c,p)}(u)] >: u \in U\},$$

where $\xi(b,p) = \{< u,\ [\gamma^-_{(b,p)}(u), \gamma^+_{(b,p)}(u)], [\zeta^-_{(b,p)}(u), \zeta^+_{(b,p)}(u)] >: u \in U\}$.

Example 6.2.12 *Consider Example 6.2.9, $(\xi_1, A_1, G_1)^c$ is given by*

$\xi_1(b_1^c,\ p_1) = \{< u_1,\ [0.2, 0.4],\ [0.1, 0.4] >,\ < u_2,\ [0.4, 0.5],\ [0.1, 0.3] >,\ < u_3,\ [0.3, 0.5],\ [0.1, 0.3] >\}$,

$\xi_1(b_2^c,\ p_1) = \{< u_1,\ [0.4, 0.5],\ [0.2, 0.3] >,\ < u_2,\ [0.2, 0.4],\ [0.1, 0.2] >,\ < u_2,\ [0.1, 0.3],\ [0.2, 0.5] >\}$.

Definition 6.2.13 *For two IVIFSE sets* (ξ_1, A_1, G_1) *and* (ξ_2, A_2, G_2) *over* U, *union of two IVIFSE sets is denoted as* $(\xi_3, A_3, G_3) = (\xi_1, A_1, G_1) \cup (\xi_2, A_2, G_2)$ *where* $A_3 = A_1 \cup A_2$ *and* $G_3 = G_1 \cup G_2$ *and for all* $\ell \in A_3$ *and* $m \in G_3$ *it is defined as follows:*

$$\xi_3(\ell,m) = \begin{cases} \xi_1(\ell,m) & \text{if } (\ell,m) \in (A_1 \times G_1) \setminus (A_2 \times G_2) \\ \xi_2(\ell,m) & \text{if } (\ell,m) \in (A_2 \times G_2) \setminus (A_1 \times G_1) \\ \{< u,\ [\gamma^-_{1(\ell,m)}(u) \vee \gamma^-_{2(\ell,m)}(u), \\ \quad \gamma^+_{1(\ell,m)}(u) \vee \gamma^+_{2(\ell,m)}(u)], & \text{if } (\ell,m) \in (A_1 \cap A_2 \times G_1 \cap G_2) \\ \quad [\zeta^-_{1(\ell,m)}(u) \wedge \zeta^-_{2(\ell,m)}(u), \\ \quad \zeta^+_{1(\ell,m)}(u) \wedge \zeta^+_{2(\ell,m)}(u)] >\}, \end{cases}$$

where $\xi_1(\ell,m) = \{< u,\ [\gamma^-_{1(\ell,m)}(u), \gamma^+_{1(\ell,m)}(u)], [\zeta^-_{1(\ell,m)}(u), \zeta^+_{1(\ell,m)}(u)] >: u \in U\}$
and $\xi_2(\ell,m) = \{< u,\ [\gamma^-_{2(\ell,m)}(u), \gamma^+_{2(\ell,m)}(u)], [\zeta^-_{2(\ell,m)}(u), \zeta^+_{2(\ell,m)}(u)] >: u \in U\}$.

Example 6.2.14 *Let* $U = \{u_1, u_2, u_3\}$, $A_1 = \{b_1,\ b_2\}$ *and* $G_1 = \{p_1,\ p_2\}$ *then IVIFSE set* (ξ_1, A_1, G_1) *is given by*

$\xi_1(b_1,\ p_1) = \{< u_1,\ [0.1, 0.4],\ [0.2, 0.4] >, < u_2,\ [0.1, 0.3],\ [0.4, 0.5] >, < u_3,\ [0.1, 0.3],\ [0.3, 0.5] >\}$,

$\xi_1(b_2,\ p_1) = \{< u_1,\ [0.2, 0.3],\ [0.4, 0.5] >, < u_2,\ [0.1, 0.2],\ [0.2, 0.4] >, < u_2,\ [0.2, 0.5],\ [0.1, 0.3] >\}$,

$\xi_1(b_1,\ p_2) = \{< u_1,\ [0.6, 0.9],\ [0.0, 0.1] >, < u_2,\ [0.4, 0.6],\ [0.2, 0.3] >, < u_2,\ [0.3, 0.5],\ [0.2, 0.5] >\}$,

$\xi_1(b_2,\ p_2) = \{< u_1,\ [0.3, 0.4],\ [0.4, 0.5] >, < u_2,\ [0.2, 0.5],\ [0.1, 0.5] >, < u_3,\ [0.3, 0.7],\ [0.2, 0.3] >\}$,

Also if $A_2 = \{b_1,\ b_2,\ b_3\}$, $G_2 = \{p_1, p_2\}$ then *IVIFSE set* (ξ_2, A_2, G_2) is given by

$\xi_2(b_1,\ p_1) = \{< u_1,\ [0.2, 0.5],\ [0.3, 0.5] >, < u_2,\ [0.2, 0.4],\ [0.4, 0.5] >, < u_3,\ [0.2, 0.5],\ [0.4, 0.5] >\}$,

$\xi_2(b_2,\ p_1) = \{< u_1,\ [0.2, 0.4],\ [0.5, 0.6] >, < u_2,\ [0.3, 0.5],\ [0.4, 0.5] >, < u_2,\ [0.4, 0.5],\ [0.4, 0.5] >\}$,

$\xi_2(b_3,\ p_1) = \{< u_1,\ [0.1, 0.3],\ [0.2, 0.4] >, < u_2,\ [0.4, 0.5],\ [0.3, 0.4] >, < u_3,\ [0.2, 0.3],\ [0.3, 0.5] >\}$,

$\xi_2(b_1,\ p_2) = \{< u_1,\ [0.8, 0.9],\ [0.0, 0.1] >, < u_2,\ [0.5, 0.7],\ [0.2, 0.3] >, < u_2,\ [0.1, 0.3],\ [0.4, 0.5] >\}$,

$\xi_2(b_2,\ p_2) = \{< u_1,\ [0.1, 0.4],\ [0.2, 0.4] >, < u_2,\ [0.1, 0.3],\ [0.4, 0.5] >, < u_3,\ [0.1, 0.2],\ [0.4, 0.6] >\}$,

$\xi_2(b_3,\ p_2) = \{< u_1,\ [0.2, 0.4],\ [0.3, 0.5] >, < u_2,\ [0.5, 0.7],\ [0.2, 0.3] >, < u_2,\ [0.2, 0.5],\ [0.1, 0.3] >\}$.

Now, $(\xi_3, A_3, G_3) = (\xi_1, A_1, G_1) \cup (\xi_2, A_2, G_2)$ is given by

$\xi_3(b_1,\ p_1) = \{< u_1,\ [0.2, 0.5],\ [0.2, 0.4] >\ ,\ < u_2,\ [0.2, 0.4],\ [0.4, 0.5] >\ ,\ < u_3,\ [0.2, 0.5],\ [0.3, 0.5] >\}$,

$\xi_3(b_2,\ p_1) = \{< u_1,\ [0.2, 0.4],\ [0.4, 0.5] >\ ,\ < u_2,\ [0.3, 0.5],\ [0.2, 0.4] >\ ,\ < u_2,\ [0.4, 0.5],\ [0.1, 0.3] >\}$,

$\xi_3(b_3,\ p_1) = \{< u_1,\ [0.1, 0.3],\ [0.2, 0.4] >\ ,\ < u_2,\ [0.4, 0.5],\ [0.3, 0.4] >\ ,\ < u_3,\ [0.2, 0.3],\ [0.3, 0.5] >\}$,

$\xi_3(b_1,\ p_2) = \{< u_1,\ [0.8, 0.9],\ [0.0, 0.1] >\ ,\ < u_2,\ [0.5, 0.7],\ [0.2, 0.3] >\ ,\ < u_2,\ [0.3, 0.5],\ [0.2, 0.5] >\}$,

$\xi_3(b_2,\ p_2) = \{< u_1,\ [0.3, 0.4],\ [0.2, 0.4] >\ ,\ < u_2,\ [0.2, 0.5],\ [0.1, 0.5] >\ ,\ < u_3,\ [0.3, 0.7],\ [0.2, 0.3] >\}$,

$\xi_3(b_3,\ p_2) = \{< u_1,\ [0.2, 0.4],\ [0.3, 0.5] >\ ,\ < u_2,\ [0.5, 0.7],\ [0.2, 0.3] >\ ,\ < u_2,\ [0.2, 0.5],\ [0.1, 0.3] >\}$.

Definition 6.2.15 *For two IVIFSE sets* (ξ_1, A_1, G_1) *and* (ξ_2, A_2, G_2) *over* U, *intersection of two IVIFSE sets is denoted as* $(\xi_3, A_3, G_3) = (\xi_1, A_1, G_1) \cap (\xi_2, A_2, G_2)$ *where* $A_3 = A_1 \cup A_2$ *and* $G_3 = G_1 \cup G_2$ *and for all* $\ell \in A_3$ *and* $m \in G_3$, *it is defined as below:*

$$\xi_3(\ell, m) = \begin{cases} \xi_1(\ell, m) & \text{if } (\ell, m) \in (A_1 \times G_1) \setminus (A_2 \times G_2) \\ \xi_2(\ell, m) & \text{if } (\ell, m) \in (A_2 \times G_2) \setminus (A_1 \times G_1) \\ \{< u,\ [\gamma^-_{1(\ell,m)}(u) \wedge \gamma^-_{2(\ell,m)}(u), \\ \quad \gamma^+_{1(\ell,m)}(u) \wedge \gamma^+_{2(\ell,m)}(u)], & \text{if } (\ell, m) \in (A_1 \cap A_2 \times G_1 \cap G_2) \\ \quad [\zeta^-_{1(\ell,m)}(u) \vee \zeta^-_{2(\ell,m)}(u), \\ \quad \zeta^+_{1(\ell,m)}(u) \vee \zeta^+_{2(\ell,m)}(u)] >\}, \end{cases}$$

where $\xi_1(\ell, m) = \{< u,\ [\gamma^-_{1(\ell,m)}(u), \gamma^+_{1(\ell,m)}(u)], [\zeta^-_{1(\ell,m)}(u), \zeta^+_{1(\ell,m)}(u)] >: u \in U\}$
and $\xi_2(\ell, m) = \{< u,\ [\gamma^-_{2(\ell,m)}(u), \gamma^+_{2(\ell,m)}(u)], [\zeta^-_{2(\ell,m)}(u), \zeta^+_{2(\ell,m)}(u)] >: u \in U\}$.

Example 6.2.16 *Consider* (ξ_1, A_1, G_1) *and* (ξ_2, A_2, G_2) *of Example 6.2.14. Now* $(\xi_3, A_3, G_3) = (\xi_1, A_1, G_1) \cap (\xi_2, A_2, G_2)$ *is given by*

$\xi_3(b_1,\ p_1) = \{< u_1,\ [0.1, 0.4],\ [0.3, 0.5] >\ ,\ < u_2,\ [0.1, 0.3],\ [0.4, 0.5] >\ ,\ < u_3,\ [0.1, 0.3],\ [0.4, 0.5] >\}$,

$\xi_3(b_2,\ p_1) = \{< u_1,\ [0.2, 0.3],\ [0.5, 0.6] >\ ,\ < u_2,\ [0.1, 0.2],\ [0.4, 0.5] >\ ,\ < u_2,\ [0.2, 0.5],\ [0.4, 0.5] >\}$,

$\xi_3(b_3,\ p_1) = \{< u_1,\ [0.1, 0.3],\ [0.2, 0.4] >\ ,\ < u_2,\ [0.4, 0.5],\ [0.3, 0.4] >\ ,\ < u_3,\ [0.2, 0.3],\ [0.3, 0.5] >\}$,

$\xi_3(b_1,\ p_2) = \{< u_1,\ [0.6, 0.9],\ [0.0, 0.1] >\ ,\ < u_2,\ [0.4, 0.6],\ [0.2, 0.3] >\ ,\ < u_2,\ [0.1, 0.3],\ [0.4, 0.5] >\}$,

$\xi_3(b_2,\ p_2) = \{< u_1,\ [0.3, 0.4],\ [0.2, 0.4] >\ ,\ < u_2,\ [0.1, 0.3],\ [0.4, 0.5] >\ ,\ < u_3,\ [0.1, 0.2],\ [0.4, 0.6] >\}$,

$\xi_3(b_3,\ p_2) = \{< u_1,\ [0.1, 0.4],\ [0.4, 0.5] >\ ,\ < u_2,\ [0.5, 0.7],\ [0.2, 0.3] >\ ,\ < u_2,\ [0.2, 0.5],\ [0.1, 0.3] >\}$.

Definition 6.2.17 *The sum of two $IVIFSE$ sets (ξ_1, A_1, G_1) and (ξ_2, A_2, G_2) over U is denoted as $(\xi_1, A_1, G_1) + (\xi_2, A_2, G_2)$ and defined as:*

$$\xi_1(\ell,m)+\xi_2(\ell,m) = \begin{cases} \{< u,\ [\gamma^-_{1(\ell,m)}(u)+\gamma^-_{2(\ell,m)}(u)-\gamma^-_{1(\ell,m)}(u)\gamma^-_{2(\ell,m)}(u), \\ \gamma^+_{1(\ell,m)}(u)+\gamma^+_{2(\ell,m)}(u)-\gamma^+_{1(\ell,m)}(u)\gamma^+_{2(\ell,m)}(u)], \\ [\zeta^-_{1(\ell,m)}(u)\zeta^-_{2(\ell,m)}(u), \zeta^+_{1(\ell,m)}(u)\zeta^+_{2(\ell,m)}(u)] >\}, \end{cases}$$

where $\xi_1(\ell,m) = \{< u,\ [\gamma^-_{1(\ell,m)}(u), \gamma^+_{1(\ell,m)}(u)], [\zeta^-_{1(\ell,m)}(u), \zeta^+_{1(\ell,m)}(u)] >: u \in U\}$
and $\xi_2(\ell,m) = \{< u,\ [\gamma^-_{2(\ell,m)}(u), \gamma^+_{2(\ell,m)}(u)], [\zeta^-_{2(\ell,m)}(u), \zeta^+_{2(\ell,m)}(u)] >: u \in U\}$.

Example 6.2.18 *Consider* $\xi_1(b_1,\ p_1) = \{< u_1,\ [0.1, 0.4],\ [0.2, 0.4] >\ ,\ < u_2,\ [0.1, 0.3],\ [0.4, 0.5] >\ ,\ < u_3,\ [0.1, 0.3],\ [0.3, 0.5] >\}$, $\xi_2(b_1,\ p_1) = \{< u_1,\ [0.2, 0.5],\ [0.3, 0.5] >\ ,\ < u_2,\ [0.2, 0.4],\ [0.4, 0.5] >\ ,\ < u_3,\ [0.2, 0.5],\ [0.4, 0.5] >\}$. *Now,* $(\xi_1, A_1, G_1)+(\xi_2, A_2, G_2)$ *is given by*

$\xi_1(b_1, p_1)+\xi_2(b_1, p_1) = \{< u_1, [0.28, 0.7], [0.06, 0.2] >, < u_2, [0.28, 0.58], [0.16, 0.25] >, < u_3,\ [0.28, 0.65],\ [0.12, 0.25] >\}$.

Definition 6.2.19 *The product of two $IVIFSE$ sets (ξ_1, A_1, G_1) and (ξ_2, A_2, G_2) over U is denoted as $(\xi_1, A_1, G_1)(\xi_2, A_2, G_2)$ and defined as:*

$$\xi_1(\ell,m)\xi_2(\ell,m) = \begin{cases} \{< u,\ [\gamma^-_{1(\ell,m)}(u)\gamma^-_{2(\ell,m)}(u), \gamma^+_{1(\ell,m)}(u)\gamma^+_{2(\ell,m)}(u)], \\ [\zeta^-_{1(\ell,m)}(u)+\zeta^-_{2(\ell,m)}(u)-\zeta^-_{1(\ell,m)}(u)\zeta^-_{2(\ell,m)}(u), \\ \zeta^+_{1(\ell,m)}(u)+\zeta^+_{2(\ell,m)}(u)-\zeta^+_{1(\ell,m)}(u)\zeta^+_{2(\ell,m)}(u)] >\}, \end{cases}$$

where $\xi_1(\ell,m) = \{< u,\ [\gamma^-_{1(\ell,m)}(u), \gamma^+_{1(\ell,m)}(u)], [\zeta^-_{1(\ell,m)}(u), \zeta^+_{1(\ell,m)}(u)] >: u \in U\}$
and $\xi_2(\ell,m) = \{< u,\ [\gamma^-_{2(\ell,m)}(u), \gamma^+_{2(\ell,m)}(u)], [\zeta^-_{2(\ell,m)}(u), \zeta^+_{2(\ell,m)}(u)] >: u \in U\}$.

Example 6.2.20 *Consider* $\xi_1(b_1,\ p_1) = \{< u_1,\ [0.1, 0.4],\ [0.2, 0.4] >\ ,\ < u_2,\ [0.1, 0.3],\ [0.4, 0.5] >\ ,\ < u_3,\ [0.1, 0.3],\ [0.3, 0.5] >\}$, $\xi_2(b_1,\ p_1) = \{< u_1,\ [0.2, 0.5],\ [0.3, 0.5] >\ ,\ < u_2,\ [0.2, 0.4],\ [0.4, 0.5] >\ ,\ < u_3,\ [0.2, 0.5],\ [0.4, 0.5] >\}$. *Now* $(\xi_1, A_1, G_1)(\xi_2, A_2, G_2)$ *is given by* $\xi_1(b_1,\ p_1)\xi_2(b_1,\ p_1) = \{< u_1,\ [0.02, 0.2],\ [0.44, 0.7] >\ ,\ < u_2,\ [0.02, 0.12],\ [0.64, 0.75] >\ ,\ < u_3,\ [0.02, 0.15],\ [0.58, 0.75] >\}$.

Definition 6.2.21 *The product of $IVIFSE$ set (ξ, A, G) with an arbitrary real number $\kappa > 0$ is denoted by $\kappa(\xi, A, G)$ and defined for all as follows:*

$\kappa\xi(b,p) = \{< u,\ [1-(1-\gamma^-_{(b,p)}(u))^\kappa, 1-(1-\gamma^+_{(b,p)}(u))^\kappa], [(\zeta^-_{(b,p)}(u))^\kappa, (\zeta^+_{(b,p)}(u))^\kappa] >: u \in U\}$.

Example 6.2.22 *Consider* $\xi_1(b_1,\ p_1) = \{< u_1,\ [0.1, 0.4],\ [0.2, 0.4] >\ ,\ < u_2,\ [0.1, 0.3],\ [0.4, 0.5] >\ ,\ < u_3,\ [0.1, 0.3],\ [0.3, 0.5] >\},\ k = 6$.

The product of IVIFSE set (ξ, A, G) *with* 6 *is denoted by* $6(\xi, A, G)$ *and given as*
$6\xi_1(b_1, p_1) = \{< u_1, [0.468559, 0.953344], [0.000064, 0.004096] >\ , < u_2, [0.468559, 0.3], [0.004096, 0.015625] >\ , < u_3, [0.486559, 0.882351], [0.000729, 0.015625] >\}$.

Definition 6.2.23 *The power of IVIFSE set* (ξ, A, G) *with an arbitrary real number* $\kappa > 0$ *is denoted by* $(\xi, A, G)^\kappa$ *and defined as follows:*

$$(\xi(b,p))^\kappa = \{< u,\ [(\gamma^-_{(b,p)}(u))^\kappa, (\gamma^+_{(b,p)}(u))^\kappa], [1-(1-\zeta^-_{(b,p)}(u))^\kappa, 1-(1-\zeta^+_{(b,p)}(u))^\kappa] >: u \in U\}.$$

Example 6.2.24 *Consider* $\xi_1(b_1,\ p_1) = \{< u_1,\ [0.1, 0.4],\ [0.2, 0.4] >\ ,\ < u_2,\ [0.1, 0.3],\ [0.4, 0.5] >\ ,\ < u_3,\ [0.1, 0.3],\ [0.3, 0.5] >\},\ k = 0.5$.

The power of IVIFSE set (ξ, A, G) *with* 0.5 *is denoted by* $(\xi, A, G)^{0.5}$ *and given as*
$\xi_1(b_1,\ p_1)^{0.5} = \{< u_1,\ [0.3162, 0.6324],\ [0.1055, 0.2254] >\ ,\ < u_2,\ [0.3162, 0.5477],\ [0.2254, 0.2928] >\ ,\ < u_3,\ [0.3162, 0.5477],\ [0.1633, 0.2928] >\}$.

Definition 6.2.25 *The score and accuracy function of an interval-valued intuitionistic fuzzy soft expert set*

$$\xi(b,p) = \{< u,\ [\gamma^-_{(b,p)}(u), \gamma^+_{(b,p)}(u)], [\zeta^-_{(b,p)}(u), \zeta^+_{(b,p)}(u)] >: u \in U\},$$

are defined respectively as follows

$$S(\xi(b,p)) = \frac{\gamma^-_{(b,p)}(u) + \gamma^+_{(b,p)}(u) - \zeta^-_{(b,p)}(u) - \zeta^+_{(b,p)}(u)}{2},$$

and

$$R(\xi(b,p)) = \frac{\gamma^-_{(b,p)}(u) + \gamma^+_{(b,p)}(u) + \zeta^-_{(b,p)}(u) + \zeta^+_{(b,p)}(u)}{2}$$

where $S(\xi(b,p)) \in [-1, 1]$ *and* $R(\xi(b,p)) \in [0, 1]$.

We can build up ranking method of *IVIFSE sets* by using score function and accuracy function. The larger score and larger accuracy indicate the greater *IVIFSE* element.

Example 6.2.26 *Consider* $\xi(b_1,\ p_1) = < u_1,\ [0.4, 0.5],\ [0.2, 0.4] >$, $\xi(b_2,\ p_1) = < u_1,\ [0.2, 0.4],\ [0.5, 0.6] >$, $\xi(b_3,\ p_1) = < u_1,\ [0.1, 0.2],\ [0.3, 0.5] >$, $\xi(b_1,\ p_2) = < u_1,\ [0.4, 0.7], [0.2, 0.3] >$, $\xi(b_2, p_2) = < u_1, [0.3, 0.5], [0.0, 0.3] >$, $\xi(b_3, p_2) = < u_1, [0.2, 0.5], [0.0, 0.3] >$.

Now calculating scores of the above $IVIFSE$ *elements, we have*

$$\begin{aligned}
Ş((b_1, p_1)) &= 0.15, \\
Ş(\xi(b_2, p_1)) &= -0.25, \\
Ş(\xi(b_3, p_1)) &= -0.25, \\
Ş(\xi(b_1, p_2)) &= 0.3, \\
Ş(\xi(b_2, p_2)) &= 0.25, \\
Ş(\xi(b_3, p_2)) &= 0.2.
\end{aligned}$$

Now arranging the scores in ascending order we have

$$\xi(b_2, p_1) = \xi(b_3, p_1) \leq \xi(b_1, p_1) \leq \xi(b_3, p_2) \leq \xi(b_2, p_2) \leq \xi(b_1, p_2).$$

Since scores of $\xi(b_2, p_1)$ *and* $\xi(b_3, p_1)$ *are equal, we calculate their accuracies*

$$\begin{aligned}
Ŗ(\xi(b_2, p_1)) &= 0.85, \\
Ŗ(\xi(b_3, p_1)) &= 0.55.
\end{aligned}$$

Hence elements are ranked as

$$\xi(b_3, p_1) \leq \xi(b_2, p_1) \leq \xi(b_1, p_1) \leq \xi(b_3, p_2) \leq \xi(b_2, p_2) \leq \xi(b_1, p_2).$$

6.3 Aggregation Operators on IVIFSE Sets

In this section, we develop new operators with interval-valued intuitionistic fuzzy soft expert sets ($IVIFSE$ *sets*).

Definition 6.3.1 *Let* $(\xi_k, A_k, G_k)_{\gamma_k, \zeta_k} = \{< [\gamma^-_{k(b,p)}, \gamma^+_{k(b,p)}], [\zeta^-_{k(b,p)}, \zeta^+_{k(b,p)}] >; k = 1, 2, ...m\}$ *be* $IVIFSE$ *sets. A mapping* $Þ^{\xi}_{\varpi} : Ķ^m_I(U) \longrightarrow Ķ_I(U)$ *is called an* $IVIFSE$ *weighted average operator if it satisfies*

$$Þ^{\xi}_{\varpi}((\xi_1, A_1, G_1), (\xi_2, A_2, G_2), ..., (\xi_m, A_m, G_m)) = \sum_{k=1}^{m} \varpi_k (\xi_k, A_k, G_k)_{\gamma_k, \zeta_k},$$

where $Ķ^m_I(U)$ *denotes* m *copies of* $IVIFSE$ *sets,* $\varpi = (\varpi_1, \varpi_2, ..., \varpi_m)^t$ *is a weight vector of* A_k, $k = 1, 2, ..., m$ *satisfying the normalized conditions* $\sum_{k=1}^{m} \varpi_k = 1$; $\varpi_k \in [0, 1]$.

If $\varpi = (1/m, 1/m, ..., 1/m)^t$ *then the* $IVIFSE$ *weighted average operator can be written as*

$$Þ^{\xi}_{\varpi}((\xi_1, A_1, G_1), (\xi_2, A_2, G_2), ..., (\xi_m, A_m, G_m)) = \frac{1}{m} \sum_{k=1}^{m} (\xi_k, A_k, G_k)_{\gamma_k, \zeta_k}.$$

In this case $Þ^{\xi}_{\varpi}$ *reduces to* $IVIFSE$ *arithmetic mean operator.*

Theorem 6.3.2 *Suppose that* $(\xi_k, A_k, G_k)_{\gamma_k,\zeta_k} = \{< [\gamma^-_{k(b,p)}, \gamma^+_{k(b,p)}], [\zeta^-_{k(b,p)}, \zeta^+_{k(b,p)}] >: k = 1, 2, ...m\}$ *are IVIFSE sets. Then, by using the IVIFSE weighted average operator aggregation is also an IVIFSE set and*

$$\flat^{\xi}_{\varpi}((\xi_1, A_1, G_1), (\xi_2, A_2, G_2), ..., (\xi_m, A_m, G_m)) = \begin{cases} < [1 - \prod_{k=1}^{m}(1-\gamma^-_{k(b,p)})^{\varpi_k}, \\ 1 - \prod_{k=1}^{m}(1-\gamma^+_{k(b,p)})^{\varpi_k}], \\ [\prod_{k=1}^{m}(\zeta^-_{k(b,p)})^{\varpi_k}, \prod_{k=1}^{m}(\zeta^+_{k(b,p)})^{\varpi_k}] >, \end{cases}$$

Proof. We prove this result by using mathematical induction. The result holds for $k = 1$, by using Definition 6.2.21, .Now we have to show that it is true for $k = 2$. By using Definitions 6.3.1 and 6.2.21, we have, $\flat^{\xi}_{\varpi}((\xi_1, A_1, G_1), (\xi_2, A_2, G_2)) = \sum_{k=1}^{2}\varpi_k(\xi_k, A_k, G_k) = \varpi_1(\xi_1, A_1, G_1) + \varpi_2(\xi_2, A_2, G_2) = < [1 - (1 - \gamma^-_{1(b,p)}(u))^{\varpi_1}, 1 - (1 - \gamma^+_{1(b,p)}(u))^{\varpi_1}], [(\zeta^-_{1(b,p)}(u))^{\varpi_1}, (\zeta^+_{1(b,p)}(u))^{\varpi_1}] > + < [1 - (1 - \gamma^-_{2(b,p)}(u))^{\varpi_2}, 1-(1-\gamma^+_{2(b,p)}(u))^{\varpi_2}], [(\zeta^-_{2(b,p)}(u))^{\varpi_2}, (\zeta^+_{2(b,p)}(u))^{\varpi_2}] > = < [1-(1-\gamma^-_{1(b,p)}(u))^{\varpi_1}+1-(1-\gamma^-_{2(b,p)}(u))^{\varpi_2}-(1-(1-\gamma^-_{1(b,p)}(u))^{\varpi_1})(1-(1-\gamma^-_{2(b,p)}(u))^{\varpi_2}), 1-(1-\gamma^+_{1(b,p)}(u))^{\varpi_1}+1-(1-\gamma^+_{2(b,p)}(u))^{\varpi_2}-(1-(1-\gamma^+_{1(b,p)}(u))^{\varpi_1})(1-(1-\gamma^+_{2(b,p)}(u))^{\varpi_2})], [(\zeta^-_{1(b,p)}(u))^{\varpi_1}(\zeta^-_{2(b,p)}(u))^{\varpi_2}, (\zeta^+_{1(b,p)}(u))^{\varpi_1}(\zeta^+_{2(b,p)}(u))^{\varpi_2}] >=< [1-(1-\gamma^-_{1(b,p)})^{\varpi_1}(1-\gamma^-_{2(b,p)})^{\varpi_2}, 1-(1-\gamma^+_{1(b,p)})^{\varpi_1}(1-\gamma^+_{2(b,p)})^{\varpi_2}], [(\zeta^-_{1(b,p)})^{\varpi_1}(\zeta^-_{2(b,p)})^{\varpi_2}, (\zeta^+_{1(b,p)})^{\varpi_1}(\zeta^+_{2(b,p)})^{\varpi_2}] >$

$$= \begin{cases} < [1 - \prod_{k=1}^{2}(1-\gamma^-_{k(b,p)})^{\varpi_k}, \\ 1 - \prod_{k=1}^{2}(1-\gamma^+_{k(b,p)})^{\varpi_k}], \\ [\prod_{k=1}^{2}(\zeta^-_{k(b,p)})^{\varpi_k}, \prod_{k=1}^{2}(\zeta^+_{k(b,p)})^{\varpi_k}] > . \end{cases}$$

Suppose this result holds for $k = n$, that is $\flat^{\xi}_{\varpi}((\xi_1, A_1, G_1), (\xi_2, A_2, G_2), ..., (\xi_n, A_n, G_n)) = \sum_{k=1}^{n}\varpi_k(\xi_k, A_k, G_k) = \varpi_1(\xi_1, A_1, G_1) + \varpi_2(\xi_2, A_2, G_2) + ... + \varpi_n(\xi_n, A_n, G_n)$

$$= \begin{cases} < [1 - \prod_{k=1}^{n}(1-\gamma^-_{k(b,p)})^{\varpi_k}, \\ 1 - \prod_{k=1}^{n}(1-\gamma^+_{k(b,p)})^{\varpi_k}], \\ [\prod_{k=1}^{n}(\zeta^-_{k(b,p)})^{\varpi_k}, \prod_{k=1}^{n}(\zeta^+_{k(b,p)})^{\varpi_k}] > \end{cases} \tag{6.1}$$

Further we show that this result holds for $k = n + 1$.

$\flat^{\xi}_{\varpi}((\xi_1, A_1, G_1), (\xi_2, A_2, G_2), ..., (\xi_{n+1}, A_{n+1}, G_{n+1})) = \sum_{k=1}^{n+1}\varpi_k(\xi_k, A_k, G_k) = \sum_{k=1}^{n}\varpi_k(\xi_k, A_k, G_k) + \varpi_{n+1}(\xi_{n+1}, A_{n+1}, G_{n+1})$

Again using Definitions 6.2.17 and 6.2.21 and Equation 6.1 we have , $\wp^{\xi}_{\varpi}((\xi_1, A_1, G_1), (\xi_2, A_2, G_2), ..., (\xi_{n+1}, A_{n+1}, G_{n+1}))$

$$= \begin{cases} <[1-\prod_{k=1}^{n+1}(1-\gamma^{-}_{k(b,p)})^{\varpi_k}, \\ 1-\prod_{k=1}^{n+1}(1-\gamma^{+}_{k(b,p)})^{\varpi_k}], \\ [\prod_{k=1}^{n+1}(\zeta^{-}_{k(b,p)})^{\varpi_k}, \prod_{k=1}^{n+1}(\zeta^{+}_{k(b,p)})^{\varpi_k}] > . \end{cases}$$

Hence proved. ■

Definition 6.3.3 *Let* $(\xi_k, A_k, G_k)_{\gamma_k,\zeta_k}$ $(k = 1, 2, ..., m)$ *be IVIFSE sets. A mapping* $\wp^{O}_{\varpi} : K^m_I(U) \longrightarrow K_I(U)$ *is called an IVIFSE ordered weighted average operator if it satisfies*

$$\wp^{O}_{\varpi}((\xi_1, A_1, G_1), (\xi_2, A_2, G_2), ..., (\xi_m, A_m, G_m)) = \sum_{i=1}^{m} \varpi_i (\xi_i, \widehat{A_i, G_i})_{\widehat{\gamma}_i,\widehat{\zeta}_i},$$

where $\varpi = (\varpi_1, \varpi_2, ..., \varpi_m)^t$ *is a weight vector of* A_k, $k = 1, 2, ..., m$ *satisfying the normalized conditions* $\sum_{i=1}^{m} \varpi_i = 1$; $\varpi_i \in [0, 1]$. $(\xi_i, \widehat{A_i, G_i})_{\widehat{\gamma}_i,\widehat{\zeta}_i} =< [\widehat{\gamma}^{-}_{i(b,p)}, \widehat{\gamma}^{+}_{i(b,p)}], [\widehat{\zeta}^{-}_{i(b,p)}, \widehat{\zeta}^{+}_{i(b,p)}] >$ *is the* $i-th$ *largest of the* m *IVIFSE sets* $(\xi_k, A_k, G_k)_{\gamma_k,\zeta_k}$ *which can be determined by using ranking method of IVIFSE sets such as score and accuracy function.*

Remark 6.3.4 *If* $\varpi = (1/m, 1/m, ..., 1/m)^t$ *then the IVIFSE ordered weighted average operator* $\wp^{O}_{\varpi}$ *degenerates to the IVIFSE arithmetic mean operator.*

Theorem 6.3.5 *Suppose that* $(\xi_k, A_k, G_k)_{\gamma_k,\zeta_k}$ $(k = 1, 2, ..., m)$ *are IVIFSE sets. Then by using the IVIFSE ordered weighted arithmetic operator aggregation is also an IVIFSE set and*

$$\wp^{O}_{\varpi}((\xi_1, A_1, G_1), (\xi_2, A_2, G_2), ..., (\xi_m, A_m, G_m)) = \begin{cases} <[1-\prod_{i=1}^{m}(1-\widehat{\gamma}^{-}_{i(b,p)})^{\varpi_i}, \\ 1-\prod_{i=1}^{m}(1-\widehat{\gamma}^{+}_{i(b,p)})^{\varpi_i}], \\ [\prod_{i=1}^{m}(\widehat{\zeta}^{-}_{i(b,p)})^{\varpi_i}, \prod_{i=1}^{m}(\widehat{\zeta}^{+}_{i(b,p)})^{\varpi_i}] >, \end{cases} \quad (6.2)$$

where $(\xi_i, \widehat{A_i, G_i})_{\widehat{\gamma}_i,\widehat{\zeta}_i} =< [\widehat{\gamma}^{-}_{i(b,p)}, \widehat{\gamma}^{+}_{i(b,p)}], [\widehat{\zeta}^{-}_{i(b,p)}, \widehat{\zeta}^{+}_{i(b,p)}] >$ *is the* $i-th$ *largest of the* m *IVIFSE sets* $(\xi_k, A_k, G_k)_{\gamma_k,\zeta_k}$ *which can be determined by using ranking method of IVIFSE sets such as score and accuracy function.*

Proof. It is straightforward by using mathematical induction and Definitions 6.3.3, 6.2.21 and 6.2.23. ■

The $IVIFSE$ weighted average operator $\flat_{\varpi}^{\xi}$ considers importance of the aggregated $IVIFSE$ *sets* themselves. The $IVIFSE$ ordered weighted arithmetic operator $\flat_{\varpi}^{O}$ concerns with position importance of the ranking order of the aggregated $IVIFSE$ *sets*. The underlying aspects of the two aggregated operators have been combined in next U.

Definition 6.3.6 *Let* $(\xi_k, A_k, G_k)_{\gamma_k,\zeta_k}$ $(k = 1, 2, ..., m)$ *be* $IVIFSE$ *sets. A mapping* $\flat_{\varpi,\omega}^{F} : K_I^m(U) \longrightarrow K_I(U)$ *is called an IVIFSE fusion weighted average operator if it satisfies*

$$\flat_{\varpi,\omega}^{F}((\xi_1, A_1, G_1), (\xi_2, A_2, G_2), ..., (\xi_m, A_m, G_m)) = \sum_{i=1}^{m} \omega_i (\xi_i, \widetilde{A_i, G_i})_{\widetilde{\gamma}_i,\widetilde{\zeta}_i},$$

where $\omega = (\omega_1, \omega_2, ..., \omega_m)^t$ *is a position weight vector. The* $IVIFSE$ *set of* $(\xi_k, A_k, G_k)_{\gamma_k,\zeta_k}$ *weighted with* $m\varpi_k$ *is denoted by*

$$\overbrace{(\xi_k, A_k, G_k)_{\gamma_k,\zeta_k}} = m\varpi_k (\xi_k, A_k, G_k)_{\gamma_k,\zeta_k},$$

where $\varpi = (\varpi_1, \varpi_2, ..., \varpi_m)^t$ *is a weight vector of* A_k , $(k = 1, 2, ..., m)$ *and* $(\xi_i, \widetilde{A_i, G_i})_{\widetilde{\gamma}_i,\widetilde{\zeta}_i}$ *is the* $i-th$ *largest of the* m $IVIFSE$ *sets* $\overbrace{(\xi_k, A_k, G_k)_{\gamma_k,\zeta_k}}$ *which can be determined by using ranking method of* $IVIFSE$ *sets such as score and accuracy function.*

Remark 6.3.7 *If* $\omega = (1/m, 1/m, ..., 1/m)^t$ *then* $IVIFSE$ *fusion weighted average operator* $\flat_{\varpi,\omega}^{F}$ *degenerates to* $IVIFSE$ *weighted average operator* $\flat_{\varpi}^{\xi}$. *If* $\varpi = (1/m, 1/m, ..., 1/m)^t$ *then* $IVIFSE$ *fusion weighted average operator* $\flat_{\varpi,\omega}^{F}$ *degenerates to the* $IVIFSE$ *ordered weighted arithmetic operator* $\flat_{\varpi}^{O}$. *So* $\flat_{\varpi,\omega}^{F}$ *is a generalization of* $\flat_{\varpi}^{\xi}$ *and* $\flat_{\varpi}^{O}$. $\flat_{\varpi,\omega}^{F}$ *concerns with both the characteristics of* $\flat_{\varpi}^{\xi}$ *and* $\flat_{\varpi}^{O}$.

Theorem 6.3.8 *Suppose that* $(\xi_k, A_k, G_k)_{\gamma_k,\zeta_k}$ $(k = 1, 2, ..., m)$ *are* $IVIFSE$ *sets. Then, by using the* $IVIFSE$ *fusion weighted average operator* $\flat_{\varpi,\omega}^{F}$ *aggregation is also an* $IVIFSE$ *set and*

$$\flat_{\varpi,\omega}^{F}((\xi_1, A_1, G_1), (\xi_2, A_2, G_2), ..., (\xi_m, A_m, G_m)) = \begin{cases} < [1 - \prod_{i=1}^{m} (1 - \widetilde{\gamma}_{i(b,p)}^{-})^{\omega_i}, \\ 1 - \prod_{i=1}^{m} (1 - \widetilde{\gamma}_{i(b,p)}^{+})^{\omega_i}], \\ [\prod_{i=1}^{m} (\widetilde{\zeta}_{i(b,p)}^{-})^{\omega_i}, \prod_{i=1}^{m} (\widetilde{\zeta}_{i(b,p)}^{+})^{\omega_i}] >, \end{cases}$$

where $(\xi_i, \widetilde{A_i, G_i})_{\widetilde{\gamma}_i,\widetilde{\zeta}_i} =< [\widetilde{\gamma}_{i(b,p)}^{-}, \widetilde{\gamma}_{i(b,p)}^{+}], [\widetilde{\zeta}_{i(b,p)}^{-}, \widetilde{\zeta}_{i(b,p)}^{+}] >$ *is the* $i-th$ *largest of the* m $IVIFSE$ *sets* $\overbrace{(\xi_k, A_k, G_k)_{\gamma_k,\zeta_k}} = m\varpi_k (\xi_k, A_k, G_k)_{\gamma_k,\zeta_k}$ *which can be determined by using ranking method of* $IVIFSE$ *sets such as score and accuracy function.*

Proof. It is straightforward by using mathematical induction and Definitions 6.3.6, 6.2.21 and 6.2.23. ■

Definition 6.3.9 *Let* $(\xi_k, A_k, G_k)_{\gamma_k,\zeta_k}$ $(k = 1, 2, ..., m)$ *be* $IVIFSE$ *sets. A mapping* $\beta_{\varpi}^{GO} : K_I^m(U) \longrightarrow K_I(U)$ *is called an* $IVIFSE$ *generalized ordered weighted average operator if it satisfies*

$$\beta_{\varpi}^{GO}((\xi_1, A_1, G_1), (\xi_2, A_2, G_2), ..., (\xi_m, A_m, G_m)) = \sqrt[r]{\sum_{i=1}^{m} \omega_i (\xi_i, A_i, G_i)^r_{\widehat{\gamma}_i,\widehat{\zeta}_i}},$$

where $\omega = (\omega_1, \omega_2, ..., \omega_m)^t$ *is a position weight vector of* A_k ; $r > 0$ *is a control parameter which can be chosen according to the given condition and* $(\xi_i, A_i, G_i)_{\widehat{\gamma}_i,\widehat{\zeta}_i} =< [\widehat{\gamma}^-_{i(b,p)}, \widehat{\gamma}^+_{i(b,p)}], [\widehat{\zeta}^-_{i(b,p)}, \widehat{\zeta}^+_{i(b,p)}] >$ *is the* $i-th$ *largest of the* m $IVIFSE$ *sets* $(\xi_k, A_k, G_k)_{\gamma_k,\zeta_k}$ *which can be determined by using ranking method of* $IVIFSE$ *sets such as score and accuracy function.*

Theorem 6.3.10 *Suppose that* $(\xi_k, A_k, G_k)_{\gamma_k,\zeta_k}$ $(k = 1, 2, ..., m)$ *are* $IVIFSE$ *sets. Then, by using the* $IVIFSE$ *generalized ordered weighted average operator* $\beta_{\varpi,\omega}^{GO}$ *aggregation is also an* $IVIFSE$ *set and*

$$\beta_{\varpi,\omega}^{GO}((\xi_1, A_1, G_1), (\xi_2, A_2, G_2), ..., (\xi_m, A_m, G_m)) = \begin{cases} < [\sqrt[r]{1 - \prod_{i=1}^{m} (1 - (\widehat{\gamma}^-_{i(b,p)})^r)^{\omega_i}}, \\ \sqrt[r]{1 - \prod_{i=1}^{m} (1 - (\widehat{\gamma}^+_{i(b,p)})^r)^{\omega_i}}], \\ [1 - \sqrt[r]{1 - \prod_{i=1}^{m} (1 - (1 - \widehat{\zeta}^-_{i(b,p)})^r)^{\omega_i}}, \\ 1 - \sqrt[r]{1 - \prod_{i=1}^{m} (1 - (1 - \widehat{\zeta}^+_{i(b,p)})^r)^{\omega_i}}] >, \end{cases}$$

where $(\xi_i, A_i, G_i)_{\widehat{\gamma}_i,\widehat{\zeta}_i} =< [\widehat{\gamma}^-_{i(b,p)}, \widehat{\gamma}^+_{i(b,p)}], [\widehat{\zeta}^-_{i(b,p)}, \widehat{\zeta}^+_{i(b,p)}] >$ *is the* $i - th$ *largest of the* m $IVIFSE$ *sets* $(\xi_k, A_k, G_k)_{\gamma_k,\zeta_k}$ *which can be determined by using ranking method of* $IVIFSE$ *sets such as score and accuracy function.*

Proof. It is straightforward by using mathematical induction and Definitions 6.3.9, 6.2.21 and 6.2.23. ■

Corollary 6.3.11 *If* $r = 1$ *then the* $IVIFSE$ *generalized ordered weighted average operator* β_{ϖ}^{GO} *degenerate to the* $IVIFSE$ *ordered weighted average operator* β_{ϖ}^{O}.

Definition 6.3.12 *Let* $(\xi_k, A_k, G_k)_{\gamma_k,\zeta_k}$ $(k = 1, 2, ..., m)$ *be* $IVIFSE$ *sets. A mapping* $\flat^{G,F}_{\varpi,\omega} : K^m_I(U) \longrightarrow K_I(U)$ *is called an* $IVIFSE$ *generalized fusion weighted averaging operator if it satisfies*

$$\flat^{G,F}_{\varpi,\omega}((\xi_1, A_1, G_1), (\xi_2, A_2, G_2), ..., (\xi_m, A_m, G_m)) = \sum_{i=1}^{m} \sqrt[r]{\omega_i \widetilde{(\xi_i, A_i, G_i)}_{\widetilde{\gamma}_i,\widetilde{\zeta}_i}}^{\,r},$$

where $\omega = (\omega_1, \omega_2, ..., \omega_m)^t$ *is a position weight vector .* $r > 0$ *is a control parameter. The* $IVIFSE$ *set of* $(\xi_k, A_k, G_k)_{\gamma_k,\zeta_k}$ *weighted with* $m\varpi_k$ *is denoted by*

$$\overbrace{(\xi_k, A_k, G_k)_{\gamma_k,\zeta_k}} = m\varpi_k(\xi_k, A_k, G_k),$$

where $\varpi = (\varpi_1, \varpi_2, ..., \varpi_m)^t$ *is a weight vector for* A_k $IVIFSE$ *sets* $(\xi_k, A_k, G_k)_{\gamma_k,\zeta_k}$ *and* $\widetilde{(\xi_i, A_i, G_i)}_{\widetilde{\gamma}_i,\widetilde{\zeta}_i}$ *is the* $i-th$ *largest of the* m $IVIFSE$ *Sets* $\overbrace{(\xi_k, A_k, G_k)_{\gamma_k,\zeta_k}}$ *which can be determined by using ranking method of* $IVIFSE$ *Sets such as score and accuracy function.*

Remark 6.3.13 *If* $\varpi = (1/m, 1/m, ..., 1/m)^t$ *then* $IVIFSE$ *generalized fusion weighted average operator* $\flat^{G,F}_{\varpi,\omega}$ *degenerates to* $IVIFSE$ *generalized ordered weighted average operator* $\flat^{GO}_{\varpi}$.

Theorem 6.3.14 *Suppose that* $(\xi_k, A_k, G_k)_{\gamma_k,\zeta_k}$ *where* $k = 1, 2, ...m$ *are* $IVIFSE$ *sets. Then, by using the* $IVIFSE$ *generalized fusion weighted average operator* $\flat^{G,F}_{\varpi,\omega}$ *aggregation is also an* $IVIFSE$ *set and*

$$\flat^{G,F}_{\varpi,\omega}((\xi_1, A_1, G_1), (\xi_2, A_2, G_2), ..., (\xi_m, A_m, G_m)) = \begin{cases} <[\sqrt[r]{1 - \prod_{i=1}^{m}(1 - (\widetilde{\gamma}^-_{i(b,p)})^r)^{\omega_i}}, \\ \sqrt[r]{1 - \prod_{i=1}^{m}(1 - (\widetilde{\gamma}^+_{i(b,p)})^r)^{\omega_i}}], \\ [1 - \sqrt[r]{1 - \prod_{i=1}^{m}(1 - (1 - \widetilde{\zeta}^-_{i(b,p)})^r)^{\omega_i}}, \\ 1 - \sqrt[r]{1 - \prod_{i=1}^{m}(1 - (1 - \widetilde{\zeta}^+_{i(b,p)})^r)^{\omega_i}}] >, \end{cases}$$

where $(\xi_i, A_i, G_i)_{\widetilde{\gamma}_i,\widetilde{\zeta}_i} =< [\widetilde{\gamma}^-_{i(b,p)}, \widetilde{\gamma}^+_{i(b,p)}], [\widetilde{\zeta}^-_{i(b,p)}, \widetilde{\zeta}^+_{i(b,p)}] >$ *is the* $i-th$ *largest of the* m $IVIFSE$ *Sets* $\overbrace{(\xi_k, A_k, G_k)_{\gamma_k,\zeta_k}} = n\varpi_k(\xi_k, A_k, G_k)_{\gamma_k,\zeta_k}$ *which can be determined by using ranking method of* $IVIFSE$ *sets such as score and accuracy function.*

Proof. It is straightforward by using mathematical induction and Definitions 6.3.12, 6.2.21 and 6.2.23. ■

Corollary 6.3.15 *If* $r = 1$*, then the* $IVIFSE$ *generalized fusion weighted average operator* $\flat^{G,F}_{\varpi,\omega}$ *degenerates to the* $IVIFSE$ *fusion weighted average operator* $\flat^{F}_{\varpi,\omega}$.

6.4 Multicriteria Decision Making Of $IVIFSE$ Sets with The $IVIFSE$ Fusion Weighted Average Operator

Suppose $U = \{u_j; j = 1, 2, ..., l\}$ be the initial universe, Ç$= \{p_i; i = 1, 2, ..., n\}$ be the set of experts, A$= \{b_k; k = 1, 2, ..., m\}$ be the set of attributes. Opinion of the experts corresponding to each attribute is represented in the form of an $IVIFSE$ set. The algorithm and process of the $IVIFSE$ fusion weighted average operator method for multi attribute decision-making with $IVIFSE$ sets can be summarized as follows.

Step 1: Utilize the evaluations of experts in the form of $IVIFSE$ sets to determine the opinions regarding the given alternatives and criteria.

Step 2: Separate the opinions of each expert.

p_i	u_1	.	.	.	u_l
b_1	$<[\gamma^-_{(b_1,p_i)}, \gamma^+_{(b_1,p_i)}][\gamma^-_{(b_1,p_i)}, \gamma^+_{(b_1,p_i)}]>$	.	.	.	$<[\gamma^-_{(b_1,p_i)}, \gamma^+_{(b_1,p_i)}][\gamma^-_{(b_1,p_i)}, \gamma^+_{(b_1,p_i)}]>$
b_2	$<[\gamma^-_{(b_2,p_i)}, \gamma^+_{(b_2,p_i)}][\gamma^-_{(b_2,p_i)}, \gamma^+_{(b_2,p_i)}]>$	.	.	.	$<[\gamma^-_{(b_2,p_i)}, \gamma^+_{(b_2,p_i)}][\gamma^-_{(b_2,p_i)}, \gamma^+_{(b_2,p_i)}]>$
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
b_m	$<[\gamma^-_{(b_m,p_i)}, \gamma^+_{(b_m,p_i)}][\gamma^-_{(b_m,p_i)}, \gamma^+_{(b_m,p_i)}]>$	.	.	.	$<[\gamma^-_{(b_m,p_i)}, \gamma^+_{(b_m,p_i)}][\gamma^-_{(b_m,p_i)}, \gamma^+_{(b_m,p_i)}]>$

Step 3: Assign weights to each criteria.

Step 4: Assign position weights vector. The purpose of this weight vector is to eliminate the effect of individual preconception on comprehensive assessment.

Step5 : Aggregate attributes by using $IVIFSE$ fusion weighted average operator.

Step 6: Find the accuracy of each member of U corresponding to each expert.

Step 7: Calculate the average accuracy of each member of U.

Step 8: Generate the non decreasing chain of these averages.

Step9: Conclusion.

Example 6.4.1 *Let* $U = \{u_1$ *=Dairy farming,* u_2 *=Fish farming,* u_3 *=Poultry farming,* u_4 *=Goat fattening farm\}* *be the set of small and medium enterprises;* Ç$=\{p_1, p_2, p_3\}$ *be the set of experts;* A$= \{b_1$ *=project cost,* b_2 *=space requirement,* b_3 *=human resource requirement\}* *be the set of attributes. Three experts evaluate some enterprise and their evaluations are expressed in the form of interval valued intuitionistic fuzzy soft expert sets IVIFSE sets. Compute the comprehensive evaluation of the experts on the enterprise by using the interval valued intuitionistic fuzzy soft expert fusion weighted average operator.*

Step 1: Utilize the evaluations of experts in the form of $IVIFSE$ sets to determine the opinions regarding the given alternatives and criteria.

$\xi(b_1, p_1) = \{< u_1, [0.4, 0.5], [0.2, 0.3] >, < u_2, [0.1, 0.2], [0.6, 0.7] >, < u_3, [0.2, 0.4], [0.5, 0.6] >, < u_4, [0.3, 0.5], [0.1, 0.5] >\}$, $\xi(b_1, p_2) = \{< u_1, [0.2, 0.4], [0.4, 0.5] >, < u_2, [0.4, 0.6], [0.1, 0.3] >, < u_3, [0.1, 0.2], [0.3, 0.5] >, < u_4, [0.3, 0.5], [0.4, 0.5] >\}$, $\xi(b_1, p_3) = \{< u_1, [0.4, 0.5], [0.3, 0.4] >, < u_2, [0.4, 0.5], [0.2, 0.4] >, < u_3, [0.4, 0.7], [0.2, 0.3] >, < u_4, [0.1, 0.3], [0.5, 0.6] >\}$, $\xi(b_2, p_1) = \{< u_1, [0.2, 0.3], [0.5, 0.6] >, < u_2, [0.6, 0.7], [0.1, 0.2] >, < u_3, [0.2, 0.4], [0.4, 0.5] >, < u_4, [0.3, 0.4], [0.2, 0.4] >\}$,$\xi(b_2, p_2) = \{< u_1, [0.4, 0.6], [0.1, 0.2] >, < u_2, [0.3, 0.5], [0.3, 0.4] >, < u_3, [0.3, 0.5], [0.1, 0.5] >, < u_4, [0.2, 0.5], [0.3, 0.4] >\}$, $\xi(b_2, p_3) = \{< u_1, [0.2, 0.4], [0.3, 0.5] >, < u_2, [0.1, 0.5], [0.4, 0.5] >, < u_3, [0.3, 0.5], [0.0, 0.3] >, < u_4, [0.4, 0.5], [0.2, 0.4] >\}$, $\xi(b_3, p_1) = \{< u_1, [0.4, 0.5], [0.1, 0.3] >, < u_2, [0.3, 0.4], [0.3, 0.5] >, < u_3, [0.1, 0.3], [0.3, 0.5] >, < u_4, [0.2, 0.3], [0.2, 0.4] >\}$,$\xi(b_3, p_2) = \{< u_1, [0.3, 0.4], [0.4, 0.5] >, < u_2, [0.4, 0.5], [0.2, 0.3] >, < u_3, [0.3, 0.4], [0.2, 0.4] >, < (u_4, [0.2, 0.4], [0.1, 0.4] >\}$, $\xi(b_3, p_3) = \{< u_1, [0.4, 0.5], [0.2, 0.3] >, < u_2, [0.3, 0.4], [0.4, 0.5] >, < u_3, [0.2, 0.5], [0.0, 0.3] >, < u_4, [0.2, 0.3], [0.2, 0.5] >\}$.

Step2:

p_1	u_1	u_2
b_1	$< [0.4, 0.5], [0.2, 0.3] >$	$< [0.1, 0.2], [0.6, 0.7] >$
b_2	$< [0.2, 0.3], [0.5, 0.6] >$	$< [0.6, 0.7], [0.1, 0.2] >$
b_3	$< [0.4, 0.5], [0.1, 0.3] >$	$< [0.3, 0.4], [0.3, 0.5] >$

p_1	u_3	u_4
b_1	$< [0.2, 0.4], [0.5, 0.6] >$	$< [0.3, 0.5], [0.1, 0.5] >$
b_2	$< [0.2, 0.4], [0.4, 0.5] >$	$< [0.3, 0.4], [0.2, 0.4] >$
b_3	$< [0.1, 0.3], [0.3, 0.5] >$	$< [0.2, 0.3], [0.2, 0.4] >$

Table 6.4.1. Opinion of expert p_1

p_2	u_1	u_2
b_1	$< [0.2, 0.4], [0.4, 0.5] >$	$< [0.4, 0.6], [0.1, 0.3] >$
b_2	$< [0.4, 0.6], [0.1, 0.2] >$	$< [0.3, 0.5], [0.3, 0.4] >$
b_3	$< [0.3, 0.4], [0.4, 0.5] >$	$< [0.4, 0.5], [0.2, 0.3] >$

p_2	u_3	u_4
b_1	$< [0.1, 0.2], [0.3, 0.5] >$	$< [0.3, 0.5], [0.4, 0.5] >$
b_2	$< [0.3, 0.5], [0.1, 0.5] >$	$< [0.2, 0.5], [0.3, 0.4] >$
b_3	$< [0.3, 0.4], [0.2, 0.4] >$	$< [0.2, 0.4], [0.1, 0.4] >$

Table 6.4.2. Opinion of expert p_2

p_3	u_1	u_2
b_1	$<[0.4,0.5],[0.3,0.4]>$	$<[0.4,0.5],[0.2,0.4]>$
b_2	$<[0.2,0.4],[0.3,0.5]>$	$<[0.1,0.5],[0.4,0.5]>$
b_3	$<[0.4,0.5],[0.2,0.3]>$	$<[0.3,0.4],[0.4,0.5]>$

p_3	u_3	u_4
b_1	$<[0.4,0.7],[0.2,0.3]>$	$<[0.1,0.3],[0.5,0.6]>$
b_2	$<[0.3,0.5],[0.0,0.3]>$	$<[0.4,0.5],[0.2,0.4]>$
b_3	$<[0.2,0.5],[0.0,0.3]>$	$<[0.2,0.3],[0.2,0.5]>$

Table 6.4.3. Opinion of expert p_3

Step3: $\varpi_k = (0.34, 0.28, 0.38)^t$ be a normalized weight vector of criteria.

Step4: $\omega = (0.25, 0.50, 0.25)^t$ be a position weight vector. The purpose of this weight vector is to eliminate the effect of individual preconception on comprehensive assessment.

Step5: Aggregate criteria by using the $IVIFSE$ fusion weighted average operator

$$\flat^F_{\varpi,\omega}((\xi_1, A_1, G_1), (\xi_2, A_2, G_2), ..., (\xi_m, A_m, G_m)) = \begin{cases} <[1-\prod_{i=1}^{m}(1-\widetilde{\gamma}^-_{i(b,p)})^{\omega_i}, \\ 1-\prod_{i=1}^{m}(1-\widetilde{\gamma}^+_{i(b,p)})^{\omega_i}], \\ [\prod_{i=1}^{m}(\widetilde{\zeta}^-_{i(b,p)})^{\omega_i}, \prod_{i=1}^{m}(\widetilde{\zeta}^+_{i(b,p)})^{\omega_i}]>, \end{cases}$$

where $(\xi_i, \widetilde{A_i, G_i})_{\widetilde{\gamma}_i,\widetilde{\zeta}_i} =< [\widetilde{\gamma}^-_{i(b,p)}, \widetilde{\gamma}^+_{i(b,p)}], [\widetilde{\zeta}^-_{i(b,p)}, \widetilde{\zeta}^+_{i(b,p)}] >$ is the $i-th$ largest of the m $IVIFSE$ $sets$ $\overbrace{(\xi_k, A_k, G_k)_{\gamma_k,\zeta_k}} = m\varpi_k(\xi_k,A_k,G_k)_{\gamma_k,\zeta_k}$ which can be determined by using the ranking method of $IVIFSE$ $sets$ such as score and accuracy function.

First, we calculate $\overbrace{(\xi_k, A_k, G_k)_{\gamma_k,\zeta_k}} = m\varpi_k(\xi_k,A_k,G_k)_{\gamma_k,\zeta_k}$ by using normalized weight vector of criteria. In this example, $m = 3$, $\overbrace{b_1} = 3(0.34) < [0.4, 0.5], [0.2, 0.3] >= 1.02 < [0.4, 0.5], [0.2, 0.3] >=< [1-(1-0.4)^{1.02}, 1-(1-0.5)^{1.02}], [0.2^{1.02}, 0.3^{1.02}] >=< [0.4061, 0.5069], [0.1937, 0.2929] >$.

p_1	u_1	u_2
$\widehat{b_1}$	$<[0.4061, 0.5069], [0.1937, 0.2929]>$	$<[0.1019, 0.2036], [0.5939, 0.6950]>$
$\widehat{b_2}$	$<[0.1709, 0.2589], [0.5586, 0.6511]>$	$<[0.5368, 0.6363], [0.1445, 0.2587]>$
$\widehat{b_3}$	$<[0.4414, 0.5462], [0.0724, 0.2535]>$	$<[0.3341, 0.4414], [0.2535, 0.4538]>$

p_1	u_3	u_4
$\widehat{b_1}$	$<[0.2036, 0.4061], [0.4931, 0.5939]>$	$<[0.3050, 0.5069], [0.0955, 0.4931]>$
$\widehat{b_2}$	$<[0.1709, 0.3489], [0.4632, 0.5586]>$	$<[0.2589, 0.3489], [0.2587, 0.4632]>$
$\widehat{b_3}$	$<[0.1132, 0.3341], [0.2535, 0.4538]>$	$<[0.2246, 0.3341], [0.1597, 0.3518]>$

Table 6.4.4.

p_2	u_1	u_2
$\widehat{b_1}$	$<[0.2036, 0.4061], [0.3927, 0.4931]>$	$<[0.4061, 0.6073], [0.0955, 0.2929]>$
$\widehat{b_2}$	$<[0.3489, 0.5368], [0.1445, 0.2587]>$	$<[0.2589, 0.4414], [0.3637, 0.4632]>$
$\widehat{b_3}$	$<[0.3341, 0.4414], [0.3518, 0.4538]>$	$<[0.4414, 0.5462], [0.1597, 0.2535]>$

p_2	u_3	u_4
$\widehat{b_1}$	$<[0.1019, 0.2036], [0.2929, 0.4931]>$	$<[0.3050, 0.5069], [0.3927, 0.4931]>$
$\widehat{b_2}$	$<[0.2589, 0.4414], [0.1445, 0.5586]>$	$<[0.1709, 0.4414], [0.3637, 0.4632]>$
$\widehat{b_3}$	$<[0.3341, 0.4414], [0.1597, 0.3518]>$	$<[0.2246, 0.4414], [0.0724, 0.3518]>$

Table 6.4.5.

p_3	u_1	u_2
$\widehat{b_1}$	$<[0.4061, 0.5069], [0.2929, 0.3927]>$	$<[0.4061, 0.5069], [0.1937, 0.3927]>$
$\widehat{b_2}$	$<[0.1709, 0.3489], [0.3637, 0.5586]>$	$<[0.0847, 0.4414], [0.4632, 0.5586]>$
$\widehat{b_3}$	$<[0.4414, 0.5462], [0.1597, 0.2535]>$	$<[0.3341, 0.4414], [0.3518, 0.4538]>$

p_3	u_3	u_4
$\widehat{b_1}$	$<[0.4061, 0.7071], [0.1937, 0.2929]>$	$<[0.1019, 0.3050], [0.4931, 0.5939]>$
$\widehat{b_2}$	$<[0.2589, 0.4414], [0.0000, 0.3637]>$	$<[0.3489, 0.4414], [0.2587, 0.4632]>$
$\widehat{b_3}$	$<[0.2246, 0.5462], [0.0000, 0.2535]>$	$<[0.2246, 0.3341], [0.1597, 0.4538]>$

Table 6.4.6.

Find scores of each of the above elements by using Definition 6.2.25 as below:

p_1	u_1	u_2	u_3	u_4
b_1	0.2132	−0.4917	−0.2387	0.1117
b_2	−0.3900	0.3850	−0.2510	−0.0571
b_3	0.3309	0.0341	−0.1300	0.0236

Table 6.4.7.

p_2	u_1	u_2	u_3	u_4
b_1	−0.1381	0.3107	−0.2403	−0.0370
b_2	0.2413	−0.0633	−0.0014	−0.1073
b_3	−0.0151	0.2872	0.1320	0.1209

Table 6.4.8.

p_3	u_1	u_2	u_3	u_4
b_1	0.1137	0.1633	0.3133	−0.3401
b_2	−0.2013	−0.2479	0.1683	0.0342
b_3	0.2872	−0.0151	0.2587	−0.0274

Table 6.4.9.

Now find the $i-th$ largest of the m $IVIFSESs$.

p_1	u_1	u_2
$\widetilde{b_1}$	$<[0.4414, 0.5462], [0.0724, 0.2535]>$	$<[0.5368, 0.6363], [0.1445, 0.2587]>$
$\widetilde{b_2}$	$<[0.4061, 0.5069], [0.1937, 0.2929]>$	$<[0.3341, 0.4414], [0.2535, 0.4538]>$
$\widetilde{b_3}$	$<[0.1709, 0.2589], [0.5586, 0.6511]>$	$<[0.1019, 0.2036], [0.5939, 0.6950]>$

p_1	u_3	u_4
$\widetilde{b_1}$	$<[0.1132, 0.3341], [0.2535, 0.4538]>$	$<[0.3050, 0.5069], [0.0955, 0.4931]>$
$\widetilde{b_2}$	$<[0.2036, 0.4061], [0.4931, 0.5939]>$	$<[0.2246, 0.3341], [0.1597, 0.3518]>$
$\widetilde{b_3}$	$<[0.1709, 0.3489], [0.4632, 0.5586]>$	$<[0.2589, 0.3489], [0.2587, 0.4632]>$

Table 6.4.10.

p_2	u_1	u_2
$\widetilde{b_1}$	$<[0.3489, 0.5368], [0.1445, 0.2587]>$	$<[0.4061, 0.6073], [0.0955, 0.2929]>$
$\widetilde{b_2}$	$<[0.3341, 0.4414], [0.3518, 0.4538]>$	$<[0.4414, 0.5462], [0.1597, 0.2535]>$
$\widetilde{b_3}$	$<[0.2036, 0.4061], [0.3927, 0.4931]>$	$<[0.2589, 0.4414], [0.3637, 0.4632]>$

p_2	u_3	u_4
$\widetilde{b_1}$	$<[0.3341, 0.4414], [0.1597, 0.3518]>$	$<[0.2246, 0.4414], [0.0724, 0.3518]>$
$\widetilde{b_2}$	$<[0.2589, 0.4414], [0.1445, 0.5586]>$	$<[0.3050, 0.5069], [0.3927, 0.4931]>$
$\widetilde{b_3}$	$<[0.1019, 0.2036], [0.2929, 0.4931]>$	$<[0.1709, 0.4414], [0.3637, 0.4632]>$

Table 6.4.11.

p_3	u_1	u_2
$\widetilde{b_1}$	$<[0.4414, 0.5462], [0.1597, 0.2535]>$	$<[0.4061, 0.5069], [0.1937, 0.3927]>$
$\widetilde{b_2}$	$<[0.4061, 0.5069], [0.2929, 0.3927]>$	$<[0.3341, 0.4414], [0.3518, 0.4538]>$
$\widetilde{b_3}$	$<[0.1709, 0.3489], [0.3637, 0.5586]>$	$<[0.0847, 0.4414], [0.4632, 0.5586]>$

p_3	u_3	u_4
$\widetilde{b_1}$	$<[0.4061, 0.7071], [0.1937, 0.2929]>$	$<[0.3489, 0.4414], [0.2587, 0.4632]>$
$\widetilde{b_2}$	$<[0.2246, 0.5462], [0.0000, 0.2535]>$	$<[0.2246, 0.3341], [0.1597, 0.4538]>$
$\widetilde{b_3}$	$<[0.2589, 0.4414], [0.0000, 0.3637]>$	$<[0.1019, 0.3050], [0.4931, 0.5939]>$

Table 6.4.12.

Further, aggregate criteria by using the $IVIFSE$ fusion weighted average operator. For example for u_1 corresponding to expert p_1, $<[1-(1-0.4414)^{0.25}(1-0.4061)^{0.50}(1-0.1709)^{0.25}, 1-(1-0.5462)^{0.25}(1-0.5069)^{0.50}(1-0.2589)^{0.25}], [0.0724^{0.25}0.1937^{0.50}0.5586^{0.25}, 0.2535^{0.25}0.2929^{0.50}0.6511^{0.25}]>=<[0.3643, 0.4652], [0.1974, 0.3450]>$.

	u_1	u_2
p_1	$<[0.3643, 0.4652], [0.1974, 0.3450]>$	$<[0.3446, 0.4517], [0.2725, 0.4387]>$
p_2	$<[0.3075, 0.4587], [0.2895, 0.4026]>$	$<[0.3912, 0.5390], [0.1725, 0.3056]>$
p_3	$<[0.3643, 0.4823], [0.2657, 0.3844]>$	$<[0.2993, 0.4585], [0.3246, 0.4610]>$

	u_3	u_4
p_1	$<[0.1736, 0.3747], [0.4111, 0.5468]>$	$<[0.2540, 0.3857], [0.2256, 0.4100]>$
p_2	$<[0.2430, 0.3896], [0.0000, 0.4823]>$	$<[0.2535, 0.4752], [0.2524, 0.4462]>$
p_3	$<[0.2828, 0.5716], [0.0000, 0.2876]>$	$<[0.2300, 0.3559], [0.2388, 0.4879]>$

Table 6.4.13. Aggregated criteria

Step6: Calculate accuracy of each member of U corresponding to each expert,

	u_1	u_2	u_3	u_4
p_1	0.6860	0.7538	0.7531	0.6377
p_2	0.7292	0.7042	0.5575	0.7137
p_3	0.7484	0.7717	0.5710	0.6563

Table 6.4.14. Accuracies

Step7: Calculate average $u_1 = 0.7212$, $u_2 = 0.7432$, $u_3 = 0.6272$, $u_4 = 0.6692$.
Step8: Generate the non decreasing chain of these averages, we get

$$u_2 > u_1 > u_4 > u_3$$

Step9: Hence u_2 is the best one.

6.5 Conclusion and Future Work

In this chapter, $IVIFSE$ *set* has been defined. In this structure at the same time we consider three features of the membership degree, nonmembership degree, and hesitancy degree. In decision analysis this structure is more flexible and realistic for dealing with ambiguity and uncertainty than the fuzzy sets. Some operations of $IVIFSE$ *Set* have been defined. We have introduced some operators for $IVIFSE$ *sets*. Through these operators we can make a good decision in multi criteria decision making. An algorithm has been developed for multi criteria decision making problems with the aid of $IVIFSE$ *sets*. The suggested technique is better than the existing techniques on the basis of experts opinion. We aim to construct decision making techniques parallel to AHP and $TOPSIS$ using this structure. We also aim to study distance, entropy measures and similarity measures for this structure.

Chapter 7

Matrix Algebra of $GSESs$, $CSESs$ and $IVIFSESs$

7.1 Introduction

Cagman et al. presented the concept of soft matrices and fuzzy soft matrices in [11] and [14]. Chetia et al. also commented on some results of intuitionistic soft matrix theory in [16]. There are certain multicriteria decision making problems in which ordinary matrix algebra work to fail due to its own operations. But by using soft matrices structure we can easily tackle these types of problems. Ordinary matrix algebra have some limitations in their laws. In order to get rid of these problems we may use soft matrices operations.

In this chapter, we first define graded soft expert matrices, cubic soft expert matrices and interval-valued intuitionistic fuzzy soft expert matrices which are representations of graded soft expert sets, cubic soft expert sets and interval-valued intuitionistic fuzzy soft expert sets. Using matrix representation the information can be stored and manipulated easily. This representation also makes the multicriteria decision making problems easy to handle. Using this representation, we can easily compare the opinion of experts in meaningful way. There are some interesting results which do not hold in ordinary matrix algebra but these results holds in graded soft expert matrices, cubic soft expert matrices and interval-valued intuitionistic fuzzy soft expert matrices, for example, commutative law with respect to product holds.

7.2 Matrix Algebra of Graded Soft Expert Sets ($GSESs$)

In this section, we define matrices algebra for graded soft expert sets. Further we discuss some operations on it and investigate several properties with respect to their operations.

Definition 7.2.1 *Let U be a finite universe set containing n alternatives, E; a set of criteria and X; a set of experts (or decision makers). Let O be a set of opinions with a given preference relation $\precsim$ among the opinions. A graded soft expert set (abbreviated as GSE set) (F, A, Y) is characterized by a mapping $F : A \times Y \longrightarrow P(U \times O)$ defined for every $e \in A$ and $p \in Y$ by $F(e,p) = \{(u_i, o_i) : i \in I\}$, where $I = \{1, 2, 3, ..., n\}$, $A \subseteq E$, $Y \subseteq X$ and $P(U \times O)$ denotes the power set of $U \times O$. Here the set of opinions O contains graded values of the given parameters i.e. the values $o_1, o_2, ..., o_n$ can be graded as $o_1 \precsim o_2 \precsim ... \precsim o_n$ which means that o_n is the most preferred value while o_1 is the least preferred one and so forth.*

Definition 7.2.2 *Let U be the initial universe, E; a set of criteria and X; a set of experts (or decision makers). and O be a set of opinions. $F(a, y) = \{(u_p, o_p) : p \in I\}$, where $I = \{1, 2, 3, ..., q\}$ is a graded soft expert set*

y	u_1	u_2	u_3	...	u_m
a_1	(a_1, u_1)	(a_1, u_2)	(a_1, u_3)	...	(a_1, u_m)
a_2	(a_2, u_1)	(a_2, u_2)	(a_2, u_3)	...	(a_2, u_m)
a_3	(a_3, u_1)	(a_3, u_2)	(a_3, u_3)	...	(a_3, u_m)
...	...	...	...	...	...
a_l	(a_l, u_1)	(a_l, u_2)	(a_l, u_3)	...	(a_l, u_m)

Table 7.2.1

If a_{ij} = opinion of expert y corresponding to pair $(a_i, u_j) = o_j$,we can define

$$\ddot{A}_y = [a_{ij}]_{l\times m} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{21} \\ \cdots & . & \cdots & \cdots \\ a_{l1} & a_{l2} & \cdots & a_{lm} \end{bmatrix}$$

it is called Graded soft expert matrix $(GSEM)$ of the Graded soft expert set $(GSES)$ of order $l \times m$ over U.

Example 7.2.3 *Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be a set of alternatives, $E = \{a_1, a_2\}$ be a set of criteria, $X = \{y_1, y_2\}$ be a set of experts and $O = \{0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$ be the set of possible grades for the given parameters. Let GSE set is given as follows* $F(a_1, y_1) = \{(u_1, 0.5), (u_2, 0.1), (u_3, 0.7), (u_4, 0.9), (u_5, 0.2)\}$, $F(a_1, y_2) = \{(u_1, 0.5), (u_2, 0.2), (u_3, 0.7), (u_4, 0.3), (u_5, 0.4)\}$, $F(a_2, y_1) = \{(u_1, 0.9), (u_2, 0.3), (u_3, 0.2), (u_4, 0.3), (u_5, 0.6)\}$, $F(a_2, y_2) = \{(u_1, 0.8), (u_2, 0.9), (u_3, 0.4), (u_4, 0.1), (u_5, 0.4)\}$.

Graded soft expert matrix ($GSEM$) *of the Graded soft expert set* ($GSES$) *corresponding to expert* y_1 *is given as* $\ddot{A}_{y_1} = \begin{bmatrix} 0.5 & 0.1 & 0.7 & 0.9 & 0.2 \\ 0.9 & 0.3 & 0.2 & 0.3 & 0.6 \end{bmatrix}_{2\times5}$

Graded soft expert matrix ($GSEM$) *of the Graded soft expert set* ($GSES$) *corresponding to expert* y_2 *is given as* $\ddot{A}_{y_2} = \begin{bmatrix} 0.5 & 0.2 & 0.7 & 0.3 & 0.4 \\ 0.8 & 0.9 & 0.4 & 0.1 & 0.4 \end{bmatrix}_{2\times5}$

Collection of all GSE *matrices is denoted by* $GSEMs_{l\times m}$.

7.2.1 Operations on GSE Matrices

Now we define some operations on GSE matrices.

Definition 7.2.4 *Let* $\ddot{A}_0 = [a_{ij}]_{l\times m} \in GSEMs_{l\times m}$. $\ddot{A}_0$ *is called zero GSE matrix if for all* i *and* j $a_{ij} = 0$.

Example 7.2.5 $\ddot{A}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times4}$.

Definition 7.2.6 *Let* $\ddot{A}_U = [a_{ij}]_{l\times m} \in GSEMs_{l\times m}$. $\ddot{A}_U$ *is called universe GSE matrix if for all* i *and* j $a_{ij} = 1$.

Example 7.2.7 $\ddot{A}_U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}_{4\times4}$.

Definition 7.2.8 *Let* $\ddot{A}_y = [a_{ij}]_{l\times m}$ *where* $a_{ij} = o_j$, $\ddot{A}_z = [b_{ij}]_{l\times m}$ *where* $b_{ij} = o_j^{'} \in GSEMs_{l\times m}$.

Product of two GSE matrices is denoted and defined as $\ddot{A}_y \otimes \ddot{A}_z = [a_{ij} \otimes b_{ij}]_{l\times m}$ *where* $a_{ij} \otimes b_{ij} = (o_j)(o_j^{'})$.

Example 7.2.9 *Let* $\ddot{A}_y = \begin{bmatrix} 0.3 & 0.3 & 0.4 & 0.3 \\ 0.2 & 0.1 & 0.9 & 0.5 \\ 0.9 & 0.4 & 0.6 & 0.7 \\ 0.5 & 0.7 & 0.8 & 0.2 \end{bmatrix}_{4\times4}$ *and* $\ddot{A}_z = \begin{bmatrix} 0.1 & 0.2 & 0.7 & 0.2 \\ 0.4 & 0.4 & 0.3 & 0.5 \\ 0.5 & 0.6 & 0.5 & 0.6 \\ 0.7 & 0.8 & 0.2 & 0.2 \end{bmatrix}_{4\times4}$

then $\ddot{A}_y \otimes \ddot{A}_z = \begin{bmatrix} 0.03 & 0.06 & 0.28 & 0.06 \\ 0.08 & 0.04 & 0.27 & 0.25 \\ 0.45 & 0.24 & 0.30 & 0.42 \\ 0.35 & 0.56 & 0.16 & 0.04 \end{bmatrix}_{4\times4}$.

Definition 7.2.10 *Let* $\ddot{A}_y = [a_{ij}]_{l\times m}$ *where* $a_{ij} = o_j$, $\ddot{A}_z = [b_{ij}]_{l\times m}$ *where* $b_{ij} = o_j^{'} \in GSEMs_{l\times m}$.

Min-Product of two GSE matrices is denoted and defined as $\ddot{A}_y \triangle \ddot{A}_z = [a_{ij} \triangle b_{ij}]_{l\times m}$ *where* $a_{ij} \triangle b_{ij} = (o_j \wedge o_j^{'})$.

Example 7.2.11 *Consider* $\ddot{A}_y$, $\ddot{A}_z$ *of Example 7.2.9, then* $\ddot{A}_y \triangle \ddot{A}_z = \begin{bmatrix} 0.1 & 0.2 & 0.4 & 0.2 \\ 0.2 & 0.1 & 0.3 & 0.5 \\ 0.5 & 0.4 & 0.5 & 0.6 \\ 0.5 & 0.7 & 0.2 & 0.2 \end{bmatrix}_{4\times 4}$.

Definition 7.2.12 *Let* $\ddot{A}_y = [a_{ij}]_{l\times m}$ *where* $a_{ij} = o_j$, $\ddot{A}_z = [b_{ij}]_{l\times m}$ *where* $b_{ij} = o_j^{'} \in GSEMs_{l\times m}$.

Max-Product of two GSE matrices is denoted and defined as $\ddot{A}_y \triangledown \ddot{A}_z = [a_{ij} \triangledown b_{ij}]_{l\times m}$ *where* $a_{ij} \triangledown b_{ij} = (o_j \vee o_j^{'})$.

Example 7.2.13 *Consider* $\ddot{A}_y$, $\ddot{A}_z$ *of Example 7.2.9, then* $\ddot{A}_y \triangledown \ddot{A}_z = \begin{bmatrix} 0.3 & 0.3 & 0.7 & 0.3 \\ 0.4 & 0.4 & 0.9 & 0.5 \\ 0.9 & 0.6 & 0.6 & 0.7 \\ 0.7 & 0.8 & 0.8 & 0.2 \end{bmatrix}_{4\times 4}$.

Definition 7.2.14 *Let* $\ddot{A}_y = [a_{ij}]_{l\times m}$ *where* $a_{ij} = o_j$, $\ddot{A}_z = [b_{ij}]_{l\times m}$ *where* $b_{ij} = o_j^{'} \in GSEMs_{l\times m}$.

Addition of two GSE matrices is denoted and defined as $\ddot{A}_y \oplus \ddot{A}_z = [a_{ij} \oplus b_{ij}]_{l\times m}$ *where* $a_{ij} \oplus b_{ij} = o_j + o_j^{'} - o_j o_j^{'}$.

Example 7.2.15 *Consider* $\ddot{A}_y$, $\ddot{A}_z$ *of Example 7.2.9, and* $\ddot{A}_z = \begin{bmatrix} 0.1 & 0.2 & 0.7 & 0.2 \\ 0.4 & 0.4 & 0.3 & 0.5 \\ 0.5 & 0.6 & 0.5 & 0.6 \\ 0.7 & 0.8 & 0.2 & 0.2 \end{bmatrix}_{4\times 4}$

then $\ddot{A}_y \oplus \ddot{A}_z = \begin{bmatrix} 0.37 & 0.44 & 0.82 & 0.44 \\ 0.52 & 0.46 & 0.93 & 0.75 \\ 0.95 & 0.76 & 0.8 & 0.88 \\ 0.85 & 0.94 & 0.84 & 0.36 \end{bmatrix}_{4\times 4}$.

Definition 7.2.16 *Let* $\ddot{A}_y = [a_{ij}]_{l\times m}$ *where* $a_{ij} = o_j \in GSEMs_{l\times m}$. *Scalar product of GSE matrix with real number* $k > 0$ *is denoted by* $k\ddot{A}_y = [ka_{ij}]$ *where* $ka_{ij} = 1 - (1 - o_j)^k$.

Example 7.2.17 *Consider* $\ddot{A}_y$ *of Example 7.2.9, let* $k = 0.9$ *then*

$$0.9\ddot{A}_y = \begin{bmatrix} 0.2745 & 0.2745 & 0.3685 & 0.2745 \\ 0.1819 & 0.0904 & 0.8741 & 0.4641 \\ 0.8741 & 0.3685 & 0.5616 & 0.6616 \\ 0.4641 & 0.6616 & 0.7650 & 0.1819 \end{bmatrix}_{4\times 4}.$$

Definition 7.2.18 *Let $\ddot{A}_y = [a_{ij}]_{l\times m}$ where $a_{ij} = o_j \in GSEMs_{l\times m}$. Power of GSE matrix is denoted by $(\ddot{A}_y)^k = [(a_{ij})^k]$ for $k > 0$ where $(a_{ij})^k = (o_j)^k$.*

Example 7.2.19 *Consider $\ddot{A}_y \in GSEMs_{l\times m}$ of Example 7.2.9, let $k = 0.34$ then*

$$(\ddot{A}_y)^{0.34} = \begin{bmatrix} 0.6640 & 0.6640 & 0.7323 & 0.6640 \\ 0.5785 & 0.4570 & 0.9648 & 0.7900 \\ 0.9648 & 0.7323 & 0.8405 & 0.8858 \\ 0.7900 & 0.8858 & 0.9269 & 0.5785 \end{bmatrix}_{4\times 4}.$$

Definition 7.2.20 *Let $\ddot{A}_y = [a_{ij}]_{l\times m}$ where $a_{ij} = o_j \in GSEMs_{l\times m}$. Complement of GSE matrix is denoted by $(\ddot{A}_y)^c = [(a_{ij})^c]$ where $(a_{ij})^c = 1 - o_j$.*

Example 7.2.21 *Consider $\ddot{A}_y$ of Example 7.2.9, then* $(\ddot{A}_y)^c = \begin{bmatrix} 0.7 & 0.7 & 0.6 & 0.7 \\ 0.8 & 0.9 & 0.1 & 0.5 \\ 0.1 & 0.6 & 0.4 & 0.3 \\ 0.5 & 0.3 & 0.2 & 0.8 \end{bmatrix}_{4\times 4}.$

Definition 7.2.22 *Let $\ddot{A}_y = [a_{ij}]_{l\times m}$ where $a_{ij} = o_j$,, $\ddot{A}_z = [b_{ij}]_{l\times m}$ where $b_{ij} = o_j' \in GSEMs_{l\times m}$. Then $\ddot{A}_y = [a_{ij}]_{l\times m}$ is a GSE sub matrix of $\ddot{A}_z = [b_{ij}]_{l\times m}$ if $a_{ij} \leq b_{ij}$ for all i and j. It is denoted by $\ddot{A}_y \Subset \ddot{A}_z$.*

7.2.2 Properties of GSE Matrices

In below we discuss some properties of GSE matrices.

Proposition 7.2.23 *Let $\ddot{A}_y = [a_{ij}]_{l\times m}$, $\ddot{A}_z = [b_{ij}]_{l\times m} \in GSEMs_{l\times m}$. Then commutative, De Morgan's, involution laws with respect to Min-Product and Max-product and double negation law also hold:*

1) $\ddot{A}_y \bigtriangledown \ddot{A}_z = \ddot{A}_z \bigtriangledown \ddot{A}_y$
2) $\ddot{A}_y \bigtriangleup \ddot{A}_z = \ddot{A}_z \bigtriangleup \ddot{A}_y$
3) $(\ddot{A}_y \bigtriangledown \ddot{A}_z)^c = (\ddot{A}_z)^c \bigtriangleup (\ddot{A}_y)^c$
4) $(\ddot{A}_y \bigtriangleup \ddot{A}_z)^c = (\ddot{A}_z)^c \bigtriangledown (\ddot{A}_y)^c$
5) $\ddot{A}_z \bigtriangledown \ddot{A}_z = \ddot{A}_z$
6) $\ddot{A}_z \bigtriangleup \ddot{A}_z = \ddot{A}_z$
7) $((\ddot{A}_z)^c)^c = \ddot{A}_z$.

Proof. Straightforward. ■

Proposition 7.2.24 *Let $\ddot{A}_y = [a_{ij}]_{l\times m}$, $\ddot{A}_z = [b_{ij}]_{l\times m}$, $\ddot{A}_t = [c_{ij}]_{l\times m} \in GSEMs_{l\times m}$. Then associative and distributive laws also hold with respect to Min-Product and Max-Product:*

$$1)\ \ddot{A}_y \bigtriangledown (\ddot{A}_z \bigtriangledown \ddot{A}_t) = (\ddot{A}_y \bigtriangledown \ddot{A}_z) \bigtriangledown \ddot{A}_t$$
$$2)\ \ddot{A}_y \bigtriangleup (\ddot{A}_z \bigtriangleup \ddot{A}_t) = (\ddot{A}_y \bigtriangleup \ddot{A}_z) \bigtriangleup \ddot{A}_t$$
$$3)\ \ddot{A}_y \bigtriangleup (\ddot{A}_z \bigtriangledown \ddot{A}_t) = (\ddot{A}_y \bigtriangleup \ddot{A}_z) \bigtriangledown (\ddot{A}_y \bigtriangleup \ddot{A}_t)$$
$$4)\ \ddot{A}_y \bigtriangledown (\ddot{A}_z \bigtriangleup \ddot{A}_t) = (\ddot{A}_y \bigtriangledown \ddot{A}_z) \bigtriangleup (\ddot{A}_y \bigtriangledown \ddot{A}_t).$$

Proposition 7.2.25 *Let $\ddot{A}_y = [a_{ij}]_{l\times m}$, $\ddot{A}_z = [b_{ij}]_{l\times m} \in GSEMs_{l\times m}$. Then Commutative and De Morgan's laws with respect to Addition and product also hold:*

$$1)\ \ddot{A}_y \oplus \ddot{A}_z = \ddot{A}_z \oplus \ddot{A}_y$$
$$2)\ \ddot{A}_y \otimes \ddot{A}_z = \ddot{A}_z \otimes \ddot{A}_y$$
$$3)\ (\ddot{A}_y \oplus \ddot{A}_z)^c = (\ddot{A}_z)^c \otimes (\ddot{A}_y)^c$$
$$4)\ (\ddot{A}_y \otimes \ddot{A}_z)^c = (\ddot{A}_z)^c \oplus (\ddot{A}_y)^c.$$

Proof. Straightforward. ■

Proposition 7.2.26 *Let $\ddot{A}_y = [a_{ij}]_{l\times m}$, $\ddot{A}_z = [b_{ij}]_{l\times m}$, $\ddot{A}_t = [c_{ij}]_{l\times m} \in GSEMs_{l\times m}$. Then associative law also hold with respect to addition and product:*

$$1)\ \ddot{A}_y \oplus (\ddot{A}_z \oplus \ddot{A}_t) = (\ddot{A}_y \oplus \ddot{A}_z) \oplus \ddot{A}_t$$
$$2)\ \ddot{A}_y \otimes (\ddot{A}_z \otimes \ddot{A}_t) = (\ddot{A}_y \otimes \ddot{A}_z) \otimes \ddot{A}_t.$$

Proof. Consider $\ddot{A}_y = [a_{ij}]_{l\times m}$, where $a_{ij} = o_j$, $\ddot{A}_z = [b_{ij}]_{l\times m}$, where $b_{ij} = o_j^{'}$, $\ddot{A}_t = [c_{ij}]_{l\times m}$, where $c_{ij} = o_j^{''}$.

1) $\ddot{A}_y \oplus (\ddot{A}_z \oplus \ddot{A}_t) = a_{ij} \oplus (b_{ij} \oplus c_{ij}) = o_j \oplus (o_j^{'} + o_j^{''} - o_j^{'}o_j^{''}) = o_j + (o_j^{'} + o_j^{''} - o_j^{'}o_j^{''}) - o_j(o_j^{'} + o_j^{''} - o_j^{'}o_j^{''}) = o_j + o_j^{'} + o_j^{''} - o_j^{'}o_j^{''} - o_jo_j^{'} - o_jo_j^{''} + o_jo_j^{'}o_j^{''} = o_j + o_j^{'} - o_jo_j^{'} + o_j^{''} - o_jo_j^{''} - o_j^{'}o_j^{''} + o_jo_j^{'}o_j^{''} = (o_j + o_j^{'} - o_j^{'}o_j) \oplus o_j^{''} = (\ddot{A}_y \oplus \ddot{A}_z) \oplus \ddot{A}_t$.

2) $\ddot{A}_y \otimes (\ddot{A}_z \otimes \ddot{A}_t) = a_{ij} \otimes (b_{ij} \otimes c_{ij}) = o_j(o_j^{'}o_j^{''}) = (o_jo_j^{'})o_j^{''} = (\ddot{A}_y \otimes \ddot{A}_z) \otimes \ddot{A}_t$. ■

Proposition 7.2.27 *Let $\ddot{A}_y = [a_{ij}]_{l\times m}$, $\ddot{A}_0 = [b_{ij}]_{l\times m}$, $\ddot{A}_U = [c_{ij}]_{l\times m} \in GSEMs_{l\times m}$. Then*

1) $\ddot{A}_0 \Subset \ddot{A}_y$	9) $\ddot{A}_U \oplus \ddot{A}_y = \ddot{A}_U$
2) $\ddot{A}_y \Subset \ddot{A}_U$	10) $\ddot{A}_U \otimes \ddot{A}_y = \ddot{A}_y$
3) $(\ddot{A}_0)^c = \ddot{A}_U$	11) $\ddot{A}_U \bigtriangledown \ddot{A}_y = \ddot{A}_U$
4) $(\ddot{A}_U)^c = \ddot{A}_0$	12) $\ddot{A}_U \bigtriangleup \ddot{A}_y = \ddot{A}_y$
5) $\ddot{A}_0 \oplus \ddot{A}_y = \ddot{A}_y$	13) $\ddot{A}_U \oplus \ddot{A}_0 = \ddot{A}_U$
6) $\ddot{A}_0 \otimes \ddot{A}_y = \ddot{A}_0$	14) $\ddot{A}_U \otimes \ddot{A}_0 = \ddot{A}_0$
7) $\ddot{A}_0 \bigtriangledown \ddot{A}_y = \ddot{A}_y$	15) $\ddot{A}_U \bigtriangledown \ddot{A}_0 = \ddot{A}_U$
8) $\ddot{A}_0 \bigtriangleup \ddot{A}_y = \ddot{A}_0$	16) $\ddot{A}_U \bigtriangleup \ddot{A}_0 = \ddot{A}_0$.

Proof. Straightforward. ■

Remark 7.2.28 *Distributive law with respect to addition over product do not hold in $GSEMs$.*

Example 7.2.29 *Let* $\ddot{A}_x = [0.5]_{1\times1}$, $\ddot{A}_y = [0.3]_{1\times1}$ *and* $\ddot{A}_z = [0.9]_{1\times1}$.
$\ddot{A}_x \oplus (\ddot{A}_y \otimes \ddot{A}_z) = [0.5] \oplus ([\ 0.3] \otimes [0.9]) = [\ 0.5] \oplus [0.27] = [0.635]$.
$(\ddot{A}_x \oplus \ddot{A}_y) \otimes (\ddot{A}_x \oplus \ddot{A}_z) = ([0.5] \oplus [0.3]) \otimes ([0.5] \oplus [0.9]) = [0.65] \otimes [0.95] = [0.6175]$.
Hence $\ddot{A}_x \oplus (\ddot{A}_y \otimes \ddot{A}_z) \neq (\ddot{A}_x \oplus \ddot{A}_y) \otimes (\ddot{A}_x \oplus \ddot{A}_z)$.

7.3 Matrix Algebra of Cubic Soft Expert Sets ($CSESs$)

In this section, we define matrices algebra for cubic soft expert sets. Further we discuss some operations on it and investigate several properties with respect to their operations.

Definition 7.3.1 *Let U be a finite universe set containing n alternatives, E; a set of criteria and X; a set of experts (or decision makers). A pair (β, E, X) is called a cubic soft expert set over U if and only if $\beta : E \times X \longrightarrow CP(U)$ is a mapping into the set of all cubic sets in U. Cubic soft expert set is denoted and defined as*

$$(\beta, E, X) = \{\beta(e, x) = \{< u, A_{(e,x)}(u), \lambda_{(e,x)}(u) >: u \in U, (e, x) \in E \times X\}.$$

where $A_{(e,x)}(u)$ is an interval valued fuzzy set and $\lambda_{(e,x)}(u)$ is a fuzzy set. Here decision makers give their opinions in the form of cubic set.

The collection of all cubic soft expert sets $CSESs$ is denoted as $\widehat{\beta}$.

Definition 7.3.2 *Let U be the initial universe, E; a set of criteria and X; a set of experts (or decision makers). $\beta(e, x) = \{< u, A_{(e,x)}(u), \lambda_{(e,x)}(u) >: u \in U, (e, x) \in E \times X$ be the cubic soft expert set.*

x	u_1	u_2	u_3	...	u_m
e_1	(e_1, u_1)	(e_1, u_2)	(e_1, u_3)	...	(e_1, u_m)
e_2	(e_2, u_1)	(e_2, u_2)	(e_2, u_3)	...	(e_2, u_m)
e_3	(e_3, u_1)	(e_3, u_2)	(e_3, u_3)	...	(e_3, u_m)
...	...	...	...	...	...
e_l	(e_l, u_1)	(e_l, u_2)	(e_l, u_3)	...	(e_l, u_m)

Table 7.3.1

If a_{ij} = *opinion of expert* x *corresponding to pair* $(e_i, u_j) =< A = [A^-, A^+], \lambda >$, *we can define* $\ddot{B}_x = [a_{ij}]_{l\times m} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{21} \\ \dots & \dots & \dots & \dots \\ a_{l1} & a_{l2} & \dots & a_{lm} \end{bmatrix}$.

it is called cubic soft expert matrix $(CSEM)$ *of the cubic soft expert set* $(CSES)$ *of order* $l \times m$ *over* U.

Collection of all cubic soft expert matrices $(CSE$ *matrices*$)$ *is denoted by* $CSEMs_{l\times m}$.

Example 7.3.3 *Let* $U = \{u_1, u_2, u_3\}$ *be the initial universe,* $E = \{e_1, e_2\}$ *be the set of attributes,* $X = \{x_1, x_2\}$ *be the set of experts. Then the cubic set* $(\beta, E, X) = \{\beta(e,x) = \{< u,\ A_{(e,x)}(u),\ \lambda_{(e,x)}(u) >; u \in U, (e,x) \in (E \times X)\}$ *in* U *is an internal cubic soft expert set.*

$\beta(e_1, x_1) = \{(u_1, [0.5, 0.8], 0.7), (u_2, [0.6, 0.9], 0.8), (u_3, [0.4, 0.7], 0.5)\}$,
$\beta(e_2, x_1) = \{(u_1, [0.4, 0.7], 0.6), (u_2, [0.7, 0.9], 0.8), (u_3, [0.3, 0.5], 0.4)\}$,
$\beta(e_1, x_2) = \{(u_1, [0.4, 0.8], 0.5), (u_2, [0.6, 0.9], 0.8), (u_3, [0.4, 0.6], 0.5)\}$,
$\beta(e_2, x_2) = \{(u_1, [0.3, 0.8], 0.4), (u_2, [0.6, 0.9], 0.7), (u_3, [0.5, 0.7], 0.6)\}$.

Cubic soft expert matrix $(CSEM)$ *of the Cubic soft expert set* $(CSES)$ *corresponding to expert* x_1 *is given as*

$$\ddot{B}_{x_1} = \begin{bmatrix} < [0.5, 0.8], 0.7 > & < [0.6, 0.9], 0.8 > & < [0.4, 0.7], 0.5 > \\ < [0.4, 0.7], 0.6 > & < [0.7, 0.9], 0.8 > & < [0.3, 0.5], 0.4 > \end{bmatrix}_{2\times 3}.$$

Cubic soft expert matrix $(CSEM)$ *of the Cubic soft expert set* $(CSES)$ *corresponding to expert* x_2 *is given as*

$$\ddot{B}_{x_2} = \begin{bmatrix} < [0.4, 0.8], 0.5 > & < [0.6, 0.9], 0.8 > & < [0.4, 0.6], 0.5 > \\ < [0.3, 0.8], 0.4 > & < [0.6, 0.9], 0.7 > & < [0.5, 0.7], 0.6 > \end{bmatrix}_{2\times 3}.$$

7.3.1 Operations on CSE Matrices

This subsection gives various operations defined on CSE matrices:

Definition 7.3.4 *Let* $\ddot{B}_0 = [a_{ij}]_{l\times m} \in CSEMs_{l\times m}$. $\ddot{B}_0$ *is called zero* CSE *matrix if for all* i *and* j $a_{ij} =< [0, 0], 0 >$.

Example 7.3.5 $\ddot{B}_0 = \begin{bmatrix} < [0,0], 0 > & < [0,0], 0 > & < [0,0], 0 > & < [0,0], 0 > \\ < [0,0], 0 > & < [0,0], 0 > & < [0,0], 0 > & < [0,0], 0 > \\ < [0,0], 0 > & < [0,0], 0 > & < [0,0], 0 > & < [0,0], 0 > \\ < [0,0], 0 > & < [0,0], 0 > & < [0,0], 0 > & < [0,0], 0 > \end{bmatrix}_{4\times 4}$.

Definition 7.3.6 *Let* $\ddot{B}_U = [a_{ij}]_{l\times m} \in CSEMs_{l\times m}$. $\ddot{B}_U$ *is called universe* CSE *matrix if for all* i *and* j $a_{ij} =< [1, 1], 1 >$.

Example 7.3.7 $\ddot{B}_U = \begin{bmatrix} <[1,1],1> & <[1,1],1> & <[1,1],1> & <[1,1],1> \\ <[1,1],1> & <[1,1],1> & <[1,1],1> & <[1,1],1> \\ <[1,1],1> & <[1,1],1> & <[1,1],1> & <[1,1],1> \\ <[1,1],1> & <[1,1],1> & <[1,1],1> & <[1,1],1> \end{bmatrix}_{4\times 4}$.

Definition 7.3.8 *Let* $\ddot{B}_N = [a_{ij}]_{l\times m} \in CSEMs_{l\times m}$. $\ddot{B}_N$ *is called null* CSE *matrix if for all* i *and* j $a_{ij} =< [1,1], 0 >$.

Example 7.3.9 $\ddot{B}_N = \begin{bmatrix} <[1,1],0> & <[1,1],0> & <[1,1],0> & <[1,1],0> \\ <[1,1],0> & <[1,1],0> & <[1,1],0> & <[1,1],0> \\ <[1,1],0> & <[1,1],0> & <[1,1],0> & <[1,1],0> \\ <[1,1],0> & <[1,1],0> & <[1,1],0> & <[1,1],0> \end{bmatrix}_{4\times 4}$.

Definition 7.3.10 *Let* $\ddot{B}_A = [a_{ij}]_{l\times m} \in CSEMs_{l\times m}$. $\ddot{B}_A$ *is called absolute* CSE *matrix if for all* i *and* j $a_{ij} =< [0,0], 1 >$.

Example 7.3.11 $\ddot{B}_A = \begin{bmatrix} <[0,0],1> & <[0,0],1> & <[0,0],1> & <[0,0],1> \\ <[0,0],1> & <[0,0],1> & <[0,0],1> & <[0,0],1> \\ <[0,0],1> & <[0,0],1> & <[0,0],1> & <[0,0],1> \\ <[0,0],1> & <[0,0],1> & <[0,0],1> & <[0,0],1> \end{bmatrix}_{4\times 4}$.

Definition 7.3.12 *Let* $\ddot{B}_{x_1} = [a_{ij}]_{l\times m}$ *where* $a_{ij} =< [A_1^-, A_1^+], \lambda_1 >$, $\ddot{B}_{x_2} = [b_{ij}]_{l\times m}$ *where* $b_{ij} =< [A_2^-, A_2^+], \lambda_2 > \in CSEMs_{l\times m}$.

Addition of two CSE *matrices is denoted and defined as* $\ddot{B}_{x_1} \oplus \ddot{B}_{x_2} = [a_{ij} \oplus b_{ij}]_{l\times m}$. *where* $a_{ij} \oplus b_{ij} =< [A_1^- + A_2^- - A_1^- A_2^-, A_1^+ + A_2^+ - A_1^+ A_2^+], \lambda_1 + \lambda_2 - \lambda_1\lambda_2 >$.

Example 7.3.13 *Let* $\ddot{B}_{x_1} = \begin{bmatrix} <[0.6,0.9],0.2> & <[0.2,0.3],0.1> \\ <[0.3,0.5],0.1> & <[0.4,0.5],0.3> \\ <[0.6,0.7],0.6> & <[0.6,0.7],0.5> \end{bmatrix}_{3\times 2}$, *and*

$\ddot{B}_{x_2} = \begin{bmatrix} <[0.7,0.9],0.5> & <[0.1,0.7],0.3> \\ <[0.8,0.9],0.7> & <[0.2,0.5],0.7> \\ <[0.2,0.7],0.6> & <[0.3,0.8],0.5> \end{bmatrix}_{3\times 2}$

then $\ddot{B}_{x_1} \oplus \ddot{B}_{x_2} = \begin{bmatrix} <[0.88,0.99],0.6> & <[0.28,0.79],0.37> \\ <[0.86,0.95],0.73> & <[0.52,0.75],0.79> \\ <[0.68,0.91],0.84> & <[0.72,0.94],0.75> \end{bmatrix}_{3\times 2}$.

Definition 7.3.14 *Let* $\ddot{B}_{x_1} = [a_{ij}]_{l\times m}$ *where* $a_{ij} =< [A_1^-, A_1^+], \lambda_1 >$, $\ddot{B}_{x_2} = [b_{ij}]_{l\times m}$ *where* $b_{ij} =< [A_2^-, A_2^+], \lambda_2 > \in CSEMs_{l\times m}$.

P-Min Product of two CSE matrices is denoted and defined as $\ddot{B}_{x_1} \wedge_P \ddot{B}_{x_2} = [a_{ij} \wedge_P b_{ij}]_{l\times m}$.

where $a_{ij} \wedge_P b_{ij} =< [A_1^- \wedge A_2^-,\ A_1^+ \wedge A_2^+], \lambda_1 \wedge \lambda_2 >$.

Example 7.3.15 *Consider* $\ddot{B}_{x_1}, \ddot{B}_{x_2}$ *of Example 7.3.13,*

$$\text{then } \ddot{B}_{x_1} \wedge_P \ddot{B}_{x_2} = \begin{bmatrix} <[0.6,0.9],0.2> & <[0.1,0.3],0.1> \\ <[0.3,0.5],0.1> & <[0.2,0.5],0.3> \\ <[0.2,0.7],0.6> & <[0.3,0.7],0.5> \end{bmatrix}_{3\times 2}.$$

Definition 7.3.16 *Let* $\ddot{B}_{x_1} = [a_{ij}]_{l\times m}$ *where* $a_{ij} =< [A_1^-,\ A_1^+],\ \lambda_1 >$, $\ddot{B}_{x_2} = [b_{ij}]_{l\times m}$ *where* $b_{ij} =< [A_2^-,\ A_2^+],\ \lambda_2 > \in CSEMs_{l\times m}$.

P-Max Product of two CSE matrices is denoted and defined as $\ddot{B}_{x_1} \vee_P \ddot{B}_{x_2} = [a_{ij} \vee_P b_{ij}]_{l\times m}$.

where $a_{ij} \vee_P b_{ij} =< [A_1^- \vee A_2^-,\ A_1^+ \vee A_2^+], \lambda_1 \vee \lambda_2 >$.

Example 7.3.17 *Consider* $\ddot{B}_{x_1}, \ddot{B}_{x_2}$ *of Example 7.3.13,*

$$\text{then } \ddot{B}_{x_1} \vee_p \ddot{B}_{x_2} = \begin{bmatrix} <[0.7,0.9],0.5> & <[0.2,0.7],0.3> \\ <[0.8,0.9],0.7> & <[0.4,0.5],0.7> \\ <[0.6,0.7],0.6> & <[0.6,0.8],0.5> \end{bmatrix}_{3\times 2}.$$

Definition 7.3.18 *Let* $\ddot{B}_{x_1} = [a_{ij}]_{l\times m}$ *where* $a_{ij} =< [A_1^-,\ A_1^+],\ \lambda_1 >$, $\ddot{B}_{x_2} = [b_{ij}]_{l\times m}$ *where* $b_{ij} =< [A_2^-,\ A_2^+],\ \lambda_2 > \in CSEMs_{l\times m}$.

R-Min Product of two CSE matrices is denoted and defined as $\ddot{B}_{x_1} \wedge_R \ddot{B}_{x_2} = [a_{ij} \wedge_R b_{ij}]_{l\times m}$.

where $a_{ij} \wedge_R b_{ij} =< [A_1^- \wedge A_2^-,\ A_1^+ \wedge A_2^+], \lambda_1 \vee \lambda_2 >$.

Example 7.3.19 *Consider* $\ddot{B}_{x_1}, \ddot{B}_{x_2}$ *of Example 7.3.13,*

$$\text{then } \ddot{B}_{x_1} \wedge_R \ddot{B}_{x_2} = \begin{bmatrix} <[0.6,0.9],0.5> & <[0.1,0.3],0.3> \\ <[0.3,0.5],0.7> & <[0.2,0.5],0.7> \\ <[0.2,0.7],0.6> & <[0.3,0.7],0.5> \end{bmatrix}_{3\times 2}.$$

Definition 7.3.20 *Let* $\ddot{B}_{x_1} = [a_{ij}]_{l\times m}$ *where* $a_{ij} =< [A_1^-,\ A_1^+],\ \lambda_1 >$, $\ddot{B}_{x_2} = [b_{ij}]_{l\times m}$ *where* $b_{ij} =< [A_2^-,\ A_2^+],\ \lambda_2 > \in CSEMs_{l\times m}$.

R-Max Product of two CSE matrices is denoted and defined as $\ddot{B}_{x_1} \vee_R \ddot{B}_{x_2} = [a_{ij} \vee_R b_{ij}]_{l\times m}$.

where $a_{ij} \vee_R b_{ij} =< [A_1^- \vee A_2^-,\ A_1^+ \vee A_2^+], \lambda_1 \wedge \lambda_2 >$.

Example 7.3.21 *Consider $\ddot{B}_{x_1}, \ddot{B}_{x_2}$ of Example 7.3.13,*

$$\text{then } \ddot{B}_{x_1} \vee_p \ddot{B}_{x_2} = \begin{bmatrix} <[0.7,0.9],0.2> & <[0.2,0.7],0.1> \\ <[0.8,0.9],0.1> & <[0.4,0.5],0.3> \\ <[0.6,0.7],0.6> & <[0.6,0.8],0.5> \end{bmatrix}_{3\times 2}.$$

Definition 7.3.22 *Let $\ddot{B}_{x_1} = [a_{ij}]_{l\times m}$ where $a_{ij} =< [A_1^-,\ A_1^+],\ \lambda_1 >$, $\ddot{B}_{x_2} = [b_{ij}]_{l\times m}$ where $b_{ij} =< [A_2^-,\ A_2^+],\ \lambda_2 > \in CSEMs_{l\times m}$.*

Product of two CSE matrices is denoted and defined as $\ddot{B}_{x_1} \otimes \ddot{B}_{x_2} = [a_{ij} \otimes b_{ij}]_{l\times m}$. where $a_{ij} \otimes b_{ij} =< [A_1^- A_2^-,\ A_1^+ A_2^+], \lambda_1\lambda_2 >$.

Example 7.3.23 *Consider $\ddot{B}_{x_1}, \ddot{B}_{x_2}$ of Example 7.3.13,*

$$\text{then } \ddot{B}_{x_1} \otimes \ddot{B}_{x_2} = \begin{bmatrix} <[0.42,0.81],0.10> & <[0.02,0.21],0.03> \\ <[0.24,0.45],0.07> & <[0.08,0.25],0.21> \\ <[0.12,0.49],0.36> & <[0.18,0.56],0.25> \end{bmatrix}_{3\times 2}.$$

Definition 7.3.24 *Let $\ddot{B}_{x_1} = [a_{ij}]_{l\times m}$ where $a_{ij} =< [A_1^-,\ A_1^+],\ \lambda_1 > \in CSEMs_{l\times m}$. Scalar Product of CSE matrices with arbitrary real number $k > 0$ is denoted and defined as $k\ddot{B}_{x_1} = [ka_{ij}]_{l\times m}$.*

where $ka_{ij} =< [1-(1-A_1^-)^k,\ 1-(1-A_1^+)^k], 1-(1-\lambda_1)^k >$.

Example 7.3.25 *Consider $\ddot{B}_{x_1}$ of Example 7.3.13, let $k = 0.4$*

$$\text{then } 0.4\ddot{B}_{x_1} = \begin{bmatrix} <[0.3068,0.6018],0.0853> & <[0.0853,0.1329],0.0412> \\ <[0.1329,0.2421],0.0412> & <[0.1848,0.2421],0.1329> \\ <[0.3068,0.3822],0.3068> & <[0.3068,0.3822],0.2421> \end{bmatrix}_{3\times 2}.$$

Definition 7.3.26 *Let $\ddot{B}_{x_1} = [a_{ij}]_{l\times m}$ where $a_{ij} =< [A_1^-,\ A_1^+],\ \lambda_1 > \in CSEMs_{l\times m}$.*

Power of CSE matrices with arbitrary real number $k > 0$ is denoted and defined as $(\ddot{B}_{x_1}) = [(a_{ij})^k]_{l\times m}$.

where $(a_{ij})^k =< [(A_1^-)^k,\ (A_1^+)^k], (\lambda_1)^k >$.

Example 7.3.27 *Consider $\ddot{B}_{x_1}$ of Example 7.3.13, let $k = 0.10$*

$$\text{then } (\ddot{B}_{x_1})^{0.10} = \begin{bmatrix} <[0.9502,0.9895],0.8513> & <[0.8513,0.8865],0.7943> \\ <[0.8865,0.9330],0.7943> & <[0.9124,0.9330],0.8865> \\ <[0.9502,0.9649],0.9502> & <[0.9502,0.9649],0.9330> \end{bmatrix}_{3\times 2}.$$

Definition 7.3.28 *Let $\ddot{B}_{x_1} = [a_{ij}]_{l\times m}$ where $a_{ij} =< [A_1^-,\ A_1^+],\ \lambda_1 > \in CSEMs_{l\times m}$.*

Complement of CSE matrices is denoted and defined as $(\ddot{B}_{x_1})^c = [(a_{ij})^c]_{l\times m}$. where $(a_{ij})^c =< [1-A_1^+,\ 1-A_1^-], 1-\lambda_1 >$.

Example 7.3.29 *Consider* $\ddot{B}_{x_1}$ *of Example 7.3.13, then*

$$(\ddot{B}_{x_1})^c = \begin{bmatrix} <[0.1, 0.4], 0.8> & <[0.7, 0.8], 0.9> \\ <[0.5, 0.7], 0.9> & <[0.5, 0.6], 0.7> \\ <[0.3, 0.4], 0.4> & <[0.3, 0.4], 0.5> \end{bmatrix}_{3\times 2}.$$

Definition 7.3.30 *Let* $\ddot{B}_{x_1} = [a_{ij}]_{l\times m}$, $\ddot{B}_{x_2} = [b_{ij}]_{l\times m} \in CSEMs_{l\times m}$. *Then* $\ddot{B}_{x_1} = [a_{ij}]_{l\times m}$ *is an* CSE *P-sub matrix of* $\ddot{B}_{x_2} = [b_{ij}]_{l\times m}$ *if* $a_{ij} \leq_P b_{ij}$, *that is,* $A_1 \leq A_2$ *and* $\lambda_1 \leq \lambda_2$ *where*

$a_{ij} =< A_1 = [A_1^-,\ A_1^+],\ \lambda_1 >$, $b_{ij} =< A_2 = [A_2^-,\ A_2^+],\ \lambda_2 >$ *for all* i *and* j. *It is denoted by* $\ddot{B}_{x_1} \Subset_P \ddot{B}_{x_2}$.

Definition 7.3.31 *Let* $\ddot{B}_{x_1} = [a_{ij}]_{l\times m}$, $\ddot{B}_{x_2} = [b_{ij}]_{l\times m} \in CSEMs_{l\times m}$. *Then* $\ddot{B}_{x_1} = [a_{ij}]_{l\times m}$ *is an* CSE *R-sub matrix of* $\ddot{B}_{x_2} = [b_{ij}]_{l\times m}$ *if* $a_{ij} \leq_R b_{ij}$, *that is,* $A_1 \leq A_2$ *and* $\lambda_1 \geq \lambda_2$ *where* $a_{ij} =< A_1 = [A_1^-, A_1^+], \lambda_1 >$, $b_{ij} =< A_2 = [A_2^-, A_2^+], \lambda_2' >$ *for all* i *and* j. *It is denoted by* $\ddot{B}_{x_1} \Subset_R \ddot{B}_{x_2}$.

7.3.2 Properties of CSE Matrices

In this subsection we check the properties and associative, commutative, distributive, De Morgans, double negation and involution laws of CSE matrices with respect to their operations.

Proposition 7.3.32 *Let* $\ddot{B}_{x_1} = [a_{ij}]_{l\times m}$, $\ddot{B}_0 = [b_{ij}]_{l\times m} \in CSEMs_{l\times m}$. *Then*

1) $\ddot{B}_{x_1} \oplus \ddot{B}_0 = \ddot{B}_{x_1}$
2) $\ddot{B}_{x_1} \otimes \ddot{B}_0 = \ddot{B}_0$
3) $\ddot{B}_0 \Subset_P \ddot{B}_{x_1}$
4) $(\ddot{B}_0)^c = \ddot{B}_U$.

Proof. Straightforward. ■

Proposition 7.3.33 *Let* $\ddot{B}_{x_1} = [a_{ij}]_{l\times m}$, $\ddot{B}_U = [b_{ij}]_{l\times m} \in CSEMs_{l\times m}$. *Then*

1) $\ddot{B}_{x_1} \oplus \ddot{B}_U = \ddot{B}_U$
2) $\ddot{B}_{x_1} \otimes \ddot{B}_U = \ddot{B}_{x_1}$
3) $\ddot{B}_{x_1} \Subset_P \ddot{B}_U$
4) $(\ddot{B}_U)^c = \ddot{B}_0$.

Proof. Straightforward. ■

The following proposition shows some of the properties of absolute $CSEM$, zero $CSEM$, universe $CSEM$ and null $CSEM$ with respect to the operation of P-Min Product, P-Max Product, R-Min Product, R-Max Product and complement.

Proposition 7.3.34 *Let* $\ddot{B}_N = [a_{ij}]_{l\times m}$, $\ddot{B}_A = [b_{ij}]_{l\times m}$,$\ddot{B}_U = [c_{ij}]_{l\times m}$, $\ddot{B}_0 = [d_{ij}]_{l\times m} \in CSEMs_{l\times m}$. *Then*

1) $(\ddot{B}_N)^c = \ddot{B}_A$
2) $(\ddot{B}_A)^c = \ddot{B}_N$
3) $(\ddot{B}_N)^c \vee_P \ddot{B}_N = \ddot{B}_U = (\ddot{B}_A)^c \vee_P \ddot{B}_A$
4) $(\ddot{B}_N)^c \wedge_P \ddot{B}_N = \ddot{B}_0 = (\ddot{B}_A)^c \wedge_P \ddot{B}_A$
5) $(\ddot{B}_N)^c \vee_R \ddot{B}_N = \ddot{B}_N = (\ddot{B}_A)^c \vee_R \ddot{B}_A$
6) $(\ddot{B}_N)^c \wedge_R \ddot{B}_N = \ddot{B}_A = (\ddot{B}_A)^c \vee_R \ddot{B}_A$
7) $(\ddot{B}_U)^c \vee_P \ddot{B}_U = \ddot{B}_U = (\ddot{B}_0)^c \vee_P \ddot{B}_0$
8) $(\ddot{B}_U)^c \wedge_P \ddot{B}_U = \ddot{B}_0 = (\ddot{B}_0)^c \wedge_P \ddot{B}_0$
9) $(\ddot{B}_U)^c \vee_R \ddot{B}_U = \ddot{B}_N = (\ddot{B}_0)^c \vee_R \ddot{B}_0$
10) $(\ddot{B}_U)^c \wedge_R \ddot{B}_U = \ddot{B}_A = (\ddot{B}_0)^c \vee_R \ddot{B}_0$.

Proof. Straightforward. ■

Proposition 7.3.35 *Let* $\ddot{B}_N = [a_{ij}]_{l\times m}$, $\ddot{B}_A = [b_{ij}]_{l\times m}$,$\ddot{B}_U = [c_{ij}]_{l\times m}$, $\ddot{B}_0 = [d_{ij}]_{l\times m} \in CSEMs_{l\times m}$. *Then*

1) $\ddot{B}_N \Subset_P \ddot{B}_U$
2) $\ddot{B}_U \Subset_R \ddot{B}_N$
3) $\ddot{B}_0 \Subset_P \ddot{B}_N$
4) $\ddot{B}_0 \Subset_R \ddot{B}_N$
5) $\ddot{B}_A \Subset_P \ddot{B}_U$
6) $\ddot{B}_A \Subset_R \ddot{B}_U$
7) $\ddot{B}_A \Subset_R \ddot{B}_0$
8) $\ddot{B}_0 \Subset_P \ddot{B}_A$
9) $\ddot{B}_A \Subset_R \ddot{B}_N$
10) $\ddot{B}_0 \Subset_P \ddot{B}_U$.

Proof. Straightforward. ■

Proposition 7.3.36 *Let* $\ddot{B}_N = [a_{ij}]_{l\times m}$, $\ddot{B}_A = [b_{ij}]_{l\times m}$,$\ddot{B}_U = [c_{ij}]_{l\times m}$, $\ddot{B}_0 = [d_{ij}]_{l\times m} \in CSEMs_{l\times m}$. *Then*

1) *If* $\ddot{B}_A \Subset_P \ddot{B}_U$ *and* $\ddot{B}_N \Subset_P \ddot{B}_U$ *then*
 a) $\ddot{B}_A \vee_P \ddot{B}_N \Subset_R \ddot{B}_U$
 b) $\ddot{B}_A \wedge_P \ddot{B}_N \Subset_R \ddot{B}_U$
 c) $\ddot{B}_A \vee_R \ddot{B}_N \Subset_R \ddot{B}_U$
 d) $\ddot{B}_A \wedge_R \ddot{B}_N \Subset_R \ddot{B}_U$
2) *If* $\ddot{B}_A \Subset_R \ddot{B}_U$ *and* $\ddot{B}_A \Subset_R \ddot{B}_N$ *then*
 a) $\ddot{B}_A \Subset_R \ddot{B}_U \wedge_P \ddot{B}_N$
 b) $\ddot{B}_A \Subset_R \ddot{B}_U \vee_P \ddot{B}_N$
 c) $\ddot{B}_A \Subset_R \ddot{B}_U \wedge_R \ddot{B}_N$
 d) $\ddot{B}_A \Subset_R \ddot{B}_U \vee_R \ddot{B}_N$
3) *If* $\ddot{B}_0 \Subset_P \ddot{B}_N$ *and* $\ddot{B}_0 \Subset_P \ddot{B}_U$ *then*
 a) $\ddot{B}_0 \Subset_P \ddot{B}_N \wedge_P \ddot{B}_U$
 b) $\ddot{B}_0 \Subset_P \ddot{B}_N \vee_P \ddot{B}_U$
 c) $\ddot{B}_0 \Subset_P \ddot{B}_N \wedge_R \ddot{B}_U$
 d) $\ddot{B}_0 \Subset_P \ddot{B}_N \vee_R \ddot{B}_U$
4) *If* $\ddot{B}_U \Subset_R \ddot{B}_N$ *and* $\ddot{B}_A \Subset_R \ddot{B}_N$ *then*
 a) $\ddot{B}_U \vee_P \ddot{B}_A \Subset_R \ddot{B}_N$
 b) $\ddot{B}_U \wedge_P \ddot{B}_A \Subset_R \ddot{B}_N$
 c) $\ddot{B}_U \vee_R \ddot{B}_A \Subset_R \ddot{B}_N$
 d) $\ddot{B}_U \wedge_R \ddot{B}_A \Subset_R \ddot{B}_N$.

Proof. Straightforward. ■

Proposition 7.3.37 *Let* $\ddot{B}_N = [a_{ij}]_{l\times m}$, $\ddot{B}_A = [b_{ij}]_{l\times m}$,$\ddot{B}_U = [c_{ij}]_{l\times m}$, $\ddot{B}_0 =$

$[d_{ij}]_{l\times m}$, $\ddot{B}_{x_1} = [e_{ij}]_{l\times m} \in CSEMs_{l\times m}$. *Then*

$$1)\ \ddot{B}_{x_1} \vee_P \ddot{B}_U = \ddot{B}_U$$
$$2)\ \ddot{B}_{x_1} \wedge_P \ddot{B}_U = \ddot{B}_{x_1}$$
$$3)\ \ddot{B}_{x_1} \vee_P \ddot{B}_0 = \ddot{B}_{x_1}$$
$$4)\ \ddot{B}_{x_1} \wedge_P \ddot{B}_0 = \ddot{B}_0$$
$$5)\ \ddot{B}_{x_1} \vee_R \ddot{B}_N = \ddot{B}_N$$
$$6)\ \ddot{B}_{x_1} \wedge_R \ddot{B}_N = \ddot{B}_{x_1}$$
$$7)\ \ddot{B}_{x_1} \vee_R \ddot{B}_A = \ddot{B}_{x_1}$$
$$8)\ \ddot{B}_{x_1} \wedge_R \ddot{B}_A = \ddot{B}_A.$$

Proof. Straightforward. ■

Next proposition shows some of the properties of absolute $CSEM$, zero $CSEM$, universe $CSEM$ and null $CSEM$ with respect to the operation of P-Min Product, P-Max Product, R-Min Product and R-Max Product.

Proposition 7.3.38 *Let* $\ddot{B}_N = [a_{ij}]_{l\times m}$, $\ddot{B}_A = [b_{ij}]_{l\times m}$,$\ddot{B}_U = [c_{ij}]_{l\times m}$, $\ddot{B}_0 = [d_{ij}]_{l\times m} \in CSEMs_{l\times m}$. *Then*

1) $\ddot{B}_N \vee_P \ddot{B}_A = \ddot{B}_U$	13) $\ddot{B}_A \vee_P \ddot{B}_U = \ddot{B}_U$
2) $\ddot{B}_N \wedge_P \ddot{B}_A = \ddot{B}_0$	14) $\ddot{B}_A \wedge_P \ddot{B}_U = \ddot{B}_A$
3) $\ddot{B}_N \vee_R \ddot{B}_A = \ddot{B}_N$	15) $\ddot{B}_A \vee_R \ddot{B}_U = \ddot{B}_U$
4) $\ddot{B}_N \wedge_R \ddot{B}_A = \ddot{B}_A$	16) $\ddot{B}_A \wedge_R \ddot{B}_U = \ddot{B}_A$
5) $\ddot{B}_N \vee_P \ddot{B}_U = \ddot{B}_U$	17) $\ddot{B}_A \vee_P \ddot{B}_0 = \ddot{B}_0$
6) $\ddot{B}_N \wedge_P \ddot{B}_U = \ddot{B}_N$	18) $\ddot{B}_A \wedge_P \ddot{B}_0 = \ddot{B}_0$
7) $\ddot{B}_N \vee_R \ddot{B}_U = \ddot{B}_N$	19) $\ddot{B}_A \vee_R \ddot{B}_0 = \ddot{B}_A$
8) $\ddot{B}_N \wedge_R \ddot{B}_U = \ddot{B}_U$	20) $\ddot{B}_A \wedge_R \ddot{B}_0 = \ddot{B}_A$
9) $\ddot{B}_N \vee_P \ddot{B}_0 = \ddot{B}_N$	21) $\ddot{B}_U \vee_P \ddot{B}_0 = \ddot{B}_U$
10) $\ddot{B}_N \wedge_P \ddot{B}_0 = \ddot{B}_0$	22) $\ddot{B}_U \wedge_P \ddot{B}_0 = \ddot{B}_0$
11) $\ddot{B}_N \vee_R \ddot{B}_0 = \ddot{B}_N$	23) $\ddot{B}_U \vee_R \ddot{B}_0 = \ddot{B}_N$
12) $\ddot{B}_N \wedge_R \ddot{B}_0 = \ddot{B}_0$	24) $\ddot{B}_U \wedge_P \ddot{B}_0 = \ddot{B}_A$

Proof. Straightforward. ■

Proposition 7.3.39 *Let* $\ddot{B}_{x_1} = [a_{ij}]_{l\times m}$, $\ddot{B}_{x_2} = [b_{ij}]_{l\times m} \in CSEMs_{l\times m}$. *Then commutative, involution with respect to P-Max Product, R-Max Product, R-Min Product*

P-Min Product and double negation laws hold:

$$\begin{array}{ll} 1)\ \ddot{B}_{x_1} \vee_P \ddot{B}_{x_2} = \ddot{B}_{x_2} \vee_P \ddot{B}_{x_1} & 6)\ \ddot{B}_{x_1} \vee_R \ddot{B}_{x_2} = \ddot{B}_{x_2} \vee_R \ddot{B}_{x_1} \\ 2)\ \ddot{B}_{x_1} \wedge_P \ddot{B}_{x_2} = \ddot{B}_{x_2} \wedge_P \ddot{B}_{x_1} & 7)\ \ddot{B}_{x_1} \wedge_R \ddot{B}_{x_2} = \ddot{B}_{x_2} \wedge_R \ddot{B}_{x_1} \\ 3)\ \ddot{B}_{x_1} \vee_P \ddot{B}_{x_1} = \ddot{B}_{x_1} & 8)\ \ddot{B}_{x_1} \vee_R \ddot{B}_{x_1} = \ddot{B}_{x_1} \\ 4)\ \ddot{B}_{x_1} \wedge_P \ddot{B}_{x_1} = \ddot{B}_{x_1} & 9)\ \ddot{B}_{x_1} \wedge_R \ddot{B}_{x_1} = \ddot{B}_{x_1}. \\ 5)\ ((\ddot{B}_{x_1})^c)^c = \ddot{B}_{x_1} & \end{array}$$

Proof. Straightforward. ■

The following result is very significant result. In ordinary matrices algebra product of two matrices is not commutative but in $CSEMs$ commutativity holds.

Proposition 7.3.40 *Let* $\ddot{B}_{x_1} = [a_{ij}]_{l\times m}$, $\ddot{B}_{x_2} = [b_{ij}]_{l\times m} \in CSEMs_{l\times m}$. *Then commutative law holds in* $CSEMs$ *with respect to the product operation.*

$$\ddot{B}_{x_1} \otimes \ddot{B}_{x_2} = \ddot{B}_{x_2} \otimes \ddot{B}_{x_1}$$

Proof. Consider $\ddot{B}_{x_1} = [a_{ij}]_{l\times m}$ where $a_{ij} =< [A_1^-,\ A_1^+],\ \lambda_1 >$, $\ddot{B}_{x_2} = [b_{ij}]_{l\times m}$ where $b_{ij} =< [A_2^-,\ A_2^+],\ \lambda_2 >$.

$\ddot{B}_{x_1} \otimes \ddot{B}_{x_2} =< [A_1^-,\ A_1^+],\ \lambda_1 > \oplus < [A_2^-,\ A_2^+],\ \lambda_2 >=< [A_1^- A_2^-,\ A_1^+ A_2^+],\ \lambda_1\lambda_2 >=< [A_2^- A_1^-,\ A_2^+ A_1^+],\ \lambda_2\lambda_1 >=< [A_2^-,\ A_2^+],\ \lambda_2 > \oplus < [A_1^-,\ A_1^+],\ \lambda_1 >= \ddot{B}_{x_2} \otimes \ddot{B}_{x_1}$. ■

The given below proposition shows that De Morgan's laws holds in $CSEMs$ under the operation of addition and product.

Proposition 7.3.41 *Let* $\ddot{B}_{x_1} = [a_{ij}]_{l\times m}$, $\ddot{B}_{x_2} = [b_{ij}]_{l\times m} \in CSEMs_{l\times m}$.

$$\begin{array}{l} 1)\ (\ddot{B}_{x_1} \oplus \ddot{B}_{x_2})^c = (\ddot{B}_{x_1})^c \otimes (\ddot{B}_{x_2})^c \\ 2)\ (\ddot{B}_{x_1} \otimes \ddot{B}_{x_2})^c = (\ddot{B}_{x_1})^c \oplus (\ddot{B}_{x_2})^c. \end{array}$$

Proof. Consider $\ddot{B}_{x_1} = [a_{ij}]_{l\times m}$ where $a_{ij} =< [A_1^-,\ A_1^+],\ \lambda_1 >$, $\ddot{B}_{x_2} = [b_{ij}]_{l\times m}$ where $b_{ij} =< [A_2^-,\ A_2^+],\ \lambda_2 >$.

1) $(\ddot{B}_{x_1} \oplus \ddot{B}_{x_2})^c = (< [A_1^-,\ A_1^+],\ \lambda_1 > \oplus < [A_2^-,\ A_2^+],\ \lambda_2 >)^c = (< [A_1^- + A_2^- - A_1^- A_2^-,\ A_1^+ + A_2^+ - A_1^+ A_2^+],\ \lambda_1 + \lambda_2 - \lambda_1\lambda_2 >)^c = < [1 - (A_1^+ + A_2^+ - A_1^+ A_2^+),\ 1 - (A_1^- + A_2^- - A_1^- A_2^-)],\ 1 - (\lambda_1 + \lambda_2 - \lambda_1\lambda_2) > = < [1 - A_1^+ - A_2^+ + A_1^+ A_2^+,\ 1 - A_1^- - A_2^- + A_1^- A_2^-],\ 1 - \lambda_1 - \lambda_2 + \lambda_1\lambda_2 > = < [(1 - A_1^+)(1 - A_2^+),\ (1 - A_1^-)(1 - A_2^-)],\ (1 - \lambda_1)(1 - \lambda_2) >=< [1 - A_1^+,\ 1 - A_1^-],\ (1 - \lambda_1) > \otimes < [1 - A_2^+,\ 1 - A_2^-],\ (1 - \lambda_2) > = (< [A_1^-,\ A_1^+],\ \lambda_1 >)^c \otimes < ([A_2^-,\ A_2^+],\ \lambda_2 >)^c = (\ddot{B}_{x_1})^c \otimes (\ddot{B}_{x_2})^c$.

2) $(\ddot{B}_{x_1} \otimes \ddot{B}_{x_2})^c = (< [A_1^-, A_1^+], \lambda_1 > \otimes < [A_2^-, A_2^+], \lambda_2 >)^c = (< [A_1^- A_2^-, A_1^+ A_2^+], \lambda_1\lambda_2 >)^c = < [1 - A_1^+ A_2^+, 1 - A_1^- A_2^-], 1 - \lambda_1\lambda_2 > = < [1 + 1 - 1 + A_1^+ - A_1^+ + A_2^+ - A_2^+ - A_1^+ A_2^+,$

$1 + 1 - 1 + A_1^- - A_1^- + A_2^- - A_2^- - A_1^- A_2^-], 1 + 1 - 1 + \lambda_1 - \lambda_1 + \lambda_2 - \lambda_2 - \lambda_1\lambda_2 > = < [2 - A_1^+ - A_2^+ - 1 + A_1^+ + A_2^+ - A_1^+ A_2^+, 2 - A_1^- - A_2^- - 1 + A_1^- + A_2^- - A_1^- A_2^-],$

$2 - \lambda_1 - \lambda_2 - 1 + \lambda_1 + \lambda_2 - \lambda_1\lambda_2 > = < [1 - A_1^+ + 1 - A_2^+ - (1 - A_1^+)(1 - A_2^+), 1 - A_1^- + 1 - A_2^- - (1 - A_1^-)(1 - A_2^-)], 1 - \lambda_1 + 1 - \lambda_2 - (1 - \lambda_1)(1 - \lambda_2) > = < [1 - A_1^+, 1 - A_1^-], 1 - \lambda_1 > \oplus < [1 - A_2^+, 1 - A_2^-], 1 - \lambda_2 > = (< [A_1^-, A_1^+], \lambda_1 >)^c \oplus (< [A_2^-, A_2^+], \lambda_2 >)^c = (\ddot{B}_{x_1})^c \oplus (\ddot{B}_{x_2})^c.$ ■

Remark 7.3.42 *Distributive law with respect to addition over product do not hold in $CSEMs$.*

Example 7.3.43 *Let* $\ddot{B}_{x_1} = [< [0.2,\ 0.6],\ 0.5 >]_{1\times 1}$, $\ddot{B}_{x_2} = [< [0.1,\ 0.4],\ 0.3 >]_{1\times 1}$ *and* $\ddot{B}_{x_3} = [< [0.4,\ 0.7],\ 0.9 >]_{1\times 1}$.

$\ddot{B}_{x_1} \oplus (\ddot{B}_{x_2} \otimes \ddot{B}_{x_3}) = [< [0.2, 0.6], 0.5 >] \oplus ([< [0.1, 0.4], 0.3 >] \otimes [< [0.4, 0.7], 0.9 >]) = [< [0.2, 0.6], 0.5 >] \oplus [< [0.04, 0.28], 0.27 >] = [< [0.232, 0.712], 0.635 >].$

$(\ddot{B}_{x_1} \oplus \ddot{B}_{x_2}) \otimes (\ddot{B}_{x_1} \oplus \ddot{B}_{x_3}) = ([< [0.2, 0.6], 0.5 >] \oplus [< [0.1, 0.4], 0.3 >]) \otimes ([< [0.2, 0.6], 0.5 >] \oplus [< [0.4, 0.7], 0.9 >]) = [< [0.28, 0.76], 0.65 >] \otimes [< [0.52, 0.88], 0.95 >] = [< [0.1456, 0.6688], 0.6175 >]$. *Hence* $\ddot{B}_{x_1} \oplus (\ddot{B}_{x_2} \otimes \ddot{B}_{x_3}) \neq (\ddot{B}_{x_1} \oplus \ddot{B}_{x_2}) \otimes (\ddot{B}_{x_1} \oplus \ddot{B}_{x_3})$.

Proposition 7.3.44 *Let* $\ddot{B}_{x_1} = [a_{ij}]_{l\times m}$, $\ddot{B}_{x_2} = [b_{ij}]_{l\times m} \in CSEMs_{l\times m}$.

1) *if* $\ddot{B}_{x_1} \Subset_P \ddot{B}_{x_2}$ *then*
a) $\ddot{B}_{x_1} \vee_P \ddot{B}_{x_2} = \ddot{B}_{x_2}$
b) $\ddot{B}_{x_1} \wedge_P \ddot{B}_{x_2} = \ddot{B}_{x_1}$
2) *if* $\ddot{B}_{x_1} \Subset_R \ddot{B}_{x_2}$ *then*
a) $\ddot{B}_{x_1} \vee_R \ddot{B}_{x_2} = \ddot{B}_{x_2}$
b) $\ddot{B}_{x_1} \wedge_R \ddot{B}_{x_2} = \ddot{B}_{x_1}$

Proof. Consider $\ddot{B}_{x_1} = [a_{ij}]_{l\times m}$ where $a_{ij} =< [A_1^-, A_1^+], \lambda_1 >$, $\ddot{B}_{x_2} = [b_{ij}]_{l\times m}$ where $b_{ij} =< [A_2^-, A_2^+], \lambda_2 >$.

1*a*) Since $\ddot{B}_{x_1} \Subset_P \ddot{B}_{x_2} \Rightarrow A_1^- \leq A_2^-, A_1^+ \leq A_2^+, \lambda_1 \leq \lambda_2$. Now $\ddot{B}_{x_1} \vee_P \ddot{B}_{x_2} =< [A_1^-, A_1^+], \lambda_1 > \vee_P < [A_2^-, A_2^+], \lambda_2 >=< [A_1^- \vee A_2^-, A_1^+ \vee A_2^+], \lambda_1 \vee \lambda_2 >=< [A_2^-, A_2^+], \lambda_2 >= \ddot{B}_{x_2}$.

1*b*) Now $\ddot{B}_{x_1} \wedge_P \ddot{B}_{x_2} =< [A_1^-, A_1^+], \lambda_1 > \wedge_P < [A_2^-, A_2^+], \lambda_2 >=< [A_1^- \wedge A_2^-, A_1^+ \wedge A_2^+], \lambda_1 \wedge \lambda_2 >=< [A_1^-, A_1^+], \lambda_1 >= \ddot{B}_{x_1}$.

$2a)$ Since $\ddot{B}_{x_1} \Subset_R \ddot{B}_{x_2} \Rightarrow A_1^- \leq A_2^-$, $A_1^+ \leq A_2^+$, $\lambda_1 \geq \lambda_2$. Now $\ddot{B}_{x_1} \vee_R \ddot{B}_{x_2} =< [A_1^-, A_1^+], \lambda_1 > \vee_R < [A_2^-, A_2^+], \lambda_2 >=< [A_1^- \vee A_2^-, A_1^+ \vee A_2^+], \lambda_1 \wedge \lambda_2 >=< [A_2^-, A_2^+], \lambda_2 >= \ddot{B}_{x_2}$.

$2b)$ Now $\ddot{B}_{x_1} \wedge_R \ddot{B}_{x_2} =< [A_1^-, A_1^+], \lambda_1 > \wedge_R < [A_2^-, A_2^+], \lambda_2 >=< [A_1^- \wedge A_2^-, A_1^+ \wedge A_2^+], \lambda_1 \vee \lambda_2 >=< [A_1^-, A_1^+], \lambda_1 >= \ddot{B}_{x_1}$. ∎

Proposition 7.3.45 *Let* $\ddot{B}_{x_1} = [a_{ij}]_{l\times m}$, $\ddot{B}_{x_2} = [b_{ij}]_{l\times m}$ *and* $\ddot{B}_{x_3} = [c_{ij}]_{l\times m} \in CSEMs_{l\times m}$. *Then Associative, De Morgan's and distributive laws also hold with respect to P-Max Product, R-Max Product, R-Min Product and P-Min Product:*

1) $(\ddot{B}_1 \vee_P \ddot{B}_{x_2}) \vee_P \ddot{B}_{x_3} = \ddot{B}_{x_1} \vee_p (\ddot{B}_{x_2} \vee_P \ddot{B}_{x_3})$
2) $(\ddot{B}_{x_1} \wedge_P \ddot{B}_{p_2}) \wedge_P \ddot{B}_{x_3} = \ddot{B}_{x_1} \wedge_P (\ddot{B}_{x_2} \wedge_P \ddot{B}_{x_3})$
3) $(\ddot{B}_1 \vee_R \ddot{B}_{x_2}) \vee_R \ddot{B}_{x_3} = \ddot{B}_{x_1} \vee_R (\ddot{B}_{x_2} \vee_R \ddot{B}_{x_3})$
4) $(\ddot{B}_{x_1} \wedge_R \ddot{B}_{p_2}) \wedge_R \ddot{B}_{x_3} = \ddot{B}_{x_1} \wedge_R (\ddot{B}_{x_2} \wedge_R \ddot{B}_{x_3})$
5) $(\ddot{B}_{x_1} \vee_P \ddot{B}_{x_2})^c = (\ddot{B}_{x_1})^c \wedge_P (\ddot{B}_{x_2})^c$
6) $(\ddot{B}_{x_1} \vee_R \ddot{B}_{x_2})^c = (\ddot{B}_{x_1})^c \wedge_R (\ddot{B}_{x_2})^c$
7) $(\ddot{B}_{x_1} \wedge_P \ddot{B}_{x_2})^c = (\ddot{B}_{x_1})^c \vee_P (\ddot{B}_{x_2})^c$
8) $(\ddot{B}_{x_1} \wedge_R \ddot{B}_{x_2})^c = (\ddot{B}_{x_1})^c \vee_R (\ddot{B}_{x_2})^c$
9) $\ddot{B}_{x_1} \wedge_P (\ddot{B}_{x_2} \vee_P \ddot{B}_{x_3}) = (\ddot{B}_{x_1} \wedge_P \ddot{B}_{x_2}) \vee_P (\ddot{B}_{x_1} \wedge_P \ddot{B}_{x_3})$
10) $\ddot{B}_{x_1} \wedge_R (\ddot{B}_{x_2} \vee_R \ddot{B}_{x_3}) = (\ddot{B}_{x_1} \wedge_R \ddot{B}_{x_2}) \vee_R (\ddot{B}_{x_1} \wedge_R \ddot{B}_{x_3})$
11) $\ddot{B}_{x_1} \vee_P (\ddot{B}_{x_2} \wedge_P \ddot{B}_{x_3}) = (\ddot{B}_{x_1} \vee_P \ddot{B}_{x_2}) \wedge_P (\ddot{B}_{x_1} \vee_P \ddot{B}_{x_3})$
12) $\ddot{B}_{x_1} \vee_R (\ddot{B}_{x_2} \wedge_R \ddot{B}_{x_3}) = (\ddot{B}_{x_1} \vee_R \ddot{B}_{x_2}) \wedge_R (\ddot{B}_{x_1} \vee_R \ddot{B}_{x_3})$.

Proof. Straightforward. ∎

7.4 Matrix Algebra of Interval Valued Intuitionistic Fuzzy Soft Expert Sets ($IVIFSESs$)

In this section, we define matrices algebra for interval valued intuitionistic fuzzy soft expert sets. Further we discuss some operations on it and investigate several properties with respect to their operations.

Definition 7.4.1 *Let* U *be the initial universe,* A *be the set of attributes and* G *be the set of experts. Interval valued intuitionistic fuzzy soft expert set* ($IVIFSE$ *set*) *is a triplet* (ξ, A, G) *which is characterized by mapping* $\xi : A \times G \longrightarrow K_I(U)$ *where the set of* *the interval-valued intuitionistic fuzzy sets on the universe set U is denoted by $Ķ_I(U)$. For $b \in Ą$ and $p \in Ģ$ we define*

$$\xi(b,p) = \{< u,\ [\gamma^-_{(b,p)}(u), \gamma^+_{(b,p)}(u)], [\zeta^-_{(b,p)}(u), \zeta^+_{(b,p)}(u)] >: u \in U\}.$$

Definition 7.4.2 *Let U be the initial universe, E; a set of criteria and X; a set of experts (or decision makers). $\xi(b,p) = \{< u,\ [\gamma^-_{(b,p)}(u), \gamma^+_{(b,p)}(u)], [\zeta^-_{(b,p)}(u), \zeta^+_{(b,p)}(u)] >: u \in U,\ (b,p) \in E \times X$ be the interval-valued intuitionistic fuzzy soft expert set.*

p	u_1	u_2	u_3	...	u_m
b_1	(b_1,u_1)	(b_1,u_2)	(b_1,u_3)	...	(b_1,u_m)
b_2	(b_2,u_1)	(b_2,u_2)	(b_2,u_3)	...	(b_2,u_m)
b_3	(b_3,u_1)	(b_3,u_2)	(b_3,u_3)	...	(b_3,u_m)
...	...	...	...	...	...
b_l	(b_l,u_1)	(b_l,u_2)	(b_l,u_3)	...	(b_l,u_m)

Table 7.4.1

If a_{ij} = opinion of expert p corresponding to pair $(b_i,u_j) =< [\gamma^-,\gamma^+],[\zeta^-,\zeta^+] >$, we can define $\ddot{C}_x = [a_{ij}]_{l\times m} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{21} \\ \dots & \dots & \dots & \dots \\ a_{l1} & a_{l2} & \dots & a_{lm} \end{bmatrix}$.

it is called interval-valued intuitionistic fuzzy soft expert matrix ($IVIFSEM$) of the interval-valued intuitionistic fuzzy soft expert set ($IVIFSES$) of order $l \times m$ over U.

Collection of all interval-valued intuitionistic fuzzy soft expert matrices ($IVIFSE$ matrices) is denoted by $IVIFSEMs_{l\times m}$.

Example 7.4.3 *Let $U = \{u_1, u_2, u_3\}$ be the initial universe. $Ą = \{b_1, b_2\}$ be the set of attributes and $Ģ = \{p_1, p_2\}$ be the set of experts. Then we can view the $IVIFSE$ Set $(\xi, Ą, Ģ)$ as consisting of opinions of experts subject to the given attributes following collection of approximations:*

$\xi(b_1,p_1) = \{< u_1,\ [0.4,0.5],[0.2,0.4] >, < u_2,\ [0.1,0.5],[0.4,0.5] >, < u_3,\ [0.3,0.4],[0.4,0.5] >,$

$\xi(b_2,p_1) = \{< u_1,\ [0.2,0.4],[0.5,0.6] >, < u_2,\ [0.2,0.4],[0.4,0.5] >, < u_3,\ [0.1,0.3],[0.3,0.5] >,$

$\xi(b_1,p_2) = \{< u_1,\ [0.4,0.7],[0.2,0.3] >, < u_2,\ [0.2,0.5],[0.2,0.3] >, < u_3,\ [0.3,0.5],[0.4,0.5] >,$

$\xi(b_2,p_2) = \{< u_1,\ [0.3,0.5],[0.0,0.3] >, < u_2,\ [0.1,0.4],[0.3,0.6] >, < u_3,\ [0.2, 0.5], [0.3, 0.4] >\}.$

Interval-valued intuitionistic fuzzy soft expert matrix ($IVIFSEM$) of the interval-valued intuitionistic fuzzy soft expert set ($IVIFSES$) corresponding to expert p_1 is given as

$$\ddot{C}_{p_1} = \begin{bmatrix} <[0.4,0.5],[0.2,0.4]> & <[0.1,0.5],[0.4,0.5]> & <[0.3,0.4],[0.4,0.5]> \\ <[0.2,0.4],[0.5,0.6]> & <[0.2,0.4],[0.4,0.5]> & <[0.1,0.3],[0.3,0.5]> \end{bmatrix}_{2\times 3}.$$

Interval-valued intuitionistic fuzzy soft expert matrix ($IVIFSEM$) of the interval-valued intuitionistic fuzzy soft expert set ($IVIFSES$) corresponding to expert p_2 is given as

$$\ddot{C}_{p_2} = \begin{bmatrix} <[0.4,0.7],[0.2,0.3]> & <[0.2,0.5],[0.2,0.3]> & <[0.3,0.5],[0.4,0.5]> \\ <[0.3,0.5],[0.0,0.3]> & <[0.1,0.4],[0.3,0.6]> & <[0.2,0.5],[0.3,0.4]> \end{bmatrix}_{2\times 3}.$$

7.4.1 Operations on $IVIFSE$ Matrices

In this section, we define operations of addition, product , Min-Product, Max-Product, complement, power and scalar product of $IVIFSE$ matrices.

Definition 7.4.4 *Let $\ddot{C}_N = [a_{ij}]_{l\times m} \in IVIFSEMs_{l\times m}$. $\ddot{C}_N$ is called null $IVIFSE$ matrix if for all i and j $a_{ij} =<[1,1],[0,0]>$.*

Example 7.4.5 $\ddot{C}_N = \begin{bmatrix} <[1,1],[0,0]> & <[1,1],[0,0]> & <[1,1],[0,0]> \\ <[1,1],[0,0]> & <[1,1],[0,0]> & <[1,1],[0,0]> \\ <[1,1],[0,0]> & <[1,1],[0,0]> & <[1,1],[0,0]> \end{bmatrix}_{3\times 3}.$

Definition 7.4.6 *Let $\ddot{C}_A = [a_{ij}]_{l\times m} \in CSEMs_{l\times m}$. $\ddot{C}_A$ is called absolute $IVIFSE$ matrix if for all i and j $a_{ij} =<[0,0],[1,1]>$.*

Example 7.4.7 $\ddot{C}_A = \begin{bmatrix} <[0,0],[1,1]> & <[0,0],[1,1]> & <[0,0],[1,1]> \\ <[0,0],[1,1]> & <[0,0],[1,1]> & <[0,0],[1,1]> \\ <[0,0],[1,1]> & <[0,0],[1,1]> & <[0,0],[1,1]> \end{bmatrix}_{3\times 3}.$

Definition 7.4.8 *Let $\ddot{C}_{p_1} = [a_{ij}]_{l\times m}$ where $a_{ij} =<[\gamma_1^-, \gamma_1^+],[\zeta_1^-, \zeta_1^+]>$, $\ddot{C}_{p_2} = [b_{ij}]_{l\times m}$ where $b_{ij} <[\gamma_2^-, \gamma_2^+],[\zeta_2^-, \zeta_2^+]> \in IVIFSEMs_{l\times m}$.*

Addition of two $IVIFSE$ matrices is denoted and defined as $\ddot{C}_{p_1} \oplus \ddot{C}_{p_2} = [a_{ij} \oplus b_{ij}]_{l\times m}$.

where $a_{ij} \oplus b_{ij} =<[\gamma_1^- + \gamma_2^- - \gamma_1^-\gamma_2^-, \gamma_1^+ + \gamma_2^+ - \gamma_1^+\gamma_2^+],[\zeta_1^-\zeta_2^-, \zeta_1^+\zeta_2^+]>$.

Example 7.4.9 *Let* $\ddot{C}_{p_1} = \begin{bmatrix} <[0.6,0.9],[0.0,0.1]> & <[0.2,0.3],[0.1,0.6]> \\ <[0.3,0.5],[0.1,0.4]> & <[0.4,0.5],[0.3,0.5]> \\ <[0.6,0.7],[0.2,0.3]> & <[0.6,0.7],[0.1,0.2]> \end{bmatrix}_{3\times 2},$

and

$$\ddot{C}_{p_2} = \begin{bmatrix} <[0.7,0.9],[0.0,0.1]> & <[0.1,0.7],[0.2,0.3]> \\ <[0.8,0.9],[0.0,0.1]> & <[0.2,0.5],[0.3,0.4]> \\ <[0.2,0.7],[0.2,0.3]> & <[0.3,0.4],[0.2,0.5]> \end{bmatrix}_{3\times 2}$$

$$then\ \ddot{C}_{p_1} \oplus \ddot{C}_{p_2} = \begin{bmatrix} <[0.88,0.99],[0.0,0.01]> & <[0.28,0.79],[0.02,0.18]> \\ <[0.86,0.95],[0.0,0.04]> & <[0.52,0.75],[0.09,.20]> \\ <[0.68,0.91],[0.04,0.09]> & <[0.72,0.82],[0.02,0.10]> \end{bmatrix}_{3\times 2}.$$

Definition 7.4.10 *Let* $\ddot{C}_{p_1} = [a_{ij}]_{l\times m}$ *where* $a_{ij} = <[\gamma_1^-, \gamma_1^+],[\zeta_1^-, \zeta_1^+]>$, $\ddot{C}_{p_2} = [b_{ij}]_{l\times m}$ *where* $b_{ij} <[\gamma_2^-, \gamma_2^+],[\zeta_2^-, \zeta_2^+]> \in IVIFSEMs_{l\times m}$.

Product of two IVIFSE matrices is denoted and defined as $\ddot{C}_{p_1} \otimes \ddot{C}_{p_2} = [a_{ij} \otimes b_{ij}]_{l\times m}$.

where $a_{ij} \otimes b_{ij} = <[\gamma_1^-\gamma_2^-, \gamma_1^+\gamma_2^+],[\zeta_1^- + \zeta_2^- - \zeta_1^-\zeta_2^-, \zeta_1^+ + \zeta_2^+ - \zeta_1^+\zeta_2^+]>$.

Example 7.4.11 *Consider* $\ddot{C}_{p_1}$, $\ddot{C}_{p_2}$ *of Example 7.4.9 then*

$$\ddot{C}_{p_1} \otimes \ddot{C}_{p_2} = \begin{bmatrix} <[0.42,0.81],[0.0,0.19]> & <[0.02,0.21],[0.28,0.72]> \\ <[0.24,0.45],[0.0,0.46]> & <[0.08,0.25],[0.51,0.70]> \\ <[0.12,0.49],[0.36,0.51]> & <[0.18,0.28],[0.28,0.60]> \end{bmatrix}_{3\times 2}.$$

Definition 7.4.12 *Let* $\ddot{C}_{p_1} = [a_{ij}]_{l\times m}$ *where* $a_{ij} = <[\gamma_1^-, \gamma_1^+],[\zeta_1^-, \zeta_1^+]>$, $\ddot{C}_{p_2} = [b_{ij}]_{l\times m}$ *where* $b_{ij} = <[\gamma_2^-, \gamma_2^+],[\zeta_2^-, \zeta_2^+]> \in IVIFSEMs_{l\times m}$.

Min-Product of two IVIFSE matrices is denoted and defined as $\ddot{C}_{p_1} \barwedge \ddot{C}_{p_2} = [a_{ij} \barwedge b_{ij}]_{l\times m}$.

where $a_{ij} \barwedge b_{ij} = <[\gamma_1^- \wedge \gamma_2^-, \gamma_1^+ \wedge \gamma_2^+],[\zeta_1^- \vee \zeta_2^-, \zeta_1^+ \vee \zeta_2^+]>$.

Example 7.4.13 *Consider* $\ddot{C}_{p_1}$, $\ddot{C}_{p_2}$ *of Example 7.4.9 then*

$$\ddot{C}_{p_1} \barwedge \ddot{C}_{p_2} = \begin{bmatrix} <[0.6,0.9],[0.0,0.1]> & <[0.1,0.3],[0.2,0.6]> \\ <[0.3,0.5],[0.1,0.4]> & <[0.2,0.5],[0.3,0.5]> \\ <[0.2,0.7],[0.2,0.3]> & <[0.3,0.4],[0.2,0.5]> \end{bmatrix}_{3\times 2}.$$

Definition 7.4.14 *Let* $\ddot{C}_{p_1} = [a_{ij}]_{l\times m}$ *where* $a_{ij} = <[\gamma_1^-, \gamma_1^+],[\zeta_1^-, \zeta_1^+]>$, $\ddot{C}_{p_2} = [b_{ij}]_{l\times m}$ *where* $b_{ij} <[\gamma_2^-, \gamma_2^+],[\zeta_2^-, \zeta_2^+]> \in IVIFSEMs_{l\times m}$.

Max-Product of two IVIFSE matrices is denoted and defined as $\ddot{C}_{p_1} \veebar \ddot{C}_{p_2} = [a_{ij} \veebar b_{ij}]_{l\times m}$.

where $a_{ij} \veebar b_{ij} = <[\gamma_1^- \vee \gamma_2^-, \gamma_1^+ \vee \gamma_2^+],[\zeta_1^- \wedge \zeta_2^-, \zeta_1^+ \wedge \zeta_2^+]>$.

Example 7.4.15 *Consider* $\ddot{C}_{p_1}$, $\ddot{C}_{p_2}$ *of Example 7.4.9 then*

$$\ddot{C}_{p_1} \veebar \ddot{C}_{p_2} = \begin{bmatrix} <[0.7,0.9],[0.0,0.1]> & <[0.2,0.7],[0.1,0.3]> \\ <[0.8,0.9],[0.0,0.1]> & <[0.4,0.5],[0.3,0.4]> \\ <[0.6,0.7],[0.2,0.3]> & <[0.6,0.7],[0.1,0.2]> \end{bmatrix}_{3\times 2}.$$

Definition 7.4.16 *Let* $\ddot{C}_{p_1} = [a_{ij}]_{l\times m}$ *where* $a_{ij} =< [\gamma_1^-, \gamma_1^+], [\zeta_1^-, \zeta_1^+] > \in IVIFSEMs_{l\times m}$.

Product of $IVIFSE$ *matrices with an arbitrary real number* $k > 0$ *is denoted and defined as* $k\ddot{C}_{p_1} = [ka_{ij}]_{l\times m}$.

where $ka_{ij} =< [1-(1-\gamma_1^-)^k, 1-(1-\gamma_1^+)^k], [(\zeta_1^-)^k, (\zeta_1^+)^k] >$.

Example 7.4.17 *Consider* $\ddot{C}_{p_1}$ *of Example 7.4.9 then let* $k = 0.7$*and then*

$$k\ddot{C}_{p_1} = \begin{bmatrix} <[0.4734, 0.8004], [0.0, 0.1995]> & <[0.1446, 0.2209], [0.1995, 0.6993]> \\ <[0.2209, 0.3844], [0.0, 0.1995]> & <[0.3006, 0.3844], [0.4305, 0.5265]> \\ <[0.4734, 0.5694], [0.3241, 0.4305]> & <[0.4734, 0.5694], [0.1995, 0.3241]> \end{bmatrix}_{3\times 2}.$$

Definition 7.4.18 *Let* $\ddot{C}_{p_1} = [a_{ij}]_{l\times m}$ *where* $a_{ij} =< [\gamma_1^-, \gamma_1^+], [\zeta_1^-, \zeta_1^+] > \in IVIFSEMs_{l\times m}$. *Complement of* $IVIFSE$ *matrices is denoted and defined as* $(\ddot{C}_{p_1})^c = [(a_{ij})^c]_{l\times m}$.

where $(a_{ij})^c =< [(\zeta_1^-)^k, (\zeta_1^+)^k, [(\gamma_1^-)^k, (\gamma_1^+)^k]] >$.

Example 7.4.19 *Consider* $\ddot{C}_{p_1}$ *of Example 7.4.9 then*

$$(\ddot{C}_{p_1})^c = \begin{bmatrix} <[0.0, 0.1], [0.6, 0.9]> & <[0.1, 0.6], [0.2, 0.3]> \\ <[0.1, 0.4], [0.3, 0.5]> & <[0.3, 0.5], [0.4, 0.5]> \\ <[0.2, 0.3].[0.6, 0.7]> & <[0.1, 0.2], [0.6, 0.7]> \end{bmatrix}_{3\times 2}.$$

Definition 7.4.20 *Let* $\ddot{C}_{p_1} = [a_{ij}]_{l\times m}$ *where* $a_{ij} =< [\gamma_1^-, \gamma_1^+], [\zeta_1^-, \zeta_1^+] > \in IVIFSEMs_{l\times m}$. *Power of* $IVIFSE$ *matrices with an arbitrary real number* $k > 0$ *is denoted and defined as* $(\ddot{C}_{p_1})^k = [(a_{ij})^k]_{l\times m}$.

where $(a_{ij})^k =< [(\gamma_1^-)^k, (\gamma_1^+)^k], [1-(1-\zeta_1^-)^k, 1-(1-\zeta_1^+)^k] >$.

Example 7.4.21 *Consider* $\ddot{C}_{p_1}$ *of Example 7.4.9 let* $k = 0.37$ *and then*

$$(\ddot{C}_{p_1})^{0.37} = \begin{bmatrix} <[0.8277, 0.9617], [0.0, 0.0382]> & <[0.5512, 0.6405], [0.0382, 0.2875]> \\ <[0.6405, 0.7737], [0.0382, 0.1722]> & <[0.7124, 0.7737], [0.1236, 0.2262]> \\ <[0.8277, 0.8763], [0.0792, 0.1236]> & <[0.8277, 0.8763], [0.0382, 0.0792]> \end{bmatrix}_{3\times 2}.$$

Definition 7.4.22 *Let* $\ddot{C}_{p_1} = [a_{ij}]_{l\times m}$ *where* $a_{ij} =< [\gamma_1^-, \gamma_1^+], [\zeta_1^-, \zeta_1^+] >$, $\ddot{C}_{p_2} = [b_{ij}]_{l\times m}$ *where* $b_{ij} =< [\gamma_2^-, \gamma_2^+], [\zeta_2^-, \zeta_2^+] > \in IVIFSEMs_{l\times m}$.

Then $\ddot{C}_{p_1} = [a_{ij}]_{l\times m}$ *is an* $IVIFSE$ *sub matrix of* $\ddot{C}_{p_2} = [b_{ij}]_{l\times m}$ *if* $a_{ij} \leq b_{ij}$ *for all* i *and* j, *that is,* $\gamma_1^- \leq \gamma_2^-$, $\gamma_1^+ \leq \gamma_2^+$ *and* $\zeta_1^- \geq \zeta_2^-$, $\zeta_1^+ \geq \zeta_2^+$.*It is denoted by* $\ddot{C}_{p_1} \Subset \ddot{C}_{p_2}$.

7.4.2 Properties of $IVIFSE$ Matrices

In this section we check the properties and associative, commutative, distributive, De Morgans, double negation and involution laws of CSE matrices with respect to their operations.

Proposition 7.4.23 *Let* $\ddot{C}_{p_1} = [a_{ij}]_{l\times m}$, $\ddot{C}_N = [b_{ij}]_{l\times m} \in IVIFSEMs_{l\times m}$. *Then*

$$1)\ \ddot{C}_{p_1} \oplus \ddot{C}_N = \ddot{C}_N$$
$$2)\ \ddot{C}_{p_1} \otimes \ddot{C}_N = \ddot{C}_{p_1}.$$

Proof. Straightforward. ■

Proposition 7.4.24 *Let* $\ddot{C}_{p_1} = [a_{ij}]_{l\times m}$, $\ddot{C}_A = [b_{ij}]_{l\times m} \in IVIFSEMs_{l\times m}$. *Then*

$$1)\ \ddot{C}_{p_1} \oplus \ddot{C}_A = \ddot{C}_{p_1}$$
$$2)\ \ddot{C}_{p_1} \otimes \ddot{C}_A = \ddot{C}_A.$$

Proof. Straightforward. ■

Proposition 7.4.25 *Let* $\ddot{B}_{x_1} = [a_{ij}]_{l\times m}$, $\ddot{B}_{x_2} = [b_{ij}]_{l\times m} \in CSEMs_{l\times m}$. *Then commutative law holds in* $IVIFSEMs$ *with respect to the product operation.*

$$\ddot{C}_{p_1} \otimes \ddot{C}_{p_2} = \ddot{C}_{p_2} \otimes \ddot{C}_{p_1}$$

Proof. Consider $\ddot{C}_{p_1} = [a_{ij}]_{l\times m}$ where $a_{ij} =< [\gamma_1^-, \gamma_1^+], [\zeta_1^-, \zeta_1^+] >$, $\ddot{C}_{p_2} = [b_{ij}]_{l\times m}$ where $b_{ij} =< [\gamma_2^-, \gamma_2^+], [\zeta_2^-, \zeta_2^+] >$.

$\ddot{C}_{p_1} \otimes \ddot{C}_{p_2} =< [\gamma_1^-, \gamma_1^+], [\zeta_1^-, \zeta_1^+] > \otimes < [\gamma_2^-, \gamma_2^+], [\zeta_2^-, \zeta_2^+] > =< [\gamma_2^-\gamma_1^-, \gamma_2^+\gamma_1^+], [\zeta_2^- + \zeta_1^- - \zeta_2^-\zeta_1^-, \zeta_2^+ + \zeta_1^+ - \zeta_2^+\zeta_1^+] > =< [\gamma_2^-, \gamma_2^+], [\zeta_2^-, \zeta_2^+] > \otimes < [\gamma_1^-, \gamma_1^+], [\zeta_1^-, \zeta_1^+] > = \ddot{C}_{p_2} \otimes \ddot{C}_{p_1}$. ■

Proposition 7.4.26 *Let* $\ddot{C}_{p_1} = [a_{ij}]_{l\times m}$, $\ddot{C}_{p_2} = [b_{ij}]_{l\times m} \in IVIFSEMs_{l\times m}$. *Then De Morgan's laws with respect to addition and product holds:*

$$1)\ (\ddot{C}_{p_1} \oplus \ddot{C}_{p_2})^c = (\ddot{C}_{p_1})^c \otimes (\ddot{C}_{p_2})^c$$
$$2)\ (\ddot{C}_{p_1} \otimes \ddot{C}_{p_2})^c = (\ddot{C}_{p_1})^c \oplus (\ddot{C}_{p_2})^c$$

Proof. Consider $\ddot{C}_{p_1} = [a_{ij}]_{l\times m}$ where $a_{ij} =< [\gamma_1^-, \gamma_1^+], [\zeta_1^-, \zeta_1^+] >$, $\ddot{C}_{p_2} = [b_{ij}]_{l\times m}$ where $b_{ij} =< [\gamma_2^-, \gamma_2^+], [\zeta_2^-, \zeta_2^+] >$.

1) $(\ddot{C}_{p_1} \oplus \ddot{C}_{p_2})^c = (< [\gamma_1^-, \gamma_1^+], [\zeta_1^-, \zeta_1^+] > \oplus < [\gamma_2^-, \gamma_2^+], [\zeta_2^-, \zeta_2^+] >)^c = (< [\gamma_1^- + \gamma_2^- - \gamma_1^-\gamma_2^-, \gamma_1^+ + \gamma_2^+ - \gamma_1^+\gamma_2^+], [\zeta_1^-\zeta_2^-, \zeta_1^+\zeta_2^+] >)^c =< [\zeta_1^-\zeta_2^-, \zeta_1^+\zeta_2^+], [\gamma_1^- + \gamma_2^- - \gamma_1^-\gamma_2^-, \gamma_1^+ + \gamma_2^+ - \gamma_1^+\gamma_2^+] > =< [\zeta_1^-, \zeta_1^+], [\gamma_1^-, \gamma_1^+] > \otimes < [\zeta_2^-, \zeta_2^+], [\gamma_2^-, \gamma_2^+] > = (\ddot{C}_{p_1})^c \otimes (\ddot{C}_{p_2})^c$.

2) $(\ddot{C}_{p_1} \otimes \ddot{C}_{p_2})^c = (< [\gamma_1^-, \gamma_1^+], [\zeta_1^-, \zeta_1^+] > \otimes < [\gamma_2^-, \gamma_2^+], [\zeta_2^-, \zeta_2^+] >)^c = (< [\gamma_1^-\gamma_2^-, \gamma_1^+\gamma_2^+], [\zeta_1^- + \zeta_2^- - \zeta_1^-\zeta_2^-, \zeta_1^+ + \zeta_2^+ - \zeta_1^+\zeta_2^+] >)^c =< [\zeta_1^- + \zeta_2^- - \zeta_1^-\zeta_2^-, \zeta_1^+ + \zeta_2^+ - \zeta_1^+\zeta_2^+], [\gamma_1^-\gamma_2^-, \gamma_1^+\gamma_2^+] > = (< [\zeta_1^-, \zeta_1^+], [\gamma_1^-, \gamma_1^+] > \oplus < [\zeta_2^-, \zeta_2^+], [\gamma_2^-, \gamma_2^+] >) = (\ddot{C}_{p_1})^c \oplus (\ddot{C}_{p_2})^c$.

■

Remark 7.4.27 *Distributive law with respect to addition over product do not hold in $IVIFSEMs$.*

Example 7.4.28 *Let* $\ddot{C}_{p_1} = [< [0.2,\ 0.6],\ [0.2,\ 0.4] >]_{1\times 1}$, $\ddot{C}_{p_2} = [< [0.1,\ 0.4],\ [0.3,\ 0.5] >]_{1\times 1}$ *and* $\ddot{C}_{p_3} = [< [0.4,\ 0.7],\ [0.1,\ 0.3] >]_{1\times 1}$.

$\ddot{C}_{p_1} \oplus (\ddot{C}_{p_2} \otimes \ddot{C}_{p_3}) = [< [0.2, 0.6], [0.2, 0.4] >] \oplus ([< [0.1, 0.4], [0.3, 0.5] >] \otimes [< [0.4, 0.7], [0.1, 0.3] >]) = [< [0.2, 0.6], [0.2, 0.4] >] \oplus [< [0.04, 0.28], [0.37, 0.65] >] = [< [0.232, 0.712], [0.074, 0.26] >]$.

$(\ddot{C}_{p_1} \oplus \ddot{C}_{p_2}) \otimes (\ddot{C}_{p_1} \oplus \ddot{C}_{p_3}) = ([< [0.2, 0.6], [0.2, 0.4] >] \oplus [< [0.1, 0.4], [0.3, 0.5] >]) \otimes ([< [0.2, 0.6], [0.2, 0.4] >] \oplus [< [0.4, 0.7], [0.1, 0.3] >]) = [< [0.28, 0.76], [0.06, 0.20] >] \otimes [< [0.52, 0.88], [0.02, 0.12] >] = [< [0.1456, 0.6688], [0.0788, 0.296] >]$.
Hence $\ddot{C}_{p_1} \oplus (\ddot{C}_{p_2} \otimes \ddot{C}_{p_3}) \neq (\ddot{C}_{p_1} \oplus \ddot{C}_{p_2}) \otimes (\ddot{C}_{p_1} \oplus \ddot{C}_{p_3})$.

Next proposition shows some of the properties of absolute $IVIFSEM$ and null $IVIFSEM$ with respect to the operation of addition, product, Min- Product and Max-Product.

Proposition 7.4.29 *Let* $\ddot{C}_N = [b_{ij}]_{l\times m}$, $\ddot{C}_A = [c_{ij}]_{l\times m} \in IVIFSEMs_{l\times m}$. *Then*

$$1)\ \ddot{C}_A \oplus \ddot{C}_N = \ddot{C}_N$$
$$2)\ \ddot{C}_A \otimes \ddot{C}_N = \ddot{C}_A$$
$$3)\ \ddot{C}_A \underline{\vee} \ddot{C}_N = \ddot{C}_N$$
$$4)\ \ddot{C}_A \overline{\wedge} \ddot{C}_N = \ddot{C}_A.$$

Proof. Straightforward. ■

Proposition 7.4.30 *Let* $\ddot{C}_{p_1} = [a_{ij}]_{l\times m}$, $\ddot{C}_{p_2} = [b_{ij}]_{l\times m} \in IVIFSEMs_{l\times m}$. *Then commutative, involution and double negation laws hold:*

$$1)\ \ddot{C}_{p_1} \underline{\vee} \ddot{C}_{p_2} = \ddot{C}_{p_2} \underline{\vee} \ddot{C}_{p_1}$$
$$2)\ \ddot{C}_{p_1} \overline{\wedge} \ddot{C}_{p_2} = \ddot{C}_{p_2} \overline{\wedge} \ddot{C}_{p_1}$$
$$3)\ \ddot{C}_{p_1} \oplus \ddot{C}_{p_2} = \ddot{C}_{p_2} \oplus \ddot{C}_{p_1}$$
$$4)\ \ddot{C}_{p_1} \otimes \ddot{C}_{p_2} = \ddot{C}_{p_2} \otimes \ddot{C}_{p_1}$$
$$5)\ \ddot{C}_{p_1} \underline{\vee} \ddot{C}_{p_1} = \ddot{C}_{p_1}$$
$$6)\ \ddot{C}_{p_1} \overline{\wedge} \ddot{C}_{p_1} = \ddot{C}_{p_1}$$
$$7)\ ((\ddot{C}_{p_1})^c)^c = \ddot{C}_{p_1}.$$

Proof. Consider $\ddot{C}_{p_1} = [a_{ij}]_{l\times m}$ where $a_{ij} =< [\gamma_1^-, \gamma_1^+], [\zeta_1^-, \zeta_1^+] >$, $\ddot{C}_{p_2} = [b_{ij}]_{l\times m}$ where $b_{ij} =< [\gamma_2^-, \gamma_2^+], [\zeta_2^-, \zeta_2^+] >$.

1) $\ddot{C}_{p_1} \veebar \ddot{C}_{p_2} = (< [\gamma_1^-, \gamma_1^+], [\zeta_1^-, \zeta_1^+] > \veebar < [\gamma_2^-, \gamma_2^+], [\zeta_2^-, \zeta_2^+] >) = < [\gamma_1^- \vee \gamma_2^-, \gamma_1^+ \vee \gamma_2^+], [\zeta_1^- \wedge \zeta_2^-, \zeta_1^+ \wedge \zeta_2^+] > = < [\gamma_2^- \vee \gamma_1^-, \gamma_2^+ \vee \gamma_1^+], [\zeta_2^- \wedge \zeta_1^-, \zeta_2^+ \wedge \zeta_1^+] > = (< [\gamma_2^-, \gamma_2^+], [\zeta_2^-, \zeta_2^+] > \veebar < [\gamma_1^-, \gamma_1^+], [\zeta_1^-, \zeta_1^+] >) = \ddot{C}_{p_2} \veebar \ddot{C}_{p_1}$.

2) $\ddot{C}_{p_1} \barwedge \ddot{C}_{p_2} = (< [\gamma_1^-, \gamma_1^+], [\zeta_1^-, \zeta_1^+] > \barwedge < [\gamma_2^-, \gamma_2^+], [\zeta_2^-, \zeta_2^+] >) = < [\gamma_1^- \wedge \gamma_2^-, \gamma_1^+ \wedge \gamma_2^+], [\zeta_1^- \vee \zeta_2^-, \zeta_1^+ \vee \zeta_2^+] > = < [\gamma_2^- \wedge \gamma_1^-, \gamma_2^+ \wedge \gamma_1^+], [\zeta_2^- \vee \zeta_1^-, \zeta_2^+ \vee \zeta_1^+] > = (< [\gamma_2^-, \gamma_2^+], [\zeta_2^-, \zeta_2^+] > \barwedge < [\gamma_1^-, \gamma_1^+], [\zeta_1^-, \zeta_1^+] >) = \ddot{C}_{p_2} \veebar \ddot{C}_{p_1}$.

In a similar manner we can prove the rest of the result by using U of addition, product, Min-Product, Max-Product and complement of matrices. ■

Proposition 7.4.31 *Let* $\ddot{C}_{p_1} = [a_{ij}]_{l\times m}$, $\ddot{C}_{p_2} = [b_{ij}]_{l\times m}$ *and* $\ddot{C}_{p_3} = [c_{ij}]_{l\times m} \in IVIFSEMs_{l\times m}$. *Then Associative, De Morgan's and distributive laws also hold with respect to Max-Product and Min-Product:*

$$
\begin{aligned}
&1)\ (\ddot{C}_{p_1} \veebar \ddot{C}_{p_2}) \veebar \ddot{C}_{p_3} = \ddot{C}_{p_1} \veebar (\ddot{C}_{p_2} \veebar \ddot{C}_{p_3})\\
&2)\ (\ddot{C}_{p_1} \barwedge \ddot{C}_{p_2}) \barwedge \ddot{C}_{p_3} = \ddot{C}_{p_1} \barwedge (\ddot{C}_{p_2} \barwedge \ddot{C}_{p_3})\\
&3)\ (\ddot{C}_{p_1} \veebar \ddot{C}_{p_2})^c = (\ddot{C}_{p_1})^c \barwedge (\ddot{C}_{p_2})^c\\
&4)\ (\ddot{C}_{p_1} \barwedge \ddot{C}_{p_2})^c = (\ddot{C}_{p_1})^c \veebar (\ddot{C}_{p_2})^c\\
&5)\ \ddot{C}_{p_1} \veebar (\ddot{C}_{p_2} \barwedge \ddot{C}_{p_3}) = (\ddot{C}_{p_1} \veebar \ddot{C}_{p_2}) \barwedge (\ddot{C}_{p_1} \veebar \ddot{C}_{p_3})\\
&6)\ \ddot{C}_{p_1} \barwedge (\ddot{C}_{p_2} \veebar \ddot{C}_{p_3}) = (\ddot{C}_{p_1} \barwedge \ddot{C}_{p_2}) \veebar (\ddot{C}_{p_1} \barwedge \ddot{C}_{p_3})
\end{aligned}
$$

Proof. Consider $\ddot{C}_{p_1} = [a_{ij}]_{l\times m}$ where $a_{ij} =< [\gamma_1^-, \gamma_1^+], [\zeta_1^-, \zeta_1^+] >$, $\ddot{C}_{p_2} = [b_{ij}]_{l\times m}$ where $b_{ij} =< [\gamma_2^-, \gamma_2^+], [\zeta_2^-, \zeta_2^+] >$, $\ddot{C}_{p_3} = [c_{ij}]_{l\times m}$ where $c_{ij} =< [\gamma_3^-, \gamma_3^+], [\zeta_3^-, \zeta_3^+] >$.

1) $(\ddot{C}_{p_1} \veebar \ddot{C}_{p_2}) \veebar \ddot{C}_{p_3} = (< [\gamma_1^-, \gamma_1^+], [\zeta_1^-, \zeta_1^+] > \veebar < [\gamma_2^-, \gamma_2^+], [\zeta_2^-, \zeta_2^+] >) \veebar < [\gamma_3^-, \gamma_3^+], [\zeta_3^-, \zeta_3^+] > = < [\gamma_1^- \vee \gamma_2^-, \gamma_1^+ \vee \gamma_2^+], [\zeta_1^- \wedge \zeta_2^-, \zeta_1^+ \wedge \zeta_2^+] > \veebar < [\gamma_3^-, \gamma_3^+], [\zeta_3^-, \zeta_3^+] >$ $=$

$< [(\gamma_1^- \vee \gamma_2^-) \vee \gamma_3^-, (\gamma_1^+ \vee \gamma_2^+) \vee \gamma_3^+], [(\zeta_1^- \wedge \zeta_2^-) \wedge \zeta_3^-, (\zeta_1^+ \wedge \zeta_2^+) \wedge \zeta_3^+] > = < [\gamma_1^- \vee (\gamma_2^- \vee \gamma_3^-), \gamma_1^+ \vee (\gamma_2^+ \vee \gamma_3^+)], [\zeta_1^- \wedge (\zeta_2^- \wedge \zeta_3^-), \zeta_1^+ \wedge (\zeta_2^+ \wedge \zeta_3^+)] > = < [\gamma_1^-, \gamma_1^+], [\zeta_1^-, \zeta_1^+] > \veebar < [\gamma_2^- \vee \gamma_3^-, \gamma_2^+ \vee \gamma_3^+], [\zeta_2^- \wedge \zeta_3^-, \zeta_2^+ \wedge \zeta_3^+] > = < [\gamma_1^-, \gamma_1^+], [\zeta_1^-, \zeta_1^+] > \veebar (< [\gamma_2^-, \gamma_2^+], [\zeta_2^-, \zeta_2^+] > \veebar < [\gamma_3^-, \gamma_3^+], [\zeta_3^-, \zeta_3^+] >) = \ddot{C}_{p_1} \veebar (\ddot{C}_{p_2} \veebar \ddot{C}_{p_3})$.

2) Similar as above.

3) $(\ddot{C}_{p_1} \veebar \ddot{C}_{p_2})^c = (< [\gamma_1^-, \gamma_1^+], [\zeta_1^-, \zeta_1^+] > \veebar < [\gamma_2^-, \gamma_2^+], [\zeta_2^-, \zeta_2^+] >)^c = (< [\gamma_1^- \vee \gamma_2^-, \gamma_1^+ \vee \gamma_2^+], [\zeta_1^- \wedge \zeta_2^-, \zeta_1^+ \wedge \zeta_2^+] >)^c = < [\zeta_1^- \wedge \zeta_2^-, \zeta_1^+ \wedge \zeta_2^+], [\gamma_1^- \vee \gamma_2^-, \gamma_1^+ \vee \gamma_2^+] > = (< [\zeta_1^-, \zeta_1^+], [\gamma_1^-, \gamma_1^+] > \barwedge < [\zeta_2^-, \zeta_2^+], [\gamma_2^-, \gamma_2^+] >) = (\ddot{C}_{p_1})^c \barwedge (\ddot{C}_{p_2})^c$.

4) $(\ddot{C}_{p_1} \bar{\wedge} \ddot{C}_{p_2})^c = (< [\gamma_1^-, \gamma_1^+], [\zeta_1^-, \zeta_1^+] > \bar{\wedge} < [\gamma_2^-, \gamma_2^+], [\zeta_2^-, \zeta_2^+] >)^c = (< [\gamma_1^- \wedge \gamma_2^-, \gamma_1^+ \wedge \gamma_2^+], [\zeta_1^- \vee \zeta_2^-, \zeta_1^+ \vee \zeta_2^+] >)^c = < [\zeta_1^- \vee \zeta_2^-, \zeta_1^+ \vee \zeta_2^+], [\gamma_1^- \wedge \gamma_2^-, \gamma_1^+ \wedge \gamma_2^+] > = (< [\zeta_1^-, \zeta_1^+], [\gamma_1^-, \gamma_1^+] > \underline{\vee} < [\zeta_2^-, \zeta_2^+], [\gamma_2^-, \gamma_2^+] >) = (\ddot{C}_{p_1})^c \underline{\vee} (\ddot{C}_{p_2})^c$.

5) $\ddot{C}_{p_1} \underline{\vee} (\ddot{C}_{p_2} \bar{\wedge} \ddot{C}_{p_3}) = < [\gamma_1^-, \gamma_1^+], [\zeta_1^-, \zeta_1^+] > \underline{\vee} (< [\gamma_2^-, \gamma_2^+], [\zeta_2^-, \zeta_2^+] > \bar{\wedge} < [\gamma_3^-, \gamma_3^+], [\zeta_3^-, \zeta_3^+] >) = < [\gamma_1^-, \gamma_1^+], [\zeta_1^-, \zeta_1^+] > \underline{\vee} < [\gamma_2^- \wedge \gamma_3^-, \gamma_2^+ \wedge \gamma_3^+], [\zeta_2^- \vee \zeta_3^-, \zeta_2^+ \vee \zeta_3^+] >$
$=$

$< [\gamma_1^- \vee (\gamma_2^- \wedge \gamma_3^-), \gamma_1^+ \vee (\gamma_2^+ \wedge \gamma_3^+)], [\zeta_1^- \wedge (\zeta_2^- \vee \zeta_3^-), \zeta_1^+ \wedge (\zeta_2^+ \vee \zeta_3^+)] > = < [(\gamma_1^- \vee \gamma_2^-) \wedge (\gamma_1^- \vee \gamma_3^-), (\gamma_1^+ \vee \gamma_2^+) \wedge (\gamma_1^+ \vee \gamma_3^+)], [(\zeta_1^- \wedge \zeta_2^-) \vee (\zeta_1^- \wedge \zeta_3^-), (\zeta_1^+ \wedge \zeta_2^+) \vee (\zeta_1^+ \wedge \zeta_3^+)] >$
$=$

$< [(\gamma_1^- \vee \gamma_2^-), (\gamma_1^+ \vee \gamma_2^+)], [(\zeta_1^- \wedge \zeta_2^-), (\zeta_1^+ \wedge \zeta_2^+)] > \bar{\wedge} < [(\gamma_1^- \vee \gamma_3^-), (\gamma_1^+ \vee \gamma_3^+)], [(\zeta_1^- \wedge \zeta_3^-), (\zeta_1^+ \wedge \zeta_3^+)] > = (< [\gamma_1^-, \gamma_1^+], [\zeta_1^-, \zeta_1^+] > \underline{\vee} < [\gamma_2^-, \gamma_2^+], [\zeta_2^-, \zeta_2^+] >) \bar{\wedge} (< [\gamma_1^-, \gamma_1^+], [\zeta_1^-, \zeta_1^+] > \underline{\vee} < [\gamma_3^-, \gamma_3^+], [\zeta_3^-, \zeta_3^+] >) = (\ddot{C}_{p_1} \underline{\vee} \ddot{C}_{p_2}) \bar{\wedge} (\ddot{C}_{p_1} \underline{\vee} \ddot{C}_{p_3})$.

6) Similar as above. ■

The following proposition states some properties of null $IVIFSEM$ with $IVIFSEM$ under the operation of Min-Product, Max-Product and complement.

Proposition 7.4.32 *Let* $\ddot{C}_{p_1} = [a_{ij}]_{l\times m}$, $\ddot{C}_N = [b_{ij}]_{l\times m} \in IVIFSEMs_{l\times m}$. *Then*

1) $\ddot{C}_{p_1} \underline{\vee} \ddot{C}_N = \ddot{C}_N$

2) $\ddot{C}_{p_1} \bar{\wedge} \ddot{C}_N = \ddot{C}_{p_1}$

3) $(\ddot{C}_N)^c = \ddot{C}_A$.

Proof. Straightforward. ■

The following proposition states some properties of absolute $IVIFSEM$ with $IVIFSEM$ under the operation of Min-Product, Max-Product and complement.

Proposition 7.4.33 *Let* $\ddot{C}_{p_1} = [a_{ij}]_{l\times m}$, $\ddot{C}_A = [b_{ij}]_{l\times m} \in IVIFSEMs_{l\times m}$. *Then*

1) $\ddot{C}_{p_1} \underline{\vee} \ddot{C}_A = \ddot{C}_{p_1}$

2) $\ddot{C}_{p_1} \bar{\wedge} \ddot{C}_A = \ddot{C}_A$

3) $(\ddot{C}_A)^c = \ddot{C}_N$.

Proof. Straightforward. ■

The next proposition states the inclusion relation between absolute $IVIFSEM$, null $IVIFSEM$ and $IVIFSEM$.

Proposition 7.4.34 *Let* $\ddot{C}_{p_1} = [a_{ij}]_{l\times m}$, $\ddot{C}_A = [b_{ij}]_{l\times m}$, $\ddot{C}_N = [c_{ij}]_{l\times m} \in IVIFSEMs_{l\times m}$. *Then*

1) $\ddot{C}_A \Subset \ddot{C}_N$

2) $\ddot{C}_{p_1} \Subset \ddot{C}_N$

3) $\ddot{C}_A \Subset \ddot{C}_{p_1}$.

Proof. Straightforward. ■

7.5 Conclusion and Future Work

In this chapter, matrix algebra has been defined for generalizations of soft expert sets. In previous chapters we have been defined the modified and the generalized form of soft expert sets. In this concept we have been given expert opinions in the form of GSE sets CSE sets and $IVIFSE$ sets. All of these structures have some specifications. we can easily represent past or future opinions of experts in one type of structure. By using these structures we have been developed matrix algebra. This type of representation is very useful in multicriteria decision making analysis. So for that purpose we have been defined GSE matrices, CSE matrices and $IVIFSE$ matrices. Also we have been defined some operations on these matrices. These matrices also satisfied some of the properties which were not hold in ordinary matrix algebra, for example, commutativity of matrices with respect to product operation. There are also a very fruitful results which hold in all type of matrices which were discussed in this chapter, that is, De Morgan's laws holds with respect to addition over product, product over addition, Min Product over Max Product, Max Product over Min Product, P-Min Product over P-Max Product. P-Max Product over P-Min Product, R-Min Product over R-Max Product. and R-Max Product over R-Min Product. Distributive law with respect to addition over product do not holds in $GSEMs$, $CSEMs$ and $IVIFSEMs$. But holds with respect to Min Product over Max Product, Max Product over Min Product, P-Min Product over P-Max Product. P-Max Product over P-Min Product, R-Min Product over R-Max Product. and R-Max Product over R-Min Product. In future we aim to develop some algorithms on these matrices for multicriteria decision making problems. Also we will develop some programming on it. We will also use this representation in AHP, ANP and $TOPSIS$.

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